A New Multidimensional Euler Scheme [1]

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Abstract

A new idea is presented to solve the multidimensional Euler equations numerically. The aim of this idea is to obtain a robust shock capturing method without the use of dimensional splitting. The starting point is the idea of the one-dimensional flux vector splitting and the homogeneity of the Euler equations. Using this concept it is shown that a different interpretation of the one-dimensional waves and the use of some physical properties lead to a decomposition of the state vector into three multidimensional waves. This idea includes most of the physical properties of the Euler equations and allows infinitely many propagation directions. Assuming a Cartesian grid and constant states within each cell a numerical scheme is derived and some test calculations are shown.

Keywords: Euler equations, multidimensional waves, dimensional splitting

Subject Classification: 35Q35, 65M06, 76M25, 76N15

Introduction

In multidimensional flow calculations most of the finite difference or finite volume methods use a one-dimensional Euler solver in multiple directions. Here the main propagation directions are the coordinate axes, i.e. the cell interfaces of the underlying grid. With this standard dimensional splitting approach the accuracy of the solution is first order. With some special modifications the order is at most two [6]. There are flow properties which can not be correctly described by these splittings, e.g. a shock diagonal to the grid.

There are some new investigations in multidimensional Euler solvers to circumvent the problem mentioned above. The idea of these codes is to determine the main propagation direction and then solve a one-dimensional problem in this direction [3], or to construct a set of elementary waves which transport the residuum of one cell to some neighboring nodes [4]. Now the propagation directions are independent of the underlying grid. But these directions have to be calculated from the data of the flowfield and especially from some of their gradients. This causes a loss of robustness of the resulting scheme.

Our method is a synthesis of the previous ones. The underlying concept is based on a decoupling of the multidimensional flux into a finite number of multidimensional elementary waves comparable to the flux vector splitting in one space dimension. These elementary waves include most of the properties of the Euler equations as there are the homogeneity of space and the invariance under a reflection. From these ideas we get a numerical scheme which allows infinitely many propagation directions in contrast to only two for the dimensional splitting. Moreover, the main part of these waves does not depend on gradients of the data in contrast to the approach above. The gradients only affect higher order terms.

Equations and notation

In the following investigations we will restrict ourself to the case of the Euler equations. Before we start with the description of the idea we introduce the notation used in this paper. The one-dimensional homogeneous Euler equations can be written in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) = 0. \tag{1}$$

The vector of the conserved quantities $\mathbf{U}$ and the physical flux function $\mathbf{F}(\mathbf{U})$ are

$$\mathbf{U} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix}. \tag{2}$$

Here ρ is the density, m is the momentum, E is the total energy, $u = m/\rho$ is the velocity and p is the pressure. Using the equation of state for an ideal gas, we obtain

$$p = (\gamma - 1)\left(E - \rho\frac{u^2}{2}\right) \tag{3}$$

for the pressure. In our case γ is a constant with $\gamma = 1.4$, the value for air.

For simplicity, we consider the Euler equations in two space dimensions, although the ideas carry over to three dimensions. The differential equations then have the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) + \frac{\partial}{\partial x}\mathbf{G}(\mathbf{U}) = 0. \tag{4}$$

The conservation equation for n, the y-component of the momentum, is added to the system. The vectors have the form

$$\mathbf{U} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho u \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

where $v = n/\rho$ is the velocity in y-direction. Equation (3) becomes

$$p = (\gamma - 1)\left(E - \rho\frac{u^2 + v^2}{2}\right).$$

To complete this collection of formulas we add the eigenvalues and eigenvectors of the Jacobian matrix of $\mathbf{F}(\mathbf{U})$ in (2). The eigenvalues are $\lambda_1 = u - c$, $\lambda_2 = u$ and $\lambda_3 = u + c$ and

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ H - uc & u^2/2 & H + uc \end{pmatrix} \tag{5}$$

is the matrix of the corresponding eigenvectors with the total enthalpy $H = (E + p)\rho$ and the speed of sound c defined by $c^2 = \gamma p/\rho$. The vector $\mathbf{R}^{-1}\mathbf{U}$ appearing in the one-dimensional flux vector splitting has the simple form

$$\mathbf{R}^{-1}\mathbf{U} = \rho\left(\frac{1}{2\gamma}, \frac{\gamma - 1}{\gamma}, \frac{1}{2\gamma}\right)^T.$$

$\mathbf{I}$ denotes the unit matrix and $\mathbf{0}$ the corresponding vector of zeros. $\mathbf{x}$ is the coordinate vector and $\mathbf{u}$ the velocity vector in several space dimensions.

