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Abstract

We establish universality and expression rate bounds for a class of neural Deep Operator Networks (DON) emulating Lipschitz (or Hölder) continuous maps $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ between (subsets of) separable Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$. The DON architecture considered uses linear encoders $\mathcal{E}$ and decoders $\mathcal{D}$ via (biorthogonal) Riesz bases of $\mathcal{X}$, $\mathcal{Y}$, and an approximator network of an infinite-dimensional, parametric coordinate map that is Lipschitz continuous on the sequence space $\ell^2(\mathbb{N})$. Unlike previous works [24, 47] which required for example $\mathcal{G}$ to be holomorphic, the present expression rate results require mere Lipschitz (or Hölder) continuity of $\mathcal{G}$. Key in the proof of the present expression rate bounds is the use of either super-expressive activations (e.g. [64, 58] and the references there) which are inspired by the Kolmogorov superposition theorem ([34] or [44, Chap.11] for a comprehensive exposition), or of nonstandard NN architectures with standard (ReLU) activations as recently proposed in [67]. We illustrate the abstract results by approximation rate bounds for emulation of a) solution operators for parametric elliptic variational inequalities, and b) Lipschitz maps of Hilbert-Schmidt operators.

keywords: Neural Networks, Operator Learning, Curse of Dimensionality, Lipschitz Continuous Operators

subject classifiction: 41A65, 68T15, 68Q32

1 Introduction

Following widespread use in data and image classification and forecasting, recent years have seen development of *Deep Neural Networks* (DNNs for short) in scientific computing as universal, and highly versatile approximation architectures, that challenge established numerical approximations such as Finite Element and Finite Difference discretizations of partial differential equations (PDEs) and integral equations. Algorithmic approaches are based on inserting DNNs into suitable (variational, weak, least squares, strong etc.) forms of the PDE under consideration. Besides the universality of DNNs which was mathematically established early on (see, e.g. [54] and references there), in recent years a much more detailed picture on the expressive power of DNNs has emerged. In particular, feed-forward DNNs with suitable architectures and activation functions can emulate practically all standard spline- and Finite Element approximation spaces commonly used in the numerical analysis of PDEs (see, e.g., [43]) and, in particular, also high-order FE spaces [50] with corresponding spectral or exponential approximation rates [52]. We refer to

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[5] and the references there for some of the algorithmic developments. A particular feature of numerical approximations of PDE solutions based on DNNs as approximation architectures that was observed in practice was the apparent insensitivity of the DNN approximation quality to the so-called "curse of dimensionality" (CoD for short). This is particularly relevant for approximating maps

$$\mathcal{G} : \mathcal{X} \to \mathcal{Y} \tag{1}$$

between (in general, infinite-dimensional) separable Hilbert spaces[1] $\mathcal{X}$ and $\mathcal{Y}$. Operators $\mathcal{G}$ as in (1) emerge for example as parameter-to-solution mappings for parametric PDEs within the field of Uncertainty Quantification (see, e.g., [48] and the references there), or in so-called digital twins of complex, physical systems governed by partial differential equations (PDEs) (see [32] and the references there). Owing to the infinite dimension of $\mathcal{X}$ and $\mathcal{Y}$ in (1), efficient numerical approximations of maps $\mathcal{G}$ are to overcome the CoD.

Several (intrinsically different) mechanisms for overcoming the CoD in DNN emulations have been identified and mathematically justified recently. This includes the seminal work of A. *Barron* [3], *Monte-Carlo path simulation* type arguments (e.g. [20, 29] and the references there), and the emulation of sparse (generalized) *polynomial chaos expansions* (e.g. [2, 17]) by DNNs (e.g. [56, 51, 57]).

Specifically, in [56, 51, 57], a parametric representation of inputs $x \in \mathcal{X}$ of $\mathcal{G}$ was used to prove DNN emulation rates for approximating $\mathcal{G}$. The construction used DNNs whose depth scales polylogarithmic in the parameter dimension, and polynomially in the DNN expression accuracy (i.e., emulation fidelity). Key in the proofs of these results is the *holomorphic* dependence of $\mathcal{G}(x)$ on the input x. The related DNN emulation results were obtained with sparsely connected, deep feedforward NNs with ReLU or smooth (e.g. sigmoidal or $\tanh(\cdot)$) activation. DNN emulation rate results that are free from the CoD for *low regularity maps* $\mathcal{G}$ between function spaces were obtained e.g. using the so-called Feynman-Kac representation of solutions of Kolmogorov PDEs in (jump-)diffusion models. These results used ReLU DNNs of moderate depth [20, 29], but the error bounds hold in a mean-square sense or only with high probability.

While quantified, parametric holomorphy of solution families of parametric PDEs has been verified in many settings (particularly in elliptic and parabolic PDEs, e.g. [27, 66, 31, 12, 23]), there are broad classes of applications where relevant maps are Hölder or Lipschitz, but not holomorphic. One purpose of the present paper is to obtain mean-square DNN expression rate bounds for *Operator Network* (ONet) emulations with architecture (3) below, of Lipschitz (and, more generally, Hölder smooth) maps $\mathcal{G}$ between separable Hilbert spaces.

1.1 Previous work for operator networks

A rather recent line of research uses so-called *Operator Networks* to emulate the possibly nonlinear input-output map $\mathcal{G}$, such as for example the coefficient-to-solution map in linear, elliptic divergence form PDEs of second order. A variety of DNN architectures has been put forward recently with the aim of efficient operator emulation, with distinct architectures tailored to the emulation of particular operators. A number of acronyms labelling these DNN classes has been coined ("deepONets" [45], Fourier Neural Operators "FNOs" [35, 41], UNet architectures combined with FNOs "U-FNOs" [62], encoders based on transformers, etc.). We refer to [35, 38, 21, 49, 42, 6] and the references there.

In this paper, we discuss an architecture that belongs to the same general category as those proposed in, for instance, [25, 45, 24]. It reduces the task of approximating $\mathcal{G}$ to that of emulating (components of) *countably-parametric maps* $G : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with DNNs: using an appropriate *encoder* $\mathcal{E}_{\mathcal{X}} : \mathcal{X} \to \ell^2(\mathbb{N})$ and *decoder* $\mathcal{D}_{\mathcal{Y}} : \ell^2(\mathbb{N}) \to \mathcal{Y}$, the map $\mathcal{G}$ in (1) allows the structural representation

$$\mathcal{G} = \mathcal{D}_{\mathcal{Y}} \circ G \circ \mathcal{E}_{\mathcal{X}}. \tag{2}$$

[1]More generally, separably-valued maps $\mathcal{G}$ into an otherwise nonseparable target space $\mathcal{Y}$ may be considered. In [35, Section 9, App. B] additional conditions on separable Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ necessary to extend the present arguments to this more general setting are discussed.

For the deepONet "branch-trunk" architecture [9, 45], expression rate bounds were first investigated in [38] for holomorphic maps $\mathcal{G}$, with inputs $x \in \mathcal{X}$ stemming from a Karhunen-Loéve expansion with exponentially decaying eigenvalues. In [24], we considered the approximation of holomorphic maps using encoders and decoders based on frame representations of the input and output in $\mathcal{X}$ and $\mathcal{Y}$. We showed that dimension-independent algebraic expression rates can be obtained depending on the smoothness of the spaces $\mathcal{X}$, $\mathcal{Y}$, for example in terms of Sobolev or Besov regularity. In this situation, the frame coefficients typically exhibit only polynomial decay rather than exponential decay. For Lipschitz continuous $\mathcal{G}$, the recent preprint [36] offers lower bounds, indicating that the PCA-net using PCA-based encoders and decoders with standard, deep feedforward neural network based approximators generally can *not* avoid the CoD and is unable to achieve algebraic convergence.

1.2 Contributions

We consider (nonlinear) maps $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ which are Lipschitz or Hölder regular between infinite-dimensional, separable Hilbert spaces. Endowing $\mathcal{X}$ and $\mathcal{Y}$ with stable (Riesz) bases and corresponding encoder/decoder pairs, such maps admit the structure (2). We prove algebraic expression rate bounds for corresponding finite-parametric DNN surrogates, which are a key step in the mathematical analysis of DNN operator surrogates as outlined e.g. in [35, Section 2]. The considered linear encoders and decoders built on Riesz-bases in $\mathcal{X}$ and $\mathcal{Y}$ accommodate a variety of currently used Operator Nets, comprising Fourier- and KL expansions. If $\mathcal{X}$ and $\mathcal{Y}$ are function spaces over domains $D \subset \mathbb{R}^d$ and $D' \subset \mathbb{R}^{d'}$ (possibly of different dimensions d and d'; for simplicity of notation, we constrain here to $D = D'$ and $d = d'$), exhibiting sufficient smoothness to allow continuous embeddings into the space of continuous functions on $\overline{D}$, also point-collocation as e.g. in the deepONet architecture [45] is in principle admitted. Furthermore, we present a universal approximation theorem for continuous (but not necessarily Hölder continuous) operators $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$, which guarantees uniform convergence on any compact subset of $\mathcal{X}$.

