Fully Discrete Multiscale Galerkin BEM

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Abstract

We analyze multiscale Galerkin methods for strongly elliptic boundary integral equations of order zero on closed surfaces in $\mathbb{R}^3$. Piecewise polynomial, discontinuous multiwavelet bases of any polynomial degree are constructed explicitly. We show that optimal convergence rates in the boundary energy norm and in certain negative norms can be achieved with "compressed" stiffness matrices containing $O(N (\log N)^2)$ nonvanishing entries where N denotes the number of degrees of freedom on the boundary manifold. We analyze a quadrature scheme giving rise to fully discrete methods. We show that the fully discrete scheme preserves the asymptotic accuracy of the scheme and that its overall computational complexity is $O(N (\log N)^4)$ kernel evaluations. The implications of the results for the numerical solution of elliptic boundary value problems in or exterior to bounded, three-dimensional domains are discussed.

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1. Introduction

Strongly elliptic boundary value problems in smooth and bounded domains $\Omega \subset \mathbb{R}^3$ can be reduced to equivalent integral equations on the boundary manifold $\Gamma = \partial\Omega$ [4, 35]. For second order elliptic systems, the solution is represented as a combination of so-called single and double layer potentials and boundary integral equations are obtained by passing with the source point to the boundary. The resulting boundary integral operators are often strongly elliptic pseudodifferential operators on the boundary manifold Γ [12]. The discretization of these integral equations by finite elements on the boundary manifold leads to the so-called *boundary element methods*. In the present paper we analyze Galerkin discretizations of a class of boundary integral operators of order zero which contains in particular the classical Fredholm equations of the second kind. We admit closed, piecewise analytic surfaces in $\mathbb{R}^3$ and require *strong ellipticity* of the boundary integral operators in the form of a Gårding inequality in $L^2(\Gamma)$. The Galerkin method is based on subspaces $V^L \subset L^2(\Gamma)$ of discontinuous, piecewise polynomials of degree $d \geq 0$. We use a particular, fully orthogonal multiwavelet basis of V^L the construction of which we perform explicitly for subdivisions based on triangles or quadrilaterals. Special cases include the *Haar wavelets* ($d = 0$) and the piecewise linear multiwavelets introduced in [25].

Following [25] we show that the stiffness matrix in the wavelet basis can be compressed to $O(N_L(\log N_L)^2)$ "essential" elements practically without affecting the asymptotic convergence rate of the scheme. This analysis assumes, however, the exact evaluation of the entries in the stiffness matrix which is unrealistic on general, curved surfaces. Hence we give here, apart from a self-contained exposition of the boundary reduction of elliptic partial differential equations, the construction of multiwavelets and the consistency analysis of the stiffness matrix compression, a new and general scheme for the approximation of the nonzero entries of the compressed Galerkin stiffness matrix by numerical quadrature. We show that with tensor product Gaussian quadratures of judiciously chosen orders and possibly geometric subdivisions of the region of integration, the asymptotic convergence rates of the Galerkin scheme based on the compressed stiffness matrix can be retained with only slight increase in computational complexity. Moreover, the resulting fully discrete scheme is *explicit* in that the compressed, numerically integrated Galerkin stiffness matrix is evaluated *directly*. This is in contrast to algorithms inspired by image compression, where first the whole Galerkin stiffness matrix is evaluated (corresponding to a full, digitized image), then a fast wavelet transform is applied to the rows and columns to change into the wavelet basis and only then small, nonzero entries are dropped (this so-called "ε-truncation" was first proposed in the context of integral operators in [3]). Clearly, this approach is still of $O(N_L^2)$ complexity and in particular the quadrature bottleneck to generate the dense Galerkin stiffness matrix makes it unattractive for large scale applications.

In contrast, our quadrature error estimations in conjunction with the consistency analysis of the compression allows to determine *a-priori* which entries of the compressed Galerkin stiffness matrix must be calculated to which accuracy and which entries can be dropped altogether. This allows to generate a-priori an appropriate sparse matrix storage scheme which handles only the $O(N_L (\log N_L)^2)$ essential entries of the compressed stiffness matrix. Further, since our multiwavelets are piecewise polynomial in local coordinates standard quadratures can be applied and different entries of the compressed stiffness matrix can be evaluated inde-

pendently of each other. This allows for the parallel evaluation and distributed storage of these elements on different processing units without communication – in contrast to compression type algorithms.

As far as we know, the present paper is the first analysis of a fully discrete multiscale Galerkin scheme in $\mathbb{R}^3$ that is not confined to a particular integral equation, but rather covers a whole class of boundary integral operators. Recent related work on the analysis of fully discrete, fast discretization schemes includes Rathsfeld [26] who considers the double layer potential equation on a polygonal boundary and a different approach to numerical quadrature.

Although we confine ourselves in the present paper essentially to classical boundary integral equations of the second kind, i.e. operators of order zero, we point out that the techniques used in our analysis are quite flexible and apply to more general situations. For example, the concepts of multiresolution analysis and the consistency analysis of the compressed Galerkin scheme can be generalized to boundary integral operators of nonzero order, *provided* a stable basis with the proper number of vanishing moments is available (the construction of specific, necessarily biorthogonal bases and the proof of their stability appear to be the principal issues here). For integral operators of orders ± 1 on polygons this was done in [23]. The compression and quadrature error analysis in Sections 3 and 4 of the present paper applies also for piecewise smooth surfaces such as polyhedra. There, however, the strong ellipticity of the boundary integral equations considered here (i.e., the validity of a Gårding inequality in L^2 on the boundary manifold) is a delicate problem (see [8]).

The main result of the paper, namely the consistent quadrature approximation of the compressed Galerkin stiffness matrix in essentially optimal complexity is likewise not confined to zero order operators on smooth surfaces.

The paper is organized as follows. In Section 2 we briefly review the reduction of elliptic boundary value problems to strongly elliptic boundary integral equations following [4, 35]. We focus on the classical, so-called *indirect method* (see, e.g., [4]). In this case, no general principle ensures strong ellipticity of the resulting boundary integral equations which are now Fredholm equations of the second kind. Thus, strong ellipticity in the form of a Gårding inequality in $L^2(\Gamma)$ for these boundary integral operators must be checked on a case by case basis via the associated *principal symbol*. As is well-known, strong ellipticity implies quasioptimal asymptotic convergence rates of Galerkin discretizations [11]. We give several examples for the boundary reduction which result in strongly elliptic boundary integral equations (for the general theory we refer to [4]).

The description and the analysis of the multiscale Galerkin discretization schemes is divided into two parts, Sections 3 and 4. In Section 3 we present, following [5, 6, 23, 25], a multiwavelet Galerkin discretization for strongly elliptic boundary integral operators of order zero. This class includes in particular all examples presented in Section 2. We give a consistency analysis showing that most of the $O((N_L)^2)$ entries in the Galerkin stiffness matrix can be neglected while essentially retaining the optimal asymptotic convergence rates of the full Galerkin scheme. This is also true in negative norms which implies superconvergence of field values at interior points obtained from inserting the Galerkin approximations to the boundary densities into the representation formula used in the boundary reduction. In this part of the analysis, we assume that all $O(N_L(\log N_L)^2)$ entries that are kept in the "compressed" Galerkin stiffness matrix are computed exactly – a rather unrealistic assumption.

The second part of our analysis, i.e. the derivation of a quadrature scheme for the direct

evaluation of the compressed Galerkin stiffness matrix, is presented in Section 4. We show how a consistent (i.e. preserving all asymptotic convergence properties established in Section 3 for the compressed Galerkin scheme), fully discrete multiwavelet Galerkin scheme can be obtained by using tensor product Gaussian quadratures of appropriate orders and, where necessary, element subdivisions [30], to approximate the entries of the compressed matrix. Special attention is paid to the evaluation of the singular integrals where the Galerkin scheme together with certain regularizing coordinate transformations as in [10] allows for a stable and accurate quadrature. The total work necessary to obtain the consistent quadrature approximation to the compressed Galerkin stiffness matrix is shown be $O(N_L(\log N_L)^3)$ kernel evaluations for the nonsingular and $O(N_L(\log N_L)^4)$ kernel evaluations for the singular integrals.

2. Strongly elliptic boundary integral equations

We consider boundary value problems for elliptic systems of second order in variational form:
Given $f \in L^2(\Omega)$, find $U \in H^1(\Omega)$ such that

$$\mathcal{L}U = 0 \qquad in\ \Omega \tag{2.1}$$

subject to the boundary conditions

$$\mathcal{B}U = f \qquad on\ \Gamma. \tag{2.2}$$

We will focus in this work on the case when $\mathcal{L}$ is a self-adjoint second order $N \times N$ matrix differential operator and $\mathcal{B}$ a boundary operator, either the trace operator γ_0 for the Dirichlet problem or the boundary operator γ_1 for the Neumann problem.

We assume that the boundary value problem (2.1) - (2.2) admits a unique weak solution in $H^1(\Omega)$.

For the operators $\mathcal{L}, \mathcal{B}$, we may also consider *exterior* boundary value problems posed in $\Omega^c = \mathbb{R}^3 \backslash \overline{\Omega}$. Here the boundary conditions must be appended by suitable *radiation conditions* at infinity in order to ensure the unique solvability of the boundary value problem in $H^1_{loc}(\Omega^c)$ (see, in particular, [13] for exterior problems in elasticity). Throughout $n(y)$ denotes the unit normal vector at $y \in \Gamma$ pointing into Ω^c.

We describe now the boundary integral equation reformulation for the boundary value problem (2.1) - (2.2). We assume that we are given a *fundamental solution* of the differential operator L in (2.1) which is a matrix function

$$G(x - y) : \mathbb{R}^3 \times \mathbb{R}^3 \backslash \{x = y\} \to \mathbb{C}^{N \times N}$$

so that

$$\mathcal{L}\,G(x - \cdot) = \delta(x - \cdot)\,I \tag{2.3}$$

holds in the sense of distributions.

For the boundary reduction, we look for w in the form of a potential with an unknown density function u on Γ. This approach will lead to boundary integral operators of order zero and is also known as the *indirect method*.

We use the following potentials.

Double layer ansatz for the Dirichlet problem:

$$U(x) = \int_\Gamma [\gamma_{1,y} G(x-y)] u(y) ds_y, \qquad x \in \Omega. \tag{2.4}$$

Taking traces in (2.4) and using the Dirichlet boundary condition, we arrive with the *jump relations* for the double layer potential (2.4) at the boundary integral equation

$$Au = \left(\frac{1}{2}I + K\right) u = f \text{ on } \Gamma, \quad (Ku)(x) = \int_\Gamma \gamma_{1,y} G(x-y) u(y) ds_y. \tag{2.5}$$

Once the boundary integral equation (2.5) is solved, the density function $u(y)$ is inserted into (2.4) and the solution $U(x)$ is obtained in Ω.

Single layer ansatz for the Neumann problem:

$$U(x) = \int_\Gamma G(x-y) u(y) ds_y, \qquad x \in \Omega. \tag{2.6}$$

Applying the natural boundary conditions γ_1 in (2.2) and letting $x \to \Gamma$, we get the boundary integral equation

$$Au = \left(\frac{1}{2}I - K'\right) u = f \text{ on } \Gamma, \quad (K'u)(x) = \int_\Gamma [\gamma_{1,x} G(x-y)] u(y) ds_y. \tag{2.7}$$

Once the boundary integral equation (2.7) is solved, the density function $u(y)$ is inserted into (2.6) and the solution $U(x)$ is obtained in Ω.

The unique solvability of the boundary integral equations (2.5), (2.7) can be ensured provided the boundary operator A is injective and satisfies, in each case, the following conditions:

1. *Continuity*:
$$\|Au\|_0 \leq C(A) \|u\|_0. \tag{2.8}$$

2. *Gårding Inequality*: There exists a compact operator $T : L^2(\Gamma) \to L^2(\Gamma)$ and a positive constant C such that
$$\text{Re} \langle (A+T)u, u \rangle \geq C \|u\|_{L^2(\Gamma)}^2 \quad \forall u \in L^2(\Gamma) \tag{2.9}$$

3. *Injectivity*:
$$Au = 0 \Rightarrow u = 0. \tag{2.10}$$

A sufficient condition of the validity of (2.9) is the positive definiteness of the *principal symbol* of the operator A which is defined, for example, in [12, 16].

We close this section with some examples of boundary value problems (2.1) - (2.2), exhibiting in each case the boundary operator γ_1, the fundamental solution G and the principal symbol of the boundary integral operators A in (2.5) and (2.7). Note that since the boundary integral operators in (2.5), (2.7) are mutually adjoint, their principal symbol matrices coincide up to Hermitean transposition.

We illustrate these considerations by some particular boundary value problems. In each case, we exhibit the fundamental solution and the boundary operator γ_1 together with the double layer kernel of the operator K in (2.5).

Example A: Boundary value problem for the Laplace equation

Here

$$\mathcal{L}U = -\Delta U, \qquad \mathcal{B}U = \gamma_1 U = \frac{\partial U}{\partial n}$$

$$G(x-y) = \frac{1}{4\pi \,|x-y|}, \qquad [\gamma_{1,y} G(x-y)] = \frac{n(y)\cdot(x-y)}{|x-y|^3}.$$

As is well-known, on smooth surfaces the double layer operator K and its adjoint K' are compact operators in $L^2(\Gamma)$ whence it follows that the principal symbol of the boundary integral operator A is $1/2$. This boundary value problem arises in many areas of engineering, so for example in electrostatic field calculations where U is the electric potential and $u(y)$ in (2.6) is the charge distribution on the electrode Γ.

Example B: Boundary value problem for the Lamé-Navier equations of linearized, three-dimensional elasticity

Here the infinitesimal displacement field $U : \Omega \to \mathbb{R}^3$ is determined from (2.1) - (2.2) with

$$\mathcal{L}U = -\mu\Delta U - (\lambda + \mu)\mathrm{grad\,div}U$$

and the Lamé-constants λ and μ are given parameters characterizing the homogeneous and isotropic material constituting the deforming body. Here γ_1 is the *traction operator* which is given explicitly on Γ by

$$\mathcal{B}U = \gamma_1 U = \lambda\,(\mathrm{div}U)\,n(y) + 2\mu\frac{\partial U}{\partial n} + \mu n \times \mathrm{curl}U.$$

The fundamental solution is

$$G(x-y) = \frac{\lambda + 3\mu}{8\pi(\lambda+2\mu)}\left\{\frac{1}{|x-y|} + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(x-y)(x-y)^\top}{|x-y|^3}\right\}$$

and

$$[\gamma_{1,y} G(x-y)] = \frac{\mu^2}{4\pi(\lambda+2\mu)}\left\{\frac{n(y)^\top (x-y)}{|x-y|^3}I + \frac{n(y)(x-y)^\top - (x-y)n(y)^\top}{|x-y|^3}\right.$$

$$\left. + \frac{2(\lambda+\mu)}{\mu}\frac{n(y)^\top(x-y)}{|x-y|^5}(x-y)(x-y)^\top\right\}.$$

To present the principal symbol, we denote for $x \in \Gamma$ by

$$N(x) = (t_1(x), t_2(x), n(x))$$

the matrix consisting of two mutually orthogonal unit tangent vectors at x to Γ and the exterior unit normal vector $n(y)$. Then the principal symbol $\sigma_0(x,\xi)$ of A in (2.5) is given by [35]

$$\sigma_0(x,\xi) = N^{-1}(x)\frac{1}{|\xi|}\begin{pmatrix} \varepsilon\,|\xi| & 0 & -\mathbf{i}\gamma\xi_1 \\ 0 & \varepsilon\,|\xi| & -\mathbf{i}\gamma\xi_2 \\ \mathbf{i}\gamma\xi_1 & \mathbf{i}\gamma\xi_2 & \varepsilon\,|\xi| \end{pmatrix}N(x)$$

where $\gamma = \frac{\mu}{\lambda+2\mu}$ and $\varepsilon = 1$ for interior and $\varepsilon = -1$ for exterior problems. Hence we see that the boundary integral equations (2.5), (2.7) are strongly elliptic in this case, too.

Example C: Oblique derivative problem

Assume that $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain. The oblique derivative problem for $L = -\Delta$ in Ω^c consists in solving $\mathcal{L}U = 0$ in Ω^c subject to the boundary condition

$$\mathcal{B}U(x) = b(x) \cdot \operatorname{grad} U(x) + \sigma(x)U(x) = f(x) \qquad x \in \Gamma \tag{2.11}$$

and the radiation condition

$$\lim_{|x| \to \infty} U(x) = 0.$$

Here $b = b(x) : \Gamma \to \mathbb{R}^3$ is a given direction field of length one, i.e. $b^\top b = 1$, depending smoothly on x and $f(x)$ is a given, sufficiently smooth function on Γ. It was proved by Giraud that this problem has a unique solution if, for example, $\sigma(x) \geq 0$ and $b(x)^\top n(x) > 0 \quad \forall x \in \Gamma$ (see, e.g., [19]).

Using the single layer ansatz (2.6) and inserting into (2.11), we get the following Cauchy-singular boundary integral equation for the unkown density $u(y)$ (see [19]):

$$\frac{u(x)}{2a(x)} + \int_\Gamma \frac{\partial G(x-y)}{\partial b(x)} u(y) ds_y + \sigma(x) \int_\Gamma G(x-y)u(y) ds_y = f(x) \quad x \in \Gamma. \tag{2.12}$$

Here $a(x) = b(x)^\top n(x)$. This equation arises for example in physical geodesy for the determination of the earth's shape from gravity measurements [20]. In [19], the principal symbol of the boundary integral operator in (2.12) is derived. We have

$$\sigma_0(x) = \frac{1}{2}\left(n(x)^\top b(x) + \mathbf{i}r(x)^\top b(x)\right) \tag{2.13}$$

where $r(x)^\top n(x) = 0$, i.e $r(x)$ is any direction in the tangent plane. Since

$$\operatorname{Re}\sigma_0(x) \geq \frac{1}{2} \inf_{x \in \Gamma} \left|b(x)^\top n(x)\right|$$

the boundary integral operator in (2.12) is strongly elliptic if the direction $b(x)$ is not tangential anywhere on Γ.

Example D: Exterior Stokes flow

Here we are interested, for example, in determining the velocity field and the pressure distribution (U, p) of Newtonian, incompressible viscous flow exterior to a smooth and bounded surface Γ in $\mathbb{R}^3$. For illustration we consider the exterior Dirichlet problem. Other cases can be handled similarly, see, e.g., [17].

