THE GEVREY CLASS IMPLICIT MAPPING THEOREM WITH APPLICATION TO UQ OF SEMILINEAR ELLIPTIC PDES

H. Harbrecht and M. Schmidlin and Ch. Schwab

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Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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HELMUT HARBRECHT, MARC SCHMIDLIN, AND CHRISTOPH SCHWAB

Abstract. This article is concerned with a regularity analysis of parametric operator equations with a perspective on uncertainty quantification. We study the regularity of mappings between Banach spaces near branches of isolated solutions that are implicitly defined by a residual equation. Under s-Gevrey assumptions on on the residual equation, we establish s-Gevrey bounds on the Fréchet derivatives of the local data-to-solution mapping. This abstract framework is illustrated in a proof of regularity bounds for a semilinear elliptic partial differential equation with parametric and random field input.

Keywords: Implicit mappings, parametric regularity, uncertainty quantification, semilinear elliptic PDEs

MSC: 35B30, 35J61, 47J07

1. Introduction

The numerical approximation of quantities of interest such as the expectation or the variance of an output functional that depends on the solution of a partial differential equation (PDE) with random input parameters is a well-established field of research. Generally, the random data is one or more coefficient appearing in the PDE or, for example by use of the domain mapping approach [42], the domain on which the PDE is posed. A common approach is to consider the random coefficients or domain mapping to be given by an affine parametric expansion depending on countably many random variables, see [6, 30, 33, 42] for example. Indeed, such a representation can be achieved for example via the Karhunen-Loève expansion of a random vector field. This then gives rise to parametric, deterministic PDEs, where the parameters are precisely the random variables, which implies that computing the expectation or variance of an output functional of the parametric, deterministic PDE may be done by evaluating countably dimensional integrals.

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To be able to truncate the dimension of the integral and to choose appropriate quadrature rules in order to approximately evaluate such integrals, it is necessary to analyse the regularity of the integrand with respect to the parameters. Different amounts of smoothness then justify the use of Monte Carlo quadrature, quasi-Monte Carlo quadratures including higher order versions as well as anisotropic sparse-grid based quadrature methods, see e.g. [10, 18, 25, 35, 43]. For example, the analytic dependence on the parameters has been shown for the second-order diffusion equation, for linear elasticity and for Stokes and Navier-Stokes equations as well as for a class of semilinear elliptic problems with either a random diffusion coefficients or on random domains in works such as [9, 11, 12, 27, 28, 30]. Moreover, the regularity of the dependence on the parameters for elliptic partial differential eigenvalue problems (EVP) has been considered in e.g. [2, 8, 21, 23, 26].

In essence, there are two proof strategies that are used in the works that consider deterministic quadrature methods: the *holomorphy* and the *real-variable inductive* arguments. While the holomorphy argument has been successfully used to show the analytic dependence on the parameters for linear PDEs as well as EVPs and nonlinear PDEs, the real-variable inductive argument only has been able to show the analytic dependence on the parameters for linear PDEs. Specifically, for the case of EVPs, the real-variable inductive argument only has been able to show suboptimal Gevrey class non-analytic dependence and, recently, it was shown in [8] that this suboptimality is an artefact of the proof strategy itself. This artefact indeed also implies that the real-variable inductive argument will at best allow to prove suboptimal Gevrey class non-analytic dependence when considering nonlinear PDEs. Indeed, in [8] the authors developed their so-named *alternative-to-factorial technique* to circumvent the deficiency of the real-variable inductive argument and prove the optimal analytic dependence on the parameters for a class of elliptic EVPs.

1.1. **Layout.** The structure of this article is as follows. We first establish an abstract framework that concerns the regularity of mappings between Banach spaces in Section 2. The main result we establish there is Theorem 6. It provides bounds on the Fréchet derivatives of a local implicit mapping defined by a residual equation, showing that the local implicit mapping can inherit the s-Gevrey smoothness of the residual equation. In other words, we establish the *s-Gevrey class implicit mapping theorem* that generalises the holomorphic implicit mapping theorem and is not only of qualitative but also of quantitative nature. Moreover, we also provide both, the qualitative and quantitative behaviour, of s-Gevrey class mappings between Banach spaces under composition in Theorems 7 and 8.

In Section 3, a model semilinear elliptic PDE is reformulated as a residual equation that relates the data, which we wish to consider as random, to the solution. By means of Theorem 6 we show that the solution depends analytically on the data. Additionally, if the input uncertainty is formulated mathematically as a random field realization of data in an s-Gevrey smooth residual equation, Theorem 8 implies the s-Gevrey smooth parametric dependence of the solution on the random input parameters.

We demonstrate then in Section 4 how the s-Gevrey smooth dependence of the solution of the model semilinear elliptic PDE on the data can also be shown accounting for possible higher spatial regulearity. Specifically, by several concrete choices of our abstract function space setting we show that the solution in H^2 and Kondrat'ev type spaces depends s-Gevrey smooth on the data. Further possible extensions and concluding remarks are stated in Section 5.

1.2. **Contributions.** The main contributions of this article are as follows:

(1) We establish the s-Gevrey class implicit mapping theorem with a proof that is achieved by means of a novel modification of the real-variable inductive argument that is distinct from the alternative-to-factorial approach in [8].
(2) We prove a result on s-Gevrey regularity of composition maps which is of independent interest.
(3) We demonstrate, for a model, semilinear elliptic PDE, that the presently developed, abstract framework directly implies novel parametric regularity results, without use of the real-variable inductive argument.

Specifically, our aim for this article is to introduce this approach, where one considers the regularity of the data-to-solution mapping and the regularity of the parameters-to-data mapping separately. We additionally note that, as the approach is based on an inductive argument, it can also be used when one has only finite smoothness.

1.3. **Notation.** We use standard multi-index notation: we denote the natural numbers including 0 by $\mathbb{N}$ and excluding 0 by $\mathbb{N}^*$. Moreover, for a sequence of natural numbers, $\boldsymbol{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}^*} \in \mathbb{N}^{\mathbb{N}^*}$, we as usual define the support of the sequence as

$$\operatorname{supp} \boldsymbol{\alpha} = \{n \in \mathbb{N}^* \mid \alpha_n \neq 0\}.$$

If $\operatorname{supp} \boldsymbol{\alpha}$ is of finite cardinality, we say that $\boldsymbol{\alpha}$ is finitely supported. The set of finitely supported sequences of natural numbers is then denoted by $\mathbb{N}_f^{\mathbb{N}^*}$, while we will refer to its elements as multi-indices. For multi-indices $\boldsymbol{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}^*}, \boldsymbol{\beta} = \{\beta_n\}_{n\in\mathbb{N}^*} \in \mathbb{N}_f^{\mathbb{N}^*}$ and a sequence of real numbers $\boldsymbol{\gamma} = \{\gamma_n\}_{n\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}$, we use the following common notation:

$$|\boldsymbol{\alpha}| := \sum_{n\in\operatorname{supp}\boldsymbol{\alpha}} \alpha_n, \qquad \boldsymbol{\alpha}! := \prod_{n\in\operatorname{supp}\boldsymbol{\alpha}} \alpha_n!,$$

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \prod_{n\in\operatorname{supp}\boldsymbol{\alpha}\cup\operatorname{supp}\boldsymbol{\beta}} \binom{\alpha_n}{\beta_n}, \qquad \boldsymbol{\gamma}^{\boldsymbol{\alpha}} := \prod_{n\in\operatorname{supp}\boldsymbol{\alpha}} \gamma_n^{\alpha_n}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, then we denote the Banach space of bounded, linear maps from $\mathcal{X}$ to $\mathcal{Y}$ as $\mathcal{B}(\mathcal{X};\mathcal{Y})$ and the space of bounded, n-linear maps from $\mathcal{X}$ to $\mathcal{Y}$ as $\mathcal{B}^n(\mathcal{X};\mathcal{Y})$. Moreover, on the product Banach space $\mathcal{X}\times\mathcal{Y}$, we use the norm

$$\big\|(x,y)\big\|_{\mathcal{X}\times\mathcal{Y}} := \max\big\{\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}\big\}.$$

In a Sobolev space on a domain G, the bracket $\langle\cdot,\cdot\rangle_G$ denotes the duality pairing extending the scalar product of the Hilbert space $L^2(G)$ by continuity.

The symbol D shall denote the Fréchet derivative, and for integer $k>0$, D^k the corresponding multilinear Fréchet derivative of order k. Equipped with a subscript, e.g. D_2, it shall denote a partial Fréchet derivative.

For a finite set A, $|A|$ shall denote the number of elements $a\in A$.

2. Regularity of implicit and composite mappings

We introduce an abstract form of nonlinear, implicit operator equation. We recap assumptions of Gevrey-regularity of the residual map R, and conditions for the validity of the implicit function theorem, ensuring the existence of a continuous data-to-solution map for the residual equation. We then state and prove the main result of this article: quantitative bounds on Fréchet derivatives of the data-to-solution map under corresponding Gevrey-regularity hypotheses of the dependence of R on the data. Additionally, we also provide quantitative bounds on the Fréchet derivatives of the composition of mappings of Gevrey-regularity.

2.1. **Residual equation.** In the rest of this section, let $\mathcal{D}$, $\mathcal{U}$ and $\mathcal{R}$ be real Banach spaces, $D\subset\mathcal{D}$ and $U\subset\mathcal{D}$ open sets and $R\colon D\times U\to\mathcal{R}$ a mapping. As in [9], the idea then is that the *residual equation*: given data $d\in D$, find $u\in U$ such that

$$R(d,u) = 0 \quad \text{in } \mathcal{R} \tag{2.1}$$

can be considered to be the general, abstract operator equation of interest. It could constitute, for example, a suitable weak form of a partial differential equation (PDE) or boundary integral equation (BIE). We are looking for suitable solutions u of (2.1) as functions of the input data $d\in\mathcal{D}$. In the domain of "Uncertainty Quantification" one wishes to quantify *uncertainty propagation*, i.e., quantitative bounds on the dependence of the solution u of (2.1) on the data d.

Specifically, we consider that d encodes uncertain input data of the PDE, which uncertainty one models as random by assuming (or computing from assimilated data) at hand a probability measure on $\mathcal{D}$ (equipped with its Borel sigma-algebra) charging D. In this setup, u signifies the unknown solution, which is a solution of (2.1) for some d, i.e. precisely when $R(d,u) = 0$, under conditions on the residual mapping R to ensure unique solvability of the residual equation (2.1) *locally*, i.e. in a neighbourhood of nominal data $d^\star$ and a neighbourhood of an associated nominal solution $u^\star$.

Let us assume that $R \in C^n(D \times U; \mathcal{R})$ for some $n \in \mathbb{N}^*$, i.e. that R is n-times continuously Fréchet differentiable on $D \times U$, that there are $d^\star \in D$ and $u^\star \in U$, which fulfil the residual equation, i.e.

$$R(d^\star, u^\star) = 0,$$

and that $\mathrm{D}_2\, R(d^\star, u^\star) \in \mathcal{B}(\mathcal{U}; \mathcal{R})$ exists and is a Banach space isomorphism. Then, by the implicit mapping theorem for Banach spaces, see e.g. [37, Chapter XIV, Theorem 2.1], there is an open neighbourhood of $d^\star$, $D^\star \subset D$, and a unique, continuous mapping $S^\star \colon D^\star \to U$ with $S^\star(d^\star) = u^\star$ such that

$$R\big(d, S^\star(d)\big) = 0 \text{ for all } d \in D^\star.$$

Furthermore, this local solution mapping $S^\star$ also inherits regularity from the residual mapping, that is $S^\star \in C^n(D^\star; \mathcal{U})$.

Naturally, if $R \in C^\infty(D \times U; \mathcal{R})$, then this implies that the local solution mapping also fulfils $S^\star \in C^\infty(D^\star; \mathcal{U})$. If in addition R is real analytic at $(d^\star, u^\star)$, then by considering the holomorphic extension of R, see [1, p. 75], which is defined on the complex couple spaces, see [40, p. 312], enables us to use the holomorphic implicit mapping theorem, see [20, Theorem 10.2.1], to prove that $S^\star$ itself locally at $d^\star$ has a holomorphic extension and thus is real analytic there. Real analyticity resp. parametric holomorphy of parametric solution maps is well-known to enable deterministic numerical interpolation and quadrature approximations which converge at rates which depend only on a suitable sparsity of the data. In particular, convergence rates are free from the so-called "curse of dimensionality", see e.g. [17, 19, 25, 43].

2.2. **Fréchet derivatives of the residual equation.** A drawback of the preceding argument is that it does not provide any additional control regarding the analyticity. However, more control is attainable by directly considering the Fréchet derivatives of the local solution mapping. We start with the equation

$$R\big(d, S^\star(d)\big) = 0$$

and take the nth Fréchet derivative of both sides with respect to d in the directions $h_1, \ldots, h_n$. For the left-hand side, we use the Faà di Bruno formula, see [14], and arrive at the equation

$$\begin{aligned}
&\sum_{\sigma\in\Pi_n} \sum_{r=1}^{n} \sum_{i\in C(n,r)} \frac{1}{r!} \operatorname{D}^r R\big(d, S^\star(d)\big) \\
&\qquad \Bigg[\frac{1}{i_1!}\big(\operatorname{D}^{i_1} \operatorname{Id}(d)[h_{\sigma(1)}, \ldots, h_{\sigma(i_1)}], \operatorname{D}^{i_1} S^\star(d)[h_{\sigma(1)}, \ldots, h_{\sigma(i_1)}]\big), \ldots, \\
&\qquad \frac{1}{i_r!}\big(\operatorname{D}^{i_r} \operatorname{Id}(d)[h_{\sigma(n-i_r+1)}, \ldots, h_{\sigma(n)}], \operatorname{D}^{i_r} S^\star(d)[h_{\sigma(n-i_r+1)}, \ldots, h_{\sigma(n)}]\big)\Bigg] = 0.
\end{aligned}$$

Here, $C(n, r)$ is the set of compositions of the natural number n into r positive integers, given by

$$C(n,r) := \left\{\big(i_1, \ldots, i_r\big) \in \big(\mathbb{N}\big)^r : \sum_{k=1}^{r} i_k = n \text{ and } i_k \in \mathbb{N}^* \text{ for all } 1 \le k \le r\right\}, \tag{2.2}$$

and Π_n is the set of all permutations of $\{1, \ldots, n\}$.

