Pre-Wavelets on Scattered Knots and from Radial Function Spaces: A Review

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Abstract

We review recent work on univariate prewavelets that are either spline prewavelets on non-equally spaced knots or prewavelets from spaces spanned by translates of radial basis functions with non-equally spaced centres. Some of the results on radial basis functions we present apply to more than one dimension as well, so long as the centres are again confined to a grid.

AMS(MOS) Subject Classification: 41A15, 41A30, 42C15, 65D15

1. Introduction.

The purpose of this article is to summarize briefly recent work on (mostly) univariate prewavelets from spline or radial basis function spaces that are based on non-uniformly distributed knots or "centres". This is by no means a comprehensive review of the state-of-the-art in wavelet research; instead we will attempt to give a reasonably complete overview about this very particular aspect of research into prewavelets. We begin with outlining the purpose and the usefulness of prewavelets. As it is well known that spline spaces are useful for constructing prewavelets, see, for instance, the book (Chui, 1992), we shall then describe why *radial basis functions* are also very useful for this.

Prewavelet expansions of $L^2(\mathbb{R})$ functions are decompositions into different phase components, much like Fourier expansions, as well as space (or time) components, unlike the familiar Fourier analysis. This is useful for signal processing, image analysis and computer vision, as well as sound analysis and the numerical treatment of partial differential equations by microlocalisation (DeVore and Lucier, 1992). Each such decomposition of a given f starts with establishing an initial approximation f_j to f and continues by approximating on successively finer scales of resolution. This initial f_j is from a linear subspace $V_j \subset L^2(\mathbb{R})$ that is an element of a bi-infinite "nested" sequence of spaces with $V_j \subset V_{j+1}$ for all integers j, from which the prewavelets stem as well. The $\{V_j\}_{j=-\infty}^{\infty}$ represent the different scales of resolution and form the *multiresolution analysis* underlying the decomposition, where one requires at a minimum that

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R}), \qquad \bigcap_{j=-\infty}^{\infty} V_j = \{0\}. \tag{1.1}$$

The desire to expand f into an *orthogonal* decomposition with respect to phase or frequency leads one to study subspaces $W_j \subset V_{j+1}$ such that W_j is orthogonal to V_j (in short: $W_j \perp V_j$) and $V_j \oplus W_j = V_{j+1}$. Thereby, indeed, an orthogonal series

$$L^2(\mathbb{R}) = \cdots + W_{-1} + W_0 + W_1 + W_2 + \cdots \tag{1.2}$$

is found by virtue of (1.1), and therefore also a decomposition of f into mutually orthogonal components, corresponding to the different frequency parts in Fourier analysis. The *prewavelets* are the generators of the W_js, their coefficients always square-summable. The prewavelets are usually required to form a Riesz basis of W_j but need not be an orthogonal basis like wavelets. (More about this in Section 3.) It is desirable that they decay quickly at infinity, so that the expansion of f's components with respect to time is *well localized*. In case the V_j contain nontrivial compactly supported functions, the prewavelets should ideally have compact support too. Since it is the function $f_j \in V_j$ that is actually decomposed, it is important to the effectiveness of the decomposition that the spaces V_j allow f_j to be a good approximation to f for a reasonably sized j. (Fast computation of the prewavelet expansions is made possible by the so-called fast wavelet transform [Chui, 1992, for instance].) This is the point where the efficacy of radial basis functions can be exploited. Indeed, radial basis function methods are well-known to be efficient and accurate for the approximation of functions (Powell, 1991, Buhmann, 1993a). Radial basis function methods were first introduced as interpolatory schemes but, because they provide such high quality approximants, they are presently used for many different approximation schemes, such as least-squares approximations (Buhmann, 1993b) and prewavelets. Among all radial functions currently in use, the multiquadric radial function $\varrho(r) = \sqrt{r^2 + c^2}$ is probably best understood and most successful in practical applications. We shall therefore focus on it in this review at various points. Indeed, several of the results we present are tailormade for spaces spanned by translates of the multiquadric function, including the case $c = 0$ which is especially important for more than one dimension.

On the other hand, the most frequent choice for the aforementioned V_j in current applications of prewavelets are still spline spaces with fixed degree and equally spaced knots (Chui, 1992, or Chui and Wang, 1992, for example) of spacing 2^{-j}, $j \in \mathbb{Z}$. This provides good approximations which are, however, always of limited smoothness. This should be contrasted with approximants from spaces V_j spanned by 2^{-j} translates of the multiquadric function which are $C^\infty(\mathbb{R})$. According to the known theory of radial functions (Buhmann, 1993c, for example), they provide essentially the same approximation order as splines give, if the centres have the same spacing as the knots of the splines. There are no non-trivial compactly supported functions in those spaces, as there are in spline spaces, but there are generating functions for the V_j from radial basis function spaces that diminish at a high algebraic rate toward infinity. Those functions replace the familiar B-splines of univariate spline spaces (see, e.g., Powell, 1990, and the recent paper by Beatson and Dyn, 1994). Improvements can also be expected if one stays with the spline prewavelet *ansatz*, but lets the knots vary and be unequally distributed according to the local smoothness properties of f. That is the reason why, before radial function prewavelets were introduced, spline prewavelets on non-equally spaced knots were investigated as a further development in the theory of prewavelets. This is the point where we start our review. After describing it in the following section, we shall in Section 3 describe the various approaches to radial function prewavelets that have been offered in the literature. This includes several papers that deal with multiple dimensions, but gridded centres, as well as work in one dimension where the centres are no longer equally spaced but satisfy a weak, natural regularity condition.