Generalization of the waves

The main idea of this numerical scheme is to use the characteristic propagation directions in each point of the flowfield and to propagate appropriate quantities along these directions. Let us consider one space dimension first. We will see that if we interpret the one-dimensional characteristic waves in a certain manner then there is no difference between the one- and the multi-dimensional case.

The homogeneity of the one-dimensional Euler equations give

$$\mathbf{F}(\mathbf{U}) = \mathbf{A}\mathbf{U} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}\mathbf{U} = \sum_{i=1}^{3} \mathbf{r}_i \lambda_i \alpha_i \tag{6}$$

with the Jacobian matrix $\mathbf{A}$ of $\mathbf{F}$, the matrix of right eigenvectors $\mathbf{R}$ in (5) and the matrix of eigenvalues $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. This property allows us to decompose a state $\mathbf{U}$ in the flowfield into the eigenvectors of $\mathbf{A}$

$$\mathbf{U} = \mathbf{R}\mathbf{R}^{-1}\mathbf{U} = \sum \mathbf{r}_i \alpha_i$$

and propagate these quantities with their characteristic speeds to get the flux $\mathbf{F}$. This kind of method is called flux vector splitting and the special form (6) is the Steger-Warming splitting. We are now able to calculate the fluxes at each cell interface (in 1-D) and advance the solution in time. In 1-D we have three different waves traveling with speeds $u-c$, u and $u+c$ as shown in Figure 1. The constant vector of conserved quantities $\mathbf{U}$ of the left hand side is decomposed in these three waves. Each wave is independent of each other, so we can treat them separately. We start with the second wave corresponding to λ_2. This wave describes the convection of the gas. The interval $I_0 = [x_{i-1/2}, x_{i+1/2}]$ moves with velocity u as shown in Figure 2. The propagation is in the flow direction. Even in several space dimensions there is only one propagation direction and so we can generalize this wave easily. In the two dimensional case the propagation of a constant state $\mathbf{U}$ in a domain Ω_0 by this wave has the form as shown in Figure 5. The shape of the cell is the same, it only moves with velocity $\mathbf{u}$ which is now a vector. The velocity $\mathbf{u}$ even can be a linear function in space. Then the shape of the domain will be destroyed. We characterize such a propagation with

Rule 1.

- *Information moves from each point* $\mathbf{x}$ *with velocity* $\mathbf{u}(\mathbf{x})$.

The information is described by a generalized version of the vector $\alpha_2 \mathbf{r}_2$. Using the multidimensional version of Dirac's delta function

$$\int_{\mathbb{R}^N} \delta(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = f(\mathbf{0})$$

we can formulate

Definition 1. *(Wave $\mathcal{U}$)*

With the function

$$\mathbf{R}_2(\mathbf{U}) := \frac{\gamma - 1}{\gamma} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho \mathbf{u}^2/2 \end{pmatrix}$$

we define wave $\mathcal{U}$ of domain Ω_0 at time $t + \Delta t$ by

$$\mathcal{U}_{\Omega_0}(\mathbf{x}, t + \Delta t) = \int_{\Omega_0} \mathbf{R}_2(\mathbf{U}(\mathbf{y}, t)) \delta(\mathbf{x} - (\mathbf{y} + \Delta t \mathbf{u}(\mathbf{y}, t))) d\mathbf{y}.$$

$\mathcal{U}_{\Omega_0}(\mathbf{x}, t + \Delta t)$ describes how $\mathbf{R}_2$ in Ω_0 is distributed aftet time Δt. Returning to the 1-D case, $\mathcal{U}(x, t + \Delta t)$ is exactly the same function as in Figure 2 extracted from the Steger-Warming splitting. But Definition 1 is for arbitrary space dimensions. The flux from a domain Ω_0 into a domain Ω_1 during time Δt is given by

$$\mathbf{F}^u_{\Omega_0 \Omega_1} = \int_{\Omega_1} \mathcal{U}_{\Omega_0}(\mathbf{x}, t + \Delta t) d\mathbf{x}.$$

We are now calculating fluxes from one domain into another. This allows us to compute fluxes between domains without a cell interface, for example the diagonal cells in $\mathbb{R}^2$, or even between cells which are not direct neighbors.