The governing equations are

$$\begin{aligned} -\nu\Delta U + \operatorname{grad} p &= 0, \quad \operatorname{div} U = 0 \text{ in } \Omega^c, \\ U &= f \text{ on } \Gamma \end{aligned} \tag{2.14}$$

where f is a prescribed velocity field on the surface of the body satisfying $\int_\Gamma f \cdot n\, ds = 0$ and $\nu > 0$ denotes the viscosity of the fluid. We require in addition that the fluid is at rest at infinity, i.e.

$$|U(x)| = o(1), \quad |\operatorname{grad} U(x)| = o(|x|^{-1})$$
$$|p(x)| = o(|x|^{-1}), \quad |\operatorname{grad} p(x)| = o(|x|^{-2}) \qquad \text{for } |x| \to \infty.$$

The fundamental velocity tensor, the so-called *Stokeslet*, is given by

$$G(x - y) = \frac{1}{8\pi\nu} \left\{ |x - y|^{-1}\, I + \frac{(x - y)\,(x - y)^\top}{|x - y|^3} \right\}.$$

We represent (U, p) as double layer potentials of an unknown density $u : \Gamma \to \mathbb{R}^3$ as follows:

$$U_i(x) = \sum_{j,k=1}^{3} \int_\Gamma u_j(y) T_{ijk}(y - x) n_k(y) ds_y, \tag{2.15}$$

$$p(x) = \sum_{j,k=1}^{3} \int_\Gamma u_j(y) \Pi_{jk}(y - x) n_k(y) ds_y \tag{2.16}$$

where

$$T_{ijk}(\hat{x}) := -\frac{3}{4\pi} \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{|\hat{x}|^5}$$

and

$$\Pi_{jk}(\hat{x}) = \frac{\nu}{2\pi} \left\{ -\frac{\delta_{jk}}{|\hat{x}|^3} + 3\frac{\hat{x}_j \hat{x}_k}{|\hat{x}|^5} \right\}.$$

Letting in (2.15) the point x tend to Γ, we obtain the boundary integral equation (2.5) with the hydrodynamic double layer potential

$$(Ku)_i(x) = \sum_{j,k=1}^{3} \int_\Gamma u_j(y) T_{ijk}(y - x) n_k(y) ds_y, \quad x \in \Gamma.$$

We observe that due to the classical estimate

$$|(x - y) \cdot n(y)| \leq C(\Gamma) |x - y|^2 \qquad x, y \in \Gamma$$

valid for smooth, bounded surfaces Γ the double layer kernel $\sum_{k=1}^{3} T_{ijk}(x - y) n_k(y)$ admits the estimate

$$\left| \sum_{k=1}^{3} T_{ijk}(x - y) n_k(y) \right| \leq C(\Gamma) |x - y|^{-1},$$

i.e. it is weakly singular and thus integrable. Moreover, on smooth surfaces the hydrodynamic double layer potential K is a compact operator on $[L^2(\Gamma)]^3$. Thus, the principal symbol of the operator $A = \frac{1}{2}I + K$ is equal to $\frac{1}{2}I$ and hence A is a strongly elliptic boundary integral operator in $[L^2(\Gamma)]^3$. This can also be seen by letting formally $\lambda \to \infty$ in the fundamental solution and the principal symbol for the elasticity problem. The proof of the injectivity of A is given in [17, Theorem 3.1].

We remark that corresponding boundary integral equations are also obtained for the *time harmonic* variants of the above boundary value problems. The positivity of the principal symbols is unchanged then since the mass term $-\rho\omega^2 U$ in the differential operator does not contribute to the leading derivatives.

In summary, in each of the above examples we can reduce the boundary value problem (2.1), (2.2) to a boundary integral equation

$$Au = f \tag{2.17}$$

for the unknown density function $u \in L^2(\Gamma)$ with a strongly elliptic pseudodifferential operator A of order zero. It is for such problems that we develop and analyze now a wavelet based Galerkin discretization scheme.

3. Multiscale Galerkin Boundary Elements

3.1. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a piecewise analytic, orientable Lipschitz boundary manifold $\Gamma = \partial\Omega$. More precisely, Γ admits a partition into N_0 disjoint open sets Γ_j, $j = 1, ..., N_0$ and there exists a covering of Γ by a collection of larger, open sets $\widetilde{\Gamma}_j$ with $\Gamma_j \subset\subset \widetilde{\Gamma}_j \subset \Gamma$, i.e.

$$\Gamma = \overline{\bigcup_{1 \leq j \leq N_0} \Gamma_j} = \bigcup_{1 \leq j \leq N_0} \widetilde{\Gamma}_j, \qquad \Gamma_j \cap \Gamma_k = \emptyset \quad j \neq k. \tag{3.1}$$

There exist local charts $\kappa_j \in C^{0,1}(\widetilde{\Gamma}_j, \mathbb{R}^2)$ which map $\widetilde{\Gamma}_j$ bijectively onto certain reference domains $\tilde{\mathcal{U}}_j^0 \subset \mathbb{R}^2$. The set $\{(\widetilde{\Gamma}_j, \kappa_j)\}$ forms a Lipschitz atlas of Γ.

We assume that each Γ_j is a curvilinear, either quadrilateral or triangular surface piece in $\mathbb{R}^3$. We can therefore in particular assume that for all quadrilateral resp. triangular Γ_j there exists a *common* reference domain $\mathcal{U}^0 \subset \tilde{\mathcal{U}}_j^0$ such that

$$\kappa_j^{-1}(\mathcal{U}^0) = \Gamma_j, \quad \kappa_j^{-1} \text{ is analytic on } \overline{\mathcal{U}^0}, \quad j = 1, ..., N_0. \tag{3.2}$$

Therefore $\mathcal{U}^0$ is either the unit triangle $\{(\xi_1, \xi_2) : -1 < \xi_1 < 1, -1 < \xi_2 < \xi_1\}$ or the unit square $\{(\xi_1, \xi_2) : -1 < \xi_i < 1, i = 1, 2\}$ in $\mathbb{R}^2$. Admissible boundaries include therefore closed C^∞-manifolds as well as polyhedra.

By do we denote the surface measure defined almost everywhere on Γ. We consider the space $L^2(\Gamma)$ of functions $u : \Gamma \to \mathbb{C}^N$ which are square integrable with respect to do. An inner product on $L^2(\Gamma)$ is given by

$$\langle u, v \rangle = \int_\Gamma u\bar{v}\,do. \tag{3.3}$$

We also consider the Sobolev spaces $H^s(\Gamma_j)$ of functions with pullback in $H^s(\mathcal{U}^0)$ endowed with the norm $H^s(\mathcal{U}^0)$ transported to Γ_j. The space of functions $u \in L^2(\Gamma)$ with $u|_{\Gamma_j} \in H^s(\Gamma_j)$ for $s > 0$ is denoted by $\prod_{j=1}^{N_0} H^s(\Gamma_j)$. Evidently, the expression

$$\|u\|_s = \left(\sum_{j=1}^{N_0} \|u\|_{H^s(\Gamma_j)} \right)^{1/2} \tag{3.4}$$

is a norm in $\prod_{j=1}^{N_0} H^s(\Gamma_j)$.

An inner product $(\,\cdot\,,\,\cdot\,)$, equivalent to $\langle\,\cdot\,,\,\cdot\,\rangle$ (i.e., giving rise to equivalent norms) in $L^2(\Gamma)$, can then be defined by

$$(u,v) = \sum_{j=1}^{N_0} \int_{\mathcal{U}^0} \left(\kappa_j^* u|_{\Gamma_j}\right) \left(\kappa_j^* \overline{v}|_{\Gamma_j}\right) d\xi_1 \, d\xi_2. \tag{3.5}$$

We are interested in the numerical solution of the operator equation (2.17) in the weak form

$$u \in L^2(\Gamma) \qquad \langle Au, v\rangle = \langle f, v\rangle \qquad \forall v \in L^2(\Gamma). \tag{3.6}$$

Here the operator A is a boundary integral operator which can be represented in the form

$$(Au)(x) = c(x)u(x) + p.v. \int_\Gamma K(x,y)u(y)do(y) \tag{3.7}$$

where $K(x,y) = \mathcal{K}(x,y,x-y)$ and the kernel $\mathcal{K}$ has the form

$$\mathcal{K}(x,y,z) = \sum_{k\le|\alpha|\le k+a} s_\alpha(x,y)z^\alpha \, |z|^{-2-k}. \tag{3.8}$$

The coefficient functions $s_\alpha(x,y)$ and $c(x)$ are analytic functions of $x \in \overline{\Gamma}_i$ and $y \in \overline{\Gamma}_j$, $i,j = 1,...,N_0$ and $a,k \in \mathbb{N}_0$. All kernels in the examples in Section 2 are of this type with $k=1$ or $k=3$.

The integral in (3.7) is in general to be understood in the Cauchy principal value sense, i.e.

$$p.v. \int_\Gamma K(x,y)u(y)do(y) = \lim_{\varepsilon\to0} \int_{\Gamma\setminus B_\varepsilon(x)} K(x,y)u(y)do(y). \tag{3.9}$$

Here $B_\varepsilon(x) = \{y \in \mathbb{R}^3 : |x-y| < \varepsilon\}$ denotes the open ball of radius ε about the point x. We assume that the kernel $K(x,y)$ is such that the limit in (3.9) exists (see [19] and Section 5.4 ahead for details).

Approximate solutions to (3.6) are obtained by the Galerkin method. Given a dense sequence $\{V^L\}_{L=0}^\infty$ of finite dimensional subspaces of $L^2(\Gamma)$, we solve

$$u^L \in V^L \quad \left\langle Au^L, v\right\rangle = \langle f, v\rangle \qquad \forall v \in V^L. \tag{3.10}$$

The Gårding inequality (2.9) and the injectivity (2.10) of the operator A, ensure the unique solvability of (3.6), provided L is sufficiently large [11]. We denote by P_L the orthogonal projection

$$P_L: L^2(\Gamma) \to V^L, \qquad ((v - P_Lv), \varphi) = 0 \quad \forall \varphi \in V^L. \tag{3.11}$$

Proposition 3.1 *Assume* (2.8) - (2.10). *Then, for every* $f \in L^2(\Gamma)$ *and sufficiently large* L, *the approximate problem* (3.10) *is stable in the sense that*

$$\left\|P_L Au^L\right\|_0 \ge C_s \left\|u^L\right\|_0 \qquad \text{for all } u^L \in V^L. \tag{3.12}$$

In particular, there exist unique solutions u^L *of* (3.10) *which converge quasioptimally to the unique solution* u *of* (3.6), *i.e.,*

$$\left\|u - u^L\right\|_0 \le C \inf_{v\in V^L} \|u - v\|_0. \tag{3.13}$$

For a proof, see e.g. [11].

Our interest is here in so-called multiscale discretizations which are based on special bases for the spaces $\{V^L\}_{L=0}^\infty$ which we define next.

3.2. Multiwavelet Basis

To define $V^l \subset L^2(\Gamma)$ we divide $\mathcal{U}^0$ into 4^l subsquares resp. congruent subtriangles $\{\mathcal{U}_k^l\}$ by successively halving the sides l times. Then we define the spaces

$$V^l = \{\, u \in L^2(\Gamma) \mid \left(\kappa_j^* u|_{\Gamma_j}\right)\big|_{\mathcal{U}_k^l} \in \Pi_d(\mathcal{U}_k^l) \quad j = 1, \ldots, N_0, k = 1, \ldots, 4^l \,\}.$$

of discontinuous, piecewise polynomials of total degree d. Here κ^* is the usual pullback operator, i.e. $\kappa^* \varphi := \varphi(\kappa^{-1}(u))$. Throughout the construction of the multiwavelet basis in this section, the notion of orthogonality is understood with respect to the inner product (3.5).

Let $\{\tilde{\varphi}_\nu\}$ $\nu = 1, \ldots, \theta_d$ with $\theta_d = (d+1)^2$ for quadrilaterals and $\theta_d = (d+1)(d+2)/2$ for triangles be an orthonormal basis of $\Pi_d(\mathcal{U}^0)$. The dimension of V^l is $N_l = \theta_d N_0 4^l$. We introduce the multiindex

$$I = (j, l, k, \nu), \qquad 1 \le j \le N_0, l \in \mathbb{N}_0, 1 \le k \le 4^l, 1 \le \nu \le \theta_d.$$

Let τ_k^l denote the affine transformation which maps $\mathcal{U}_k^l$ to $\mathcal{U}^0$ and define the function $\varphi_I \colon \Gamma \to \mathbb{R}$ by

$$\kappa_j^* \left(\varphi_{(j,k,l,\nu)}\big|_{\Gamma_j}\right) = \begin{cases} 2^l \tilde{\varphi}_\nu \circ \tau_k^l & \text{in } \mathcal{U}_k^l \\ 0 & \text{otherwise} \end{cases}, \qquad \varphi_{(j,k,l,\nu)}\big|_{\Gamma \setminus \Gamma_j} = 0.$$

For $l \in \mathbb{N}_0$ an orthonormal basis of V^l is given by the functions $\{\, \varphi_I \mid I \in \mathcal{I}_l \,\}$ where

$$\mathcal{I}_l = \{\, (j, l, k, \nu) \mid j = 1, \ldots, N_0, \ k = 1, \ldots, 4^l, \ \nu = 1, \ldots, \theta_d \,\}.$$

Obviously, the spaces V^l form a hierarchy, i.e.,

$$V^0 \subset V^1 \subset \cdots \subset V^l \subset V^{l+1} \subset \cdots$$

As usual, we define a sequence of spaces W^l as orthogonal complement with respect to $(\cdot, \cdot)$ of V^{l-1} in V^l:

$$W^l := \{\, \psi \in V^l \mid (\varphi, \psi) = 0 \quad \forall \varphi \in V^{l-1} \,\}. \tag{3.14}$$

Then $V^{l+1} = V^l \oplus W^{l+1}$ and we obtain the multilevel splitting

$$V^L := W^0 \oplus W^1 \oplus \cdots \oplus W^L \tag{3.15}$$

where $W^0 := V^0$. Hence every function $u^L \in V^L$ admits a unique decomposition

$$u^L = w^0 + w^1 + \cdots + w^L, \qquad w^l \in W^L, l = 0, \ldots, L. \tag{3.16}$$

Let $P_{-1} = 0$. Then $w^l = (P_l - P_{l-1})u^L$ in (3.16).

To obtain an orthonormal basis for W^l we proceed similarly as for V^l. First we consider the space $\tilde{W}^1$ of discontinuous, piecewise polynomials of total degree d on the four $\mathcal{U}_k^1$, $k = 1, \ldots, 4$ which are orthogonal in $L^2(\mathcal{U}^0)$ on all polynomials of total degree d on $\mathcal{U}^0$. Let $\tilde{\psi}_1, \ldots, \tilde{\psi}_{3\theta_d}$ denote an orthonormal basis of $\tilde{W}^1$.

We define the functions $\psi_I \colon \Gamma \to \mathbb{R}$ for $l \geq 1$ by

$$\kappa_j^* \left(\psi_{(j,l,k,\nu)} \big|_{\Gamma_j} \right) = \begin{cases} 2^{l-1} \tilde{\psi}_\nu \circ \tau_k^l & \text{in } \mathcal{U}_k^l \\ 0 & \text{otherwise} \end{cases}, \qquad \psi_{(j,l,k,\nu)} \big|_{\Gamma \backslash \Gamma_j} = 0. \tag{3.17}$$

For $l = 0$ we use the basis functions

$$\psi_I = \varphi_I \quad \text{for } I = (j, 0, k, \nu). \tag{3.18}$$

Then for $l \in \mathbb{N}_0$ an orthonormal basis of W^l is given by the functions $\{\, \psi_I \mid I \in \mathcal{J}_l \,\}$ where

$$\mathcal{J}_l := \{\, (j, l, k, \nu) \mid 1 \leq j \leq N_0, 1 \leq k \leq 4^l, 1 \leq \nu \leq 3\theta_d \,\} \quad \text{for } l \geq 1$$
$$\mathcal{J}_0 := \mathcal{I}_0. \tag{3.19}$$

Remark 3.1 The multiwavelets ψ_I in (3.17) are in fact fully orthonormal with respect to the inner product (3.5). They are, moreover, piecewise polynomials in local coordinates. This will be essential in the quadrature error analysis in Section 5 ahead.

By (3.15) an orthonormal basis of V^L for $L \in \mathbb{N}_0$ is given by

$$\{\, \psi_I \mid I \in \mathcal{J}_0 \cup \ldots \cup \mathcal{J}_L \,\}.$$

The L^2-projectors P_L onto V^L admit the explicit representation

$$u^L = P_L u = \sum_{I \in \mathcal{I}_L} (u, \varphi_I) \, \varphi_I \tag{3.20}$$

and we have for w^l in (3.16)

$$w^l = (P_l - P_{l-1})u = \sum_{J \in \mathcal{J}_l} (u, \psi_J) \, \psi_J . \tag{3.21}$$

Therefore the norm of functions in $L^2(\Gamma)$ can be characterized by the multiwavelet expansion coefficients.

Proposition 3.2 *For every* $u \in L^2(\Gamma)$*, there holds*

$$\|u\|_{L^2(\Gamma)}^2 \sim \sum_{l=0}^{\infty} \sum_{J \in \mathcal{J}_l} |(u, \psi_J)|^2 \tag{3.22}$$

where $\sim$ *denotes the equivalence of norms.*

Moreover, the higher order Sobolev norms of smoother functions can be estimated by properly weighted sums of multiwavelet coefficients.

Proposition 3.3 *Let* $0 \leq s \leq d + 1$*. Then for every* $u \in \Pi_{j=1}^{N_0} H^s(\Gamma_j)$ *and every* $L \in \mathbb{N}$

$$\sum_{l=0}^{L} 2^{2ls} \sum_{J \in \mathcal{J}_l} |(u, \psi_J)|^2 \leq C \sum_{l=0}^{L} 2^{2ls} \|w^l\|_0^2 \leq C \, L^\nu \sum_{j=0}^{N_0} \|u\|_{H^s(\Gamma_j)}^2 \tag{3.23}$$

where $\nu = 0$ *for* $0 \leq s < d + 1$ *and* $\nu = 1$ *for* $s = d + 1$*.*

12

The proof is obtained exactly as that of [25, Proposition 4.2].

We use the multiwavelet basis (3.18), (3.17) in the Galerkin equations (3.10). To this end, we write u^L in the form

$$u^L = \sum_{l=0}^{L} \sum_{J \in \mathcal{J}_l} u_J^L \psi_J, \qquad u_J^L = \left(u^L, \psi_J \right)$$

and denote by $\vec{u} = (u_J^L)_{J \in \mathcal{J}_l; l=0,\ldots,L}$ the vector of unknown coefficients of u^L. It is determined by the linear system

$$\sum_{l=0}^{L} \sum_{J \in \mathcal{J}_l} \langle \psi_I, A\psi_J \rangle \, u_J^L = \langle \psi_I, f \rangle, \quad I \in \mathcal{J}_{l'}, l' = 0, \ldots, L. \tag{3.24}$$

We denote the $N_L \times N_L$ matrix by A^L, i.e.,

$$A_{IJ}^L := \langle \psi_I, A\psi_J \rangle, \quad I \in \mathcal{J}_{l'}, J \in \mathcal{J}_l, l, l' = 0, \ldots, L. \tag{3.25}$$

Then (3.24) corresponds to the linear system

$$A^L \vec{u} = \vec{f}$$

where $\vec{f} = (\langle \psi_I, f \rangle)_{I \in \mathcal{J}_{l'}, l'=0,\ldots,L}$. Note that A^L is not symmetric in general. The condition numbers of the sequence $\{A^L\}$ of matrices is bounded:

Proposition 3.4 *There exists $\kappa^* \in \mathbb{R}$ such that for all L*

$$\mathrm{cond}_2(A^L) \leq \kappa^*.$$

Proof: This follows from the stability (3.12) and the norm equivalence (3.22). $\qquad \square$

3.3. Consistency Analysis for the Compressed Galerkin Scheme

The wavelet basis $\{\psi_I\}$ defined in (3.18), (3.17) has vanishing moments in local coordinates. More precisely, for all $J \in \mathcal{J}_l, l \geq 1$,

$$\int_{\mathcal{U}^0} (\psi_J|_{\Gamma_j} \circ \chi_j)(s) s^\alpha \, ds = 0 \quad \text{for } |\alpha| \leq d. \tag{3.26}$$

The vanishing moment property (3.26) implies the smallness of certain entries of the matrix A^L. This is due to

$$\left| D_x^\alpha D_y^\beta K(x, y) \right| \leq \frac{c(\alpha, \beta, \Gamma)}{|x - y|^{2+|\alpha|+|\beta|}} \tag{3.27}$$

(here D_x^α, D_y^β are Cartesian derivatives in $\mathbb{R}^3$ acting on a smooth extension of K to a tubular neighborhood of Γ) which is an immediate consequence of (3.8). By S_J we denote the set

$$S_J = \{ x \in \Gamma \mid \psi_J(x) \neq 0 \} \tag{3.28}$$

and we define $d_{JJ'} = \mathrm{dist}(S_J, S_{J'})$. Then there holds

Lemma 3.1 *The entries $A_{J,J'}^L$ in the Galerkin matrix (3.25) with $d_{JJ'} > 0$ satisfy*

$$\left| A_{JJ'}^L \right| \leq C d_{JJ'}^{-2(d+2)} 2^{-(d+2)(l+l')}. \tag{3.29}$$

The proof follows by (3.26) and Taylor expansion of the kernel about the barycenters of S_J and $S_{J'}$ (see, e.g., [25] for details).