2. Spline prewavelets on non-equally spaced knots.

The initiative to study univariate spline prewavelets when the knots of the splines are scattered came from (Buhmann and Micchelli, 1992). Starting with two prescribed knot sequences

$$\mathbf{x} = \Big\{ \cdots < x_{-1} < x_0 < x_1 < x_2 < \cdots \Big\} \subset \mathbb{R}$$

and

$$\underline{\tau} = \Big\{ \cdots < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots \Big\} \subset \mathbb{R},$$

where $\tau_{2i-1} = x_i$, $i \in \mathbb{Z}$, so that $\mathbf{x} \subset \underline{\tau}$, they construct prewavelets that span the orthogonal complement W_0 of V_0 in V_1, where

$$\left.\begin{aligned} V_0 &:= \Big\{ \sum_{j=-\infty}^{\infty} c_j B_j^{\mathbf{c}} \;\Big|\; c = \{c_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) \Big\}, \\ V_1 &:= \Big\{ \sum_{j=-\infty}^{\infty} c_j B_j^{\mathbf{f}} \;\Big|\; c = \{c_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) \Big\}, \end{aligned}\right\} \tag{2.1}$$

and $B_j^{\mathbf{c}}$ and $B_j^{\mathbf{f}}$ are the B-splines of fixed degree n on the knot sequences $\mathbf{x}$ and $\underline{\tau}$, respectively. Their supports are $[x_j, x_{j+n+1}]$ and $[\tau_j, \tau_{j+n+1}]$, respectively. They are assumed to be normalized so that they form a partition of unity. It is clear from this how to construct in general a whole nested sequence of V_js for a given nested sequence of knots $\{\underline{\tau}^j\}_{j=-\infty}^{\infty}$, say, with $\underline{\tau}^{j-1} \subset \underline{\tau}^j$ for all integers j.

Using in particular the fact that V_0 and V_1 are precisely those splines of degree n on $\mathbf{x}$ and $\underline{\tau}$, respectively, that are also in $L^2(\mathbb{R})$, the following two theorems are established.

Theorem 1. *There is a unique (up to a sign) sequence of functions $\{\psi_k\}_{k=-\infty}^{\infty}$ in V_1 and orthogonal to V_0 of minimal support $[x_k, x_{k+2n+1}]$ and $\|\psi_k\|_2 = 1$ each which also admit representations*

$$B_j^{\mathbf{f}} = \sum_{k=-\infty}^{\infty} A_{jk}\psi_k + \sum_{k=-\infty}^{\infty} B_{jk}B_k^{\mathbf{c}}, \qquad j \in \mathbb{Z}. \tag{2.2}$$

They have no open interval of zeros within their support. If the $\underline{\tau}$ are N-periodic for some $N \in \mathbb{N}$, then the coefficients in this representation (2.2) *decay exponentially. Therefore, every $g \in V_1$ can be written as a sum $g_1 \in W_0$ and $g_2 \in V_0$, i.e. $V_1 = W_0 \oplus V_0$, whence the $\{\psi_k\}_{k=-\infty}^{\infty}$ are* prewavelets*. Here*

$$W_0 := \left\{ \sum_{j=-\infty}^{\infty} c_j\psi_j \,\middle|\, c = \{c_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) \right\}. \tag{2.3}$$

The assumption of periodicity is, in practice, no real restriction, because matrix multiplications have to be performed in the fast wavelet transform at each level of the prewavelet decomposition or reconstruction of the function, and one does not want to have infinitely many different multiplications of this kind. More specifically, for a given $g \in V_1$, $g = \sum_{j=-\infty}^{\infty} c_j B_j^{\mathbf{f}}$, the coefficients of its prewavelet part g_1 are $\sum_{j=-\infty}^{\infty} c_j A_{jk}$, $k \in \mathbb{Z}$, and those of its part g_2 in V_0 are $\sum_{j=-\infty}^{\infty} c_j B_{jk}$, $k \in \mathbb{Z}$. Since these series are infinite, it is useful for computations to have coefficients A_{jk}, B_{jk} that repeat periodically when j varies, instead of infinitely many different ones.

Theorem 2. *If the $\underline{\tau}$ satisfy a bounded global mesh ratio condition*

$$\sup_{i,j\in\mathbb{Z}} \frac{\tau_{i+1} - \tau_i}{\tau_{j+1} - \tau_j} < \infty, \tag{2.4}$$

there is a sequence of prewavelets $\{\psi_k\}_{k=-\infty}^{\infty}$ in V_1 and orthogonal to V_0 that decay exponentially and which also admit representations (2.2) *with exponentially decaying coefficients.*

The case where *several* knots are inserted into each interval (x_j, x_{j+1}) when passing from V_0 to V_1 is considered in the paper too. Regarding the prewavelets of Theorem 1, the following fact about the sign changes of the prewavelets is also shown in the paper:

Proposition 3. *Suppose $\psi \in C([x_k, x_{k+\theta}])$ has a zero set of measure zero on $[x_k, x_{k+\theta}]$ and is orthogonal to all $B_j^{\mathbf{c}}$ defined above. Then ψ has at least $n+\theta$ sign changes in $(x_k, x_{k+\theta})$ and, if it has exactly that number of sign changes, they lie in the intervals $(x_{k-n+j-1}, x_{k+j})$, $j = 1, 2, \ldots, n+\theta$, respectively.*

It is a corollary of this proposition and the variation diminishing property of B-splines that the ψ_k of Theorem 1 have exactly $3n+1$ sign changes each and that their coefficients, when the prewavelets are viewed as a linear combination of $B_j^{\mathbf{f}}$s, alternate in sign.

