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ABSTRACT. We study the non-Hermitian skin effect in a three-dimensional system of finitely many subwavelength resonators with an imaginary gauge potential. We introduce a discrete approximation of the eigenmodes and eigenfrequencies of the system in terms of the eigenvectors and eigenvalues of the so-called gauge capacitance matrix $\mathcal{C}_N^\gamma$, which is a dense matrix due to long-range interactions in the system. Based on translational invariance of this matrix and the decay of its off-diagonal entries, we prove the condensation of the eigenmodes at one edge of the structure by showing the exponential decay of its pseudo-eigenvectors. In particular, we consider a range-k approximation to keep the long-range interaction to a certain extent, thus obtaining a k-banded gauge capacitance matrix $\mathcal{C}_{N,k}^\gamma$. Using techniques for Toeplitz matrices and operators, we establish the exponential decay of the pseudo-eigenvectors of $\mathcal{C}_{N,k}^\gamma$ and demonstrate that they approximate those of the gauge capacitance matrix $\mathcal{C}_N^\gamma$ well. Our results are numerically verified. In particular, we show that long-range interactions affect only the first eigenmodes in the system. As a result, a tridiagonal approximation of the gauge capacitance matrix, similar to the nearest-neighbour approximation in quantum mechanics, provides a good approximation for the higher modes. Moreover, we also illustrate numerically the behaviour of the eigenmodes and the stability of the non-Hermitian skin effect with respect to disorder in a variety of three-dimensional structures.

Keywords. Non-Hermitian systems, non-Hermitian skin effect, subwavelength resonators, imaginary gauge potential, stability, Toeplitz matrix.

AMS Subject classifications. 35B34, 35P25, 35C20, 81Q12.

1. Introduction

The non-Hermitian skin effect is the phenomenon whereby a large proportion of the bulk eigenmodes of a non-Hermitian system are localised at one edge of an open chain of resonators. It has applications in topological photonics, phononics, and other condensed matter systems [18, 19, 25, 32]. It is one of the most promising and exciting research topics in physics in recent years [15, 23, 24, 27, 31, 33, 35].

In our recent work [2], the non-Hermitian skin effect in the subwavelength regime is studied using first-principle mathematical analysis for one-dimensional systems of subwavelength resonators. An imaginary gauge potential (in the form of a first-order directional derivative) is added inside the resonators to break Hermiticity. Explicit asymptotic expressions for the subwavelength eigenfrequencies and eigenmodes are derived using a gauge capacitance matrix formulation of the problem (which is a reformulation of the standard capacitance matrices that are commonplace in Hermitian subwavelength physics and electrostatics). Moreover, the exponential decay of eigenmodes and their accumulation at one edge of the structure (the non-Hermitian skin effect) is shown to be induced by the Fredholm index of an associated Toeplitz operator. In [3], the robustness of the non-Hermitian skin effect in one dimension with respect to random imperfections in the system is quantified. Moreover, the competition between the non-Hermitian skin effect and Anderson localisation is illustrated.

The generalisation of our previous work to higher-dimensional systems is important. Although it is well-known that the skin effect can be experimentally realised in three-dimensional systems [35, 36], there is no mathematical framework to theoretically support the condensation effect in three dimensions. In this paper, we consider three-dimensional systems of subwavelength resonators with imaginary gauge potentials. As in the one-dimensional case, by using an asymptotic methodology, we can approximate the subwavelength eigenfrequencies and eigenmodes of a finite chain of resonators by the eigenvalues of a so-called *gauge capacitance matrix* $\mathcal{C}_N^\gamma$. The main difference compared to the one-dimensional case is that the three-dimensional gauge capacitance matrix is not tridiagonal. Its off-diagonal entries account for long-range interactions, which is inherent in three-dimensional systems. Nevertheless, inspired by the nearest-neighbour approximation in quantum mechanics, we consider a range-k approximation to keep the long-range interaction to a certain extent, thus obtaining a k-banded gauge capacitance matrix $\mathcal{C}_{N,k}^\gamma$. Using techniques for the Toeplitz matrices and operators, we prove the exponential decay of the pseudo-eigenvectors of such k-banded matrix and demonstrate that they approximate those of the gauge capacitance matrix $\mathcal{C}_N^\gamma$ well. We numerically verify our results by showing that the long-range interactions affect only the first eigenmodes in the system. A tridiagonal (nearest-neighbour) approximation of the gauge capacitance matrix, provides a good approximation for the higher modes, which successively improves as we increase the range of interactions. Finally, we illustrate numerically the behaviour of the eigenmodes and the stability of the non-Hermitian skin effect with respect to disorder in a variety of structures.

The paper is organised as follows. In Section 2, we present the mathematical setup of the problem. Section 3 is devoted to the derivations of the fundamental solution of a (non-reciprocal) Helmholtz operator with an imaginary gauge potential and the analysis of the associated layer potentials. In Section 4 we introduce our discrete approximation of the problem which provides approximations of the eigenfrequencies and eigenmodes of a finite chain of subwavelength resonators in terms of the eigenvalues and eigenvectors of the gauge capacitance matrix. Given this discrete formulation, we study in Section 5 the properties of the gauge capacitance matrix and show that it can be approximated by a Toeplitz matrix. In Section 6, we prove the exponential decay of the pseudo-eigenvectors of the gauge capacitance matrix and deduce the condensation of the subwavelength eigenmodes at one edge of the structure. In Section 7, we numerically illustrate our main findings in this paper for a variety of subwavelength resonator systems. Moreover, we show the robustness of the skin effect with respect to changes in the position of the resonators. The paper ends with some concluding remarks. Appendix A is devoted to the characterisation of the spectra and pseudo-spectra of perturbed banded Toeplitz matrices.

2. Problem setting

We will consider an array of material inclusions that are identical in size and shape and which have an imaginary gauge potential in their interior. To be more precise, we let $D \subset [0, 1) \times \mathbb{R}^2$ be a simply connected, bounded domain of class $C^{1,s}$ for some $0 < s < 1$. Then, we consider a one-dimensional array given by $\mathcal{D} := \bigcup_{i=1}^N D_i$, where D_i is the translated domain $D_i := D + (i, 0, 0)^\top$, with the superscript $\top$ denoting the transpose.

A resonant frequency $\omega \in \mathbb{C}$ is such that $\mathrm{Re}(\omega) > 0$ and there exists a non-trivial associated eigenmode u solution to

$$
\begin{cases}
\Delta u + \omega^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}, \\
\Delta u + \omega^2 u + \gamma \partial_1 u = 0 & \text{in } \mathcal{D}, \\
u|_+ - u|_- = 0 & \text{on } \partial\mathcal{D}, \\
\delta \dfrac{\partial u}{\partial \nu}\bigg|_+ - \dfrac{\partial u}{\partial \nu}\bigg|_- = 0 & \text{on } \partial\mathcal{D}, \\
u \text{ satisfies an outgoing radiation condition.}
\end{cases}
\tag{2.1}
$$

Here, ν is the outgoing normal to ∂D, $|_+$ and $|_-$ denote the limits from the outside and inside of $\mathcal{D}$. $\delta > 0$ is a non-dimensional material contrast parameter that will be assumed to be small, such that the system is in a high-contrast regime. Finally, the first-order directional derivative with coefficient $\gamma \neq 0$ corresponds to an imaginary gauge potential within the inclusions. This term breaks the time-reversal symmetry of the problem and is the crucial mechanism responsible for the condensation effects that will be observed here. For the outgoing radiation condition (which reduces to the Sommerfeld radiation condition for ω real), we refer the reader to [5, 20, 30].

We are interested in the subwavelength regime. We look for the subwavelength resonant frequencies and their associated eigenmodes for the system of resonators $\mathcal{D}$ within this regime, which we characterise by satisfying $\omega \to 0$ as $\delta \to 0$. In order to approximate these subwavelength resonant frequencies and eigenmodes, we perform an asymptotic analysis in a high-contrast limit, given by $\delta \to 0$. This limit recovers subwavelength resonant frequencies and eigenmodes, while keeping the characteristic size of the resonators fixed. As it will be seen in Section 4, it leads to a discrete approximation of (2.1) in terms of a so-called gauge capacitance matrix. By investigating the Toeplitz structure of the gauge capacitance matrix in Section 5, we will prove in Section 6 the condensation of the subwavelength eigenmodes at one edge of the structure $\mathcal{D}$.

3. Fundamental solutions and single layer potentials

Let G^ω be the outgoing fundamental solution of the Helmholtz operator $\Delta + \omega^2$. Let $\mathcal{S}_D^\omega$ be the associated single layer potential

$$\mathcal{S}_D^\omega : \phi \in L^2(\partial D) \mapsto \int_{\partial D} G^\omega(x-y)\phi(y)\mathrm{d}\sigma(y).$$

We define the Neumann-Poincaré operator associated to G^ω to be the operator $\mathcal{K}_D^{\omega,*} : L^2(\partial D) \to L^2(\partial D)$ given by

$$\mathcal{K}_D^{\omega,*}[\phi](x) = \int_{\partial D} \frac{\partial}{\partial \nu_x} G^\omega(x-y)\phi(y)\mathrm{d}\sigma(y), \quad x \in \partial D, \phi \in L^2(\partial D). \tag{3.1}$$

We recall the following well-known results regarding $\mathcal{S}_D^\omega[\phi]$; see, for instance, [13, 26].

LEMMA 3.1 (Jump conditions). *For the single layer potential $\mathcal{S}_D^\omega$, it holds*

$$\mathcal{S}_D^\omega[\phi]|_+ = \mathcal{S}_D^\omega[\phi]|_-, \quad \phi \in L^2(\partial D). \tag{3.2}$$

The associated Neumann–Poincaré operator $\mathcal{K}_D^{\omega,*}$ satisfies

$$\left.\frac{\partial}{\partial \nu}\mathcal{S}_D^\omega[\phi]\right|_\pm = \left(\pm\frac{1}{2}I + \mathcal{K}_D^{\omega,*}\right)[\phi], \quad \phi \in L^2(\partial D). \tag{3.3}$$

It is also well-known that $\mathcal{S}_D^0 : L^2(\partial D) \to H^1(\partial D)$ is invertible. Here, H^1 is usual Sobolev space of square-integrable functions whose weak derivative is square integrable.

We now turn to the operator with an imaginary gauge potential $\Delta + \omega^2 + \gamma\partial_{x_1}$. The following result provides a fundamental solution for this operator.

PROPOSITION 3.2. *Introduce*

$$G_\gamma^\omega(x) = -\frac{\exp(-\gamma x_1/2 + \mathrm{i}\sqrt{\omega^2 - \gamma^2/4}|x|)}{4\pi|x|}. \tag{3.4}$$

Then, G_γ^ω satisfies

$$\Delta G_\gamma^\omega + \omega^2 G_\gamma^\omega + \gamma\partial_{x_1} G_\gamma^\omega = \delta_0 \qquad in \ \mathbb{R}^3. \tag{3.5}$$

Proof. Observe that $F_\gamma^\omega(x) = -\frac{\exp(\mathrm{i}\sqrt{\omega^2-\gamma^2/4}|x|)}{4\pi|x|}$ is a fundamental solution of the Helmholtz operator $\Delta + \omega^2 - \gamma^2/4$. Letting

$$G_\gamma^\omega(x) = F_\gamma^\omega(x)e^{-\gamma x_1/2}, \tag{3.6}$$

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we have

$$\left(\Delta + \omega^2 - \gamma^2/4\right) F_\gamma^\omega(x) = e^{\gamma x_1/2} \left(\Delta + \omega^2 + \gamma \partial_{x_1}\right) G_\gamma^\omega(x) \tag{3.7}$$

$$= \delta_0(x), \tag{3.8}$$

which proves the claim. ∎

Observe that G_γ^ω does not satisfy the vanishing boundary condition as $x \to \infty$; nevertheless, as we will use G_γ^ω to represent the solution inside the compact domains D_i, the behaviour of G_γ^ω as $x \to \infty$ does not affect the solution.

We now define the single layer potential associated to G_γ^ω by

$$\widetilde{\mathcal{S}}_{D,\gamma}^\omega : \phi \in L^2(\partial D) \mapsto \int_{\partial D} G_\gamma^\omega(x-y)\phi(y)\mathrm{d}\sigma(y). \tag{3.9}$$

Analogously to (3.1), we also define the Neumann-Poincaré operator associated to (3.4) by

$$\widetilde{\mathcal{K}}_{D,\gamma}^{\omega,*}[\phi](x) = \int_{\partial D} \frac{\partial}{\partial \nu_x} G_\gamma^\omega(x-y)\phi(y)\mathrm{d}\sigma(y), \quad x \in \partial D, \phi \in L^2(\partial D). \tag{3.10}$$

Since G_γ^ω has the same singularity as G^ω at the origin, similarly to Lemma 3.1, we can obtain the following jump conditions for the normal derivative of $\widetilde{\mathcal{S}}_{D,\gamma}^\omega$ across ∂D.

LEMMA 3.3. *The Neumann-Poincaré operator $\widetilde{\mathcal{K}}_{D,\gamma}^{\omega,*}$ satisfies*

$$\frac{\partial}{\partial \nu} \widetilde{\mathcal{S}}_{D,\gamma}^\omega[\phi]\bigg|_\pm = \left(\pm \frac{1}{2}I + \widetilde{\mathcal{K}}_{D,\gamma}^{\omega,*}\right)[\phi]. \tag{3.11}$$

The following result will be of use to us.