By simplifying the terms with $r = 1$ and reordering the equation, we arrive at the following formula for the nth Fréchet derivative of $S^\star$,

$$\begin{aligned}
&\operatorname{D}^n S^\star(d)[h_1, \ldots, h_n] \\
&\quad = -\Big(\operatorname{D}_2 R\big(d, S^\star(d)\big)\Big)^{-1} \Bigg[\operatorname{D}_1 R\big(d, S^\star(d)\big)\big[\operatorname{D}^n \operatorname{Id}(d)[h_1, \ldots, h_n]\big] \\
&\qquad + \sum_{\sigma\in\Pi_n} \sum_{r=2}^{n} \sum_{i\in C(n,r)} \frac{1}{r!} \operatorname{D}^r R\big(d, S^\star(d)\big) \\
&\qquad \Bigg[\frac{1}{i_1!}\big(\operatorname{D}^{i_1} \operatorname{Id}(d)[h_{\sigma(1)}, \ldots, h_{\sigma(i_1)}], \operatorname{D}^{i_1} S^\star(d)[h_{\sigma(1)}, \ldots, h_{\sigma(i_1)}]\big), \ldots, \\
&\qquad \frac{1}{i_r!}\big(\operatorname{D}^{i_r} \operatorname{Id}(d)[h_{\sigma(n-i_r+1)}, \ldots, h_{\sigma(n)}], \operatorname{D}^{i_r} S^\star(d)[h_{\sigma(n-i_r+1)}, \ldots, h_{\sigma(n)}]\big)\Bigg]\Bigg].
\end{aligned}$$

Importantly, taking the inverse of $\operatorname{D}_2 R\big(d, S^\star(d)\big)$ is possible when d is close enough to $d^\star$, because continuity of $\operatorname{D}_2 R$ and $S^\star$ imply that $\operatorname{D}_2 R\big(d, S^\star(d)\big)$ then also necessarily is a Banach space isomorphism. In particular, for $n = 1$ and for $h \in \mathcal{D}$,

$$\operatorname{D} S^\star(d)[h] = -\Big(\operatorname{D}_2 R\big(d, S^\star(d)\big)\Big)^{-1} \Big[\operatorname{D}_1 R\big(d, S^\star(d)\big)[h]\Big], \tag{2.3}$$

while for $n \geq 2$ and for $h_i \in \mathcal{D}$, $i = 1, \ldots, n$,

(2.4)
$$
\begin{aligned}
&\mathrm{D}^n S^\star(d)[h_1, \ldots, h_n] \\
&\quad = -\Big(\mathrm{D}_2 R\big(d, S^\star(d)\big)\Big)^{-1} \Bigg[\sum_{\sigma \in \Pi_n} \sum_{r=2}^{n} \sum_{i \in C(n,r)} \frac{1}{r!} \mathrm{D}^r R\big(d, S^\star(d)\big) \\
&\qquad \Bigg[\frac{1}{i_1!} \big(\mathrm{D}^{i_1} \mathrm{Id}(d)[h_{\sigma(1)}, \ldots, h_{\sigma(i_1)}], \mathrm{D}^{i_1} S^\star(d)[h_{\sigma(1)}, \ldots, h_{\sigma(i_1)}]\big), \ldots, \\
&\qquad\quad \frac{1}{i_r!} \big(\mathrm{D}^{i_r} \mathrm{Id}(d)[h_{\sigma(n-i_r+1)}, \ldots, h_{\sigma(n)}], \mathrm{D}^{i_r} S^\star(d)[h_{\sigma(n-i_r+1)}, \ldots, h_{\sigma(n)}]\big) \Bigg] \Bigg].
\end{aligned}
$$

2.3. **Regularity estimates.** With the formulas for the Fréchet derivatives of the local solution mapping at hand, we now consider the case, where the residual map R is locally s-Gevrey at $(d^\star, u^\star)$ for some $s \in \mathbb{R}_{\geq 1}$.

2.3.1. *Assumptions.* We list and comment on the assumptions behind our results.

Assumption 1. (1) We *assume* existence of open, nonempty neighbourhoods $\tilde{D} \subset D$ of $d^\star$ and $\tilde{U} \subset U$ of $u^\star$ as well as two numbers $\varsigma, \digamma \in \mathbb{R}_{\geq 0}$ such that

$$
\big\| \mathrm{D}^n R(d, u) \big\|_{\mathcal{B}^n(\mathcal{D} \times \mathcal{U}; \mathcal{R})} \leq (n!)^s \varsigma \digamma^n \tag{2.5}
$$

holds for some $s \geq 1$ and for all $d \in \tilde{D}$, $u \in \tilde{U}$ and $n \in \mathbb{N}$. Possibly reducing the size of the set $D^\star$ in (2.6) and using the continuity of $S^\star$, we from here on assume that we have $D^\star \subset \tilde{D}$ and $S^\star(D^\star) \subset \tilde{U}$. In the analytic case, i.e. when $s = 1$, Pringsheim's Theorem, see e.g. [7, p. 169], states that assumption (2.5) is equivalent to R being real analytic at $(d^\star, u^\star)$.

(2) We assume that the norms of the inverse of the Banach space isomorphisms $\mathrm{D}_2 R\big(d, S^\star(d)\big)$ are bounded uniformly for all $d \in D^\star$ by a uniform constant $\alpha \in \mathbb{R}_{>0}$, that is,

$$
\left\| \Big(\mathrm{D}_2 R\big(d, S^\star(d)\big)\Big)^{-1} \right\|_{\mathcal{B}(\mathcal{R}; \mathcal{U})} \leq \alpha \tag{2.6}
$$

holds for all $d \in D^\star$. Note that this bound is a local stability estimate for the linearisations of the residual equation. By continuity of $\mathrm{D}_2 R$ and possibly reducing the size of the set $D^\star$, such a uniform bound is achievable precisely when we have that $\mathrm{D}_2 R(d^\star, u^\star)$ is a Banach space isomorphism. Without loss of generality, we require $\alpha \geq 1$, $\varsigma \geq 1$ and $\digamma \geq 1$.

2.3.2. *Combinatorial results.* The proofs of our main result on s-Gevrey regularity of the data-to-solution map $d \mapsto S^\star(d)$ in (2.6) depend in an essential manner on several combinatorial facts which we recapitulate here for the readers' convenience. We remind of the definition (2.2) of $C(n, r)$.

Lemma 2. *Let $n \in \mathbb{N}$, then for any $r \in \{1, \ldots, n\}$ and any $i \in C(n, r)$, we have the combinatorial inequality*

$$r! \prod_{j=1}^{r} i_j! \leq n!.$$

Proof. For the proof, one simply notes that the right-hand side counts the number of ways to permute the list

$$1, 2, 3, \ldots, n$$

without any restrictions, while the left-hand side counts the number of permutations achievable if one segments the list into r sublists of lengths $i_1, \ldots, i_r$, then first permutes the elements in each sublist and then permutes the sublists themselves. $\square$

In view of the type of bounds appearing, defining s-Gevrey smoothness, we consider bounds for a particular sequence $(\kappa_n)_{n \in \mathbb{N}^*}$.

Lemma 3. *The sequence $(\kappa_n)_{n \in \mathbb{N}^*}$ recursively defined by*

$$\kappa_n = \sum_{r=2}^{n} \sum_{i \in C(n,r)} \prod_{j=1}^{r} \kappa_{i_j} \tag{2.7}$$

for $n \geq 2$ and $\kappa_1 = 1$ are the Schröder–Hipparchus numbers, also sometimes called the little Schröder numbers or the super-Catalan numbers. The κ_n are bounded by

$$\kappa_n \leq c_\kappa^{n-1} \quad \text{for all } n \in \mathbb{N}^*, \quad \text{where } c_\kappa := 3 + \sqrt{8}.$$

This bound is optimal in the sense that if $c_1, c_2 \in \mathbb{R}_{>0}$ are two constants for which $\kappa_n \leq c_1 c_2^{n-1}$ for all $n \in \mathbb{N}$, then $c_1 \geq 1$ and $c_2 \geq c_\kappa$.

Proof. The κ_n are precisely the Schröder–Hipparchus numbers as defined in [38], see e.g. [39]. Furthermore, in [13, p. 57] it is shown that the Schröder–Hipparchus numbers satisfy the three-term recursion

$$\kappa_{n+1} = \frac{6n-3}{n+1}\kappa_n - \frac{n-2}{n+1}\kappa_{n-1}, \quad n \geq 2, \quad \kappa_1 = \kappa_2 = 1.$$

We now prove that

$$\kappa_{n+1} \leq c_\kappa \kappa_n$$

holds for all $n \geq 1$ by induction. For $n = 1$, this is obviously true. Therefore, we assume that $n \geq 2$. The induction hypothesis implies that $\kappa_{n-1} \geq \kappa_n / c_\kappa$ holds. Inserting this into the three-term recursion yields

$$\kappa_{n+1} \leq \left(\frac{6n-3}{n+1} - \frac{1}{c_\kappa} \frac{n-2}{n+1} \right) \kappa_n .$$

Straightforward calculation gives

$$c_\kappa^2 - c_\kappa \left(\frac{6n-3}{n+1} \right) + \left(\frac{n-2}{n+1} \right) = \frac{24 + 9\sqrt{8}}{n+1} \geq 0.$$

Hence,

$$\frac{6n-3}{n+1} - \frac{1}{c_\kappa} \frac{n-2}{n+1} \leq c_\kappa$$

holds, which proves $\kappa_{n+1} \leq c_\kappa \kappa_n$ and concludes the induction. Obviously, this now implies the bound $\kappa_n \leq c_\kappa^{n-1}$.

Lastly, let $c_1, c_2 \in \mathbb{R}_{>0}$ be two constants for which the bound $\kappa_n \leq c_1 c_2^{n-1}$ holds. Then, inserting $n = 1$ directly yields $1 = \kappa_1 \leq c_1$. In [34, p. 539] it is proven that the asymptotic behaviour of the Schröder–Hipparchus numbers is given by

$$\kappa_n \sim \frac{1}{4} \sqrt{\frac{\sqrt{18} - 4}{\pi}} n^{-3/2} c_\kappa^n ,$$

which clearly induces that $c_2 \geq c_\kappa$ must hold. □

2.3.3. *s-Gevrey regularity.* We start by establishing a first growth bound on the differentials of the data-to-solution mapping $S^\star$.

Lemma 4. *Under Assumption 1, the Fréchet derivatives of the local solution mapping $S^\star$ are bounded as follows: with the constants $\alpha, \varsigma, F \geq 1$ from* (2.5),

$$\forall n \in \mathbb{N}^* : \sup_{d \in D^\star} \left\| \mathrm{D}^n S^\star(d) \right\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \leq (n!)^s \alpha^{2n-1} \varsigma^{2n-1} F^{3n-2} \kappa_n . \tag{2.8}$$

Here, the sequence $(\kappa_n)_{n \in \mathbb{N}^*}$ is recursively defined as in (2.7).

Proof. We first note that obviously $\kappa_n \geq 1$ holds for all $n \in \mathbb{N}^*$. With this, we consider $n = 1$. Then, taking the norm of equation (2.3) and inserting the bounds yields

$$\begin{aligned} \left\| \mathrm{D}\, S^\star(d) \right\|_{\mathcal{B}(\mathcal{D};\mathcal{U})} &\leq \left\| \Big(\mathrm{D}_2\, R\big(d, S^\star(d)\big) \Big)^{-1} \right\|_{\mathcal{B}(\mathcal{R};\mathcal{U})} \left\| \mathrm{D}_1\, R\big(d, S^\star(d)\big) \right\|_{\mathcal{B}(\mathcal{D};\mathcal{R})} \\ &\leq \alpha 1! \varsigma F = (1!)^s \alpha \varsigma F \kappa_1 , \end{aligned}$$

proving the assertion for $n = 1$.

The rest of the proof is by induction: Let $n \geq 2$, then taking the norm of equation (2.4) leads to

$$\begin{aligned}
&\big\|\mathrm{D}^n S^\star(d)\big\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \\
&\quad\leq \left\|\Big(\mathrm{D}_2 R\big(d, S^\star(d)\big)\Big)^{-1}\right\|_{\mathcal{B}(\mathcal{R};\mathcal{U})} \sum_{\sigma\in\Pi_n}\sum_{r=2}^{n}\sum_{i\in C(n,r)} \frac{1}{r!}\Big\|\mathrm{D}^r R\big(d, S^\star(d)\big)\Big\|_{\mathcal{B}^r(\mathcal{D}\times\mathcal{U};\mathcal{R})} \\
&\qquad\qquad \prod_{j=1}^{r}\frac{1}{i_j!}\max\Big(\big\|\mathrm{D}^{i_j}\,\mathrm{Id}(d)\big\|_{\mathcal{B}^{i_j}(\mathcal{D};\mathcal{D})}, \big\|\mathrm{D}^{i_j} S^\star(d)\big\|_{\mathcal{B}^{i_j}(\mathcal{D};\mathcal{U})}\Big).
\end{aligned}$$

Next, by inserting all the bounds and noting that we have

$$\big\|\mathrm{D}^k\,\mathrm{Id}(d)\big\|_{\mathcal{B}^k(\mathcal{D};\mathcal{D})} \leq 1 \leq (k!)^s \alpha^{2k-1}\varsigma^{2k-1}F^{3k-2}\kappa_k$$

for all $k \in \mathbb{N}^*$, we can calculate

$$\begin{aligned}
&\big\|\mathrm{D}^n S^\star(d)\big\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \\
&\quad\leq \alpha \sum_{\sigma\in\Pi_n}\sum_{r=2}^{n}\sum_{i\in C(n,r)} \frac{1}{r!}(r!)^s\varsigma F^r \prod_{j=1}^{r}\frac{1}{i_j!}(i_j!)^s\alpha^{2i_j-1}\varsigma^{2i_j-1}F^{3i_j-2}\kappa_{i_j} \\
&\quad= n!\sum_{r=2}^{n}\sum_{i\in C(n,r)} (r!)^{s-1}\alpha\varsigma F^r \prod_{j=1}^{r}(i_j!)^{s-1}\alpha^{2i_j-1}\varsigma^{2i_j-1}F^{3i_j-2}\kappa_{i_j} \\
&\quad= n!\sum_{r=2}^{n}\alpha^{2n-r+1}\varsigma^{2n-r+1}F^{3n-r}\sum_{i\in C(n,r)}(r!)^{s-1}\prod_{j=1}^{r}(i_j!)^{s-1}\kappa_{i_j}.
\end{aligned}$$

For any $r \in \{1, \ldots, n\}$ and any $i \in C(n, r)$, we have

$$(r!)^{s-1}\prod_{j=1}^{r}(i_j!)^{s-1} \leq (n!)^{s-1}$$

by Lemma 2. Hence, by inserting this, we arrive at the asserted bound,

$$\begin{aligned}
\big\|\mathrm{D}^n S^\star(d)\big\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} &\leq (n!)^s\alpha^{2n-1}\varsigma^{2n-1}F^{3n-2}\sum_{r=2}^{n}\sum_{i\in C(n,r)}\prod_{j=1}^{r}\kappa_{i_j} \\
&= (n!)^s\alpha^{2n-1}\varsigma^{2n-1}F^{3n-2}\kappa_n.
\end{aligned}$$

□

Remark 5. We note that the proof for the bounds

$$\big\|\mathrm{D}^n S^\star(d)\big\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \leq (n!)^s\alpha^{2n-1}\varsigma^{2n-1}F^{3n-2}\kappa_n$$

only requires the bounds

$$\big\|\mathrm{D}^n R\big(d, S^\star(d)\big)\big\|_{\mathcal{B}^n(\mathcal{D}\times\mathcal{U};\mathcal{R})} \leq (n!)^s\varsigma F^n$$

and (2.6) to hold at the specific $d \in D^\star$ one is considering.