The goal of the paper (Lyche and Mørken, 1992) is to extend this work by searching for minimal support bases for the spaces W_0 that are the orthogonal complement of V_0 in V_1; more generally than Buhmann and Micchelli, they consider *non-decreasing* knot sequences $\mathbf{x}$ and $\underline{\tau}$ that are subsets of each other and may contain knots repeatedly, i.e. knots of multiplicity higher than one. They construct minimally supported prewavelets for such knot sequences and include explicit formulae using determinants. There is a concrete algorithm given in that paper to construct such prewavelets and it is shown that the thereby obtained prewavelets are nonzero within their support, except for a specified number of sign changes. We mention, finally in this section, the work by Lemarié (1992), where also nested sets of spline spaces whose knots satisfy global bounded mesh ratio conditions (2.4) are used to construct multiresolution analyses.

3. Radial basis function prewavelets.

The research pursued in this field is manifold. There are, firstly, several papers on prewavelets from radial basis function spaces spanned by a radial function and its integer translates, even in more than one dimension. On the other hand, there are approaches to prewavelets from radial function spaces where the centres of the radial functions are scattered (so far only in one dimension). This appears particularly suitable because radial functions show most of their effectiveness when the data are not any longer confined to grids, several alternative methods (in the simplest case: tensor product [spline] methods) being readily available for the regular grid case.

We begin with discussing results of the paper by Micchelli, Rabut and Utreras (1991), whose work was extended by Micchelli (1992), where the construction of multivariate prewavelets is performed for spaces spanned by shifts of fundamental solutions of iterated Laplace equations. Those functions belong to the class of radial basis functions most widely studied and have the form

$$\phi(r) = \begin{cases} r^{2\lambda-n} & \text{if } n \text{ is odd,} \\ r^{2\lambda-n}\log r & \text{if } n \text{ is even,} \end{cases} \qquad r = \|x\| \geqslant 0, \tag{3.1}$$

where $\lambda > \frac{1}{2}n$ is an integer, $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^n$. Actually, their work is more general, in that not just radial functions as in (3.1) are treated, but fundamental solutions of general elliptic partial differential operators. We shall restrict ourselves, however, to (3.1) for the purpose of illustration.

The main reason why this *ansatz* works is that those radial functions (unlike, for example, the multiquadric radial function) have distributional Fourier transforms which are reciprocals of even order homogeneous polynomials with no roots except at zero. Indeed, up to a nonzero constant whose value is unimportant here, $\hat{\phi}$ corresponding to (3.1) is $\|\cdot\|^{-2\lambda}$. The transforms are therefore analytic in a tube about the real axis except at the origin. This has three very important consequences. Firstly, one can construct quickly decaying finite differences of these radial function, because symmetric differencing in the real domain amounts to multiplying the Fourier transform of the radial function by an even order trigonometric polynomial with roots at zero. The trigonometric polynomial resolves the singularity of $\hat{\phi}$ at zero (e.g. Buhmann, 1993c) if the differences are of high enough order. The rate at which these differences decay depends only on the order of contact of the trigonometric polynomial and $\|\cdot\|^{-2\lambda}$ at the origin. It can therefore be arbitrarily high. For instance, a difference Φ of $\phi(\|\cdot\|)$ can be conveniently defined by its Fourier transform as follows:

$$\hat{\Phi}(y) = \frac{\left(\sum_{s=1}^{n} \sin^2\left(\frac{1}{2}y_s\right)\right)^{\lambda}}{\left\|\frac{1}{2}y\right\|^{2\lambda}}, \qquad y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n. \tag{3.2}$$

Note especially that $\hat{\Phi} \in C(\mathbb{R}^n)$. It can be shown that this function Φ satisfies the decay estimate $|\Phi(x)| = O\big((1 + \|x\|)^{-n-2}\big)$, so that, in particular, $\Phi \in L^2(\mathbb{R}^n)$.

Secondly, these differences are able to generate a multiresolution analysis because, on one hand, they satisfy a *refinement equation* and, on the other hand, can approximate $L^2(\mathbb{R}^n)$ functions arbitrarily closely if translated by integers and dilated by powers of two. We want to explain this point in detail. The multiresolution analysis has the form $\{V_j\}_{j=-\infty}^{\infty}$, where the V_j are defined by

$$V_j := \left\{ \sum_{k\in\mathbb{Z}^n} c_k \Phi(2^j \cdot -k) \;\middle|\; c = \{c_k\}_{k\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n) \right\}. \tag{3.3}$$

That the differences satisfy a refinement equation means that there exists $a \in \ell^1(\mathbb{Z}^n)$ such that

$$\Phi(x) = \sum_{j\in\mathbb{Z}^n} a_j \Phi(2x - j), \qquad x \in \mathbb{R}^n,$$

where $a = \{a_j\}_{j\in\mathbb{Z}^n}$. This implies in particular that the spaces defined in (3.3) are nested, as desired. (For this, $a \in \ell^1(\mathbb{Z}^n)$ is not a necessary condition but suffices.) The refinement equation is satisfied due to the homogeneity of $\hat{\phi}$. In our case, the a_j are a constant multiple of the Fourier coefficients of $\theta(2\cdot)/\theta$, where $\theta : \mathbb{T}^n \to \mathbb{R}$ is the numerator in (3.2). Here $\mathbb{T} = [-\pi, \pi]$. The first condition of (1.1) holds because the translates and dilates of Φ provide approximations to at most linearly growing f, say, of the simple form

$$Q_{2^{-j}} f(x) = \sum_{k\in\mathbb{Z}^n} f(k2^{-j}) \Phi(2^j x - k), \qquad x \in \mathbb{R}^n,$$

that are exact on linear polynomials f and converge uniformly to f as $j \to \infty$ for f from a class that is dense in $L^2(\mathbb{R}^n)$. The second condition of (1.1) holds because the translates and dilates of Φ form Riesz bases of the V_j which we will explain now.