PROPOSITION 3.4. *The operator $\widetilde{\mathcal{S}}_{D,\gamma}^0 : L^2(\partial D) \to L^2(\partial D)$ is injective and hence has a left inverse*

Proof. We assume that $\widetilde{\mathcal{S}}_{D,\gamma}^0[\phi](x) = 0$ for $x \in \partial D$. For $x \in \mathbb{R}^3$, we define u and v as

$$u(x) = \widetilde{\mathcal{S}}_{D,\gamma}^0[\phi](x) \quad \text{and} \quad v(x) = e^{\gamma x_1/2}u(x).$$

Since $\Delta u + \gamma \partial_{x_1} u = 0$, we find that v satisfies the *modified* Helmholtz equation

$$\begin{cases} \Delta v - \dfrac{\gamma^2}{4}v = 0, \\ v|_{\partial D} = 0, \\ v(x) = O\left(|x|^{-1}\right) \quad \text{as } x \to \infty. \end{cases} \tag{3.12}$$

The modified Helmholtz equation is well-known to have a unique solution; indeed, if we multiply (3.12) with $\bar{v}$ and integrate by parts, we have

$$\int_{\mathbb{R}^3} \left(|\nabla v|^2 + \frac{\gamma^2}{2}|v|^2\right)\mathrm{d}\sigma = 0,$$

which shows that $v(x) = 0$, and hence $u(x) = 0$. Finally, from Lemma 3.3, we have that

$$\phi = \frac{\partial u}{\partial \nu}\bigg|_+ - \frac{\partial u}{\partial \nu}\bigg|_- = 0,$$

which proves the claim. ∎

4. Gauge capacitance matrix approximation

We are looking for non-trivial solutions to the non-reciprocal Helmholtz problem (2.1) in the subwavelength regime. Problem (2.1) can be reformulated using layer potential techniques. In fact, there exist density functions ψ and ϕ such that the solution u of (2.1) can be represented as

$$u(x) = \begin{cases} \widetilde{\mathcal{S}}^{\omega}_{\mathcal{D},\gamma}[\psi], & x \in \mathcal{D}, \\ \mathcal{S}^{\omega}_{\mathcal{D}}[\phi], & x \in \mathbb{R}^3 \backslash \mathcal{D}. \end{cases} \tag{4.1}$$

Hence, for ω small enough, (2.1) is equivalent to

$$\mathcal{A}(\omega,\delta)\begin{pmatrix} \psi \\ \phi \end{pmatrix} = 0, \tag{4.2}$$

with $\mathcal{A}(\omega,\delta) : L^2(\partial\mathcal{D}) \times L^2(\partial\mathcal{D}) \to H^1(\partial\mathcal{D}) \times L^2(\partial\mathcal{D})$ being given by

$$\mathcal{A}(\omega,\delta) = \begin{pmatrix} \widetilde{\mathcal{S}}^{\omega}_{\mathcal{D},\gamma} & -\mathcal{S}^{\omega}_{\mathcal{D}} \\ -\frac{1}{2}I + \widetilde{\mathcal{K}}^{\omega,*}_{\mathcal{D},\gamma} & -\delta(\frac{1}{2}I + \mathcal{K}^{\omega,*}_{\mathcal{D}}) \end{pmatrix}. \tag{4.3}$$

Note that for $\omega = \delta = 0$, we have

$$\mathcal{A}(0,0) = \begin{pmatrix} \widetilde{\mathcal{S}}^{0}_{\mathcal{D},\gamma} & -\mathcal{S}^{0}_{\mathcal{D}} \\ -\frac{1}{2}I + \widetilde{\mathcal{K}}^{0,*}_{\mathcal{D},\gamma} & 0 \end{pmatrix}. \tag{4.4}$$

In the next result, we show that $-\frac{1}{2}I + \widetilde{\mathcal{K}}^{0,*}_{\mathcal{D},\gamma}$ has a non-trivial kernel. The main idea of the capacitance approximation is that the subwavelength resonant modes arise as perturbations of these kernel functions for small (but non-zero) δ.

LEMMA 4.1. *If $\mathcal{D}$ consists of N connected components D_i, i.e., $\mathcal{D} = \bigcup_{i=1}^{N} D_i$, then the kernel of the operator $-\frac{1}{2}I + \widetilde{\mathcal{K}}^{0,*}_{\mathcal{D},\gamma} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ is N-dimensional and spanned by*

$$\psi_i = (\widetilde{\mathcal{S}}^{0}_{\mathcal{D},\gamma})^{-1}[\chi_{\partial D_i}], \ i = 1, \dots, N. \tag{4.5}$$

Here (as usual), $\chi_{\partial D_i}$ denotes the indicator function of ∂D_i.

Proof. Let ψ_i be defined by (4.5). Introduce $u = \widetilde{\mathcal{S}}^{0}_{\mathcal{D},\gamma}[\psi_i]$. From (3.5), the function u satisfies the following equation:

$$\Delta u + \gamma \partial_1 u = 0 \quad \text{in } D_i. \tag{4.6}$$

Moreover, $u = 1$ on ∂D_i. Denote by $v(x) = u(x) - 1$ in D_i. Then, by integrating by parts over D_i, we have

$$\begin{aligned} 0 &= \int_{D_i} \left(v\Delta v + \gamma v \partial_{x_1} v \right) \, \mathrm{d}x \\ &= \int_{\partial D_i} v\frac{\partial v}{\partial \nu} \, \mathrm{d}\sigma - \int_{D_i} |\nabla v|^2 \, \mathrm{d}x + \frac{\gamma}{2} \int_{D_i} \partial_{x_1} v^2 \, \mathrm{d}x \\ &= \int_{\partial D_i} v\frac{\partial v}{\partial \nu} \, \mathrm{d}\sigma - \int_{D_i} |\nabla v|^2 \, \mathrm{d}x + \frac{\gamma}{2} \int_{\partial D_i} \nu_1 v^2 \, \mathrm{d}\sigma, \end{aligned} \tag{4.7}$$

where ν_1 is the first component of the outgoing normal to ∂D_i. Since v vanishes on ∂D_i, the first and third terms in (4.7) are zero. We can conclude that $\nabla v = 0$ and $v = 0$. It follows that $u = 1$ in D_i. Therefore, by the jump relation (3.11),

$$\frac{\partial u}{\partial \nu}\bigg|_{-} = \left(-\frac{1}{2}I + \widetilde{\mathcal{K}}^{\gamma,*}_{\mathcal{D},\gamma} \right) [\psi_i] = 0.$$

Conversely, assume that $\psi \in L^2(\partial\mathcal{D})$ satisfies

$$\left(-\frac{1}{2}I + \widetilde{\mathcal{K}}^{\gamma,*}_{\mathcal{D},\gamma} \right) [\psi] = 0.$$

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Then, $u = \widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma}[\psi]$ satisfies (4.6) in $\mathcal{D}$ together with the Neumann boundary condition $\partial u/\partial\nu|_- = 0$ on $\partial\mathcal{D}$. Let $F(x) = e^{\gamma x_1}\nabla u(x)$ for $x \in \mathcal{D}$. Then F satisfies the following curl-div system of equations in $\mathcal{D}$ with a vanishing normal component on $\partial\mathcal{D}$:

$$
\begin{cases}
\nabla \cdot F = 0 & \text{in } \mathcal{D}, \\
\nabla \times (e^{-\gamma x_1} F) = 0 & \text{in } \mathcal{D}, \\
F \cdot \nu = 0 & \text{on } \partial\mathcal{D}.
\end{cases}
\tag{4.8}
$$

From the uniqueness of a solution to (4.8) (see, for instance, [1, Sect. 2.2]), it follows that u is constant inside each connected component D_i of $\mathcal{D}$. Therefore,

$$
\psi_i = \sum_{i=1}^{N} c_i (\widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma})^{-1}[\chi_{\partial D_i}],
$$

for some constants c_i. ∎

The following result follows from Lemma 4.1.

LEMMA 4.2. *The kernel of the operator* $-\frac{1}{2}I + \widetilde{\mathcal{K}}^0_{\mathcal{D},\gamma} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ *is spanned by*

$$
\phi_i = e^{\gamma x_1}\chi_{\partial D_i}, \; i = 1, \ldots, N.
\tag{4.9}
$$

Proof. For any $\psi \in L^2(\partial\mathcal{D})$, by using the jump conditions (3.11), we have

$$
\begin{aligned}
\left\langle \psi, \left(-\frac{1}{2}I + \widetilde{\mathcal{K}}^0_{\mathcal{D},\gamma}\right)[\phi_i] \right\rangle &= \int_{\partial D_i} e^{\gamma x_1} \left.\frac{\partial}{\partial\nu}\right|_- \widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma}[\psi] \, d\sigma \\
&= \int_{D_i} \left(e^{\gamma x_1}\Delta\widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma}[\psi] + \nabla(e^{\gamma x_1}) \cdot \nabla\widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma}[\psi] \right) dx \\
&= \int_{D_i} \left(e^{\gamma x_1}\Delta\widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma}[\psi] + \gamma e^{\gamma x_1}\partial_{x_1}\widetilde{\mathcal{S}}^0_{\mathcal{D},\gamma}[\psi] \right) dx \\
&= 0,
\end{aligned}
\tag{4.10}
$$

where $\langle \, , \, \rangle$ denotes the L^2 scalar product on ∂D_i.

On the other hand, since for $\gamma = 0$, $-\frac{1}{2}I + \mathcal{K}^0_{\mathcal{D}} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ is Fredholm of index zero and $\widetilde{\mathcal{K}}^0_{\mathcal{D},\gamma} - \mathcal{K}^0_{\mathcal{D}} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ is compact (as G^0_γ and G^0 have the same singularity at the origin), the operator $-\frac{1}{2}I + \widetilde{\mathcal{K}}^0_{\mathcal{D},\gamma} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ is also Fredholm of index zero. From Lemma 4.1, it then follows that the dimension of the kernel of $-\frac{1}{2}I + \widetilde{\mathcal{K}}^0_{\mathcal{D},\gamma}$ is N. This completes the proof. ∎

We now introduce the gauge capacitance matrix, which allows us to reduce the problem of finding the subwavelength eigenfrequencies and eigenmodes to a finite-dimensional eigenvalue problem.

Definition 4.1 (Gauge capacitance matrix). *The gauge capacitance matrix* $\mathcal{C}^\gamma_N = ((\mathcal{C}^\gamma_N)_{i,j})$, $i, j = 1, \ldots, N$, *reads*

$$
(\mathcal{C}^\gamma_N)_{i,j} = -\frac{\delta}{\int_{D_i} e^{\gamma x_1} \, dx} \int_{\partial D_i} e^{\gamma x_1}(\mathcal{S}^0_{\mathcal{D}})^{-1}[\chi_{\partial D_j}](x) \, dx.
\tag{4.11}
$$

The following results hold.

Theorem 4.3 (Discrete approximations).   *(i) The N subwavelength eigenfrequencies ω_i of (2.1) satisfy, as $\delta \to 0$,*

$$
\omega_n = \sqrt{\lambda_n} + O(\delta),
\tag{4.12}
$$

where $(\lambda_n)_{1 \leq n \leq N}$ are the eigenvalues of $\mathcal{C}^\gamma_N$;

(ii) Let v_n be the eigenvector of $\mathcal{C}_N^\gamma$ associated to λ_n. Then the normalised resonant mode u_n associated to the resonant frequency ω_n is given by

$$u_n(x) = \begin{cases} v_n \cdot \mathbf{S}^{\omega_n}(x) + O(\delta) & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}, \\ v_n \cdot \widetilde{\mathbf{S}}^{\omega_n}(x) + O(\delta) & \text{in } D_i, \end{cases} \tag{4.13}$$

where

$$\widetilde{\mathbf{S}}^{\omega_n}(x) = \begin{pmatrix} \widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^{\omega_n}[\psi_1](x) \\ \cdots \\ \widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^{\omega_n}[\psi_N](x) \end{pmatrix}, \tag{4.14}$$

with $\psi_i = (\widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^0)^{-1}[\chi_{\partial D_i}]$ for $i = 1, \ldots, N$.

Proof. We make use of the functional approach first introduced in [14] (see also [7]). In view of (4.4), by using Lemma 4.1, we have the kernel basis functions of $\mathcal{A}(\omega, \delta)$ at $\delta = \omega = 0$:

$$\ker(\mathcal{A}(0,0)) = \text{span}\{\Psi_i\}_{i=1,\ldots,N}, \ \Psi_i = \begin{pmatrix} (\widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^0)^{-1}[\chi_{\partial D_i}] \\ (\mathcal{S}_{\mathcal{D}}^0)^{-1}[\chi_{\partial D_i}] \end{pmatrix}. \tag{4.15}$$

For ω in a small neighbourhood $\mathcal{V}$ of 0, we have the following pole-pencil decomposition:

$$\left(-\frac{1}{2}I + \widetilde{\mathcal{K}}_{\mathcal{D},\gamma}^{\omega,*} \right)^{-1} = \sum_{j=1}^N \frac{\langle \phi_j, \cdot \rangle \, \psi_j}{\omega^2} + \mathcal{R}(\omega), \tag{4.16}$$

where the right- and left-kernel functions ψ_i and ϕ_i are normalised such that

$$\frac{1}{\omega^2} \left\langle \phi_j, \left(-\frac{1}{2} + \widetilde{\mathcal{K}}_{\mathcal{D},\gamma}^{\omega,*} \right) [\psi_i] \right\rangle = \delta_{ij}, \tag{4.17}$$

and the operator-valued function $\omega \mapsto \mathcal{R}(\omega)$ is holomorphic on $\mathcal{V}$. Here, we can take $(\phi_j)_{j=1}^N$

$$\phi_j = C_j e^{\gamma x_1} \chi_{\partial D_j}, \tag{4.18}$$

for normalisation factors C_j. We have

$$\left\langle \phi_j, \left(-\frac{1}{2} + \widetilde{\mathcal{K}}_{\mathcal{D},\gamma}^{\omega,*} \right) [\psi_i] \right\rangle = \int_{\partial D_j} C_j e^{\gamma x_1} \left. \frac{\partial}{\partial \nu} \right|_- \widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^\omega[\psi_i] \, d\sigma$$

$$= C_j \int_{D_j} \left(e^{\gamma x_1} \Delta \widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^\omega[\psi_i] + \gamma e^{\gamma x_1} \partial_{x_1} \widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^\omega[\psi_i] \right) dx \tag{4.19}$$

$$= -\delta_{ij} \omega^2 C_j \int_{D_j} e^{\gamma x_1} \, dx.$$

It then follows that

$$\phi_j = -\frac{1}{\int_{D_j} e^{\gamma x_1} \, dx} e^{\gamma x_1} \chi_{\partial D_j}. \tag{4.20}$$

Hence we have the kernel functions

$$\ker(\mathcal{A}^*(0,0)) = \text{span}\{\Phi_i\}_{i=1,\ldots,N}, \quad \Phi_i = -\frac{1}{\int_{D_i} e^{\gamma x_1} \, dx} \begin{pmatrix} 0 \\ e^{\gamma x_1} \chi_{\partial D_i} \end{pmatrix}. \tag{4.21}$$

For small δ, we can write

$$\mathcal{A}(\omega, \delta) = \mathcal{A}(\omega, 0) + \mathcal{L}(\omega, \delta). \tag{4.22}$$

We have

$$0 = \mathcal{A}(\omega, 0)^{-1} \mathcal{A}(\omega, \delta)[\Phi] = \mathcal{A}(\omega, 0)^{-1}(\mathcal{A}(\omega, 0) + \mathcal{L})[\Phi]. \tag{4.23}$$

This leads to

$$\left(I + \frac{K\mathcal{L}}{\omega^2} + \mathcal{R}\mathcal{L} \right) [\Phi] = 0, \tag{4.24}$$

or equivalently,

$$(\omega^2 I + K\mathcal{L} + \omega^2 \mathcal{R}\mathcal{L})[\Phi] = 0, \tag{4.25}$$

where $K = \sum_{i=1}^{N} \langle \phi_i, \cdot \rangle \, \psi_i$. Since $\mathcal{L} = \mathcal{L}_0 + \widehat{L}$ with $\mathcal{L}_0$ being independent of ω and $\widehat{L}$ satisfying (in the operator norm) $\|\widehat{L}\| = O(\omega\delta)$ uniformly in $\omega \in \mathcal{V}$, the subwavelength resonances are approximately the square roots of the eigenvalues of $-K\mathcal{L}_0$ restricted to $\ker(\mathcal{A}(0,0))$. This is given by the gauge capacitance matrix $\mathcal{C}_N^\gamma$. Finally, (4.14) follows from the representation formula (4.1). ∎

REMARK 4.2. *Theorem 4.3 can easily be generalised to a setting with general material parameters. Assume that the resonator D_i has wave speed v_i and contrast parameter δ_i. Correspondingly, let v be the wave speed in the surrounding medium. Assume that $\delta_i = O(\delta)$, for $i = 1, \ldots, N$, and $0 < \delta \ll 1$. For such a system of resonators, the resonance problem reads*

$$
\begin{cases}
\Delta u + \dfrac{\omega^2}{v^2} u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}, \\[2mm]
\Delta u + \dfrac{\omega^2}{v_i^2} u + \gamma \partial_1 u = 0 & \text{in } D_i, \quad i = 1, \ldots, N, \\[2mm]
u|_+ - u|_- = 0 & \text{on } \partial \mathcal{D}, \\[2mm]
\delta_i \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D_i, \quad i = 1, \ldots, N, \\[2mm]
u \text{ satisfies an outgoing radiation condition.}
\end{cases}
\tag{4.26}
$$

It can be easily seen that the gauge capacitance matrix $\mathcal{C}_N^\gamma = \left((\mathcal{C}_N^\gamma)_{i,j} \right), i, j = 1, \ldots, N$, defined by

$$
(\mathcal{C}_N^\gamma)_{i,j} = -\frac{\delta_i v_i^2}{\int_{D_i} e^{\gamma x_1} \, \mathrm{d}x} \int_{\partial D_i} e^{\gamma x_1} (\mathcal{S}_{\mathcal{D}}^0)^{-1} [\chi_{\partial D_j}] \mathrm{d}x,
\tag{4.27}
$$

provides as in Theorem 4.3 leading-order approximations to the N subwavelength resonant frequencies of (4.26) and their associated resonant eigenmodes. In fact, the N subwavelength resonant frequencies of (4.26) and their associated eigenmodes are respectively approximated by (4.12) and (4.13), where $(\lambda_n)_{1 \le n \le N}$ and $(v_n)_{1 \le n \le N}$ are respectively the eigenvalues and associated eigenvectors of $\mathcal{C}_N^\gamma$ defined by (4.27).

