

Accuracy Barriers of Three Time Level Difference Schemes for Hyperbolic Equations

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Dedicated to Heinz-Otto Kreiss on his 60th Birthday

Abstract

A basic assumption for the interior scheme when solving hyperbolic mixed initial boundary value problems is that it satisfies the von Neumann stability condition. Here we show that this condition limits the order of accuracy a scheme with a given difference stencil can have. The proofs use order stars.

Keywords: difference schemes, hyperbolic equations, initial boundary value problem, mixed initial value problems, Neumann stability, accuracy barriers, order stars

Mathematics Subject Classification: 65M10

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1 Introduction

In their famous paper [2], Gustafsson, Kreiss and Sundström have developed a theory for the stability of difference schemes for hyperbolic initial boundary value problems. As basic assumption they request always that the “interior” scheme be stable in the von Neumann sense for the pure Cauchy problem [2, Assumption 5.1]. In the present paper we investigate whether this von Neumann stability condition limits the error order p of difference schemes. We shall show that for Courant numbers μ with $|\mu| \in (0, \frac{1}{2})$ and explicit difference schemes, where at least on one side the stencil has the same number of points on each old time level, such a stability barrier holds. In fact under the above assumptions we are able to prove that

$$(1.1) \quad p \leq 2R,$$

where R is the number of downwind points to the left of the characteristic through the point on the new time level. This is a partial confirmation of a conjecture made in 1985, which predicts the following bound for the order p of a stable scheme

$$(1.2) \quad p \leq 2 \min\{R, S\},$$

see [7]. Here R is defined as above and S is similarly the number of upwind points. This means that a stable scheme of order p needs to have on each side of the characteristic at least $\lceil \frac{p}{2} \rceil$ points in the stencil. (Here $\lceil \alpha \rceil$ denotes the smallest integer which is not smaller than α .) If $p = 1$, this conjecture reduces to the Courant-Friedrichs-Lewy condition. Therefore the bound (1.2) has the flavour of being an extension of the Courant-Friedrichs-Lewy condition. The conjecture was proved in [7] for two-time level schemes, while many examples in support of (1.2) for multi-time level schemes have been given in [4,5,6].

In the present paper we give typical results and outline the proofs. The proofs are done using so called order stars, which have been introduced in [10]. However in the present context one has the difficulty that the order star has to be defined on the Riemann surface of an algebraic function and the “comparison” function has a logarithmic singularity. Observe that in [3] the same comparison function has been used.

In Section 2 we give the exact notation and state the main results. In Section 3 we give an outline of the proofs. It should be noted that in the

present paper we shall not strive for full generality but limit ourselves to typical results. In a forthcoming paper [8] we shall give more details and attempt to squeeze more results out of the techniques presented here.

2 Main results

In the GKS theory [2] one always assumes that the difference scheme for the interior is von Neumann stable for the initial value problem

$$(2.1) \quad \begin{aligned} \frac{\partial}{\partial t} u(t, x) &= c \frac{\partial}{\partial x} u(t, x), \quad x \in \mathbf{R}, t \geq 0 \\ u(0, x) &= u_0(x) \text{ given.} \end{aligned}$$

Such a scheme can be described as follows. Let Δt and Δx denote the step sizes in the time and space variables, and $\mu = c \frac{\Delta t}{\Delta x}$ the Courant number. If $u_{n,m}$ is meant to be an approximation of $u(n\Delta t, m\Delta x)$ then an explicit three-time level scheme has the form

$$(2.2) \quad \begin{aligned} u_{n+2,m} + \sum_{i=0}^1 \sum_{j=-r}^s a_{ij} u_{n+i,m+j} &= 0 \\ m = 0, \pm 1, \pm 2, \dots, \quad n = 2, 3, \dots \end{aligned}$$