That the translates and scales of Φ can form a Riesz basis of each V_j for a suitable θ is the third consequence of the shape of $\hat{\phi}$. Namely, there exist constants $0 < \mu_j \leqslant M_j < \infty$ so that

$$\mu_j \|c\| \leqslant \Big\| \sum_{k\in\mathbb{Z}^n} c_k \Phi(2^j \cdot -k) \Big\|_2 \leqslant M_j \|c\|, \qquad c = \{c_k\}_{k\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n), \tag{3.4}$$

for all $j \in \mathbb{Z}$. Here $\|\cdot\|$ is the $\ell^2(\mathbb{Z}^n)$, $\|\cdot\|_2$ the $L^2(\mathbb{R}^n)$ norm. This is an important property that is usually incorporated into the requirements of multiresolution analysis as well. Indeed, it can be viewed as a suitable replacement for the orthonormality condition that is imposed on the translates of *wavelets* (as compared to prewavelets) at each level j of scaling. Indeed, if the k translates in (3.4) are orthonormal, $\mu_j = M_j \equiv 1$ are possible in (3.4). The reason why (3.4) holds here is that $\hat{\phi}$ has no zero and that our chosen θ exactly matches its singularity at zero without having any further zeros. For $j = 0$, the upper and the lower bounds in (3.4) are the maximum and the minimum of

$$\sum_{k\in\mathbb{Z}^n} |\hat{\Phi}(t + 2\pi k)|^2, \qquad t \in \mathbb{T}^n,$$

respectively. For other js, (3.4) follows from scaling. That (3.4) and the decay of Φ lead to the second property in (1.1) is shown in the paper by Micchelli, Rabut and Utreras and is not spelled out here, although it is not a difficult argument.

In this set-up, the prewavelets are found which are described in our next theorem. Their construction is related to the construction that led to Theorem 2 and to the work in (Chui and Wang, 1992).

Theorem 4. *Let Φ be as above and define ψ_0 by its Fourier transform*

$$\hat{\psi}_0(y) = \|y\|^{-2\lambda} \frac{|\hat{\Phi}(\frac{1}{2}y)|^2}{\sum_{k\in\mathbb{Z}^n} |\hat{\Phi}(\frac{1}{2}y + 2\pi k)|^2}, \qquad y \in \mathbb{R}^n.$$

Further let E *be the set of corners of the unit cube in* $\mathbb{R}^n$ *and define*

$$\psi_{\mathrm{e}}(y) = \psi_0\big(y - \tfrac{1}{2}\mathrm{e}\big), \qquad y \in \mathbb{R}^n,\ \mathrm{e} \in \mathrm{E}. \tag{3.5}$$

Define finally

$$W_{j,\mathrm{e}} := \Big\{ \sum_{k\in\mathbb{Z}^n} c_k \psi_{\mathrm{e}}(2^j \cdot - k) \,\Big|\, c = \{c_k\}_{k\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n) \Big\}, \qquad \mathrm{e} \in \mathrm{E} \setminus \{0\},\ j \in \mathbb{Z}, \tag{3.6}$$

and

$$W_j := \bigoplus_{\mathrm{e}\in\mathrm{E}\setminus\{0\}} W_{j,\mathrm{e}}, \qquad j \in \mathbb{Z}.$$

Then we have $W_j \perp W_\ell$ for all integers $j \neq \ell$ and

$$L^2(\mathbb{R}^n) = \overline{\bigoplus_{j=-\infty}^{\infty} W_j}.$$

In Micchelli (1992), this work is generalized by combining it with results on prewavelets from "box-" (or "cube-") splines and by admitting scaling factors other than 2. (In fact, scaling by integer matrices M with $|\det M| > 1$ is admitted and it is also required that $\lim_{j\to\infty} M^{-j} = 0$. This is usually achieved by requiring that all of M's eigenvalues are larger than one in modulus. In this event, $|\det M| - 1$ prewavelets and their integer shifts span each W_j.). Utreras (1993) constructs prewavelets in the same way as in Theorem 4 and points to the important fact that no multiresolution analysis can be found for shifts of (3.1), i.e. when ϕ is replaced by $\phi(\sqrt{r^2+c^2})$ for $c > 0$. More generally, he shows that it is the exponential (that is, too fast) decay of the Fourier transform $\hat{\phi}$ of the resulting basis function ϕ that prohibits the existence of a multiresolution analysis generated by shifts and scales of ϕ or a linear combination of translates of ϕ. (It is the refinement equation that fails.) This applies for example to the multiquadric radial function for odd n and $\lambda = \frac{1}{2}(n+1)$. This statement, however, only holds if the multiresolution analysis is required to be *stationary*, i.e. results from the dilates and shifts of just *one* function as in (3.3). On the other hand, several authors have studied *non-stationary* multiresolution analyses, where the generating function of each V_j may be a different $L^2(\mathbb{R}^n)$ function Φ_j, say. Hence, the V_j are of the form

$$V_j := \left\{ \sum_{k\in\mathbb{Z}^n} c_k \Phi_j(\cdot - M^{-j}k) \;\middle|\; c = \{c_k\}_{k\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n) \right\},$$