Our main result, Theorem 5.7, states that there exist *finite-parametric approximator surrogates* $\widetilde{G}$ of the countably-parametric maps G in (2) which, upon insertion into (2), result in finite-parametric neural operators $\widetilde{\mathcal{G}}$ offering (on "higher-regularity" inputs from subspaces of $\mathcal{X}$) algebraic consistency orders with $\mathcal{G}$ that are free of the CoD. This is achieved by leveraging recent progress in the construction of DNN approximations for scalar functions of many variables of low regularity. These networks are either of standard feedforward architecture, but leverage superior expressive power of DNNs by invoking *nonstandard activations* (termed "super-expressive" e.g. in [58, 64] inspired by the Kolmogorov-Arnold superposition theorem, e.g. [34, 44] and the references there) or they are based on *nonstandard architectures* allowing a higher degree of connectivity than DNNs of plain feedforward type (e.g. [67]). As such, our results do not contradict the recent lower bounds for ReLU-activated, feedforward DONs stated e.g. in [38, 36]. We remark that computational adaptation of finite-parametric activation functions in, e.g., so-called "PiNNs", has been observed to be computationally effective in applications in [30]. By resorting to such nonstandard DNN architectures and activations in the emulation of the (component functions of) map G in (2), we identify sufficient conditions in order for the maps $\mathcal{G}$ to be emulated by DNNs with accuracy that behaves algebraically in terms of the number of neurons. We finally observe that the present setting (2) is a particular case of a number of other DON architectures (e.g. [35, 38]). The DONs analyzed can also be viewed as a building block of the recently featured "Nonlocal Neural Operators" (NNOs) in [37].

1.3 Notation

Throughout, and unless explicitly stated otherwise, $\mathcal{X}$ and $\mathcal{Y}$ shall denote separable Hilbert spaces of infinite dimension. For any $p \in [1, \infty]$, let $\ell^p(\mathbb{N})$ denote the space of all p-summable, real-valued sequences over $\mathbb{N}$. The Borel σ-algebra of any metric space $(\mathcal{Z}, d_{\mathcal{Z}})$ is generated by the open sets in $\mathcal{Z}$ and denoted by $\mathcal{B}(\mathcal{Z})$. For any σ-finite and complete measure space $(E, \mathcal{E}, \mu)$, a Banach space $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$, and summability exponent $p \in [1, \infty]$, we define the Lebesgue-Bochner spaces

$$L^p(E, \mu; \mathcal{Z}) := \{\varphi : E \to \mathcal{Z} : \ \varphi \text{ is strongly measurable and } \|\varphi\|_{L^p(E,\mu;\mathcal{Z})} < \infty\},$$

where

$$\|\varphi\|_{L^p(E,\mu;\mathcal{Z})} := \begin{cases} \left(\int_E \|\varphi(x)\|_{\mathcal{Z}}^p \mu(dx)\right)^{1/p}, & p \in [1, \infty) \\ \operatorname*{ess\,sup}_{x \in E} \|\varphi(x)\|_{\mathcal{Z}}, & p = \infty. \end{cases}$$

In case that $\mathcal{Z} = \mathbb{R}$, we use the shorthand notation $L^p(E, \mu) := L^p(E, \mu; \mathbb{R})$. In case that $E \subseteq \mathbb{R}^d$ is a subset of Euclidean space, we assume $\mathcal{E} = \mathcal{B}(E)$ and μ is the Lebesgue measure, and write $L^p(E) := L^p(E, \mu; \mathbb{R})$, unless stated otherwise. We further denote by $\|\cdot\|_2$ the Euclidean norm on $E \subseteq \mathbb{R}^d$.

1.4 Layout

This paper is organized as follows: In Section 2 we briefly present the setting of our consistency analysis, which consists of linear N-term encoder/decoder pairs in the domain and the range of the operator under consideration. Section 3 provides a universal approximation theorem valid for the approximation of continuous maps $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$. To show convergence rates, in Section 4 we first discuss preliminary results regarding the dimension truncation of the encoded input and output. We use these results in Section 5 to give our main result, which shows algebraic convergence rates for the approximation of Lipschitz-continuous operators. We comment on the extension to e.g. Hölder continuity in Section 6.

To illustrate the scope of our abstract results, in Section 7 we prove ONet expression rate bounds for particular classes of Lipschitz mappings $\mathcal{G}$ covered by our setting: in Section 7.1, we consider solution maps to parametric elliptic variational inequalities as arise e.g. in optimal stopping, optimal control, and in contact problems in mechanics. In Section 7.2, we obtain ONet expression rate bounds for Lipschitz maps of Hilbert-Schmidt operators acting on separable Hilbert spaces.

2 Setting

We adopt the setting from [24]. Let $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}})$ and $(\mathcal{Y}, \langle\cdot,\cdot\rangle_{\mathcal{Y}})$ denote two separable Hilbert spaces over $\mathbb{R}$. We consider neural network emulations of infinite-dimensional (non-linear) operators $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$. To this end, we rewrite $\mathcal{G}$ in the form $\mathcal{G} = \mathcal{D}_Y \circ G \circ \mathcal{E}_{\mathcal{X}}$, where $\mathcal{E}_{\mathcal{X}} : \mathcal{X} \to \ell^2(\mathbb{N})$ and $\mathcal{D}_{\mathcal{Y}} : \ell^2(\mathbb{N}) \to \mathcal{Y}$ are linear *encoder* and *decoder*, respectively, and $G : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is an infinite-parametric map. We employ linear encoders or "analysis operators" that convert function space inputs from $\mathcal{X}$ with suitable representation systems to coefficient sequences such as e.g. Fourier coefficients w.r.t. a fixed orthonormal basis of $\mathcal{X}$ (such as, e.g., principal component representations with respect to a Karhunen-Loève (KL) basis corresponding to a covariance operator of a probability measure on $\mathcal{X}$, leading to the so-called "PCA-ONet", see e.g. [36]), while our decoders perform the converse "synthesis" operation w.r.t. to another fixed representation system in $\mathcal{Y}$. We do not insist on orthogonal representation systems in $\mathcal{X}$ or in $\mathcal{Y}$, and rather admit Riesz bases of $\mathcal{X}, \mathcal{Y}$ and their analysis and synthesis operators as encoders and decoders. We build finite-parametric deep operator network surrogates of the general architecture

$$\widetilde{\mathcal{G}} := \mathcal{D}_Y \circ \widetilde{G} \circ \mathcal{E}_{\mathcal{X}}, \tag{3}$$

Theorem 3.1 implies in particular universality of the DON where the *approximator network* $\widetilde{G} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is a neural network emulation of the infinite-parametric, Lipschitz-continuous maps G, that allow for an efficient approximation of $\mathcal{G}$ on suitable sets $S \subseteq \mathcal{X}$. More precisely, we consider convergence of the mean-squared error

$$\left(\int_S \|\mathcal{G}(x) - \widetilde{\mathcal{G}}(x)\|_{\mathcal{Y}}^2 \, \mu(dx)\right)^{1/2}, \tag{4}$$

where μ is an appropriate measure on $(S, \mathcal{B}(S))$.

2.1 Encoders and Decoders

We admit *linear en- and decoders* built from Riesz bases as representation systems in $\mathcal{X}$ and $\mathcal{Y}$, comprising in particular Fourier-, Wavelet- and KL-bases for input and output parametrization. The following notion of a Riesz basis is one of several equivalent definitions. It follows as a consequence of [10, Definition 3.6.1] and [10, Theorem 3.6.6]. We refer to [10, Section 3.6] for details.

Definition 2.1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a separable Hilbert space. A complete sequence $\mathbf{\Psi}_{\mathcal{H}} = (\psi_i, i \in \mathbb{N}) \subset \mathcal{H}$ is called *Riesz basis of* $\mathcal{H}$ if there exists constants[2] $0 < \lambda_{\mathcal{H}} \leq \Lambda_{\mathcal{H}} < \infty$ such that for all $\boldsymbol{c} = (c_i, i \in \mathbb{N}) \in \ell^2(\mathbb{N})$ there holds

$$\lambda_{\mathcal{H}} \left\|\boldsymbol{c}\right\|_{\ell^2(\mathbb{N})}^2 = \lambda_{\mathcal{H}} \sum_{i\in\mathbb{N}} c_i^2 \leq \left\| \sum_{i\in\mathbb{N}} c_i\psi_i \right\|_{\mathcal{H}}^2 \leq \Lambda_{\mathcal{H}} \sum_{i\in\mathbb{N}} c_i^2 = \Lambda_{\mathcal{H}} \left\|\boldsymbol{c}\right\|_{\ell^2(\mathbb{N})}^2 . \tag{5}$$

For any Riesz basis $\mathbf{\Psi}_{\mathcal{H}} = (\psi_i, i \in \mathbb{N})$, there exists another (unique for given $\mathbf{\Psi}_{\mathcal{H}} = (\psi_i, i \in \mathbb{N})$) Riesz basis $\widetilde{\mathbf{\Psi}}_{\mathcal{H}} = (\widetilde{\psi}_i, i \in \mathbb{N})$, called *dual basis* or *biorthogonal system* to $\mathbf{\Psi}_{\mathcal{H}}$, such that there holds

$$f = \sum_{i\in\mathbb{N}} \langle f, \widetilde{\psi}_i\rangle_{\mathcal{H}}\psi_i \quad \text{for all } f \in \mathcal{H} \qquad \text{and} \qquad \langle \psi_i, \widetilde{\psi}_j\rangle_{\mathcal{H}} = \delta_{ij} \quad \text{for all } i, j \in \mathbb{N},$$

see e.g. [10, Theorem 3.6.2]. If $\mathbf{\Psi}_{\mathcal{H}}$ is an orthonormal basis (ONB) of $\mathcal{H}$, then $\widetilde{\mathbf{\Psi}}_{\mathcal{H}} = \mathbf{\Psi}_{\mathcal{H}}$ and $\lambda_{\mathcal{H}} = \Lambda_{\mathcal{H}} = 1$.