We note that the preceding proof essentially has the form of what is called the real-variable inductive argument in [8]. However, since we are considering Fréchet derivatives, we are able to forgo the use of multi-indices. Moreover, the bounds in equation (2.8) can be rewritten as

$$\left\|\mathrm{D}^n S^\star(d)\right\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \leq (n!)^s \frac{1}{\alpha\varsigma F^2}\left(\alpha^2\varsigma^2 F^3\right)^n \kappa_n.$$

Combining Lemmas 4 and 3 proves:

Theorem 6. *The Fréchet derivatives of the local data-to-solution map $S^\star$ are bounded as follows:*

$$\forall n \in \mathbb{N}^* : \sup_{d\in D^\star}\left\|\mathrm{D}^n S^\star(d)\right\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \leq (n!)^s \tilde{\varsigma}\tilde{F}^n. \tag{2.9}$$

Herein, with $\alpha, \varsigma, F \geq 1$ as defined in (2.5) *and* (2.6) *above,*

$$\tilde{\varsigma} := \frac{1}{c_\kappa \alpha\varsigma F^2} \quad \textit{and} \quad \tilde{F} := c_\kappa\alpha^2\varsigma^2 F^3.$$

Theorem 6 establishes the s-Gevrey smoothness of the local data-to-solution map $S^\star : D^\star \to \mathcal{U}$ and, thus, that an *s-Gevrey class implicit mapping theorem* holds for all $s \in \mathbb{R}_{\geq 1}$. In particular, the Gevrey regularity index $s \geq 1$ in the regularity assumption (2.5) on the residual equation is inherited by the data-to-solution map. Furthermore, Theorem 6 reestablishes for the particular case $s = 1$ the real analyticity of the local solution mapping $S^\star$ (without the use of holomorphy arguments as e.g. in [9]) using Pringsheim's Theorem, yielding a purely real analytic proof of the real analytic implicit mapping theorem. Indeed in this case, the bounds (2.9) provide quantitative bounds on the domain of analyticity of $S^\star$. By using the Cauchy–Hadamard formula for example, we know that the radius of convergence for the Taylor series of $S^\star$ at every $d \in D^\star$ is at least

$$\frac{1}{\tilde{F}} = \frac{1}{c_\kappa\alpha^2\varsigma^2 F^3}. \tag{2.10}$$

2.4. **Gevrey-regularity of composite and parametric mappings.** Since in applications the quantity of interest may not always simply be the solution itself but rather some other derived quantity, we now additionally supply bounds for the Fréchet derivatives of the composition of Gevrey-regular mappings.

Theorem 7. *Let $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ be real Banach spaces and consider mappings $M_1 \colon X_1 \to \mathcal{X}_2$ and $M_2 \colon X_2 \to \mathcal{X}_3$, where $X_1 \subset \mathcal{X}_1$ and $X_2 \subset \mathcal{X}_2$ are open and $M_1(X_1) \subset X_2$ holds. Furthermore, let $x_1 \in X_1$ and set $x_2 := M_1(x_1)$. If M_j is s_j-Gevrey at an $x_j \in X_j$ for some $s_j \in \mathbb{R}_{\geq 1}$, i.e. there is an open neighbourhood*

$N_j \subset X_j$ of x_j and constants $\mu_j, \nu_j \in \mathbb{R}_{\geq 0}$ such that

$$\forall n \in \mathbb{N}^* : \sup_{x \in N_j} \big\| \mathrm{D}^n M_j(x) \big\|_{\mathcal{B}^n(\mathcal{X}_j;\mathcal{X}_{j+1})} \leq (n!)^{s_j} \mu_j \nu_j^n$$

holds, where $j \in \{1,2\}$. Then, $M := M_2 \circ M_1 \colon X_1 \to \mathcal{X}_3$ is s-Gevrey at an x_1 for $s := \max\{s_1, s_2\}$. Specifically, for $N := N_1 \cap M_1^{-1}(N_2)$ and

$$\mu = \mu_2 \frac{\nu_2 \mu_1}{\nu_2 \mu_1 + 1} \quad \text{and} \quad \nu = (\nu_2 \mu_1 + 1)\nu_1,$$

we have

$$\forall n \in \mathbb{N}^* : \sup_{x \in N} \big\| \mathrm{D}^n M(x) \big\|_{\mathcal{B}^n(\mathcal{X}_1;\mathcal{X}_3)} \leq (n!)^s \mu \nu^n .$$

Proof. As both, M_1 and M_2, are infinitely Fréchet differentiable on N_1 and N_2, respectively, it is clear that M also is infinitely Fréchet differentiable on N. Thus, the Faà di Bruno formula gives us the following formula for the nth derivative of M at an $x \in N$,

$$\begin{aligned}
&\mathrm{D}^n M(x)[h_1, \dots, h_n] \\
&\quad = \sum_{\sigma \in \Pi_n} \sum_{r=1}^{n} \sum_{i \in C(n,r)} \frac{1}{r!} \mathrm{D}^r M_2\big(M_1(x)\big) \\
&\qquad\quad \left[\frac{1}{i_1!} \mathrm{D}^{i_1} M_1(d)[h_{\sigma(1)}, \dots, h_{\sigma(i_1)}], \dots, \frac{1}{i_r!} \mathrm{D}^{i_r} M_1(d)[h_{\sigma(n-i_r+1)}, \dots, h_{\sigma(n)}] \right].
\end{aligned}$$

Now, by taking the norm and inserting the bounds, we arrive at

$$\begin{aligned}
\big\| \mathrm{D}^n M(x) \big\|_{\mathcal{B}^n(\mathcal{X}_1;\mathcal{X}_3)} &\leq n! \sum_{r=1}^{n} \sum_{i \in C(n,r)} (r!)^{s_2 - 1} \mu_2 \nu_2^r \prod_{j=1}^{r} (i_j!)^{s_1 - 1} \mu_1 \nu_1^{i_j} \\
&\leq (n!)^s \mu_2 \nu_1^n \sum_{r=1}^{n} \nu_2^r \mu_1^r \sum_{i \in C(n,r)} 1,
\end{aligned}$$

where we have also used Lemma 2. Since $|C(n,r)| = \binom{n-1}{r-1}$, we finally have

$$\big\| \mathrm{D}^n M(x) \big\|_{\mathcal{B}^n(\mathcal{X}_1;\mathcal{X}_3)} \leq (n!)^s \mu_2 \nu_1^n \sum_{r=1}^{n} \binom{n-1}{r-1} \nu_2^r \mu_1^r = (n!)^s \mu_2 \nu_2 \mu_1 (\nu_2 \mu_1 + 1)^{n-1} \nu_1^n,$$

which proves the assertion. □

In applications, the data is often given in a parametrised fashion. Therefore, we also provide bounds for mixed partial derivatives of arbitrary order for the composition of mappings, where the outer mapping is Gevrey-regular and the inner mapping has Gevrey-regular mixed partial derivatives of arbitrary order.

Theorem 8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces and consider mappings $P\colon \mathcal{P} \to \mathcal{X}$ and $M\colon X \to \mathcal{Y}$, where $\mathcal{P} \subset \mathbb{R}^{\mathbb{N}^*}$ and $X \subset \mathcal{X}$ are open and $P(\mathcal{P}) \subset X$ holds. Furthermore, consider a parameter $\mathbf{y} \in \mathcal{P}$ and set $x := P(\mathbf{y})$. Assume in addition that P has s_P-Gevrey mixed partial derivatives of arbitrary order at $\mathbf{y}$ with weight $\boldsymbol{\gamma} \in \mathbb{R}^{\mathbb{N}^*}$ and M is s_M-Gevrey at an x for some $s_P, s_M \in \mathbb{R}_{\geq 1}$, i.e. there are open neighbourhoods $N_P \subset \mathcal{P}$ of $\mathbf{y}$ and $N_M \subset X$ of x and constants $\mu_P, \nu_P, \mu_M, \nu_M \in \mathbb{R}_{\geq 0}$ such that*

$$\forall \boldsymbol{\alpha} \in \mathbb{N}_f^{\mathbb{N}^*} \setminus \{\mathbf{0}\} : \ \sup_{\mathbf{y} \in N} \big\| \partial^{\boldsymbol{\alpha}} P(\mathbf{y}) \big\|_{\mathcal{X}} \leq \big(|\boldsymbol{\alpha}|!\big)^{s_P} \mu_P \nu_P^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

and

$$\forall n \in \mathbb{N}^* : \ \sup_{x \in N_M} \big\| \mathrm{D}^n M(x) \big\|_{\mathcal{B}^n(\mathcal{X};\mathcal{Y})} \leq (n!)^{s_M} \mu_M \nu_M^n$$

hold.

Then, $Q := M \circ P\colon \mathcal{P} \to \mathcal{Y}$ has s-Gevrey mixed partial derivatives of arbitrary order at $\mathbf{y}$ with weight $\boldsymbol{\gamma}$ for $s := \max\{s_P, s_M\}$. Specifically, for $N := N_P \cap P^{-1}(N_M)$ and

$$\mu = \mu_M \frac{\nu_M \mu_P}{\nu_M \mu_P + 1} \quad \text{and} \quad \nu = (\nu_M \mu_P + 1)\nu_P,$$

we have

$$\forall \boldsymbol{\alpha} \in \mathbb{N}_f^{\mathbb{N}^*} \setminus \{\mathbf{0}\} : \ \sup_{\mathbf{y} \in N} \big\| \partial^{\boldsymbol{\alpha}} Q(\mathbf{y}) \big\|_{\mathcal{Y}} \leq \big(|\boldsymbol{\alpha}|!\big)^{s} \mu \nu^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} .$$

Proof. As $Q = M \circ P$ and M is s_M-Gevrey and P has s_P-Gevrey mixed partial derivatives of arbitrary order, it follows that Q also has mixed partial derivatives of arbitrary order.

For $\boldsymbol{\alpha} \neq \mathbf{0}$, according to the Faà di Bruno formula, we have

$$\partial^{\boldsymbol{\alpha}} Q(\mathbf{y}) = \boldsymbol{\alpha}! \sum_{r=1}^{|\boldsymbol{\alpha}|} \frac{1}{r!} \sum_{\boldsymbol{\beta} \in C(\boldsymbol{\alpha}, r)} \mathrm{D}^r M\big(P(\mathbf{y})\big)\big[\partial^{\boldsymbol{\beta}_1} P(\mathbf{y}), \ldots, \partial^{\boldsymbol{\beta}_r} P(\mathbf{y})\big] \prod_{j=1}^{r} \frac{1}{\boldsymbol{\beta}_j!},$$

where $C(\boldsymbol{\alpha}, r)$ is the set of multi-index compositions of a multi-index $\boldsymbol{\alpha}$ into r non-vanishing multi-indices, given by

$$C(\boldsymbol{\alpha}, r) := \bigg\{ \big(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r\big) \in \big(\mathbb{N}_f^{\mathbb{N}^*}\big)^r : \sum_{j=1}^{r} \boldsymbol{\beta}_j = \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta}_j \neq \mathbf{0} \text{ for all } 1 \leq j \leq r \bigg\}.$$

Taking the norm and inserting the bounds yields

$$\begin{aligned}
\big\|\partial^{\boldsymbol{\alpha}} Q(\mathbf{y})\big\|_{\mathcal{Y}} &\leq \boldsymbol{\alpha}! \sum_{r=1}^{|\boldsymbol{\alpha}|} \frac{1}{r!} \sum_{\boldsymbol{\beta}\in C(\boldsymbol{\alpha},r)} (r!)^{s_M} \mu_M \nu_M^r \prod_{j=1}^{r} \frac{\big(|\boldsymbol{\beta}_j|!\big)^{s_P}}{\boldsymbol{\beta}_j!} \mu_P \nu_P^{|\boldsymbol{\beta}_j|} \boldsymbol{\gamma}^{\boldsymbol{\beta}_j} \\
&= \mu_M \nu_P^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \sum_{r=1}^{|\boldsymbol{\alpha}|} \nu_M^r \mu_P^r (r!)^{s-1} \boldsymbol{\alpha}! \sum_{\boldsymbol{\beta}\in C(\boldsymbol{\alpha},r)} \prod_{j=1}^{r} \frac{\big(|\boldsymbol{\beta}_j|!\big)^{s-1} |\boldsymbol{\beta}_j|!}{\boldsymbol{\beta}_j!} \\
&\leq \big(|\boldsymbol{\alpha}|!\big)^{s-1} \mu_M \nu_P^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \sum_{r=1}^{|\boldsymbol{\alpha}|} \nu_M^r \mu_P^r \boldsymbol{\alpha}! \sum_{\boldsymbol{\beta}\in C(\boldsymbol{\alpha},r)} \prod_{j=1}^{r} \frac{|\boldsymbol{\beta}_j|!}{\boldsymbol{\beta}_j!},
\end{aligned}$$

where we have used the combinatorial inequality from Lemma 2. Then, using the identity

$$\boldsymbol{\alpha}! \sum_{C(\boldsymbol{\alpha},r)} \prod_{j=1}^{r} \frac{|\boldsymbol{\beta}_j|!}{\boldsymbol{\beta}_j!} = |\boldsymbol{\alpha}|! \binom{|\boldsymbol{\alpha}|-1}{r-1}$$

from [31, Lemma 1] finally gives us the asserted bound

$$\begin{aligned}
\big\|\partial^{\boldsymbol{\alpha}} Q(\mathbf{y})\big\|_{\mathcal{Y}} &\leq \big(|\boldsymbol{\alpha}|!\big)^{s} \mu_M \nu_P^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \sum_{r=1}^{|\boldsymbol{\alpha}|} (\nu_M \mu_P)^r \binom{|\boldsymbol{\alpha}|-1}{r-1} \\
&= \big(|\boldsymbol{\alpha}|!\big)^{s} \mu_M \nu_M \mu_p (\nu_M \mu_p + 1)^{|\boldsymbol{\alpha}|-1} \nu_P^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}. \qquad \square
\end{aligned}$$

3. Semilinear elliptic PDE on random domains with polynomial nonlinearities

To demonstrate the application of Theorems 6 and 8 in the context of uncertainty quantification for PDEs with random data, we consider a semilinear, elliptic model problem with polynomial nonlinearity. For this purpose, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, on which randomness for the parametric problem shall be modeled.

3.1. **Problem formulation.** Consider the following semilinear elliptic PDE with random data (coefficients, source term and physical domain) of the following, generic form: for $\omega \in \Omega$,

$$\tag{3.1} \begin{cases} -\operatorname{div}\big(a[\omega](\mathbf{x}) \nabla u[\omega](\mathbf{x})\big) + b[\omega](\mathbf{x}) \mathfrak{N}\big(u[\omega](\mathbf{x})\big) = f[\omega](\mathbf{x}) & \text{for } \mathbf{x} \in G[\omega], \\ u[\omega](\mathbf{x}) = 0 & \text{for } \mathbf{x} \in \Gamma_{\mathrm{D}}[\omega], \\ \big\langle \nabla u[\omega](\mathbf{x}), \mathbf{n}[\omega](\mathbf{x}) \big\rangle = g[\omega](\mathbf{x}) & \text{for } \mathbf{x} \in \Gamma_{\mathrm{N}}[\omega]. \end{cases}$$

The domain $G[\omega]$ is assumed to be a bounded nonempty subset of $\mathbb{R}^m$ with dimension $m \in \mathbb{N}^*$.

We next introduce $q \in \mathbb{R}_{\geq 2}$ dependent on m, with the following restrictions

$$\begin{cases} q < \infty, & \text{when } m \in \{1,2\}, \\ q \leq \frac{2m}{m-2}, & \text{else.} \end{cases}$$

Owing to the Sobolev embedding theorem, this choice guarantees that we have the continuous (but not necessarily compact) embedding $H^1\big(G[\omega]\big) \subset L^q\big(G[\omega]\big)$. With this, we now require that $\mathfrak{N}$ in (3.1) is a polynomial of degree at most $\lfloor q-1 \rfloor$ with $\mathfrak{N}(0) = 0$, i.e.

$$\mathfrak{N}(\zeta) = \sum_{j=1}^{\lfloor q-1 \rfloor} \theta_j \zeta^j \tag{3.2}$$

holds for all $\zeta \in \mathbb{R}$ for some coefficients $\theta_j \in \mathbb{R}$. Clearly, this implies that $\mathfrak{N}$ fulfils the polynomial growth bound

$$\forall \zeta \in \mathbb{R}: \quad \big|\mathfrak{N}(\zeta)\big| \leq c_{\mathfrak{N}}\big(1 + |\zeta|^{q-1}\big) \tag{3.3}$$

for some constant $c_{\mathfrak{N}}$. In addition, we assume that $\mathfrak{N}$ is monotone, in the sense that it fulfils

$$\forall \zeta, \zeta' \in \mathbb{R}: \quad \big(\mathfrak{N}(\zeta) - \mathfrak{N}(\zeta')\big)(\zeta - \zeta') \geq 0. \tag{3.4}$$

As we shall show, under positivity assumptions on the parametric coefficient $a[\omega](\mathbf{x})$ and for non-negative coefficient $b[\omega](\mathbf{x})$ in (3.1), the boundary value problem (3.1) gives rise to a well-posed monotone operator equation in (a subspace of) the Hilbertian Sobolev space $H^1\big(G[\omega]\big)$.