The generalization of the two sonic waves is more complicated and we will use another property of the Euler equations. The solution of (1) is invariant with respect to the transformation of space $\tilde{x} \to -x$. Information traveling with velocity λ_3 in the solution is moving with velocity $-\lambda_3 = -u - c = \tilde{u} - c = \tilde{\lambda}_3$ after the transformation (see Fig. 3).

So in one space dimension these two waves seem to have something in common. If we examine the eigenvectors of $\mathbf{A}$ in (5), we see that the terms in $\mathbf{r}_1$ and $\mathbf{r}_3$ have either the same or the opposite sign. We now want to describe both propagation directions in one wave or at least in two waves which contain both directions. Then we generalize them replacing two directions by all directions. We will characterize this propagation with

Rule 2.

- *Information moves from each point in all directions with velocity c, the speed of sound, relative to the motion of gas.*
- *The mass of information is conserved.*

The part of $\mathbf{r}_1$ and $\mathbf{r}_3$ with the same sign can easily be distributed in this way. For an initially constant function, Figure 4 shows the new shape after time Δt in the 1-D case. In Rule 2 we are not restricted to a finite number of propagation directions. At this point we can deal with another main property of the Euler equations. The solution of (4) is invariant with respect to a rotation of space and from this we get infinitely many propagation directions. Small disturbances move along the Mach cone. In two space dimensions (or more) Rule 2 enables us to construct a multidimensional wave. Now we are able to introduce this property of the Euler equations into a new numerical scheme. Note that this property is not preserved in the dimensional splitting approach. Putting these considerations into a mathematical formulation we get

Definition 2. *(Wave C)*

With the functions

$$\mathbf{R}_1(\mathbf{U}) := \frac{1}{\gamma}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ \rho H\end{pmatrix}, \qquad \mathbf{g}(\mathbf{x},t,\Delta t) = \mathbf{x} + \Delta t(\mathbf{u}(\mathbf{x},t) + \mathbf{n}\, c(\mathbf{x},t))$$

we define wave $\mathcal{C}$ of domain Ω_0 at time $t+\Delta t$ by

$$\mathcal{C}_{\Omega_0}(\mathbf{x}, t+\Delta t) = \frac{1}{|O|}\int_O\int_{\Omega_0} \mathbf{R}_1(\mathbf{U}(\mathbf{y},t))\delta(\mathbf{x}-\mathbf{g}(\mathbf{y},t,\Delta t))d\mathbf{y}dO.$$

O is the surface of the N-dimensional unit ball and $\mathbf{n}$ is the outer normal to this surface. $\mathcal{C}_{\Omega_0}(\mathbf{x}, t+\Delta t)$ describes how $\mathbf{R}_1$ in Ω_0 is distributed after time Δt. The intergation over the whole surface O takes into account the propagation in all directions. Figure 7 shows the support of the wave $\mathcal{C}$ at time $t+\Delta t$ if all the quantities are constant in Ω_0. In this case $\mathbf{R}_1$ is independent of the space variable $\mathbf{x}$ and the integral reduces to

$$h^c(\mathbf{x}, t+\Delta t) := \frac{1}{|O|}\int_O\int_{\Omega_0} \delta(\mathbf{x}-\mathbf{g}(\mathbf{y},t))d\mathbf{y}dO. \tag{7}$$

Figure 8 shows the shape of h^c for $\Delta tc = 0.4$ ($\Delta x = 1$). The flux of this wave is obtained in the same manner as for wave $\mathcal{U}$. We get

$$\mathbf{F}^c_{\Omega_0\Omega_1} = \int_{\Omega_1} \mathcal{C}_{\Omega_0}(\mathbf{x}, t+\Delta t)d\mathbf{x}. \tag{8}$$

With wave $\mathcal{C}$ we described only one part of the eigenvectors in (5). We still have to propagate the part with opposite sign in $\mathbf{r}_1$ and $\mathbf{r}_3$. If we interpret these quantities as vectors pointing in opposite directions, we can characterize this propagation by

Rule 3.