For the remainder of this article, we fix Riesz bases $\mathbf{\Psi}_{\mathcal{X}} = (\psi_i, i \in \mathbb{N}) \subset \mathcal{X}$ and $\mathbf{\Psi}_{\mathcal{Y}} = (\eta_j, j \in \mathbb{N}) \subset \mathcal{Y}$ for $\mathcal{X}$ and $\mathcal{Y}$, respectively, and denote their corresponding dual bases by $\widetilde{\mathbf{\Psi}}_{\mathcal{X}} = (\widetilde{\psi}_i, i \in \mathbb{N}) \subset \mathcal{X}$ and $\widetilde{\mathbf{\Psi}}_{\mathcal{Y}} = (\widetilde{\eta}_j, j \in \mathbb{N}) \subset \mathcal{Y}$. The associated Riesz constants from (5) are denoted by

$$0 < \lambda_{\mathcal{X}} \leq \Lambda_{\mathcal{X}} < \infty \,, \quad \text{ and } \quad 0 < \lambda_{\mathcal{Y}} \leq \Lambda_{\mathcal{Y}} < \infty \,.$$

For given Riesz bases $\mathbf{\Psi}_{\mathcal{X}}$ of $\mathcal{X}$ and $\mathbf{\Psi}_{\mathcal{Y}}$ of $\mathcal{Y}$, we define the encoder/decoder pairs

$$\mathcal{E}_{\mathcal{X}} : \mathcal{X} \to \ell^2(\mathbb{N}), \quad x \mapsto (\langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}}, i \in \mathbb{N}), \qquad \mathcal{D}_{\mathcal{X}} : \ell^2(\mathbb{N}) \to \mathcal{X}, \quad \boldsymbol{c} \mapsto \sum_{i\in\mathbb{N}} c_i\psi_i, \tag{6}$$

and

$$\mathcal{E}_{\mathcal{Y}} : \mathcal{Y} \to \ell^2(\mathbb{N}), \quad y \mapsto (\langle y, \widetilde{\eta}_j\rangle_{\mathcal{Y}}, j \in \mathbb{N}), \qquad \mathcal{D}_{\mathcal{Y}} : \ell^2(\mathbb{N}) \to \mathcal{Y}, \quad \boldsymbol{c} \mapsto \sum_{j\in\mathbb{N}} c_j\eta_j. \tag{7}$$

On the entire spaces, the encoders/decoders are boundedly invertible mappings and it holds

$$\mathcal{D}_{\mathcal{H}} \circ \mathcal{E}_{\mathcal{H}} = I_{\mathcal{H}} \quad \text{ for } \quad \mathcal{H} \in \{\mathcal{X}, \mathcal{Y}\}.$$

We mention that explicit, "Finite-Element-like" *constructions of piecewise polynomial biorthogonal systems in polytopal domains* are available, see e.g. [13, 8, 14, 43].

2.2 Smoothness Scales $\mathcal{X}^s$, $\mathcal{Y}^t$

Our convergence rate analysis will be obtained on (in general compact) subsets $\mathcal{X}^s \subset \mathcal{X}$ and $\mathcal{Y}^t \subset \mathcal{Y}$ of inputs / output pairs which admit extra regularity. We postulate that, in terms of the Riesz bases $\mathbf{\Psi}_{\mathcal{X}}$ and $\mathbf{\Psi}_{\mathcal{Y}}$, this regularity takes the form of weighted summability of the corresponding sequences of expansion coefficients. To formalize this condition, we next define a scale of Hilbert spaces depending on a smoothness parameter characterizing this type of coefficient decay. Typical instances of such "smoothness spaces" are Sobolev and Besov spaces with p-integrable weak derivatives (e.g. [59] and the references

[2] The constants $\lambda_{\mathcal{H}}$ and $\Lambda_{\mathcal{H}}$ depend on the Riesz basis $\mathbf{\Psi}_{\mathcal{H}}$ but not on $\mathcal{G}$. They quantify the sensitivity of the approximator error $G - \tilde{G}$ on the output error. For conciseness, we do not explicitly indicate this dependence in our notation.

there), or the Cameron-Martin space of the covariance operator of a Gaussian measure on $\mathcal{X}$ or $\mathcal{Y}$ (see [46]).

Let then $\boldsymbol{w} = (w_i, i \in \mathbb{N}) \subset (0,1]$ be a non-increasing sequence of weights such that $\boldsymbol{w} \in \ell^{1+\varepsilon}(\mathbb{N})$ for all $\varepsilon > 0$. The latter condition is sufficient to derive the truncation error rates in Proposition 4.4 for arbitrary small $\delta > 0$. With this sequence, following [24, Section 2], for all $s, t \geq 0$ we introduce Hilbert spaces $\mathcal{X}^s \subset \mathcal{X}$, $\mathcal{Y}^t \subset \mathcal{Y}$ via their norms

$$\|x\|_{\mathcal{X}^s}^2 := \sum_{i\in\mathbb{N}} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}}^2 w_i^{-2s}, \qquad \|y\|_{\mathcal{Y}^t}^2 := \sum_{j\in\mathbb{N}} \langle y, \widetilde{\eta}_j\rangle_{\mathcal{Y}}^2 w_j^{-2t}. \tag{8}$$

In order to streamline the presentation, we utilize the same sequence $\boldsymbol{w}$ of weights w_j to characterize $\mathcal{X}^s$ and $\mathcal{Y}^t$, but all of our subsequent results remain valid for distinct weighting sequences $\boldsymbol{w}_{\mathcal{X}}, \boldsymbol{w}_{\mathcal{Y}} \in \ell^{1+\varepsilon}(\mathbb{N})$. We further note that $\mathcal{X}^s = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}^s} < \infty\}$ equipped with the scalar product $\langle x, x'\rangle_{\mathcal{X}^s} := \sum_{i\in\mathbb{N}} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}} \langle x', \widetilde{\psi}_i\rangle_{\mathcal{X}} w_i^{-2s}$ is a separable Hilbert space with Riesz basis $(w_i^s \psi_i, i \in \mathbb{N})$, see e.g. [24, Lemma 2.9 and Remark 2.10].

3 Universality

A universality result for the approximation of functionals mapping from compact subsets of $C([a,b])$ or $L^p([a,b])$ to $\mathbb{R}$ using neural networks was already established in the pioneering work of Chen and Chen in 1993 [9]. More recently, a universal approximation theorem for Lipschitz continuous $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ utilizing the PCA-net architecture was proven in [37]. Universality in the infinite width limit was shown in the $L^2(\mathcal{X}, \mu)$ sense for measures μ possessing finite fourth moments, with the KL basis of the covariance of μ, and relaxed later to require only finite second moments for μ and $\mathcal{G}$ being a μ-measurable map [36, Theorem 3.1].

We start our present approximation rate analysis by proving an universal approximation theorem which is valid for any continuous $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$, with uniform convergence on compact subsets of $\mathcal{X}$. To state the result, we introduce the set of admissible activation functions as in [40]:

$$\mathcal{A} := \big\{\sigma \in L^\infty_{\mathrm{loc}}(\mathbb{R}) \text{ is not polynomial and the closure of the points of discontinuity has Lebesgue measure } 0\big\}.$$

ONets $\widetilde{\mathcal{G}}$ as in (3) with the (components of the) approximator $\widetilde{G}$ being a feedforward σ-NN with activation $\sigma \in \mathcal{A}$ are universal.

Theorem 3.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be two separable Hilbert spaces, let $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ be continuous and let $\sigma \in \mathcal{A}$. Then there exists a sequence of operator nets $\widetilde{\mathcal{G}}_n : \mathcal{X} \to \mathcal{Y}$, $n \in \mathbb{N}$, with architecture* (3) *such that*

$$\forall x \in \mathcal{X} : \quad \lim_{n\to\infty} \widetilde{\mathcal{G}}_n(x) = \mathcal{G}(x) \ .$$

The convergence is uniform on every compact subset of $\mathcal{X}$.