Here a_{ij} are real coefficients depending in general on μ , $a_{ij} = a_{ij}(\mu)$, and $-r \leq s$, $a_{i,-r} a_{i,s} \neq 0$ for $i = 0, 1$. The coefficients a_{ij} determine two very important features of the scheme, namely the order and the von Neumann condition. It is convenient to introduce the characteristic function

$$(2.3) \quad \Phi(z, w) = w^2 + a_1(z)w + a_0(z).$$

where

$$(2.4) \quad a_i(z) := \sum_{j=-r}^s a_{ij} z^j, \quad i = 0, 1.$$

Taking as usual Fourier transforms in space of (2.2) leads immediately to the

Definition 2.1 The scheme (2.2) for the pure initial value problem (2.1) is said to satisfy the von Neumann condition if

$$(2.5) \quad \left. \begin{aligned} \Phi(z, w) &= 0 \\ |z| &= 1 \end{aligned} \right\} \Rightarrow \begin{cases} |w| \leq 1 \text{ and if} \\ |w| = 1, \text{ then } w \text{ is a simple root.} \end{cases}$$

For more details see for example one of the references [1,2,6].

A scheme (2.2) has *error order* p if and only if for any smooth solution $u(t, x)$ of (2.2) one has

$$u(t+2\Delta t, x) + \sum_{i=0}^1 \sum_{j=-r}^s a_{ij} u(t+i\Delta t, x+j\Delta x) = C \frac{\partial^{p+1}}{\partial x^{p+1}} u(t, x) (\Delta x)^{p+1} + O((\Delta x)^{p+2})$$

(2.6) if $\Delta x \rightarrow 0$.

Since we are interested in schemes with positive order only, we assume that $\Phi(1, 1) = 0$.

Proposition 2.2 ([6,9]) *Let a scheme (2.2) with the Courant number μ be stable and satisfy $\Phi(1, 1) = 0$. Then the order is p if and only if the algebraic function w given by $\Phi(z, w(z)) \equiv 0$ has exactly one branch $w_1(z)$, which is analytic in a neighbourhood of $z = 1$ and satisfies*

$$(2.7) \quad z^\mu - w_1(z) = O((z-1)^{p+1}) \quad \text{as } z \rightarrow 1.$$

□

The exact solution is constant along the characteristics, $x = -ct + \text{const}$. Hence if μ is such that the characteristic through the point (t_{n+2}, x_m) passes through a further point of the difference stencil then one can construct an exact scheme. However if $2\mu \notin \mathbf{Z}$ this cannot happen and then it is easy to show that the highest possible order is

$$(2.8) \quad p = 2r + 2s + 1,$$

see [6]. To formulate the main results we denote the number of stencil points to the left of the characteristic through the point (t_{n+2}, x_m) by R and introduce S as the number of points to the right of this characteristic. If we restrict ourselves to the cases $0 < |\mu| < \frac{1}{2}$ these numbers are

$$R = \begin{cases} 2r & \text{if } -\frac{1}{2} < \mu < 0 \\ 2r + 2 & \text{if } 0 < \mu < \frac{1}{2} \end{cases}$$

and

$$S = \begin{cases} 2s + 2 & \text{if } -\frac{1}{2} < \mu < 0 \\ 2s & \text{if } 0 < \mu < \frac{1}{2} \end{cases}$$

R, S are also called the number of downwind points, upwind points respectively. We are now in a position to state the main theorems.

Theorem 2.3 *Assume that $\mu \in (-\frac{1}{2}, 0)$ and that the scheme is stable. Then one has*

$$(2.9) \quad p \leq 2R.$$

□

Remark 2.4 In the proof we shall see that we don't use any information from the upwind side of the characteristic and therefore the result is true if the stencil extends to the upwind side not uniformly on the time levels t_n and t_{n+1} , i.e. $a_{1s} \cdot a_{0s} = 0$ is allowed.

Using a standard symmetry argument gives the following

Corollary 2.5 *Assume that $\mu \in (0, \frac{1}{2})$ and that the scheme is stable. Then one has*

$$p \leq 2S.$$

□

The next theorem gives an order barrier in terms of the number of downwind points in the case $\mu \in (0, \frac{1}{2})$.

Theorem 2.6 *Assume that $\mu \in (0, \frac{1}{2})$ and $r \geq 1$ and that the scheme is stable. Then one has*

$$p \leq 2S.$$

□

The assumption $r \geq 1$ is a technical one. It prevents that $z = 0$ is a branch point of the algebraic function $w(z)$. We believe that the theorem is true even if $r = 0$.