We will now show that most of the $N_L \times N_L$ entries of the stiffness matrix $\{A_{JJ'}\}$ can be replaced by zero without affecting the convergence rates of the resulting "compressed Galerkin scheme". To this end we introduce the following *truncation strategy*:

$$\tilde{A}_{JJ'} := \begin{cases} A_{JJ'} & \text{if } d_{JJ'} \leq \delta_{l,l'} \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

where $\{\delta_{l,l'}\}$ is a matrix of truncation parameters at our disposal.

To estimate the consistency error thus introduced we define the block matrices

$$\mathbf{A}_{l,l'} := \{A_{J,J'}\}_{J \in \mathcal{J}_l, J' \in \mathcal{J}_{l'}},$$

with $\mathcal{J}_l, \mathcal{J}_{l'}$ as in (3.19). Analogously we define $\tilde{\mathbf{A}}_{l,l'}$, $l, l' = 0, \ldots, L$. These are submatrices of the respective stiffness matrices. In the following lemma we estimate the effect of the truncation (3.30) on each block.

Lemma 3.2

$$\left\| \mathbf{A}_{l,l'} - \tilde{\mathbf{A}}_{l,l'} \right\|_\infty \leq C \delta_{l,l'}^{-2(d+2)} 2^{-(d+2)l} 2^{-dl'} \max\{\delta_{l,l'}^2, 2^{-2l'}\}, \tag{3.31}$$

$$\left\| \mathbf{A}_{l,l'} - \tilde{\mathbf{A}}_{l,l'} \right\|_1 \leq C \delta_{l,l'}^{-(d+2)l} 2^{-dl} 2^{-(d+2)l'} \max\{\delta_{l,l'}^2, 2^{-2l}\}. \tag{3.32}$$

Proof: We have with Lemma 3.1

$$\left\| \mathbf{A}_{l,l'} - \tilde{\mathbf{A}}_{l,l'} \right\|_\infty = \max_{J \in \mathcal{J}_l} \sum_{\substack{J' \in \mathcal{J}_{l'} \\ d_{JJ'} \geq \delta_{l,l'}}} |A_{J,J'}| \leq C \max_{J \in \mathcal{J}_l} \sum_{\substack{J' \in \mathcal{J}_{l'} \\ d_{JJ'} \geq \delta_{l,l'}}} d_{JJ'}^{-2(d+2)} 2^{-(d+2)(l+l')}.$$

We estimate the terms from the $S_{J'}$ closest to S_J directly and majorize the remaining terms by an integral

$$
\begin{aligned}
\left\| \mathbf{A}_{l,l'} - \tilde{\mathbf{A}}_{l,l'} \right\|_\infty &\leq C \operatorname{dist}(S_J, S_{J'})^{-2(d+2)} 2^{-(d+2)(l+l')} \\
&\quad + \int_{\substack{x \in \Gamma \\ \operatorname{dist}(x, S_J) \geq \delta_{l,l'}}} \operatorname{dist}(S_J, S_{J'})^{-2(d+2)} 2^{-(d+2)l} 2^{-dl'} \, d\sigma_x \\
&\leq C \left(\operatorname{dist}(S_J, S_{J'})^{-2(d+2)} 2^{-(d+2)(l+l')} + \operatorname{dist}(S_J, S_{J'})^{-2(d+1)} 2^{-(d+2)l} 2^{-dl'} \right).
\end{aligned}
$$

The estimate for $\left\| \mathbf{A}_{l,l'} - \tilde{\mathbf{A}}_{l,l'} \right\|_1$ follows in the same way with J and J' interchanged. $\qquad \square$

We estimate next the number of nonzero elements in $\tilde{\mathbf{A}}_{l,l'}$, denoted by $\mathcal{N}(\tilde{\mathbf{A}}_{l,l'})$.

Lemma 3.3

$$\mathcal{N}(\tilde{\mathbf{A}}_{l,l'}) \leq N_{l-1} N_{l'-1} \min\{C(2^{-2l} + 2^{-2l'} + \delta_{l,l'}^2), 1\} \tag{3.33}$$

Proof: We note that for each ψ_J, $J \in \mathcal{J}_l$ there are at most $1 + (2^{-2l} + C\delta_{l,l'}^2)2^{-2l'}$ values of $J' \in \mathcal{J}_l$ such that $\mathrm{dist}(S_J, S_{J'}) < \delta_{l,l'}$. $\qquad\square$

The stiffness matrices $A^L, \tilde{A}^L$ with respect to the multi-wavelet basis $\{\psi_J\}$ define finite dimensional operators $\mathcal{A}^L, \tilde{\mathcal{A}}^L : V^L \to (V^L)'$ where $(V^L)'$ denotes the dual space of V^L. We have the following consistency estimate for the difference between these operators.

Theorem 3.1 *Let $s, \tilde{s} \in [0, d+1]$ and assume that*

$$\alpha \geq \frac{s+d+1}{2(d+1)} \quad , \quad \tilde{\alpha} \geq \frac{\tilde{s}+d+1}{2(d+1)} \ . \tag{3.34}$$

Assume further that the truncation parameters $\{\delta_{l,l'}\}$ in (3.30) satisfy

$$\delta_{l,l'} \geq \max\{a2^{-L}2^{\alpha(L-l)}2^{\tilde{\alpha}(L-l')}, 2^{-l}, 2^{-l'}\} \ . \tag{3.35}$$

Then, for $u \in H^s(\Gamma), \tilde{u} \in H^s(\Gamma)$ there holds

$$\left| \left\langle (\mathcal{A}^L - \tilde{\mathcal{A}}^L)P_L u, P_L \tilde{u} \right\rangle \right| \leq C a^{-2(d+1)} L^\nu N_L^{-(s+\tilde{s})/2} \|u\|_s \|\tilde{u}\|_{\tilde{s}} \tag{3.36}$$

where $\nu = \nu_1 + \nu_2$ and

$$\nu_1 = \begin{cases} 0 & \text{if } \frac{s+d+1}{2(d+1)} < \alpha \text{ and } \frac{\tilde{s}+d+1}{2(d+1)} < \tilde{\alpha} \\ 1 & \text{otherwise} \end{cases} \ ,$$

$\nu_2 = 0$ if $s < d+1$ and $\tilde{s} < d+1$, $\nu_2 = 1$ if $s = d+1$ and $\tilde{s} = d+1$, and $\nu_2 = \frac{1}{2}$ otherwise.

Proof: Using Proposition 3.3 we find

$$\left| \left\langle (\mathcal{A}^L - \tilde{\mathcal{A}}^L)P_L u, P_L \tilde{u} \right\rangle \right| \leq C L^{\nu_2} N_L^{-(s+\tilde{s})/2} \|u\|_s \|\tilde{u}\|_{\tilde{s}} \|E^L\|_2 \ ,$$

where the matrix E^L is now given by

$$E_{JJ'}^L = 2^{s(L-l)}2^{\tilde{s}(L-l')}(A_{JJ'}^L - \tilde{A}_{JJ'}^L) \ .$$

We estimate $\|E^L\|_2$ using the Schur-Lemma (see, for example [18], p. 269) with $\gamma_J = 2^{-l}$. Recall that $\tilde{\mathbf{A}}_{l,l'}$ and $\mathbf{A}_{l,l'}$ denote the blocks of $\tilde{A}^L$ and A^L corresponding to the levels l and l'. We estimate with Lemma 3.2 and (3.35)

$$
\begin{aligned}
\sum_{J' \in \mathcal{J}_{l'}} |E_{JJ'}^L|\gamma_{J'} &= \sum_{l'=0}^{L} 2^{s(L-l)}2^{\tilde{s}(L-l')}2^{-l'}\|\mathbf{A}_{l,l'} - \tilde{\mathbf{A}}_{l,l'}\|_\infty \\
&\leq C \sum_{l'=0}^{L} 2^{s(L-l)}2^{\tilde{s}(L-l')}\delta_{l,l'}^{-2(d+2)}2^{-(d+2)l}2^{-(d+1)l'}\max\{\delta_{l,l'}^2, 2^{-2l'}\} \\
&\leq C \sum_{l'=0}^{L} 2^{s(L-l)}2^{\tilde{s}(L-l')}\delta_{l,l'}^{-2(d+2)}2^{-(d+2)l}2^{-(d+1)l'}\delta_{l,l'}^2 \ ,
\end{aligned}
$$

since $\delta_{l,l'} \geq 2^{-l'}$ by (3.35). Furthermore, (3.35) gives

$$
\begin{aligned}
\sum_{J' \in \mathcal{J}_{l'}} |E_{JJ'}^L| \gamma_{J'} &\leq C \sum_{l'=0}^{L} 2^{s(L-l)} 2^{\tilde{s}(L-l')} 2^{-(d+1)l} 2^{-(d+1)l'} 2^{-l} \\
&\qquad \min\{ a^{-2(d+1)} 2^{2(d+1)L} 2^{-2(d+1)\alpha(L-l)} 2^{-2(d+1)\tilde{\alpha}(l-l')}, 2^{2(d+1)l}, 2^{2(d+1)l'} \} \\
&\leq Ca^{-2(d+1)} 2^{-l} \sum_{l'=0}^{L} 2^{(s+d+1-2(d+1)\alpha)(L-l)} 2^{(\tilde{s}+d+1-2(d+1)\tilde{\alpha})(L-l')} \\
&\leq CL^{\nu_1} a^{-2(d+1)} \gamma_J .
\end{aligned}
$$

The estimate for the column sum follows completely analogously. $\qquad \square$

The consistency estimate Theorem 3.1 allows to show that the compressed Galerkin scheme (3.30) has essentially the same asymptotic convergence rate as the original one while the number of nonzero entries in $\tilde{A}^L$ is considerably smaller than N_L^2.

Theorem 3.2 *Let $\nu(t) = 0$ for $0 \leq t < d+1$ and $\nu(t) = 3/2$ for $t = d+1$. Assume (3.35) with $1 > \alpha > (s+d+1)/(2d+2)$ for $s < d+1$ and $\alpha = 1$ for $s = d+1$, and analogously for $\tilde{\alpha}$. Let L be sufficiently large. Then there holds*

1. *If the constant a in (3.35) is sufficiently large, the compressed Galerkin scheme*

$$
\tilde{\mathcal{A}}^L \tilde{u}^L = P_L f \tag{3.37}
$$

   is stable, i.e.

$$
\left\| \tilde{\mathcal{A}}^L v^L \right\|_0 \geq c \left\| v^L \right\|_0 \qquad \forall v^L \in V^L. \tag{3.38}
$$

2. *Assume in addition that $f, u \in H^s(\Gamma)$, $0 \leq s \leq d+1$. Then*

$$
\left\| u - \tilde{u}^L \right\|_0 \leq C (\log N_L)^{\nu(s)} N_L^{-s/2} \|u\|_s . \tag{3.39}
$$

3. *Assume further that that for any $g \in H^{\tilde{s}}(\Gamma)$ the solution φ of the adjoint equation $A^* \varphi = g$ belongs to $H^{\tilde{s}}(\Gamma)$ for some $0 \leq \tilde{s} \leq d+1$. Then*

$$
\left| \left\langle u - \tilde{u}^L, g \right\rangle \right| \leq C (\log N_L)^{\nu(s)+\nu(\tilde{s})} N_L^{-(s+\tilde{s})/2} \|u\|_s \|g\|_{\tilde{s}} . \tag{3.40}
$$

Proof:

1. It is well known that the Gårding inequality (2.9) implies for L sufficiently large the *discrete inf-sup condition* which can be written as

$$
\left\| v^L \right\|_0 \leq c_s \left\| A v^L \right\|_{(V^L)'} \qquad \forall v^L \in V^L \tag{3.41}
$$

   Using (3.36) with $v^L \in V^L$ and $s = \tilde{s} = 0$ we obtain with (3.41)

$$
\left\| \tilde{\mathcal{A}}^L v^L \right\|_{(V^L)'} \geq \left\| A v^L \right\|_{(V^L)'} - \left\| (\tilde{\mathcal{A}}^L - A) v^L \right\|_{(V^L)'} \geq c_s^{-1} \left\| v^L \right\|_0 - Ca^{-2(d+1)} \left\| v^L \right\|_0
$$

   This gives for sufficiently large a

$$
\left\| v^L \right\|_0 \leq C \left\| \tilde{\mathcal{A}}^L v^L \right\|_{(V^L)'} \qquad \forall v^L \in V^L \tag{3.42}
$$

2. We have

$$\left\|u - \tilde{u}^L\right\|_0 \le \|u - P_L u\| + \left\|P_L u - \tilde{u}^L\right\|$$

Using (3.42) and $\left\langle \tilde{\mathcal{A}}^L u^L, v^L \right\rangle = \left\langle Au, v^L \right\rangle$ for $v^L \in V^L$ we obtain

$$\left\|P_L u - \tilde{u}^L\right\|_0 \le C\left\|\tilde{\mathcal{A}}^L(P_L u - \tilde{u}^L)\right\|_{(V^L)'} = C\left\|\tilde{\mathcal{A}}^L P_L u - Au\right\|_{(V^L)'}$$

yielding

$$\left\|u - \tilde{u}^L\right\|_0 \le \|u - P_L u\|_0 + C\|A(u - P_L u)\|_{(V^L)'} + C\left\|(A - \tilde{\mathcal{A}}^L)P_L u\right\|_{(V^L)'}$$

The first two terms are estimated using the approximation property and the continuity of A. The estimate for the third term follows from (3.36) with $\tilde{s} = 0$ and $P^L v^L = v^L$:

$$\left|\left\langle (A - \tilde{\mathcal{A}}^L)P_L u, v^L \right\rangle\right| \le C a^{-2(d+1)} L^{\nu(s)} N_L^{-s/2} \|u\|_s \left\|v^L\right\|_0$$

3. Let $\varphi^L := P_L \varphi$, then

$$\left|\left\langle u - \tilde{u}^L, g \right\rangle\right| = \left|\left\langle A(u - \tilde{u}^L), \varphi \right\rangle\right| = \left|\left\langle A(u - \tilde{u}^L), \varphi - \varphi^L \right\rangle\right| + \left|\left\langle A(u - \tilde{u}^L), \varphi^L \right\rangle\right|$$

The first term can be estimated by $C\left\|u - \tilde{u}^L\right\|_0 \|\varphi - P_L\varphi\|_0$ which gives the desired bound using (3.39) and the regularity of φ. For the second term we have

$$
\begin{aligned}
\left\langle A(u - \tilde{u}^L), \varphi^L \right\rangle &= \left\langle (\tilde{\mathcal{A}}^L - A)\tilde{u}^L, \varphi^L \right\rangle \\
&= \left\langle (\tilde{\mathcal{A}}^L - A)(\tilde{u}^L - P_L u), P_L\varphi \right\rangle + \left\langle (\tilde{\mathcal{A}}^L - A)P_L u, P_L\varphi \right\rangle
\end{aligned}
$$

The second term on the right hand side can be estimated by (3.36). Since $\tilde{u}^L - P_L u \in V^L$ we have for the first term using (3.36) with $s = 0$

$$
\begin{aligned}
\left|\left\langle (\tilde{\mathcal{A}}^L - A)P_L(\tilde{u}^L - P_L u), P_L\varphi \right\rangle\right| &\le C N_L^{-\tilde{s}/2} L^{\nu(\tilde{s})} \left\|\tilde{u}^L - P_L u\right\|_0 \|\varphi\|_{\tilde{s}} \\
&\le C N_L^{-\tilde{s}/2} L^{\nu(\tilde{s})} \left(\left\|\tilde{u}^L - u\right\|_0 + \|u - P_L u\|_0\right) \|\varphi\|_{\tilde{s}}
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Thus, up to logarithms, the compressed Galerkin scheme preserves the optimal convergence rates of the original Galerkin scheme without compression.

Remark 3.2 Since the boundary integral equation (2.17) was obtained from the boundary value problem (2.1), (2.2), an approximate solution $\tilde{U}^L$ of (2.1), (2.2) can be obtained by inserting $\tilde{u}^L$ into the potential ansatz (2.4), (2.6). For the resulting error $\left|U(x) - \tilde{U}^L(x)\right|$ in the solution $U(x)$ of the elliptic boundary value problem at any interior point $x \in \Omega$ the estimate (3.40) (with $s = \tilde{s} = d + 1$ and smooth boundary Γ) implies

$$\left|U(x) - \tilde{U}^L(x)\right| \le C(x) \left(\log N_L\right)^3 N_L^{-(d+1)} \|f\|_{d+1} \tag{3.43}$$

i.e. *twice* the convergence rate achieved for the boundary density in $L^2(\Gamma)$.

Analogous convergence estimates hold also for derivatives of the solution for which representation formulas can be obtained by differentiating the representations (2.4), (2.6) with respect to $x \in \Omega$. For example, if one uses the simplest Haar wavelet ($d = 0$) for the boundary integral equation (2.6) of the Stokes problem (2.14), the pressure representation formula (2.16) yields approximate pressures $\tilde{p}^L$ which converge pointwise at interior points with the rate $h^2 (\log h)^3$ where h denotes the boundary meshwidth.

Finally we note that in general the constant $C(x)$ in (3.43) blows up as $x \to \Gamma$. Nevertheless, with appropriate postprocessing the full rate of convergence $(\log N_L)^3 N_L^{-(d+1)}$ can be recovered also at the boundary [33, 34].

Remark 3.3 For $\delta_{l,l'}$ as in (3.35) and $\alpha, \tilde{\alpha} \leq 1$, the number of nonvanishing entries $\mathcal{N}(\tilde{A}^L)$ in the compressed stiffness matrix $\tilde{A}^L$ is bounded by

$$\mathcal{N}(\tilde{A}^L) = \begin{cases} O(N_L(\log N_L)^2) & \text{if} \quad \alpha = \tilde{\alpha} = 1, \\ O(N_L \log N_L) & \text{otherwise.} \end{cases} \tag{3.44}$$

Remark 3.4 Although our construction of multiwavelet works for piecewise smooth surfaces Γ, we emphasize that then the solution u and the auxiliary functions φ in Theorem 3.2, point 3., belong only to $H^s(\Gamma)$ for some s which is (possibly substantially) smaller than $d + 1$, even if the data f and g are piecewise smooth. This is of course due to the edge and vertex singularities induced by the unsmoothness of Γ. The corresponding reduced convergence rate can be compensated, however, by employing properly graded subdivisions of Γ rather than the quasiuniform ones used here. For an analysis of a multiscale Galerkin scheme on such graded meshes on polygons, we refer to [24].

Remark 3.5 We will use Theorem 3.1 and 3.2 to analyze the effect of the quadrature error. We will choose the quadrature in such a way that the matrix after truncation and quadrature still satisfies (3.31), (3.32), (3.35). Since only these inequalities are needed the statements of Theorem 3.2 will also hold for the fully discrete method with truncation and quadrature.

4. Quadrature error analysis

So far we assumed that the entries $A_{JJ'}$ of the compressed stiffness matrix $\tilde{A}^L$ are computed exactly. Except in very special circumstances, however, only approximate values $\hat{A}_{JJ'}$ that must be obtained by numerical quadrature are available. In this section we present and analyze a quadrature strategy which a) preserves the asymptotic convergence rates of the compressed scheme in Theorem 3.2 and b) essentially retains the asymptotic complexity (3.44) of the compressed scheme. This will be achieved by tensor product *Gaussian* quadrature formulas of properly selected orders. The case of triangular elements is treated by conical product rules.

Our purpose, then, is to determine a family of Gaussian quadrature rules $Q_{JJ'}$ to compute approximations $\hat{A}_{JJ'} = Q_{JJ'} \tilde{A}_{JJ'}$ of the nonvanishing entries $\tilde{A}_{JJ'}$ of the compressed stiffness matrix $\tilde{A}^L$ such that (3.38) - (3.40) are preserved. We will show that this can be done with $O(N_L (\log N_L)^3)$ kernel evaluations in the far-field, i.e. all offdiagonal entries of the compressed stiffness matrix and with $O(N_L (\log N_L)^4)$ kernel evaluations for the singular integrals. The

quadrature error analysis will utilize the consistency analysis introduced for the compression error analysis (cf. Remark 3.5).

