Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

The conjugated vs. the unconjugated infinite element method for the Helmholtz equation in exterior domains

K. Gerdes

Research Report No. 96-11
July 1996

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

The conjugated vs. the unconjugated infinite element method for the Helmholtz equation in exterior domains

K. Gerdes

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 96-11 July 1996

Dedicated to Prof. J. Tinsley Oden on the occasion of his 60th birthday

Abstract

This work is devoted to a convergence study of infinite element (IE) discretizations for the Helmholtz equation in exterior domains. The different behavior of the *conjugated* and the *unconjugated* IE formulation is analyzed in context of

1.) a formulation following a mathematical existence theory by Leis,

2.) a formulation not based on an available existence theory following Burnett.

Four variational formulations are presented and the differences in implementing them are discussed. The effect of using or not using the *complex conjugate* in the weak formulation is carefully studied. The numerical and theoretical results clearly indicate which of the four presented formulations is the most efficient that can still give reliable results.

1 Introduction

This paper is motivated by the new concept on various infinite elements by Burnett [3], Astley et al. [1], Babuska and Shirron [2] and our earlier work on infinite elements [7, 5, 8].

The central problem deals with the scattering of acoustic waves on elastic or rigid (the simplified case) objects. The mathematical formulation consists of the Helmholtz equation in the exterior domain accompanied by the Sommerfeld radiation condition and Neumann boundary condition on the boundary of the scatterer (rigid scattering). In the elastic scattering case the Helmholtz equation is coupled to equations describing the behavior of the structure (elasto- or viscoelasto-dynamics).

One popular method to solve the scattering problem is to replace the Helmholtz equation *and* the Sommerfeld radiation condition with an *equivalent* boundary integral equation, giving rise later on to an appropriate variational formulation and a Boundary Element (BE) approximation. The boundary equation can be formulated directly on the surface of the scatterer or some auxiliary smooth surface surrounding the object. In the latter case the BE approximation on the truncating surface has to be coupled with a Finite Element (FE) approximation for the interior domain - in between the scatterer and the surface.

This approach, although more expensive, avoids problems with the integration of singular kernels arising from BE approximations on non-smooth boundaries (corners, edges) or degeneration of the formulation in the case of non-convex scatterers (screen-like problems). Implementations can be highly specialized for a fixed class of smooth truncating boundaries.

In both cases mentioned above, whether the BEM is applied to the surface of the scatterer or an auxiliary surface, the application of the BE approach turns out expensive for large wave numbers. Numerical examples for the rigid scattering with $k = 20$ using the BEM on a parallel machine were described in [6]. The rigid scattering problem on a unit sphere with $k = 20$ was solved in [7] using the IEM. In spite of the important differences among the BEM and the IEM –the first works for arbitrary geometries, while the latter, semi-analytic, is specialized for spherical domains– a comparison is useful to illustrate the issue of cost and explains why we have been motivated to further study the subject.

The specific IE approach presented in [3] is not based on a mathematical theory, but Burnett's numerical results indicate that the method works well. The key to the success in his method is, that the weak formulation is symmetric (in the real sense) and that he does not use the complex conjugate over the second argument in the bilinear form. Babuska and Shirron [2] have shown that the method converges with this setting near the surface of the scatterer, but that the far field solution diverges. Therefore, the far field solution has to be computed with the Helmholtz integral formula, whereas the conjugated IEM delivers a reliable solution in the whole computational domain.

The differences that have been reported about the various IEM have motivated the present study. The goal is to compare the different formulations on the same benchmark problem, the scattering of a plane wave on the surface of the unit sphere. Further, we will analyze the convergence properties of the IEM and compare them with existing results. The main merrit of this paper will be to give an overview of the convergence properties of the existing IEM and to state which of the formulations is based on a mathematical theory. The analysis presented here will allow the scientist to choose the optimal IEM for a given application.

The content of this paper is outlined as follows. In section 2 the rigid scattering problem and the various IE formulations are introduced and numerical results for the rigid scattering on a unit sphere are presented. Section 3 presents an analysis of the stability of the methods, followed by an analysis of the convergence properties in sections 4 and 5, and the conclusions in section 6.

2 IEM for the Helmholtz Equation in the Exterior Spherical Domain

We begin with a discussion of the exterior boundary-value problem for the Helmholtz equation. In particular, we present the four possible variational formulations mentioned above, and discuss the corresponding modifications in the element computations.

2.1 Classical Formulation of the Problem

Given domain $\Omega^e \subset \mathbb{R}^3$, exterior to the unit sphere, we wish to find a function $u = u(\boldsymbol{x})$ satisfying:

- the Helmholtz equation in the exterior domain

$$-\Delta u - k^2 u = 0 \qquad \text{in} \quad \Omega^e, \tag{2.1}$$

where k is the wave number,

- a Neumann boundary condition on the sphere

$$\nabla_n u = g \qquad \text{for} \quad |\boldsymbol{x}| = 1, \tag{2.2}$$

- and the Sommerfeld radiation condition at infinity

$$\left| \frac{\partial u}{\partial n} - iku \right| = O\left(\frac{1}{r^2}\right). \tag{2.3}$$

2.2 Existence Theory and Variational Formulations

The first step is to introduce a truncated exterior domain Ω^e_γ

$$\Omega^e_\gamma = \Omega^e \cap \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,:\, |\boldsymbol{x}| < \gamma \right\}.$$

For the exterior spherical domain Ω^e_γ reduces simply to the annulus

$$\Omega^e_\gamma = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,:\, 1 < |\boldsymbol{x}| < \gamma \right\}.$$

Next, we multiply the Helmholtz equation by a test function in the unconjugated version or by the complex conjugate of a test function v in the conjugated version, integrate it over the truncated domain Ω^e_γ, and integrate by parts. Using the Neumann boundary condition on $\partial\Omega^e_\gamma$ yields the formulations

$$\int_{\Omega^e_\gamma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega^e_\gamma - k^2 \int_{\Omega^e_\gamma} u\, v \, d\Omega^e_\gamma - \int_{S_\gamma} \frac{\partial u}{\partial n} v \, dS_\gamma = \int_{\partial\Omega^e} g\, v \, dS, \tag{2.4}$$

$$\int_{\Omega^e_\gamma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \overline{v} \, d\Omega^e_\gamma - k^2 \int_{\Omega^e_\gamma} u\, \overline{v} \, d\Omega^e_\gamma - \int_{S_\gamma} \frac{\partial u}{\partial n} \overline{v} \, dS_\gamma = \int_{\partial\Omega^e} g\, \overline{v} \, dS, \tag{2.5}$$

respectively, where S_γ is the truncated sphere with radius $r = \gamma$. The Sommerfeld radiation condition (2.3) can now be rewritten in the form

$$\frac{\partial u}{\partial r} = iku + \varphi(\boldsymbol{x}), \tag{2.6}$$

where $\varphi(\boldsymbol{x}) = O\left(r^{-2}\right)$ is an *unknown* function. Next, we build it into the variational formulations (2.4) and (2.5) by substituting formula (2.6) for $\partial u/\partial n = \partial u/\partial r$, in the corresponding boundary term,

$$\int_{\Omega^e_\gamma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega^e_\gamma - k^2 \int_{\Omega^e_\gamma} u\, v \, d\Omega^e_\gamma - ik \int_{S_\gamma} uv \, dS_\gamma = \int_{\partial\Omega^e} g\, v \, dS + \int_{S_\gamma} \varphi v \, dS_\gamma, \tag{2.7}$$

$$\int_{\Omega^e_\gamma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \overline{v} \, d\Omega^e_\gamma - k^2 \int_{\Omega^e_\gamma} u \, \overline{v} \, d\Omega^e_\gamma - ik \int_{S_\gamma} u\overline{v} \, dS_\gamma = \int_{\partial\Omega^e} g \, \overline{v} \, dS + \int_{S_\gamma} \varphi \overline{v} \, dS_\gamma, \tag{2.8}$$

respectively. We emphasize that function φ in formulations (2.7) and (2.8) is unknown.

The next step is to consider the limiting process as γ being extended to infinity. In the limit, the boundary term involving function φ should be eliminated and all the improper integrals should make sense. It is also important to remember that the leading term in the solution u is of the form

$$\frac{\exp(ikr)}{r}.$$

Consequently, both u and its gradient $\boldsymbol{\nabla} u$ are *not* L^2-integrable over the exterior domain. A remedy to this problem is to employ different test functions of order $O\left(r^{-3}\right)$. This allows the integrals to be interpreted in the usual Lebesgue sense. The problem is, however, that not only the integral on S_γ involving the function φ disappears in the limit, but the one involving function u as well. In other words, this particular choice of the test function *does not* allow the retention of the Sommerfeld radiation condition into the weak formulation. A solution proposed by Leis [10] is to include the Sommerfeld condition directly in the definition of the spaces. This leads to the definition of the following weighted Sobolev space

$$H^+_{1,w}(\Omega^e) = \left\{ u : \|u\|^+_{1,w} < \infty \right\} \tag{2.9}$$

with the norm $\|u\|^+_{1,w}$ corresponding to the inner product

$$(u,v)^+_{1,w} = \int_{\Omega^e} w \, u \, \overline{v} + w \, \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \overline{v} \, d\Omega^e + \int_{\Omega^e} \left(\frac{\partial u}{\partial r} - iku \right) \overline{\left(\frac{\partial v}{\partial r} - ikv \right)} \, d\Omega^e. \tag{2.10}$$

Two particular weights are of interest, $w = \frac{1}{r^2}$ and a "dual" weight $w^* = r^2$. The unconjugated and conjugated Leis variational formulations read now as follows

$$\left\{ \begin{array}{l} \text{Find } u \in H^+_{1,w}(\Omega^e) \quad \text{such that} \\ \displaystyle\int_{\Omega^e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega^e - k^2 \int_{\Omega^e} u \, v \, d\Omega^e = \int_{\partial\Omega^e} g \, v \, dS \quad \forall \, v \in H^+_{1,w^*}(\Omega^e), \end{array} \right. \tag{2.11}$$

$$\left\{ \begin{array}{l} \text{Find } u \in H^+_{1,w}(\Omega^e) \quad \text{such that} \\ \displaystyle\int_{\Omega^e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \overline{v} \, d\Omega^e - k^2 \int_{\Omega^e} u \, \overline{v} \, d\Omega^e = \int_{\partial\Omega^e} g \, \overline{v} \, dS \quad \forall \, v \in H^+_{1,w^*}(\Omega^e). \end{array} \right. \tag{2.12}$$

An alternative procedure has been proposed by Burnett in [3], for which we have to represent both solution u and test function v in the form

$$
\begin{aligned}
u(r,\theta,\phi) &= \frac{\exp(ikr)}{r} u_0(\theta,\phi) + U(r,\theta,\phi), \\
v(r,\theta,\phi) &= \frac{\exp(ikr)}{r} v_0(\theta,\phi) + V(r,\theta,\phi),
\end{aligned}
\tag{2.13}
$$

where functions $u_0(\theta,\phi)$ and $v_0(\theta,\phi)$ are frequently known as the "radiation patterns", and functions $U(r,\theta,\phi)$, $V(r,\theta,\phi)$ are from $H^1(\Omega^e)$, i.e. both U, V and their gradients $\boldsymbol{\nabla} U$, $\boldsymbol{\nabla} V$ are square-integrable. Function u of this form satisfies *automatically* the Sommerfeld radiation condition.