Clearly a remark similar to Remark 2.4 holds here too as well as a corollary corresponding to Corollary 2.5. One can collect the results in the following

Corollary 2.7 *Assume that $|\mu| < \frac{1}{2}$, $r \geq 1$ and $s \geq 1$. Then for a stable scheme one has the bound*

$$p \leq 2 \min\{R, S\}.$$

□

3 Order stars and proofs of the theorems

As already indicated earlier we shall not strive for full generality but use the theory tailored exactly to the theorems we want to prove.

3.1 Properties of algebraic function w

As has been shown in [6] we can always assume that Φ is irreducible. The algebraic function w , satisfying $\Phi(z, w(z)) \equiv 0$ is double-valued, consisting in general for each z of two values $w_1(z), w_2(z)$. It can be made single valued by introducing the Riemann surface M , i.e.

$$M = \{(z, w) \in \bar{\mathbf{C}} \times \bar{\mathbf{C}} \mid \Phi(z, w) = 0\}.$$

It is known that M is a closed and connected set. Even so none of the variables z and w has more prominence than the other, we will in our notation have a tendency to use z as “independent variable” and w as dependent variable. Thus we shall often prefer to work with a double valued function $w_1(z), w_2(z)$ and say that these represent two sheets, one “above” the other, which interact at a finite number of branch points z_i , where $w_1(z_i) = w_2(z_i)$ and cuts connecting these branch points.

Remark 3.1 Branch points of w

The branch points of w can occur at $0, \infty$ and the points z_i where $a_1^2(z_i) - 4a_0(z_i) = 0$. Since the coefficients of this polynomial equation are real, the branch points will be either real, or they will occur in complex conjugate pairs. Branch cuts along which the sheets of M are connected, can therefore, without loss of generality be taken to fall on the real axis or to be orthogonal and symmetric with respect to the real axis. However sometimes it gives “simpler” pictures if one uses pairs of conjugate complex cuts.

Remark 3.2 If a scheme is stable, the corresponding algebraic function cannot have a branch point at $z = 1$ (see (2.5)). The sheet of M on which the point $z = 1, w = 1$ occurs, is called the *principal sheet*, the other one the *secondary sheet*. Since there is not a unique way of making the cuts this notion is in principle a local property in a neighborhood of $z = 1$. However in describing examples it is helpful to introduce the following notation, once the cuts are made. All points on M which can be reached from $(1,1)$ without

crossing a cut are said to be on the principle sheet and the other ones, except for points on the cuts, are said to belong to the secondary sheet.

Remark 3.3 Poles of w

Since only explicit difference schemes are considered with $a_{1,-r} \neq 0$, there is at most one point on M where z is finite and w has a pole. In fact this happens at $z = 0$ and this occurs if and only if $r \geq 1$. Moreover the pole has order r .

Remark 3.4 Zeros of w

Since $a_{0,-r} \neq 0$ the finite points where w becomes zero are identical with the zeros of $a_0(z)$. In particular $z = 0$ is not a zero of w .

Observe that if $r > 0$ and $a_{1,-r} \neq 0$ then the algebraic function has no branch point at $z = 0$.

3.2 Order Stars on the Riemann surface

An order star is defined on the Riemann surface M of the algebraic function w in the following way. Define the function φ by

$$\varphi(z, w) = z^{-\mu}w, \quad (z, w) \in M$$

and the *order star* Ω by

$$\Omega = \{(z, w) \in M : |\varphi(z, w)| > 1\}.$$

Because of the factor $z^{-\mu}$ the function φ is multiple-valued on M . However, the order star Ω , being defined by means of the modulus of φ , is again well defined on M . Ω^c denotes the complement of Ω , i.e. $\Omega^c = M \setminus \Omega$. Since the coefficients a_{ij} are real, it is easily verified that Ω is symmetric with respect to the real axis.

The order and stability of a difference scheme, which were interpreted in Section 2 as properties of the function w associated with the scheme, can now be expressed as properties of the corresponding order star. We give without proof those properties which are standard results in investigations involving order stars (e.g. [10]).