The reason that we first only consider polynomial nonlinearities $\mathfrak{N}$ here stems from the mapping properties of the Nemyckii operator associated with $\mathfrak{N}$. Specifically, for $1 \leq p' \leq p \leq \infty$, it is known that the Nemyckii operator $N\colon L^p \to L^{p'}$, defined by

$$N(u) := \mathfrak{N} \circ u, \tag{3.5}$$

is infinitely Fréchet differentiable only if $\mathfrak{N}$ is a polynomial with a small enough degree or $p = \infty$ and in this case its Fréchet derivatives simply are given by

$$\mathrm{D}^n N(u)[u_1, \ldots, u_n] := (\mathfrak{N}^{(n)} \circ u) \cdot u_1 \cdots u_n, \tag{3.6}$$

see [3, Theorems 3.12, 3.15 and 3.16]. Therefore, for $m \geq 2$, as $H^1 \not\subset L^\infty$, one can in general only consider the polynomial nonlinearities $\mathfrak{N}$ that we allow above, if one is interested in the analytic or s-Gevrey smooth dependence of the solution of (3.1) on the data[1] (coefficients, source term and physical domain).

[1]Note carefully that we show such dependence for data-to-solution maps between function spaces in G of *finite smoothness*.

If additional Sobolev regularity of weak solutions is available, parametric solution regularity can hold for more general nonlinearities as we will discuss in Section 4.

3.2. **Parametric domain.** Before providing a detailed statement, we clarify the notion of "random domain" $G[\omega]$ in (3.1). To this end, we adopt the random domain mapping approach as introduced in [42] and applied e.g. in [9, 30, 33] and the references there. To formulate it, we assume at hand a Lipschitz domain $\hat{G} \subset \mathbb{R}^m$, referred to as *reference domain*[2], a disjoint decomposition of its (Lipschitz-) boundary $\hat{\Gamma} = \partial\hat{G}$ into two measurable sets, $\hat{\Gamma} = \hat{\Gamma}_{\mathrm{D}} \cup \hat{\Gamma}_{\mathrm{N}}$, a parametric domain mapping $\mathbf{V} \colon \square \to C^1(\hat{G}; \mathbb{R}^m)$ with $\square := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}^*}$ and random parameters $\mathbf{Y} \colon \Omega \to \square$. We assume that $\mathbf{V}[\mathbf{y}]$ is a C^1-diffeomorphism and fulfils the uniformity condition

$$\big\|\mathbf{V}[\mathbf{y}]\big\|_{C^1(\hat{G};\mathbf{V}[\mathbf{y}](\hat{G}))} \leq c_{\mathbf{V}} \quad \text{and} \quad \big\|(\mathbf{V}[\mathbf{y}])^{-1}\big\|_{C^1(\mathbf{V}[\mathbf{y}](\hat{G});\hat{G})} \leq c_{\mathbf{V}} \tag{3.7}$$

for every $\mathbf{y} \in \square$, where $c_{\mathbf{V}} \geq 1$ is a fixed constant. We also assume that $\hat{\Gamma}_{\mathrm{D}}$ has non-zero surface measure, so that a Poincaré inequality holds on the Sobolev spaces (3.8) on $\hat{G}$. In an abuse of notation, we will consider any function defined over $\square$ to also be defined over Ω, by evaluating it at $\mathbf{Y}(\omega)$ for all $\omega \in \Omega$.

With this, we set

$$G[\mathbf{y}] := V[\mathbf{y}](\hat{G}), \quad \Gamma_{\mathrm{D}}[\mathbf{y}] := V[\mathbf{y}](\hat{\Gamma}_{\mathrm{D}}) \quad \text{and} \quad \Gamma_{\mathrm{N}}[\mathbf{y}] := V[\mathbf{y}](\hat{\Gamma}_{\mathrm{N}})$$

as well as

$$\begin{aligned} a[\mathbf{y}](\mathbf{x}) &:= \hat{a}\Big(\big(\mathbf{V}[\mathbf{y}]\big)^{-1}(\mathbf{x})\Big), & b[\mathbf{y}](\mathbf{x}) &:= \hat{b}\Big(\big(\mathbf{V}[\mathbf{y}]\big)^{-1}(\mathbf{x})\Big), \\ f[\mathbf{y}](\mathbf{x}) &:= \hat{f}\Big(\big(\mathbf{V}[\mathbf{y}]\big)^{-1}(\mathbf{x})\Big) \quad \text{and} & g[\mathbf{y}](\mathbf{x}) &:= \hat{g}\Big(\big(\mathbf{V}[\mathbf{y}]\big)^{-1}(\mathbf{x})\Big), \end{aligned}$$

where $\hat{a} \in L^\infty(\hat{G})$, $\hat{b} \in L^\infty(\hat{G})$, $\hat{f} \in H^{-1}_{\mathrm{D}}(\hat{G})$ and $\hat{g} \in H^{-1/2}(\hat{\Gamma}_{\mathrm{N}})$. Note that this means that we are effectively defining the boundary decomposition as well as the functions a, b, f and g in Lagrangian coordinates. Clearly, the choices directly imply that $a[\mathbf{y}] \in L^\infty\big(G[\mathbf{y}]\big)$, $b[\mathbf{y}] \in L^\infty\big(G[\mathbf{y}]\big)$, $f[\mathbf{y}] \in H^{-1}_{\mathrm{D}}\big(G[\mathbf{y}]\big)$ and $g[\mathbf{y}] \in H^{-1/2}\big(\Gamma_{\mathrm{N}}[\mathbf{y}]\big)$ for every $\mathbf{y} \in \square$. Here, $H^{-1}_{\mathrm{D}}\big(G[\mathbf{y}]\big)$ denotes the dual of

$$H^1_{\mathrm{D}}\big(G[\mathbf{y}]\big) := \big\{v \in H^1\big(G[\mathbf{y}]\big) : v(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \Gamma_{\mathrm{D}}[\mathbf{y}]\big\}. \tag{3.8}$$

We also assume that $\hat{a}$ and $\hat{b}$ fulfil the ellipticity and non-negativity condition

$$\underline{a} := \operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \hat{a}(\mathbf{x}) > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \hat{b}(\mathbf{x}) \geq 0, \tag{3.9}$$

respectively. For later convenience, we introduce the constant $c_a := \min\{1, \underline{a}\}$.

[2]The "reference domain" corresponds to the notion of "reference configuration" in continuum mechanics. It is, in general, distinct from the "nominal domain" in shape-uncertainty quantification.

Owing to the Sobolev embedding $H^1\big(G[\mathbf{y}]\big) \subset L^q\big(G[\mathbf{y}]\big)$, the nonlinear form

$$H^1\big(G[\mathbf{y}]\big) \times H^1\big(G[\mathbf{y}]\big) \to \mathbb{R}, (w, v) \mapsto \big\langle b[\mathbf{y}]N(w), v\big\rangle_{G[\mathbf{y}]}$$

is well-defined as $\mathfrak{N}$ is a polynomial of at most degree $\lfloor q-1 \rfloor$ and thus its Nemyckii operator N from (3.5) is well-defined as $\mathfrak{N}$ fulfils the polynomial growth bound (3.3). Then, it is straightforward to see that the variational formulation of (3.1) for every $\mathbf{y} \in \square$ reads: find $u[\mathbf{y}] \in H^1_{\mathrm{D}}\big(G[\mathbf{y}]\big)$ so that for all $v \in H^1_{\mathrm{D}}\big(G[\mathbf{y}]\big)$, we have

$$\big\langle a[\mathbf{y}] \nabla u[\mathbf{y}], \nabla v\big\rangle_{G[\mathbf{y}]} + \big\langle b[\mathbf{y}]N\big(u[\mathbf{y}]\big), v\big\rangle_{G[\mathbf{y}]} = \big\langle f[\mathbf{y}], v\big\rangle_{G[\mathbf{y}]} + \big\langle g[\mathbf{y}], v\big\rangle_{\Gamma_{\mathrm{N}}[\mathbf{y}]}. \tag{3.10}$$

3.3. **Domain mapping approach.** By utilising that $\mathbf{V}[\mathbf{y}]$ is a C^1-diffeomorphism, we can pull back the spatially weak formulation (3.10) by considering

$$\hat{u}[\mathbf{y}](\mathbf{x}) := u[\mathbf{y}]\big(\mathbf{V}[\mathbf{y}](\mathbf{x})\big).$$

Then, we have that $\hat{u}[\mathbf{y}] \in H^1_{\mathrm{D}}(\hat{G})$ for every $\mathbf{y} \in \square$ and for all $v \in H^1_{\mathrm{D}}\big(G[\mathbf{y}]\big)$ fulfils

$$\begin{aligned}\big\langle \tilde{\mathbf{A}}[\mathbf{y}] \nabla \hat{u}[\mathbf{y}], \nabla(v \circ \mathbf{V}[\mathbf{y}])\big\rangle_{\hat{G}} &+ \big\langle \tilde{b}[\mathbf{y}]N(\hat{u}[\mathbf{y}]), v \circ \mathbf{V}[\mathbf{y}]\big\rangle_{\hat{G}} \\ &= \big\langle \tilde{f}[\mathbf{y}], v \circ \mathbf{V}[\mathbf{y}]\big\rangle_{\hat{G}} + \big\langle \tilde{g}[\mathbf{y}], v \circ \mathbf{V}[\mathbf{y}]\big\rangle_{\hat{\Gamma}_{\mathrm{N}}}\end{aligned}$$

with

$$\begin{aligned}\tilde{\mathbf{A}}[\mathbf{y}](\mathbf{x}) &= \det\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)^{-1}\hat{a}(\mathbf{x})\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)^{-\mathsf{T}},\\ \tilde{b}[\mathbf{y}](\mathbf{x}) &= \det\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)\hat{b}(\mathbf{x}),\\ \tilde{f}[\mathbf{y}](\mathbf{x}) &= \det\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)\hat{f}(\mathbf{x})\\ \text{and} \quad \tilde{g}[\mathbf{y}](\mathbf{x}) &= \Big\|\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)^{-\mathsf{T}}\mathbf{n}(\mathbf{x})\Big\|_2 \det\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)\hat{g}(\mathbf{x}),\end{aligned}$$

where $\mathbf{J}[\mathbf{y}](\mathbf{x}) = \mathrm{D}_{\mathbf{x}}\big(\mathbf{V}[\mathbf{y}]\big)(\mathbf{x})$. Note that we have made use of the fact here that $\mathbf{V}[\mathbf{y}] : \hat{G} \to G[\mathbf{y}]$ is a C^1-diffeomorphism for every $\mathbf{y} \in \square$. This implies that $\det\big(\mathbf{J}[\mathbf{y}](\mathbf{x})\big)$ has the same sign for all $\mathbf{x}$; without loss of generality, we assume that it is positive, i.e. orientation preserving.

As the map $H^1_{\mathrm{D}}\big(G[\mathbf{y}]\big) \to H^1_{\mathrm{D}}(\hat{G}),\ v \mapsto v \circ \mathbf{V}[\mathbf{y}]$ is an isomorphism for every $\mathbf{y} \in \square$, we can replace the $v \circ \mathbf{V}[\mathbf{y}]$ terms with $v \in H^1_{\mathrm{D}}\big(G[\mathbf{y}]\big)$ for some $\hat{v}$ with $\hat{v} \in H^1_{\mathrm{D}}(\hat{G})$. We thus arrive at the spatially weak formulation for the pullback: $\hat{u}[\mathbf{y}] \in H^1_{\mathrm{D}}(\hat{G})$ for every $\mathbf{y} \in \square$ fulfils

$$\big\langle \tilde{\mathbf{A}}[\mathbf{y}] \nabla \hat{u}[\mathbf{y}], \nabla \hat{v}\big\rangle_{\hat{G}} + \big\langle \tilde{b}[\mathbf{y}]N(\hat{u}[\mathbf{y}]), \hat{v}\big\rangle_{\hat{G}} = \big\langle \tilde{f}[\mathbf{y}], \hat{v}\big\rangle_{\hat{G}} + \big\langle \tilde{g}[\mathbf{y}], \hat{v}\big\rangle_{\hat{\Gamma}_{\mathrm{N}}} \tag{3.11}$$

for all $\hat{v} \in H^1_{\mathrm{D}}(\hat{G})$. It is straightforward to see that we have $\tilde{\mathbf{A}}[\mathbf{y}] \in L^\infty(\hat{G}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\tilde{b}[\mathbf{y}] \in L^\infty(\hat{G})$, $\tilde{f}[\mathbf{y}] \in H^{-1}_{\mathrm{D}}(\hat{G})$ and $\tilde{g}[\mathbf{y}] \in H^{-1/2}(\hat{\Gamma}_{\mathrm{N}})$ for every $\mathbf{y} \in \square$. Moreover, for every $\mathbf{y} \in \square$, $\tilde{\mathbf{A}}[\mathbf{y}]$ and $\tilde{b}[\mathbf{y}]$ retain their ellipticity and non-negativity condition

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \min_{\mathbf{v}\in\mathbb{R}^m\setminus\{\mathbf{0}\}} \frac{\mathbf{v}^{\mathsf{T}}\tilde{\mathbf{A}}[\mathbf{y}](\mathbf{x})\mathbf{v}}{\mathbf{v}^{\mathsf{T}}\mathbf{v}} \geq c_a c_{\mathbf{V}}^{-m-2} > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \tilde{b}[\mathbf{y}](\mathbf{x}) \geq 0,$$

respectively.

3.4. **Residual operator equation.** To cast equation (3.11) into the abstract setting of Section 2 on a fixed domain $\hat{G}$, we choose

$$\begin{aligned}\mathcal{D} &:= L^\infty(\hat{G};\mathbb{R}^{m\times m}_{\mathrm{sym}}) \times L^\infty(\hat{G}) \times H^{-1}_{\mathrm{D}}(\hat{G}) \times H^{-1/2}(\hat{\Gamma}_{\mathrm{N}}),\\ \mathcal{U} &:= H^1_{\mathrm{D}}(\hat{G}),\\ \mathcal{R} &:= H^{-1}_{\mathrm{D}}(\hat{G}) = \mathcal{U}'\end{aligned}$$

and the subset of admissible data

$$D := D_{\mathbf{A}} \times D_b \times H^{-1}_{\mathrm{D}}(\hat{G}) \times H^{-1/2}(\hat{\Gamma}_{\mathrm{N}}),$$

where

$$D_{\mathbf{A}} := \left\{\mathbf{A} \in L^\infty(\hat{G};\mathbb{R}^{m\times m}_{\mathrm{sym}}) : \operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \min_{\mathbf{v}\in\mathbb{R}^m\setminus\{\mathbf{0}\}} \frac{\mathbf{v}^\mathsf{T}\mathbf{A}(\mathbf{x})\mathbf{v}}{\mathbf{v}^\mathsf{T}\mathbf{v}} \geq c_a c_{\mathbf{V}}^{-m-2} =: c_{\mathbf{A}}\right\}$$

and $D_b := \big\{b \in L^\infty(\hat{G}) : \operatorname{ess\,inf}_{\mathbf{x}\in\hat{G}} b(\mathbf{x}) \geq 0\big\}$. For the sake of legibility, we associate the data $d \in \mathcal{D}$ to be given by the tuple $(\mathbf{A}, b, f, g)$. We also extend this to modifications of d, for example $d_1 = (\mathbf{A}_1, b_1, f_1, g_1)$.