Surprisingly, upon substituting formulas (2.13) into (2.4) and (2.5) and cancelling out terms involving the radiation patterns, one can pass to the limit with $\gamma \to \infty$. In this case, the unconjugated and conjugated variational formulations following the Burnett approach read as

$$
\left\{
\begin{array}{l}
\text{Find } u \in H^+_{1,w}(\Omega^e) \text{ such that} \\
\displaystyle\int_{\Omega^e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega^e - k^2 \int_{\Omega^e} u\, v \, d\Omega^e - ik \lim_{\gamma\to\infty} \int_{S_\gamma} uv \, dS_\gamma \\
\displaystyle = \int_{\partial\Omega^e} g\, v\, dS \qquad \forall\, v \in H^+_{1,w}(\Omega^e),
\end{array}
\right.
\tag{2.14}
$$

$$
\left\{
\begin{array}{l}
\text{Find } u \in H^+_{1,w}(\Omega^e) \text{ such that} \\
\displaystyle\int_{\Omega^e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \overline{v} \, d\Omega^e - k^2 \int_{\Omega^e} u\, \overline{v} \, d\Omega^e - ik \lim_{\gamma\to\infty} \int_{S_\gamma} u\overline{v} \, dS_\gamma \\
\displaystyle = \int_{\partial\Omega^e} g\, \overline{v}\, dS \qquad \forall\, v \in H^+_{1,w}(\Omega^e),
\end{array}
\right.
\tag{2.15}
$$

respectively.

The integrands in formulations (2.14) and (2.15) are understood in the Cauchy Principle Value sense discussed above.

2.3 Separation of Variables

The resulting equations in θ and ϕ are exactly the same as for the Laplace equation, compare [7, 8]. The only difference occurs in the radial direction, where the corresponding equation is now the Bessel equation (see also [11])

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r)}{\partial r}\right) + \left(k^2 - \frac{\lambda^2}{r^2}\right) f(r) = 0.
\tag{2.16}
$$

Solutions to (2.16) are the spherical Hankel functions of the first and second kind $h_n^{(1)}(kr)$, $h_n^{(2)}(kr)$ (see, e.g. [11] for a definition). The Sommerfeld radiation boundary condition eliminates the spherical Hankel functions of the second kind, so that the final solution to the Helmholtz equation can be represented in the form

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} h_n(kr) P_n^m(\cos\theta)\left(A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)\right), \tag{2.17}$$

where $P_n^m(\cos\theta)$ are the Legendre functions, compare [11].

Assuming that the Neumann boundary condition (2.2) is given by

$$g(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\left(\tilde{A}_{nm}\cos(m\phi) + \tilde{B}_{nm}\sin(m\phi)\right), \tag{2.18}$$

then the orthogonality properties of $P_n^m(\cos\theta)$, sin and cos functions result in the following relationship between the coefficients in the solution and the coefficients in the boundary condition (for details compare [7, 8])

$$A_{nm} = -\frac{\tilde{A}_{nm}}{\left.\frac{\partial h_n(kr)}{\partial r}\right|_{r=1}}, \qquad B_{nm} = -\frac{\tilde{B}_{nm}}{\left.\frac{\partial h_n(kr)}{\partial r}\right|_{r=1}}. \tag{2.19}$$

The spherical Hankel functions of the first kind $h_n(kr)$ have the following representation [11]

$$h_n(kr) = \sum_{m=0}^{n} \frac{\exp(ikr)}{r^{m+1}} \frac{\exp\left(-i\frac{\pi}{2}(n+1)\right)}{k(2k)^m} i^m \left(n+\frac{1}{2}, m\right) \tag{2.20}$$

with

$$\left(n+\frac{1}{2}, m\right) = \begin{cases} 1 & m = 0 \\ \prod_{k=1}^{m}(n+k)\cdot\prod_{k=1}^{m}\frac{(n-m+k)}{k} & m \geq 1. \end{cases}$$

2.4 Definition of the *hp*-infinite element

The infinite elements for the unconjugated formulations are constructed and defined in a manner similar to the infinite elements for the conjugated formulations. The only differences result from the use of the complex conjugate and different powers in the denominator of the radial shape functions.

The trial functions for approximating the solution u are now

$$\psi_j(r) = \frac{\exp(ikr)}{r^j}, \qquad j \geq 1, \tag{2.21}$$

and, for the test functions in the Leis formulation,

$$\tilde{\psi}_j(r) = \frac{\exp{(ikr)}}{r^{j+2}}, \qquad j \geq 1, \tag{2.22}$$

and, for the test functions in the Burnett formulation,

$$\tilde{\psi}_j(r) = \frac{\exp{(ikr)}}{r^{j}}, \qquad j \geq 1. \tag{2.23}$$

Note the different powers in r for the trial and test functions in the Leis formulation. The complex conjugate has to be used over the test functions in the conjugated IE formulations.

The use of the sesquilinear formulation eliminates the need of integration of the oscillatory components $\exp(ikr)$. Indeed, the complex conjugate of the test function is

$$\overline{\psi}_j(r) = \frac{\exp(-ikr)}{r^{j+2}}, \quad j \geq 1,$$

and, upon introducing into the variational formulation, the exponential term cancels out. In the bilinear (unconjugated) formulations this is not the case and the exponential integral has to be evaluated (see [9]).

2.5 Difference (formal) between the Conjugated and Unconjugated Versions of the IEM

The formulations following Leis, introduced in (2.11) and (2.12) are based on a different space setting. The difference between the conjugated and the unconjugated versions has recently been studied for a 2D model problem in [2]. Here we point out a few formal differences between the conjugated and unconjugated versions. We start our analysis in the context of the Burnett variational formulations. In particular, two issues are of interest:

- Lebesgue integrability
- elimination of the additional surface integral resulting from the Sommerfeld radiation condition

The analysis starts again from the weak formulation on the truncated domain. Taking (formally) the limit on both sides of the variational equation (2.8), we get

$$\begin{aligned} & \lim_{\gamma\to\infty}\left(\int_{\Omega_\gamma}\boldsymbol{\nabla}u\boldsymbol{\nabla}\overline{v}\,d\Omega_\gamma - k^2\int_{\Omega_\gamma}u\overline{v}\,d\Omega_\gamma - ik\int_{S_\gamma}u\overline{v}\,dS_\gamma\right) \\ = & \int_S g\overline{v}\,dS + \lim_{\gamma\to\infty}\int_{S_\gamma}\varphi\overline{v}\,dS_\gamma. \end{aligned} \tag{2.24}$$

In the case of the unconjugated version, the complex conjugate over function v has to be dropped. The following analysis shows that, in either case, only the not integrable terms, i.e. the terms that are not well defined, cancel each other out. In particular, the additional surface integral needs to cancel out in the weak form without the complex conjugate, but it does not need to cancel out in the weak form with the complex conjugate.

Functions u and v are of the form

$$u = u(r,\theta,\phi) = \sum_{n=1}^{N}\frac{\exp(ikr)}{r^n}f_n(\theta,\phi), \tag{2.25}$$

where $f_n(\theta,\phi)$ are some functions of the angular variables that need not be specified for this analysis.

Using the summation convention, we simplify the notation to

$$\begin{aligned} u &= \frac{\exp(ikr)}{r^n}f_n(\theta,\phi) = \frac{\exp(ikr)}{r^n}f_n, \\ v &= \frac{\exp(ikr)}{r^m}f_m(\theta,\phi) = \frac{\exp(ikr)}{r^m}f_m. \end{aligned} \tag{2.26}$$

Case 1: Sesquilinear form formulation

Substituting (2.26) into (2.24) and using the definition of $\boldsymbol{\nabla}$ in spherical coordinates gives

$$\begin{aligned} & \lim_{\gamma\to\infty}\int_{\Omega_\gamma}\frac{\partial}{\partial r}\left(\frac{\exp(ikr)}{r^n}\right)\frac{\partial}{\partial r}\left(\frac{\exp(-ikr)}{r^m}\right)f_n\overline{f}_m \\ & \qquad +\frac{1}{r^2}\frac{1}{r^{n+m}}\left(\frac{\partial f_n}{\partial\theta}\frac{\partial\overline{f}_m}{\partial\theta}+\frac{\partial f_n}{\partial\phi}\frac{\partial\overline{f}_m}{\partial\phi}\right) - k^2\frac{1}{r^{n+m}}f_n\overline{f}_m\,d\Omega_\gamma \\ - & \; ik\lim_{\gamma\to\infty}\int_{S_\gamma}\frac{1}{\gamma^{n+m}}f_n\overline{f}_m\,dS_\gamma \\ = & \int_S g\overline{v}\,dS + \lim_{\gamma\to\infty}\int_{S_\gamma}\varphi\overline{v}\,dS_\gamma. \end{aligned}$$

In the following, the right hand side and the terms involving the angular derivatives are neglected, since these terms are obviously Lebesgue integrable for all $n, m \geq 1$.