The proof is given in Appendix A. We remark that Theorem 3.1 implies in particular universality of the DON architecture (3) for maps between the Hilbertian Sobolev spaces $H^s(D)$ and $H^{s'}(D')$ for $s, s' \in \mathbb{R}$.

Remark 3.2. We observe that the expressions (6), (7) for the coefficient sequences $\boldsymbol{c}$ are "*nonlocal*" in terms of the function space inputs x and y. Non-locality was highlighted in [37] as important prerequisite for universality of a number of DON architectures. In their analysis, the authors also examined the possibility of using neural network approximations for the encoder and decoder mappings when $\mathcal{X}$ and $\mathcal{Y}$ are function spaces on bounded domains. Although it is technically possible to emulate $\mathcal{D}_{\mathcal{Y}}$ and $\mathcal{E}_{\mathcal{X}}$ using DNNs in our context, we opted to not elaborate on this, in order to not overload the presentation.

4 Dimension Truncation

Taking the cue from [24], within the architecture (2), we approximate $\mathcal{G}$ on finite-parametric subspaces of $\mathcal{X}$. This implies that encoding will contain a form of "dimension-truncation" where only a finite number N of parameters in otherwise infinite-parametric, equivalent representations of inputs from $\mathcal{X}$ are retained. Importantly, due to the index-dependent relative importance of the component maps $(g_j, j \in \mathbb{N})$ of the parametric map G in (2). In Section 4.1 we introduce the sets of admissible inputs for the ensuing expression rate analysis. We characterize in particular higher regularity of inputs and outputs in terms of weighted summability of the coefficient sequences resulting from encoding in $\mathcal{X}$ and $\mathcal{Y}$.

4.1 Smoothness Classes

Let $r > 0$, $s > \frac{1}{2}$ and define the "cubes"

$$C_r^s(\mathcal{X}) := \left\{ x \in \mathcal{X} : \mathcal{E}_{\mathcal{X}}(x) \in \bigtimes_{i \in \mathbb{N}} [-rw_i^s, rw_i^s] \right\} = \left\{ x \in \mathcal{X} : \sup_{i \in \mathbb{N}} \left| \langle x, \widetilde{\psi}_i \rangle_{\mathcal{X}} w_i^{-s} \right| \le r \right\}.$$

Note that for any $s' \in [0, s - \frac{1}{2})$ there holds $C_r^s(\mathcal{X}) \subset \mathcal{X}^{s'}$ by [24, Remark 3.2]. Further, let

$$B_r(\mathcal{X}^s) := \{ x \in \mathcal{X}^s : \|x\|_{\mathcal{X}^s} \le r \}$$

denote the closed ball with radius $r > 0$ in $\mathcal{X}^s$. Then, for any $\varepsilon > 0$ and with $r_\varepsilon^2 := r^2 \sum_{i \in \mathbb{N}} w_i^{1+2\varepsilon} \in (0, \infty)$ it holds

$$B_r(\mathcal{X}^s) \subseteq C_r^s(\mathcal{X}) \subseteq B_{r_\varepsilon}(\mathcal{X}^{s - \frac{1}{2} - \varepsilon}). \tag{9}$$

Hence, error bounds on approximations of $\mathcal{G}$ on $C_r^s(\mathcal{X})$ trivially imply bounds on the closed ball $B_r(\mathcal{X}^s)$.

We fix the following assumption to derive finite-dimensional surrogates for $\mathcal{G}$.

Assumption 4.1. There exist $s > \frac{1}{2}$, t, $r > 0$ and a constant $L_{\mathcal{G}} > 0$ such that $\mathcal{G}[C_r^s(\mathcal{X})] \in \mathcal{Y}^t$ and

$$\|\mathcal{G}(x) - \mathcal{G}(x')\|_{\mathcal{Y}^t} \le L_{\mathcal{G}} \|x - x'\|_{\mathcal{X}}, \quad x, x' \in C_r^s(\mathcal{X}). \tag{10}$$

Remark 4.2. The global Lipschitz continuity in Assumption 4.1 implies the linear growth bound

$$\|\mathcal{G}(x)\|_{\mathcal{Y}^t} \le C(1 + \|x\|_{\mathcal{X}}), \quad x \in C_r^s(\mathcal{X}), \quad \text{with} \quad C := \max(L_{\mathcal{G}}, \|\mathcal{G}(0)\|_{\mathcal{Y}^t}). \tag{11}$$

4.2 Decoding Dimension Truncation

For $N \in \mathbb{N}$ define the *restriction operator* $\mathcal{R}_N$ for sequences in $\ell^2(\mathbb{N})$ via

$$\mathcal{R}_N : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad \boldsymbol{c} = (c_i, i \in \mathbb{N}) \mapsto (c_1, \dots, c_N, 0, 0, \dots).$$

We define the *N-term output-truncated* approximation $\mathcal{G}_N : \mathcal{X} \to \mathcal{Y}$ of $\mathcal{G}$ via

$$\mathcal{G}_N := \mathcal{D}_{\mathcal{Y}} \circ \mathcal{R}_N \circ \mathcal{E}_{\mathcal{Y}} \circ \mathcal{G}. \tag{12}$$

Proposition 4.3. *Under Assumption 4.1 there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$*

$$\sup_{x \in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N(x)\|_{\mathcal{Y}} \le C w_{N+1}^t.$$

Proof. For any $x \in C_r^s(\mathcal{X})$ and $\delta > 0$ it holds by Assumption 4.1 that

$$
\begin{aligned}
\|\mathcal{G}(x) - \mathcal{G}_N(x)\|_{\mathcal{Y}}^2 &= \left\| \sum_{j>N} \langle \mathcal{G}(x), \widetilde{\eta}_j \rangle_{\mathcal{Y}} \eta_j \right\|_{\mathcal{Y}}^2 \\
&\leq \Lambda_{\mathcal{Y}} \sum_{j>N} \langle \mathcal{G}(x), \widetilde{\eta}_j \rangle_{\mathcal{Y}}^2 w_j^{2t-2t} \\
&\leq \Lambda_{\mathcal{Y}} w_{N+1}^{2t} \sum_{j>N} \langle \mathcal{G}(x), \widetilde{\eta}_j \rangle_{\mathcal{Y}}^2 w_j^{-2t} \\
&\leq \Lambda_{\mathcal{Y}} w_{N+1}^{2t} \|\mathcal{G}(x)\|_{\mathcal{Y}^t}^2 ,
\end{aligned}
$$

where the second inequality holds since $\boldsymbol{w}$ is a sequence of decreasing positive real numbers. Furthermore, by (11), since $\boldsymbol{\Psi}_{\mathcal{Y}}$ is a Riesz basis and due to $s > \frac{1}{2}$, it holds

$$
\|\mathcal{G}(x)\|_{\mathcal{Y}^t}^2 \leq C^2(1 + \|x\|_{\mathcal{X}})^2 \leq C^2 \left(1 + \Lambda_{\mathcal{X}} \sum_{i \in \mathbb{N}} \langle x, \widetilde{\psi}_i \rangle_{\mathcal{X}}^2 \right) \leq C^2 \left(1 + \Lambda_{\mathcal{X}} r^2 \sum_{i \in \mathbb{N}} w_i^{2s} \right)
$$

where the last term is finite and independent of $x \in C_r^s(\mathcal{X})$. This shows the claim. □

4.3 Encoding Dimension Truncation

Assume that we have fixed a truncation index N for $\mathcal{G}_N$ in Subsection 4.2. We now choose *component-dependent truncations of the encoded inputs* $M_j \in \mathbb{N}$ for component j of G and define corresponding scalar-valued mappings

$$
g_j : \ell^2(\mathbb{N}) \to \mathbb{R}, \quad \boldsymbol{c} \mapsto \langle [\mathcal{G} \circ \mathcal{D}_{\mathcal{X}} \circ \mathcal{R}_{M_j}](\boldsymbol{c}), \widetilde{\eta}_j \rangle_{\mathcal{Y}} = \left\langle \mathcal{G}\left(\sum_{i=1}^{M_j} c_i \psi_i \right), \widetilde{\eta}_j \right\rangle_{\mathcal{Y}}, \quad j = 1, \dots, N \,. \tag{13}
$$

We then define the multi-index (of truncation indices) $\mathbf{M} := (M_1, \dots, M_N) \in \mathbb{N}^N$ and the *input-truncated approximation* $\mathcal{G}_N^{\mathbf{M}} : \mathcal{X} \to \mathcal{Y}$ to $\mathcal{G}_N$ via

$$
\mathcal{G}_N^{\mathbf{M}} : \mathcal{X} \to \mathcal{Y}, \quad x \mapsto \mathcal{D}_{\mathcal{Y}}\left(([g_1 \circ \mathcal{E}_{\mathcal{X}}](x), \dots, [g_N \circ \mathcal{E}_{\mathcal{X}}](x), 0, 0, \dots) \right) = \sum_{j=1}^{N} g_j(\mathcal{E}_{\mathcal{X}}(x)) \eta_j. \tag{14}
$$

We observe that (14) has the "encoder-decoder" structure that appears in a number of recently considered DON architectures. We refer to the nonlocal neural operators [37, Eqn. (2.5) and App. A.4], [36], to the deep ONets [35, Eqn. (33)], and to the references in [35] for further DON architectures. The next result has general expression rate bounds for (14) which will apply in particular to the mentioned settings.