5. Toeplitz structure of the gauge capacitance matrix

In this section, in order to demonstrate the non-Hermitian skin effect in three-dimensional systems of subwavelength resonators, we derive some properties of the gauge capacitance matrix $\mathcal{C}_N^\gamma$. In particular, we show that it is, up to a small perturbation, a Toeplitz matrix. In order to do that, we need to establish some convergence results, characterising the matrix in the limit when the number of resonators goes to infinity. This is a generalisation of the approach used in [8]. In particular, we look at periodic chain structures with only one single resonator in the unit cell. This structure is equivalent to equidistant chains of resonators. Formally, we let $l_1 \in \mathbb{R}^3$ denote the lattice vector generating the linear lattice

$$
\Lambda := \{ m l_1 | m \in \mathbb{Z} \}.
\tag{5.1}
$$

Without loss of generality, assume that l_1 is aligned with the x_1-axis. Denote by $Y \subset \mathbb{R}^3$ the fundamental domain of the lattice. The dual lattice of Λ, denoted by Λ^*, is generated by α_1, satisfying $\alpha_1 \cdot l_1 = 2\pi$. The Brillouin zone Y^* is defined as

$$
Y^* := (\mathbb{R} \times \{0_{\mathbb{R}^2}\})/\Lambda^*.
\tag{5.2}
$$

As before, we let D denote the periodically repeated resonator D consider an infinite structure of resonators

$$
\bigcup_{m \in \Lambda} (D + m).
\tag{5.3}
$$

For any function $f \in L^2(\mathbb{R}^3)$, the Floquet transform of f is defined as

$$\mathcal{F}[f](x, \alpha) := \sum_{m \in \Lambda} f(x - m) e^{i\alpha \cdot m}, \quad x, \alpha \in \mathbb{R}^3. \tag{5.4}$$

Extending the set of equations (2.1) to the infinite structure and applying the Floquet transform with $u^\alpha(x) := \mathcal{F}[u](x, \alpha)$, we obtain

$$\begin{cases} \Delta u^\alpha + \omega^2 u^\alpha = 0 & \text{in } Y \backslash D, \\ \Delta u^\alpha + \omega^2 u^\alpha + \gamma \partial_1 u^\alpha = 0 & \text{in } D, \\ u^\alpha|_+ - u^\alpha|_- = 0 & \text{on } \partial D, \\ \delta \dfrac{\partial u^\alpha}{\partial \nu}\bigg|_+ - \dfrac{\partial u^\alpha}{\partial \nu}\bigg|_- = 0 & \text{on } \partial D, \\ u^\alpha(x_1, x_2, x_3) & \text{is } \alpha\text{-quasiperiodic in } x_1, \\ u^\alpha(x_1, x_2, x_3) & \text{satisfies the } \alpha\text{-quasiperiodic outgoing radiation} \\ & \quad \text{condition as } \sqrt{x_2^2 + x_3^2} \to \infty. \end{cases} \tag{5.5}$$

We refer to [11] for a precise statement of the α-quasiperiodic radiation condition. We can proceed as in the usual case and define the quasiperiodic function outside the resonators as

$$G^{\alpha,k}(x, y) := -\sum_{m \in \Lambda} \frac{e^{ik|x-y-m|}}{4\pi|x-y-m|} e^{i\alpha \cdot m}, \tag{5.6}$$

together with the quasiperiodic single layer potential

$$\mathcal{S}_D^{\alpha,k}[\phi](x) := \int_{\partial D} G^{\alpha,k}(x, y) \phi(y) \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3, \tag{5.7}$$

for $\phi \in L^2(\partial D)$. The series in (5.6) converges uniformly for x and y in compact sets of $\mathbb{R}^3$, $x \neq y$, and $k \neq |\alpha + q|$ for all $q \in \Lambda^*$. Repeating the procedures in Section 3, we can generalise the concept of quasiperiodic capacity first introduced in [12] to (5.5) and define the gauge capacitance matrix as follows.

Definition 5.1. *If $\alpha \neq 0$, the gauge quasiperiodic capacity is given by*

$$\widehat{\mathcal{C}}^{\alpha,\gamma} = -\frac{\delta}{\int_D e^{\gamma x_1} \, \mathrm{d}x} \int_{\partial D} e^{\gamma x_1} (\mathcal{S}_D^{\alpha,0})^{-1}[\chi_{\partial D}](x) \, \mathrm{d}\sigma(x). \tag{5.8}$$

The above introduced gauge capacity is a "dual-space" capacitance representation. The infinitely periodic system with only one resonator in a unit cell is equivalent to an infinite chain of equidistant resonators which has a "real-space" capacitance representation through the inverse Floquet-Bloch transformation (for details, see [6, 8]) given by

$$\mathcal{C}_{i,j}^\gamma = \frac{1}{|Y^*|} \int_{Y^*} \widehat{\mathcal{C}}^{\alpha,\gamma} e^{-i\alpha(i-j)} \mathrm{d}\alpha. \tag{5.9}$$

It is clear from the definition above that $\mathcal{C}^\gamma$ is a Toeplitz matrix. We now circle back to the finite system. Let $\mathcal{D}_N$ denote a finite chain of N equidistant resonators and $\mathcal{C}_N^\gamma$ be the corresponding gauge capacitance matrix. We can extend $\mathcal{D}_N$ to a larger chain $\mathcal{D}_R$ with R resonators and denote by $\mathcal{C}_R^\gamma$ the corresponding gauge capacitance matrix. We then let $\widetilde{\mathcal{C}}_N^\gamma$ be the embedded $N \times N$-size centre block of $\mathcal{C}_R^\gamma$. We call $\widetilde{\mathcal{C}}_N^\gamma$ the truncated gauge capacitance matrix with size $N \times N$. Following the same steps as those in [6], we have the following theorem.

Theorem 5.1. *For $R \to \infty$, we have for every $i, j = 1, \dots, N$, that*

$$\lim_{R \to \infty} (\widetilde{\mathcal{C}}_N^\gamma)_{i,j} = \mathcal{C}_{i,j}^\gamma. \tag{5.10}$$

Proof. As shown in [6], for given $\phi \in L^2(D)$, we have

$$\mathcal{S}_D^\alpha[\phi] = \mathcal{S}_{\mathcal{D}_R}[\phi] + \mathcal{R}^\alpha[\phi], \tag{5.11}$$

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where $\mathcal{R}^\alpha = o(1)$ as $R \to \infty$. From the Neumann series, we have

$$(\mathcal{S}_D^\alpha)^{-1}[\chi_{\partial D}] = \mathcal{S}_{\mathcal{D}_R}^{-1}[\chi^\alpha] + o(1), \tag{5.12}$$

with

$$\chi^\alpha = \sum_{|m| < N l_1} \chi_{\partial D_i} e^{i\alpha \cdot m}. \tag{5.13}$$

Finally, we have as $N \to \infty$ that

$$
\begin{aligned}
\mathcal{C}_{ij}^\gamma &= -\frac{1}{|Y^*|} \int_{Y^*} \frac{\delta}{\int_D e^{\gamma x_1}\, \mathrm{d}x} \int_{\partial D} e^{\gamma x_1} (\mathcal{S}_D^{\alpha,0})^{-1}[\chi^\alpha] e^{-i\alpha(i-j)} \mathrm{d}\sigma \mathrm{d}\alpha + o(1) \\
&= \frac{\delta}{\int_{D_i} e^{\gamma x_1}\mathrm{d}x} \int_{\partial D_i} e^{\gamma x_1} \mathcal{S}_{\mathcal{D}_R}^{-1}[\chi_{\partial D_j}]\mathrm{d}\sigma + o(1) \tag{5.14} \\
&= (\widetilde{\mathcal{C}}_N^\gamma)_{ij} + o(1). \qquad\blacksquare
\end{aligned}
$$

The next results show that the matrix $\mathcal{C}_N^\gamma$ has a Toeplitz structure up to some small perturbation.

LEMMA 5.2. *We have the following estimate of the matrix entries:*

$$\left| (\mathcal{C}_N^\gamma)_{i,j} - (\widetilde{\mathcal{C}}_N^\gamma)_{i,j} \right| \leq \frac{\delta K}{(1 + \min(i, N-i))(1 + \min(j, N-j))}, \tag{5.15}$$

for some constant K independent of N.

Proof. Defining x_1^i to be the x_1-coordinate of the center of D_i, we can compute

$$
\begin{aligned}
\left| (\mathcal{C}_N^\gamma)_{i,j} - (\widetilde{\mathcal{C}}_N^\gamma)_{i,j} \right| &= \left| \frac{\delta}{\int_{D_i} e^{\gamma x_1}\mathrm{d}x} \int_{\partial D_i} e^{\gamma x_1} \left(\mathcal{S}_{\mathcal{D}_N}^{-1} - \mathcal{S}_{\mathcal{D}_R}^{-1} \right) [\chi_{\partial D_j}]\mathrm{d}\sigma \right| \\
&= \left| \frac{\delta}{\int_{D_i} e^{\gamma(x_1 - x_1^i)}\mathrm{d}x} \int_{\partial D_i} e^{\gamma(x_1 - x_1^i)} \left(\mathcal{S}_{\mathcal{D}_N}^{-1} - \mathcal{S}_{\mathcal{D}_R}^{-1} \right) [\chi_{\partial D_j}]\mathrm{d}\sigma \right| \tag{5.16} \\
&\leq \frac{\delta K}{(1 + \min(i, N-i))(1 + \min(j, N-j))},
\end{aligned}
$$

for some constant K independent of N. Here, the last inequality follows from the proof of [8, Lemma 3.1]. $\qquad\blacksquare$

LEMMA 5.3. *We have the following estimate:*

$$\left| (\mathcal{C}_N^\gamma)_{i,j} - \mathcal{C}_{i,j}^\gamma \right| \leq \frac{\delta K}{(1 + \min(i, N-i))(1 + \min(j, N-j))}, \quad i, j = 1, \ldots, N, \tag{5.17}$$

for some constant K independent of N.

Proof. From Theorem 5.1, it follows that for every $\varepsilon > 0$, there exists a large enough R so that $\left| (\widetilde{\mathcal{C}}_N^\gamma)_{i,j} - \mathcal{C}_{i,j}^\gamma \right| < \varepsilon$. The desired estimate follows from Lemma 5.2 and the triangular inequality. $\qquad\blacksquare$

The following classical result allows us to describe the decay property of the entries of the gauge capacitance matrix. We refer, for instance, to [22] for its proof.

LEMMA 5.4. *If $f \in BV(\mathbb{T})$, then for the n-th Fourier coefficient $\widehat{f}(n)$ it holds*

$$|\widehat{f}(n)| \leq \frac{\mathrm{Var}_{\mathbb{T}}(f)}{2\pi |n|}, \quad n \neq 0. \tag{5.18}$$

Here, $\mathbb{T}$ is the unit circle, $BV(\mathbb{T})$ is the set of functions of bounded variation on $\mathbb{T}$, and $\mathrm{Var}_{\mathbb{T}}(f)$ is the total variation of f.

Finally, from Lemma 5.4 we obtain the following decay estimate for the entries of the gauge capacitance matrix.

PROPOSITION 5.5. *For $i \neq j$, there exists a constant K, independent of N, so that*

$$|\mathcal{C}_{i,j}^{\gamma}| \leq \frac{\delta K}{|i-j|}. \tag{5.19}$$

Proof. We first note that by exactly the same arguments as those in [9, Section 3.3], we can show that $\alpha \mapsto \widehat{\mathcal{C}}^{\alpha,\gamma}$ is piecewise differentiable whose derivative is absolutely integrable over Y^*. It follows that it has bounded variation. By definition (5.9), $\mathcal{C}_{i,j}^{\gamma}$ is the $(i-j)$-th Fourier coefficient of $\widehat{\mathcal{C}}^{\alpha,\gamma}$, so Lemma 5.4 shows that

$$|\mathcal{C}_{i,j}^{\gamma}| \leq \frac{\delta K}{|i-j|}. \tag{5.20}$$

∎

Finally, we generalise this decay estimate to the finite gauge capacitance matrix. We will use the following classical notion of the (regular) capacitance matrix.

Definition 5.2. *The capacitance matrix $C_N = (C_N)_{i,j}$, for $i,j = 1, \ldots, N$, is defined by*

$$(C_N)_{i,j} = -\int_{\partial D_i} (\mathcal{S}_{\mathcal{D}_N}^0)^{-1}[\chi_{\partial D_j}](x) \, d\sigma(x). \tag{5.21}$$

It is well-known that the diagonal coefficients $(C_N)_{i,i}$ are positive while the off-diagonal coefficients $(C_N)_{i,j}, i \neq j$ are negative [11]. The capacitance coefficients are equivalently given by (see, for example, [7])

$$(C_N)_{i,j} = -\int_{\partial D_i} \frac{\partial V_j(x)}{\partial \nu} \, d\sigma(x), \tag{5.22}$$

where V_j is given by the solution to the following problem:

$$\begin{cases} \Delta V_j = 0, & x \in \mathbb{R}^3 \setminus \mathcal{D}_N, \\ V_j(x) = \chi_{\partial D_i}(x), & x \in \partial \mathcal{D}_N, \\ V_j(x) = O(|x|^{-1}), & x \to \infty. \end{cases} \tag{5.23}$$

We begin with the following lemma, which states that the capacitance coefficient associated to two given resonators can only increase if we add additional resonators.