Substituting spherical coordinates for Ω_γ and S_γ (the jacobian for S_γ is γ^2) yields

$$\begin{aligned}
& \lim_{\gamma\to\infty} \int_1^\gamma \frac{k^2}{r^{n+m-2}} + \frac{ik(n-m)}{r^{n+m-1}} + \frac{nm}{r^{n+m}} \, dr \int_S f_n \overline{f}_m \, dS \\
- \quad & \lim_{\gamma\to\infty} k^2 \int_1^\gamma \frac{1}{r^{n+m-2}} \, dr \int_S f_n \overline{f}_m \, dS - ik \lim_{\gamma\to\infty} \int_S \frac{1}{\gamma^{n+m-2}} f_n \overline{f}_m \, dS.
\end{aligned}$$

It can be seen that for $n = m = 1$ the integrand $ik(n-m)/r^{n+m-1}$ is zero, and that this term is integrable for all n, m with $n + m > 2$. Interpreting the integrals in r in a Cauchy Principal Value (CPV) sense, we notice that the two not-integrable integrands k^2/r^{n+m-2} cancel each other out, i.e.

$$\lim_{\gamma\to\infty} \left(\int_1^\gamma \frac{k^2}{r^{n+m-2}} \, dr \int_S f_n \overline{f}_m \, dS - \int_1^\gamma \frac{k^2}{r^{n+m-2}} \, dr \int_S f_n \overline{f}_m \, dS \right) = 0 \;\forall\; n, m.$$

The additional surface integral yields

$$-ik \lim_{\gamma\to\infty} \int_S \frac{1}{\gamma^{n+m-2}} f_n \overline{f}_m \, dS = \begin{cases} 0 & n+m > 2 \\ -ik \displaystyle\int_S f_1 \overline{f}_1 \, dS & n+m = 2. \end{cases}$$

Here one could think that this term would cancel out with the term

$$\lim_{\gamma\to\infty} \int_1^\gamma \frac{ik(n-m)}{r^{n+m-1}} \, dr \int_S f_n \overline{f}_m \, dS,$$

for $n = 2$ and $m = 1$. But this would require that $f_2 = f_1$ and that is, in general, *not* the case. This does not cause any problems since all remaining terms, including the additional surface integral, are well defined and Lebesgue integrable.

Case 2: Bilinear form formulation

Substituting in exactly the same way as before we obtain

$$\begin{aligned}
& \lim_{\gamma\to\infty} \int_{\Omega_\gamma} \frac{\partial}{\partial r} \left(\frac{\exp(ikr)}{r^n} \right) \frac{\partial}{\partial r} \left(\frac{\exp(ikr)}{r^m} \right) f_n f_m \\
& \qquad + \frac{1}{r^2} \frac{\exp(i2kr)}{r^{n+m}} \left(\frac{\partial f_n}{\partial \theta} \frac{\partial f_m}{\partial \theta} + \frac{\partial f_m}{\partial \phi} \frac{\partial f_m}{\partial \phi} \right) - k^2 \frac{\exp(i2kr)}{r^{n+m}} f_n f_m \, d\Omega_\gamma \\
- \quad & ik \lim_{\gamma\to\infty} \int_{S_\gamma} \exp(i2k\gamma) \frac{1}{\gamma^{n+m}} f_n f_m \, dS_\gamma \\
= \quad & \int_S g \overline{v} \, dS + \lim_{\gamma\to\infty} \int_{S_\gamma} \varphi \overline{v} \, dS_\gamma.
\end{aligned}$$

Again, as before, the right hand side and the terms involving the angular derivatives are neglected, due to their Lebesgue integrability $\forall\ n, m \geq 1$.

Carrying on the differentiation on the left-hand side, we get

$$\begin{aligned} & \lim_{\gamma\to\infty}\int_1^\gamma \exp(i2kr)\left(\frac{-k^2}{r^{n+m-2}} - \frac{ik(n+m)}{r^{n+m-1}} + \frac{nm}{r^{n+m}}\right) dr \int_S f_n f_m\, dS \\ - & \lim_{\gamma\to\infty} k^2 \int_1^\gamma \frac{\exp(i2kr)}{r^{n+m-2}}\, dr \int_S f_n f_m\, dS - ik \lim_{\gamma\to\infty}\int_S \frac{\exp(i2k\gamma)}{\gamma^{n+m-2}} f_n f_m\, dS. \end{aligned}$$

This time the not integrable terms in the body integral do *not* cancel each other out. Only the additional surface integral will make them vanish. Interpreting the r integrals again in a CPV sense we obtain

$$\begin{aligned} & \lim_{\gamma\to\infty}\int_1^\gamma \exp(i2kr)\frac{-2k^2}{r^{n+m-2}}\, dr \int_S f_n f_m\, dS \\ + & \lim_{\gamma\to\infty}\int_1^\gamma \exp(i2kr)\left(\frac{-ik(n+m)}{r^{n+m-1}} + \frac{nm}{r^{n+m}}\right) dr \int_S f_n f_m\, dS \\ - & ik \lim_{\gamma\to\infty}\frac{\exp(i2k\gamma)}{\gamma^{n+m-2}}\int_S f_n f_m\, dS. \end{aligned}$$

The second integral term can now be integrated $\forall\ n, m \geq 1$ using the sine and cosine integrals [9]. Similarly, the first integral can be evaluated provided that $n+m > 2$. The additional surface integral is zero $\forall\ n, m$ with $n+m > 2$.

The only term that has to be taken care of, is the first and the third integral for $n = m = 1$, i.e.

$$\begin{aligned} & \lim_{\gamma\to\infty}\int_1^\gamma \exp(i2kr)\frac{-2k^2}{r^{1+1-2}}\, dr \int_S f_1 f_1\, dS - ik \lim_{\gamma\to\infty}\frac{\exp(i2k\gamma)}{\gamma^{1+1-2}}\int_S f_1 f_1\, dS \\ = & \lim_{\gamma\to\infty}\left(\int_1^\gamma -2k^2\exp(i2kr)\, dr - ik\exp(i2k\gamma)\right)\int_S f_1 f_1\, dS \\ = & \lim_{\gamma\to\infty}\left(-2k^2\,\frac{1}{i2k}\exp(i2kr)\Big|_1^\gamma - ik\exp(i2k\gamma)\right)\int_S f_1 f_1\, dS \\ = & \lim_{\gamma\to\infty}\left(-\frac{k}{i}\exp(i2k\gamma) + \frac{k}{i}\exp(i2k) - ik\exp(i2k\gamma)\right)\int_S f_1 f_1\, dS \\ = & \lim_{\gamma\to\infty}\frac{k}{i}\exp(i2k)\int_S f_1 f_1\, dS \\ = & \frac{k}{i}\exp(i2k)\int_S f_1 f_1\, dS. \end{aligned}$$

In the last steps, only the fact that $-1/i = i$ is used. Again, as before, all integrals or limits can be defined in a CPV sense, i.e. all not integrable terms and the additional surface integral have vanished. In this case, the additional surface integral coming from the Sommerfeld radiation condition is not well defined in the limit, but combined with the corresponding singular body integrals cancels out, and makes the total sum well defined as well.

It should be noted that the original variational formulation (2.24) is the *same* in both cases, except for the complex conjugate. The difference introduced by the complex conjugate being present or not translates into the surface integral being well defined or not, which makes only a small difference in the actual implementation of the weak formulation.

The above analysis can also be applied to the conjugated and unconjugated Leis weak formulation (2.11) and (2.12), but since in this approach all integrals are well defined due to the different weighted Sobolev space setting, it *does not* make any formal difference whether one does or does not use the complex conjugate, except in the presence of the complex conjugate sign over the second argument.

2.6 Implementation Details for the Helmholtz Equation

The implementation of the IEM is analogous to a 2D FE implementation. The integration of the shape functions in the radial direction involves the following integrals in the conjugated formulation

$$\int_1^\infty \frac{\exp(ikr)\exp(-ikr)}{r^j}\,dr = \frac{1}{j-1}, \quad j \geq 2, \tag{2.27}$$

and in the unconjugated version terms of the form

$$\int_1^\infty \frac{\exp(ikr)\exp(ikr)}{r^j}\,dr = \int_1^\infty \frac{\exp(2ikr)}{r^j}\,dr, \tag{2.28}$$

which can be evaluated using the exponential integral [9] by using the representation

$$\int_1^\infty \frac{\exp(2ikr)}{r^n}\,dr = \sum_{l=1}^{n-1} \frac{-(-i2k)^{l-1}\exp(2ik)}{\prod_{\hat{l}=1}^{l} -n+\hat{l}} + \frac{(-i2k)^{n-1}}{\prod_{\hat{l}=1}^{n-1} -n+\hat{l}} \int_1^\infty \frac{\exp(2ikr)}{r} dr, \tag{2.29}$$

where $n \geq 1$, $\sum_{l=1}^{0} := 0$ and $\prod_{\hat{l}=1}^{0} := 1$. The representation (2.29) can be easily verified by induction.

2.7 Scattering of a Plane Wave by a Rigid Sphere

The three-dimensional incident plane wave can be decomposed into the spherical harmonics as follows (see [7, 8])

$$p^{inc}(r,\theta) = P_{inc} \exp ikx = P_{inc} \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\theta) j_n(kr), \tag{2.30}$$

where $x = r\sin\theta\cos\phi$, P_{inc} is the amplitude of the incident wave, $P_n(\cos\theta)$ denotes the Legendre polynomial of degree n and $j_n(kr)$ is the n-th spherical Bessel function of the first kind.

The incident wave is scattered by the rigid unit sphere and the goal is to find the scattered wave p^s. The condition that relates the incident and the scattered wave for the rigid obstacle is

$$\nabla_n \left(p^{inc} + p^s\right) = 0 \qquad \text{on the surface of the scatterer.} \tag{2.31}$$

Then the scattered wave is given by (for details see [7, 8])

$$p^s = \sum_{n=0}^{\infty} h_n(kr) P_n^m(\cos\theta) A_n, \tag{2.32}$$

with

$$A_n = \frac{-P_{inc}(2n+1) i^n \left.\frac{\partial j_n(kr)}{\partial r}\right|_{r=1}}{\left.\frac{\partial h_n(kr)}{\partial r}\right|_{r=1}} \forall \quad n \geq 1. \tag{2.33}$$

2.8 Error Calculations

In the previous work on infinite elements [7, 8] we have used the weighted H^1-norm

$$\|u - u_h\|_1^2 = \int_\Omega \frac{1}{r^2} |u - u_h|^2 \, d\Omega + \int_\Omega \frac{1}{r^2} \left|\boldsymbol{\nabla}(u - u_h)\right|^2 \, d\Omega \tag{2.34}$$

to measure the error between the exact solution u and the numerical solution u_h. This norm (2.34) is consistent with the mathematical theory by Leis [10] for the Helmholtz equation and gave satisfactory results because the conjugated IEM converged in the whole exterior domain. From the stability and convergence analysis presented in this work and the analysis presented in [2] we know that the unconjugated IEM will fail in the far field. Therefore it will not make any sense to use the weighted H^1-norm to measure

	conjugated Leis IEM			conjugated Burnett IEM		
p	$N=1$	$N=3$	$N=6$	$N=1$	$N=3$	$N=6$
2	100	100	89.15	100	100	100
3	100	100	95.94	100	100	98.85
4	100	87.64	27.43	100	100	31.96
5	100	84.00	9.20	100	100	14.71
6	100	88.89	4.00	100	100	8.85
7	100	89.25	3.04	100	100	7.68
8	100	89.31	2.96	100	100	7.59

Table 1: Rigid scattering of a plane wave with $k = 10$. $\|u - u_h\|_\infty$ in percent for the conjugated IEM with $N = 1, 3, 6$ radial shape functions.

the error for the unconjugated IEM. We are interested in comparing the performance of the conjugated and unconjugated IEM in the near field. Therefore, consistent with [2], we will use the L^∞-norm on the surface of the unit sphere to measure the error, i.e.

$$\|u - u_h\|_\infty = \sup_{\boldsymbol{x} \in S} |u(\boldsymbol{x}) - u_h(\boldsymbol{x})|. \tag{2.35}$$

We will compute (2.35) by replacing $\sup_{\boldsymbol{x} \in S}$ with the maximum over all Gausspoints for each element in S.