where, as in Micchelli (1992), a general scaling matrix M with integer entries is admitted. The nesting property of the V_j is ensured by the non-stationary refinement condition

$$\Phi_j(x) = \sum_{k\in\mathbb{Z}^n} a_k \Phi_{j+1}(x - M^{-j-1}k), \qquad x \in \mathbb{R}^n, \tag{3.7}$$

with $a \in \ell^1(\mathbb{Z}^n)$. For illustration, we concentrate on the work by Stöckler (1993) first. He uses only an infinite sequence of nested spaces $V_0 \subset V_1 \subset \cdots$ and has two additional requirements on Φ_j, $j \in \mathbb{Z}_+$. The first one is that the translates of those functions be Riesz bases uniformly for each V_j, i.e. that the inequalities

$$\hat{\mu}\|c\| \leqslant \Big\| \sum_{k\in\mathbb{Z}^n} c_k \Phi_j(\cdot - M^{-j}k) \Big\|_2 \leqslant \hat{M}\|c\|, \qquad c = \{c_k\}_{k\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n),\ j \in \mathbb{Z}_+,$$

be satisfied for positive finite constants $\hat{\mu}$ and $\hat{M}$ that are the same for all j. The second one is

$$\sum_{k\in\mathbb{Z}^n} \big|\big(\Phi_j * (\Phi_j(-\cdot))\big)(x+k)\big| < \infty \tag{3.8}$$

almost everywhere (a.e.) for $x \in \mathbb{R}^n$. Here, $*$ denotes convolution. A general theorem is given that guarantees the existence of $|\det M| - 1$ prewavelets for each j whose translates span W_j. In the special case $n = 2$ and $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ – so that $\det M = 2$ and only one prewavelet is required –, prewavelets for radial function spaces can be found if $\hat{\phi}$ is positive almost everywhere and satisfies

$$\frac{\hat{\phi}(\|y + 2\pi k\|)}{\hat{\phi}(\|y\|)} = O\bigg(\frac{1 + \|y\|}{\|y + 2\pi k\|}\bigg)^m, \qquad k \in \mathbb{Z}^2, \quad \|y\| \leqslant \pi|k|, \tag{3.9}$$

a.e. for some $m > 1$. Here $|\cdot|$ is the ℓ^1 norm. This holds, for instance, for the shifted versions of (3.1) that have been mentioned above, exactly *because* their Fourier transforms decay so fast. We let $\hat{\Phi}_j := \hat{\phi}(\|\cdot\|)/\sqrt{\sigma_j}$, where

$$\sigma_j(y) = 2^j \sum_{k\in\mathbb{Z}^2} \hat{\phi}(\|y + 2\pi M^{jT} k\|)^2, \qquad t \in \mathbb{T}^2.$$

Because the translates of Φ_j are orthonormal, they are, in particular, Riesz and $\hat{\mu} = \hat{M} = 1$. It can also be shown that they decay fast enough to satisfy (3.8). Then we define

$$\hat{\psi}_j(y) = \exp(-\mathrm{i}\kappa \cdot M^{-(j+1)T} y) \sqrt{\frac{\sigma_{j+1}(y + 2\pi M^{jT}\kappa)}{\sigma_j(y)}}\, \hat{\Phi}_{j+1}(y), \qquad y \in \mathbb{R}^2.$$

The $\mathbb{Z}^2$ translates of ψ_j span W_j with $V_{j+1} = W_j \oplus V_j$. Here, κ is the vector $(1,0)$ which gives $\mathbb{Z}^2 = M^T\mathbb{Z}^2 \oplus M^T(\kappa + \mathbb{Z}^2)$.

A general treatise on prewavelets is given in (de Boor, DeVore and Ron, 1993). While their approach deals only with equally spaced centres and scaling by 2 instead of a general

M, it does apply to all the radial basis function spaces that have been mentioned so far in this section. No assumptions about stationarity of the nested spaces are made, so we have spaces V_j that are the $L^2(\mathbb{R}^n)$ closure of the span of $2^{-j}k$, $k \in \mathbb{Z}^n$, translates of a function Φ_j each. They are assumed to be nested, however, and to satisfy (1.1), but no refinement equation like (3.7) with *summable* coefficients is required. Of course, the Φ_j must be in $L^2(\mathbb{R}^n)$, but they need not have a concrete rate of decay at infinity. For radial basis functions, differencing gives the square integrability as before. The authors show the sufficiency of $\operatorname{supp} \hat{\Phi}_j = \mathbb{R}^n$ in order that the W_j are finitely generated shift-invariant spaces, i.e. generated by the multi-integer translates of just finitely many functions, and that there are specific basis functions ψ_{e}, $\mathrm{e} \in \mathrm{E} \setminus \{0\}$, in V_{j+1} that generate the desired spaces W_j. This support assumption holds for all radial basis functions that are mentioned in this review. It suffices to study V_0, V_1 and W_0, because the other spaces are obtained by dilation. An instance of the basis functions that produce W_0 are the expressions

$$\hat{\psi}_{\mathrm{e}} = \hat{\Phi}_{\mathrm{e}} - \widehat{P\Phi_{\mathrm{e}}}, \qquad \mathrm{e} \in \mathrm{E} \setminus \{0\},$$

where $\Phi_{\mathrm{e}} = \Phi_0(\cdot + \frac{1}{2}\mathrm{e})$ and $P : V_1 \to V_0$ is the orthogonal projector. The translates of these ψ_{e} may not, however, satisfy a Riesz stability property. It is generally true that, if the $\mathbb{Z}^n$ shifts of $\psi \in V_1$ generate V_1, the integer shifts of $\psi(\cdot + \frac{1}{2}\mathrm{e})$, $\mathrm{e} \in \mathrm{E} \setminus \{0\}$, are a basis for W_0 if additionally

$$\sum_{j \in \mathbb{Z}^n} \hat{\psi}(\cdot + 4\pi j)\hat{\Phi}_0(\cdot + 4\pi j)$$

is 2π-periodic. In one dimension, we are led to the following result that specifies the prewavelets.