Proposition 4.4. *Under Assumption 4.1, for any $\delta > 0$ exists a constant $C > 0$, such that for any $\mathbf{M} \in \mathbb{N}^N$ there holds*

$$
\sup_{x \in C_r^s(\mathcal{X})} \left\| \mathcal{G}_N(x) - \mathcal{G}_N^{\mathbf{M}}(x) \right\|_{\mathcal{Y}} \leq C \max_{j=1,\dots,N} w_j^{t - \frac{1}{2} - \delta} w_{M_j+1}^{s - \frac{1}{2} - \delta}. \tag{15}
$$

In case that $\mathbf{M} = (M, \dots, M) \in \mathbb{N}^N$ for some $M \in \mathbb{N}$,

$$
\sup_{x \in C_r^s(\mathcal{X})} \left\| \mathcal{G}_N(x) - \mathcal{G}_N^{\mathbf{M}}(x) \right\|_{\mathcal{Y}} \leq C w_M^{s - \frac{1}{2} - \delta}. \tag{16}
$$

Proof. For any $x \in C_r^s(\mathcal{X})$ it holds by (14) and (13) that

$$\begin{aligned}\left\|\mathcal{G}_N(x) - \mathcal{G}_N^{\mathbf{M}}(x)\right\|_{\mathcal{Y}}^2 &= \left\|\sum_{j=1}^N \langle \mathcal{G}(x), \widetilde{\eta}_j\rangle_{\mathcal{Y}} \eta_j - g_j(\mathcal{E}_{\mathcal{X}}(x))\eta_j\right\|_{\mathcal{Y}}^2 \\ &\leq \Lambda_{\mathcal{Y}} \sum_{j=1}^N \left|\langle \mathcal{G}(x), \widetilde{\eta}_j\rangle_{\mathcal{Y}} - g_j(\mathcal{E}_{\mathcal{X}}(x))\right|^2 \qquad (17) \\ &= \Lambda_{\mathcal{Y}} \sum_{j=1}^N \left\langle \mathcal{G}(x) - \mathcal{G}\left(\sum_{i=1}^{M_j} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}} \psi_i\right), \widetilde{\eta}_j\right\rangle_{\mathcal{Y}}^2 w_j^{2t-2t}.\end{aligned}$$

Now, for arbitrary $\mathbf{M} = (M_j)_{j=1}^N \in \mathbb{N}^N$, the Lipschitz continuity of $\mathcal{G}$ on $C_r^s(\mathcal{X})$ yields for any $\delta > 0$ that

$$\begin{aligned}\left\|\mathcal{G}_N(x) - \mathcal{G}_N^{\mathbf{M}}(x)\right\|_{\mathcal{Y}}^2 &\leq \Lambda_{\mathcal{Y}} \sum_{j=1}^N w_j^{2t} \left\|\mathcal{G}(x) - \mathcal{G}\left(\sum_{i=1}^{M_j} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}} \psi_i\right)\right\|_{\mathcal{Y}^t}^2 \\ &\leq \Lambda_{\mathcal{Y}} L_{\mathcal{G}}^2 \sum_{j=1}^N w_j^{2t} \left\|\sum_{i>M_j} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}} \psi_i\right\|_{\mathcal{X}}^2 \\ &\leq \Lambda_{\mathcal{Y}} L_{\mathcal{G}}^2 \sum_{j=1}^N w_j^{2t} \Lambda_{\mathcal{X}} \left|\sum_{i>M_j} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}}^2\right| \\ &\leq \Lambda_{\mathcal{Y}} L_{\mathcal{G}}^2 \Lambda_{\mathcal{X}} r^2 \sum_{j=1}^N w_j^{2t} \sum_{i>M_j} w_i^{2s} \\ &\leq C \sum_{j=1}^N w_j^{2t} w_{M_j+1}^{2(s-\frac{1}{2}-\delta)} \sum_{i>M_j} w_i^{1+2\delta} \\ &\leq C \max_{j=1,\dots,N} w_j^{2(t-\frac{1}{2}-\delta)} w_{M_j+1}^{2(s-\frac{1}{2}-\delta)},\end{aligned}$$

which shows (15).

To show (16), fix $M \in \mathbb{N}$ and $\mathbf{M} = (M \dots, M)$. Then, using (17) and (5), for any $\delta > 0$

$$\begin{aligned}\left\|\mathcal{G}_N(x) - \mathcal{G}_N^{\mathbf{M}}(x)\right\|_{\mathcal{Y}}^2 &\leq \Lambda_{\mathcal{Y}} \lambda_{\mathcal{Y}}^{-1} \left\|\mathcal{G}(x) - \mathcal{G}\left(\sum_{i=1}^{M} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}} \psi_i\right)\right\|_{\mathcal{Y}^t}^2 \\ &\leq \Lambda_{\mathcal{Y}} \lambda_{\mathcal{Y}}^{-1} L_{\mathcal{G}}^2 \left\|\sum_{i>M_j} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}} \psi_i\right\|_{\mathcal{X}}^2 \\ &\leq \Lambda_{\mathcal{Y}} \lambda_{\mathcal{Y}}^{-1} L_{\mathcal{G}}^2 \Lambda_{\mathcal{X}} \left|\sum_{i>M} \langle x, \widetilde{\psi}_i\rangle_{\mathcal{X}}^2\right| \\ &\leq C w_{M+1}^{2(s-\frac{1}{2}-\delta)}.\end{aligned}$$

This concludes the proof. ∎

Remark 4.5. Proposition 4.4 suggests the following strategy to choose the truncation parameters M_j:

- $t > \frac{1}{2}$: For $\delta > 0$ small enough it holds $t - \frac{1}{2} - \delta > 0$ and thus $w_j^{t-\frac{1}{2}-\delta} \leq 1$ for all $j \in \mathbb{N}$. Hence (15) is sharper than (16), and a choice for $\mathbf{M}$ minimizing $\sum_{j=1}^{N} M_j$ while not increasing the best upper bound in (15)-(16) is obtained if $w_j^{t-\frac{1}{2}-\delta} w_{M_j+1}^{s-\frac{1}{2}-\delta} \sim$ const.

- $t \leq \frac{1}{2}$: In this case $t - \frac{1}{2} - \delta < 0$ and thus $w_j^{t-\frac{1}{2}-\delta} \geq 1$ for all $j \in \mathbb{N}$. Hence (16) is sharper than (15), and therefore a choice for $\mathbf{M}$ minimizing $\sum_{j=1}^{N} M_j$ while not increasing the best upper bound in (15)-(16) is obtained if $M_j = M$ for all j.

5 Deep Operator Surrogates

The dimension truncation in $\mathcal{X}$ and $\mathcal{Y}$ from the previous section effectively yields a finite-dimensional approximation $\mathcal{G}_N^{\mathbf{M}}$ to $\mathcal{G}$. In the next step, we replace the dimension-truncated, finite-parametric, nonlinear coordinate map $\mathcal{G}_N^{\mathbf{M}}$ by an *approximator*, i.e., by neural network surrogate maps, and bound the resulting overall approximation error. We further estimate the number of parameters (degrees of freedom) in the network, that are necessary to achieve a prescribed error tolerance $\varepsilon > 0$.

We fix the following assumption on the approximation of d-variate Lipschitz functions.

Assumption 5.1. Let $d \in \mathbb{N}$ and $f : [0,1]^d \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant $L_f > 0$.

Then, for any $\varepsilon \in (0,1]$, there exists a (neural network) surrogate $\widetilde{f} : [0,1]^d \to \mathbb{R}$ with at most $\mathcal{O}(d^\alpha \varepsilon^{-\beta}(1 + \log(d) + |\log(\varepsilon)|)^\kappa)$ many parameters such that

$$\left\| f - \widetilde{f} \right\|_{L^2([0,1]^d)} \leq L_f \varepsilon.$$

The constants $\alpha \geq 1$ and $\beta, \kappa \geq 0$ and the hidden constant in $\mathcal{O}(\cdot)$ are independent of d, L_f and ε.

Assumption 5.1 holds for several DNN architectures and activations. A (non-exhaustive) collection of examples is provided in Table 1.