LEMMA 5.6. *Let $\mathcal{D}_N$ and $\mathcal{D}_{N+1}$ be collections of N and $N+1$ resonators, such that $\mathcal{D}_{N+1} = \mathcal{D}_N \cup D_{N+1}$. For all $1 < i, j < N$ we then have*

$$(C_N)_{i,j} \leq (C_{N+1})_{i,j}.$$

Proof. Let $V_{N,j}$ and $V_{N+1,j}$, respectively, be the unique solutions to the problems

$$\begin{cases} \Delta V_{N,j} = 0, & x \in \mathbb{R}^3 \setminus \mathcal{D}_N, \\ V_{N,j}(x) = \chi_{\partial D_j}(x), & x \in \partial \mathcal{D}_N, \\ V_{N,j}(x) = O(|x|^{-1}), & x \to \infty, \end{cases} \qquad \begin{cases} \Delta V_{N+1,j} = 0, & x \in \mathbb{R}^3 \setminus \mathcal{D}_{N+1}, \\ V_{N+1,j}(x) = \chi_{\partial D_j}(x), & x \in \partial \mathcal{D}_{N+1}, \\ V_{N+1,j}(x) = O(|x|^{-1}), & x \to \infty. \end{cases}$$

From the maximum principle, we observe that $0 < V_{N,j}(x), V_{N+1,j}(x) < 1$. Then we define $u(x) = V_{N,j}(x) - V_{N+1,j}(x)$. Note that u satisfies

$$\begin{cases} \Delta u = 0, & x \in \mathbb{R}^3 \setminus \mathcal{D}_{N+1}, \\ u(x) = 0, & x \in \partial \mathcal{D}_N, \\ 0 < u(x) < 1, & x \in \partial \mathcal{D}_{N+1}, \\ u(x) = O(|x|^{-1}), & x \to \infty. \end{cases}$$

Again, by the maximum principle we have $0 < u(x) < 1$ in $\mathbb{R}^3 \setminus \mathcal{D}_{N+1}$. Together with the boundary condition $u(x) = 0$ on $\partial \mathcal{D}_N$, this implies that

$$\frac{\partial u}{\partial \nu} \geq 0, \quad x \in \partial \mathcal{D}_N.$$

From (5.22), we then have

$$(C_{N+1})_{i,j} - (C_N)_{i,j} = \int_{\partial D_i} \frac{\partial u}{\partial \nu} \mathrm{d}\sigma \geq 0,$$

which proves the claim. ∎

We now have the following decay estimate of the gauge capacitance coefficients.

PROPOSITION 5.7. *Let $d_{i,j}$ be the minimal distance between D_i and D_j. For $i \neq j$, there exists a constant K, depending only on the shape of D_i and D_j, such that $|(\mathcal{C}_N)_{i,j}^\gamma| \leq \delta K d_{i,j}^{-1}$. In particular, we have a constant K', independent of N, such that*

$$|(\mathcal{C}_N)_{i,j}^\gamma| \leq \frac{\delta K'}{|i-j|}. \tag{5.24}$$

Proof. We begin by observing that

$$\begin{aligned}
|(\mathcal{C}_N)_{i,j}^\gamma| &= \left| \frac{\delta_i v_i^2}{\int_{D_i} e^{\gamma x_1} \mathrm{d}x} \int_{\partial D_i} e^{\gamma x_1} \psi_j \mathrm{d}\sigma \right| \\
&\leq \delta K_1 \int_{\partial D_i} |\psi_j| \mathrm{d}\sigma,
\end{aligned} \tag{5.25}$$

for some constant K_1. We moreover have

$$\psi_j = \left(\frac{1}{2} + \mathcal{K}_{\mathcal{D}_N}^{0,*} \right) [\psi_j] = \frac{\partial}{\partial \nu} V_j(x),$$

where V_j is defined as in (5.23). In the same way as in the proof of Lemma 5.6, the maximum principle and boundary conditions implies that $\partial_\nu V_j \geq 0$ on $\partial \mathcal{D}_N \setminus \partial D_j$. From (5.25), it therefore follows that

$$\left| (\mathcal{C}_N)_{i,j}^\gamma \right| \leq \delta K_1 \left| (C_N)_{i,j} \right|, \tag{5.26}$$

where $(C_N)_{i,j}$ are the regular capacitance coefficients, defined as in (5.22). Let now $\mathcal{D}_2 = D_i \cup D_j$ be the resonator structure consisting solely of the two components D_i and D_j, and let C_2 be the 2×2 capacitance matrix associated to $\mathcal{D}_2$. Following [10, Lemma 4.3], we then have

$$|(C_2)_{1,2}| < \frac{K_2}{d_{i,j}},$$

for some constant K_2. Moreover, since $(C_N)_{i,j}$ and $(C_2)_{1,2}$ are both negative, we have from Lemma 5.6 that

$$|(C_N)_{i,j}| \leq |(C_2)_{1,2}| < \frac{K_2}{d_{i,j}}.$$

This, combined with (5.26), proves the claim. ∎

Figure 5.1 shows the decay rate of the entries of the gauge capacitance matrix, which is in accordance with Proposition 5.7. We remark that the decay rate is close to $|i-j|^{-r}$ for $r \approx 2$. The precise decay rate depends on the geometry of the resonators. In the dilute limit (when the resonators are asymptotically small), the decay rate will approach $r = 1$ [10]. Here, we use the multipole method, as described in [12, Appendix C], to compute the associated gauge capacitance matrix.

6. Non-Hermitian skin effect

As shown in item (ii) of Theorem 4.3, at leading-order in δ, each subwavelength resonant mode u_n of (2.1) is determined by an eigenvector of the capacitance matrix $\mathcal{C}_N^\gamma$. Thus, the non-Hermitian skin effect is related to the exponential decay of the eigenvectors of the gauge capacitance matrix $\mathcal{C}_N^\gamma$. Although a precise characterisation of the exponential decay of
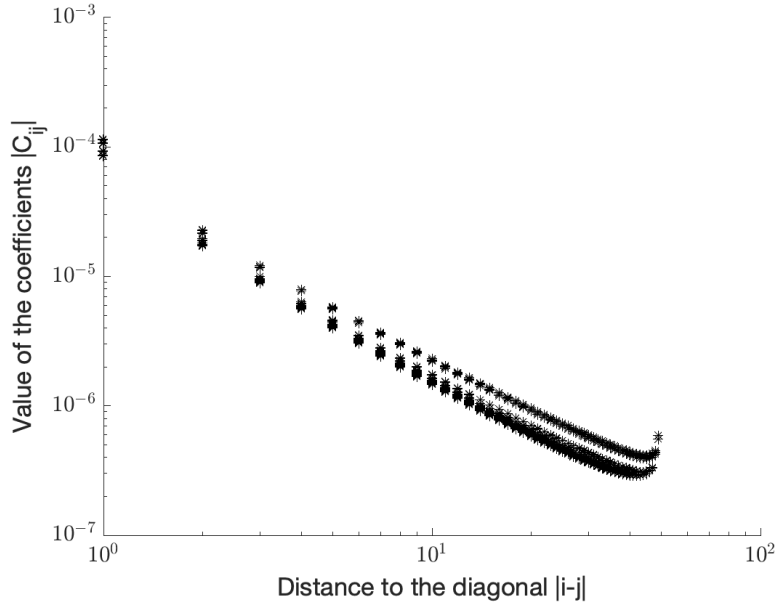
FIGURE 5.1. Decay of the entries of the gauge capacitance matrix for $N = 100$ and $\gamma = 1$ in *log-log* scale. The x-axis represents the index $|i - j|$ and the y-axis displays the absolute value of $(C_N^\gamma)_{i,j}$.

eigenvectors of a dense Toeplitz matrix remains an open problem, we can demonstrate the exponential decay of the pseudo-eigenvectors $\mathbf{v}$ of $\mathcal{C}_N^\gamma$ satisfying

$$\frac{\left\| \left(\mathcal{C}_N^\gamma - \lambda I \right) \mathbf{v} \right\|_2}{\left\| \mathbf{v} \right\|_2} < \varepsilon$$

for a certain pseudo-eigenvalue λ and small ε. Here, I denotes the identity matrix. Demonstrating the exponential decay of the pseudo-eigenvectors of $\mathcal{C}_N^\gamma$ is still non-trivial. A standard approach is to make use of a tridiagonal approximation similar to the nearest-neighbour approximation in quantum mechanics, where we only consider interactions between neighbouring resonators. This results in a tridiagonal gauge capacitance matrix $\mathcal{C}_{N,2}^\gamma$ defined as in (6.1) for $k = 2$. The non-Hermitian skin effect for such model has been thoroughly discussed in [2, 4]. However, due to the long-range interactions between the resonators, the elements of $\mathcal{C}_N^\gamma$ decay slowly as shown in (5.24), making the nearest-neighbour approximation inaccurate. For instance, the first several modes of $\mathcal{C}_N^\gamma$ in Figure 6.4 differ considerably from those of $\mathcal{C}_{N,2}^\gamma$. A straightforward generalisation of the nearest-neighbour approximation is a range-k approximation, where we consider the interactions of neighbouring k-resonators. This results in a k-banded gauge capacitance matrix $\mathcal{C}_{N,k}^\gamma$ defined by (6.1). In the next subsection, we shall characterise the exponential decay of the pseudo-eigenvectors of $\mathcal{C}_{N,k}^\gamma$ and in Subsection 6.2 we shall show that these pseudo-eigenvectors approximate well the pseudo-eigenvectors of the matrix $\mathcal{C}_N^\gamma$, given large enough k and N.

6.1. Exponential decay of pseudo-eigenvectors of k-banded gauge capacitance matrices

We first define a k-banded gauge capacitance matrix $\mathcal{C}_{N,k}^\gamma$ from $\mathcal{C}_N^\gamma$ as

$$\left(\mathcal{C}_{N,k}^\gamma \right)_{i,j} = \begin{cases} \left(\mathcal{C}_N^\gamma \right)_{i,j}, & \text{for } |i - j| < k, \\ 0, & \text{for } |i - j| \geq k, \end{cases} \quad 1 \leq i, j \leq N, \tag{6.1}$$

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and demonstrate the exponential decay of its pseudo-eigenvectors. We let, for $|i - j| < k$,

$$a_{j-i} = \mathcal{C}_{i,j}^{\gamma} \tag{6.2}$$

with $\mathcal{C}_{i,j}^{\gamma}$ being defined by (5.9). The matrix whose $(i, j)^{\text{th}}$ entry is a_{i-j} is a Toeplitz matrix, whose symbol is given by

$$f(z) = \sum_{j=-(k-1)}^{k-1} a_j z^j. \tag{6.3}$$

Define $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$. Let $I(f(\mathbb{T}), \lambda)$ be the winding number of $f(\mathbb{T})$ around λ in the usual positive (counterclockwise) sense.

Define

$$\xi(i, j) = \sum_{q=i}^{j} \frac{1}{q}. \tag{6.4}$$

Then by Abel's criteria, $\sum_{j=1}^{\infty} \frac{\xi(j+k_1, j+k_2)^2}{j^2}$ is convergent for any $k_1 < k_2$ and $j + k_1 \geq 1$. The following result on the pseudo-eigenvectors of $\mathcal{C}_{N,k}^{\gamma}$ holds.

Theorem 6.1. *For any $\varepsilon > 0$, we can choose an integer $k > 0$ so that*

$$\max \left(\frac{1}{k}, \quad \sum_{j=k+1}^{+\infty} \frac{\xi(j+2-k, j+k)^2}{(1+j)^2} \right) \leq \varepsilon$$

with $\xi(\cdot)$ being defined by (6.4). Let λ be any complex number with $I(f(\mathbb{T}), \lambda) \neq 0$ for f defined by (6.3). For some $\rho < 1$ and any sufficiently large $N > 4k$, there exist nonzero pseudo-eigenvectors $\mathbf{v}^{(N)}$ of $\mathcal{C}_{N,k}^{\gamma}$ with $\left|\left|\mathbf{v}^{(N)}\right|\right|_2 = 1$ satisfying

$$\left|\left| \left(\mathcal{C}_{N,k}^{\gamma} - \lambda \right) \mathbf{v}^{(N)} \right|\right|_2 < \max \left(C_1, C_2 N^{k-1} \right) \rho^{N-2k+2} + \delta K \sqrt{10\varepsilon} \tag{6.5}$$

such that

$$\frac{\left| \left(\mathbf{v}^{(N)} \right)_j \right|}{\max_j \left| \left(\mathbf{v}^{(N)} \right)_j \right|} \leq \begin{cases} C_3 \rho^{j-1}, & \text{if } I(f(\mathbb{T}), \lambda) > 0, \\ C_3 \rho^{N-j}, & \text{if } I(f(\mathbb{T}), \lambda) < 0, \end{cases} \quad 1 \leq j \leq N, \tag{6.6}$$

where C_1, C_2, C_3 are independent of N. In particular, the constant ρ can be taken to be any number so that $1 > \rho \geq r$ with $I(f(\mathbb{T}_r), \lambda) > 0$ or $\frac{1}{\rho} \geq \frac{1}{r} > 1$ with $I\left(f\left(\mathbb{T}_{\frac{1}{r}}\right), \lambda\right) < 0$.