Theorem 5. *Suppose* $\Phi_j \in L^2(\mathbb{R})$, $j = 0, 1$. *Assume* $\operatorname{supp} \hat{\Phi}_0 = \operatorname{supp} \hat{\Phi}_1 = \mathbb{R}$ *and define*

$$\hat{\psi} = \exp(-\tfrac{1}{2}i\cdot)\hat{\Phi}_1 \sum_{j \in \mathbb{Z}} \hat{\Phi}_0\big(\cdot + 4\pi(j + \tfrac{1}{2})\big)\hat{\Phi}_1\big(\cdot + 4\pi(j + \tfrac{1}{2})\big).$$

If $\hat{\psi} \in L^2(\mathbb{R})$, *then its inverse transform* ψ *has* $\mathbb{Z}$ *translates that generate* W_0. *If the integer shifts of* Φ_0 *and the half-integer shifts of* Φ_1 *have a Riesz basis property, so do the integer shifts of* ψ.

In several dimensions, the following result is true which is originally due to Riemenschneider and Shen (1991) but de Boor, DeVore and Ron give a new proof.

Theorem 6. *Suppose* supp $\hat{\Phi}_j = \mathbb{R}^n$, $j = 0, 1$, *and that each* Φ_j *is real-valued and symmetric about* 0. *Let*

$$B := \sum_{j \in \mathbb{Z}^n} \hat{\Phi}_0(\cdot + 4\pi j)\hat{\Phi}_1(\cdot + 4\pi j).$$

Suppose Φ_0*'s multi-integer shifts satisfy a Riesz stability condition and that the half multi-integer shifts of* Φ_1 *do as well. Assume finally that there is a 1–1 map* $\alpha : \frac{1}{2}\mathrm{E} \setminus \{0\} \to 2\pi\mathrm{E} \setminus \{0\}$ *that satisfies*

(i) $\exp(\mathrm{i}\frac{1}{2}\mathrm{e} \cdot \alpha(\frac{1}{2}\mathrm{e})) = -1$ *for all* $\mathrm{e} \in \mathrm{E} \setminus \{0\}$,

(ii) $\exp(\mathrm{i}\frac{1}{2}(\mathrm{e}_1 - \mathrm{e}_2) \cdot \alpha(\frac{1}{2}(\mathrm{e}_1 - \mathrm{e}_2))) = -1$ *for all* $\mathrm{e}_1, \mathrm{e}_2 \in \mathrm{E} \setminus \{0\}$ *with* $\mathrm{e}_1 \neq \mathrm{e}_2$.

Then the integer shifts of the functions ψ_e *that are defined by*

$$\hat{\psi}_\mathrm{e} = \exp(-\mathrm{i}\mathrm{e}\cdot)B(\cdot + \alpha(\tfrac{1}{2}\mathrm{e}))\hat{\Phi}_1, \qquad \mathrm{e} \in \mathrm{E} \setminus \{0\},$$

provide a Riesz basis for W_0.

It should be noted that (i) and (ii) above can be met if and only if $n = 1, 2$ or 3.

In the setting of both theorems above, general W_j, $j \in \mathbb{Z}$, can be found so that

$$L^2(\mathbb{R}^n) = \overline{\bigoplus_{j=-\infty}^{\infty} W_j}$$

if (1.1) is true. Characterisations of the multiresolution properties (1.1) are provided too. The first property is fulfilled by the above construction if and only if

$$\bigcup_{j \in \mathbb{Z}} \operatorname{supp} \hat{\Phi}_j = \mathbb{R}^n$$

which is one of our requirements anyway. (The second condition of (1.1) is true if the nested set of spaces V_j is stationary, but $\{0\}$ may have to be replaced by a set of dimension at most one otherwise.) Examples that fulfill this requirement exist in abundance: all radial basis function that we mention in this article have globally ($\mathbb{R}^n$) supported Fourier transforms.

In the author's paper (1993b), a different tack is pursued. There, univariate prewavelets from spaces spanned by (integer) translates of multiquadric and related functions are constructed. In order to get a square-integrable basis function first, derivatives of multiquadric functions are taken and convolved with B-splines. This is the same as taking divided differences of the radial function but is more amenable to analysis because one can make use of the positivity of ϕ's Fourier transform. After all, convolution of functions in the real domain means function multiplication in the Fourier domain. This fact is extensively used in the proofs of the results that follow. The work considers, in fact, not just decompositions of $L^2(\mathbb{R})$, but (discrete) Sobolev spaces of arbitrary integer order, although we only formulate the results for continuous Sobolev spaces $H_2^m(\mathbb{R})$, $m \in \mathbb{Z}_+$, here. That is the space of all $L^2(\mathbb{R})$ functions all of whose derivatives up to order m are also in $L^2(\mathbb{R})$. It shall be equipped with the norm that is the sum of the $L^2(\mathbb{R})$ norms of the function and all those derivatives. Let $\Phi \in C^m(\mathbb{R})$ with

$$\left|\Phi^{(\ell)}(x)\right| = O(|x|^{-1-\varepsilon}), \qquad x \to \pm\infty, \quad \ell = 0, 1, \ldots, m, \tag{3.10}$$