The outline of the section is as follows: in Section 4.1 we collect some classical derivative-free error estimates for Gaussian Quadrature in one dimension and generalize them by a tensor product argument. In Section 4.2 we investigate the analyticity of our integrands. Particular attention is paid to the size of the region of analyticity. Section 4.3 then discusses the strategy for the numerical evaluation of the regular integrals and its complexity. Section 4.4 contains an analysis of the quadrature of the singular integrals arising on the diagonals of the blocks $\mathbf{A}_{l,l'}$.

Throughout this section we denote by $G^n f$ the n-point Gauss-Legendre quadrature applied to $f(x)$ in $[-1,1]$. Further, we denote by $\mathcal{E}_\rho \subset \mathbb{C}$ the closed ellipse with foci at $z = \pm 1$ and with semiaxis sum $\rho > 1$.

4.1. Quadrature error estimates for analytic functions

In this subsection we collect some known error estimates for Gaussian quadrature formulas for analytic functions. They have been used in the analysis of the discretization error for second kind boundary integral equations in [14]. We begin with a classical derivative-free quadrature error estimate in one dimension.

Proposition 4.1 Let $f(x)$ be analytic in $[-1,1]$ and admit an analytic continuation $f(z)$ into the closed ellipse $\mathcal{E}_\rho \subset \mathbb{C}$ with foci at ± 1 and semiaxis sum $\rho > 1$. Then

$$|E^n f| = |If - G^n f| \le C\rho^{-2n} \max_{z \in \partial \mathcal{E}_\rho} |f(z)|. \tag{4.1}$$

This estimate goes back to Davis, see, e.g. [7, Eqn. (4.6.1.11)]. Higher dimensional analogs of it can be obtained by a tensor product construction.

Proposition 4.2 Let $f \in C(\overline{\Omega_1 \times \Omega_2})$. Define

$$If = I_1 I_2 f := \int\limits_{\Omega_1} \int\limits_{\Omega_2} f(\xi_1, \xi_2) d\xi_1 d\xi_2$$

and let

$$Q_i g := \sum_{j=1}^{N_i} w_j^{(i)} g(\xi_j^{(i)}), \qquad i = 1, 2$$

denote quadratures with $w_j^{(i)} > 0$ and $\xi_j^{(i)} \in \bar{\Omega}_i$, $i = 1, 2$. Then

$$|Ef| = |(I - Q_1 Q_2) f| \le |\Omega_1| \max_{\xi_1 \in \bar{\Omega}_1} |(I_2 - Q_2) f(\xi_1, \cdot)| + |\Omega_2| \max_{\xi_2 \in \bar{\Omega}_2} |(I_1 - Q_1) f(\cdot, \xi_2)|. \tag{4.2}$$

Proof: Let $Q = Q_1 Q_2$ denote any tensor product formula. We write

$$\begin{aligned}
(I - Q) f &= (I_1 I_2 - Q_1 Q_2) f = (I_1 I_2 - I_1 Q_2 + I_1 Q_2 - Q_1 Q_2) f \\
&= I_1 [(I_2 - Q_2) f] + Q_2 [(I_1 - Q_1) f]
\end{aligned}$$

and estimate

$$
\begin{aligned}
|(I - Q)\, f| \;&\leq\; |\Omega_1| \max_{x_1 \in \overline{\Omega}_1} |(I_2 - Q_2)\, f(x_1, \cdot)| + \sum_{j=1}^{N_2} w_j^{(2)} \left| (I_1 - Q_1)\, f(\cdot, x_j^{(2)}) \right| \\
&\leq\; |\Omega_1| \max_{x_1 \in \overline{\Omega}_1} |(I_2 - Q_2)\, f(x_1, \cdot)| + |\Omega_2| \max_{x_2 \in \overline{\Omega}_2} |(I_1 - Q_1)\, f(\cdot, x_2)|
\end{aligned}
$$

since $|\Omega_2| = \sum_{j=1}^{N_2} w_j^{(2)}$ by our assumption on the positivity of the weights $w_j^{(2)}$. $\qquad\square$

¿From Propositions 4.1 and 4.2 we deduce the basic derivative-free quadrature error estimate which we will use below.

Proposition 4.3 *Let $B = [-1,1]$, $\Omega_1 = \Omega_2 = B \times B$ and $f(u,u') : \Omega_1 \times \Omega_2 \to \mathbb{R}$. Assume further that for every $u \in \Omega_1$, $f(u,u')$ admits an analytic continuation as a function of u' to $\mathcal{E}_{\rho_2}^2 \subset \mathbb{C}^2$ for some $\rho_2 > 1$ with*

$$
M_2 := \max_{u \in B \times B} \left\{ \max_{u_1' \in B} \max_{u_2' \in \partial \mathcal{E}_{\rho_2}} |f(u,u')| + \max_{u_1' \in \partial \mathcal{E}_{\rho_2}} \max_{u_2' \in B} |f(u,u')| \right\} < \infty \tag{4.3}
$$

and that for every $u' \in \Omega_2$, $f(u,u')$ admits an analytic continuation as a function of u to $\mathcal{E}_{\rho_1}^2 \subset \mathbb{C}^2$ for some $\rho_1 > 1$ with

$$
M_1 := \max_{u' \in B \times B} \left\{ \max_{u_1 \in B} \max_{u_2 \in \partial \mathcal{E}_{\rho_1}} |f(u,u')| + \max_{u_1 \in \partial \mathcal{E}_{\rho_1}} \max_{u_2 \in B} |f(u,u')| \right\} < \infty \tag{4.4}
$$

Then, for every $n_1, n_2 \in \mathbb{N}$,

$$
\left| \left(I - G_{u_1}^{n_1} G_{u_2}^{n_1} G_{u_1'}^{n_2} G_{u_2'}^{n_2} \right) f \right| \leq C \left\{ \rho_1^{-2n_1} M_1 + \rho_2^{-2n_2} M_2 \right\}. \tag{4.5}
$$

Proof: We apply Proposition 4.2 and must therefore estimate the quadrature error for the double integral

$$
\max_{u \in \overline{\Omega}_1} \left| \left(I_2 - G_{u_1'}^{n_2} G_{u_2'}^{n_2} \right) f(u, \cdot) \right|
$$

and its counterpart with indices "1" and "2" interchanged. Applying Proposition 4.2 once more to each of these two quadrature errors, we get that

$$
\begin{aligned}
|Ef| \;\leq\; & 2\, |\Omega_1| \max_{u \in B \times B} \left\{ \max_{u_1' \in B} \left| \int_B f(u; u_1', u_2')\, du_2' - G_{u_2'}^{n_2} f(u; u_1', \cdot) \right| \right. \\[2ex]
& \qquad\qquad\qquad \left. + \max_{u_2' \in B} \left| \int_B f(u; u_1', u_2')\, du_1' - G_{u_1'}^{n_2} f(u; \cdot, u_2') \right| \right\} \\[3ex]
+ \; & 2\, |\Omega_2| \max_{u' \in B \times B} \left\{ \max_{u_1 \in B} \left| \int_B f(u_1, u_2; u')\, du_2 - G_{u_2}^{n_1} f(u_1, \cdot; u') \right| \right. \\[2ex]
& \qquad\qquad\qquad \left. + \max_{u_2 \in B} \left| \int_B f(u_1, u_2; u')\, du_1 - G_{u_1}^{n_1} f(\cdot, u_2; u') \right| \right\}.
\end{aligned}
$$

We apply Proposition 4.1 to each of the four one-dimensional quadrature errors in the above bounds. This yields (4.5) and an analogous bound for M_2. $\qquad\square$

4.2. Analyticity of the kernel in local coordinates

To apply the error estimate (4.5) to the compressed Galerkin scheme, we must investigate the analyticity of the kernel $K(x, y)$ in local coordinates, i.e. the analyticity of

$$\tilde{K}_{jj'}(u, u') = K(\kappa_j^{-1}(u), \kappa_{j'}^{-1}(u')) \quad 1 \leq j, j' \leq N_0 \tag{4.6}$$

(the dependence of $\tilde{K}$ on j and j' will not be explicitly indicated when it is clear from the context).

Lemma 4.1
i) For every $u \in \overline{\mathcal{U}^0}$, $\tilde{K}_{jj'}(u, v)$ is a real analytic function of

$$v \in \overline{\mathcal{U}^0} \backslash \{u\} \quad \text{if } j = j', \quad v \in \overline{\mathcal{U}^0} \quad \text{otherwise} .$$

It admits, for every $u' \in \overline{\mathcal{U}}_0$, $u' \neq u$, an analytic continuation (for convenience again denoted by $\tilde{K}_{jj'}(u, v)$) for

$$v \in \mathcal{B}(u', r) = \{w \in \mathbb{C}^2 : (u' - w)^\top \overline{(u' - w)} < r^2\}. \tag{4.7}$$

Here $0 < r = \hat{\gamma} \left| \kappa_j^{-1}(u) - \kappa_{j'}^{-1}(u') \right|$ and the constant $\hat{\gamma} > 0$ depends only on the global shape of Γ and on the domains of analyticity of the charts $\{\kappa_j\}$ and of the functions s_α in (3.8).
ii) Conversely, for every $u' \in \overline{\mathcal{U}^0}$, $\tilde{K}_{jj'}(v, u')$ is a real analytic function of

$$v \in \overline{\mathcal{U}^0} \backslash \{u'\} \quad \text{if } j = j', \quad v \in \overline{\mathcal{U}^0} \quad \text{otherwise}$$

and admits, for every $u' \neq u \in \overline{\mathcal{U}}_0$, an analytic continuation for $v \in \mathcal{B}(u', r)$ with r as above.

Proof: It is sufficient to prove only the first part of the lemma since the second part is completely analogous. We show first that, for given $u, u' \in \overline{\mathcal{U}}_0$, the kernel $\tilde{K}_{jj'}(u, v)$ is *real* analytic in $\mathcal{B}(u', r) \cap \mathbb{R}^2$.

By assumption, the charts $\kappa_j(u)$ are real analytic functions of $u \in \overline{\mathcal{U}}_0$ and bijective. Therefore, with $x = \kappa_j(u)$, $y = \kappa_{j'}(u')$ the numerators $s_\alpha(x, y)$ in (3.8) are, as compositions of analytic functions, real analytic for $u, u' \in \overline{\mathcal{U}}_0$. Their domains of analyticity are determined by the domains of analyticity of the charts $\kappa_j(u)$ and of s_α.

Next, we consider the analyticity of

$$\left| \kappa_j^{-1}(u) - \kappa_{j'}^{-1}(u') \right|.$$

We distinguish several cases.
Case i) $j = j'$.
Since the charts $(\kappa_j)^{-1} : \overline{\mathcal{U}^0} \to \overline{\Gamma_j}$ are bijective, there exists a global constant γ such that

$$0 < \gamma^{-1} \leq \frac{\left| \kappa_j^{-1}(u) - \kappa_j^{-1}(u') \right|}{|u - u'|} \leq \gamma \quad \text{for } u, u' \in \overline{\mathcal{U}}_0, 1 \leq j \leq N_0.$$

Further, for $x \neq y$ the Euclidean distance $|x - y| = \left\{ (x - y)^\top (x - y) \right\}^{1/2}$ is real analytic in x for fixed y and vice versa. Since the charts $(\kappa_j)^{-1}$ are analytic on $\overline{\mathcal{U}^0}$, for fixed $u, u' \in \overline{\mathcal{U}}_0$ the expression $\left| \kappa_j^{-1}(u) - \kappa_j^{-1}(v) \right|$ is an analytic function of $v \in B(u', r) \cap \mathbb{R}^2$ for $r < \min\{ \hat{\hat{\gamma}}, \gamma \left| \kappa_j^{-1}(u) - \kappa_j^{-1}(u') \right| \}$ where $\hat{\hat{\gamma}} > 0$ depends only on the domains of analyticity of the chart κ_j^{-1}.

The assertion of the lemma follows from the definition (3.8) of the kernel if $\gamma \left| \kappa_j^{-1}(u) - \kappa_j^{-1}(u') \right| \leq \hat{\hat{\gamma}}$ (take $\hat{\gamma} = \gamma$). Otherwise, i.e. when

$$0 < \hat{\hat{\gamma}} \leq \gamma \left| \kappa_j^{-1}(u) - \kappa_{j'}^{-1}(u') \right|,$$

select $\hat{\gamma} \leq \gamma$ such that $\hat{\hat{\gamma}} \geq \hat{\gamma} |x - y|$. Hence the asserted selection of r with $\hat{\gamma} = \min\{ \gamma, \hat{\hat{\gamma}}/\operatorname{diam}(\Omega) \}$ works in all cases. Finally, since $\tilde{K}_{jj'}(u, v)$ is a composition of real analytic functions, it admits a complex analytic extension (e.g. via the classic power-series argument) as a function of $v \in B(u', r)$.

Case ii) $j \neq j'$ and $\operatorname{dist}(\Gamma_j, \Gamma_{j'}) \geq \hat{\hat{\gamma}} > 0$. Here the assertion of the theorem is true for all $u, u' \in \overline{\mathcal{U}^0}$, since the kernel is nonsingular and real analytic in $u' \in \overline{\mathcal{U}^0}$ for every $u \in \overline{\mathcal{U}^0}$ and vice versa. Notice that the value of r in (4.7) must possibly be reduced depending on $\hat{\hat{\gamma}}$.

Case iii) $j \neq j'$ and $\operatorname{dist}(\Gamma_j, \Gamma_{j'}) = 0$. This is the case when two surface pieces are adjacent, i.e. $\overline{\Gamma_j} \cap \overline{\Gamma_{j'}}$ is either a line or a vertex. Fix $u, u' \in \overline{\mathcal{U}^0}$ such that

$$\left| \kappa_j^{-1}(u) - \kappa_{j'}^{-1}(u') \right| > 0.$$

Then it follows as above that for fixed $u \in \overline{\mathcal{U}^0}$ the function $\tilde{K}_{jj'}(u, v)$ is analytic in $v \in B(u', r)$ where r is as in (4.7) and vice versa. □

The numerical quadrature rules $Q_{JJ'}$ are constructed on the reference domain $\mathcal{U}^0$, i.e the unit square $S = (-1, 1)^2$ or the unit triangle $T = \{(x_1, x_2) : -1 < x_1 < 1, -1 < x_2 < x_1\}$. To this end, the kernel $\tilde{K}_{jj'}(u, u')$ is mapped from $\mathcal{U}_k^l \times \mathcal{U}_{k'}^{l'}$ to the reference domain $\mathcal{U}^0$ via the affine transformations

$$\tau_k^l : \mathcal{U}_k^l \longrightarrow \mathcal{U}^0.$$

Denote by

$$\tilde{K}_{JJ'}(u, u') := \tilde{K}_{jj'} \left((\tau_k^l)^{-1}(u), (\tau_{k'}^{l'})^{-1}(u') \right) \tag{4.8}$$

the transported kernel and let

$$\tilde{d}_{JJ'} := \operatorname{dist}\left(\Gamma(J), \Gamma(J') \right), \quad \Gamma(J) := (\tau_k^l \circ \kappa_j)^{-1}(\mathcal{U}^0) \tag{4.9}$$

denote the Euclidean distance of the images $\Gamma(J)$, $\Gamma(J')$ of $\mathcal{U}^0$ under the respective coordinate transformations.

We consider first the case where $\mathcal{U}^0 = S$, i.e. we have *quadrilateral elements*.

Lemma 4.2 *Assume that $\tilde{d}_{JJ'} > 0$ and that $\mathcal{U}^0 = (-1, 1)^2$. Then there exists a constant $\gamma > 0$ which depends only on the kernel, the boundary Γ and its parametrization such that*
  i) for every $u_1, u_2, u_2' \in [-1, 1]$, $\tilde{K}_{JJ'}(u, u')$ admits an analytic extension to $u_1' \in \mathcal{E}_{\rho'}$ with

$$\rho' = 1 + \gamma 2^{l'} \tilde{d}_{JJ'} \tag{4.10}$$

and

$$\max_{u_1,u_2,u_2'\in[-1,1]} \max_{u_1'\in\mathcal{E}_{\rho'}} \left|\tilde{K}_{JJ'}(u,u')\right| \leq M/\left(\tilde{d}_{JJ'}\right)^2. \tag{4.11}$$

Analogously, for every $u_1,u_2,u_1'\in[-1,1]$, $\tilde{K}_{JJ'}(u,u')$ admits an analytic extension to $u_2'\in\mathcal{E}_{\rho'}$ with ρ' as in (4.10) and

$$\max_{u_1,u_2,u_1'\in[-1,1]} \max_{u_2'\in\mathcal{E}_{\rho'}} \left|\tilde{K}_{JJ'}(u,u')\right| \leq M/\left(\tilde{d}_{JJ'}\right)^2. \tag{4.12}$$

   ii) Conversely, for every $u_1',u_2',u_2\in[-1,1]$, $\tilde{K}_{JJ'}(u,u')$ admits an analytic extension to $u_1\in\mathcal{E}_\rho$ with

$$\rho = 1 + \gamma 2^l \tilde{d}_{JJ'} \tag{4.13}$$

and

$$\max_{u_1',u_2',u_2\in[-1,1]} \max_{u_1\in\mathcal{E}_\rho} \left|\tilde{K}_{JJ'}(u,u')\right| \leq M/\left(\tilde{d}_{JJ'}\right)^2. \tag{4.14}$$

Analogously, for every $u_1',u_2',u_1\in[-1,1]$, $\tilde{K}_{JJ'}(u,u')$ admits an analytic extension to $u_2\in\mathcal{E}_\rho$ with ρ as in (4.13) and

$$\max_{u_1',u_2',u_1\in[-1,1]} \max_{u_2\in\mathcal{E}_\rho} \left|\tilde{K}_{JJ'}(u,u')\right| \leq M/\left(\tilde{d}_{JJ'}\right)^2. \tag{4.15}$$

Here the constant M is independent of J and J'.

Proof: Assume first that $l = l' = 0$, i.e. τ_k^l is affine and does not change the area. Then the assertion (i) follows from Lemma 4.1, since for every point $u \in \overline{\mathcal{U}^0}$ and every $u' \in \overline{\mathcal{U}^0}$ there exists $B(u',r)$ such that $\tilde{K}_{JJ'}(u,v)$ is an analytic function for $v \in B(u',r)$. Therefore we can select $\mathcal{E}_{\rho'}^2$ such that

$$\overline{\mathcal{U}^0} \subset\subset \overline{\mathcal{E}_{\rho'}^2} \subset\subset \bigcup_{u'\in\overline{\mathcal{U}^0}} B(u', r(u, u'))$$

The analyticity of $\tilde{K}_{JJ'}(u,v)$ and its homogeneity implies also the bound (4.11) for $l' = 0$.
    The proof of (ii) is analogous.
    The case $l, l' > 0$ is then obtained by a scaling argument. $\qquad\square$

We reduce the case that $\mathcal{U}^0 = T$ of *triangular elements* to the case where $\mathcal{U}^0 = S = (-1,1)^2$ via the degenerate mapping (sometimes also called the "Duffy-transformation") $u = \Phi(\xi)$, $u' = \Phi(\xi')$ given by

$$\Phi(\xi) = \begin{pmatrix} \xi_1 \\ -1 + (\xi_1 + 1)(\xi_2 + 1)/2 \end{pmatrix}. \tag{4.16}$$

We define in this case the transformed kernel by

$$\bar{K}_{JJ'}(\xi,\xi') := \tilde{K}_{jj'}\left((\tau_k^l)^{-1}\circ\Phi(\xi), (\tau_{k'}^{l'})^{-1}\circ\Phi(\xi')\right). \tag{4.17}$$

Note that an application of $n \times n$ tensor product Gaussian quadrature to $\bar{K}_{JJ'}(\xi,\xi')$ in the unit square corresponds to using a conical product rule for $\tilde{K}_{jj'}$ in the triangle.
    To estimate the quadrature error on triangles, we need an analog of Lemma 4.2 for the transformed kernel (4.17).