Proof. We decompose the k-banded matrix $\mathcal{C}_{N,k}^{\gamma}$ as

$$\mathcal{C}_{N,k}^{\gamma} = A + M,$$

where the entries of the matrices A and M are respectively given by

$$A_{i,j} = \begin{cases} \left(\mathcal{C}_{N,k}^{\gamma} \right)_{i,j}, & 1 \leq i \leq k, \text{ or } N - (k-1) \leq i \leq N, \\ \mathcal{C}_{i,j}^{\gamma}, & |i - j| < k, k < i < N - (k-1), \\ 0, & |i - j| \geq k, \end{cases} \tag{6.7}$$

and

$$M_{i,j} = \begin{cases} 0, & 1 \leq i \leq k, \text{ or } N - (k-1) \leq i \leq N, \\ \left(\mathcal{C}_{N,k}^{\gamma} \right)_{i,j} - \mathcal{C}_{i,j}^{\gamma}, & |i - j| < k, k < i < N - (k-1), \\ 0, & |i - j| \geq k. \end{cases} \tag{6.8}$$

Note that, based on the definition of $\mathcal{C}_{i,j}^{\gamma}$ in (5.9), A is a k-banded perturbed Toeplitz matrix like the matrix in (A.3). By Theorem A.3, we have that for any complex number λ with $I(f(\mathbb{T}), \lambda) \neq 0$, where f is defined by (6.3) and for some $\rho < 1$ and sufficiently large $N > 4k$, there exist nonzero pseudo-eigenvectors $\mathbf{v}^{(N)}$ with $\left|\left|\mathbf{v}^{(N)}\right|\right|_2 = 1$ satisfying

$$\left|\left| (A - \lambda) \mathbf{v}^{(N)} \right|\right|_2 \leq \max \left(C_1, C_2 N^{k-1} \right) \rho^{N-2k+2},$$

such that

$$\frac{\left|\left(\mathbf{v}^{(N)}\right)_j\right|}{\max_j \left|\left(\mathbf{v}^{(N)}\right)_j\right|} \leq \begin{cases} C_3 \rho^{j-1}, & \text{if } I(f(\mathbb{T}),\lambda) > 0, \\ C_3 \rho^{N-j}, & \text{if } I(f(\mathbb{T}),\lambda) < 0, \end{cases} \quad 1 \leq j \leq N, \qquad (6.9)$$

where C_1, C_2, C_3 are independent of N. To prove (6.5), we estimate $\left\|M\mathbf{v}^{(N)}\right\|_2$. By the definition of M, we have $\left(M\mathbf{v}^{(N)}\right)_j = 0$ for $1 \leq j \leq k, N - (k-1) \leq j \leq N$. For $k+1 \leq j \leq \lfloor \frac{N}{2} \rfloor - k$, we have

$$\left|\left(M\mathbf{v}^{(N)}\right)_j\right| = \left|\sum_{q=j+1-k}^{j+k-1} M_{j,q}\left(\mathbf{v}^{(N)}\right)_q\right| \leq \sum_{q=j+1-k}^{j+k-1} |M_{j,q}| \left|\left(\mathbf{v}^{(N)}\right)_q\right|$$

$$\leq \sum_{q=j+1-k}^{j+k-1} \frac{\delta K}{(1+j)(1+q)} \quad \left(\text{by Lemma 5.3, } \left\|\mathbf{v}^{(N)}\right\|_2 = 1, \text{ and } N > 4k\right)$$

$$= \frac{1}{1+j} \sum_{q=j+1-k}^{j+k-1} \frac{\delta K}{1+q} = \frac{\delta K}{1+j}\xi(j+2-k, j+k),$$

where $\xi(j+2-k, j+k)$ is defined as in (6.4). Therefore,

$$\sum_{j=k+1}^{\lfloor \frac{N}{2} \rfloor - k} \left|\left(M\mathbf{v}^{(N)}\right)_j\right|^2 \leq (\delta K)^2 \sum_{j=k+1}^{\lfloor \frac{N}{2} \rfloor} \frac{\xi(j+2-k, j+k)^2}{(1+j)^2} < (\delta K)^2 \varepsilon,$$

where the last inequality is from the condition on k. Similarly, we can also prove that

$$\sum_{j=\lfloor \frac{N}{2} \rfloor + k}^{N-k} \left|\left(M\mathbf{v}^{(N)}\right)_j\right|^2 < (\delta K)^2 \varepsilon.$$

For $\lfloor \frac{N}{2} \rfloor - k + 1 \leq j \leq \lfloor \frac{N}{2} \rfloor + k - 1$, we have

$$\left|\left(M\mathbf{v}^{(N)}\right)_j\right| = \left|\sum_{q=j+1-k}^{j+k-1} M_{j,q}\left(\mathbf{v}^{(N)}\right)_q\right| \leq \sum_{q=j+1-k}^{j+k-1} |M_{j,q}| \left|\left(\mathbf{v}^{(N)}\right)_q\right|$$

$$\leq \sum_{q=j+1-k}^{j+k-1} \frac{\delta K}{(1+\lfloor \frac{N}{2} \rfloor - k + 1)(1+\lfloor \frac{N}{2} \rfloor - k + 1)} \quad \left(\text{by Lemma 5.3, } \left\|\mathbf{v}^{(N)}\right\|_2 = 1, N > 4k\right)$$

$$\leq \frac{2\delta K}{k} \quad (\text{since } N > 4k).$$

It follows that

$$\sum_{j=\lfloor \frac{N}{2} \rfloor - k + 1}^{\lfloor \frac{N}{2} \rfloor + k - 1} \left|\left(M\mathbf{v}^{(N)}\right)_j\right|^2 \leq \frac{8(\delta K)^2}{k} \leq 8(\delta K)^2 \varepsilon.$$

Combining all above estimates, we have

$$\left\|M\mathbf{v}^{(N)}\right\|_2 < \delta K \sqrt{10\varepsilon},$$

and

$$\left\|\left(\mathcal{C}_{N,k}^\gamma - \lambda\right)\mathbf{v}^{(N)}\right\|_2 \leq \left\|(A-\lambda)\mathbf{v}^{(N)}\right\|_2 + \left\|M\mathbf{v}^{(N)}\right\|_2$$
$$< \max\left(C_1, C_2 N^{k-1}\right) \rho^{N-2k+2} + \delta K \sqrt{10\varepsilon}.$$

∎

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We have demonstrated exponential decay of the pseudo-eigenvectors of the k-banded matrix $\mathcal{C}_{N,k}^{\gamma}$ for sufficiently large N. In particular, by Theorem 6.1, when the winding number $I(f(\mathbb{T}), \lambda)$ for the corresponding symbol f is non-zero, then there must be pseudo-eigenvectors of λ with certain exponential decay. On the other hand, we stress that the exponential decay property of the pseudo-eigenvectors may not be valid for general Toeplitz matrices; see [29].

Finally, we numerically illustrate the results in Theorem 6.1. In particular, Figures 6.1 and 6.2 show respectively the symbol functions of the k-banded Toeplitz matrices $\mathcal{C}_{N,k}^{\gamma}$ for $\gamma = \pm 1$ evaluated on the unit circle $\mathbb{T}$. As predicted by Theorem 6.1, all the points inside the circle are the pseudo-eigenvalues of $\mathcal{C}_{N,k}^{\gamma}$ for large enough N, including the black dots, which are the eigenvalues of the gauge capacitance matrix $\mathcal{C}_{N}^{\gamma}$. Figure 6.1b shows the eigenvectors of $\mathcal{C}_{N}^{\gamma}$ corresponding to the black and red dots. In particular, the localised eigenmodes in gray are the pseudo-eigenmodes of $\mathcal{C}_{N,k}^{\gamma}$. As seen from Figures 6.1 and 6.2, the winding numbers and the positions of the pseudo-eigenvalues correctly predict the exponential decay of the corresponding pseudo-eigenmodes. As seen in red in Figures 6.1b and 6.2b, the non-localised mode corresponds to the eigenvalue of $\mathcal{C}_{N}^{\gamma}$ outside the enclosed region with non-zero winding.

(A) Symbol function $f(\mathbb{T})$ and eigenvalues (asterisks) plotted in the complex plane.

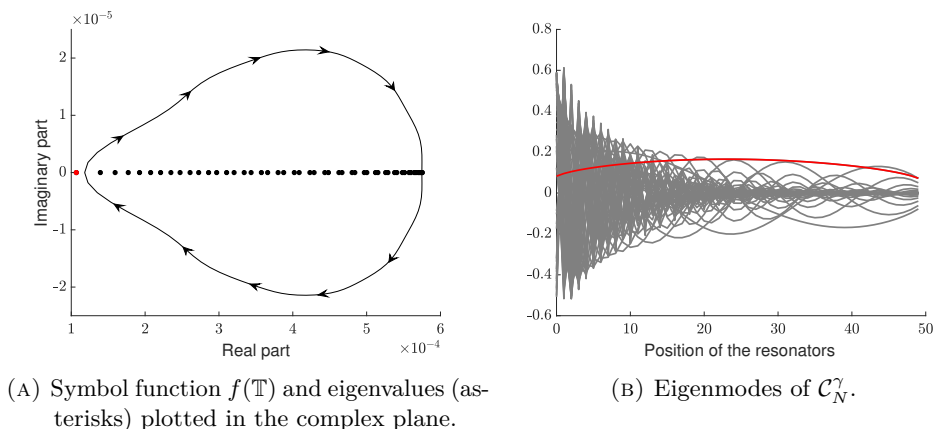(B) Eigenmodes of $\mathcal{C}_{N}^{\gamma}$.

FIGURE 6.1. Chain of resonators with $N = 50$ and $\gamma = 1$; symbol function of $\mathcal{C}_{N,k}^{\gamma}$ showing a winding number equals to -1. Here, we choose $k = 10$.
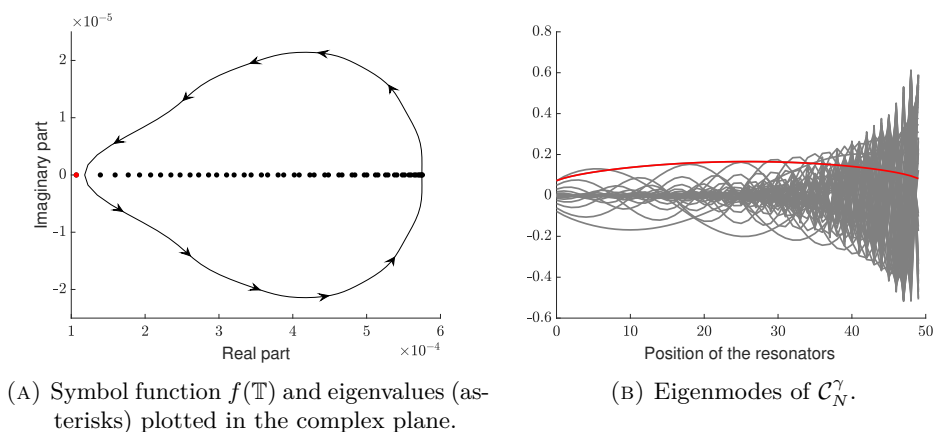

(A) Symbol function $f(\mathbb{T})$ and eigenvalues (asterisks) plotted in the complex plane.

(B) Eigenmodes of $\mathcal{C}_{N}^{\gamma}$.

FIGURE 6.2. Chain of resonators with $N = 50$ and $\gamma = -1$; symbol function of $\mathcal{C}_{N,k}^{\gamma}$ showing a winding number equals to 1. Here, we choose $k = 10$.

6.2. Exponential decay of pseudo-eigenvectors of gauge capacitance matrices

In Subsection 6.1, we have demonstrated the exponential decay of the pseudo-eigenvectors of the k-banded gauge capacitance matrix $\mathcal{C}_{N,k}^\gamma$ corresponding to λ so that $I(f(\mathbb{T}), \lambda) \neq 0$ and numerically illustrated it. In this section, we demonstrate that, for sufficiently large k, the pseudo-eigenvectors of $\mathcal{C}_{N,k}^\gamma$ having the exponential decay property are also pseudo-eigenvectors of $\mathcal{C}_N^\gamma$. In particular, we have the following theorem.

Theorem 6.2. *For any $\varepsilon_1 > 0$, we can choose an integer $k > 0$ so that*

$$\max \left(\frac{1}{k}, \sum_{j=k}^{+\infty} \frac{1}{j^2} \right) < \varepsilon_1.$$

Then, for any unit pseudo-eigenvector of $\mathcal{C}_{N,k}^\gamma$ satisfying

$$\left\| \left(\mathcal{C}_{N,k}^\gamma - \lambda I \right) \mathbf{v}^{(N)} \right\|_2 \leq \varepsilon_2, \tag{6.10}$$

and

$$\frac{\left| \left(\mathbf{v}^{(N)} \right)_j \right|}{\max_j \left| \left(\mathbf{v}^{(N)} \right)_j \right|} \leq C\rho^{j-1}, j = 1, \ldots, N, \ or \ \frac{\left| \left(\mathbf{v}^{(N)} \right)_j \right|}{\max_j \left| \left(\mathbf{v}^{(N)} \right)_j \right|} \leq C\rho^{N-j}, j = 1, \ldots, N, \tag{6.11}$$

for some constant C and $0 < \rho < 1 - \varepsilon_1$, we have

$$\left\| (\mathcal{C}_N^\gamma - \lambda I) \mathbf{v}^{(N)} \right\|_2 \leq \varepsilon_2 + \delta K C \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_1 - \rho} \sqrt{\varepsilon_1} + \varepsilon_1 \rho^k \frac{1}{1 - \rho} \sqrt{\frac{1}{1 - \rho^2}} \right). \tag{6.12}$$

Proof. By Proposition 5.7, we have

$$|(\mathcal{C}_N^\gamma)_{i,j}| \leq \frac{\delta K}{|i - j|}, \tag{6.13}$$

for some constant K. We decompose $\mathcal{C}_N^\gamma$ as follows

$$\mathcal{C}_N^\gamma = \mathcal{C}_{N,k}^\gamma + L + R,$$

where the matrices L and R are respectively defined by

$$(L)_{ij} = \begin{cases} (\mathcal{C}_N^\gamma)_{i,j}, & i - j \geq k, \\ 0, & i - j < k, \end{cases}$$

and

$$(R)_{ij} = \begin{cases} (\mathcal{C}_N^\gamma)_{i,j}, & i - j \leq -k, \\ 0, & i - j > -k. \end{cases}$$

In order to prove (6.12), we estimate $\left\| L\mathbf{v}^{(N)} \right\|_2$ and $\left\| R\mathbf{v}^{(N)} \right\|_2$. Since $\left\| \mathbf{v}^{(N)} \right\|_2 = 1$ implies $\max_j \left| \left(\mathbf{v}^{(N)} \right)_j \right| \leq 1$, (6.11) gives

$$\left| \left(\mathbf{v}^{(N)} \right)_j \right| \leq C\rho^{j-1}, 1 \leq j \leq N, \ or \ \left| \left(\mathbf{v}^{(N)} \right)_j \right| \leq C\rho^{N-j}, 1 \leq j \leq N. \tag{6.14}$$

First, for $\mathbf{v}^N$ satisfying

$$\left| \left(\mathbf{v}^{(N)} \right)_j \right| \leq C\rho^{j-1}, \quad 1 \leq j \leq N, \tag{6.15}$$

we calculate $\left\| L\mathbf{v}^{(N)} \right\|_2$. Note that the entries of $L\mathbf{v}^{(N)}$ are given by

$$(L\mathbf{v}^{(N)})_j = \begin{cases} 0, & j = 1, \ldots, k, \\ \displaystyle\sum_{q=k+1}^{j} (\mathcal{C}_N^\gamma)_{j,q-k} \left(\mathbf{v}^{(N)} \right)_{q-k}, & j = k+1, \ldots, N. \end{cases} \tag{6.16}$$

Define

$$s := \min_{p \geq k} \min_{q=1,2,\cdots} \left(\frac{p}{p+q} \right)^{\frac{1}{q}}. \tag{6.17}$$

It is not hard to see that

$$1 - \varepsilon_1 < \frac{k}{k+1} \leq s \leq 1. \tag{6.18}$$

By (6.13) and (6.15), we have for $j = k+1, \ldots, N$,

$$\left| (L\mathbf{v}^{(N)})_j \right| = \left| \sum_{q=k+1}^{j} (\mathcal{C}_N^\gamma)_{j,q-k} \left(\mathbf{v}^{(N)} \right)_{q-k} \right| \leq \sum_{q=k+1}^{j} \left| (\mathcal{C}_N^\gamma)_{j,q-k} \right| \left| \left(\mathbf{v}^{(N)} \right)_{q-k} \right|$$

$$\leq \delta KC \sum_{q=k+1}^{j} \frac{1}{j-q+k} \rho^{q-k-1} = \delta KC \sum_{q=k+1}^{j} \frac{1}{q-1} \rho^{j-q}$$

$$= \delta KC \sum_{q=k+1}^{j} \frac{1}{q-1} s^{j-q} (\rho/s)^{j-q}$$

$$= \frac{\delta KC}{j-1} \left(\sum_{q=k+1}^{j} \frac{j-1}{q-1} s^{j-q} (\rho/s)^{j-q} \right)$$

$$= \frac{\delta KC}{j-1} \left(\sum_{q=k+1}^{j} \frac{q-1+j-q}{q-1} s^{j-q} (\rho/s)^{j-q} \right)$$

$$\leq \frac{\delta KC}{j-1} \left(\sum_{q=k+1}^{j} (\rho/s)^{j-q} \right) \quad \text{(by (6.17))}$$

$$\leq \frac{\delta KC}{j-1} \sum_{q=0}^{\infty} \left(\frac{\rho}{1-\varepsilon_1} \right)^q \quad \text{(by (6.18))}$$

$$= \frac{\delta KC}{j-1} \frac{1-\varepsilon_1}{1-\varepsilon_1-\rho}.$$

Thus, combining the above estimates, we obtain that

$$\left\| L\mathbf{v}^{(N)} \right\|_2 \leq \delta KC \frac{1-\varepsilon_1}{1-\varepsilon_1-\rho} \sqrt{\sum_{j=k+1}^{N} \frac{1}{(j-1)^2}} \leq \delta KC \frac{1-\varepsilon_1}{1-\varepsilon_1-\rho} \sqrt{\varepsilon_1}, \tag{6.19}$$

where the last inequality is from the condition on k in the theorem.