Lemma 3.5 *If a scheme is stable then $\Omega \cap \partial\Delta = \emptyset$ where $\partial\Delta$ denotes the boundary of the “unit disk” $\Delta = \{(z, w) \in M : |z| < 1\}$.* □

Lemma 3.6 *A scheme (2.2) has order p if and only if at the point $z = 1$ on the principal sheet of M the order star consists of $p + 1$ sectors of angle $\frac{\pi}{p+1}$, separated by $p + 1$ sectors of Ω^c , each with the same angle. \square*

A subset A (with boundary ∂A) of Ω is said to be an Ω -component if $\partial A \subset \partial\Omega$ and A is connected. Ω^c -components are defined similarly. An Ω -component is said to be of multiplicity m if it contains m Ω -sectors at $z = 1$ on the principal sheet, similarly for Ω^c -components.

The relationship between the multiplicity of an Ω -component and the total order of the poles of φ that it contains, plays an important role in analyses involving order stars. These two concepts are related using the argument principle (see [10]). In order to do this we must make sure that z^μ can be defined uniquely in regions which contain $z = 0$. Thus we need to make at most two cuts L_i on M connecting the zero points $(0, w_i^0)$ to the infinity points (∞, w_i^∞) , $i = 1, 2$. In the present context we shall choose L_1 and L_2 such that they have an identical projection onto the z -plane and do not meet a branch point of $w(z)$. These cuts should not be confused with the cuts mentioned earlier which had been used only to represent M on “top of the z -plane”.

Clearly Ω consists of different components. For stable schemes, these are completely inside or completely outside Δ .

In the following we shall give bounds for the multiplicity of such Ω -components inside Δ . To do this we need two lemmata to integrate φ'/φ along the cuts L_i .

Lemma 3.7 *Assume $w(z)$ has no branch point at $z = 0$. Let $-\alpha$ be the leading exponent of $\varphi(z, w_1(z))$ at $z = 0$, where $w_1(z)$ is a branch of $w(z)$, i.e.*

$$\varphi(z, w_1(z)) = z^{-\alpha}\Psi(z), \quad \Psi(0) \neq 0.$$

Let γ be a circular path with centre $z = 0$, a radius sufficiently small and which encircles 0 in a clockwise direction. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\varphi'(z, w_1(z))}{\varphi(z, w_1(z))} dz = \alpha.$$

\square

Lemma 3.8 *Assume that γ is a line segment on one of the cuts L_1 or L_2 used to uniquely define z^μ . Let γ^+ and γ^- denote both sides of γ traversed in opposite direction. Then*

$$\int_{\gamma^+} \frac{\varphi'(z, w_1(z))}{\varphi(z, w_1(z))} dz = - \int_{\gamma^-} \frac{\varphi'(z, w_1(z))}{\varphi(z, w_1(z))} dz.$$

□

Observe that γ has different values on γ^+ and γ^- since z^μ has different values.

One classifies now the possible Ω -components in Δ and shows for each class a bound for its multiplicity in terms of the leading exponent of φ at $z = 0$. Using the argument principle one can easily show the following

Lemma 3.9 *A bounded Ω -component contains at least one of the points $z = 0$ of M where the leading exponent of φ is negative.* □

Hence we only have to consider Ω -components which contain either one point $(0, w_i^0)$ or both points $(0, w_i^0)$, $i = 1, 2$.

Proposition 3.10 *Let Ω_1 be an Ω -component containing exactly one point with $z = 0, (0, w_1^0)$ say. Let $-\alpha$ be the leading exponent of φ at $(0, w_1^0)$. Then the multiplicity m of Ω_1 satisfies*

$$m \leq 2[\alpha].$$

□

Outline of the proof of Proposition 3.10. Since M can be very complicated one has to consider many cases. In fact in [8] we consider more than 20 different cases. As an illustration we show here the simplest one Fig. 1a and two of the most complicated ones Fig. 1b and 1c.

Let us start with the simplest case. Assume that Ω_1 lies completely on the principal sheet. In order to uniquely define z^μ we make the cut L_1 as indicated in Fig. 1a. Let γ consist of $\gamma_0 = \partial\Omega_1$, the path γ_c^+, γ_c^- along the cut L_1 and γ_r a small circle around 0 as indicated in Fig. 1a. We integrate φ'/φ along γ .