Lemma 4.3 *For the kernel $\bar{K}_{JJ'}(\xi, \xi')$ in (4.17) statements i) and ii) of Lemma 4.2 remain true for $\tilde{K}_{JJ'}(\xi, \xi')$, with possibly different constants γ and M in (4.10) - (4.15).*

Proof: As in the proof of Lemma 4.2, we first consider $l = l' = 0$.

As before, we obtain from Lemma 4.1 the analyticity of the kernel $\tilde{K}_{JJ'}(u, u')$ defined in (4.8) on $\overline{T} \times \overline{T}$. It therefore remains to show the following: if $f(u)$ is analytic in $\overline{T}$ then $f(\Phi(\xi))$ is an analytic function of $\xi \in \overline{S}$. If f is analytic in $\overline{T}$, for every $u_0 \in \overline{T}$ there exists $R_0 > 0$ such that

$$f(u) = \sum_{\alpha \in \mathbb{N}_0^2} \frac{1}{\alpha!} f^{(\alpha)}(u_0)(u - u_0)^\alpha \quad \forall |u - u_0| < R_0. \tag{4.18}$$

We show that $(f \circ \Phi)(\xi)$ can be expanded into a convergent power series about any point $\xi_0 \in \overline{T}$ for which $u_0 = \Phi(\xi_0)$. To this end, observe that

$$
\begin{aligned}
R_0^2 > |u - u_0|^2 &= (\xi_1 - \xi_{01})^2 + \frac{1}{4} \left[(\xi_1 + 1)(\xi_2 - \xi_{02}) + (\xi_{02} + 1)(\xi_1 - \xi_{01}) \right]^2 \\
&\geq \frac{1}{4} \Big[4(\xi_1 - \xi_{01})^2 + (\xi_1 + 1)^2(\xi_2 - \xi_{02})^2 - \varepsilon(\xi_1 + 1)^2(\xi_2 - \xi_{02})^2 \\
&\qquad - \varepsilon^{-1}(\xi_{02} + 1)^2(\xi_1 - \xi_{01})^2 + (\xi_{02} + 1)^2(\xi_1 - \xi_{01})^2 \Big] \\
&\geq \frac{1}{4} \left[\frac{8}{3}(\xi_1 - \xi_{01})^2 + (\xi_1 + 1)^2(\xi_2 - \xi_{02})^2/4 \right]
\end{aligned}
$$

where we selected $\varepsilon = 3/4$.

This shows that inserting $u = \Phi(\xi)$ into (4.18) yields a power series for $f(\Phi(\xi))$ which converges for sufficiently small $|\xi - \xi_0|$. Now the assertion follows for $l = l' = 0$ as in the proof of Lemma 4.2.

For $l, l' > 0$, we use once again a scaling argument. $\qquad\square$

Lemmas 4.2 and 4.3 will be used when the surface is subdivided either only into quadrilaterals or only into triangles. The arguments in the proof can, however, also be used in the case of mixed partitions of Γ consisting of both quadrilaterals and of triangles.

Corollary 4.1 *Statements i) and ii) of Lemma 4.2 remain also valid for the kernels*

$$K_{JJ'}(\xi, \xi') := \tilde{K}_{jj'} \left((\tau_k^l)^{-1}(\xi), (\tau_{k'}^{l'})^{-1} \circ \Phi(\xi') \right),$$

and

$$K_{JJ'}(\xi, \xi') := \tilde{K}_{jj'} \left((\tau_k^l)^{-1} \circ \Phi(\xi), (\tau_{k'}^{l'})^{-1}(\xi') \right).$$

4.3. Quadrature for the nonsingular integrals

We analyze the quadrature for the non-singular integrals, i.e. for those entries

$$A_{JJ'}^L = \int\limits_\Gamma \int\limits_\Gamma K(x, y)\psi_J(x)\overline{\psi_{J'}(y)}\, do(x)\, do(y) \tag{4.19}$$

of the stiffness matrix for which

$$\text{dist}\,(\text{supp}\psi_J, \text{supp}\psi_{J'}) > 0 \tag{4.20}$$

(note that (4.20) implies that $\langle c\psi_J, \psi_{J'}\rangle = 0$.) According to (3.17), every $\psi_J(x)$ is a scaled and transported copy of some $\tilde{\psi}_\nu$, a piecewise polynomial basis function of $\tilde{W}^1$.

As indicated in Remark 3.5 we will determine the number of Gauss points in such a way that the consistency estimates (3.31), (3.32) still hold for each block. This and corresponding estimates for the singular integrals in the next section imply, via Theorems 3.1 and 3.2, optimal (up to logarithmic terms) convergence rates of the fully discrete solution in the boundary energy norms as well as at interior points of the domain.

Finally, we estimate the complexity of evaluating the nonsingular integrals. The numerical evaluation of the singular integrals is the topic of the following section.

4.3.1. Basic error estimate

In local coordinates, the multiwavelets $\psi_J(x)$ are piecewise polynomial functions. This allows to derive quadrature error estimates from corresponding results for piecewise polynomial density functions.

Lemma 4.4 *Let $\mathcal{U}^0 = (-1, 1)^2$. Let $\lambda, \lambda' \in \mathbb{N}_0$ and $J = (j, \lambda, k, \nu) \in \mathcal{J}_\lambda$, $J' = (j', \lambda', k', \nu') \in \mathcal{J}_{\lambda'}$ be such that $\tilde{d}_{JJ'} > 0$. Let further*

$$\pi(v) = \sum_{0 \le \alpha_1, \alpha_2 \le d} c_\alpha v^\alpha, \quad \tilde{\pi}(v) = \sum_{0 \le \alpha_1, \alpha_2 \le d'} \tilde{c}_\alpha v^\alpha$$

be polynomials of seperate degrees d and d', respectively.

Denote by $|do(v)| = |\partial\kappa_j(v)/\partial v|^{-1}$ the surface element at $x = \kappa_j^{-1}(v)$ and define

$$f_{JJ'}(u, u') := \tilde{K}_{JJ'}(u, u')\pi(u)\tilde{\pi}(u') \left|do\left((\tau_k^\lambda)^{-1}(u)\right)\right| \left|do\left((\tau_{k'}^{\lambda'})^{-1}(u')\right)\right|. \quad (4.21)$$

Then the following quadrature error estimate holds:

$$\left|\left(I - G_{u_1}^{n_1} G_{u_2}^{n_1} G_{u_1'}^{n_2} G_{u_2'}^{n_2}\right) f_{JJ'}\right| \le C 2^{-2\lambda - 2\lambda'} \|\pi\| \|\tilde{\pi}\| \tilde{d}_{JJ'}^{-2} \left\{\rho_1^{-2n_1+d} + \rho_2^{-2n_2+d'}\right\}. \quad (4.22)$$

Here $\|\pi\| = \sum_{|\alpha| \le d} |c_\alpha|$ and $\|\tilde{\pi}\| = \sum_{|\alpha| \le d'} |\tilde{c}_\alpha|$. Further, $\rho_1 = 1 + \gamma 2^\lambda \tilde{d}_{JJ'}$, $\rho_2 = 1 + \gamma 2^{\lambda'} \tilde{d}_{JJ'}$ with $\tilde{d}_{JJ'}$ as in (4.9). The constant C in (4.22) is independent of J and J'.

Proof: We apply the error estimate (4.5). To this end, we must verify the analyticity assumption and obtain an estimate of the constants ρ_i and M_i, $i = 1, 2$, in (4.5).

Consider the integrand $f_{JJ'}$ defined in (4.21) for arbitrary, fixed $u_1, u_2, u_1' \in [-1, 1]$ as a function of u_2' only. Assume first that $\lambda = \lambda' = 0$. Due to the analyticity of the charts κ_j, the surface element $|do(v')|$ is analytic in $v_2' \in [-1, 1]$ and thus admits an analytic extension to $v_2' \in \mathcal{E}_{\rho_2}$ for some $\rho_2 > 1$.

Since $\tilde{\pi}(u')$ is, for fixed $u_1' \in [-1, 1]$, a polynomial in u_2' of degree $\le d$, it is sufficient to refer to Lemma 4.2 to show the analyticity of $f_{JJ'}(u, u')$ in (4.21) as a function of u_2'. The analyticity of $f_{JJ'}$ in the remaining three variables is seen in the same fashion. This completes the proof for the case $\lambda = \lambda' = 0$.

The case that $\lambda > 0$ and/or $\lambda' > 0$ is deduced from the above one by a scaling argument, keeping in mind the growth of π and of $\tilde{\pi}$ for large complex arguments and the bounds for M_1 and for M_2 derived in Lemma 4.2. $\qquad\square$

The following lemma presents a corresponding estimate for triangles.

Lemma 4.5 *Let $\mathcal{U}^0 = T = \{(x_1, x_2) : -1 < x_1 < 1, -1 < x_2 < x_1\}$. With the notations as in Lemma 4.4, define*

$$\tilde{f}_{JJ'}(v, v') := f_{JJ'}(\Phi(v), \Phi(v'))(v_1 + 1)(v_1' + 1). \qquad (4.23)$$

Let

$$\pi(u) = \sum_{0 \leq |\alpha| \leq d} c_\alpha u^\alpha, \quad \tilde{\pi}(u) = \sum_{0 \leq |\alpha| \leq d'} \tilde{c}_\alpha u^\alpha, \quad u \in \mathcal{U}^0$$

be polynomials of total degrees d, resp. d' on $\mathcal{U}^0$. Then the following quadrature error estimate holds:

$$\left| \left(I - G_{v_1}^{n_1} G_{v_2}^{n_1} G_{v_1'}^{n_2} G_{v_2'}^{n_2} \right) \tilde{f}_{JJ'} \right| \leq C 2^{-2\lambda - 2\lambda'} \|\pi\| \|\tilde{\pi}\| \tilde{d}_{JJ'}^{-2} \left\{ \rho_1^{-2n_1 + d + 1} + \rho_2^{-2n_2 + d' + 1} \right\} \qquad (4.24)$$

where we denote, as before, $\|\pi\| = \sum_{0 \leq |\alpha| \leq d} |c_\alpha|$ and $\|\tilde{\pi}\|$ is defined analogously. The constant C in (4.24) is independent of n_1, n_2 and J, J'.

Proof: We proceed exactly as in the proof of Lemma 4.4 but use Lemma 4.3 in addition to obtain the analyticity of $\tilde{f}_{JJ'}$ as a function of v_i, v_i' after applying the transformation Φ (note that the Gaussian rules applied to the transformed integrand in (4.24) correspond to a use of conical product rules). It remains to verify the bounds for M_i and ρ_i corresponding to $\tilde{f}_{JJ'}(v, v')$. The bounds for the kernel follow from Lemma 4.3 whereas the analytic extensions of the surface elements $do(x)$ are bounded independently of J, J' by a scaling argument. It remains to consider the growth of $\pi(\Phi(v))$, $\tilde{\pi}(\Phi(v'))$ for large complex $|v|$, $|v'|$. To this end observe that

$$\pi(\Phi(v)) = \sum_{0 \leq \alpha_1, \alpha_2 \leq d} \hat{c}_\alpha v^\alpha$$

for certain coefficients $\hat{c}_\alpha$, since $\pi(u)$ is a polynomial of total degree d. Thus $\pi(\Phi(v))$ is, for fixed $|v_1| \leq 1$, growing as $O(|v_2|^d)$ for large $|v_2|$ and vice versa. The polynomial $\tilde{\pi}(\Phi(v))$ is discussed analogously. Taking into account the Jacobians $|do(x)|$ composed with the affine maps τ_k^λ and the additional linear term $(v_1 + 1)(v_1' + 1)$ stemming from the Jacobians of $\Phi(v)$ and $\Phi(v')$, the assertion follows. $\qquad \square$

4.3.2. Quadrature strategy

Based on Lemmas 4.4 and 4.5 we are now in position to estimate the quadrature error for those entries $A_{JJ'}$ of the compressed stiffness matrix $\tilde{A}^L$ for which the integrand is nonsingular, i.e. for which the distance $\tilde{d}_{JJ'}$ between the respective boundary elements defined in (4.9) is positive. The singular case will be discussed in the next section.

To define the quadratures, we recall Remark 3.1, i.e that the multiwavelets ψ_J are, in local coordinates, piecewise polynomial functions. Consequently, we apply on each of the four pieces of $\operatorname{supp}\psi_J$ a tensor product Gaussian quadrature rule or a conical product rule if the boundary elements are triangular. Throughout, we denote by n_1 the number of Gauss points used in the quadrature over $\Gamma(J)$ and by n_2 the number of Gauss points used for the quadrature over $\Gamma(J')$. In either case, we denote the resulting quadrature rule by $Q_u^{n_1} Q_{u'}^{n_2}$. Quadrature error estimates will be obtained by applying Lemmas 4.4 and 4.5 to each piece.

We saw in Lemmas 4.4 and 4.5 that Gaussian quadratures exhibit exponential convergence, provided the integrands can be extended analytically to $\mathcal{E}_\rho \supset [-1, 1]$. The convergence rate

ρ is, by Lemma 4.2, basically determined by $\tilde{d}_{JJ'}/\max\{\mathrm{diam}(\Gamma(J)),\mathrm{diam}(\Gamma(J'))\}$. If $\rho > 1$ were independent of J, J', uniform exponential convergence would result. When all boundary elements $\Gamma(J)$ are of approximately the same size (quasiuniform partitions), this can indeed be shown. For the multiscale discretizations under consideration here, however, the quantity $\tilde{d}_{JJ'}/\max\{\mathrm{diam}(\Gamma(J)),\mathrm{diam}(\Gamma(J'))\}$ can become arbitrarily small, for example in the block $\mathbf{A}_{L0}$, and hence ρ arbitrarily close to 1. The resulting degeneracy in the convergence rates ρ_i in (4.22), (4.24) can be compensated by subdivision of the larger element, as we show in Theorem 4.2 below.

Let us first introduce the notation

$$\langle d \rangle = \begin{cases} d & \text{if } \Gamma(J) \text{ is quadrilateral,} \\ d+1 & \text{if } \Gamma(J) \text{ is triangular.} \end{cases} \tag{4.25}$$

This is convenient since the quadrature error bounds (4.22) and (4.24) can both be expressed in the form

$$C2^{-2\lambda-2\lambda'}\|\pi\|\,\|\tilde{\pi}\|\,\tilde{d}_{JJ'}^{-2}\left\{\rho_1^{-2n_1+\langle d\rangle}+\rho_2^{-2n_2+\langle d'\rangle}\right\}. \tag{4.26}$$

We consider first the case where the distance between boundary elements $\Gamma(J)$ and $\Gamma(J')$ is larger than $\max\{\mathrm{diam}(\Gamma(J)),\Gamma(J'))\}$.

Theorem 4.1 *Let $J = (j, l, k, \nu)$ and $J' = (j', l', k', \nu')$ with $l \geq l'$ be such that*

$$\tilde{d}_{JJ'} \geq \gamma^{-1}2^{-l'} \tag{4.27}$$

where $\tilde{d}_{JJ'}$ is as in (4.9) and let $f_{JJ'}(u, u')$ be as in (4.21). Let approximations $\hat{A}_{JJ'}$ to all nonzero entries $A_{JJ'}$ of $\tilde{\mathbf{A}}_{ll'}$ for which (4.27) holds be computed by

$$\hat{A}_{JJ'} = Q_u^{n_1}Q_{u'}^{n_2}g_{JJ'}(u, u') \tag{4.28}$$

where

$$g_{JJ'}(u, u') = \begin{cases} f_{JJ'}(u, u') & \text{if } \Gamma(J), \Gamma(J') \text{ quadrilateral,} \\ f_{JJ'}(\Phi(u), u') & \text{if } \Gamma(J) \text{ triangular and } \Gamma(J') \text{ quadrilateral,} \\ f_{JJ'}(u, \Phi(u')) & \text{if } \Gamma(J) \text{ quadrilateral and } \Gamma(J') \text{ triangular,} \\ f_{JJ'}(\Phi(u), \Phi(u')) & \text{if } \Gamma(J), \Gamma(J') \text{ triangular.} \end{cases}$$

Here the product Gaussian quadratures are applied to each of the four pieces on which the multiwavelets ψ_J and $\tilde{\psi}_{J'}$ are, in local coordinates, polynomial functions. We denote by $\hat{\mathbf{A}}_{ll'}$ the block of elements obtained from numerical quadrature. Assume further that the nonzero entries of $\tilde{\mathbf{A}}_{ll'}$ for which (4.27) is violated are computed exactly.

Then there hold the block consistency estimates (3.31) and (3.32) with $\alpha = \tilde{\alpha} = 1$, i.e.

$$\left\|\tilde{\mathbf{A}}_{l,l'} - \hat{\mathbf{A}}_{l,l'}\right\|_\infty \leq C2^{-(d+1)(2L-l'-l)}2^{l-l'} \tag{4.29}$$

$$\left\|\tilde{\mathbf{A}}_{l,l'} - \hat{\mathbf{A}}_{l,l'}\right\|_1 \leq C2^{-(d+1)(2L-l'-l)}2^{l-l'} \tag{4.30}$$

provided we select the number of Gauss points in (4.28) *according to*

$$n_1 \geq \Psi(a, \langle d \rangle, J, J'), \qquad n_2 \geq \Psi(a, \langle d \rangle, J', J) \tag{4.31}$$

where

$$\Psi(a, d, J, J') = \frac{d}{2} + \frac{2 \log_2 \gamma + l - l' + (d+2)(2L - l - l')}{2 \log_2 \left(1 + \gamma 2^{l+1} \tilde{d}_{JJ'} \right)}.$$

Proof: We apply Lemmas 4.4 and 4.5 with $\lambda = l$, $\lambda' = l'$ and $\pi = \psi_J$ and $\tilde{\pi} = \tilde{\psi}_{J'}$ to each of the four pieces on which ψ_j and $\tilde{\psi}_{J'}$ are, in local coordinates, polynomials. Due to the way the ψ_J are normalized, we have

$$\|\pi\| = O(2^l), \qquad \|\tilde{\pi}\| = O(2^{l'}).$$

Hence we have, for J, J' such that (4.27) holds, the error estimate

$$\left| A_{JJ'} - \hat{A}_{JJ'} \right| \leq C d_{JJ'}^{-2} 2^{-l-l'} \left\{ \left(1 + \gamma 2^l \tilde{d}_{JJ'} \right)^{-2n_1 + \langle d \rangle} + \left(1 + \gamma 2^{l'} \tilde{d}_{JJ'} \right)^{-2n_2 + \langle d \rangle} \right\}. \tag{4.32}$$

We observe that in each row of $\tilde{\mathbf{A}}_{l,l'}$ there are at most $Ca^2 2^{(L-l)}$ nonzero elements. To achieve the error bounds (4.29), (4.30) we require therefore

$$\left| A_{JJ'} - \hat{A}_{JJ'} \right| \leq C 2^{-(\langle d \rangle + 2)(2L - l - l')} \quad C \text{ ind. of } J, J'.$$

Comparison with (4.32) gives

$$\begin{aligned}
\left(1 + \gamma 2^l \tilde{d}_{JJ'} \right)^{-2n_1 + \langle d \rangle} &\leq 2^{-(\langle d \rangle + 2)(2L - l - l')} 2^{l+l'} \tilde{d}_{JJ'}^2 \\
&= 2^{-(\langle d \rangle + 2)(2L - l - l')} 2^{l+l'} \left(\gamma 2^l \tilde{d}_{JJ'} \right)^2 \gamma^{-2} 2^{-2l}
\end{aligned}$$

Taking $\log_2$ on both sides and simplifying, we obtain

$$n_1 \geq \frac{\langle d \rangle}{2} - 1 + \frac{2 \log_2 \gamma + l - l' + (\langle d \rangle + 2)(2L - l - l')}{2 \log_2 \left(1 + \gamma 2^l \tilde{d}_{JJ'} \right)}.$$

This is the asserted bound for n_1. The bound for n_2 is obtained completely analogously, with l and l' interchanged, however. $\qquad \square$

By Remark 3.5, the choice (4.31) will ensure the preservation of (3.38)-(3.40) under quadrature.