2.9 Convergence Rates

In the following, convergence rates are presented for the rigid scattering of a plane wave on the unit sphere. The corresponding exact solution is given by (2.32). The question is how does the number of shape functions in the radial direction affect the approximation of the exact solution. The p-convergence rates are studied in terms of the L^∞-error norm as a function of the order of approximation in the radial direction. In all examples the wave number k is set to 10. Tables 1, 2 show the error $\|u - u_h\|_\infty$ for the conjugated and unconjugated IEM for p varying from 2 to 8 and for the number of radial shape functions $N = 1, 3, 6$. The corresponding convergence rates are presented in Figure 1.

From Tables 1 and 2 we clearly see that the conjugated IEM will only give reliable results, if up to six radial shape functions are used. The unconjugated Burnett IEM can also provide good results if only three radial shape functions are used.

On the other hand, from [7, 8] we know that the conjugated IEM will provide a reliable solution in the whole exterior domain, provided N large enough. The unconjugated

	unconjugated Leis IEM			unconjugated Burnett IEM		
p	$N=1$	$N=3$	$N=6$	$N=1$	$N=3$	$N=6$
2	100	100	88.99	100	100	100
3	100	100	100	100	100	100
4	61.30	67.82	26.86	80.26	33.23	29.08
5	52.63	53.31	8.37	58.63	14.04	9.07
6	54.60	39.75	3.25	59.31	8.74	3.40
7	54.87	42.73	1.85	59.63	7.35	2.31
8	55.25	41.37	1.79	60.24	7.20	2.20

Table 2: Rigid scattering of a plane wave with $k = 10$. $\|u - u_h\|_\infty$ in percent for the unconjugated IEM with $N = 1, 3, 6$ radial shape functions.

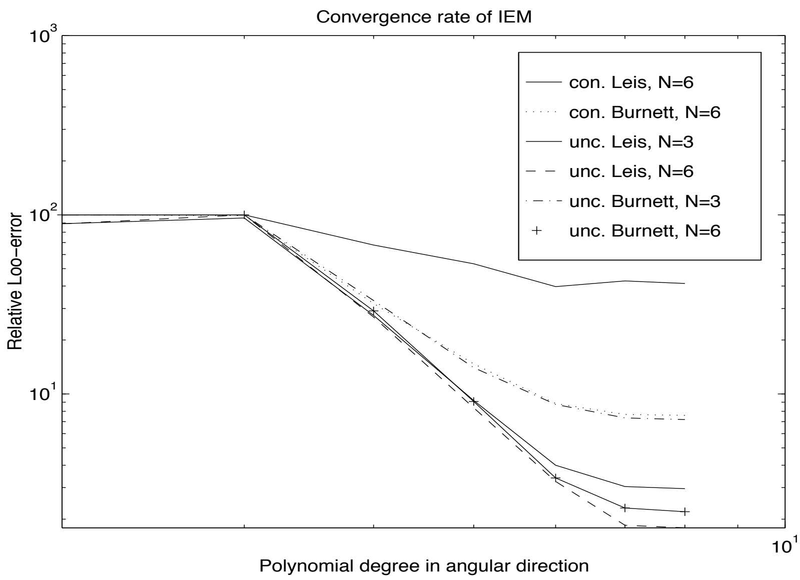


Figure 1: Convergence Rates for the IEM with $N = 3, 6$ and $p = 2, \cdots, 8$.

Burnett IEM will provide with less cost an accurate solution on the surface of the unit sphere, but the Helmholtz integral formula [2] has to be employed in order to compute the far field solution.

From the numerical experiments presented here and those in [7, 8] we clearly see that the conjugated Leis IEM should be used if the near and far field solution is of interest. The unconjugated Burnett IEM is most efficient if only the near field solution and the far field solution at a few points is of interest. We also note that the unconjugated Leis IEM does not yield equally good results with only three radial shape functions as the unconjugated Burnett IEM. The reason for this is that the unconjugated Leis IEM is not symmetric in the real sense, whereas the unconjugated Burnett IEM is symmetric.

3 A Stability Analysis for the Helmholtz Equation in the Exterior Spherical Domain

This section addresses the stability of the proposed IEM. The continuous stability constant is computed for axisymmetric solutions to the scattering problem for both the unconjugated Leis and Burnett IE formulations. The stability constants are then compared to the stability constants for the conjugated IEM described in [7, 8].

3.1 Continuous LBB Constant for the unconjugated Leis IE Formulation

Given weighted Sobolev spaces V_ρ and V_{ρ^*}, with weights $\rho = 1/r^2$ and $\rho^* = \rho^{-1} = r^2$, consider an abstract variational problem of the form

$$\begin{cases} \text{Find } u \in V_\rho & \text{such that} \\ b(u,v) = l(v) & \forall\, v \in V_{\rho^*}, \end{cases} \tag{3.1}$$

where $b(u,v)$ is a continuous bilinear form on $V_\rho \times V_{\rho^*}$ and $l(v)$ is a continuous linear form on V_{ρ^*}.

The bilinear form b defines a linear operator B prescribed on V_ρ with values in the topological dual V'_{ρ^*}

$$B : V_\rho \to V'_{\rho^*} \quad < Bu, v >= b(u,v) \quad \forall\, u \in V_\rho,\ v \in V_{\rho^*}, \tag{3.2}$$

by which (3.1) can be rewritten in the operator form

$$u \in V_\rho, \quad Bu = l. \tag{3.3}$$

Consistently with the classical theory of linear operators in Banach spaces, operator B is postulated to be *bounded below*

$$\|Bu\|_{V'_{\rho^*}} \geq \gamma \|u\|_{V_\rho}, \tag{3.4}$$

where $\|\cdot\|_{V_\rho}$ is the norm in the weighted Sobolev space V_ρ and $\|\cdot\|_{V'_{\rho^*}}$ is the norm in the weighted dual Sobolev space V'_{ρ^*}. The optimal (largest) constant γ is known as the LBB (Ladyzenskaya-Babuska-Brezzi) constant

$$\gamma = \inf_{u\neq 0} \frac{\|Bu\|_{V'_{\rho^*}}}{\|u\|_{V_\rho}}. \tag{3.5}$$

Introducing the Riesz operator

$$R : V_{\rho^*} \to V'_{\rho^*} \qquad < Ru, v >= (u, v)_{\rho^*} \quad \forall\, u, v \in V_{\rho^*}, \tag{3.6}$$

where $(\cdot,\cdot)_{\rho^*}$ denotes the inner product in the space V_{ρ^*}, allows to eliminate the dual norm in (3.5)

$$\begin{aligned} \gamma^2 &= \inf_{u\neq 0} \frac{\|Bu\|^2_{V'_{\rho^*}}}{\|u\|^2_{V_\rho}} = \inf_{u\neq 0} \frac{\|R^{-1}Bu\|^2_{V_{\rho^*}}}{\|u\|^2_{V_\rho}} = \inf_{\|u\|^2_{V_\rho}=1} \|R^{-1}Bu\|^2_{V_{\rho^*}} \\ &= \inf_{(u,\overline{u})_\rho=1} \left(R^{-1}Bu, \overline{R^{-1}Bu}\right)_{\rho^*}. \end{aligned} \tag{3.7}$$

Next, the application of the standard Lagrange multiplier technique, leads to the eigenvalue problem

$$\left\{ \begin{array}{ll} \text{Find } u \in V_\rho,\ \gamma^2 \in I\!R & \text{such that} \\ 2\Re\left(R^{-1}Bu, \overline{R^{-1}B\delta u}\right)_{\rho^*} = \gamma^2 2\Re(u, \overline{\delta u})_\rho & \forall\, \delta u \in V_\rho \\ (u, \overline{u})_\rho = 1. & \end{array} \right. \tag{3.8}$$

Finally, it is convenient to rewrite (3.8) as a system of two equations. Introducing an auxiliary variable

$$V_{\rho^*} \ni u^a = R^{-1}Bu \tag{3.9}$$

yields

$$\left\{ \begin{array}{lcll} \Re\left(R^{-1}Bu, \overline{\delta u^a}\right)_{\rho^*} &=& \Re\left(u^a, \overline{\delta u^a}\right)_{\rho^*} & \forall\, \delta u^a \in V_{\rho^*} \\ \Re\left(u^a, \overline{R^{-1}B\delta u}\right)_{\rho^*} &=& \gamma^2 \Re\left(u, \overline{\delta u}\right)_\rho & \forall\, \delta u \in V_\rho. \end{array} \right. \tag{3.10}$$

Now, recalling the definitions of operators B and R yields

$$\begin{cases} \Re\, b\left(u, \overline{\delta u^a}\right) &= \Re\left(u^a, \overline{\delta u^a}\right)_{\rho^*} \quad \forall\, \delta u^a \in V_{\rho^*} \\ \Re\, b\left(\overline{\delta u}, u^a\right) &= \gamma^2 \Re\left(u, \overline{\delta u}\right)_{\rho} \quad \forall\, \delta u \in V_{\rho}, \end{cases} \tag{3.11}$$

where

$$\begin{aligned} (u, \overline{v})_{\rho} &= \int_S \int_1^{\infty} \frac{1}{r^2} \left(u\overline{v} + \boldsymbol{\nabla} u \boldsymbol{\nabla} \overline{v}\right) r^2\, dr dS \\ &= \int_S \int_1^{\infty} \left(u\overline{v} + \boldsymbol{\nabla} u \boldsymbol{\nabla} \overline{v}\right)\, dr dS, \\ (u, \overline{v})_{\rho^*} &= \int_S \int_1^{\infty} r^2 \left(u\overline{v} + \boldsymbol{\nabla} u \boldsymbol{\nabla} \overline{v}\right) r^2\, dr dS \\ &= \int_S \int_1^{\infty} r^4 \left(u\overline{v} + \boldsymbol{\nabla} u \boldsymbol{\nabla} \overline{v}\right)\, dr dS, \\ b(u, v) &= \int_S \int_1^{\infty} \left(\boldsymbol{\nabla} u \boldsymbol{\nabla} v - k^2 uv\right) r^2\, dr dS. \end{aligned} \tag{3.12}$$