- *From each point a momentum wave moves in all directions with velocity c relative to the gas.*

Applying this to the 1-D case the momentum wave is scalar and "all directions" are represented by the different signs. Figure 6 shows the distribution of a constant function after time Δt according to this rule. Again we can easily generalize this to several space dimensions. After fitting some constants, e.g. the amplitude of this wave is not determined by Rule 3 in contrast to the wave $\mathcal{C}$, we can put the mathematical formulation in

Definition 3. *(Wave $\mathcal{C}^-$)*

With $\mathbf{I}$ *the unity matrix and* $\mathbf{0}$ *the vector of zeros in* $I\!R^N$, $\mathbf{g}$ *from Def. 2 and the* $N \times (N+2)$ *matrix function*

$$\mathbf{L}_3(\mathbf{U}) := \frac{\rho c}{\gamma} \begin{pmatrix} \mathbf{0}^T \\ \mathbf{I} \\ \mathbf{u}^T \end{pmatrix}$$

we define the wave $\mathcal{C}^-$ *of domain* Ω_0 *at time* $t + \Delta t$ *by*

$$\mathcal{C}^-_{\Omega_0}(\mathbf{x}, t + \Delta t) = \frac{N}{|O|} \int_O \int_{\Omega_0} \mathbf{L}_3(\mathbf{U}(\mathbf{y}, t))\mathbf{n}\,\delta(\mathbf{x} - \mathbf{g}(\mathbf{y}, t, \Delta t))d\mathbf{y}dO.$$

$\mathcal{C}^-_{\Omega_0}(\mathbf{x}, t + \Delta t)$ describes how momentum and energy are distributed after time Δt due to the pressure in Ω_0. Figure 9 shows the support of $\mathcal{C}^-$ for constant states and Figure 10 shows $|h^{c-}|$ for the same time as h^c. Analogous to (7), h^{c-} is given as

$$h^{c-}(\mathbf{x}, t + \Delta t) := \frac{N}{|O|} \int_O \int_{\Omega_0} \mathbf{n}\,\delta(\mathbf{x} - \mathbf{g}(\mathbf{y}, t))d\mathbf{y}dO.$$

The flux from domain Ω_0 to Ω_1 with wave $\mathcal{C}^-$ is

$$\mathbf{F}^{c-}_{\Omega_0\Omega_1} = \int_{\Omega_1} \mathcal{C}^-_{\Omega_0}(\mathbf{x}, t + \Delta t)d\mathbf{x}. \tag{9}$$

Thus the total flux is given by

$$\mathbf{F}_{\Omega_0\Omega_1} = \mathbf{F}^u_{\Omega_0\Omega_1} + \mathbf{F}^c_{\Omega_0\Omega_1} + \mathbf{F}^{c-}_{\Omega_0\Omega_1}.$$

For a given finite volume discretization of the space $I\!R^N$ we can calculate the mean value of a cell at time $t + \Delta t$:

$$\mathbf{U}^{n+1}_{\Omega_0} = \mathbf{U}^n_{\Omega_0} - \frac{1}{V_{\Omega_0}} \left(\sum_{l \neq 0} \mathbf{F}_{\Omega_0\Omega_l} - \mathbf{F}_{\Omega_l\Omega_0} \right)$$

Ω_0 is the domain of the desired cell and Ω_j, $j \neq 0$ are the domains of the other cells. If we only consider the next neighbors we have to sum over $3^N - 1$ domains.

Up to now we made neihter any assumption on the dimension of the space nor on the shape of the functions, e.g. density, velocity or speed of sound. To obtain a numerical scheme using Definitions 1 – 3 we assume constant states within the cells and restrict ourself to a Cartesian grid. With these assumptions the integration of the delta function in Def. 1 – 3 can be done analytically and we get a robust numerical scheme called the method of transport.

Numerical results and conclusions

We tested the resulting numerical method in two space dimensions. At first we computed the solution of various two-dimensional Riemann problems, for moderate initial data, i.e. when the ratio of maximal to minimal density is lower than 5. The results are comparable to those obtained with dimensional splitting. The solution of these and more Riemann problems computed with high accuracy are shown in [5].

A more interesting test case in view of hypersonic flow is the situation with a free stream Mach number of 25 and a source term within the flow field. The density ratio is now more than 100. Even with large time steps (CFL number ≈ 1) no unphysical values occur in the flow field and the shock is captured well. For comparison, the standard Van Leer flux vector splitting was not able to do this. The CFL number had to be reduced to obtain a solution in this case.

The disadvantage of this method is due to geometrical reasons. The Mach cone in two space dimensions are circles and so parts of the waves $\mathcal{C}$ and $\mathcal{C}^-$ have circles as boundaries (see Figures 7 and 8). In [1] the calculation of the waves is done in detail. It points out that the waves are complicated functions of roots and inverse trigonometric functions. So the integration in (8) and (9) to obtain the fluxes needs a lot of computational work. Therefore the resulting method of transport is $6-8$ time slower than a comparable first order method with dimensional splitting. Since there is no reduction of the time step for hard problems the scheme is only $3-4$ times slower.