Now, we define the residual operator $R\colon D \times \mathcal{U} \to \mathcal{R}$ by setting

$$\big(R(d,u)\big)(v) := \big\langle \mathbf{A}\,\nabla u, \nabla v\big\rangle_{\hat{G}} + \big\langle bN(u), v\big\rangle_{\hat{G}} - \big\langle f, v\big\rangle_{\hat{G}} - \big\langle g, v\big\rangle_{\hat{\Gamma}_{\mathrm{N}}} \tag{3.12}$$

for all $v \in \mathcal{U}$. Note that this is justified since $\mathcal{U}' = \mathcal{R}$ and the right-hand side of equation (3.12) is linear in v. With this residual operator R, equation (3.11) can be restated as the residual equation

$$R\big(\tilde{d}[\mathbf{y}], \hat{u}[\mathbf{y}]\big) = 0,$$

where $\tilde{d}\colon \square \to D$ is the *parameters-to-data mapping* given by

$$\tilde{d}[\mathbf{y}] := \big(\tilde{\mathbf{A}}[\mathbf{y}], \tilde{b}[\mathbf{y}], \tilde{f}[\mathbf{y}], \tilde{g}[\mathbf{y}]\big). \tag{3.13}$$

Before we can discuss the regularity of mapping that sends the data to a solution in Section 3.5 and the parametric regularity of solutions with the parametric data $\tilde{d}[\mathbf{y}]$ in Section 3.6, we here consider the solvability of the residual equation

$$\forall d \in D : u \in \mathcal{U} \text{ such that } R(d,u) = 0 \text{ in } \mathcal{R}.$$

First, we establish strong monotonicity of the nonlinear operator $w \mapsto R(d,w)$.

Lemma 9. *Let $\mathfrak{N}$ satisfy the polynomial growth bound* (3.3) *and the monotonicity* (3.4). *Then, for every $d \in D$, the operator $\mathcal{U} \to \mathcal{R}$, $w \mapsto R(d,w)$ is strongly monotone in the sense of* [16, Definition 11.1] *with constant $c_{PF}^{-2} c_{\mathbf{A}} > 0$, i.e.*

$$\big(R(d,w_1) - R(d,w_2)\big)(w_1 - w_2) \geq c_{PF}^{-2} c_{\mathbf{A}} \|w_1 - w_2\|^2_{\mathcal{U}}$$

holds for all $w_1, w_2 \in \mathcal{U}$*, where* $c_{PF} > 1$ *is the Poincaré-Friedrichs constant satisfying*

$$\|v\|_{\mathcal{U}}^2 \leq c_{PF}^2 \big\langle \nabla v, \nabla v \big\rangle_{\hat{G}}$$

for all $v \in \mathcal{U}$*.*

Proof. We note that, by the the stated values for q, we have the continuous Sobolev embedding $H^1(\hat{G}) \subset L^q(\hat{G})$. Therefore, the polynomial growth bound (3.3) implies that the residual map $\mathcal{U} \to \mathcal{R},\ w \mapsto R(d,w)$ is continuous. Hence, we are left to verify monotonicity. Obviously, we have

$$\begin{aligned}\big(R(d,w_1) - R(d,w_2)\big)&(w_1 - w_2)\\ &= \big\langle \mathbf{A}\nabla u_1 - \mathbf{A}\nabla u_2, \nabla u_1 - \nabla u_2 \big\rangle_{\hat{G}} + \big\langle bN(u_1) - bN(u_2), u_1 - u_2 \big\rangle_{\hat{G}}.\end{aligned}$$

The asserted strong monotonicity thus follows, as by the assumed ellipticity (3.9)

$$\big\langle \mathbf{A}\nabla u_1 - \mathbf{A}\nabla u_2, \nabla u_1 - \nabla u_2 \big\rangle_{\hat{G}} \geq c_{\mathbf{A}} \big\langle \nabla(u_1 - u_2), \nabla(u_1 - u_2) \big\rangle_{\hat{G}}$$

holds and

$$\big\langle bN(u_1) - bN(u_2), u_1 - u_2 \big\rangle_{\hat{G}} \geq 0$$

holds by (3.4) and the (assumed) non-negativity of the coefficient $b(\hat{x})$ in $\hat{G}$. $\square$

This now directly yields the following result.

Lemma 10. *Let* $\mathfrak{N}$ *satisfy the polynomial growth bound* (3.3) *and the monotonicity* (3.4)*. Then, for every* $d \in D$*, there exists a unique* $u \in \mathcal{U}$ *which fulfils the residual equation* $R(d,u) = 0$*. In addition, we have the injectivity bound*

$$\big\|u\big\|_{\mathcal{U}} \leq 2c_{PF}^2 c_{\mathbf{A}}^{-1} \|d\|_{\mathcal{D}}. \tag{3.14}$$

Proof. As $\mathcal{U}$ is a real, separable Hilbert space and $\mathcal{R}$ is its dual and, for every $d \in D$, the operator $\mathcal{U} \to \mathcal{R},\ u \mapsto R(d,u)$ is strongly monotone, the existence theorem on monotone operator equations, [16, Theorem 11.2], implies the existence and uniqueness of a $u \in \mathcal{U}$ which fulfils the residual equation $R(d,u) = 0$. For the bound, we calculate by using the strong monotonicity

$$c_{\mathrm{PF}}^{-2} c_{\mathbf{A}} \|u\|_{\mathcal{U}}^2 \leq \big(R(d,u) - R(d,0)\big)(u - 0) \leq \big\|R(d,0)\big\|_{\mathcal{R}} \|u\|_{\mathcal{U}}.$$

Then, as

$$\begin{aligned}\Big|\big(R(d,0)\big)(v)\Big| &= \Big|-\big\langle f, v \big\rangle_{\hat{G}} - \big\langle g, v \big\rangle_{\hat{\Gamma}_{\mathrm{N}}}\Big|\\ &\leq 2\max\big\{\|f\|_{H_{\mathrm{D}}^{-1}(\hat{G})}, \|g\|_{H^{-1/2}(\hat{\Gamma}_{\mathrm{N}})}\big\} \|v\|_{\mathcal{U}} \leq 2\|d\|_{\mathcal{D}} \|v\|_{\mathcal{U}}\end{aligned}$$

holds for every $v \in \mathcal{U}$, the asserted bound follows. $\square$

Therefore, there exists a unique *data-to-solution mapping* $S\colon D \to \mathcal{U}$ such that the equation

$$R\big(d, S(d)\big) = 0 \text{ in } \mathcal{R}$$

is fulfilled for all $d \in D$. Moreover, we know that S maps bounded nonempty subsets $B \subset D$ to bounded nonempty subsets $S(B) \subset \mathcal{U}$ and it is straightforward to show that it is indeed not only continuous but even locally Lipschitz continuous. Now, the solutions of (3.11) can be stated as $\hat{u}[\mathbf{y}] = S\big(\tilde{d}[\mathbf{y}]\big)$, where we call $\hat{u}\colon \square \to H^1_{\mathrm{D}}(\hat{G})$ the *parameters-to-solution mapping.*

Using the first Fréchet derivative of N from (3.6), the first Fréchet derivative of R immediately implies in addition the following result concerning the linear maps $\mathrm{D}_2\, R(d,u) \in \mathcal{B}(\mathcal{U};\mathcal{R})$.

Proposition 11. *Let $\mathfrak{N}$ be a polynomial nonlinearity satisfying* (3.2) *and the monotonicity* (3.4)*. Then, for all $d \in D$, $u \in \mathcal{U}$ and $v \in \mathcal{U}$, we have*

$$\big(\mathrm{D}_2\, R(d,u)[u_1]\big)(v) = \big\langle \mathbf{A} \nabla u_1, \nabla v \big\rangle_{\hat{G}} + \big\langle b \,\mathrm{D}\, N(u)[u_1], v \big\rangle_{\hat{G}}.$$

Hence, $\mathrm{D}_2\, R(d,u) \in \mathcal{B}(\mathcal{U};\mathcal{R})$ *is strongly monotone with constant $c_{PF}^{-2} c_{\mathbf{A}}$ and, therefore, a Banach space isomorphism with*

$$\Big\| \big(\mathrm{D}_2\, R(d,u)\big)^{-1} \Big\|_{\mathcal{B}(\mathcal{R};\mathcal{U})} \leq c_{PF}^2 c_{\mathbf{A}}^{-1}.$$

3.5. **Regularity of the data-to-solution mapping.** We shall focus on the regularity of the data-to-solution mapping S. To this end, we first consider the structure of the residual operator R. A first observation is that the polynomial nonlinearity $\mathfrak{N}$ implies a polynomial structure of the residual operator R.

Lemma 12. *Let $\mathfrak{N}$ be a polynomial nonlinearity satisfying* (3.2)*. Then, the residual operator $R\colon D \times \mathcal{U} \to \mathcal{R}$ is a continuous polyomial map between the Banach spaces $\mathcal{D} \times \mathcal{U}$ and $\mathcal{R}$. In particular, R is real analytic everywhere and its first Fréchet derivative is characterised by*

$$\begin{aligned}\Big(\mathrm{D}\, R(d,u)\big[(d_1,u_1)\big]\Big)(v) &= \big\langle \mathbf{A}_1 \nabla u + \mathbf{A} \nabla u_1, \nabla v \big\rangle_{\hat{G}} \\ &\quad + \big\langle b_1 N(u) + b \,\mathrm{D}\, N(u)[u_1], v \big\rangle_{\hat{G}} - \big\langle f_1, v \big\rangle_{\hat{G}} - \big\langle g_1, v \big\rangle_{\hat{\Gamma}_{\mathrm{N}}},\end{aligned}$$

its second Fréchet derivative by

$$\begin{aligned}\Big(\mathrm{D}^2\, R(d,u)\big[(d_1,u_1),(d_2,u_2)\big]\Big)(v) &= \big\langle \mathbf{A}_1 \nabla u_2 + \mathbf{A}_2 \nabla u_1, \nabla v \big\rangle_{\hat{G}} \\ &\quad + \big\langle b \,\mathrm{D}^2\, N(u)[u_1,u_2] + b_1 \,\mathrm{D}\, N(u)[u_2] + b_2 \,\mathrm{D}\, N(u)[u_1], v \big\rangle_{\hat{G}}\end{aligned}$$

and, for $n \in \mathbb{N}_{\geq 3}$, its nth Fréchet derivative by

$$\Big(\mathrm{D}^n\, R(d,u)\big[(d_1,u_1),(d_2,u_2),\ldots,(d_n,u_n)\big]\Big)(v)$$
$$= \left\langle b\,\mathrm{D}^n\, N(u)[u_1,u_2,\ldots,u_n] + \sum_{\sigma\in\Pi_n} b_{\sigma(1)} N^{(n-1)}(u)[u_{\sigma(2)},\ldots,u_{\sigma(n)}], v\right\rangle_{\hat{G}}.$$

Note that we thus especially have that $\mathrm{D}^n\, R(d,u) = 0$ *holds for* $n \geq \lfloor q-1 \rfloor + 2$.

Proof. It is straightforward to see that the first, third and fourth terms on the right-hand side of the definition

$$\big(R(d,u)\big)(v) := \big\langle \mathbf{A}\,\nabla u, \nabla v\big\rangle_{\hat{G}} + \big\langle bN(u), v\big\rangle_{\hat{G}} - \big\langle f, v\big\rangle_{\hat{G}} - \big\langle g, v\big\rangle_{\hat{\Gamma}_{\mathrm{N}}}$$

are continuous polynomial maps between $\mathcal{D}\times\mathcal{U}$ and $\mathcal{R}$. For the second term, it is clear that the map

$$L^\infty(\hat{D}) \times L^q(\hat{D}) \to L^{q/(q-1)}(\hat{D}), \quad (b,u) \mapsto bN(u)$$

is a continuous polynomial map between $L^\infty(\hat{G}) \times L^q(\hat{G})$ and $L^{q/(q-1)}(\hat{D})$. As we have $\mathcal{U} = H^1_{\mathrm{D}}(\hat{G}) \subset L^q(\hat{G})$ and by duality $L^{q/(q-1)}(\hat{G}) \subset H^{-1}_{\mathrm{D}}(\hat{G}) = \mathcal{R}$, it is also a continuous polynomial between $L^\infty(\hat{G}) \times \mathcal{U}$ and $\mathcal{R}$. Thus, R is a continuous polynomial between $\mathcal{D}$ and $\mathcal{R}$. Lastly, the characterisations of the derivatives of R now simply may be calculated algebraically and by using (3.6). ☐

The characterisation of the Fréchet derivatives of R can be used to derive bounds of $\mathrm{D}^n\, R$. However, the fact that R is a continuous polynomial map trivially implies the following assertion.

Proposition 13. *Let* $\mathfrak{N}$ *be a polynomial nonlinearity satisfying* (3.2)*. Then, for any bounded nonempty subsets* $B \subset D$ *and* $V \subset \mathcal{U}$ *there exists a constant* $\varsigma \geq 1$ *such that*

$$\big\|\mathrm{D}^n\, R(d,u)\big\|_{\mathcal{B}^n(\mathcal{D}\times\mathcal{U};\mathcal{R})} \leq n!\varsigma$$

holds for all $n \in \mathbb{N}$ *and all* $d \in B$ *and* $u \in V$*. Note that* $\big\|\mathrm{D}^n\, R(d,u)\big\|_{\mathcal{B}^n(\mathcal{D}\times\mathcal{U};\mathcal{R})} = 0$ *holds for* $n \geq \lfloor q-1 \rfloor + 2$.

With these bounds at hand, we arrive at the bounds for the Fréchet derivatives of the data-to-solution mapping S.

Theorem 14. *Let* $\mathfrak{N}$ *be a polynomial nonlinearity satisfying* (3.2) *and the monotonicity* (3.4) *and let* $\alpha := c_{PF}^2 c_{\mathbf{A}}^{-1}$*. Then, for any bounded nonempty subset* $B \subset D$ *there exists a constant* $\varsigma \geq 1$ *such that for any* $d \in B$ *we have*

$$\big\|\mathrm{D}^n\, S(d)\big\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \leq\; n!\tilde{\varsigma}\tilde{\digamma}^n$$

for all $n \in \mathbb{N}^*$*, where*

$$\tilde{\varsigma} = \frac{1}{c_\kappa \alpha\varsigma} \quad \textit{and} \quad \tilde{\digamma} = c_\kappa \alpha^2 \varsigma^2.$$

Proof. Simply apply Theorem 6 with $s = 1$, upon noting that the Propositions 13 and 11 together with equation (3.14) and Remark 5 guarantee that its premises are fulfilled after setting $\digamma := 1$. $\square$

3.6. **Regularity of the parameters-to-data and the parameters-to-solution mapping.** Having shown the regularity of the data-to-solution mapping S, we next consider the smoothness of the parameters-to-data mapping $\tilde{d}$ in (3.13) and of the parameters-to-solution mapping $\hat{u}$ stemming from (3.11). To this end, we make the following assumption:

Assumption 15. The parametric domain mapping $\mathbf{V}\colon \square \to C^1(\hat{G};\mathbb{R}^m)$ satisfies (3.7) and admits bounded mixed partial derivatives of arbitrary order in the sense that, for some constants $\mu_{\mathbf{V}}, \kappa_{\mathbf{V}} > 0$, $s \geq 1$ and a $\boldsymbol{\gamma} \in \ell^1(\mathbb{N}^*)$, these satisfy

$$\left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \mathbf{V}[\mathbf{y}]\right\|_{C^1(\hat{G};\mathbb{R}^m)} \leq \left(|\boldsymbol{\alpha}|!\right)^s \mu_{\mathbf{V}} \kappa_{\mathbf{V}}^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

for all $\mathbf{y} \in \square$ and all multi-indices $\boldsymbol{\alpha}$.