$\hat{\Phi}(t) > 0$, for all $t \in \mathbb{R}$, and $\hat{\Phi}(0) = 1$ be given. The quantity ε is fixed and positive. We restrict ourselves here to the example $\Phi(x) = \hat{\kappa}\phi^{(2\lambda)}(x)$, where $2\lambda = n+1$, $\hat{\kappa}$ is a suitable normalization parameter, $\phi(x) = \tilde{\phi}(\sqrt{x^2+c^2})$, $c \geqslant 0$, and $\tilde{\phi}$ is one of the functions $\tilde{\phi}(r) = r^{2\lambda-1}$, $\lambda \in \mathbb{N}$, but the theory in (Buhmann, 1993b) is more general. Nevertheless, this covers the multiquadric example (for $\lambda = 1$). We consider the functions

$$\left.\begin{aligned} C_j &:= B_j^{\mathbf{c}} * \Phi, \quad j \in \mathbb{Z}, \\ F_j &:= B_j^{\mathbf{f}} * \Phi, \quad j \in \mathbb{Z}. \end{aligned}\right\} \tag{3.11}$$

The B-splines are the same as those in Section 2. Thus, C_j and F_j are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, because the B-splines are and because $\Phi \in L^1(\mathbb{R})$. We now define V_0 and V_1 as

$$\left.\begin{aligned} V_0 &:= \left\{ \sum_{j=-\infty}^{\infty} c_j C_j \;\middle|\; c = \{c_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) \right\}, \\ V_1 &:= \left\{ \sum_{j=-\infty}^{\infty} c_j F_j \;\middle|\; c = \{c_j\}_{j=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) \right\}. \end{aligned}\right\} \tag{3.12}$$

Then, for $\underline{\tau} = \frac{1}{2}\mathbb{Z}$, the following theorem is true.

Theorem 7. *Let Φ be as above and let $P : V_1 \to V_0$ be the orthogonal projection with respect to the Sobolev inner product on $H_2^m(\mathbb{R})$. Then $\psi_\ell := F_\ell - PF_\ell$, $\ell \in \mathbb{Z}$, is a prewavelet, i.e. it satisfies*

(a) *$\psi_\ell \in V_1$, $\psi_\ell \perp V_0$ with respect to the Sobolev inner product on $H_2^m(\mathbb{R})$,*

(b) *$V_1 = V_0 \oplus W_0$.*

It is furthermore true that the prewavelet decays at least as fast as

$$|\psi_\ell(x)| = O\big((1+|x-\ell|)^{-3}\big), \qquad x \in \mathbb{R}.$$

In contrast to this work, (Buhmann, 1994) covers the case when $\mathbf{x}$ and $\underline{\tau}$ are no longer equally spaced. They still have to satisfy a global bounded mesh ratio condition (2.4). The spaces V_0, V_1 and W_0 are defined as above. The assumptions on Φ are the same except that (3.10) is only demanded for $m = 0$, and we are back in $L^2(\mathbb{R})$, but $\varepsilon > n$ is needed additionally, as well as $\Phi \geqslant 0$. The prewavelets are now constructed on the basis of functions

$$\Psi_\ell(x) = \sum_{k=-\infty}^{\infty} d_k^\ell F_k^{2n+1}(x), \qquad x \in \mathbb{R}, \tag{3.13}$$

where $F_k^{2n+1} = B_k^{\mathbf{f},2n+1} * \Phi$ and $B_k^{\mathbf{f},2n+1}$ are the B-splines on the knot sequence $\underline{\tau}$ just as above but with degree $2n+1$ and support $[\tau_k, \tau_{k+2n+2}]$. The coefficients d_k^ℓ of these functions are supposed to be such that

$$\big(\Phi(-\cdot) * \Psi_\ell\big)(\tau_{j+n+1}) = \begin{cases} 1 & \text{if } \ell = j \text{ and } n \text{ odd}, \\ 1 & \text{if } \ell = j+1 \text{ and } n \text{ even}, \\ 0 & \text{otherwise}, \end{cases} \qquad j, \ell \in \mathbb{Z}. \tag{3.14}$$

Then we define $\psi_\ell := \Psi_{2\ell}^{(n+1)}$. This is in V_1 because $(n+1)$-st derivatives of B-splines of degree $2n+1$ are expressible as finite linear combinations of B-splines of degree n.

We show that the desired orthogonality conditions $\psi_\ell \perp V_0$ hold by integration by parts: Suppose (3.14) holds for a suitable $d^\ell = \{d_k^\ell\}_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$. Thus, using the fact that C_j and $\Psi_{2\ell}$ and all their derivatives up to degree $n+1$ are integrable and vanish at infinity,

$$\Big\langle B_j^{\mathbf{c}} * \Phi, \sum_{k=-\infty}^{\infty} d_k^{2\ell} B_k^{\mathbf{f},2n+1^{(n+1)}} * \Phi \Big\rangle = (-1)^{n+1} \Big\langle \sum_{k=-\infty}^{\infty} \mu_k^j \Phi(\cdot - x_k), \Psi_{2\ell} \Big\rangle = 0, \quad j \in \mathbb{Z},$$

(where $\langle \cdot,\cdot \rangle$ is the standard $L^2(\mathbb{R})$ inner product) because the $(n+1)$-st derivative of $B_j^{\mathbf{c}}$ is a finite collection of delta functions $\delta(\cdot - x_k)$, $k \in \mathbb{Z}$, the μ_k^j being some real coefficients. The central result is as follows.