Showing the consistency of this method one notices that a lot of simplifications are possible. Thus we can reduce the computational effort drastically. As shown in [2] this simple multidimensional Euler solver is only 20 % slower than a standard method but as robust as the original method of transport. Because of the time step reduction for dimensional splitting methods, this simplified method is nearly two times faster.

The main advantage of the method of transport is of theoretical nature. The generality of the definition of the waves allows a better understanding of multidimensional phenomena. It may be comparable to the Godunov method in one space dimension where most of the physics is included in the numerical scheme but it is not the most efficient one.

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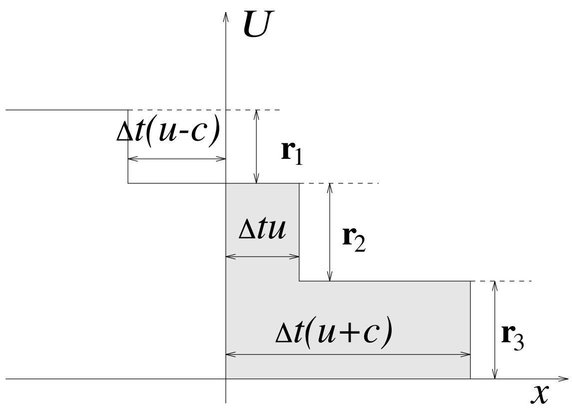


Figure 1: The Steger-Warming flux over a cell boundary.

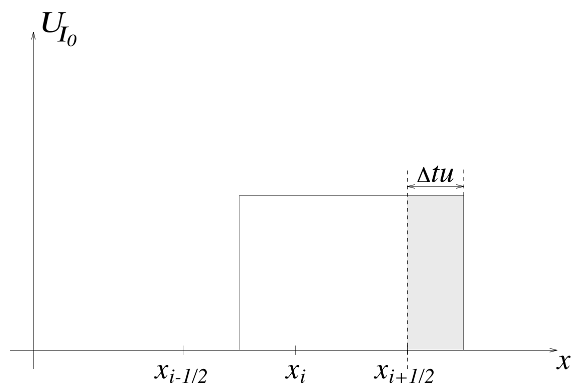


Figure 2: The flux from interval I_0 to the right neighbor with the convection.

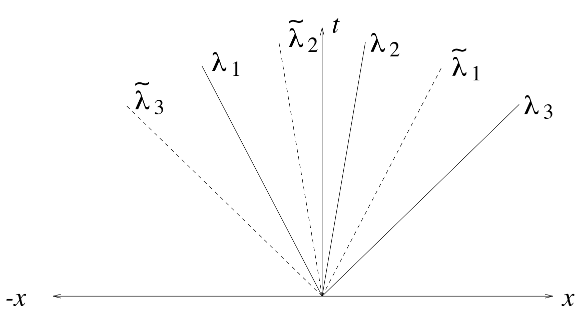


Figure 3: Characteristic lines before (λ) and after the reflection ($\tilde{\lambda}$).

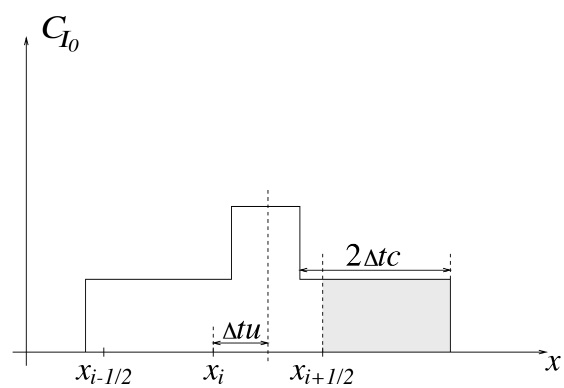
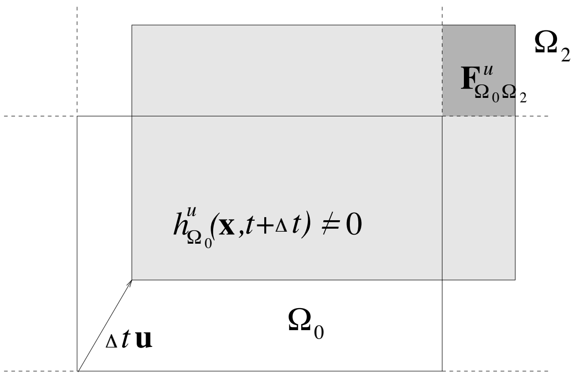


Figure 4: The distribution of wave $\mathcal{C}$ at time $t + \Delta t$ for constant $\mathbf{R}_1$ in I_0.