Architecture	Activations	# of parameters	Assumption 5.1
Feedforward DNNs [63]	ReLU	$\mathcal{O}(d^2(1+\log(d))^2\varepsilon^{-d}(1+\lvert\log(\varepsilon)\rvert)^2)$	✗
NestNets [67]	ReLU	$\mathcal{O}(\mathfrak{h}^2 d^{2+d/(2(\mathfrak{h}+1))}\varepsilon^{-d/(\mathfrak{h}+1)})$ for any $\mathfrak{h} \in \mathbb{N}$	(✓)
FLES [58]	$\lfloor\cdot\rfloor$, 2^x, sin	$\mathcal{O}(d(1+\log(d)+\lvert\log(\varepsilon)\rvert))$	✓
Deep Fourier [65]	ReLU, sin	$\mathcal{O}(d(1+\lvert\log(\varepsilon)\rvert)^2)$	✓

Table 1: Complexity of several neural network architectures to approximate Lipschitz functions $f : [0,1]^d \to \mathbb{R}$ for given dimension $d \in \mathbb{N}$ in $L^2([0,1]^d)$ with tolerance $\varepsilon \in (0,1]$. The parameter $\mathfrak{h} \in \mathbb{N}$ in the second row signifies the "height" of a NestNet architecture, see also Example 5.2.

Example 5.2 (NestNets)**.** In [67] the authors introduced a novel neural network architecture, dubbed *NestNets*. The proposed networks augment the classical, two-dimensional feedforward network (fully connected with size parameters width and depth) by a third dimension, termed "height" in [67], and indexed by $\mathfrak{h} \in \mathbb{N}$. Classical feedforward networks are contained as NestNets of height $\mathfrak{h} = 1$. By a bit-extraction technique, the authors showed that strict ReLU activated NestNets can express high-dimensional Hölder-continuous functions $f : [0,1]^d \to \mathbb{R}$ in L^p for $p < \infty$ without the CoD, *in terms of the number of neurons constituting the NestNet.* Moreover, by increasing the height $\mathfrak{h}$ of the network, one obtains in principle arbitrary fast algebraic rates of convergence with respect to the number of neurons which are free of the CoD, cf. Table 1. In the NestNet architecture of [67], super-expressivity holds with strictly ReLU activated NestNets. I.e., super-expressivity is afforded by architecture, specifically the vastly larger connectivity of NestNets with height $\mathfrak{h} > 1$, rather than by more sophisticated activations. However, in one forward pass through a NestNet with height $\mathfrak{h} \geq 2$, the parameters are applied repeatedly to transform the input, with the number of floating points operations in a forward evaluation of the NestNet growing exponentially in $\mathfrak{h}$. Thus, strictly speaking, *NestNets overcome the CoD with respect to the number of neurons, but they do not lift the curse of dimensionality with respect to the number of floating point operations.*

Remark 5.3 (NestNets and Skip Connections)**.** NestNets of height $\mathfrak{h} > 1$ could be viewed as extreme cases of feedforward NNs with skip connections. Insertion of skip connections in, for example, feedforward DONs has been *empirically*, i.e., in numerical tests, found to enhance DON expressivity significantly. See, e.g. [4, Section 2.4].

Example 5.4 (Superexpressive activations)**.** In [58] the authors introduce so-called *Floor-Exponential-Step* (FLES) networks with three hidden layers and a combination of floor ($x \mapsto \lfloor x \rfloor$), exponential $x \mapsto 2^x$ and binary step units $x \mapsto \mathbb{1}_{x \geq 0}$ as activation functions. Relying on a similar bit extraction technique as for NestNets, the authors show that for any Lipschitz function $f : [0,1]^d \to \mathbb{R}$ there is a FLES-NN $\widetilde{f} : [0,1]^d \to \mathbb{R}$ with at most $\mathcal{O}(d+N)$ parameters such that

$$\|f - \widetilde{f}\|_{L^\infty([0,1]^d)} \leq 6 L_f \sqrt{d} 2^{-N}, \quad N \in \mathbb{N},$$

see [58, Corollary 1.2].

For fixed $\varepsilon \in (0,1]$ and $d \in \mathbb{N}$, let $N_\varepsilon := \left\lceil \log_2(6\sqrt{d}) + |\log_2(\varepsilon)| \right\rceil \in \mathbb{N}$, such that there holds $6\sqrt{d}2^{-N_\varepsilon} \leq \varepsilon$. Then, there is a $C > 0$ such that there holds

$$N_\varepsilon \leq 1 + \log_2(6\sqrt{d}) + |\log_2(\varepsilon)| \leq C(1 + \log(d) + |\log(\varepsilon)|).$$

The total number of parameters $\widetilde{N} \in \mathbb{N}$ of the FLES-NN $\widetilde{f}$ is bounded by

$$\widetilde{N} \leq C(d+N) \leq C(d + 1 + \log(d) + |\log(\varepsilon)|) \leq Cd(1 + \log(d) + |\log(\varepsilon)|).$$

Consequently, Assumption 5.1 holds for FLES feedforward NNs with $\alpha = \kappa = 1$ and $\beta = 0$.

This "superexpressivity" has been generalized for various elementary activation functions in [64]. Therein, the author shows that the approximation technique for FLES in [58] may be transferred to several classes of elementary, smooth activations. The results in [64] are stated for general, uniformly continuous functions $f : [0,1]^d \to \mathbb{R}$ with respect to the supremum norm, thus no rates are derived. However, since the line of proof closely follows [58], one would expect the same exponential rates for feed forward networks with super expressive smooth activations. On a further note, Yarotski emphasizes in [64, Section 3] that most standard activation functions like ReLU, tanh, sigmoid, binary step units, etc. are *not* superexpressive.

To construct the surrogate operator networks, we approximate for each j the parametric co-ordinate maps $g_j : \ell^2(\mathbb{N}) \to \mathbb{R}$ in (13). In order to "restrict" g_j to the finite-dimensional domain $\mathbb{R}^{M_j}$, we introduce the restriction maps

$$\mathfrak{g}_j : \mathbb{R}^{M_j} \to \mathbb{R}, \quad (c_1, \ldots, c_n) \mapsto \left\langle \mathcal{G}\left(\sum_{i=1}^{M_j} c_i \psi_i\right), \widetilde{\eta}_j \right\rangle_{\mathcal{Y}}, \qquad j = 1, \ldots, N. \tag{18}$$

We further introduce the *projection operator* Π_{M_j} via

$$\Pi_{M_j} : \ell^2(\mathbb{N}) \to \mathbb{R}^{M_j}, \quad \boldsymbol{c} \mapsto (c_1, \ldots, c_{M_j}),$$

to obtain the identity

$$g_j(\boldsymbol{c}) = \mathfrak{g}_j(\Pi_{M_j}(\boldsymbol{c})), \quad c \in \ell^2(\mathbb{N}). \tag{19}$$

In the next step, we establish the Lipschitz continuity of $\mathfrak{g}_j$.

Lemma 5.5. *Let Assumption 4.1 hold and let $\mathfrak{g}_j : \mathbb{R}^{M_j} \to \mathbb{R}$ be defined as in* (18) *for a given truncation index $M_j \in \mathbb{N}$. Then, for any c, $c' \in \bigtimes_{i=1}^{M_j}[-rw_i^s, rw_i^s] \subset \mathbb{R}^{M_j}$ there holds*

$$|\mathfrak{g}_j(c) - \mathfrak{g}_j(c')| \leq L_{\mathcal{G}} w_j^t \sqrt{\Lambda_{\mathcal{X}}} \|c - c'\|_2. \tag{20}$$

Proof. Fix c, $c' \in \bigtimes_{i=1}^{M_j}[-rw_i^s, rw_i^s]$. Then with $x := \sum_{i=1}^{M_j} c_i \psi_i$, $x' := \sum_{i=1}^{M_j} c_i' \psi_i \in C_r^s(\mathcal{X})$ holds $c = \Pi_{M_j}(\mathcal{E}_{\mathcal{X}}(x))$ and $c' = \Pi_{M_j}(\mathcal{E}_{\mathcal{X}}(x'))$. Thus

$$|\mathfrak{g}_j(c) - \mathfrak{g}_j(c')| = \left|\mathfrak{g}_j([\Pi_{M_j} \circ \mathcal{E}_{\mathcal{X}}](x)) - \mathfrak{g}_j([\Pi_{M_j} \circ \mathcal{E}_{\mathcal{X}}](x'))\right| = |g_j(\mathcal{E}_{\mathcal{X}}(x)) - g_j(\mathcal{E}_{\mathcal{X}}(x'))|$$