Theorem 8. *The functions* (3.13) *exist as desired and they are such that* $\psi_\ell \in V_1$ *and* $V_1 = V_0 \oplus W_0$, *where* W_0 *is defined in* (2.3), *if*

$$\max\left[\tfrac{1}{2}, \sup_{j\in\mathbb{Z}} \frac{\tau_{j+2n+2} - \tau_j}{2n+2} \sup_{t\in\mathbb{R}} \sum_{k=-\infty}^{\infty} \Phi(t+\tau_k)\right] < \\ 2 \inf_{k\in\mathbb{Z}} \int_0^\infty (\Phi(-\cdot) * \Phi)(z) B_k^{\mathbf{f},2n+1}(\tau_{k+n+1} + z)\, dz, \tag{3.15}$$

$$\sup_{k\in\mathbb{Z}} \frac{\tau_{2k+2n+1} - \tau_{2k-1}}{n+1} < 2 \min\Big\{1, \inf_{k\in\mathbb{Z}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\Phi}(y)|^2 |\widehat{B_k^{\mathbf{c}}}(y)|^2\, dy\Big\}. \tag{3.16}$$

In other words, the ψ_ℓ *are* prewavelets, *since they generate* (2.3) *with* $W_0 \perp V_0$ *and* $V_1 = V_0 \oplus W_0$. *Further, these prewavelets satisfy the summability properties*

$$\sum_{j=-\infty}^{\infty} |\psi_\ell(x+\tau_j)| \leqslant \text{const.} < \infty, \qquad x \in \mathbb{R}, \quad \ell \in \mathbb{Z}, \tag{3.17}$$

uniformly in x *and* ℓ *and, uniformly in* x,

$$\sum_{\ell=-\infty}^{\infty} |\psi_\ell(x)| \leqslant \text{const.} < \infty, \qquad x \in \mathbb{R}. \tag{3.18}$$

We remark that there exist sets $\underline{\tau}$ that satisfy the conditions (3.15)–(3.16) of the theorem, because, for instance, any small enough pertubation of $\underline{\tau} = \mathbb{Z}$ does for the multiquadric function $\Phi = \frac{1}{2}\varrho''$, $\varrho(r) = \sqrt{r^2+c^2}$ if c is small enough.

The following theorem is easily established using the positivity of Φ's Fourier transform and the fact that the Fourier transform as an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry.

Theorem 9. *Let* $\{\underline{\tau}^j\}_{j=-\infty}^{\infty}$ *be a sequence of nested knot sequences in* $\mathbb{R}$ *that each satisfy a global bounded mesh ratio condition and become dense in* $\mathbb{R}$ *for* $j \to \infty$. *Suppose also* $\sup_{k\in\mathbb{Z}} \tau_{k+1}^j - \tau_k^j \to \infty$, $j \to -\infty$. *Let* V_j *be defined according to this setting and in analogy to* (3.12). *Then* (1.1) *holds.*

Corollary 10. *Given the set-up of* Theorems 7 *or* 8, *we can form prewavelets* $\{\psi_{\ell j}\}_{\ell=-\infty}^{\infty}$, $j \in \mathbb{Z}$, *in* V_{j+1} *and corresponding spaces* W_j *spanned by sums of the prewavelets with square-summable coefficients such that* $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, *and*

$$H_2^m(\mathbb{R}) = \overline{\bigoplus_{j=-\infty}^{\infty} W_j}$$

for the set-up of Theorem 7 *and*

$$L^2(\mathbb{R}) = \overline{\bigoplus_{j=-\infty}^{\infty} W_j}$$

for Theorem 8.

In the recent paper by Jetter and Stöckler (1994), a Riesz basis property for a set of functions which result from a univariate *symmetric* preconditioning of a radial basis function ϕ with a positive generalised Fourier transform $\hat{\phi}$ is found that satisfies certain lower bounds on compact sets. Concretely, the basis functions take on the following form – up to normalization –, where we use the Fourier transform to define them:

$$\hat{\Phi}_j = \sqrt{\hat{\phi}}\hat{B}_j^{\mathbf{c}},$$

where $B_j^{\mathbf{c}}$ are the same B-splines as used before. The points $\mathbf{x}$ have to satisfy $x_j \to \pm\infty$ as $j \to \pm\infty$ and a uniform minimal separation distance property. Hence the inner product matrices with entries $\langle \Phi_j, \Phi_k \rangle$ correspond to preconditioned interpolation matrices with entries $\phi(x_j - x_k)$, where the preconditioning is performed by taking divided differences of degree $n+1$ both with respect to rows and columns. Jetter and Stöckler prove the important result that the spectrum of this Gram matrix is bounded below and above, so that the Φ_j are a Riesz basis for the space they generate.

In (Chui, Ward and Stöckler, 1994), radial basis function wavelets for equally spaced centres in one dimension are constructed and particular attention is given to the size of their time/frequency localization windows. The construction is of the same type as the construction of prewavelets for non-stationary multiresolution analyses of Stöckler outlined above. Apart from the important estimates for the size of their time/frequency localization windows, it is shown that the prewavelets on all scales $j = 0, 1, \ldots$ together with the generator of V_0 define an orthonormal basis of $L^2(\mathbb{R})$, and estimates are given for the distance between any f in the Sobolev space $H_2^m(\mathbb{R})$ (the m is the same as in (3.9)) and each individual V_j.

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