Theorem 4.1 addressed only the case that (4.27) holds. As explained above, however, many elements in off-diagonal blocks as e.g. $\tilde{\mathbf{A}}_{L0}$ are such that (4.27) does not hold. This will be overcome by binary subdivision of the larger of the two panels $\Gamma(J)$ and $\Gamma(J')$ until (4.27) is satisfied for all subelements. We assume w.l.o.g that $l > l'$ and that $\mathrm{diam}(\Gamma(J')) \geq \mathrm{diam}(\Gamma(J))$. The following lemma shows how to construct the required subdivision of $\Gamma(J')$.

Lemma 4.6 *Assume* $\mathrm{diam}(\Gamma(J')) \geq \mathrm{diam}(\Gamma(J))$, (4.20) *and that* (4.27) *does not hold, i.e. that*

$$0 < \tilde{d}_{JJ'} < \gamma^{-1} 2^{-l'}. \tag{4.33}$$

Then there exists a binary partition $\{\Gamma(\tilde{J}) : \tilde{J} \in \Lambda(J, J')\}$ *of* $\Gamma(J')$ *such that* $\tilde{d}_{J\tilde{J}} \geq \gamma^{-1} 2^{-\tilde{l}}$ *for all* $\tilde{J} \in \Lambda(J, J')$. *Moreover, the number of subelements* $\Gamma(\tilde{J}) \subset \Gamma(J')$ *is bounded by*

$$|\Lambda(J, J')| \leq C(\gamma)(l - l'). \tag{4.34}$$

Proof: The triangulation of $\Gamma(J')$ is given by $\Lambda(J, J') = \mathcal{F}_J(J')$ where the function $\mathcal{F}_J$ is recursively defined as follows:

$$\mathcal{F}_J(\tilde{J}) := \begin{cases} \{\tilde{J}\} & \text{if } \tilde{d}_{J\tilde{j}} \geq \gamma^{-1} 2^{-\tilde{l}} \\ \bigcup_{k=1}^{4} \mathcal{F}_J(\tilde{J}_k) & \text{otherwise} \end{cases} \tag{4.35}$$

where $\tilde{J}_k$, $k = 1, \ldots, 4$ denote the indices $(\tilde{\jmath}, \tilde{l} + 1, k, \tilde{\nu})$ of the four subtriangles of triangle $\tilde{J}$. One verifies that the above recursion terminates due to the assumption $\tilde{d}_{JJ'} > 0$. The bound (4.34) is evident in the case the surface piece Γ_j is planar, e.g. in the case of a polyhedron where $C(\gamma)$ is bounded by 12. For general curved boundaries, it might be larger, depending on the curvature present, but is always bounded independently of l and l'. $\qquad \square$

The smallest subelements $\Gamma(\tilde{J})$ generated by the above procedure will belong to level $l^* = O(-\log_2(d_{JJ'}))$, i.e. they result from at most $O(l^* - l')$ bisection steps applied to $\Gamma(J')$.

Remark 4.1 Evidently, all subdivisions can be performed in the parameter domain $\mathcal{U}^0$ where each subelement can be easily identified and stored.

Remark 4.2 If $l = l'$, i.e. for the diagonal blocks, no subdivisions are needed to ensure asymptotically, i.e. as $L \to \infty$, an optimal convergence rate of the fully discrete scheme. In practice, however, the decision whether or not to subdivide the regions of integration should be based on the geometric distance to the singularity versus the element diameter. In this way greater robustness for irregular geometries is achieved.

Theorem 4.2 *Let $J = (j, l, k, \nu)$ and $J' = (j', l', k', \nu')$ with $l > l'$ be such that (4.33) holds where $\tilde{d}_{JJ'}$ is as in (4.9).*

Let approximations $\hat{A}_{JJ'}$ to the corresponding nonzero entries $A_{JJ'}$ of $\tilde{\mathbf{A}}_{ll'}$ be computed by the variable order, composite quadrature rule

$$\hat{A}_{JJ'} = \sum_{\tilde{J} \in \Lambda(J, J')} Q_u^{n_1(\tilde{l})} Q_{u'}^{n_2(\tilde{l})} g_{J\tilde{j}}(u, u')$$

where the integrand $g_{JJ'}(u, u')$ is defined in (4.28) and the quadrature orders satisfy

$$n_1(J, J', \tilde{l}) \geq \frac{\langle d \rangle}{2} + \frac{\log_2(\tilde{l} - l') + (\langle d \rangle + 2)(2L - l - l') + l' - l - 2\langle d \rangle}{2\gamma(l + 1 - \tilde{l})} \tag{4.36}$$

and

$$n_2(J, J', \tilde{l}) \geq \frac{\log_2(\tilde{l} - l') + (\langle d \rangle + 2)(2L - l - l') + l' - l - 2\langle d \rangle}{2 \log_2(1 + 2\gamma)}. \tag{4.37}$$

Let the remaining nonzero entries of $\tilde{\mathbf{A}}_{l,l'}$ be computed exactly. Then the consistency estimates (4.29) and (4.30) hold.

Proof: We apply Lemmas 4.4 and 4.5 to the quadrature errors for each of the pairs $J, \tilde{J}$, $\tilde{J} \in \Lambda(J, J')$ which results in the error bound

$$B_{J\tilde{j}} = C 2^{l' - l - 2\tilde{l}} 2^{2\tilde{l}} \left\{ (1 + \gamma 2^{l+1-\tilde{l}})^{-2n_1} 2^{(l-1-\tilde{l})\langle d \rangle} + (1 + 2\gamma)^{-2n_2} 2^{-\langle d \rangle} \right\}.$$

We require, in order for the accumulated quadrature error to be of the proper size for the consistency estimates (4.29) and (4.30), analogous to the proof of Theorem 4.1 that

$$B_{J\tilde{J}} \leq \frac{1}{\tilde{l} - l'} 2^{-(\langle d \rangle + 2)(2L - l' - l)}.$$

This yields the requirement

$$\left(1 + \gamma 2^{l+1-\tilde{l}}\right)^{-2n_1} \leq (\tilde{l} - l')^{-1} 2^{-(\langle d \rangle + 2)(2L-l-l') + l - l' - (l-1-\tilde{l})\langle d \rangle}$$

and

$$(1 + 2\gamma)^{-2n_2} \leq (\tilde{l} - l')^{-1} 2^{-(\langle d \rangle + 2)(2L-l-l') + l - l' + \langle d \rangle}$$

from where we get (4.36) and (4.37). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

4.3.3. Complexity of the non-singular quadrature

We estimate the complexity of the numerical evaluation of the nonsingular integrals in the compressed stiffness matrix $\tilde{A}^L$ using the Gaussian quadratures as described in Theorems 4.1 and 4.2, respectively.

Theorem 4.3 *The computational complexity for the numerical integration of the $O\left((N_L(\log N_L)^2\right)$ nonvanishing, nonsingular entries of the compressed stiffness matrix $\tilde{A}^L$ with the quadrature strategies in Theorems 4.1 and 4.2, respectively, is bounded by $O\left((N_L(\log N_L)^3\right)$ kernel evaluations in local coordinates.*

Proof: In the proof, we will utilize the following estimate which follows immediately from integration by parts and asymptotic expansions of the exponential integral function (see, e.g., [22]):

$$\int_1^x \frac{t^\ell}{(\log t)^k} dt \leq C(k, \ell) \frac{x^{\ell+1}}{(\log x)^k} \qquad k, \ell \in \mathbb{N}_0, \quad x \geq 1. \tag{4.38}$$

Since the asserted work estimate is asymptotic and since $O(\cdot)$ may depend on the geometry of the domain and on the value of the constants γ and a in (4.31), we assume for our estimation that $\gamma = a = 1$ (this only affexts the constants in the work estimate). We will also assume without loss of generality that $l \geq l'$. Throughout, C will denote a generic constant which is independent of l, l', L but which may depend on other parameters, as e.g. d, Ω etc.

We consider first the work W_1 corresponding to the case where no element is subdivided, i.e. the situation of Theorem 4.1. Here (4.27) and the truncation criteria (3.30) and (3.35) (with $\alpha = \tilde{\alpha} = 1$) implying with our assumption that

$$2^{-l'} \leq d_{JJ'} \leq 2^{L-l-l'}. \tag{4.39}$$

The total work for the integrals in block $\tilde{\mathbf{A}}_{l,l'}$ satisfying (4.39) is given by

$$W_1^{l,l'} = \sum_{J,J'} (n_1)^2 (n_2)^2 \sim 2^{2l'} \sum_{2^{-l'} \leq d_{J,J'} \leq 2^{L-l-l'}} \left(\psi(1, \langle d \rangle, J, J')\right)^2 \left(\psi(1, \langle d \rangle, J', J)\right)^2$$

kernel evaluations. Now, for fixed J', $d_{JJ'} = 2^{-l}r$ where $r(i,j) = \text{dist}((i,j), \{i = j\})$. Estimating the sum by an integral we get

$$W_1^{l,l'} \leq C2^{2l'} \int\limits_{2^{l-l'} \leq r \leq 2^{L-l'}} \left[\frac{d}{2} + \frac{l - l' + (d+2)(2l - l - l')}{2(1 + \log_2 r)}\right]^2 \cdot$$

$$\left[\frac{d}{2} + \frac{l' - l + (d+2)(2l - l - l')}{2(1 + l' - l + \log_2 r)}\right]^2 r\,dr.$$

Changing variables $\tilde{r} = 2^{l'-l+1}r$, $r = 2^{l-l'-1}\tilde{r}$ yields

$$W_1^{l,l'} \leq C2^{2l'} \int\limits_{2 \leq \tilde{r} \leq 2^{L-l+1}} \left[d + \frac{l - l' + (d+2)(2L - l - l')}{l - l' + \log_2 \tilde{r}}\right]^2 \cdot$$

$$\left[d + \frac{l' - l + (d+2)(2l - l - l')}{\log_2 \tilde{r}}\right]^2 2^{2(l-l'+1)}\tilde{r}\,dr.$$

To apply the Cauchy-Schwarz inequality we estimate

$$W_{11}^{l,l'} := 2^{2l} \int\limits_{2 \leq \tilde{r} \leq 2^{L-l+1}} \left[d + \frac{l - l' + (d+2)(2L - l - l')}{l - l' + \log_2 \tilde{r}}\right]^4 \tilde{r}\,d\tilde{r}.$$

Using $(p + q)^4 \leq C_1(p^4 + q^4)$, we get

$$W_{11}^{l,l'} \leq Cd^4 2^{2L} + \int\limits_{\tilde{r}=2}^{2^{L-l+1}} \left(\frac{a}{b + \log_2 \tilde{r}}\right)^4 \tilde{r}\,d\tilde{r}$$

with $a = l - l' + (d+2)(2L - l - l') \geq 0$ and $b = l - l' \geq 0$ independent of $\tilde{r}$. Substituting $\hat{r} = 2^b\tilde{r}$, we estimate

$$\int\limits_{\tilde{r}=2}^{2^{L-l+1}} \left(\frac{a}{b + \log_2 \tilde{r}}\right)^4 \tilde{r}\,d\tilde{r} = \int\limits_{\hat{r}=2^{b+1}}^{2^{b+L-l+1}} \left(\frac{a}{\log_2 \hat{r}}\right)^4 2^{-2b}\hat{r}\,d\hat{r} \leq a^4 2^{-2b} \int\limits_{\hat{r}=1}^{2^{b+L-l+1}} \frac{\hat{r}}{(\log_2 \hat{r})^4}\,d\hat{r}.$$

Using (4.38) with $\ell = 1$, $k = 4$ and substituting back, we get

$$\int\limits_{\tilde{r}=2}^{2^{L-l+1}} \left(\frac{a}{b + \log_2 \tilde{r}}\right)^4 \tilde{r}\,d\tilde{r} \leq C \left[\frac{l - l' + (d+2)(2L - l - l')}{l - l' + L - l + 1}\right]^4 2^{2(L-l+1)}$$

and using $p^4 + q^4 \leq C_2(p + q)^4$ for $p, q > 0$, we arrive at

$$W_{11}^{l,l'} \leq C2^{2l} \left[d + \frac{2L - l - l' + l - l'}{l - l' + \log_2(2^{L-l+1})}\right]^4 2^{2(L-l+1)}$$

$$= C2^{2L} \left[d + \frac{2(L - l')}{L - l' + 1}\right]^4.$$

Further, using again (4.38) we obtain analogously

$$
\begin{aligned}
W_{12}^{l,l'} &:= 2^{2l} \int\limits_{2 \le \tilde{r} \le 2^{L-l+1}} \left[d + \frac{l' - l + (d+2)(2L - l - l')}{\log_2 \tilde{r}} \right]^4 \tilde{r} d\tilde{r} \\
&\le C 2^{2L} \left[d + \frac{2(L-l)}{L-l+1} \right]^4 .
\end{aligned}
$$

The total quadrature work for entries which are computed according to Theorem 4.1 is therefore bounded by

$$
W_1 \le C \sum_{l=0}^{L} \sum_{l'=0}^{l} W_1^{l,l'}.
$$

Since $W_1^{l,l'} = W_1^{l',l}$ and $W_1^{l,l'} \le C \left(W_{11}^{l,l'} \right)^{1/2} \left(W_{12}^{l,l'} \right)^{1/2}$, we obtain first that

$$
W_1^{l,l'} \le C 2^{2L} \left[d^4 + \left(\frac{2(L-l)}{L-l+1} \right)^2 \left(\frac{2(L-l')}{L-l'+1} \right)^2 \right] \le C 2^{2L} \left[d^4 + \frac{(L-l)^4 + (L-l')^4}{(L-l+1)^2(L-l'+1)^2} \right].
$$

Summing over all blocks, the first d^4-term results in an $O(L^2 N_L)$ bound. It remains to estimate the second term.

$$
\begin{aligned}
W_1 &\le C 2^{2L} \sum_{l=0}^{L} \sum_{l'=0}^{l} \frac{[(L-l) + (L-l')]^4}{(L-l'+1)^2 (L-l+1)^2} \\
&\le C 2^{2L} \sum_{l=0}^{L} \frac{(L-l)^4}{(L-l+1)^2} \left\{ \sum_{l'=0}^{l} \frac{1}{(L+1-l')^2} \right\} + \frac{1}{(L+1-l)^2} \left\{ \sum_{l'=0}^{l} \frac{(L-l')^4}{(L+1-l')^2} \right\} \\
&\le C 2^{2L} \sum_{l=0}^{L} \frac{(L-l)^4}{(L+1-l)^3} + \frac{L^3}{(L+1-l)^2} \\
&\le C 2^{2L} L^3.
\end{aligned}
$$

Consider next the work W_2 for those entries $\tilde{A}_{J,J'}^L$ which are computed as specified in Theorem 4.2, i.e. by subdividing $\Gamma(J')$ as described in Lemma 4.6. In this case we may assume that $l > l'$. Then we have, under our assumptions, that the quadrature work $W_2^{l,l'}$ in block $\tilde{\mathbf{A}}_{l,l'}$ is bounded by

$$
W_2^{l,l'} := \sum_{2^{-l} \le d_{JJ'} \le 2^{-l'}} \sum_{\tilde{k}=l'+1}^{l} (n_1(\tilde{k}))^2 (n_2(\tilde{k}))^2
$$

kernel evaluations with $n_i(k) = n_i(J, J', k)$ as in (4.36), (4.37). Here $k^* := \tilde{k} - l' \le l - l'$ is the depth of the binary refinement of the larger boundary element $\Gamma(J')$ (cf. Lemma 4.6). We estimate the quadrature work in block $\tilde{\mathbf{A}}_{l,l'}$ as follows:

$$
\begin{aligned}
W_2^{l,l'} &\le C 2^{2l'} \sum_{j_*=0}^{l-l'} 2^{2(l-l')-j_*} \sum_{\tilde{k}=l'+1}^{j_*+l'} \left(d + \frac{z}{l+1-\tilde{k}} \right)^2 z^2 \\
&= C 2^{2l} \sum_{j_*=0}^{l-l'} 2^{-j_*} \sum_{k^*=1}^{j_*} \left(d + \frac{z}{l-l'+1-k^*} \right)^2 z^2
\end{aligned}
$$

where $z = \log_2 j_* + (d+2)(2L - l - l')$. Changing the order of summation gives

$$
\begin{aligned}
W_2^{l,l'} &\leq C 2^{2l} \sum_{k_*=1}^{l-l'} \sum_{j_*=k_*}^{l-l'} 2^{-j_*} \left(d + \frac{z}{l - l' + 1 - k_*} \right)^2 z^2 \\
&= C 2^{2l} \sum_{k_*=1}^{l-l'} 2^{-k_*} \sum_{j=0}^{l-l'-k_*} 2^{-j} \left(d + \frac{z}{l - l' + 1 - k_*} \right)^2 z^2 \\
&\leq C 2^{2l} \sum_{k_*=1}^{l-l'} 2^{-k_*} \left(d + \frac{\tilde{z}}{l - l' + 1 - k_*} \right)^2 \tilde{z}^2 \\
&= C 2^{2l} \sum_{k=1}^{l-l'} 2^{-(l-l'+1-k)} \left(d + \frac{\tilde{z}}{k} \right)^2 \tilde{z}^2
\end{aligned}
$$

where $\tilde{z} = \log_2(l - l') + (d+2)(2l - l - l')$ is independent of k. Hence we get the bound

$$
W_2^{l,l'} \leq C 2^{l+l'} \sum_{k=1}^{l-l'} 2^k \left(d + \frac{\tilde{z}}{k} \right)^2 \tilde{z}^2.
$$

Expanding the square, passing from sums to integrals and using (4.38) with $k = 1$ and $\ell = 0$ yields

$$
\begin{aligned}
W_2^{l,l'} &\leq C 2^{2l} \left[d + \frac{\tilde{z}}{l - l'} \right]^2 \tilde{z}^2 \\
&\leq C 2^{2l} \left[d + \frac{\log_2(l - l') + (2L - l - l')}{l - l'} \right]^2 \left(\log_2(l - l') + (2L - l - l') \right)^2.
\end{aligned}
$$

Summing over all blocks gives

$$
\begin{aligned}
W_2 &= \sum_{l,l'=0}^{L} W_2^{l,l'} \leq C \sum_{l=0}^{L} \sum_{l'=0}^{l} W_2^{l,l'} \\
&\leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l-1} \left(1 + \frac{2L - l - l'}{l - l'} \right)^2 \left(\log_2(l - l') + (2L - l - l') \right)^2 \\
&\leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l-1} \left(\log_2(l - l') \right)^2 + (2L - l - l')^2 + \frac{(2L - l - l')^4}{(l - l')^2} \\
&\leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l-1} (2(L - l) + (l - l'))^4 (l - l')^{-2} + (2(L - l) + (l - l'))^2 \\
&\leq C \sum_{l=0}^{L} 2^{2l} \left\{ (L - l)^4 \sum_{l'=0}^{l-1} (l - l')^{-2} + \sum_{l'=0}^{l-1} (l - l')^2 + (l - 1)(L - l)^2 \right\} \\
&\leq C L^3 2^{2L}.
\end{aligned}
$$

$\square$

Remark 4.3 In our asymptotic work estimates we used only certain simplified bounds for the number of quadrature points. In computational practice, however, the full expressions (4.31), (4.36) and (4.37) should be used. Note also that with a less sophisticated selection of n_1, n_2, one can still ensure the asymptotic convergence rates for the fully discrete scheme, but at the expense of higher powers of $\log N_L$ in the work estimates.

4.4. Quadrature of the singular integrals

We turn now to the quadrature of the remaining entries of the compressed stiffness matrix $\tilde{A}^L$, i.e. those $A_{JJ'}$ for which

$$\text{dist}(\text{supp}\psi_J, \text{supp}\psi_{J'}) = 0. \tag{4.40}$$

Throughout this section, we assume w.l.o.g. that $l \geq l'$. We distinguish *three basic cases*:
a) $\Gamma(J) \subseteq \Gamma(J')$, b) $\overline{\Gamma(J)}$ and $\overline{\Gamma(J')}$ share an edge and c) $\overline{\Gamma(J)}$ and $\overline{\Gamma(J')}$ share a vertex.