In what follows, we restrict the analysis to the axisymmetric case only (no dependence upon ϕ), and use the following representations

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n &&= \sum_{n=0}^{\infty} h_n(kr) P_n(\cos\theta) A_n, \\ \delta u &= \sum_{n=0}^{\infty} \delta u_n &&= \sum_{n=0}^{\infty} h_n(kr) P_n(\cos\theta) \tilde{A}_n, \\ u^a &= \sum_{n=0}^{\infty} u_n^a &&= \sum_{n=0}^{\infty} \frac{1}{r^2} h_n(kr) P_n(\cos\theta) A_n^a, \\ \delta u^a &= \sum_{n=0}^{\infty} \delta u_n^a &&= \sum_{n=0}^{\infty} \frac{1}{r^2} h_n(kr) P_n(\cos\theta) \tilde{A}_n^a. \end{aligned} \tag{3.13}$$

The computations for determining the stability constant γ are very similar to those presented in [7] and result in the final modal system of equations

$$\begin{bmatrix} a & -b \\ -\gamma_n^2 c & d \end{bmatrix} \begin{bmatrix} A_n \\ A_n^a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.14}$$

where $a = \Re b(u_n, \overline{\delta u_n^a})$, $b = \Re(u_n^a, \overline{\delta u_n^a})_{\rho^*}$, $c = \Re(u_n, \overline{\delta} u_n)_\rho$ and $d = \Re b(\overline{\delta u_n}, u_n^a)$.

The sufficient and necessary condition for a nontrivial solution to exist is

$$\det \begin{bmatrix} a & -b \\ -\gamma_n^2 c & d \end{bmatrix} = ad - \gamma_n^2 cb = 0. \tag{3.15}$$

Solving for γ_n^2 yields

$$\gamma_n^2 = \frac{ad}{cb}. \tag{3.16}$$

The global LBB constant is now equal to the infimum of the modal constants γ_n

$$\gamma^2 = \inf_{n=0,1,\ldots} \gamma_n^2. \tag{3.17}$$

For the first ten modes and wave number k less than 20 the modal LBB constants are displayed in Fig. 2, where the “x-axis” shows the wave number and the “y-axis” shows the value of the modal LBB constant.

Remark: The dependence of the stability constant γ upon the wave number k is clearly of order $\gamma = O\left(1/k^2\right)$.

3.2 Continuous LBB Constant for the unconjugated Burnett IE Formulation

The LBB constant for the unconjugated Burnett formulation is derived in the same fashion as the unconjugated Leis LBB constant. The only differences occur due to the fact that now the space $V_{\rho*}$ is equal to the space V_ρ, and that the integrals in the bilinear form of the Burnett formulation have to be interpreted in a Cauchy Principal Value sense.

The equations for the modal LBB constant γ_n are similar to (3.15). For the first ten modes and wave number k less than 20 the modal LBB constants are displayed in Figure 3.

Remark: The dependence of the stability constant γ upon the wave number k is clearly of order $\gamma = O\left(1/k^2\right)$.

3.3 Comparison of the LBB Constants for the conjugated and conjugated IEM

The LBB constant for the conjugated Leis and Burnett IEM had been computed in [7, 8]. The result was that in the conjugated cases the LBB constants behave like $\gamma = O\left(1/k\right)$. The above analysis shows that the LBB constants in the unconjugated cases behave like $\gamma = O(1/k^2)$. Therefore, the conjugated IE versions are one order more stable in k than the unconjugated IEM.

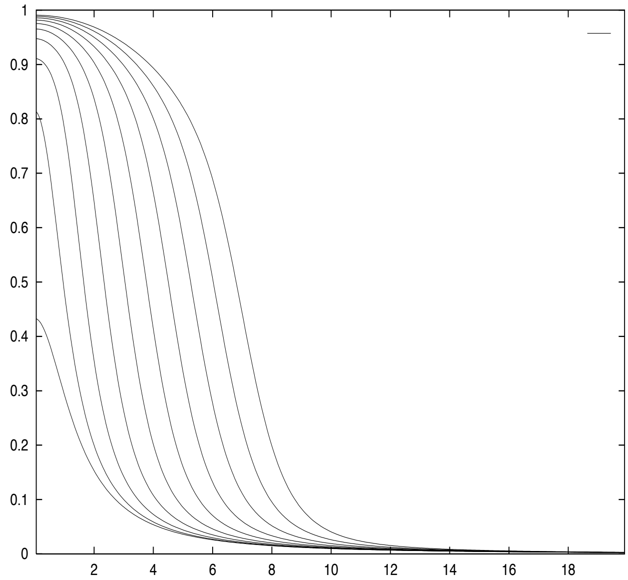


Figure 2: Modal LBB-constants for the unconjugated Leis IE formulation for the first ten modes. The pointwise infimum corresponds to the global LBB constant.

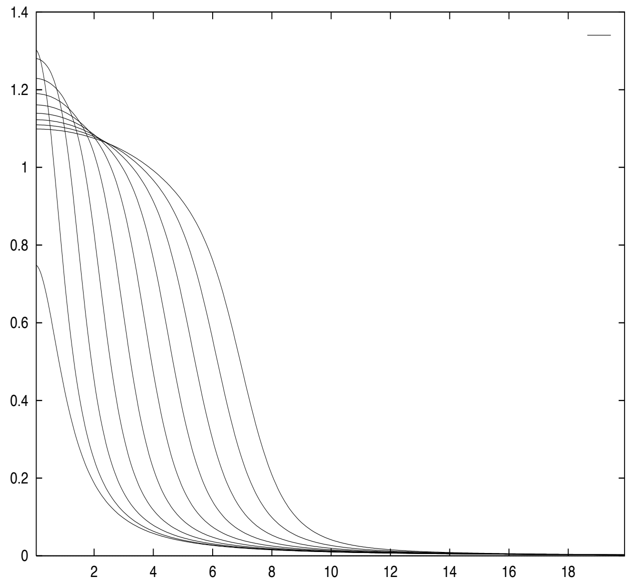


Figure 3: Modal LBB-constants for the unconjugated Burnett IE formulation for the first ten modes. The pointwise infimum corresponds to the global LBB constant.

4 Convergence of the Unconjugated Leis Infinite Element

In this section, we consider convergence of the unconjugated Leis IEM for the Helmholtz equation. The analysis is similar to the convergence study of the conjugated IEM. The reader is referred to [5, 8] for more details.

In particular, we look for the solution $u = u(\boldsymbol{x}, r)$ of problem (2.1) in the standard form

$$u(\boldsymbol{x}, r) = Y(\boldsymbol{x})R(r) \qquad \boldsymbol{x} \in S, \quad r > 1, \tag{4.1}$$

where $S = \{\boldsymbol{x} \in \mathbb{R}^3, |\boldsymbol{x}| = 1\}$.

Separation of variables for the Helmholtz equation leads now to

$$\frac{(r^2R')'}{R} + k^2r^2 = -\frac{\Delta_S Y}{Y} = c, \tag{4.2}$$

and, consequently, to the eigenvalue problem for the Laplace-Beltrami operator Δ_S

$$\Delta_S Y + cY = 0, \tag{4.3}$$

leading to the sequence of eigenvalues

$$c = c_n = n(n+1), \qquad n = 0, 1, 2, \ldots, \tag{4.4}$$

and the corresponding eigenspaces spanned by the spherical harmonics Y_{nm}, $m = 0, \ldots,$ $2n$, $n = 0, 1, 2, \ldots$. Note that, except for $n = 0$, all eigenvalues are *multiple* eigenvalues with multiplicity $2n + 1$.

The corresponding equation in the r-direction reduces now to the Bessel equation

$$(r^2R')' + \left(k^2r^2 - n(n+1)\right) R = 0, \tag{4.5}$$

with the solution represented in terms of the spherical Hankel functions

$$R(r) = R_n(r) = A_n h_n^1(kr) + B_n h_n^2(kr). \tag{4.6}$$

As before, the Sommerfeld radiation condition eliminates the Hankel functions of the second kind and, by superposition, the final form of the solution is

$$u(\boldsymbol{x}, r) = \sum_{n=1}^{\infty} \left(\sum_{m=0}^{2n} U_{nm} Y_{nm}(\boldsymbol{x}) \right) h_n^1(kr), \tag{4.7}$$

where the Hankel functions have been normalized to satisfy the Neumann boundary condition

$$-\frac{\partial}{\partial r}\left(h_n^1(kr)\right)(1) = 1. \tag{4.8}$$

Recalling that the spherical harmonics Y_{nm} constitute an orthonormal basis in $L^2(S)$ and assuming $g \in L^2(S)$, one can calculate the coefficients U_{nm} from

$$U_{nm} = (g, Y_{nm})_{L^2(S)}. \tag{4.9}$$

In what follows, the normalized Hankel functions of the first kind will be denoted by $X_n(r)$, $X_n(r) = h_n^1(kr)$.

The analysis proceeds as follows. First, an approximate solution u^N for the exact solution u (compare (4.7)) is sought in the form

$$u^N(\boldsymbol{x}, r) = \sum_{n=0}^{N} X_n(r)\, u_n(\boldsymbol{x}). \tag{4.10}$$

The test functions are approximated now using different functions in the r-direction

$$v^N(\boldsymbol{x}, r) = \sum_{n=0}^{N} \hat{X}_n(r)\, v_n(\boldsymbol{x}) = \sum_{n=0}^{N} \frac{X_n(r)}{r^2}\, v_n(\boldsymbol{x}), \tag{4.11}$$

with $\hat{X}_n(r) = X_n(r)/r^2$ to guarantee the Lebesgue integrability of the involved integrals (compare Section 2.4).