holds by (19). Furthermore,

$$\begin{aligned}
|g_j(\mathcal{E}_{\mathcal{X}}(x)) - g_j(\mathcal{E}_{\mathcal{X}}(x'))|^2 &= \left\langle \mathcal{G}\left(\sum_{i=1}^{M_j} \langle x, \widetilde{\psi}_i \rangle_{\mathcal{X}} \psi_i\right) - \mathcal{G}\left(\sum_{i=1}^{M_j} \langle x', \widetilde{\psi}_i \rangle_{\mathcal{X}} \psi_i\right), \widetilde{\eta}_j \right\rangle_{\mathcal{Y}}^2 \\
&\leq w_j^{2t} \left\| \mathcal{G}\left(\sum_{i=1}^{M_j} \langle x, \widetilde{\psi}_i \rangle_{\mathcal{X}} \psi_i\right) - \mathcal{G}\left(\sum_{i=1}^{M_j} \langle x', \widetilde{\psi}_i \rangle_{\mathcal{X}} \psi_i\right) \right\|_{\mathcal{Y}_t}^2 \\
&\leq w_j^{2t} L_{\mathcal{G}}^2 \left\| \sum_{i=1}^{M_j} \langle x - x', \widetilde{\psi}_i \rangle_{\mathcal{X}} \psi_i \right\|_{\mathcal{X}} \\
&\leq w_j^{2t} L_{\mathcal{G}}^2 \Lambda_{\mathcal{X}} \sum_{i=1}^{M_j} \langle x - x', \widetilde{\psi}_i \rangle_{\mathcal{X}}^2 \\
&= w_j^{2t} L_{\mathcal{G}}^2 \Lambda_{\mathcal{X}} \sum_{i=1}^{M_j} \left|\Pi_{M_j}(\mathcal{E}_{\mathcal{X}}(x))_i - \Pi_{M_j}(\mathcal{E}_{\mathcal{X}}(x'))_i\right|^2 \\
&= w_j^{2t} L_{\mathcal{G}}^2 \Lambda_{\mathcal{X}} \|c - c'\|_2^2 .
\end{aligned}$$

□

Lemma 5.6. *Let Assumption 5.1 hold, let $M \in \mathbb{N}$ be arbitrary and define $D_M := \bigtimes_{i=1}^{M}[-rw_i^s, rw_i^s]$. Denote by λ the univariate Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and by $\mu_M := \bigotimes_{i=1}^{M} \frac{\lambda}{2rw_i^s}$ the uniform probability measure on $D_M \subset \mathbb{R}^M$. Further, assume $g : D_M \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L_g > 0$. Then, for any $\varepsilon \in (0, 1]$, there exists a neural network $\widetilde{g} : D_M \to \mathbb{R}$ with at most $\mathcal{O}(M^\alpha \varepsilon^{-\beta}(1 + \log(M) + |\log(\varepsilon)|)^\kappa)$ parameters, such that*

$$\|g - \widetilde{g}\|_{L^2(D_M, \mu_M)} \leq L_g 2rw_1^s \varepsilon.$$

Proof. We first translate the unit cube $[0, 1]^M$ to D_M with the linear bijection

$$T : [0, 1]^M \to D_M, \quad (u_1, \cdots, u_M) \mapsto r\left((2u_1 - 1)w_1^s, \cdots, (2u_M - 1)w_M^s\right).$$

Set $g_T := g \circ T : [0, 1]^M \to \mathbb{R}$. Clearly, for all u, $u' \in [0, 1]^M$

$$|g_T(u) - g_T(u')| \leq L_g \|T(u) - T(u')\|_2 = L_g 2r \left(\sum_{i=1}^{M} |u_i - u_i'|^2 w_i^{2s}\right)^{1/2} \leq L_g 2rw_1^s \|u - u'\|_2.$$

By Assumption 5.1, for any $\varepsilon \in (0, 1]$ and for any finite $M \in \mathbb{N}$ exists a neural network $\widetilde{g}_T : [0, 1]^M \to \mathbb{R}$ with at most $C(M^\alpha \varepsilon^{-\beta}(1 + \log(M) + |\log(\varepsilon)|)^\kappa)$ parameters, where the constant $C > 0$ is independent of ε and M, and

$$\|g_T - \widetilde{g}_T\|_{L^2([0,1]^M)} \leq L_g 2rw_1^s \varepsilon.$$

Denote by λ^M the Lebesgue measure on $([0,1]^M, \mathcal{B}([0,1]^M))$. For the pushforward measure $T_\#\lambda^M := \lambda^M \circ T^{-1}$ on $(D_M, \mathcal{B}(D_M))$ and $\widetilde{g} := \widetilde{g}_T \circ T^{-1}$ there holds

$$\|g_T - \widetilde{g}_T\|_{L^2([0,1]^M)} = \|g \circ T - \widetilde{g} \circ T\|_{L^2([0,1]^M)} = \|g - \widetilde{g}\|_{L^2(D_M, T_\#\lambda^M)}.$$

Since $T_\#\lambda^M = \bigotimes_{i=1}^M \frac{\lambda}{2rw_i^s} = \mu_M$ it thus follows that

$$\|g - \widetilde{g}\|_{L^2(D_M,\mu_M)} \leq L_g 2rw_1^s\varepsilon.$$

The linear transformation T^{-1} introduces M additional parameters, regardless of ε. But since $\varepsilon \leq 1$ and $\alpha \geq 1$, it holds that $\widetilde{g}$ has at most $\mathcal{O}(M^\alpha \varepsilon^{-\beta}(1 + \log(M) + |\log(\varepsilon)|)^\kappa)$ parameters. □

To bound the mean-squared error of the overall approximation, we introduce the uniform product probability measure $\mathcal{P}_U := \bigotimes_{i\in\mathbb{N}} \frac{\lambda}{2}$ on $U := [-1,1]^{\mathbb{N}}$, where λ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and U is equipped with the product Borel σ-algebra $\mathcal{B}([-1,1]^{\mathbb{N}})$. We further define the random variable $\sigma_r^s : U \to \mathcal{X}$ on $(U, \mathcal{B}([-1,1]^{\mathbb{N}}), \mathcal{P}_U)$ via

$$\sigma_r^s : U \to \mathcal{X}, \quad \mathfrak{u} \mapsto r \sum_{i\in\mathbb{N}} w_i^s \mathfrak{u}_i \psi_i,$$

and note that $C_r^s(\mathcal{X}) \subseteq \sigma_r^s[U]$. On the other hand, $\mathcal{E}_\mathcal{X}(\sigma_r^s(\mathfrak{u})) = (rw_i^s\mathfrak{u}_i, i \in \mathbb{N}) \in \bigtimes_{i\in\mathbb{N}}[-rw_i^s, rw_i^s]$ since $\mathbf{\Psi}_\mathcal{X}$ is a Riesz basis, and thus $C_r^s(\mathcal{X}) = \sigma_r^s[U]$. Hence, the pushforward measure $(\sigma_r^s)_\#\mathcal{P}_U$ is supported on $C_r^s(\mathcal{X})$. Our main result gives a bound on ε-complexity of expression for Lipschitz maps, in terms of the number $\mathcal{N}_{para}$ of neurons which are sufficient for ε-consistency of the DON.

Theorem 5.7. *Let Assumptions 4.1 and 5.1 hold and fix $\delta > 0$ (arbitrarily small). Then, for any $\varepsilon \in (0,1]$ exists a neural network $\widetilde{G} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with at most $\mathcal{N}_{para} \in \mathbb{N}$ parameters such that*

$$\|\mathcal{G} - \mathcal{D}_\mathcal{Y} \circ \widetilde{G} \circ \mathcal{E}_\mathcal{X}\|_{L^2(C_r^s(\mathcal{X}),(\sigma_r^s)_\#\mathcal{P}_U;\mathcal{Y})} \leq \varepsilon,$$

and , for some constant $C > 0$ independent of ε

$$\mathcal{N}_{para} \leq C \begin{cases} \varepsilon^{-\alpha/(s-1/2)-(2+\beta)/2t-\delta}(1 + |\log(\varepsilon)|)^\kappa, & t \leq \frac{1}{2}, \\ \varepsilon^{-\alpha/(s-1/2)-1/t-\beta-\delta}(1 + |\log(\varepsilon)|)^\kappa, & t > \frac{1}{2}, \end{cases} \tag{21}$$

where $C = C(\delta, L_\mathcal{G}) > 0$ is independent of ε.

Proof. We first prove the claim in case that $t \leq \frac{1}{2}$.

Fix $\varepsilon \in (0,1]$ and recall from (14) that $\mathcal{G}_N^{\mathbf{M}} = \sum_{j=1}^N g_j(\mathcal{E}_\mathcal{X}(\cdot))\eta_j$ for any N and $\mathbf{M} \in \mathbb{N}^N$. We construct $\widetilde{\mathcal{G}}$ by first choosing appropriate truncation indices N and $\mathbf{M} \in \mathbb{N}^N$ to obtain $\mathcal{G}_N^{\mathbf{M}}$ such that $\sup_{x\in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N^{\mathbf{M}}(x)\|_\mathcal{Y} \leq \frac{\varepsilon}{2}$. Then we substitute each map g_j in $\mathcal{G}_N^{\mathbf{M}}$ by an appropriate neural network surrogate to achieve an overall error of at most ε.