Now, combining Theorem 8 together with the results found in [28, 30, 31], cf. especially [31, Lemmas 3 and 4], immediately imply smoothness of the parameters-to-data mapping $\tilde{d}$. Moreover, as the parametric domain mapping $\mathbf{V}$ is a bounded map by considering $\boldsymbol{\alpha} = \mathbf{0}$, this is also true for the parameters-to-data mapping $\tilde{d}$. Therefore, we also have smoothness of the parameters-to-solution mapping $\hat{u}$ by combining Theorems 8 and 14, as $\tilde{d}(\square) \subset D$ holds. Specifically, this gives the following result:

Theorem 16. *Let $\mathfrak{N}$ be a polynomial nonlinearity satisfying* (3.2) *and the monotonicity* (3.4) *and suppose that Assumption 15 holds. Then, both the parameters-to-data mapping $\tilde{d}\colon \square \to D$ and the parameters-to-solution mapping $\hat{u}\colon \square \to H^1_{\mathrm{D}}(\hat{G})$ have bounded mixed partial derivatives of arbitrary order and there exist constants $\mu_{\tilde{d}}, \kappa_{\tilde{d}} \geq 1$ and $\mu_{\hat{u}}, \kappa_{\hat{u}} \geq 1$ such that*

$$\left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \tilde{d}[\mathbf{y}]\right\|_{\mathcal{D}} \leq \left(|\boldsymbol{\alpha}|!\right)^s \mu_{\tilde{d}} \kappa_{\tilde{d}}^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

and

$$\left\|\partial^{\boldsymbol{\alpha}} \hat{u}[\mathbf{y}]\right\|_{H^1_{\mathrm{D}}(\hat{G})} \leq \left(|\boldsymbol{\alpha}|!\right)^s \mu_{\hat{u}} \kappa_{\hat{u}}^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

hold for all $\mathbf{y} \in \square$ and all multi-indices $\boldsymbol{\alpha}$.

We note that Theorems 14 and 16 themself do not actually rely on the fact that $d \in D$ or $\tilde{d}[\mathbf{y}]$ stem from a domain mapping. Specifically, the bounds of the mixed partial derivatives of arbitrary order of $\hat{u}$ are true for any parameters-to-data mapping $\tilde{d}$ that satisfies the bounds of the mixed partial derivatives of arbitrary order given in Theorem 16, as long as $\tilde{d}(\square) \subset D$ holds. Therefore, Theorem 16 is also applicable when considering the semilinear elliptic PDE on a deterministic domain but with random coefficients, as was considered in [10, 27] for example. Indeed, Theorems 14 and 16 relate to [10, Section 2.3 and especially Remark 2.6] when $s = 1$. Theorem 14 implies that it is not necessary to restrict the data d to a set $B \subset D$ that is compact in $\mathcal{D}$ but that its boundedness suffices in order to provide an explicit description of where a holomorphic extension can be defined by simply using the Cauchy–Hadamard formula, see equation (2.10).

4. Higher spatial regularity and non-polynomial nonlinearities

Up to this point, we have considered the solutions in their "energy" variational space. However, it is known that analytic or Gevrey parametric regularity with higher order spatial regularity of the solutions is mandatory for achieving dimension-independent convergence rates using multilevel quadrature or collocation methods, see [24, 29, 31, 36, 41] for example. Therefore, to demonstrate how higher spatial regularity of solutions is also handled within the framework of Section 2, we consider particular cases of the semilinear PDE (3.1). Especially, by leveraging the higher spatial regularity of the solution, we also will be able to consider non-polynomial analytic as well as s-Gevrey nonlinearities $\mathfrak{N}$ here.

For this, we assume that $\hat{G} \subset \mathbb{R}^m$ has a $C^{1,1}$-smooth boundary, that $\Gamma_{\mathrm{N}} = \emptyset$, so that $H^1_{\mathrm{D}} = H^1_0$ holds, and that the parametric domain mapping fulfils $\mathbf{V}\colon \square \to C^{1,1}(\hat{G}; \mathbb{R}^m)$. We assume that $\mathbf{V}[\mathbf{y}]$ is a $C^{1,1}$-isomorphism and fulfils the uniformity condition

$$\big\|\mathbf{V}[\mathbf{y}]\big\|_{C^{1,1}(\hat{G};\mathbf{V}[\mathbf{y}](\hat{G}))} \leq c_{\mathbf{V}} \quad \text{and} \quad \big\|(\mathbf{V}[\mathbf{y}])^{-1}\big\|_{C^{1,1}(\mathbf{V}[\mathbf{y}](\hat{G});\hat{G})} \leq c_{\mathbf{V}} \tag{4.1}$$

for every $\mathbf{y} \in \square$, where $c_{\mathbf{V}} \geq 1$ is a fixed constant. Moreover, we restrict ourselves to the cases where $m \in \{1, 2, 3\}$.

We shall also assume that $\hat{a} \in W^{1,\infty}(\hat{G})$, $\hat{b} \in L^\infty(\hat{G})$, and $\hat{f} \in L^2(\hat{G})$, which means that $a[\mathbf{y}] \in W^{1,\infty}\big(G[\mathbf{y}]\big)$, $b[\mathbf{y}] \in L^\infty\big(G[\mathbf{y}]\big)$ and $f[\mathbf{y}] \in L^2\big(G[\mathbf{y}]\big)$ for every $\mathbf{y} \in \square$. It is straightforward to see that these assumptions imply $\tilde{\mathbf{A}}[\mathbf{y}] \in W^{1,\infty}\big(\hat{G}; \mathbb{R}^{d\times d}_{\mathrm{sym}}\big)$, $\tilde{b}[\mathbf{y}] \in L^\infty(\hat{G})$ and $\tilde{f}[\mathbf{y}] \in L^2_{\mathrm{D}}(\hat{G})$ for every $\mathbf{y} \in \square$.

Lastly, we assume that $\mathfrak{N}\colon \mathbb{R} \to \mathbb{R}$ is monotonically increasing, i.e. that (3.4) holds, and fulfils $\mathfrak{N}(0) = 0$. However, instead of requiring that it fulfils (3.2), we require the stronger polynomial growth bound that $\mathfrak{N}$ fulfils

$$\forall \zeta \in \mathbb{R}: \quad \big|\mathfrak{N}(\zeta)\big| \leq c_{\mathfrak{N}}\big(1 + |\zeta|^{q/2}\big) \tag{4.2}$$

for finite constants $c_{\mathfrak{N}} > 0$ and $q > 0$ and that $\mathfrak{N}$ is s-Gevrey for some fixed $s \geq 1$, i.e. for every compact $K \subset \mathbb{R}$, there exists a constant $c_{\mathfrak{N},K} > 0$ such that for all $n \in \mathbb{N}$

$$\sup_{\zeta \in K} |\mathfrak{N}^{(n)}(\zeta)| \leq (c_{\mathfrak{N},K})^{n+1}(n!)^s\,. \tag{4.3}$$

holds.

We observe that (4.2) is a global condition on $\mathfrak{N}$, whereas (4.3) are localised to compacta $K \subset \mathbb{R}$. As (3.3), condition (4.2) ensures global existence of weak solutions. It also ensures their H^2 regularity, uniformly with respect to the data: (4.2) ensures that one has the continuous Nemyckii operator associated with $\mathfrak{N}$,

$$L^q\big(G[\mathbf{y}]\big) \to L^2\big(G[\mathbf{y}]\big),\ u \mapsto \mathfrak{N} \circ u,$$

and that there holds the continuous embedding $H_0^1\big(G[\mathbf{y}]\big) \subset L^q\big(G[\mathbf{y}]\big)$, by the Sobolev embedding theorem, and uniformly for every $\mathbf{y} \in \square$.

On the other hand, condition (4.3) will be sufficient to ensure that we have an s-Gevrey smooth Nemyckii operator associated with $\mathfrak{N}$ defined as

$$N\colon L^\infty\big(G[\mathbf{y}]\big) \to L^\infty\big(G[\mathbf{y}]\big) : u \mapsto \mathfrak{N} \circ u, \tag{4.4}$$

which we will use in combination with the continuous embedding $H^2\big(G[\mathbf{y}]\big) \subset L^\infty\big(G[\mathbf{y}]\big)$, being valid uniformly with respect to $\mathbf{y} \in \square$, that we have by the Sobolev embedding theorem.

Example 1. *We provide examples for the nonlinear term* $\mathfrak{N}$.

(1) *A first valid example for a nonlinearity* $\mathfrak{N}$ *is the cubic nonlinearity* $\mathfrak{N}(\zeta) = \zeta^3$. *Evidently,* (4.3) *is valid with* $s = 1$. *Also,* (4.2) *holds with* $q = 6$ *and* $\mathfrak{N}'(\zeta) = 3\zeta^2 \geq 0$ *from which* (3.4) *follows.*

(2) *A second example for* $\mathfrak{N}$ *is*

$$\mathfrak{N}(\zeta) = \frac{\zeta^3}{1 + \exp(-1/\zeta^2)}\,, \quad \zeta \neq 0.$$

For $\zeta \to 0$, *the definition of* $\mathfrak{N}(\zeta)$ *is completed with the corresponding limits such as* $\zeta^k \exp(-1/\zeta^2) \to 0$ *for any finite* k. *One verifies that* $\zeta \mapsto \mathfrak{N}(\zeta)$ *is smooth, but not analytic, and that for* $0 \neq \zeta \in \mathbb{R}$

$$\mathfrak{N}'(\zeta) = \frac{3\zeta^2}{(1 + \exp(-1/\zeta^2))} \left[1 - \frac{1}{3}\zeta^{-2} \exp(-\zeta^{-2})(1 + \exp(-1/\zeta^2))^{-1}\right]\,,$$

so that $\mathfrak{N}'(\zeta) > 0$ *for all* $\zeta \in \mathbb{R}$ *whence the monotonicity of* $\mathfrak{N}$ *in* (3.4) *follows. Evidently, then also the growth condition* (4.2) *holds with* $q = 6$. *Furthermore,* $\mathfrak{N}$ *is* s*-Gevrey regular with* $s \geq 3/2$.

(3) *An example of an analytic, nonpolynomial nonlinearity is*

$$\mathfrak{N}(\zeta) = 2 + \tanh(\zeta) = 2 + \frac{\exp(\zeta) - \exp(-\zeta)}{\exp(\zeta) + \exp(-\zeta)}\,.$$

Then $\mathfrak{N}(\zeta) \in [1, 3]$ *and* $\mathfrak{N}$ *is analytic (i.e.,* 1*-Gevrey) at all* $\zeta \in \mathbb{R}$. *Due to*

$$\mathfrak{N}'(\zeta) = 1/\cosh^2(\zeta) = 4/(\exp(\zeta) + \exp(-\zeta))^2 > 0$$

for all $\zeta \in \mathbb{R}$*, it satisfies* (3.4) *and also* (4.2) *with* $q = 0$.

(4) *A nonlinearity which is* not *covered is* $\mathfrak{N}(\zeta) = \exp(\zeta)$ *which appears in mathematical models of combustion, for example. While* $\mathfrak{N}$ *is analytic (condition* (4.3) *even holds with* $s = 0$*) and monotone, the polynomial growth condition* (4.2) *cannot be satisfied.*

4.1. **Residual Equation.** In view of equation (3.11) and Section 2, we choose

$$\begin{aligned} \mathcal{D} &:= W^{1,\infty}\big(\hat{G};\mathbb{R}^{d\times d}_{\mathrm{sym}}\big) \times L^\infty(\hat{G}) \times L^2(\hat{G}), \\ \mathcal{U} &:= H^1_0(\hat{G}) \cap H^2(\hat{G}), \\ \mathcal{R} &:= L^2(\hat{G}) \end{aligned} \tag{4.5}$$

and the subset of admissible data

$$D := D_{\mathbf{A}} \times D_b \times L^2(\hat{G}),$$

where

$$D_{\mathbf{A}} := \left\{ \mathbf{A} \in W^{1,\infty}\big(\hat{G};\mathbb{R}^{m\times m}_{\mathrm{sym}}\big) : \operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \min_{\mathbf{v}\in\mathbb{R}^m\setminus\{\mathbf{0}\}} \frac{\mathbf{v}^\intercal \mathbf{A}(\mathbf{x})\mathbf{v}}{\mathbf{v}^\intercal\mathbf{v}} \geq c_a c_{\mathbf{V}}^{-m-2} =: c_{\mathbf{A}} \right\}$$

and $D_b := \big\{ b \in L^\infty\big(\hat{G}\big) : \operatorname{ess\,inf}_{\mathbf{x}\in\hat{G}} b(\mathbf{x}) \geq 0 \big\}$. For the sake of legibility, we will associate the data variable $d \in \mathcal{D}$ to be given by $(\mathbf{A}, b, f)$ and also extend this to modifications of d, i.e. $d_1 = (\mathbf{A}_1, b_1, f_1)$.

Now, we define the residual operator $R\colon D \times \mathcal{U} \to \mathcal{R}$ by setting

$$R(d, u) := -\operatorname{div}(\mathbf{A}\,\nabla\, u) + bN(u) - f. \tag{4.6}$$

With this residual operator R, equation (3.11) can be restated as the residual equation

$$R\big(\tilde{d}[\mathbf{y}], \hat{u}[\mathbf{y}]\big) = 0 \text{ in } \mathcal{R},$$

where $\tilde{d}\colon \square \to D$ is the *paramaters-to-data mapping* given by

$$\tilde{d}[\mathbf{y}] := \big(\tilde{\mathbf{A}}[\mathbf{y}], \tilde{b}[\mathbf{y}], \tilde{f}[\mathbf{y}]\big).$$

In this formulation, we now first consider the solvability of the residual equation

$$R(d, u) = 0$$

for a given $d \in D$ and unknown $u \in \mathcal{U}$.