Substituting now (4.10) and (4.11) into the unconjugated Leis formulation (2.11) yields the approximate problem

$$\begin{cases} \text{Find } (u_1, \ldots, u_N) \in \boldsymbol{H}^1(S) \text{ such that} \\ \displaystyle\sum_{n=0}^{N} \int_1^\infty \frac{1}{r^2} X_n X_m \, dr \int_S \boldsymbol{\nabla}_S\, u_n \boldsymbol{\nabla}_S\, \overline{v}\, dS \\ \quad \displaystyle + \left(\int_1^\infty X_n' \left(\frac{X_m}{r^2}\right)' r^2\, dr - k^2 \int_1^\infty X_n X_m \, dr \right) \int_S u_n v \, dS \\ \displaystyle = X_m(1) \int_S g v\, dS \qquad \forall\, v \in H^1(S), \qquad m = 0, \ldots, N, \end{cases} \tag{4.12}$$

where $\boldsymbol{H}^1(S) = H^1(S) \times \cdots \times H^1(S)$ (N-times) with $H^1(S)$ denoting the Sobolev space of order unity on the sphere S.

System (4.12) is next discretized using a FE approximation, resulting in a fully discrete solution u_h^N, where the error can be estimated using the triangle inequality

$$\|u_h^N - u\| \le \|u_h^N - u^N\| + \|u^N - u\|. \tag{4.13}$$

4.1 Convergence of u^N to u

A relevant observation concerning the effect of the multiple eigenvalues has to be made. Decomposing the solution u, for a fixed r, into a sum of eigenvectors of the Laplace-Beltrami operator

$$u(\boldsymbol{x}, r) = \sum_{l=0}^{\infty} u^l(r) Y_l(\boldsymbol{x}) \,, \qquad -\Delta_S Y_l = l(l+1) Y_l, \tag{4.14}$$

and doing the same with data g,

$$g = \sum_{l=0}^{\infty} g_l, \tag{4.15}$$

one can reduce the considered boundary-value problem into a sequence of problems corresponding to each of the eigenspaces. Next, one can select a $L^2(S)$-orthogonal basis within each of the l-th eigenspace with g_l as one of the vectors. This further reduces the problem to $2l+1$ independent scalar problems, with the spectral components of the solution corresponding to other than g_l vectors, simply vanishing. This suggests starting with the representation

$$u(\boldsymbol{x}, r) = \sum_{l=0}^{\infty} u^l(r) Y_l(\boldsymbol{x}) \tag{4.16}$$

where $Y_l(\boldsymbol{x}) = g_l / \|g_l\|_{L^2(S)}$. Selecting next for the test function

$$v(\boldsymbol{x}, r) = v(r) Y_j(\boldsymbol{x}) \tag{4.17}$$

and using the $L^2(S)$-orthogonality of functions $Y_l(\boldsymbol{x})$, we obtain a sequence of *decoupled* equations for each spectral component $u^l(r)$

$$\begin{cases} u^l \in H^1(1,\infty) \\ \displaystyle\int_1^\infty \frac{\partial u^j}{\partial r} \frac{\partial v}{\partial r} r^2 \, dr + j(j+1) \int_1^\infty u^j \, v \, dr - k^2 \int_1^\infty u^j \, v r^2 \, dr \\ = v(1) \|g_j\|_{L^2(S)} \qquad \forall \, v \in H^1_4(1,\infty), \end{cases} \tag{4.18}$$

where $H^1(1,\infty)$ is the regular Sobolev space of order 1 and $H^1_4(1,\infty)$ is the weighted Sobolev space with the norm

$$\|v\|^2_{1,4} = \int_1^\infty r^4 \left(|v|^2 + |v'|^2 \right) dr. \tag{4.19}$$

Notice that the choice of the "one-dimensional" spaces is consistent with the choice of weighted spaces in (2.9). Similarly, assuming in (4.10)

$$u_n(\boldsymbol{x}) = \sum_{l=0}^{\infty} u_n^l Y_l(\boldsymbol{x}), \quad u_n^l \in \mathbb{C}, \tag{4.20}$$

and selecting $v = Y_j(\boldsymbol{x})$ in (4.12), we obtain

$$\begin{aligned}&\sum_{n=0}^{N} \left\{ j(j+1) \int_1^\infty \frac{1}{r^2} X_n X_m \, dr + \int_1^\infty X_n' \left(\frac{X_m}{r^2} \right)' r^2 \, dr - k^2 \int_1^\infty X_n X_m \, dr \right\} u_n^j \\ &= X_m(1) \|g^j\|_{L^2(S)}, \qquad m = 0, \ldots, N.\end{aligned} \tag{4.21}$$

The main observation is now that *equations* (4.21) *correspond to the Petrov-Galerkin approximation of* (4.18) using the "spectral" approximation

$$\begin{aligned} u^j(r) &\approx u_N^j(x) = \sum_{n=0}^{N} u_n^j X_n(r), \\ v(r) &\approx \sum_{m=0}^{N} v_m^j \frac{X_m(r)}{r^2}. \end{aligned} \tag{4.22}$$

The represention of the H_w^1-norm in terms of the spectral components yields

$$\begin{aligned} &\|u - u_N\|_{1,w}^2 \\ = &\sum_{j=1}^{\infty} \int_1^\infty |(u^j - u_N^j)'|^2 \, dr + j(j+1) \int_1^\infty \frac{1}{r^2} |u^j - u_N^j|^2 \, dr \\ &\quad + \int_1^\infty |u^j - u_N^j|^2 \, dr. \end{aligned} \tag{4.23}$$

This suggests introducing the *component energy norm*

$$\|u\|_{j,E}^2 = \int_1^\infty |u'|^2 \, dr + j(j+1) \int_1^\infty \frac{1}{r^2} |u|^2 \, dr + \int_1^\infty |u|^2 \, dr \tag{4.24}$$

and results in the final formula for the error in the form

$$\|u - u_N\|_{1,w}^2 = \sum_{j=N+1}^{\infty} \|X^j - X_N^j\|_{j,E}^2 \, \|g_j\|_{L^2(S)}^2, \tag{4.25}$$

where X_N^j is the solution of system (4.21) with $\|g_j\|_{L^2(S)} = 1$.

It should be emphasized that, even though the problem is three-dimensional, the error analysis is reduced to the investigation of a 1D problem and approximation properties of functions X_j, where the X_j are the spherical Hankel functions of the first kind. Thus, the three-dimensional context does not add to the complexity of the problem.

Remark: As each of the spherical Hankel functions $h_n^1(kr)$ can be represented as a linear combination of functions $\exp(ikr)/r^j$, $j = 1, \ldots, n+1$, the Hankel functions used in approximation (4.10) can be replaced with function $\exp(ikr)/r^j \times \exp(-ik)/(j - ik)$. The choice of the "shape functions" affects the matrices in the approximate problem (4.21) but it will not change the solutions X_N^j. Consequently the entire analysis remains the same.

4.2 Convergence of u_h^N to u^N

Problem (4.12) is defined on the sphere, i.e. on the $\boldsymbol{H}^1(S)$ space. The corresponding sesquilinear form can be split into a positive definite part and a compact perturbation. Consequently, the standard convergence analysis (see e.g. [4]) applies.

5 Convergence of the Unconjugated Burnett Infinite Element

The convergence of the unconjugated Burnett IEM is analyzed similarly to the previous section. Approximation (4.10) is now used for both solution u and test function v

$$\begin{aligned} u^N(\boldsymbol{x}, r) &= \sum_{n=0}^{N} X_n(r)\, u_n(x), \\ v^N(\boldsymbol{x}, r) &= \sum_{n=0}^{N} X_n(r)\, v_n(x). \end{aligned} \tag{5.1}$$

Selecting $v = X_m(r)v(\boldsymbol{x})$ and substituting (5.1) into (2.7), then the approximate problem is obtained in the form

$$\begin{aligned} &\sum_{n=0}^{N} \lim_{\gamma\to\infty} \int_1^\gamma X_n X_m \, dr \int_S \boldsymbol{\nabla}_S u_n \boldsymbol{\nabla}_S v \, dS \\ &+ \lim_{\gamma\to\infty} \left(\int_1^\gamma \left(X_n' X_m' - k^2 X_n X_m \right) r^2 \, dr - ik\gamma^2 X_n(\gamma) X_m(\gamma) \right) \int_S u_n v \, dS \\ = \; & X_m(1) \int_S g v \, dS \qquad \forall\, v \in H^1(S), \quad m = 0, \ldots, N. \end{aligned} \tag{5.2}$$

The first limit is finite, since both X_n and X_m are of order $O(1/r)$. If one breaks the integral under the second limit into two parts and attempts to pass with $\gamma \to \infty$, then both limits will yield ∞. However, if one *first* recalls the equation for X_n and integrates by parts, then simply,

$$\begin{aligned}
& \int_1^\gamma \left(X_n' X_m' - k^2 X_n X_m \right) r^2 \, dr - ik\gamma^2 X_n(\gamma) X_m(\gamma) \\
= & \int_1^\gamma - \left(r^2 X_n' \right)' X_m \, dr - X_n'(1) X_m(1) - k^2 \int_1^\gamma X_n X_m r^2 \, dr \\
& + \gamma^2 \left(X_n'(\gamma) - ik X_n(\gamma) \right) X_m(\gamma) \\
= & -n(n+1) \int_1^\gamma X_n X_m \, dr - X_n'(1) X_m(1) + \gamma^2 \left(X_n'(\gamma) - ik X_n(\gamma) \right) X_m(\gamma).
\end{aligned} \tag{5.3}$$

Consequently, passing with $\gamma \to \infty$ yields

$$\begin{aligned}
& \sum_{n=0}^N \int_1^\infty X_n X_m \, dr \int_S \boldsymbol{\nabla}_S u_n \boldsymbol{\nabla}_S v \, dS \\
& + \left(-n(n+1) \int_1^\infty X_n X_m \, dr + X_m(1) \right) \int_S u_n v \, dS \\
= & \; X_m(1) \int_S g v \, dS \qquad \forall \, v \in H^1(S), \quad m = 0, \ldots, N.
\end{aligned} \tag{5.4}$$

5.1 Convergence of u^N to u

We continue to apply the same spectral analysis as before, i.e. assuming $Y_l(\boldsymbol{x}) = g^l / \|g^l\|_{L^2(S)}$, start with the spectral representation for the exact solution

$$u(\boldsymbol{x}, r) = \sum_{j=0}^\infty u^j(r) Y_j(\boldsymbol{x}) \tag{5.5}$$

and then select the test function v as

$$v(\boldsymbol{x}, r) = v(r) Y_l(\boldsymbol{x}). \tag{5.6}$$

This results in the equation for $u^l(r)$ of the form

$$\begin{aligned}
& \lim_{\gamma \to \infty} \left\{ \int_1^\gamma \frac{\partial u^l}{\partial r} \frac{\partial v}{\partial r} r^2 \, dr + l(l+1) \int_1^\gamma u^l \, v \, dr - ik\gamma^2 u^l(\gamma) \, v(\gamma) \right. \\
& \left. \qquad -k^2 \int_1^\gamma u^l \, v \, r^2 \, dr \right\} \\
= & \; v(1) \|g^l\|_{L^2(S)}
\end{aligned} \tag{5.7}$$

and the same appropriate space setting for u^l and v which will make the problem well defined. The functional setting is important from the point of view of the convergence analysis, not the solution itself. The solution is *known* and is simply $X^l(r) \|g_l\|_{L^2(S)}$.