4.4.1. Treatment of point functionals

Before discussing the singular integrals, we consider the terms

$$\langle c\psi_J, \psi_{J'}\rangle \tag{4.41}$$

which arise from the point functional $c(x)u(x)$ in (3.7). In cases b) and c) we have obviously that $\langle c\psi_j, \psi_{J'}\rangle = 0$. Due to $l \geq l'$ we have $\Gamma(J) \subseteq \Gamma(J')$ and hence that

$$\langle c\psi_J, \psi_{J'}\rangle = \int_{\Gamma(J)} c(x)\psi_J(x)\psi_{J'}(x)do(x). \tag{4.42}$$

We begin our analysis with the observation that $\kappa_j^* \circ c|_{\overline{\Gamma}_j}$ is an analytic function of $u \in \mathcal{U}^0 = (-1,1)^2$ (we discuss only the quadrilateral case since the error analysis for triangular $\mathcal{U}^0$ is, after using Duffy coordinates, completely analogous). A change of variables in (4.42) together with the definition (3.17) of the ψ_J yields

$$\langle c\psi_J, \psi_{J'}\rangle = 2^{l'-l-2} \int_{\mathcal{U}^0} c \circ \left(\tau_k^\ell \circ \kappa_j\right)^{-1} \tilde{\psi}_\nu \tilde{\psi}_{\nu'} \, |do(u)| \, du. \tag{4.43}$$

We approximate the double integral with an n-point Gaussian quadrature rule and use the derivative-free error estimate of Proposition 4.1.

Theorem 4.4 *If the double integral (4.43) is approximated by an $n \times n$ point tensor product Gaussian quadrature with*

$$n = n(d, L, l, l') \geq d + \frac{(d+1)(2L - l - l') - 2(l - l')}{2\log_2(1 + \gamma 2^l)} \tag{4.44}$$

the block consistency estimates (4.29), (4.30) are preserved and the total quadrature work for the point functionals is bounded by $CN_L (\log N_L)^3$ integrand evaluations.

Proof: Let

$$f_{JJ'}(u) := 2^{l'-l} c \circ \left(\tau_k^\ell \circ \kappa_j\right)^{-1} \tilde{\psi}_\nu \tilde{\psi}_{\nu'} \, |do| \, du.$$

Since $\tilde{\psi}_\nu$ and $\tilde{\psi}_{\nu'}$ are polynomials on $\mathcal{U}^0$ and $c \circ \kappa_j^{-1}$, $|do|$ are analytic on $\overline{\mathcal{U}^0}$, the integrand in (4.43) is, for fixed $u_1 \in (-1,1)$, analytically extendable into $\mathcal{E}_\rho \supset [-1,1]$ with $\rho = 1 + \gamma 2^\ell$ and

likewise for u_2. The maximum M of the extended integrand on $\partial \mathcal{E}_\rho$ is, as before, bounded by $C\rho^{2d}$. Hence, using Propositions 4.1 and 4.2 we get

$$\left| \left(I - G^n_{u_1} G^n_{u_2} \right) f_{JJ'} \right| \le C 2^{l'-l} \left(1 + \gamma 2^l \right)^{-2n+2d}. \tag{4.45}$$

Analogous error bounds hold also for $\mathcal{U}^0$ a triangle if we map it to a square with (4.16), provided d is replaced with $\langle d \rangle$ then.

As explained in Remark 3.5 we want to ensure that the errors caused by (4.45) satisfy the estimates (3.31), (3.32) To this end observe that in each row/column of a block $\tilde{\mathbf{A}}_{l,l'}$ of the compressed stiffness matrix there are at most a finite, fixed number of entries of the form (4.41). Therefore the required block consistencies (4.29), (4.30) hold if the error bound (4.45) is of this order, i.e. if

$$2^{l'-l} \left(1 + \gamma 2^l \right)^{-2n+2d} \le 2^{-(d+1)(2L-l-l')} 2^{l-l'}.$$

After elementary manipulations we obtain (4.44) for the order of the Gaussian quadrature.

The work per each entry of type (4.41) is n^2 kernel evaluations. In block $\tilde{\mathbf{A}}_{l,l'}$ the diagonal has $O(2^{2l})$ entries, hence the total quadrature work for the point functionals is bounded by

$$W_c \le C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l} \left(n(d,L,l,l') \right)^2 \le C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l} \left[d^2 + \frac{(L-l')^2}{(l + \log_2 \gamma)^2} \right].$$

Elementary estimates yield the asserted bound for the work. $\qquad \square$

4.4.2. The case $\Gamma(J) \subseteq \Gamma(J')$

This case arises on the diagonals of the blocks $\mathbf{A}_{l,l'}$. Discarding the point functionals, we must evaluate

$$A_{JJ'} = \lim_{\varepsilon \to 0} \int_{\substack{(x,y)\in\Gamma(J)\times\Gamma(J') \\ |x-y|\ge\varepsilon}} K(x,y)\psi_J(x)\psi_{J'}(y)do(y)do(x). \tag{4.46}$$

¿From the rule for variable substitution in principal value integrals (see, e.g., [19, 31]), we get

$$\begin{aligned}
A_{JJ'} &= 2^{l+l'-2} \int_{u\in\mathcal{U}^l_k} c_0(u)(\tilde{\psi}_\nu \circ \tau^l_k)(\tilde{\psi}_{\nu'} \circ \tau^{l'}_{k'})du \\
&+ 2^{l+l'-2} \lim_{\varepsilon \to 0} \int_{\substack{(u,u')\in\mathcal{U}^l_k\times\mathcal{U}^{l'}_{k'} \\ |u-u'|\ge\varepsilon}} \tilde{K}_{jj'}(u,u')(\tilde{\psi}_\nu \circ \tau^l_k)(\tilde{\psi}_{\nu'} \circ \tau^{l'}_{k'}) \, |do(u)| \, |do(u')| \, du'du.
\end{aligned} \tag{4.47}$$

Here the point functional $c_0(u)$ is analytic with domain of analyticity independent of l, l'. Its quadrature can therefore be treated exactly as in Theorem 4.4.

Remark 4.4 If the charts κ_j, $\kappa_{j'}$ are local tangential coordinates to the surface, we have $c_0(u) = 0$. This is the case considered in [10].

We denote henceforth by $A_{JJ'}$ the double integral in (4.47).

Remark 4.5 For the evaluation of the singular integrals it is sufficient to consider the case where both regions of integration are of the same size.

This is evidently so for $l = l'$. If, however, $l > l'$, the larger patch $\mathcal{U}_{k'}^{l'}$ has to be dyadically refined toward $\overline{\mathcal{U}_k^l} \cap \overline{\mathcal{U}_{k'}^{l'}}$ with $l - l'$ levels, as in Lemma 4.6. This yields a series of near singular integrals which can be integrated exactly as described in Theorem 4.2. Therefore only (a bounded number of) singular integrals over $\Gamma(\tilde{J}) \subset \Gamma(J')$, $\tilde{J} = (j', l, \tilde{k}, \nu') \in \Lambda(J, J')$ for which $\overline{\Gamma(\tilde{J})} \cap \overline{\Gamma(J)} \neq \emptyset$ remain to be analyzed. For these integrals, however, the sizes of $\Gamma(\tilde{J})$ and $\Gamma(J)$ are (asymptotically) equal, i.e. $|\Gamma(J)| \sim \left|\Gamma(\tilde{J})\right| \sim 2^{-2l}$.

We will show now that for the singular double integrals (4.47) with equal sized domains of integration the singularity can effectively be *removed* by coordinate transformations.

According to Remark 4.5, we may translate/rotate $\mathcal{U}_k^l = \mathcal{U}_{\tilde{k}}^l$ to $\mathcal{U}_0^l = \left\{u : 0 \leq u_1 \leq h = 2^{-l}, 0 < u_2 < u_1\right\}$ so that it remains to evaluate

$$B_{J\tilde{J}} = 2^{l+l'} \lim_{\varepsilon \to 0} \int\limits_{\substack{(u,u') \in \mathcal{U}_0^l \times \mathcal{U}_0^l \\ |u-u'| \geq \varepsilon}} H(u, u' - u) du du' \tag{4.48}$$

where we defined

$$H(u, u' - u) = \tilde{K}_{jj'}(u, u')(\tilde{\psi}_\nu \circ \tau_k^l)(u)(\tilde{\psi}_{\nu'} \circ \tau_{k'}^{l'})(u') \left|do(u)\right| \left|do(u')\right|.$$

Note that $\tilde{K}_{jj'}$ and $|do|$ in $H(u, v)$ are independent of l.

The first transformation consists in introducing the relative coordinates [10]

$$\begin{matrix} p = u' - u \\ q = u \end{matrix} \quad \leftrightarrow \quad \begin{matrix} u' = p + q \\ u = q \end{matrix} \tag{4.49}$$

Elementary manipulations show that the region $\mathcal{U}_0^l \times \mathcal{U}_0^l$ of integration becomes the union of 6 domains in $\mathbb{R}^4$, i.e.

$$\overline{\mathcal{U}_0^l \times \mathcal{U}_0^l} = \bigcup_{\mu=1}^{6} \mathcal{V}_\mu, \qquad \mathcal{V}_\mu = \mathcal{V}_{\mu,1} \times \mathcal{V}_{\mu,2} \times \mathcal{V}_{\mu,3} \times \mathcal{V}_{\mu,4}$$

with $p_i \in \mathcal{V}_{\mu,i}$, $i = 1, 2$ and $q_i \in \mathcal{V}_{\mu,i+2}$, $i = 1, 2$, and

$$\begin{matrix}
\mathcal{V}_{1,1} = [-h, 0] & \mathcal{V}_{1,2} = [-h, p_1] & \mathcal{V}_{1,3} = [-p_2, h] & \mathcal{V}_{1,4} = [-p_2, q_1] \\
\mathcal{V}_{2,1} = [-h, 0] & \mathcal{V}_{2,2} = [p_1, 0] & \mathcal{V}_{2,3} = [-p_1, h] & \mathcal{V}_{2,4} = [-p_2, q_1 + p_1 - p_2] \\
\mathcal{V}_{3,1} = [-h, 0] & \mathcal{V}_{3,2} = [0, p_1 + h] & \mathcal{V}_{3,3} = [p_2 - p_1, h] & \mathcal{V}_{3,4} = [0, q_1 + p_1 - p_2] \\
\mathcal{V}_{4,1} = [0, h] & \mathcal{V}_{4,2} = [p_1 - h] & \mathcal{V}_{4,3} = [-p_2, 1 - p_1] & \mathcal{V}_{4,4} = [-p_2, q_1] \\
\mathcal{V}_{5,1} = [0, h] & \mathcal{V}_{5,2} = [0, p_1] & \mathcal{V}_{5,3} = [0, h - p_1] & \mathcal{V}_{5,4} = [0, q_1] \\
\mathcal{V}_{6,1} = [0, h] & \mathcal{V}_{6,2} = [p_1, h] & \mathcal{V}_{6,3} = [p_2 - p_1, h - p_1] & \mathcal{V}_{6,4} = [0, q_1 + p_1 - p_2]
\end{matrix} \tag{4.50}$$

where $h = 2^{-l}$. Therefore

$$B_{J\tilde{J}} = 2^{l+l'} \lim_{\varepsilon \to 0} \sum_{\mu=1}^{6} \int\limits_{\substack{\mathcal{V}_{\mu,1} \times \mathcal{V}_{\mu,2} \times \mathcal{V}_{\mu,3} \times \mathcal{V}_{\mu,4} \\ |p| > \varepsilon}} H(q, p) dp dq. \tag{4.51}$$

Next we map $\mathcal{V}_{\mu,1} \times \mathcal{V}_{\mu,2}$ to $\mathcal{U}_0^l$ and transform q via

$$\hat{p} = \begin{cases} p + (h, h)^\top & \mu = 1, \\ -p & \mu = 2, \\ p + (h, 0)^\top & \mu = 3, \\ -p + (h, 0)^\top & \mu = 4, \\ p & \mu = 5, \\ -p + (h, h)^\top & \mu = 6 \end{cases} \qquad \hat{q} = \begin{cases} \hat{p} - (h, h)^\top + q & \mu = 1, \\ q - \hat{p} & \mu = 2, \\ \hat{p} - (h, 0)^\top + q & \mu = 3, \\ q & \mu = 4, \\ q & \mu = 5, \\ q & \mu = 6 \end{cases} \tag{4.52}$$

This shows that

$$B_{J\tilde{J}} = 2^{l+l'} \lim_{\varepsilon \to 0} \int\limits_{\substack{\mathcal{U}_0^l \\ |\hat{p}| > \varepsilon}} \left(H_1(\hat{p}) + H_2(\hat{p}) + H_3(\hat{p}) \right) d\hat{p} \tag{4.53}$$

where

$$\begin{aligned} H_1(\hat{p}) &= \int\limits_0^{h-\hat{p}_1} \int\limits_0^{\hat{q}_1} \left(H(\hat{q}, \hat{p}) + H(\hat{q} + \hat{p}, -\hat{p}) \right) d\hat{q}, \\ H_2(\hat{p}) &= \int\limits_{\hat{p}_1 - \hat{p}_2}^{\hat{p}_1} \int\limits_0^{\hat{q}_1 - \hat{p}_1 + \hat{p}_2} \left(H(\hat{q}, (h, h)^\top - \hat{p}) + H(\hat{q} + (h, h)^\top - \hat{p}, \hat{p} - (h, h)^\top) \right) d\hat{q}, \\ H_3(\hat{p}) &= \int\limits_{\hat{p}_2}^{\hat{p}_1} \int\limits_{\hat{q}_1 - (\hat{p}_1 - \hat{p}_2)}^{\hat{q}_1} \left(H(\hat{q}, (h, 0)^\top - \hat{p}) + H(\hat{q} + (h, 0)^\top - \hat{p}, \hat{p} - (h, 0)^\top) \right) d\hat{q}. \end{aligned} \tag{4.54}$$

The functions $H_i(\hat{p})$ have singularities in the corners of $\mathcal{U}_0^l$. The purpose of the transformations was to render these singularities weakly singular, i.e. the leading singularity is cancelled out. This is a consequence of the following lemma.

Lemma 4.7 *There exists $k_{jj'}(u, v)$, analytic in $u \in \overline{\mathcal{U}^0}$ for every fixed $v \neq 0$ and homogeneous of degree -2 in $|v|$ such that*

$$H(u, v) = k_{jj'}(u, v) + R_{jj'}(u, v). \tag{4.55}$$

Here $k_{jj'}$ is antisymmetric w.r. to v, i.e.

$$k_{jj'}(u, v) = -k_{jj'}(u, -v), \qquad v \neq 0 \tag{4.56}$$

and the remainder $R_{jj'}(u, v)$ corresponds to a weakly singular boundary integral operator. In particular, $R_{jj'}(u, v)$ is analytic in $u \in \overline{\mathcal{U}^0}$ for every fixed $v \neq 0$ and analytic in $v \neq 0$ for every $u \in \overline{\mathcal{U}^0}$ and satisfies the esstimate $|R_{jj'}(u, v)| \leq C |v|^{-1}$.

Proof: The decomposition (4.55) is a consequence of Taylor expansions of the smooth parts of $H(u, v)$ about $v = 0$ and of the pseudohomogeneity of the kernel $K(x, y)$ in local coordinates (see [31]). The antisymmetry (4.56) of $k_{jj'}$ follows from $K(x, y) = \mathcal{K}(x, y, x - y)$ with $\mathcal{K}$ as in (3.8). $\qquad\square$

Remark 4.6 The antisymmetry (4.56) implies in particular the Tricomi-Giraud-Mikhlin condition

$$\int\limits_{|v|=1} k_{jj'}(u, v) = 0.$$

Notice however that (4.56) is actually a stronger condition which is, nevertheless, always satisfied for zero order boundary integral operators arising in the boundary reduction of second order elliptic PDE, as was shown in [15].

The idea is now that due to the antisymmetry (4.56) the integrand $H(u, v)$ in the definition (4.54) can be replaced by $R_{jj'}(u, v)$. Hence the $H_i(\hat{p})$ are in fact weakly singular only at the vertices of $\mathcal{U}_0^l$.

After the tranformations (4.49) and (4.52) the integral is ready for numerical qudrature. For convenience of exposition, we will state the detailed result only for $H_1(\hat{p})$ in (4.53), the other two cases are treated in exactly the same fashion.

Theorem 4.5 *Let* $\Phi(\xi) = (\xi_1, \xi_1\xi_2)^\top : \mathcal{U}_0^l \to (0, h) \times (0, 1)$ *denote the Duffy transformation. Then*

$$B_{J\tilde{J}}^1 = 2^{l+l'} \int\limits_{\mathcal{U}_0^l} H_1(\hat{p})d\hat{p} = \int\limits_0^h \int\limits_0^1 \int\limits_0^{h-\xi_1} \int\limits_0^{\hat{q}_1} f(\xi_1, \xi_2; \hat{q}_1, \hat{q}_2)d\hat{q}_2 d\hat{q}_1 d\xi_2 d\xi_1$$

where the integrand

$$f = \xi_1 \left\{ H(\hat{q}, \Phi(\xi)) + H(\hat{q} + \Phi(\xi), -\Phi(\xi)) \right\}$$

is independent of l *and* l' *and can be analytically extended in each variable past the region of integration.*

Let $G_{\xi_1}^{n_1} G_{\xi_2}^{n_2} G_{\hat{q}_1}^{m_1} G_{\hat{q}_2}^{n_1} f$ *denote the quadrature approximation to* $B_{J\tilde{J}}^1$ *by properly scaled Gaussian quadrature formulas with*

$$n_1 \geq d + \frac{(d+1)(2L - l - l')}{2(l+1)} + \frac{l'}{l+1} \tag{4.57}$$

resp.

$$n_2 \geq d + \frac{(d+1)(2L - l - l')}{2\log_2 \delta} + \frac{l'}{\log_2 \delta} \tag{4.58}$$

nodes where $\delta \in (1, 4.62)$ *depends only on the regularity of the chart* κ_j *(for regular charts it is uniformly bounded away from 1; a precise value was obtained for general surfaces in [32]).*

Then the block consistency estimates (4.29), (4.30) are preserved and the total quadrature work for the singular integrals (4.48) is bounded by $C N_L (\log N_L)^2$ *kernel evaluations.*

Proof: The function $H(\hat{q}, \hat{p}) + H(\hat{q} + \hat{p}, -\hat{p})$ is analytic for $\hat{p} \neq 0$. It remains therefore to investigate the situation at $\hat{p} = 0$. We write

$$H(\hat{q}, \hat{p}) + H(\hat{q} + \hat{p}, -\hat{p}) = H(\hat{q}, \hat{p}) + H(\hat{q}, -\hat{p}) + \hat{p} \cdot (\partial_1 H)(\zeta(\hat{p}, \hat{q}), -\hat{p})$$

where $\zeta(\hat{p}, \hat{q})$ is analytic. By Lemma 4.7, this yields

$$H(\hat{q}, \hat{p}) + H(\hat{q} + \hat{p}, -\hat{p}) = R_{jj'}(\hat{q}, \hat{p}) + R_{jj'}(\hat{q}, -\hat{p}) + \hat{p} \cdot (\partial_1 H)(\zeta(\hat{p}, \hat{q}), -\hat{p}).$$

Thus the integrand is weakly singular at $\hat{p} = 0$.

The analyticity of the integrand f with respect to $\hat{q}$ follows from that of the kernel $R_{jj'}$ of Lemma 4.7 and the charts, whereas the analyticity with respect to ξ can be shown as in [32], Theorem 1.

Next we estimate the quadrature error using a tensor product argument as in the proof of Proposition 4.3. We use, however, for the n_1-point quadrature error the estimate

$$\left| \int_0^h f(x)\pi_p(x/h)dx - G^{n_1}f \right| \leq Ch^{2n_1+1-p}, \tag{4.59}$$

where π_p is a polynomial of degree at most p independent of h.