Figure 1: Ω -components in Δ containing exactly one zero point.

One observes that the argument of φ decreases along $\partial\Omega_1$. Whenever one of the “loops” $\gamma_i, i = 1, 2, \dots, m-1$, returns to $(1, 1)$ the argument must have decreased by at least -2π . Using the argument principle and the Lemmata 3.7 and 3.8 one obtains

$$0 = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi'(z, w)}{\varphi(z, w)} dz = \frac{1}{2\pi i} \underbrace{\int_{\gamma_0}}_{<0} + \frac{1}{2\pi i} \underbrace{\left(\int_{\gamma_c^+} + \int_{\gamma_c^-} \right)}_{=0} + \frac{1}{2\pi i} \underbrace{\int_{\gamma_r}}_{=\alpha} + \frac{1}{2\pi i} \underbrace{\int_{\gamma_1+\gamma_2+\dots+\gamma_{m-1}}}_{\leq-(m-1)}$$

Hence

$$m - 1 < \alpha$$

and thus

$$m \leq \lceil \alpha \rceil.$$

Remark 3.11 Since the argument of φ is dropping along γ_0 and after returning to $(1,1)$ it differs from the starting value by an integer multiple of 2π , one finds the useful formula

$$(3.1) \quad \frac{1}{2\pi i} \int_{\gamma_0+\gamma_c^++\gamma_c^-+\gamma_r} \frac{\varphi'}{\varphi} dz \leq \begin{cases} \lfloor \alpha \rfloor & \text{if } \alpha \notin \mathbf{Z} \\ \alpha - 1 & \text{if } \alpha \in \mathbf{Z} \end{cases}$$

Here $\lfloor \alpha \rfloor$ denotes the largest integer not exceeding α .

Let us look at the more complicated Ω -component depicted in Fig. 1b. Observe that the cut L_1 passes through the point $(1,1)$. Hence the argument used before, that one returns to the same function value of φ when coming back to $(1,1)$ is only true when one ends up on the same side of this cut. Hence one obtains

$$0 \leq \lfloor \alpha \rfloor - \frac{m-2}{2} \quad \text{if } \alpha \notin \mathbf{Z}$$

and therefore

$$(3.2) \quad m \leq 2\lfloor \alpha \rfloor + 2 = 2\lceil \alpha \rceil \quad \text{if } \alpha \notin \mathbf{Z}$$

For $\alpha \in \mathbf{Z}$ one obtains the same formula.

Let us look as a last example at the Ω -component depicted in Fig. 1c. The integral along γ_1 can be treated as in the first example. However the curves γ_2 and γ_3 “loop” around $z = 0$ on the secondary sheet and thus one

has to make similar cuts there. Let β be the leading exponent of φ on this secondary sheet. Using formula (3.1) gives

$$\frac{1}{2\pi i} \int_{\gamma_3} + \frac{1}{2\pi i} \left(\int_{\gamma_c^+} + \int_{\gamma_c^-} \right) + \frac{1}{2\pi i} \int_{\gamma_r} \leq \begin{cases} \lfloor \beta \rfloor & \text{if } \beta \notin \mathbf{Z} \\ \beta - 1 & \text{if } \beta \in \mathbf{Z} \end{cases}$$

When integrating along γ_2 one has to observe that the integration on the small circle around $z = 0$ goes in opposite direction. Hence one obtains

$$\frac{1}{2\pi i} \int_{\gamma_2} + \frac{1}{2\pi i} \left(\int_{\gamma_c^+} + \int_{\gamma_c^-} \right) + \frac{1}{2\pi i} \int_{-\gamma_r} \leq \begin{cases} \lfloor -\beta \rfloor & \text{if } \beta \notin \mathbf{Z} \\ -\beta - 1 & \text{if } \beta \in \mathbf{Z} \end{cases} .$$

Since

$$\lfloor \beta \rfloor + \lfloor -\beta \rfloor = -1 \quad \text{if } \beta \notin \mathbf{Z}$$

we obtain for the two loops γ_2 and γ_3 together with the cuts and circles around 0 the bound -1. Collecting these estimates gives for a component of the type depicted in Fig. 1c

$$0 \leq \lfloor \alpha \rfloor - \frac{m-1}{2} .$$

Thus one has

$$(3.3) \quad m \leq 2\lfloor \alpha \rfloor + 1 .$$

This is even sharper than (3.2). In fact treating all cases one finds that the bound (3.2) is the worst possible.