Similarly assuming,

$$u_n(\boldsymbol{x}) = \sum_{j=1}^{\infty} u_n^j Y_l(\boldsymbol{x}), \quad u_n^j \in \mathbb{C}, \tag{5.8}$$

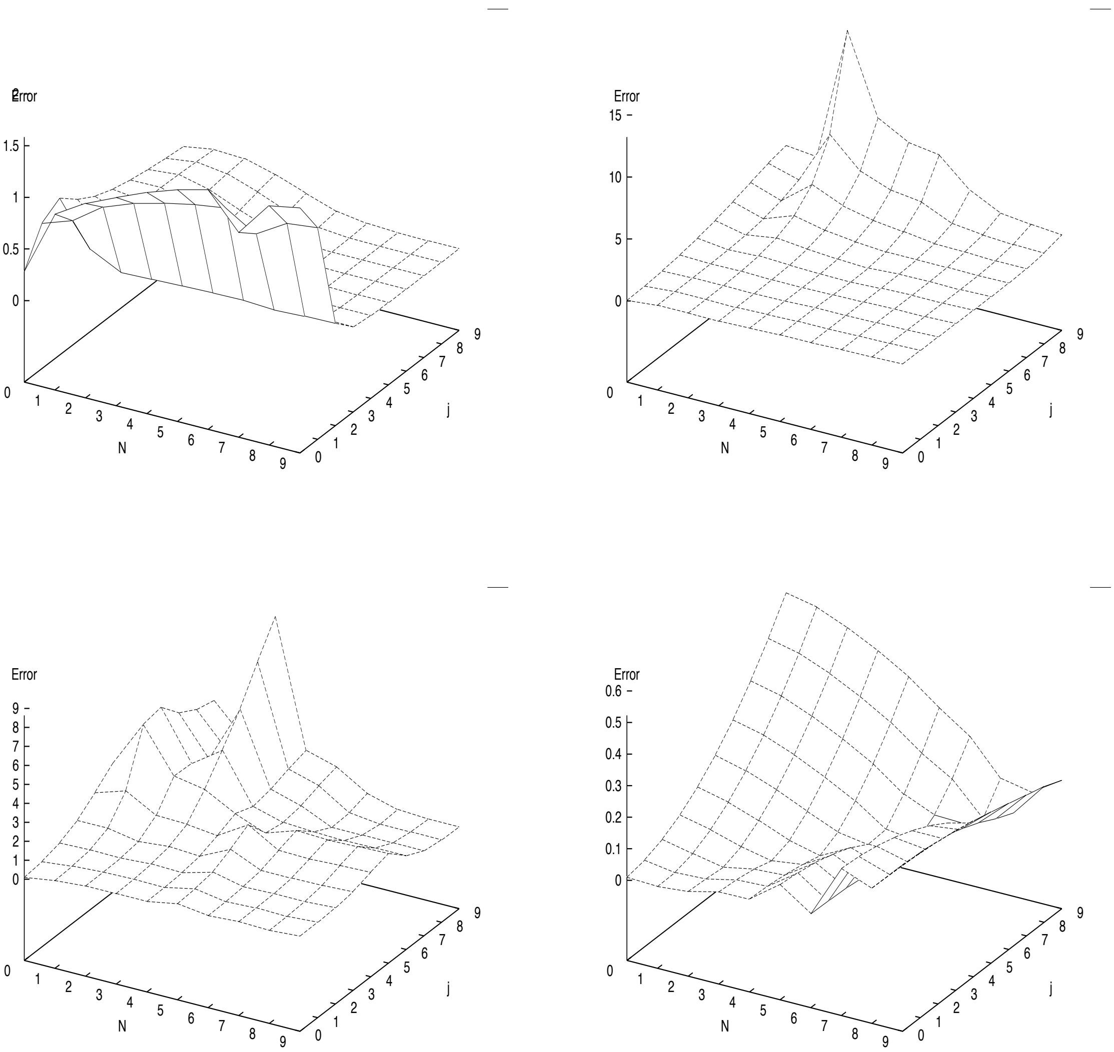
and setting $v = Y_l(\boldsymbol{x})$ in (5.4)

$$\begin{aligned} &\sum_{n=0}^{N} \left\{ (l(l+1) - n(n+1)) \int_1^{\infty} X_n X_m \, dr + X_m(1) \right\} u_n^l \\ = \;& X_m(1) \|g^l\|_{L^2(S)} \qquad m = 0, \ldots, N. \end{aligned} \tag{5.9}$$

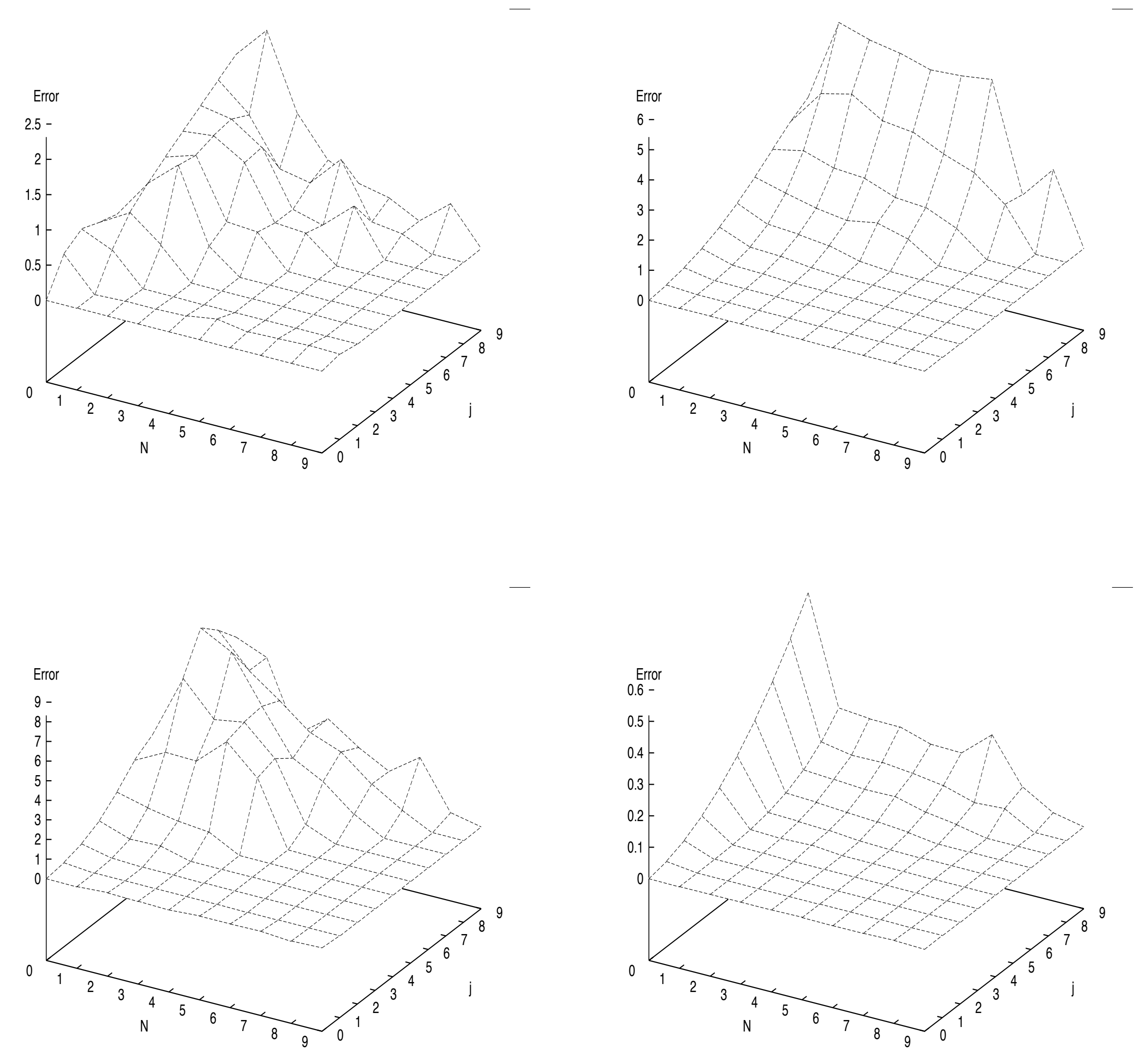
As before, the approximate solutions X_N^l are introduced and the representation of the error in form (4.25) is obtained. The only difference between the methods is in the ways in which the approximations X_N^l are calculated.