Theorem 17. *Let* $\mathfrak{N}$ *satisfy the polynomial growth bound* (4.2) *and the monotonicity* (3.4)*. Then, for every* $d \in D$*, there exists a unique* $u_d \in \mathcal{U}$ *that fulfils the residual equation* $R(d, u) = 0$ *in* $\mathcal{R}$*. Moreover, for any bounded nonempty subset* $B \subset D$*, there exists a constant* $c_B > 0$ *such that for all* $d \in B$ *holds*

$$\|u_d\|_{\mathcal{U}} \leq c_B \|d\|_{\mathcal{D}}. \tag{4.7}$$

Proof. We let $\mathcal{X} = H_0^1(\hat{G})$, then we define the operator $T_d\colon \mathcal{X} \to \mathcal{X}'$ as

$$\big(T_d(u)\big)(v) := \big\langle \mathbf{A}\nabla u, \nabla v\big\rangle_{\hat{G}} + \big\langle bN(u), v\big\rangle_{\hat{G}} - \big\langle f, v\big\rangle_{\hat{G}}.$$

For $u_1, u_2 \in \mathcal{U}$, we have

$$\begin{aligned}&\big(T_d(u_1) - T_d(u_2)\big)(u_1 - u_2)\\ &\qquad = \big\langle \mathbf{A}\nabla u_1 - \mathbf{A}\nabla u_2, \nabla u_1 - \nabla u_2\big\rangle_{\hat{G}} + \big\langle bN(u_1) - bN(u_2), u_1 - u_2\big\rangle_{\hat{G}}.\end{aligned}$$

Since

$$\big\langle \mathbf{A}\nabla u_1 - \mathbf{A}\nabla u_2, \nabla u_1 - \nabla u_2\big\rangle_{\hat{G}} \geq c_{\mathbf{A}}\big\langle \nabla(u_1 - u_2), \nabla(u_1 - u_2)\big\rangle_{\hat{G}}$$

holds by ellipticity and

$$\big\langle bN(u_1) - bN(u_2), u_1 - u_2\big\rangle_{\hat{G}} \geq 0$$

holds by monotonicity of N_K and non-negativity of b, we have that T_d is strongly monotone with

$$\big(T_d(u_1) - T_d(u_2)\big)(u_1 - u_2) \geq c_{\mathrm{PF}}^{-2} c_{\mathbf{A}} \|u_1 - u_2\|_{\mathcal{X}}^2.$$

Here, $c_{\mathrm{PF}} > 1$ is the Poincaré-Friedrichs constant satisfying

$$\|v\|_{\mathcal{X}}^2 \leq c_{\mathrm{PF}}^2 \big\langle \nabla v, \nabla v\big\rangle_{\hat{G}}$$

for all $v \in \mathcal{X}$. Hence, as $\mathcal{X}$ is a real, separable Hilbert space for every $d \in D$, the theorem on monotone operator equations, see [16, Theorem 11.2], implies the existence and uniqueness of a u_d that fulfils the operator equation $T_d(u_d) = 0$. Moreover, using the strong monotonicity, we have

$$c_{\mathrm{PF}}^{-2} c_{\mathbf{A}} \|u_d - 0\|_{\mathcal{X}}^2 \leq \big(T_d(u_d) - T_d(0)\big)(u_d - 0) \leq \big\langle -T_d(0)\big\rangle(u_d),$$

which yields that

$$\|u_d\|_{\mathcal{X}} \leq c_{\mathrm{PF}}^2 c_{\mathbf{A}}^{-1} \|f\|_{L^2(\hat{G})} \leq c_{\mathrm{PF}}^2 c_{\mathbf{A}}^{-1} \|d\|_{\mathcal{D}}.$$

Next, we set $w_d := u_d$ and note that w_d fulfils the equation

$$\big\langle \mathbf{A}\nabla w_d, \nabla v\big\rangle_{\hat{G}} = \big\langle f - bN(u_d), v\big\rangle_{\hat{G}}.$$

As u_d is in $L^q(\hat{G})$, we have that the term $f - bN(u_d)$ is an element of $L^2(\hat{G})$ with

$$\begin{aligned}\big\|f - bN(u_d)\big\|_{L^2(\hat{G})} &\leq \|f\|_{L^2(\hat{G})} + c_{\mathfrak{N}} \|b\|_{L^\infty(\hat{G})} \big(1 + \|u_d\|_{L^q(\hat{G})}^{q/2}\big)\\ &\leq \|d\|_{\mathcal{D}} \big(1 + c_{\mathfrak{N}} + c_{\mathfrak{N}} c_{\mathrm{PF}}^q c_{\mathbf{A}}^{-q/2} \|d\|_{\mathcal{D}}^{q/2}\big).\end{aligned}$$

Therefore, by elliptic regularity, see [22, Theorem 8.12], we know that $w_d \in H^2(\hat{G})$ with

$$\|w_d\|_{H^2(\hat{G})} \leq c_{\mathrm{er}} \|d\|_{\mathcal{D}} \big(1 + c_{\mathfrak{N}} + c_{\mathfrak{N}} c_{\mathrm{PF}}^q c_{\mathbf{A}}^{-q/2} \|d\|_{\mathcal{D}}^{q/2}\big), \tag{4.8}$$

where c_{er} only depends on m, $\hat{G}$, $c_{\mathbf{A}}$ and an upper bound for $\|\mathbf{A}\|_{W^{1,\infty}(\hat{G})}$.

Obviously, we thus have that $u_d \in H_0^1(\hat{G}) \cap H^2(\hat{G})$ and that u_d indeed fulfils the residual equation

$$R(d, u_d) = 0.$$

Moreover, for any bounded nonempty subset $B \subset D$, the assertion follows by setting

$$c_B \geq c_{\mathrm{er}}\big(1 + c_{\mathfrak{N}} + c_{\mathfrak{N}} c_{\mathrm{PF}}^q c_{\mathbf{A}}^{-q/2} K^{q/2}\big)$$

with $K = \sup_{d \in B} \|d\|_{\mathcal{D}}$ and where c_{er} is chosen with the upper bound K for $\|\mathbf{A}\|_{W^{1,\infty}(\hat{G})}$. $\square$

We thus know that there exists a unique, global data-to-solution mapping $S \colon D \to \mathcal{U}$ such that the equation

$$R\big(d, S(d)\big) = 0$$

is fulfilled for all $d \in D$. Moreover, we know that S maps bounded nonempty subsets $B \subset D$ to bounded nonempty subsets $S(B) \subset \mathcal{U}$. Hence, the solutions of (3.11) here can be stated as $\hat{u}[\mathbf{y}] = S\big(\tilde{d}[\mathbf{y}]\big)$, where $\hat{u} \colon \square \to H_0^1(\hat{G}) \cap H^2(\hat{G})$ is the parameters-to-solution mapping.

4.2. **Regularity of the data-to-solution mapping.** We now focus on the regularity of the data-to-solution mapping S. To this end, we first consider the regularity of the Nemyckii operator $N \colon L^\infty(\hat{G}) \to L^\infty(\hat{G})$. Indeed, between these spaces it turns out that N inherits the differentiability and smoothness of $\mathfrak{N}$.

Lemma 18. *Assume that $\hat{G}$ is a bounded Lipschitz domain and the s-Gevrey regularity* (4.3) *of the nonlinearity $\mathfrak{N}$. Then, the Nemyckii operator*

$$N \colon L^\infty(\hat{G}) \to L^\infty(\hat{G}), u \mapsto \mathfrak{N} \circ u$$

is s-Gevrey and its Fréchet derivatives are given by

$$\mathrm{D}^n N(u)[u_1, \ldots, u_n](\mathbf{x}) = \mathfrak{N}^{(n)}\big(u(\mathbf{x})\big) \cdot u_1(\mathbf{x}) \cdots u_n(\mathbf{x}).$$

Moreover, for any bounded nonempty subset $V \subset L^\infty(\hat{G})$, there exist constants $\varsigma, F \geq 1$ such that

$$\big\|\mathrm{D}^n N(u)\big\|_{\mathcal{B}^n(L^\infty(\hat{G}); L^\infty(\hat{G}))} \leq (n!)^s \varsigma F^n$$

holds for all $n \in \mathbb{N}$ and all $u \in V$.

Proof. We define the Nemyckii operators $N_n \colon L^\infty(\hat{G}) \to L^\infty(\hat{G}),\ u \mapsto \mathfrak{N}^{(n)} \circ u$ for all $n \in \mathbb{N}$. For any arbitrary open bounded nonempty subset $V \subset L^\infty(\hat{G})$, there exists a $K > 0$ such that $\|u\|_{L^\infty(\hat{G})} < K$ holds for all $u \in V$. As $[-K, K]$ is compact in $\mathbb{R}$, there exist two constants $\varsigma, F \geq 1$ such that

$$\big|\mathfrak{N}^{(n)}(\zeta)\big| \leq (n!)^s \varsigma F^n$$

holds for all $n \in \mathbb{N}$ and all $\zeta \in [-K, K]$, by s-Gevrey smoothness of $\mathfrak{N}$.

Now, consider any $u \in V$ and $h \in L^\infty(\hat{G})$ with $u + h \in V$ and $h \neq 0$. Thus, we can calculate

$$\begin{aligned}
&\big\| N_n(u+h) - N_n(u) - N_{n+1}(u)h \big\|_{L^\infty(\hat{G})} \\
&\qquad\qquad = \sup_{\mathbf{x}\in\hat{G}} \Big| \mathfrak{N}^{(n)}\big(u(\mathbf{x}) + h(\mathbf{x})\big) - \mathfrak{N}^{(n)}\big(u(\mathbf{x})\big) - \mathfrak{N}^{(n+1)}\big(u(\mathbf{x})\big)h(\mathbf{x}) \Big|
\end{aligned}$$

Applying Taylor's formula for $\mathfrak{N}^{(n)}$ yields

$$\mathfrak{N}^{(n)}\big(u(\mathbf{x}) + h(\mathbf{x})\big) = \mathfrak{N}^{(n)}\big(u(\mathbf{x})\big) + \mathfrak{N}^{(n+1)}\big(u(\mathbf{x})\big)h(\mathbf{x}) + \frac{1}{2!}\mathfrak{N}^{(n+2)}\big(\xi_{\mathbf{x}}\big)\big(h(\mathbf{x})\big)^2,$$

where $\xi_{\mathbf{x}}$ lies in the convex hull of $u(\mathbf{x})$ and $u(\mathbf{x}) + h(\mathbf{x})$. However, we have that $\xi_{\mathbf{x}} \in [-K, K]$ holds, and therefore also

$$\left| \frac{1}{2!}\mathfrak{N}^{(n+2)}\big(\xi_{\mathbf{x}}\big) \right| \leq \frac{\big((n+2)!\big)^s}{2!} \varsigma \digamma^{n+2}.$$

This proves that we have

$$\frac{\big\| N_n(u+h) - N_n(u) - N_{n+1}(u)h \big\|_{L^\infty(\hat{G})}}{\|h\|_{L^\infty(\hat{G})}} \leq \frac{\big((n+2)!\big)^s}{2!} \varsigma \digamma^{n+2} \|h\|_{L^\infty(\hat{G})}.$$

Hence, this implies that $N_n \colon L^\infty(\hat{G}) \to L^\infty(\hat{G})$ is Fréchet differentiable with its derivative given by

$$\mathrm{D}\, N_n(u)[h] = N_{n+1}(u)h.$$

Noting that $N_0 = N$, we thus inductively have that N is infinitely Fréchet differentiable for any $u \in V$ and its derivatives are given by

$$\mathrm{D}^n\, N(u)[u_1, \ldots, u_n] = N_n(u)u_1 \cdots u_n.$$

As V can be chosen as an open bounded ball around any $u \in L^\infty(\hat{G})$, this indeed shows that N is infinitely Fréchet differentiable everywhere.

Finally, noting that

$$\big\| N_n(u) \big\|_{L^\infty(\hat{G})} \leq (n!)^s \varsigma \digamma^n$$

holds for all $n \in \mathbb{N}$ and all $u \in V$ implies the final assertion, which in turn proves the s-Gevrey smoothness of N. $\square$

Using the regularity of the Nemyckii operator $N \colon L^\infty(\hat{G}) \to L^\infty(\hat{G})$ in Lemma 18 and the continuous embeddings $\mathcal{U} \subset L^\infty(\hat{G})$ and $L^\infty(\hat{G}) \subset \mathcal{R}$ valid for the choices (4.5) and $m \leq 3$, implies the following regularity of the residual operator R.

Proposition 19. *Let $\mathfrak{N}$ satisfy the polynomial growth bound* (4.2) *and the derivative bounds* (4.3).

Then, $R\colon D \times \mathcal{U} \to \mathcal{R}$ is s-Gevrey smooth between the Banach spaces $\mathcal{D} \times \mathcal{U}$ and $\mathcal{R}$ as in (4.5)*. Indeed, for any bounded nonempty subsets $B \subset \mathcal{D}$ and $V \subset \mathcal{U}$, there exist constants $\varsigma, \digamma \geq 1$ such that*

$$\left\|\mathrm{D}^n R(d,u)\right\|_{\mathcal{B}^n(\mathcal{D}\times\mathcal{U};\mathcal{R})} \leq (n!)^s \varsigma \digamma^n$$

holds for all $n \in \mathbb{N}$ and all $d \in B$ and $u \in V$.

Proof. We recall that, per equation (4.6), R is given by

$$\big(R(d,u)\big)(v) := \big\langle \mathbf{A}\nabla u, \nabla v\big\rangle_{\hat{G}} + \big\langle bN(u), v\big\rangle_{\hat{G}} - \big\langle f, v\big\rangle_{\hat{G}}.$$

Obviously, the first term in $R(d,u)$ amounts to a bounded bilinear form

$$\mathfrak{A} : \mathcal{D} \times \mathcal{U} \to \mathcal{R}, (d,u) \mapsto \Big(v \mapsto \big\langle \mathbf{A}\nabla u, \nabla v\big\rangle_{\hat{G}}\Big)$$

and, therefore, is an analytic mapping and thus s-Gevrey smooth for every $s \geq 1$.

Similarily, the third term in $R(d,u)$ amounts to a bounded linear form

$$\mathfrak{F} : \mathcal{D} \times \mathcal{U} \to \mathcal{R}, (d,u) \mapsto \Big(v \mapsto \big\langle f, v\big\rangle_{\hat{G}}\Big),$$

which also is an analytic mapping and thus s-Gevrey smooth for every $s \geq 1$.

As the Nemyckii operator $N\colon L^\infty(\hat{G}) \to L^\infty(\hat{G})$ is s-Gevrey smooth per Lemma 18, by using the Leibniz formula, the mapping

$$\times N\circ : L^\infty(\hat{G}) \times L^\infty(\hat{G}) \to L^\infty(\hat{G}), (b,u) \mapsto bN(u)$$

with $\times$ denoting the bilinear map of pointwise a.e. multiplication of pairs of elements in $L^\infty(\hat{G})$ (which is continuous, cf. e.g. [4, Prop. 1.1]) also is s-Gevrey smooth.

With the continuity of the linear embedding maps

$$\iota_{\mathcal{U}} : \mathcal{U} \subset L^\infty(\hat{G}) \text{ and } \iota_{\mathcal{R}}^* : L^\infty(\hat{G}) \subset \mathcal{R},$$

it follows that the composite mapping $\mathfrak{M} := \iota_{\mathcal{R}}^* \circ (\times N\circ) \circ \iota_{\mathcal{U}}$

$$\mathfrak{M} : \mathcal{D} \times \mathcal{U} \to \mathcal{R}, (d,u) \mapsto \Big(v \mapsto \big\langle bN(u), v\big\rangle_{\hat{G}}\Big)$$

is s-Gevrey smooth.

By linearity of differentials, the sum $\mathfrak{A} + \mathfrak{M} + \mathfrak{F} : \mathcal{D} \times \mathcal{U} \to \mathcal{R}$ is s-Gevrey smooth.

The fact that one can find constants $\varsigma, \digamma \geq 1$ for any bounded nonempty subsets $B \subset \mathcal{D}$ and $V \subset \mathcal{U}$ follows by simple bookkeeping of the constants using Lemma 18. This completes the proof. ∎

Moreover, concerning the inverse of $\mathrm{D}_2 R(d,u) \in \mathcal{B}(\mathcal{U};\mathcal{R})$, we have the following result.