Next, for $\mathbf{v}^{(N)}$ satisfying

$$\left| \left(\mathbf{v}^{(N)} \right)_j \right| \leq C \rho^{N-j}, \quad 1 \leq j \leq N, \tag{6.20}$$

we estimate $\left\| L\mathbf{v}^{(N)} \right\|_2$. The entries of $L\mathbf{v}^{(N)}$ are given by (6.16). By (6.13) and (6.20), we similarly have, for $j = k+1, \ldots, N$, that

$$\left| (L\mathbf{v}^{(N)})_j \right| = \left| \sum_{q=k+1}^{j} (\mathcal{C}_N^\gamma)_{j,q-k} \left(\mathbf{v}^{(N)} \right)_{q-k} \right| \leq \sum_{q=k+1}^{j} \left| (\mathcal{C}_N^\gamma)_{j,q-k} \right| \left| \left(\mathbf{v}^{(N)} \right)_{q-k} \right|$$

$$\leq \delta KC \sum_{q=k+1}^{j} \frac{1}{j-q+k} \rho^{N-(q-k)} = \delta KC \sum_{q=k+1}^{j} \frac{1}{q-1} \rho^{N-1-(j-q)}.$$

Further estimating, we have

$$\left| (L\mathbf{v}^{(N)})_j \right| \le \delta KC \sum_{q=k+1}^{j} \frac{1}{q-1} \rho^{N-1-(j-q)}$$

$$\le \frac{\delta KC}{k} \rho^{N-1-j+(k+1)} \left(\sum_{q=k+1}^{j} \frac{k}{q-1} \rho^{q-(k+1)} \right)$$

$$\le \frac{\delta KC}{k} \rho^{N-1-j+(k+1)} \left(\sum_{q=k+1}^{j} \rho^{q-(k+1)} \right)$$

$$\le \frac{\delta KC}{k} \rho^{N-1-j+(k+1)} \frac{1}{1-\rho}.$$

Therefore, combining the above estimates yields

$$\left\| L\mathbf{v}^{(N)} \right\|_2 \le \frac{\delta KC}{k} \frac{1}{1-\rho} \sqrt{\sum_{j=k+1}^{N} \rho^{2(N-1-j+(k+1))}}$$

$$\le \frac{\delta KC}{k} \rho^k \frac{1}{1-\rho} \sqrt{\frac{1}{1-\rho^2}} \tag{6.21}$$

$$\le \delta KC \varepsilon_1 \rho^k \frac{1}{1-\rho} \sqrt{\frac{1}{1-\rho^2}},$$

where the last inequality is from the condition on k in the theorem.

In the same fashion, we can prove that for $\mathbf{v}^{(N)}$ satisfying (6.15), we have

$$\left\| R\mathbf{v}^{(N)} \right\|_2 \le \delta KC \varepsilon_1 \rho^k \frac{1}{1-\rho} \sqrt{\frac{1}{1-\rho^2}},$$

and for $\mathbf{v}^{(N)}$ satisfying (6.20), we have

$$\left\| R\mathbf{v}^{(N)} \right\|_2 \le \delta KC \frac{1-\varepsilon_1}{1-\varepsilon_1-\rho} \sqrt{\varepsilon_1}.$$

Therefore, we arrive at

$$\left\| (\mathcal{C}_N^\gamma - \lambda I)\mathbf{v}^{(N)} \right\|_2 \le \left\| \left(\mathcal{C}_{N,k}^\gamma - \lambda I \right) \mathbf{v}^{(N)} \right\|_2 + \left\| L\mathbf{v}^{(N)} \right\|_2 + \left\| R\mathbf{v}^{(N)} \right\|_2$$

$$\le \varepsilon_2 + \delta KC \left(\frac{1-\varepsilon_1}{1-\varepsilon_1-\rho} \sqrt{\varepsilon_1} + \varepsilon_1 \rho^k \frac{1}{1-\rho} \sqrt{\frac{1}{1-\rho^2}} \right),$$

which completes the proof. ∎

Theorem 6.2 elucidates that, for sufficiently large $k > 0$, an exponentially decaying pseudo-eigenvector of the k-banded matrix $\mathcal{C}_{N,k}^\gamma$ is also a pseudo-eigenvector of the gauge capacitance matrix $\mathcal{C}_N^\gamma$. Theorems 6.1 and 6.2 indicate that the pseudo-eigenvectors/eigenvectors of the gauge capacitance matrix are exponentially decaying when $I(f(\mathbb{T}), \lambda)$ is not zero with f being the symbol associated with a sufficiently large k-banded submatrix. In particular, together with the numerical results in Figures 6.1 and 6.2, it is shown that most of the eigenmodes of $\mathcal{C}_N^\gamma$ have the exponential decay property, as most of the eigenvalues of $\mathcal{C}_N^\gamma$ lies inside $f(\mathbb{T})$. This verifies the non-Hermitian skin effect of the resonating system (2.1).

Figure 6.3 shows the first 20 eigenvectors $\mathbf{v}_j$'s of the gauge capacitance matrix $\mathcal{C}_N^\gamma$ (black line) and the corresponding 20 pseudo-eigenvectors $\mathbf{v}_j(\varepsilon)$'s of the k-banded matrix $\mathcal{C}_{N,k}^\gamma$ for $k = 10$ (blue line). We observe that, as predicted by Theorem 6.2, pseudo-eigenvectors (and hence, eigenvectors) with stronger exponential decay can be better approximated by corresponding pseudo-eigenvectors of the k-banded matrix.

In Figure 6.4, we compare the eigenvectors of the tridiagonal capacitance matrix $\mathcal{C}_{N,2}^\gamma$ and those of $\mathcal{C}_N^\gamma$. It is first shown that although the nearest-neighbour approximation is insufficient
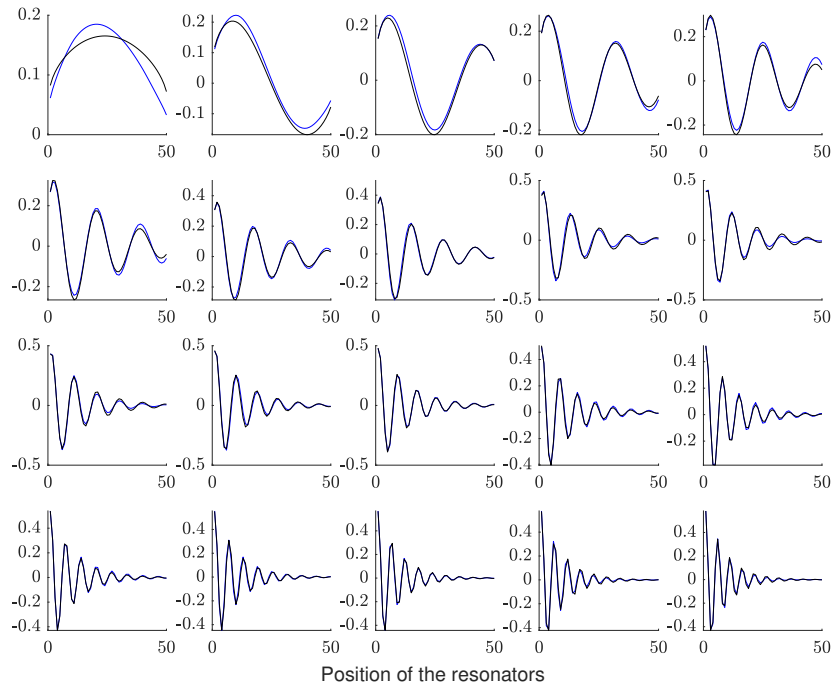
FIGURE 6.3. The first 20 eigenmodes of the gauge capacitance matrix $\mathcal{C}_N^\gamma$ are in black. And the blue modes are the pseudo-eigenmodes of the 10-banded matrix $\mathcal{C}_{N,k}^\gamma$. The eigenmodes are normalized.

to approximate all the exponentially decaying eigenvectors of $\mathcal{C}_N^\gamma$, it does approximate the ones decaying fast enough. Secondly, we observe that, for the first several modes, due to long-range interactions, the non-Hermitian skin effect in three-dimensional systems of resonators is less pronounced than that predicted by the nearest-neighbour approximation. Thus long-range interactions mainly affect the first several eigenmodes in systems of subwavelength resonators.

7. Numerical illustrations of the non-Hermitian skin effect

In this section, we provide a variety of further numerical illustrations of the skin effect. We begin by studying the stabiltiy of the skin effect in the presence of disorder. We also compute the skin effect in two-dimensional lattice structures.

7.1. Numerical simulations of the non-Hermitian skin effect in chains of subwavelength resonators

In Figure 7.1, we consider a chain of 100 identical equidistant spherical resonators in one line aligned with the x_1-axis. We plot all the eigenmodes in one graph and demonstrate the condensation of the eigenvectors at one edge of the chain. The condensation becomes more pronounced with increasing γ. Furthermore, changing the sign of γ changes the direction of condensation.

We say that an eigenmode is condensated if the ratio between the norm of its restriction to the first 20% resonators and its entire norm is greater than 80%. We then count the number of condensated eigenmodes and compute the ratio to N (the total number of eigenmodes). In Figure 7.2, we plot the portion of condensanted eigenmodes when increasing γ or N.
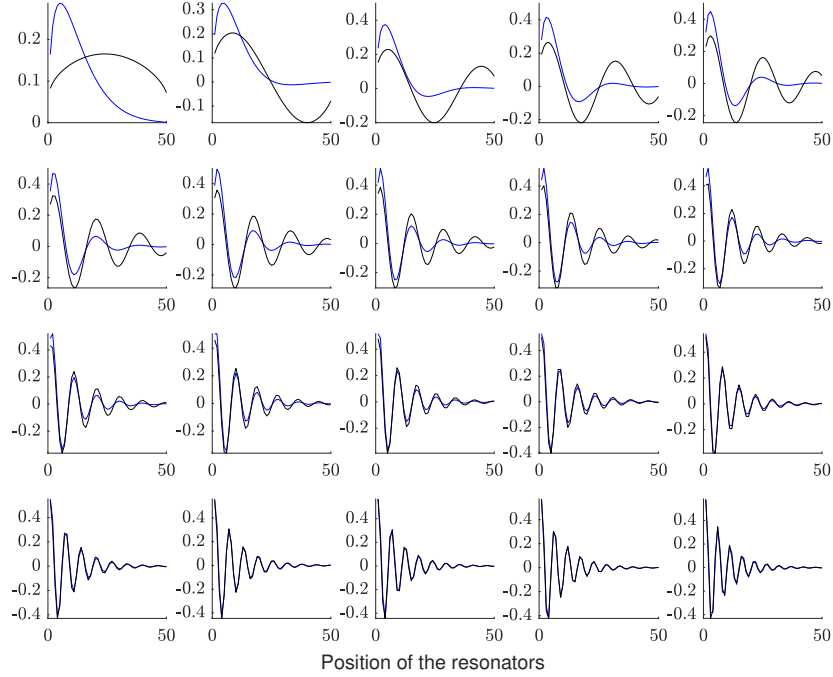
FIGURE 6.4. The first 20 modes of the tridiagonal part of the matrix (in blue) and the full matrix (in black).

(A) Eigenmodes for $\gamma = 0.5$. (B) Eigenmodes for $\gamma = 1$. (C) Eigenmodes for $\gamma = -1$.
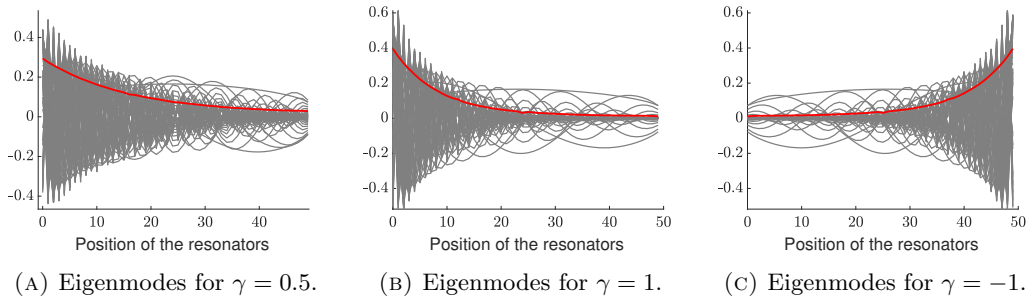
FIGURE 7.1. We plot all the eigenmodes for different values of γ. The red line symbolises the average of the absolute value of the amplitudes.

To further quantify the non-Hermitian skin effect, we introduce the concept of degree of condensation.