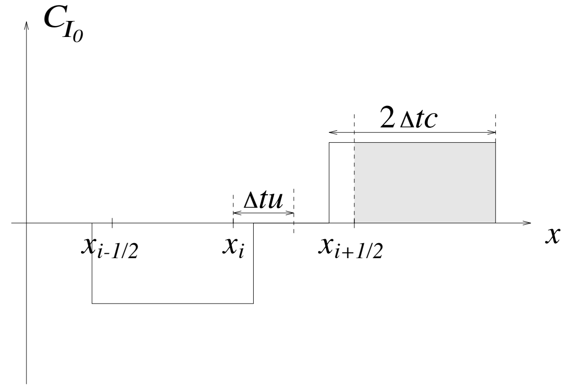
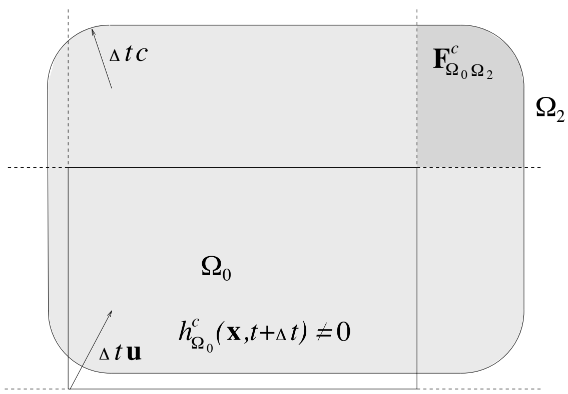


Figure 5: The support of wave $\mathcal{U}$ at time $t + \Delta t$ with constant states in domain Ω_0.

Figure 6: The distribution of wave $\mathcal{C}^-$ at time $t + \Delta t$ for constant $\mathbf{L}_3$ in I_0.



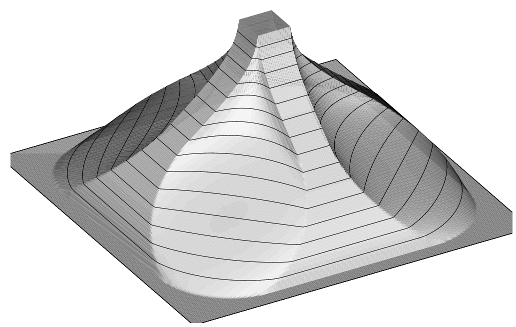
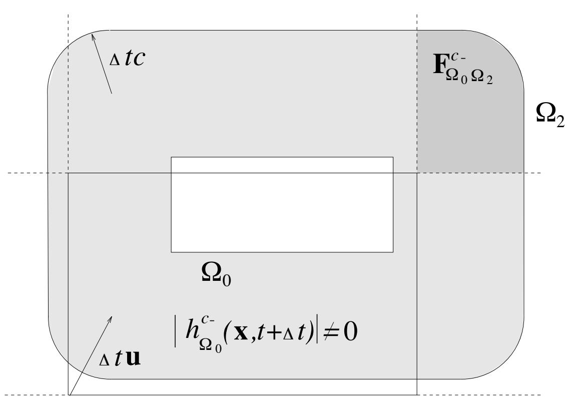


Figure 7: The support of wave $\mathcal{C}$ at time $t + \Delta t$ with constant states in domain Ω_0.

Figure 8: The distribution of wave $\mathcal{C}$ at time $t + \Delta t$ for constant $\mathbf{R}_1$ in Ω_0.



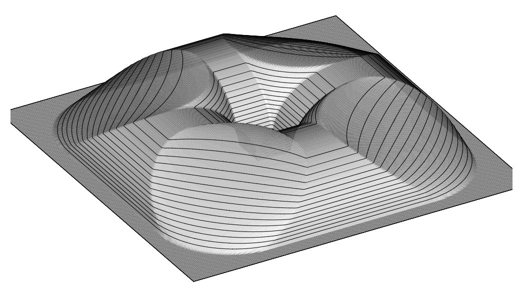


Figure 9: The support of wave $\mathcal{C}^-$ at time $t + \Delta t$ with constant states in domain Ω_0.

Figure 10: The distribution of wave $\mathcal{C}^-$ at time $t + \Delta t$ for constant $\mathbf{L}_3$ in Ω_0.

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