Proposition 20. *Let $\mathfrak{N}$ satisfy the polynomial growth bound* (4.2) *and the monotonicity* (3.4)*. Then,* $\mathrm{D}_2\, R(d,u) \in \mathcal{B}(\mathcal{U};\mathcal{R})$ *is a Banach space isomorphism for any* $d \in D$ *and* $u \in \mathcal{U}$*. Indeed, for any bounded nonempty subsets* $B \subset D$ *and* $V \subset \mathcal{U}$*, there exists a constant* $\alpha \geq 1$ *such that*

$$\left\| \big(\mathrm{D}_2\, R(d,u)\big)^{-1} \right\|_{\mathcal{B}(\mathcal{R};\mathcal{U})} \leq \alpha$$

holds for all $d \in B$ *and* $u \in V$*.*

Proof. For any $d \in D$ and $u \in \mathcal{U}$, we have that $\mathrm{D}_2\, R(d,u) \in \mathcal{B}(\mathcal{U};\mathcal{R})$ is given by

$$\mathrm{D}_2\, R(d,u)[u_1] = -\operatorname{div}(\mathbf{A}\,\nabla\, u_1) + b\,\mathrm{D}\, N(u)[u_1].$$

Thus, we consider the affine residual equation defined by the residual $T\colon D \times \mathcal{U} \times L^\infty(\hat{G}) \to \mathcal{R}$ given by

$$T\big((d,u,w),u_1\big) := -\operatorname{div}(\mathbf{A}\,\nabla\, u_1) + b\,\mathrm{D}\, N(u)[u_1] - w.$$

Now, completely analogous arguments as used in the proof of Theorem 17 prove the stated assertions. □

With these bounds at hand, we arrive at the bounds for the Fréchet derivatives of the data-to-solution mapping S.

Theorem 21. *Let $\mathfrak{N}$ satisfy the polynomial growth bound* (4.2) *and the monotonicity* (3.4)*. Then, for any bounded nonempty subset* $B \subset D$*, there exist constants* $\varsigma \geq 1$*,* $\digamma \geq 1$ *and* $\alpha \geq 1$ *such that, for any* $d \in B$*, we have*

$$\left\| \mathrm{D}^n\, S(d) \right\|_{\mathcal{B}^n(\mathcal{D};\mathcal{U})} \leq\ (n!)^s \tilde{\varsigma} \tilde{\digamma}^n$$

for all $n \in \mathbb{N}^*$*, where*

$$\tilde{\varsigma} = \frac{1}{c_\kappa \alpha \varsigma} \quad \textit{and} \quad \tilde{\digamma} = c_\kappa \alpha^2 \varsigma^2 \digamma^3.$$

Proof. Simply apply Theorem 6, after noting that the Propositions 19 and 20 together with equation (4.7) and Remark 5 guarantee that its premises are fulfilled. □

4.3. **Regularity of the parameters-to-data and the parameters-to-solution mapping.** Having shown the regularity of the data-to-solution mapping S, we next consider the smoothness of the parameters-to-data mapping $\tilde{d}$ and of the parameters-to-solution mapping $\hat{u}$. To this end, we make the following assumption:

Assumption 22. The parametric domain mapping $\mathbf{V}\colon \square \to C^{1,1}(\hat{G};\mathbb{R}^m)$ satisfies (4.1) and admits bounded mixed partial derivatives of arbitrary order in the sense that, for some constants $\mu_{\mathbf{V}}, \kappa_{\mathbf{V}} > 0$, $s \geq 1$ and a $\boldsymbol{\gamma} \in \ell^1(\mathbb{N}^*)$, these satisfy

$$\left\| \partial^{\boldsymbol{\alpha}}\, \mathbf{V}[\mathbf{y}] \right\|_{C^{1,1}(\hat{G};\mathbb{R}^m)} \leq \big(|\boldsymbol{\alpha}|!\big)^s \mu_{\mathbf{V}} \kappa_{\mathbf{V}}^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

for all $\mathbf{y} \in \square$ and all multi-indices $\boldsymbol{\alpha}$.

Now, combining Theorem 8 together with the results found in [31, Lemmas 3 and 4] immediately imply smoothness of the parameters-to-data mapping $\tilde{d}$. Moreover, as the parametric domain mapping $\mathbf{V}$ is a bounded map by considering $\boldsymbol{\alpha} = \mathbf{0}$, this is also true for the parameters-to-data mapping $\tilde{d}$. Therefore, we also have smoothness of the parameters-to-solution mapping $\hat{u}$ by combining Theorems 8 and 14, as $\tilde{d}(\square) \subset D$ holds. Specifically, this gives the following result:

Theorem 23. *Let $\mathfrak{N}$ be a polynomial nonlinearity satisfying* (3.2)*, and the monotonicity* (3.4) *and suppose that Assumption 22 holds. Then, both, the parameters-to-data mapping $\tilde{d}\colon \square \to D$ and the parameters-to-solution mapping $\hat{u}\colon \square \to H^1_{\mathrm{D}}(\hat{G})$, have bounded mixed partial derivatives of arbitrary order and there exist constants $\mu_{\tilde{d}}, \kappa_{\tilde{d}} \geq 1$ and $\mu_{\hat{u}}, \kappa_{\hat{u}} \geq 1$ such that*

$$\left\|\partial^{\boldsymbol{\alpha}} \tilde{d}[\mathbf{y}]\right\|_{\mathcal{D}} \leq \left(|\boldsymbol{\alpha}|!\right)^s \mu_{\tilde{d}} \kappa_{\tilde{d}}^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

and

$$\left\|\partial^{\boldsymbol{\alpha}} \hat{u}[\mathbf{y}]\right\|_{H^2(\hat{G})} \leq \left(|\boldsymbol{\alpha}|!\right)^s \mu_{\hat{u}} \kappa_{\hat{u}}^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

hold for all $\mathbf{y} \in \square$ and all multi-indices $\boldsymbol{\alpha}$.

4.4. **Nonsmooth reference domain.** The regularity shift (4.8) is under the assumption of homogeneous Dirichlet boundary conditions on $\partial\hat{G}$ that itself is assumed to be $C^{1,1}$-smooth. For mixed boundary conditions and/or polytopal domain, the H^2 regularity shift in $\hat{G}$ in (4.8) is known to fail in general. A regularity shift only holds in larger, corner-weighted (in dimension $m = 2$) or in corner-edge weighted (in dimension $m = 3$) $H^2(\hat{G})$ spaces of Kondrat'ev type. With this choice of spaces in the abstract setting (4.5), the abstract theory from Section 2 will also apply.

To demonstrate this, we choose to assume that

$$m = 2 \quad \text{and} \quad \hat{G} \text{ is a polygon with finite set } \mathcal{C} \text{ of corner points } \boldsymbol{c}.$$

Then, we require the hilbertian Kondrat'ev spaces $\mathcal{K}^k_a(\hat{G})$ given for $k \in \mathbb{N}_0$ and $a \in \mathbb{R}$ by

$$\mathcal{K}^k_a(\hat{G}) := \{u : \hat{G} \to \mathbb{R} \mid \rho_{\mathcal{C}}^{|\boldsymbol{\alpha}|-a} \partial^{\boldsymbol{\alpha}} u \in L^2(\hat{G}),\ |\boldsymbol{\alpha}| \leq k\}.$$

To also specify the data regularity in $\hat{G}$, we introduce

$$\mathcal{W}^{k,\infty}(\hat{G}) := \{u : \hat{G} \to \mathbb{R} \mid \rho_{\mathcal{C}}^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} u \in L^\infty(\hat{G}),\ |\boldsymbol{\alpha}| \leq k\}.$$

Here, $\rho_{\mathcal{C}}(x) > 0$ in $\hat{G}$ denotes the product of the distance of $x \in \hat{G}$ to the corners:

$$\rho_{\mathcal{C}}(x) := \prod_{\boldsymbol{c} \in \mathcal{C}} |x - \boldsymbol{c}|, \quad x \in \hat{G}.$$

Evidently, $W^{k,\infty}(\hat{G}) \subseteq \mathcal{W}^{k,\infty}(\hat{G})$ holds for all $k \in \mathbb{N}_0$.

We now recall the following result from [5, Theorem 1.1].

Proposition 24. *Assume $m = 2$ and that $\hat{G}$ is a polygon with a finite number of straight sides. Assume further in* (3.1) *$\hat{\Gamma}_N = \emptyset$, i.e. homogeneous Dirichlet boundary conditions on $\hat{\Gamma}_D$, and that*[3] *$b \in \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R})$.*

Then, the differential operator $P\colon w \mapsto \operatorname{div}(\mathbf{A}\nabla w) + bw$ is an isomorphism $P \in \mathcal{L}_{is}(\mathcal{K}^2_{1+a}(\hat{G}), \mathcal{K}^0_{a-1}(\hat{G}))$ for $a \in (-a_0, a_0)$ for some $a_0 > 0$ (depending on $\mathbf{A}$ and on the corner-angles of $\hat{G}$). Moreover, the inverse $P^{-1} \in \mathcal{L}_{is}(\mathcal{K}^0_{a-1}(\hat{G}),\ \mathcal{K}^2_{1+a}(\hat{G}))$ depends analytically on the data

$$d = (\mathbf{A}, b, f) \in \mathcal{D}_r = \Big(\mathcal{W}^{1,\infty}(\hat{G};\mathbb{R}^{m\times m}_{sym})\Big) \times \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R}) \times \mathcal{K}^0_{a-1}(\hat{G}).$$

Next, we recall that $\mathcal{K}^2_{1+a}(\hat{G})$ is (at least) continuously embedded into $L^\infty(\hat{G})$, when $a \geq 0$ and $m = 2$, see [15, Theorem 27 (i)]. Therefore, we set the regularity spaces as

$$\begin{aligned}
\mathcal{D}_a &:= \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R}^{m\times m}_{\text{sym}}) \times \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R}) \times \mathcal{K}^0_{a-1}(\hat{G}),\\
\mathcal{U}_a &:= H^1_0(\hat{G}) \cap \mathcal{K}^2_{1+a}(\hat{G}),\\
\mathcal{R}_a &:= \mathcal{K}^0_{a-1}(\hat{G})
\end{aligned} \tag{4.9}$$

and the subset of admissible data

$$D_a := D_{\mathbf{A}} \times D_b \times \mathcal{K}^0_{a-1}(\hat{G}),$$

where

$$D_{\mathbf{A}} := \left\{ \mathbf{A} \in \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R}^{m\times m}_{\text{sym}}) : \operatorname*{ess\,inf}_{\mathbf{x}\in\hat{G}} \min_{\mathbf{v}\in\mathbb{R}^m\setminus\{\mathbf{0}\}} \frac{\mathbf{v}^{\mathsf{T}}\mathbf{A}(\mathbf{x})\mathbf{v}}{\mathbf{v}^{\mathsf{T}}\mathbf{v}} \geq c_a c_{\mathbf{V}}^{-m-2} =: c_{\mathbf{A}} \right\}$$

and $D_b := \big\{ b \in \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R}) : \operatorname{ess\,inf}_{\mathbf{x}\in\hat{G}} b(\mathbf{x}) \geq 0 \big\}$.

With these definitions and results at hand, one can now obtain analogous results as in Subsections 4.1 and 4.2. Therefore, one again is in the abstract setting of Section 2 and obtains the following result.

Theorem 25. *Assume $m = 2$, $\hat{G} \subset \mathbb{R}^2$ is a polygon and $a \in [0, a_0)$ with the constant a_0 as in Proposition 24. Furthermore, let $\mathfrak{N}$ satisfy the polynomial growth bound* (4.2)*, the monotonicity* (3.4) *and the derivative bounds* (4.3)*.*

Then, there is a unique global data-to-solution mapping $S\colon D_a \to \mathcal{U}_a$ such that the equation

$$R\big(d, S(d)\big) = 0 \text{ in } \mathcal{R}_a$$

[3]Actually, $\rho_{\mathcal{C}}^2 b \in \mathcal{W}^{1,\infty}(\hat{G};\mathbb{R})$ is sufficient.

is fulfilled for all $d \in D_a$ and for any bounded nonempty subset $B_a \subset D_a$ there exist constants $\tilde{\varsigma}, \tilde{F} \geq 1$ such that

$$\forall d \in B_r\ \forall n \in \mathbb{N}: \quad \left\| \mathrm{D}^n\, S(d) \right\|_{\mathcal{B}^n(\mathcal{D}_a;\mathcal{U}_a)} \leq (n!)^s \tilde{\varsigma} \tilde{F}^{-n}$$

holds for all $n \in \mathbb{N}$ and $d \in B_a$.

Remark 26. Theorem 25 was formulated for homogeneous Dirichlet boundary conditions on all of $\partial\hat{G}$. For mixed boundary conditions as formulated in (3.1) with $|\hat{\Gamma}_D| > 0$ corresponding results hold, referring to [5, Theorem 5.4].

5. Conclusion

In the present article, we investigated the regularity of mappings between Banach spaces. Our main result here extends the implicit mapping theorem known for finite smoothness and holomorphy to the real analytic and Gevrey class situation, and is not only qualitative but is able to give quantitative bounds on the Fréchet derivatives of the implicit mapping using quantitative bounds on the Fréchet derivatives of the residual mapping, i.e. the mapping which is used to define it. Moreover, we also supplied results that qualitatively and quantitatively cover the regularity of the composition of mappings between Banach spaces and the composition of a mapping between Banach spaces and a (possibly nonlinear) countable parametric expansion with values in a Banach space.

Applying the quantitative version of the real analytic or the Gevrey class implicit mapping theorem to residual equations for partial differential equations amounts to a new methodology to prove regularity of the dependence of PDE solutions on their data. In particular, combining this with the quantitative regularity shown for the composition with a parametric expansion yields the type of parametric regularity results for PDE solutions that depend on inputs, which are represented in an affine-parametric manner in terms of some frame in the data space, as for example is common in uncertainty quantification. We illustrated the proposed approach for the specific example of a semilinear elliptic PDE defined in a random domain. Naturally, the scope of the present approach for uncertainty quantification is wider: it applies also to other possibly nonlinear PDEs like the p-Laplacian, Navier-Stokes equations [12] or related eigenvalue problems [8]. The present regularity bounds can constitute the basis for a numerical analysis of various discretisation strategies such as sparse-grid collocation, polynomial chaos approximation, and Smolyak and Quasi-Monte Carlo quadrature in the parametric domain of the parametric problem. Corresponding Sobolev regularity results in corner-weighted spaces in $\hat{G}$ are available in [32]. Details of single-level and multi-level algorithms for the quantification of uncertainty in nonlinear operator equations will be developed elsewhere.

Furthermore, the (quantitative versions of the) real analytic and the Gevrey class implicit mapping theorems as well as the theorems covering the composition of Gevrey class mappings are likely to be useful also beyond the field of uncertainty quantification.

Lastly, we note that further investigation into the quantitative bounds of the real analytic and Gevrey class implicit mapping theorems is of future interest. Especially, the question here is, if the falling factorial technique used in [8] or some other technique enables one to prove sharper bounds for the derivative, or if the bounds we have achieved here are optimal.

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Helmut Harbrecht and Marc Schmidlin
Departement Mathematik und Informatik
Universität Basel
Spiegelgasse 1, 4051 Basel, Schweiz.

Email address: {helmut.harbrecht,marc.schmidlin}@unibas.ch

Christoph Schwab
Seminar für Angewandte Mathematik
ETH Zürich
Rämistrasse 101, 8092 Zürich, Schweiz.

Email address: christoph.schwab@sam.math.ethz.ch