5.2 Comparison of the Infinite Element Concepts

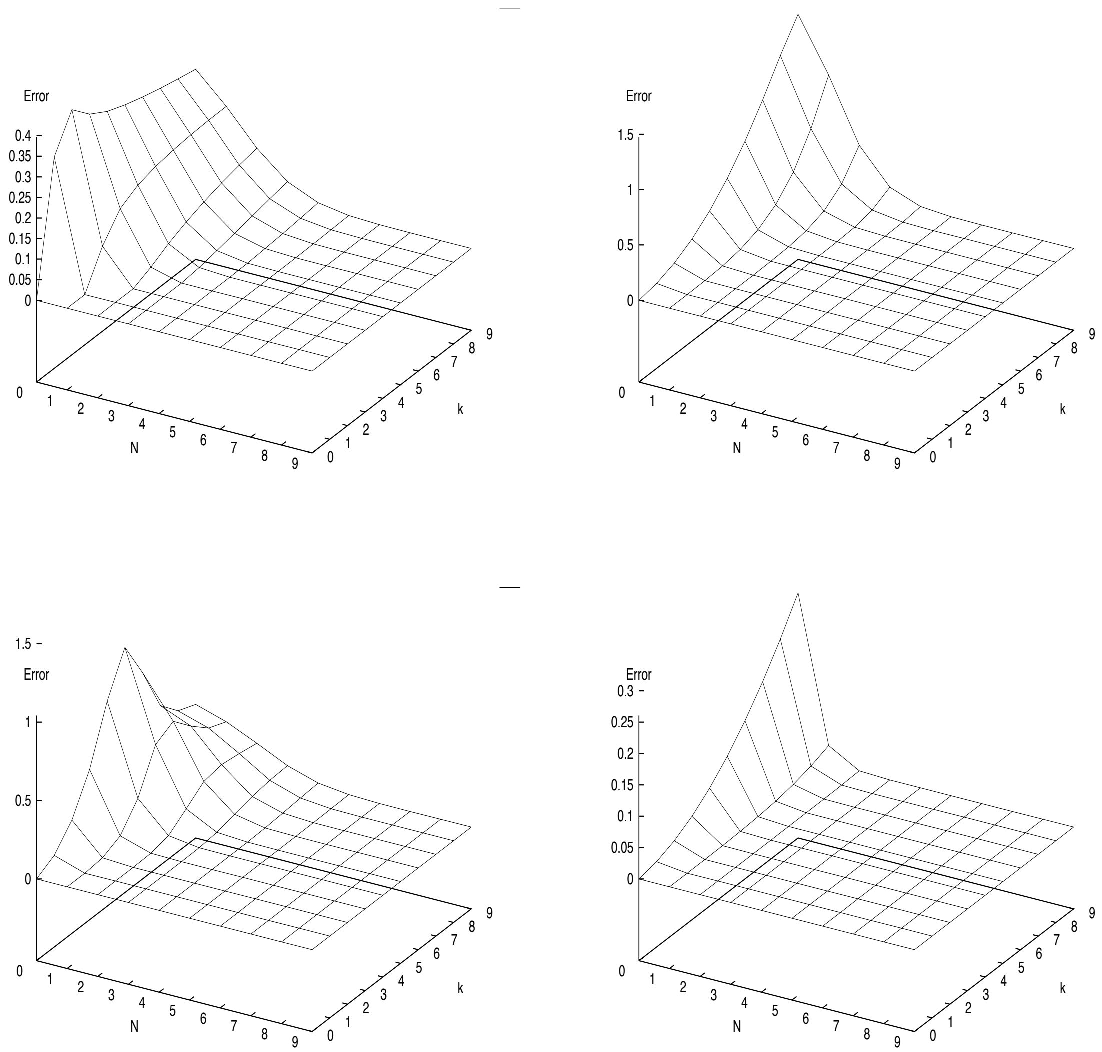
The weighted Sobolev space norm (2.10) has been suggested by the existence theory, as seen in [10]. Once there is an agreement upon the use of the norm, it becomes clear that the quality of the particular infinite element approximation is controlled by the "spectral" errors $\|X^l - X_N^l\|_{l,E}^2$. It should be emphasized that in (4.25) one has an *equality* sign. Thus, in order to compare the discussed concepts of the unconjugated and conjugated infinite elements one can simply calculate the errors with X_N^l obtained by solving systems (4.21) or (5.9). The calculations are done for four different values of frequency $k = 1, 5, 10, 50$ and summarized in Figures 3 to 10. The same calculations were already done for the conjugated formulations in [5, 8].



Figures 3-6: Unconjugated Leis formulation. Approximation error $\|X^j - X_N^j\|_{j,E}$, $j, N \leq 9$ for $k = 1, 5, 10, 50$.



Figures 7-10: Unconjugated Burnett formulation. Approximation error $\|X^j - X_N^j\|_{j,E}$, $j, N \leq 9$ for $k = 1, 5, 10, 50$.



Figures 11-14: Best Approximation Error E_N^j, $j, N \leq 9$ for $k = 1, 5, 10, 50$.

The question of the possible existence of a better, "ideal" formulation yielding smaller errors can easily be answered by comparing the presented errors with the *best approximation spectral errors*

$$E_N^l = \inf_{v \in \mathrm{span}\{X_1, \ldots, X_N\}} \|X^l - v\|_{l,E}.$$

Figures 11-14 show the best approximation error for the same values of k, compare [5]. In [5] we found out that the approximation errors of the conjugated Leis and the conjugated Burnett IEM are practically of the same order of magnitude as the best approximation error. From Figures 3-14 we see that the approximation errors for the unconjugated IEM do not behave that well anymore. For the given examples, $k = 1, 5, 10, 50$, the approximation error is well above the best approximation error. This indicates immediately that the unconjugated versions will not give reliable results, if the error is measured in the whole exterior domain. All these remarks, of course, are relative to the *particular* choice of the norm, in our case - the weighted Sobolev norm. The fact that the unconjugated IEM do not give an accurate approximation in the far field had also been reported in [2].

5.3 A Final Error Estimate for $\|u - u_N\|$

We postulate the following *conjecture* (compare [5, 8])

$$\exists\, c > 0 : \quad \|X^l - X^l_N\|_{k,E} \leq c \quad \forall\, l, N. \tag{5.10}$$

Making use of the spectral characterization of functions $g \in H^r(S)$ in terms of the spherical harmonics

$$\|g\|^2_{H^r(S)} \sim \sum_{n=0}^{\infty} n^{2r} g^{2n}, \tag{5.11}$$

where g^n denote the spectral components of function g, we obtain the final estimate in the form

$$\|u - u_N\|_{H^1_w(\Omega)} \leq \frac{c}{(N+1)^r} \|g\|_{H^r(S)}. \tag{5.12}$$

6 Conclusions

This work has investigated the effect of the complex conjugate on the quality of the approximation of the IEM. It has shown that the numerical solution obtained by the two unconjugated versions of the IEM converge in the near field. The analysis in [7] has shown that the two conjugated IEM do converge in the entire exterior domain. From the present analysis we see that the unconjugated IEM *does converge pointwise and more rapidly* than the conjugated IEM at the boundary of the scatterer and in the near field, provided that the unconjugated formulation is symmetric. This result is consistent with the results in [2]. It is evident that the conjugated Leis IEM is efficient if the solution to the rigid scattering problem is needed in the whole exterior domain. If only the near field solution is needed than the unconjugated Burnett IEM provides the near field solution much faster than the conjugated IEM.

Based on the analysis presented here we conclude that the conjugated Leis and unconjugated Burnett IEM should be used for the solution of exterior problems. Which of these formulations is more adequate for a given application depends on where in the exterior domain the solution is needed.

References

[1] R. J. Astley, G. J. Macaulay and J. P. Coyette, "Mapped Wave Envelope Elements for Acoustical Radiation and Scattering", *Journal of Sound and Vibration*, vol. 170, no. 1, pp. 97-118, 1994.

[2] I. Babuska and J. J. Shirron, "Solution of Exterior Helmholtz Problems using Finite and Infinite Elements", Dissertation preprint

[3] D. S. Burnett, "A Three-Dimensional Acoustic Infinite Element Based on a Prolate Spheroidal Multipole Expansion", *Journal of the Acoustical Society of America*, vol. 96, pp. 2798-2816, 1994.

[4] L. Demkowicz, "Asymptotic Convergence in Finite and Boundary Element Methods: Part1: Theoretical Results", *Computers and Mathematics with Applications*, vol. 27, no. 12, pp. 69-84, 1994.

[5] L. Demkowicz and K. Gerdes, "Convergence of the Infinite Element Methods for the Helmholtz Equation", *TICAM Report 95-07.*

[6] P. Geng, J. T. Oden and R. A. van de Geijn, "Massively Parallel Computation for Acoustical Scattering Problems using Boundary Element Methods", *TICAM Report 94-10*, August 1994.

[7] K. Gerdes and L. Demkowicz, "Solution of 3D Laplace and Helmholtz Equations in Exterior Domains Using *hp* Infinite Element Methods", submitted May 1995.

[8] K. Gerdes, "Solution of the 3D Laplace and Helmholtz Equation in Exterior Domains of Arbitrary Shape using HP-Finite-Infinite Elements", Doctoral Dissertation, The University of Texas at Austin, 1996.

[9] M. D. Greenberg, **Foundation of Applied Mathematics**, Prentice Hall, 1978.

[10] R. Leis, **Initial Boundary Value Problems in Mathematical Physics**, Teubner, 1986.

[11] P. M. Morse and J. Feshbach, **Methods of Theoretical Physics**, vols 1 and 2, McGraw-Hill, New York, 1953.

Research Reports

No.	Authors	Title
96-11	K. Gerdes	The conjugated vs. the unconjugated infinite element method for the Helmholtz equation in exterior domains
96-10	J. Waldvogel	Symplectic Integrators for Hill's Lunar Problem
96-09	A.-T. Morel, M. Fey, J. Maurer	Multidimensional High Order Method of Transport for the Shallow Water Equations
96-08	A.-T. Morel	Multidimensional Scheme for the Shallow Water Equations
96-07	M. Feistauer, C. Schwab	On coupled problems for viscous flow in exterior domains
96-06	J.M. Melenk	A note on robust exponential convergence of finite element methods for problems with boundary layers
96-05	R. Bodenmann, H.J. Schroll	Higher order discretisation of initial-boundary value problems for mixed systems
96-04	H. Forrer	Boundary Treatment for a Cartesian Grid Method
96-03	S. Hyvönen	Convergence of the Arnoldi Process when applied to the Picard-Lindelöf Iteration Operator
96-02	S.A. Sauter, C. Schwab	Quadrature for hp-Galerkin BEM in $\mathbb{R}^3$
96-01	J.M. Melenk, I. Babuška	The Partition of Unity Finite Element Method: Basic Theory and Applications
95-16	M.D. Buhmann, A. Pinkus	On a Recovery Problem
95-15	M. Fey	The Method of Transport for solving the Euler-equations
95-14	M. Fey	Decomposition of the multidimensional Euler equations into advection equations
95-13	M.D. Buhmann	Radial Functions on Compact Support
95-12	R. Jeltsch	Stability of time discretization, Hurwitz determinants and order stars
95-11	M. Fey, R. Jeltsch, A.-T. Morel	Multidimensional schemes for nonlinear systems of hyperbolic conservation laws
95-10	T. von Petersdorff, C. Schwab	Boundary Element Methods with Wavelets and Mesh Refinement
95-09	R. Sperb	Some complementary estimates in the Dead Core problem