Definition 7.1. *Consider a chain of N spherical resonators along the x_1-direction with centers at distinct x_1-coordinates $\{x_1{}^i\}_{i=1,\dots,n}$. For every eigenmode v, we define the degree of condensation as the vector $d_v \in \mathbb{R}^N$, where*

$$(d_v)_i = \frac{\|v|_{\{x \le x_1^i\}}\|_2}{\|v\|_2} \quad \text{for } i = 1, \dots, N. \tag{7.1}$$

Here, the vector $v|_{\{x \le x_1^i\}}$ has the same first i entries as v but all the others are set to 0.
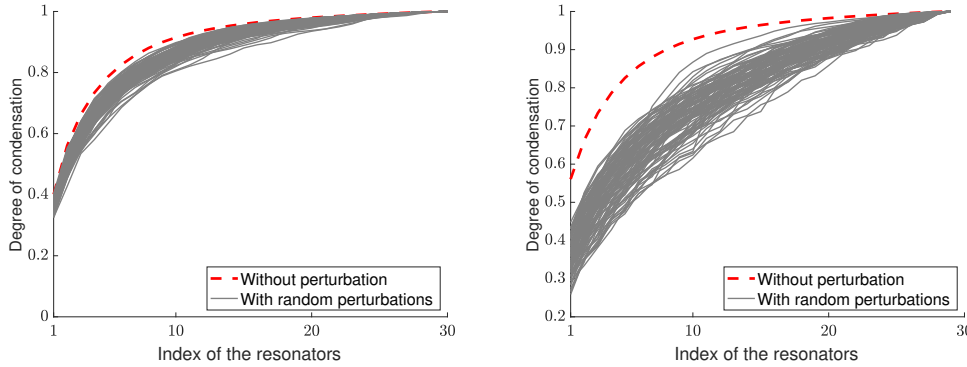
We now consider in Figure 7.3 the stability of the non-Hermitian skin effect in terms of the positions of the resonators. Define the uniformly distributed random variables $\varepsilon_i \sim \mathcal{U}_{[-\varepsilon,\varepsilon]}$

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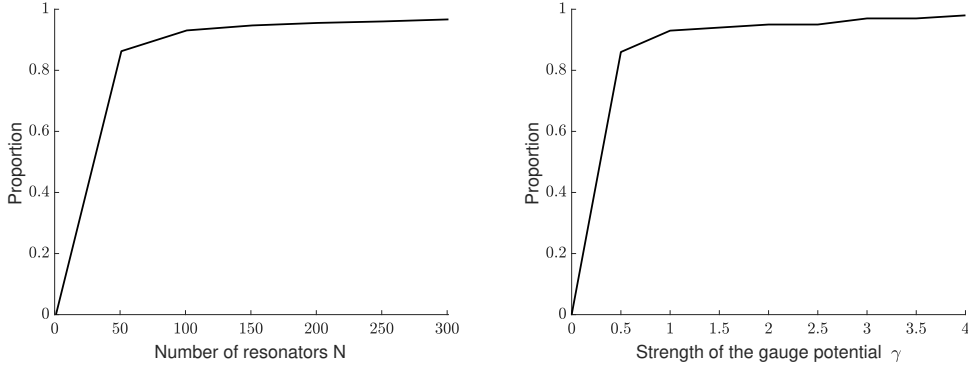
(A) Proportion of condensated eigenmodes with $\gamma = 1$ and increasing N.

(B) Proportion of condensated eigenmodes with $N = 100$ and increasing γ.

FIGURE 7.2. Proportions of condensated eigenmodes.

for $i = 1, \ldots, N$. We perturb the center of the resonators x_1^i to $x_1^i(1 + \varepsilon_i)$ and repeat the experiment 100 times while fixing $\gamma = 1$. For each perturbation, we compute the average degrees of condensation. The stability result is similar to the one in the one-dimensional case [2].

(A) Average degrees of condensation with $\varepsilon = 0.1$, computed over 100 realisations.

(B) Average degrees of condensation with $\varepsilon = 0.2$, computed over 100 realisations.

FIGURE 7.3. Average degrees of condensation for different strengths of geometric disorder. Grey lines are the randomly perturbed modes while the red line represents the unperturbed case.

Next, we consider in Figure 7.4 the stability of the non-Hermitian skin effect in terms of γ. Define the uniformly distributed random variables $\varepsilon_i \sim \mathcal{U}_{[-\varepsilon, \varepsilon]}$ for $i = 1, \ldots, N$. We perturb γ to $\gamma(1 + \varepsilon_i)$ in the ith resonator and compute the average degrees of condensation over 100 runs while fixing the equidistant resonator structure. Again, the stability result is similar to the one in the one-dimensional case.
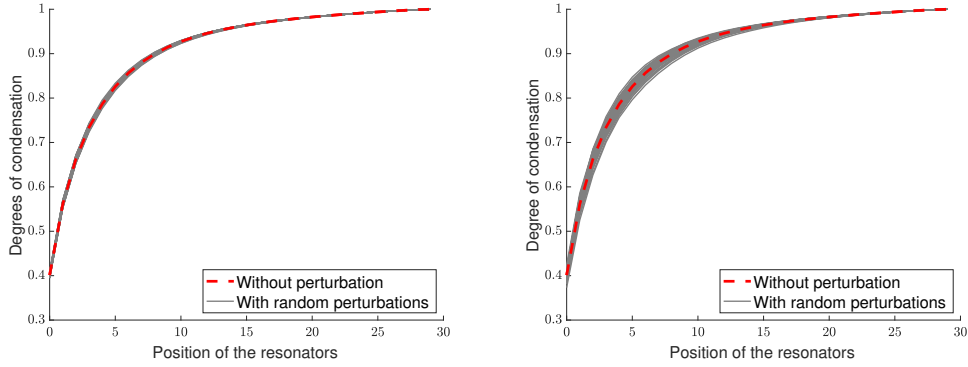
(A) Average degree of condensation with $\varepsilon = 0.1$, (B) Average degree of condensation with $\varepsilon = 0.2$, computed over 100 random realisations. computed over 100 realisations.

FIGURE 7.4. Average degree of condensation for different strengths of disorder in γ. Grey lines are the randomly perturbed modes while the red line represents the unperturbed case.

7.2. Numerical simulations of the non-Hermitian skin effect in other three-dimensional structures

Until now, we have only considered the non-Hermitian skin effect in chains of subwavelength resonators (i.e., structures with a one-dimensional lattice). In this section, we numerically demonstrate that the non-Hermitian skin effect also occurs in structures with higher dimension of the lattice. Without loss of generality, we can always, after translations and rotations, assume that the factor γ aligns with the x_1-axis. Hence, the mathematical model (2.1) still holds.

In Figure 7.5, we consider a rectangle structure with two chains of resonators. We observe that 90% of the eigenmodes are localised in the first 10 resonators.

(A) A rectangle structure with two lines and 100 resonators in each line.

(B) Degrees of condensation for all the eigenmodes of the rectangle for $\gamma = 1$. The red line represents the average amplitude.
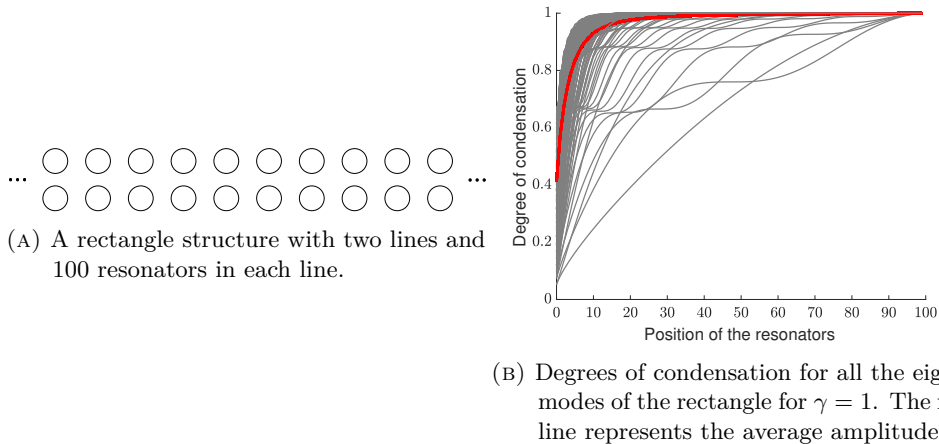
FIGURE 7.5. Numerical simulations of the non-Hermitian skin effect in a double chain system of resonators.

Finally, in Figure 7.6 we consider a rhombus structure with multiple lines. We observe that 90% of the eigenmodes are localised on the left 20% of the structure. The sudden jump of the degrees of condensation is due to the edge effects in each line.
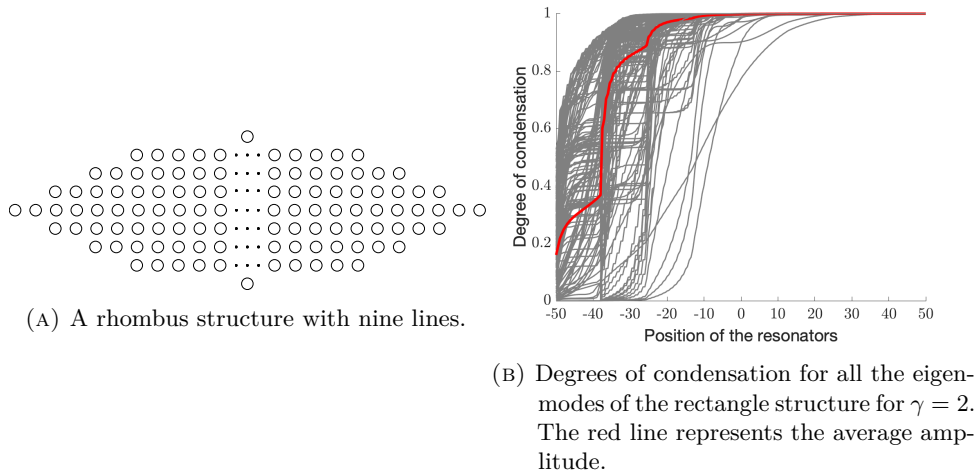
(A) A rhombus structure with nine lines.

(B) Degrees of condensation for all the eigenmodes of the rectangle structure for $\gamma = 2$. The red line represents the average amplitude.

FIGURE 7.6. Numerical simulations of the non-Hermitian skin effect in a rhombus of resonators.

8. Concluding remarks

In this paper, we have first introduced a discrete formulation for computing the subwavelength eigenfrequencies and eigenmodes of a system of subwavelength resonators with an imaginary gauge potential supported inside the resonators. This approximation is based on the so-called gauge capacitance matrix $\mathcal{C}_N^\gamma$. Unlike the one-dimensional case, due to long-range interactions in the system, the gauge capacitance matrix is dense. We have considered a range-k approximation to keep the long-range interactions to a certain extent, thus obtaining a k-banded gauge capacitance matrix $\mathcal{C}_{N,k}^\gamma$. By proving exponential decay of the pseudo-vectors of the matrix through Toeplitz matrix theory, we have deduced the condensation of the eigenmodes at one edge of the structure. We have illustrated this non-Hermitian skin effect in a variety of examples and illustrated its stability with respect to imperfections in the system. Our results give mathematical foundations of the skin effect in non-Hermitian systems with long-range coupling in three dimensions.

A challenging further problem is to generalise these results to dimer structures as we did in the one-dimensional case in [4]. Another challenging problem is to prove that when the strength of the disorder increases, there is a competition between the non-Hermitian skin effect and Anderson localisation in the bulk as recently shown numerically in the one-dimensional case in [3]. In connection with this, it would be important to prove that all the eigenvalues of the gauge capacitance matrix are real (as illustrated in Figures 6.1 and 6.2) and random perturbations of the positions of the resonators or/and the parameter γ inside the resonators leave them on the real axis but make some of them jump outside the region with non-trivial winding numbers.

Appendix A. Spectra and pseudo-spectra of Toeplitz matrices and operators

In this section, we first recall some basic results on the spectra and pseudo-spectra of Toeplitz matrices, Toeplitz operators, and Laurent operators. Then we characterise the pseudo-spectra of banded Toeplitz matrices with perturbations. This characterisation is useful for studying the gauge capacitance matrix defined in (4.11).

A.1. Spectra and pseudo-spectra of Toeplitz matrices

An $N \times N$ Toeplitz matrix is a matrix whose entries are constant along diagonals:

$$\mathbf{A}_N = \begin{pmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_{-1} & a_0 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & a_0 & a_1 \\ a_{-(N-1)} & \cdots & a_{-1} & a_0 \end{pmatrix}. \tag{A.1}$$

A semi-infinite matrix of the same form is known as a Toeplitz operator, and a doubly infinite matrix of this kind is a Laurent operator and we denote them by $\mathbf{A}$. The symbol of a Toeplitz matrix, Toeplitz operator, or Laurent operator is the function

$$f(z) = \sum_k a_k z^k$$

with $\sum_k |a_k| < +\infty$. This symbol is important for deriving many properties of the Toeplitz matrices, Toeplitz operators, and Laurent operators. For example, it fully determines the spectrum Λ of a Toeplitz or a Laurent operator $\mathbf{A}$. Define $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $I(f(\mathbb{T}), \lambda)$ as the winding number of $f(\mathbb{T})$ about λ in the usual positive (counterclockwise) sense. We have the following result.

Theorem A.1. *Let $\mathbf{A}$ and f be as described above. The following holds:*
(i) If $\mathbf{A}$ is a Laurent operator, then $\Lambda(\mathbf{A}) = f(\mathbb{T})$;
(ii) If $\mathbf{A}$ is a Toeplitz operator, then $\Lambda(\mathbf{A}) = f(\mathbb{T}) \cup \{\lambda \in \mathbb{C} : I(f(\mathbb{T}), \lambda) \neq 0\}$.

Note that the winding number $I(f(\mathbb{T}), \lambda)$ or index of the continuous curve $f(\mathbb{T})$ about the point $\lambda \in \mathbb{C}$ is also given by

$$I(f(\mathbb{T}), \lambda) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f'(z)}{f(z) - \lambda} dz, \quad \lambda \notin f(\mathbb{T}).$$

Consequently, if $\lambda \in f(\mathbb{T})$, then $I(f(\mathbb{T}), \lambda)$ is undefined.

A specific property of the Toeplitz operator $\mathbf{A}$ is that, for the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue λ with $I(f(\mathbb{T}), \lambda) \neq 0$, the amplitude of its entries a_j's decreases or increases as $j \to +\infty$. As stated in [28, Eq. (3.4)]:

If $I(f(S), \lambda) > 0$, then the right eigenvectors of $\mathbf{A}$ are geometrically decreasing;

If $I(f(S), \lambda) < 0$, then the right eigenvectors of $\mathbf{A}$ are geometrically increasing.

This is known to be the topological origin of the skin effect in non-Hermitian physical systems with complex gauge transforms [2].

For characterising the spectrum of Toeplitz matrices, no results similar to those in Theorem A.1 are available in the literature. Moreover, a detailed characterisation of general Toeplitz matrices seems to be out of reach. The only known results are asymptotic characterisations of eigenpairs of some Hermitian Toeplitz matrices and Toeplitz matrices with specific symbols; see [16] for a survey in this regard. We also refer the reader to some representations of eigenpairs of tridiagonal Toeplitz matrices [17, 21, 34], which led to our recent demonstration of the skin effect for finitely periodic systems of subwavelength resonators with complex gauge transformations in the one-dimensional case [2, 4].

On the other hand, there are some results for the pseudo-spectra of banded and semi-banded Toeplitz matrices. In order to state them, we introduce the concept of pseudo-eigenvalue.

Definition A.1. *Given $\varepsilon > 0$, the number $\lambda \in \mathbb{C}$ is a pseudo-eigenvalue of a matrix $\mathbf{A}$ if any of the following equivalent conditions is satisfied:*

(i) λ is an eigenvalue of $\mathbf{A} + \mathbf{E}$ for some $\mathbf{E} \in \mathbb{C}^{N \times N}$ with $||\mathbf{E}|| \leqslant \varepsilon$;
(ii) $\exists\, u \in \mathbb{C}^N, ||u|| = 1$, such that $||(\mathbf{A} - \lambda I)u|| \leqslant \varepsilon$;
(iii) $||(\lambda I - \mathbf{A})^{-1}|| \geqslant \varepsilon^{-1}$;
(iv) $\sigma_{\min}(\lambda I - \mathbf{A}) \leqslant \varepsilon$, where $\sigma_{\min}(\lambda I - \mathbf{A})$ is the smallest singular value of $\lambda I - \mathbf{A}$.