For the n_2-point quadrature we use, as before, Proposition 4.1. This yields the error bound

$$E = \left| B_{J\tilde{J}}^1 - G_{\xi_1}^{n_1} G_{\xi_2}^{n_2} G_{\hat{q}_1}^{n_1} G_{\hat{q}_2}^{n_1} f \right| \leq C \left\{ 2^{-(l+1)(2(n_1-d)+1)} + \delta^{-2(n_2-d)} \right\} 2^{l+l'}.$$

Here $C = C(n_1, d)$ is independent of l, l', n_2 and $\delta > 1$ depends only on the domains of analyticity of κ_j and of s_α in (3.8). ¿From the analysis for the consistency error due to the matrix compression, we must have (cf. the proof of Theorem 4.4)

$$E \leq C 2^{-(d+1)(2L-l-l')} 2^{l-l'}.$$

This gives the asserted quadrature orders (4.57), (4.58).

The work estimate is obtained as follows.

$$\begin{aligned}
W_s^1 &\leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l} (n_1)^3 n_2 \\
&\leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l} \left(1 + \frac{2L-l}{l+1} \right)^3 (1 + (2L-l)) \\
&\leq C \sum_{l=0}^{L} 2^{2l} \left\{ 2Ll + (2L-l)^4/(l+1)^2 \right\} \\
&\leq CL \sum_{l=0}^{L} 2^{2l} l + CL^4 \sum_{l=0}^{L} 2^{2l}(l+1)^{-2} \leq CL^2 2^{2L}
\end{aligned}$$

where we used (4.38) with $\ell = 1$ and $k = 2$. $\qquad\qquad\square$

4.4.3. $\Gamma(J)$ and $\Gamma(J')$ share an edge

By Remark 4.5 it is once again sufficient to consider $\Gamma(J)$ and $\Gamma(\tilde{J}) \subseteq \Gamma(J')$ belonging to the same level l. By translation and rotation of $\mathcal{U}_k^l$ and $\mathcal{U}_{\tilde{k}}^l$ this case can be further reduced to

$$B_{J\tilde{J}} = 2^{l+l'} \int_0^h \int_0^{u_1} \int_0^h \int_{-u_1}^0 H(u, u'-u)du du' \tag{4.60}$$

where $h = 2^{-l}$ and the integrand function $H(u, u' - u)$ is defined in (4.48). The key to the regularization is again an appropriate variable transformation which we present in the following lemma.

Lemma 4.8 *There holds*

$$
\int_0^h \int_0^{u_1} \int_0^h \int_{-u_1}^0 H(u_1, u_2; u_1' - u_1, u_2' - u_2) du_1 du_2 du_1' du_2' =
$$

$$
\int_0^1 \int_0^1 \int_0^h \xi^2 \left\{ \int_\xi^h \left[H\left(\zeta - \xi\eta(1-\theta), \xi\eta\theta; \xi\eta(1-\theta), -\xi(1+\eta\theta)\right) + \right.\right.
$$

$$
H\left(\zeta - \xi(1-\eta), \xi\eta; \xi(1-\eta), -\xi(1+\eta-\theta)\right) \right] d\zeta +
$$

$$
\int_\xi^h \left[H\left(\zeta, \xi(1-\theta); -\xi(1-\eta), -\xi(1+\eta-\theta)\right) + \right.
$$

$$
\left. \left. H\left(\zeta, \xi; -\xi\eta(1-\theta), -\xi(1+\eta\theta)\right) \right] d\zeta \right\} d\xi d\eta d\theta
$$

(4.61)

Proof:[28, Chap. 3.3] We map $\mathcal{U}_k^l = \{u' : 0 < u_1' < h, -u_1' < u_2' < 0\}$ onto $\mathcal{U}_k^l$ by setting $u' = (w_1, -w_2)^\top$. Hence

$$
u' - u = \begin{pmatrix} w_1 - u_1 \\ -(w_2 + u_2) \end{pmatrix}
$$

and

$$
B_{J\tilde{J}} = \int_{\mathcal{U}_k^l} \int_{\mathcal{U}_{\tilde{k}}^l} H(u, u' - u) du du' = B_{J\tilde{J}}^1 + B_{J\tilde{J}}^2
$$

where

$$
B_{J\tilde{J}}^1 = \int_0^h \int_0^{u_1} \int_0^{u_1} \int_0^{w_1} H(u_1, u_2; w_1 - u_1, -(w_2 + u_2)) dw du
$$

and

$$
B_{J\tilde{J}}^2 = \int_0^h \int_0^{w_1} \int_0^{w_1} \int_0^{u_1} H(u_1, u_2; w_1 - u_1, -(w_2 + u_2)) dw du.
$$

Setting $\tilde{w}_1 = u_1 - w_1$ in $B_{J\tilde{J}}^1$ and $\tilde{u}_1 = w_1 - u_1$ in $B_{J\tilde{J}}^2$ yields

$$
B_{J\tilde{J}}^1 = \int_0^h \int_0^{u_1} \int_0^{u_1} \int_0^{u_1 - \tilde{w}_1} H(u_1, u_2; -\tilde{w}_1, -(w_2 + u_2)) dw_2 d\tilde{w}_1 du_1 du_2,
$$

$$
B_{J\tilde{J}}^2 = \int_0^h \int_0^{w_1} \int_0^{w_1} \int_0^{w_1 - \tilde{u}_1} H(w_1 - \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) dw_2 dw_1 d\tilde{u}_1 du_2.
$$

In both terms thus the singular arguments are independent of u_1 resp. w_1, i.e. the integrand is analytic in these variables. We exchange the order of integration to move these integrations to the innermost position.

Since both terms are analogous, we concentrate now only on $B^2_{J\tilde{J}}$. The region D of integration can be split into three subdomains, resulting in $B^2_{J\tilde{J}} = B^{2,1}_{J\tilde{J}} + B^{2,2}_{J\tilde{J}} + B^{2,3}_{J\tilde{J}}$ with

$$B^{2,1}_{J\tilde{J}} = \int\limits_0^h \int\limits_0^{w_2} \int\limits_0^{w_2-\tilde{u}_1} K_1(w_2, \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) du_2 d\tilde{u}_1 dw_2,$$

$$B^{2,2}_{J\tilde{J}} = \int\limits_0^h \int\limits_0^{w_2} \int\limits_{w_2-\tilde{u}_1}^{h-\tilde{u}_1} K_2(w_2, \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) du_2 d\tilde{u}_1 dw_2,$$

$$B^{2,3}_{J\tilde{J}} = \int\limits_0^h \int\limits_{w_2}^h \int\limits_0^{h-\tilde{u}_1} K_2(w_2, \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) du_2 d\tilde{u}_1 dw_2$$

where we set

$$K_1(w_2, \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) = \int\limits_{w_2}^h H(w_1 - \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) dw_1$$

and

$$K_2(w_2, \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) = \int\limits_{\tilde{u}_1+u_2}^h H(w_1 - \tilde{u}_1, u_2; \tilde{u}_1, -(w_2 + u_2)) dw_1.$$

We substitute variables as follows: In $B^{2,1}_{J\tilde{J}}$:

$$\begin{pmatrix} \check{w}_2 \\ \check{u}_1 \\ \check{u}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_2 \\ \tilde{u}_1 \\ u_2 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} w_2 \\ \tilde{u}_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \check{w}_2 \\ \check{u}_1 \\ \check{u}_2 \end{pmatrix},$$

in $B^{2,2}_{J\tilde{J}}$:

$$\begin{pmatrix} \check{w}_2 \\ \check{u}_1 \\ \check{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_2 \\ \tilde{u}_1 \\ u_2 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} w_2 \\ \tilde{u}_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \check{w}_2 \\ \check{u}_1 \\ \check{u}_2 \end{pmatrix},$$

and in $B^{2,3}_{J\tilde{J}}$:

$$\begin{pmatrix} \check{w}_2 \\ \check{u}_1 \\ \check{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} w_2 \\ \tilde{u}_1 \\ u_2 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} w_2 \\ \tilde{u}_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \check{w}_2 \\ \check{u}_1 \\ \check{u}_2 \end{pmatrix}.$$

This yields

$$B^{2,1}_{J\tilde{J}} = \int\limits_0^h \int\limits_0^{\check{w}_2} \int\limits_0^{\check{u}_1} K_1(\check{w}_2, \check{u}_1 - \check{u}_2, \check{u}_2; \check{u}_1 - \check{u}_2, -(\check{w}_2 + \check{u}_2)) d\check{u}_2 d\check{u}_1 d\check{w}_2,$$

$$B_{J\tilde{J}}^{2,2} = \int\limits_0^h \int\limits_0^{\check{w}_2} \int\limits_0^{\check{u}_2} K_2(\check{w}_2 - \check{u}_1, \check{w}_2 - \check{u}_2, \check{u}_2; \check{w}_2 - \check{u}_2, -(\check{w}_2 - \check{u}_1 + \check{u}_2)) d\check{u}_1 d\check{u}_2 d\check{w}_2$$

and

$$B_{J\tilde{J}}^{2,3} = \int\limits_0^h \int\limits_0^{\check{w}_2} \int\limits_0^{\check{u}_2} K_2(\check{w}_2 - \check{u}_2, \check{w}_2 - \check{u}_1, \check{u}_1; \check{w}_2 - \check{u}_1, -(\check{w}_2 + \check{u}_1 - \check{u}_2)) d\check{u}_1 d\check{u}_2 d\check{w}_2.$$

Swapping the innermost integrations in $B_{J\tilde{J}}^{2,3}$ and exchanging the variables $\check{u}_1$ and $\check{u}_2$ in $B_{J\tilde{J}}^{2,2}$, we can combine $B_{J\tilde{J}}^{2,2}$ and $B_{J\tilde{J}}^{2,3}$ into a single term, resulting in

$$
\begin{aligned}
B_{J\tilde{J}}^2 \;=\; & \int\limits_0^h \int\limits_0^{\check{w}_2} \int\limits_0^{\check{u}_1} K_1(\check{w}_2, \check{u}_1 - \check{u}_2, \check{u}_2; \check{u}_1 - \check{u}_2, -(\check{w}_2 + \check{u}_2)) d\check{u}_2 d\check{u}_1 d\check{w}_2 \\
+\; & \int\limits_0^h \int\limits_0^{\check{w}_2} \int\limits_0^{\check{w}_2} K_2(\check{w}_2 - \check{u}_2, \check{w}_2 - \check{u}_1, \check{u}_1; \check{w}_2 - \check{u}_1, -(\check{w}_2 + \check{u}_1 - \check{u}_2)) d\check{u}_2 d\check{u}_1 d\check{w}_2.
\end{aligned}
$$

Applying the triangular coordinates

$$\check{w}_2 = \xi, \check{u}_1 = \xi\eta, \check{u}_2 = \xi\eta\theta$$

in the first integral and

$$\check{w}_2 = \xi, \check{u}_1 = \xi\eta, \check{u}_2 = \xi\theta$$

in the second integral yields, after backsubstituting K_1 and K_2, the first two terms in (4.61).

The substitutions for $B_{J\tilde{J}}^1$ are exactly the same, with u and w interchanged, and yield the last two terms in (4.61). $\qquad\square$

The main point of the transformation (4.61) is that for H as defined in (4.48) and kernels as in (3.8), the integrand on the right hand side of (4.61) is analytic in all variables of integration. Thus, standard Gaussian quadratures will yield the consistency required by the Galerkin scheme.

Theorem 4.6 *Denote by $f(w_1, \xi, \eta, \theta)$ the integrand in (4.61). Let $G_\zeta^{n_1} G_\xi^{n_1} G_\eta^{n_2} G_\theta^{n_2} f$ denote the quadrature approximation to $B_{J\tilde{J}}$ by properly scaled Gaussian quadrature formulas with*

$$n_1 \geq 2d + \frac{(d+1)(2L - l - l')}{2(l+1)} + \frac{l'}{l+1} \tag{4.62}$$

resp.

$$n_2 \geq 2d + \frac{(d+1)(2L - l - l')}{2\log_2 \delta} + \frac{l'}{\log_2 \delta} \tag{4.63}$$

nodes where δ is as in Theorem 4.5.

Then the block consistency estimates (4.29), (4.30) are preserved and the total quadrature work for the singular integrals (4.60) is bounded by $C N_L (\log N_L)^3$ kernel evaluations.

Proof: ¿From the definition of H in (4.48) we see that $|do(u)|$, $|do(u')|$ are, in the $(\zeta, \xi, \eta, \theta)$ coordinates, analytic functions with domains of analyticity independent of l and l'.

The product $\tilde{\psi}_\nu \tilde{\psi}_{\nu'}$ is, in triangular coordinates, a polynomial of the form

$$\sum_{\alpha_i \leq 4d} c_\alpha \zeta^{\alpha_1} \xi^{\alpha_2} \eta^{\alpha_3} \theta^{\alpha_4}.$$

We claim that the integrand is analytic in the triangular coordinates and independent of the level l. The independence of l follows directly from its definition. The analyticity as a function of (η, θ) for $\xi \neq 0$ follows from $H(u; v)$ being analytic in u and also in $v \neq 0$, since $\eta(1-\theta)$ and $1+\eta\theta$ resp. $1-\eta$ and $1+\eta-\theta$ do not vanish simultaneously for any $(\eta, \theta) \in [0,1]^2$. This also shows the analyticity with respect to ξ provided $\xi \neq 0$. The analyticity at $\xi = 0$ (and thus the uniform analyticity in ξ) follows from $H(u; v)$ being pseudohomogeneous of degree -2 in v. Using the pseudohomogeneous expansion of $H(u; v)$ with respect to local polar coodinates in v (see, e.g., [31]), we see that the factor ξ^2 introduced by the triangular coordinates cancels the singularity of H. This shows that the integrand is regular at $\xi = 0$. Analyticity at $\xi = 0$ is then shown as in [32].

To estimate the quadrature error, we use a tensor product argument. For the double integration in $(\zeta, \xi) \in (0, h)^2$, we use the error estimate (4.45) and for the integration in $(\eta, \theta) \in (0, 1)^2$, we use Proposition 4.1. This is analogous to what was done in the proof of Theorem 4.5. We get the error estimate

$$E = \left| B_{J\tilde{J}} - G_\zeta^{n_1} G_\xi^{n_1} G_\eta^{n_2} G_\theta^{n_2} f \right| \leq C 2^{l+l'} \left(2^{-(l+1)(2n_1+1-4d)} + \delta^{-2(n_2-2d)} \right).$$

Matching this error bound with the block consistency estimates yields the lower bounds for quadrature orders n_1 and n_2.

The work estimate is obtained as in the proof of Theorem 4.5, bearing in mind that we now have to sum up $2^{2l}(n_1)^2(n_2)^2$ over all blocks (thus the power of $l+1$ in the denominator is reduced by one resulting in one additional power of L). $\qquad \square$

4.4.4. $\Gamma(J)$ and $\Gamma(J')$ share a vertex

The final case where supports contain a common vertex is also treated using a special coordinate transformation due to [10, 28]. We refer to [10, Section 2.3].

One introduces first a new coordinate $w = u' - u$ and obtains, after careful examination of the limits of integration, an equivalent integral

$$B_{J\tilde{J}} = 2^{l+l'} \int_{\mathcal{D}} f(\eta) d\eta$$

over the domain

$$\mathcal{D} = \{\eta \in \mathbb{R}^4 : 0 \leq \eta_1 \leq h; 0 \leq \eta_2 \leq \eta_1; 0 \leq \eta_3 \leq \eta_1; 0 \leq \eta_4 \leq \eta_3\}$$

Now, however, 4-dimensional triangle coordinates are introduced, i.e.

$$\eta_1 = \xi, \quad \eta_i = \xi\theta_i, \quad i = 2, 3, 4. \tag{4.64}$$

This yields

$$B_{J\tilde{J}} = 2^{l+l'} \int_0^h \int_0^1 \int_0^1 \int_0^{\theta_2} \xi^3 f(\xi, \xi\theta_1, \xi\theta_2, \xi\theta_3) d\xi d\theta.$$

We can now estimate the error as in Theorems 4.5, 4.6 using the tensor product argument of Proposition 4.2.

This will yield again lower bounds for n_1, the number of nodes for the ξ integration and for n_2, the number of nodes for the θ integration of the form (4.62), (4.63), with $2d$ replaced by $3d$.

The quadrature work is estimated as follows:

$$
\begin{aligned}
W_s^1 & \leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l} n_1 (n_2)^3 \\
& \leq C \sum_{l=0}^{L} 2^{2l} \sum_{l'=0}^{l} \left(1 + \frac{2L - l}{l + 1} \right) (1 + (2L - l))^3 \\
& \leq C L^4 \sum_{l=0}^{L} 2^{2l} \leq C L^4 2^{2L}.
\end{aligned}
$$

Thus we have shown

Theorem 4.7 *The singular integral where $\Gamma(J)$ and $\Gamma(J')$ share a vertex can, after reduction to integration domains of equal size as described in Remark 4.5, be treated with the coordinate transformations described in [10, Section 2.3]. The product Gaussian integration of the regularized integrands needs, to ensure the block consistency estimates (4.29), (4.30) and thus the asymptotic convergence rates, $O(N_L (\log N_L)^4)$ kernel evaluations.*

Remark 4.7 We point out that this type of element in the stiffness matrix is the only one that requires $O(N_L (\log N_L)^4)$ kernel evaluations. Note that the regularizing factor ξ^3 introduced by the coordinates (4.64) is actually not necessary to render the transformed integrand analytic – ξ^2 would suffice here. We therefore conjecture that the $O(N_L (\log N_L)^4)$ complexity could be reduced with a different regularization.

5. Conclusion

A multiwavelet basis for L^2 on curved, piecewise smooth surfaces $\Gamma \subset \mathbb{R}^3$ has been constructed. The multiwavelet families are fully orthogonal and the construction applies for arbitrary degree of approximation $d \in \mathbb{N}_0$.

We have shown that stiffness matrices for corresponding multiscale Galerkin discretizations for a class of strongly elliptic boundary integral operators of order zero on piecewise smooth surfaces in $\mathbb{R}^3$ can be compressed to $O(N_L (\log N_L)^2)$ nonvanishing entries while retaining essentially (up to logarithmic terms) the full asymptotic convergence rates of the uncompressed scheme, even in negative norms.

The location and required accuracy of the nonzero entries in the compressed stiffness matrix can be determined a-priori and explicitly, thus bypassing the need to generate the full, dense stiffness matrix prior to compression.

Due to our multiwavelets being piecewise polynomial in local coordinates, standard tensor product Gaussian quadrature can be used for the computation of the entries in the compressed stiffness matrix. We have given explicit estimates for the required quadrature orders and shown that the approach of [10, 28] for the quadrature of the singular integrals can be used

in a multiscale context as well. The total work for all quadratures, except for one type of singular integral, was estimated to be $O(N_L(\log N_L)^3)$ kernel evaluations.

The condition number of the compressed stiffness matrix was shown to be bounded, so the iterative solution with standard methods, as e.g. Richardson iteration or generalized conjugate gradient methods can be obtained with an accuracy comparable to the discretization error in $O(N_L(\log N_L)^4)$ operations, provided we start on level 0 and use the approximation on level l as initial guess on level $l+1$ for $l = 0, \ldots, L-1$.

The consistency and quadrature error analysis presented here is quite flexible and applies also to Galerkin schemes for strongly elliptic operators of nonzero order *provided* piecewise polynomial, biorthogonal multiwavelet bases for $H^{\pm 1/2}(\Gamma)$ with the appropriate number of vanishing moments are available. We mention further that using families of biorthogonal multiwavelets with the same approximation order, but a higher number of vanishing moments than our family will allow to remove some of the logarithms in the complexity estimates [29]. As long as these families are piecewise polynomial, our quadrature error analysis will still apply.

We emphasize in closing that no computational experience with the method presented here has yet been obtained. Although the work estimates obtained here are slightly better than corresponding ones for, say, the panel clustering Galerkin method, it must be borne in mind that they are asymptotic. They have little predictive value for the performance of an actual implementation of the method which depends on many other factors as well.

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46

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