We are now able to prove Theorem 2.3.

3.3 Proof of Theorem 2.3

If $r = 0$ one has $R = 0$. Hence by the Courant-Friedrichs-Lewy condition the scheme cannot be convergent, i.e. it is impossible that one has simultaneously $p > 1$ and stability. Assume now that $r > 0$, i.e. $r \geq 1$. Hence $w(z)$ has two branches at $z = 0$ with leading exponents $-r$ and 0. Thus the leading exponents of φ at $z = 0$ are

$$-r - \mu < 0$$

and

$$-\mu > 0 .$$

Since the scheme is stable there is a clear distinction between the Ω components in and outside Δ . From Lemma 3.9 follows that there is exactly one Ω -component Ω_1 inside Δ . The leading exponent of φ at $z = 0$ is $-\alpha = -r - \mu$ and the other zero point $(0, w_2^0)$ is not part of Ω_1 . Hence Proposition 3.10 can be applied. This gives

$$m \leq 2\lceil r + \mu \rceil = 2r = R.$$

Here we used that $\mu \in (-\frac{1}{2}, 0)$. Hence the number of Ω -sectors inside Δ is R . One can have at most $R + 1$ Ω sectors outside. By Lemma 3.6 we obtain the desired result.

3.4 Proof of Theorem 2.6

Since $\mu \in (0, \frac{1}{2})$ the leading exponents of φ at $z = 0$ are

$$-\alpha_1 = -r - \mu < 0$$

and

$$-\alpha_2 = -\mu < 0.$$

Since the scheme is stable there is a clear distinction between the Ω -components inside and outside Δ . One has now the possibility that there are two different Ω -components, each one containing exactly one point $z = 0$, or exactly one Ω -component containing both points $(0, w_i^0)$, $i = 1, 2$.

Again one has to classify all Ω -components containing both zero points $(0, w_i^0)$, $i = 1, 2$. A simple example is given in Fig. 2a. Let us add to this component $m - 2$ closed curves $\gamma_3, \dots, \gamma_m$ which are of the same shape as γ_1 but lie outside γ_1 . Using the above introduced techniques gives the estimate

$$(3.4) \quad m \leq 2\lfloor \alpha_1 \rfloor + 2\lfloor \alpha_2 \rfloor + 2.$$

A large portion of classes of Ω -components is obtained by “connecting” two Ω -components with one “zero-point” through branch cuts. The worst possible case is found by combining the two examples of Fig. 1b and 1c, see Fig. 2b.

Using the bounds (3.2) and (3.3) gives

$$(3.5) \quad m \leq 2\lfloor \alpha_1 \rfloor + 2\lfloor \alpha_2 \rfloor + 3.$$

Figure 2: Examples of Ω -components containing two zero points.

This bound is not sharp enough to prove Theorem 2.6 and in fact one can improve the bound. Let δ_i be the noninteger part of α_i , i.e.

$$\delta_i = \alpha_i - \lfloor \alpha_i \rfloor.$$

Then (3.5) can be replaced by

$$(3.6) \quad m \leq 2\lfloor \alpha_1 \rfloor + 2\lfloor \alpha_2 \rfloor + 1 + 2\lfloor \delta_1 + \delta_2 \rfloor.$$

Substitution of the leading exponents α_i in the two worst bounds (3.4) and (3.6) gives the estimates

$$(3.7) \quad m \leq 2\lfloor r + \mu \rfloor + 2\lfloor \mu \rfloor + 2 = 2r + 2 = R$$

and

$$(3.8) \quad m \leq 2\lfloor r + \mu \rfloor + 2\lfloor \mu \rfloor + 1 + 2\lfloor 2\mu \rfloor = 2r + 1 < R.$$

Here we used that $\mu \in (0, \frac{1}{2})$. The rest of the proof is exactly the same as in the proof of Theorem 2.3.

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