Given $\varepsilon > 0$, the set of all pseudo-eigenvalues of $\mathbf{A}$, the pseudo-spectrum, is denoted by $\Lambda_\varepsilon(\mathbf{A})$ or simply Λ_ε.

An l-banded Toeplitz matrix is defined as $\mathbf{A}_N$ in (A.1) with $a_j = 0$ for $|j| > l$ and is denoted by $\mathbf{A}_{N,l}$. To state the result on its pseudo-spectrum, we need to introduce the sets Ω_r and Ω^R defined by

$$\Omega_r = \{z \in \mathbb{C} : I\left(f\left(\mathbb{T}_r\right), z\right) > 0\}, \quad \Omega^R = \{z \in \mathbb{C} : I\left(f\left(\mathbb{T}_R\right), z\right) < 0\}, \qquad \text{(A.2)}$$

where $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$. The following theorem ([28, Theorem 3.2]) characterises the pseudo-eigenvalues of an l-banded Toeplitz matrix.

Theorem A.2. *Let $\mathbf{A}$ be a banded Toeplitz operator with bandwidth l, i.e., $a_k = 0$ for $|k| > l$. Let $f(z)$ be the symbol of $\mathbf{A}$, and let $\mathbf{A}_N$ denote the $N \times N$ Toeplitz matrix defined by (A.1). Then, for any $r < 1$ and $\rho > r$, we have*

$$\Omega_r \cup \Omega^{1/r} \cup (\Lambda(\mathbf{A}) + \Delta_\varepsilon) \subseteq \Lambda_\varepsilon\left(\mathbf{A}_{N,l}\right) \quad \text{for} \quad \varepsilon = C\rho^N,$$

where $\Delta_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$ and C is a constant independent of N.

A.2. Pseudo-spectra of banded Toeplitz matrices with perturbations

Note that our gauge capacitance matrix is a Toeplitz matrix with perturbations on the non-zero elements. Here, we consider and analyse the pseudo-spectrum of an l-banded Toeplitz matrix with perturbations on the corner blocks. More precisely, we define the l-banded Toeplitz matrix with the first and last l rows perturbed as follows:

$$\widehat{\mathbf{A}}_{N,l} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \qquad \text{(A.3)}$$

where B_1, B_2, and B_3 are respectively of size $(l-1) \times N$, $(N-2l+2) \times N$, and $(l-1) \times N$ and are given by

$$\begin{pmatrix} a_0 + b_{0,1} & \cdots & a_{l-1} + b_{l-1,1} & 0 & \cdots & \cdots & \cdots & \cdots \\ a_{-1} + b_{-1,2} & \cdots & a_{l-2} + b_{l-2,2} & a_{l-1} + b_{l-1,2} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{2-l} + b_{2-l,l-1} & \cdots & a_0 + b_{0,l-1} & a_1 + b_{1,l-1} & \cdots & a_{l-1} + b_{l-1,l-1} & 0 & \cdots \end{pmatrix},$$

$$\begin{pmatrix} a_{-l+1} & a_{-l+2} & \cdots & a_{l-1} & 0 & \cdots & 0 \\ 0 & a_{-l+1} & a_{-l+2} & \cdots & a_{l-1} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{-l+1} & a_{-l+2} & \cdots & a_{l-1} \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & \cdots & 0 & a_{1-l} + b_{1-l,N-l+2} & \cdots & \cdots & \cdots & a_{l-3} + b_{l-3,N-l+2} & a_{l-2} + b_{l-2,N-l+2} \\ 0 & \cdots & 0 & 0 & \ddots & \ddots & \ddots & \cdots & a_{l-3} + + b_{l-3,N-l+3} \\ \vdots & \ddots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a_{1-l} + b_{1-l,N} & \cdots & a_0 + b_{0,N} \end{pmatrix}.$$

We generalise Theorem A.2 to the perturbed l-banded Toeplitz $\widehat{\mathbf{A}}_{N,l}$ as follows.

Theorem A.3. *Let $f(z)$ be the symbol defined by $\sum_{q=1-l}^{l-1} a_q z^q$, and let $\widehat{\mathbf{A}}_{N,l}$ be the $N \times N$ l-banded perturbed Toeplitz matrix in (A.3). Then, for any $r < 1$ and $\rho > r$, we have*

$$\Omega_r \cup \Omega^{1/r} \subseteq \Lambda_\varepsilon\left(\widehat{\mathbf{A}}_{N,l}\right) \quad for \quad \varepsilon = \max\left(C_1, C_2 N^{l-1}\right) \rho^{N-2l+2}, \tag{A.4}$$

where $\Omega_r, \Omega^{\frac{1}{r}}$ are defined as in (A.2) and C_1, C_2 are constants independent of N. Moreover, there exist non-zero pseudo-eigenvectors $\mathbf{v}^{(N)}$ satisfying

$$\frac{\left\|\left(\widehat{\mathbf{A}}_{N,l} - \lambda I\right) \mathbf{v}^{(N)}\right\|_2}{\left\|\mathbf{v}^{(N)}\right\|_2} \leq \max\left(C_1, C_2 N^{l-1}\right) \rho^{N-2l+2}, \tag{A.5}$$

such that

$$\frac{\left|\left(\mathbf{v}^{(N)}\right)_j\right|}{\max_j \left|\left(\mathbf{v}^{(N)}\right)_j\right|} \leq \begin{cases} C_3 \rho^{j-1}, & if \ \lambda \in \Omega_r, \\ C_3 \rho^{N-j}, & if \ \lambda \in \Omega^{\frac{1}{r}}, \end{cases} \quad 1 \leq j \leq N, \tag{A.6}$$

where C_3 is independent of N.

Proof. By symmetry, $\Omega^{1/r}$ must satisfy an estimate of type (A.4) if Ω_r does. Thus, we only need to prove that $\Omega_r \subseteq \Lambda_\varepsilon\left(\widehat{\mathbf{A}}_{N,l}\right)$.

The idea is to construct geometrically decreasing pseudo-eigenvectors. Given any $r < 1$, let $\lambda \in \Omega_r$ be arbitrary. Assume without loss of generality that $a_{-l+1} \neq 0$. The symbol is

$$f(z) = \sum_{q=-l+1}^{l-1} a_q z^q.$$

Then $f(z)$ has a pole of order exactly $l-1$ at $z = 0$. Since $I\left(f\left(\mathbb{T}_r\right), \lambda\right) \geqslant 1$ (by the definition of Ω_r), it follows by the argument principle that the equation $f(z) = \lambda$ has at least l solutions in $\{z \in \mathbb{C} : |z| < r\}$, counted with their multiplicities. Let $z_1, \ldots, z_l$ be any l of these solutions. Assume for the moment that the z_j's are distinct. Corresponding to each z_j is a vector $u_j = \left(1, z_j^1, z_j^2, \ldots, z_j^{(N-1)}\right)^T$. Since $\lambda = f(z_j) = \sum_{q=-l+1}^{l-1} a_q z_j^q$, we can compute that

$$\left(\left(\lambda I - \widehat{\mathbf{A}}_{N,l}\right)u_j\right)_q = 0, \quad l \leq q \leq N - l,$$

$$\left(\left(\lambda I - \widehat{\mathbf{A}}_{N,l}\right)u_j\right)_q = \sum_{k=-l+1}^{2-q} a_k z_j^k z_j^{q-1} + \sum_{k=1-q}^{l-1} -b_{k,q} z_j^k z_j^{q-1}, \quad 1 \leq q \leq l - 1,$$

and

$$\left(\left(\lambda I - \widehat{\mathbf{A}}_{N,l}\right)u_j\right)_q = \sum_{k=N-q+1}^{l-1} a_k z_j^k z_j^{q-1} + \sum_{k=1-l}^{N-q} -b_{k,q} z_j^k z_j^{q-1}, \quad N - l + 2 \leq q \leq N.$$

Thus, we have

$$\left(\lambda I - \widehat{\mathbf{A}}_{N,l}\right)u_j = z_j^{N-2l+2} L u_j + z_j^{-N+2l-2} R u_j, \tag{A.7}$$

where L is given by

$$\begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
-b_{1-l,\zeta} & \cdots & \cdots & \cdots & \cdots & -b_{l-2,\zeta} & a_{l-1} & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & -b_{1-l,N-1} & \cdots & -b_{0,N-1} & -b_{1,N-1} & a_2 & \cdots & a_{l-1} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & -b_{1-l,N} & \cdots & -b_{0,N} & a_1 & \cdots & \cdots & a_{l-1} & \cdots & 0
\end{pmatrix}$$

with $\zeta = N - l + 2$ and R is given by

$$
\begin{pmatrix}
0 & \cdots & a_{1-l} & \cdots & \cdots & a_{-1} & -b_{0,1} & -b_{1,1} & \cdots & -b_{l-1,1} & \cdots & \cdots & 0 \\
0 & \cdots & 0 & a_{1-l} & \cdots & a_{-2} & -b_{-1,2} & -b_{0,2} & -b_{1,2} & \cdots & -b_{l-1,2} & \cdots & 0 \\
0 & \cdots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \ddots & \ddots & a_{1-l} & -b_{2-l,l-1} & \cdots & \cdots & \cdots & \cdots & \cdots & -b_{l-1,l-1} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0
\end{pmatrix} .
$$

Since there are l linearly independent vectors $R_{1:l-1} z_j^{-N+2l-2} u_j$ of length $l-1$, we can find l complex numbers c_j so that

$$\sum_{j=1}^{l} c_j R_{1:l-1} z_j^{-N+2l-2} u_j = \mathbf{0}, \tag{A.8}$$

and let

$$\mathbf{v} = \sum_{j=1}^{l} c_j u_j.$$

Then $z_j^{-N+2l-2} R u_j = \mathbf{0}$ in (A.7) and we have

$$\left(\lambda I - \widehat{\mathbf{A}}_{N,l} \right) \mathbf{v} = L \sum_{j=0}^{l} z_j^{N-2l+2} c_j u_j. \tag{A.9}$$

We now need to relate the norm of the right-hand side of this equation to $\|\mathbf{v}\|_2$. To do this, we write $\mathbf{v} = Uc$, where U is the $N \times l$ Vandermonde matrix whose columns are the vectors u_j, and $c = (c_1, \ldots, c_l)^\top$. We let D be the diagonal matrix with elements $z_1^{N-2l+2}, \ldots, z_l^{N-2l+2}$. Then, we have

$$\|UDc\|_2 = \left\| UDU^\dagger Uc \right\|_2 = \left\| UDU^\dagger \mathbf{v} \right\|_2 \leqslant \kappa(U) \|D\|_2 \|\mathbf{v}\|_2 ,$$

where $U^\dagger$ denotes the pseudo-inverse of U and $\kappa(U) = \sigma_1/\sigma_l$ is its condition number. Since $\|D\|_2 \leqslant r^{N-2l+2}$, it follows that (A.9) implies

$$\frac{\left\| \left(\lambda I - \widehat{\mathbf{A}}_{N,l} \right) \mathbf{v} \right\|_2}{\|\mathbf{v}\|_2} \leqslant r^{N-2l+2} \kappa(U) \|L\|_2 . \tag{A.10}$$

This completes the proof under the assumption that the roots $z_1, \ldots, z_l$ are distinct. We can handle the case of multiple roots as follows. If some of the roots $z_0, \ldots, z_l$ are confluent at some points $\lambda \in \overline{\Omega}_r$, we then analyse the problem by confluent Vandermonde matrix. More precisely, assuming that the root z_j has multiplicity k, we have

$$f(z) - \lambda = P(z)(z - z_j)^k,$$

which yields

$$f^{(1)}(z_j) = 0, \ f^{(2)}(z_j) = 0, \ \ldots, \ f^{(k-1)}(z_j) = 0. \tag{A.11}$$

Now, we consider $g(z) = f(z)z^s, s = l-1, l, l+1, \ldots$ By (A.11), we have

$$g^{(1)}(z_j) = f^{(1)}(z_j)z_j^s + f(z_j)s z_j^{s-1} = f(z_j)s z_j^{s-1} = \lambda s z_j^{s-1}$$

$$g^{(2)}(z_j) = \cdots = \lambda s(s-1) z_j^{s-2}$$

$$\vdots$$

$$g^{(k-1)}(z_j) = \cdots = \lambda s(s-1) \ldots (s-k+2) z_j^{s-(k-1)}.$$

Since $f(z)z^s = \sum_{q=-l+1}^{l-1} a_q z^{q+s}$, for $s \geq l-1$, the above equalities give

$$\lambda s z_j^{s-1} = \sum_{q=-l+1}^{l-1} a_q (q+s) z_j^{q+s-1}$$

$$\lambda s(s-1) z_j^{s-2} = \sum_{q=-l+1}^{l-1} a_q (q+s)(q+s-1) z_j^{q+s-2}$$

$$\vdots$$

$$\lambda s(s-1)\ldots(s-k+2) z_j^{s-(k-1)} = \sum_{q=-l+1}^{l-1} a_q (q+s)(q+s-1)\ldots(q+s-k+2) z_j^{q+s-(k-1)}.$$

This implies that we can use

$$\begin{pmatrix} 1 \\ z_j \\ z_j^2 \\ \vdots \\ \vdots \\ z_j^{(N-1)} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2z_j \\ 3z_j^2 \\ \vdots \\ \vdots \\ (N-1)z_j^{(N-2)} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (k-1)! \\ k(k-1)\cdots 2z_j \\ \vdots \\ (N-1)\cdots(N-(k-1))z_j^{N-k-2} \end{pmatrix},$$

and other vectors corresponding to other roots z_j's to construct the pseudo-eigenvector. Instead, we consider to use

$$\begin{pmatrix} 1 \\ z_j \\ z_j^2 \\ \vdots \\ \vdots \\ z_j^{(N-1)} \end{pmatrix}, \begin{pmatrix} 0 \\ z_j \\ 2z_j^2 \\ 3z_j^3 \\ \vdots \\ \vdots \\ (N-1)z_j^{(N-1)} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (k-1)!z_j^{k-1} \\ k(k-1)\cdots 2z_j^k \\ \vdots \\ (N-1)\cdots(N-(k-1))z_j^{N-1} \end{pmatrix} \tag{A.12}$$

to construct the pseudo-eigenvector.

The above discussion shows that by considering the confluent Vandermonde matrix, we can always find such l linearly independent vectors like (A.12) to construct the pseudo-eigenvector. Then an analysis involving confluent Vandermonde matrices yields a bound analogous to (A.10) except with r^{N-2l+2} replaced by an algebraically growing factor at worst as $N^l r^{N-2l+2}$. This proves (A.5).

Now, we prove (A.6). As the pseudo-eigenvector is constructed by $\mathbf{v} = \sum_{j=1}^{l} c_j u_j$ with c_j's satisfying (A.8) and u_j's from (A.12), by the form of vectors in (A.12), we obtain

$$\frac{|\mathbf{v}_j|}{\max_j |\mathbf{v}_j|} \leq C_3 \rho^{j-1}, \ 1 \leq j \leq N, \text{ if } \lambda \in \Omega_r,$$

with C_3 being independent of N.

$\blacksquare$

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