1.) *Input and output truncation:* Propositions 4.4 and 4.3 show that for any $\delta_0 \in (0, s - \frac{1}{2})$ there exists $C_{L_\mathcal{G},\delta_0} > 0$ such that for $\mathbf{M} = (M, \ldots, M)$ where $M \in \mathbb{N}$ and $N \in \mathbb{N}$ there holds

$$\sup_{x\in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N^{\mathbf{M}}(x)\|_\mathcal{Y} \leq C_{L_\mathcal{G},\delta_0}\left(w_{N+1}^t + w_{M+1}^{s-\frac{1}{2}-\delta_0}\right). \tag{22}$$

As the weight sequence $\boldsymbol{w} \in (0,1]^{\mathbb{N}}$ is non-increasing with $\boldsymbol{w} \in \ell^{1+\widehat{\varepsilon}}(\mathbb{N})$ for any $\widehat{\varepsilon} > 0$, there exists for any $\varepsilon_0 > 0$ a $C_0 \geq 1$ such that $w_i \leq C_0 i^{-1+\varepsilon_0}$ for all $i \in \mathbb{N}$, and thus, $C_{L_\mathcal{G},\delta_0} w_{N+1}^t \leq C_{L_\mathcal{G},\delta_0} C_0^t N^{-t(1-\varepsilon_0)}$. Hence, we may fix some (arbitrary small) $\delta_0 \in (0, \frac{s}{2} - \frac{1}{4})$ and $\varepsilon_0 \in (0, \frac{\delta_0}{s-1/2-\delta_0})$, and let

$$N := \left\lceil \left(\frac{\varepsilon}{4C_{L_\mathcal{G},\delta_0} C_0^t}\right)^{-\frac{1}{t(1-\varepsilon_0)}} \right\rceil \in \mathbb{N}. \tag{23}$$

Note that by the choice of δ_0, ε_0 there holds $\varepsilon_0 < 1$ and $(s - \frac{1}{2} - \delta_0)(1 - \varepsilon_0) > s - \frac{1}{2} - 2\delta_0 > 0$. Since $t \leq \frac{1}{2}$, we thus set $\mathbf{M} = (M, \ldots, M) \in \mathbb{N}^N$ with

$$M := \left\lceil \left(\frac{\varepsilon}{4C_{L_\mathcal{G},\delta_0} C_0^{s-\frac{1}{2}-\delta_0}} \right)^{-\frac{1}{s-1/2-2\delta_0}} \right\rceil \geq \left\lceil \left(\frac{\varepsilon}{4C_{L_\mathcal{G},\delta_0} C_0^{s-\frac{1}{2}-\delta_0}} \right)^{-\frac{1}{(s-1/2-\delta_0)(1-\varepsilon_0)}} \right\rceil. \tag{24}$$

Combing the choices of N and $\mathbf{M} = (M, \ldots, M)$ with (22) and $w_i \leq C_0 i^{-1+\varepsilon_0}$ then shows

$$\sup_{x \in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N^{\mathbf{M}}(x)\|_\mathcal{Y} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{25}$$

2.) *Neural network surrogates for g_j:* It holds by (19) that $\mathcal{G}_N^{\mathbf{M}} = \sum_{j=1}^N g_j(\mathcal{E}_\mathcal{X}(\cdot))\eta_j = \sum_{j=1}^N \mathfrak{g}_j(\Pi_{M_j} \circ \mathcal{E}_\mathcal{X}(\cdot))\eta_j$. Furthermore, Lemma 5.5 shows that each $\mathfrak{g}_j : \mathbb{R}^{M_j} \to \mathbb{R}$ is Lipschitz continuous on the subset $D_{M_j} := \bigtimes_{i=1}^{M_j}[-rw_i^s, rw_i^s] \subset \mathbb{R}^{M_j}$ with Lipschitz constant given by $L_\mathcal{G} w_j^t \sqrt{\Lambda_\mathcal{Y}}$. Lemma 5.6 then in turn shows that for any fixed $j = 1, \ldots, N$ and for any $\varepsilon_j \in (0, 1]$ there exists an approximation $\widetilde{\mathfrak{g}}_j : D_{M_j} \to \mathbb{R}$ of $\mathfrak{g}_j$, such that

$$\|\mathfrak{g}_j - \widetilde{\mathfrak{g}}_j\|_{L^2(\mathcal{D}_{M_j}, \mu_{M_j})} \leq L_\mathcal{G} w_j^t \sqrt{\Lambda_\mathcal{Y}} 2rw_1^s \varepsilon_j, \tag{26}$$

where $\mu_{M_j} := \bigotimes_{i=1}^{M_j} \frac{\lambda}{2rw_i^s}$ denotes the uniform probability measure on D_{M_j}. Furthermore, the DNN $\widetilde{\mathfrak{g}}_j$ uses at most $\mathcal{O}(M_j^\alpha \varepsilon_j^{-\beta}(1 + \log(M_j) + |\log(\varepsilon_j)|)^\kappa)$ parameters.

With this at hand we define the neural network surrogate

$$\widetilde{\mathcal{G}} : \mathcal{X} \to \mathcal{Y}, \quad x \mapsto \sum_{j=1}^N \widetilde{\mathfrak{g}}_j(\Pi_{M_j} \mathcal{E}_\mathcal{X}(x))\eta_j. \tag{27}$$

To bound the error in this surrogate, we observe that for any $x \in C_r^s(\mathcal{X})$ it holds that

$$\begin{aligned} \left\|\mathcal{G}_N^{\mathbf{M}}(x) - \widetilde{\mathcal{G}}(x)\right\|_\mathcal{Y}^2 &= \left\| \sum_{j=1}^N \left(\mathfrak{g}_j(\Pi_{M_j}\mathcal{E}_\mathcal{X}(x)) - \widetilde{\mathfrak{g}}_j(\Pi_{M_j}\mathcal{E}_\mathcal{X}(x)) \right) \eta_j \right\|_\mathcal{Y}^2 \\ &\leq \Lambda_\mathcal{Y} \sum_{j=1}^N \left| \mathfrak{g}_j(\Pi_{M_j}\mathcal{E}_\mathcal{X}(x)) - \widetilde{\mathfrak{g}}_j(\Pi_{M_j}\mathcal{E}_\mathcal{X}(x)) \right|^2. \end{aligned}$$

Furthermore, we have $\Pi_{M_j}\mathcal{E}_\mathcal{X}(x) = \Pi_{M_j}\mathcal{E}_\mathcal{X}(\sigma_r^s(\mathfrak{u}))$ for some (in general non-unique) $\mathfrak{u} \in U$. In addition, if $\mathfrak{u} \sim \mathcal{P}_U$ then $\Pi_{M_j}\mathcal{E}_\mathcal{X}(\sigma_r^s(\mathfrak{u})) = r(w_1^s \mathfrak{u}_1, \ldots, w_{M_j}^s \mathfrak{u}_{M_j}) \sim \bigotimes_{i=1}^{M_j} \frac{\lambda}{2rw_i^s} = \mu_{M_j}$. Therefore, (26) yields with $C_{L_\mathcal{G}} := L_\mathcal{G} \Lambda_\mathcal{Y}^{\frac{3}{2}} 2rw_1^s$ that

$$\left\|\mathcal{G}_N^{\mathbf{M}} - \widetilde{\mathcal{G}}\right\|_{L^2(C_r^s(\mathcal{X}), (\sigma_r^s)_\# \mathcal{P}_U; \mathcal{Y})}^2 \leq \Lambda_\mathcal{Y} \sum_{j=1}^N \|\mathfrak{g}_j - \widetilde{\mathfrak{g}}_j\|_{L^2(D_{M_j}, \mu_{M_j})}^2 \leq C_{L_\mathcal{G}}^2 \sum_{j=1}^N w_j^{2t} \varepsilon_j^2.$$

We then obtain

$$\begin{aligned} \left\|\mathcal{G}_N^{\mathbf{M}} - \widetilde{\mathcal{G}}\right\|_{L^2(C_r^s(\mathcal{X}), (\sigma_r^s)_\# \mathcal{P}_U; \mathcal{Y})}^2 &\leq C_{L_\mathcal{G}}^2 \left(\max_{j=1,\ldots,N} w_j^{2(t-\frac{1}{2}-\delta_0)} \varepsilon_j^2 \right) \sum_{j=1}^N w_j^{1+2\delta_0} \\ &\leq C_{L_\mathcal{G}}^2 C_0^{1+2\delta_0} \zeta(1+2\delta_0) \left(\max_{j=1,\ldots,N} w_j^{2(t-\frac{1}{2}-\delta_0)} \varepsilon_j^2 \right), \end{aligned}$$

where $\zeta : (1,\infty) \to (0,\infty)$ denotes the Riemann zeta function. Now let $C_{\delta_0} := C_0^{\frac{1}{2}+\delta_0}\zeta(1+2\delta_0)^{\frac{1}{2}}$ and

$$\varepsilon_j := \min\left(\frac{w_j^{-(t-\frac{1}{2}-\delta_0)}\varepsilon}{2C_{L_\mathcal{G}}C_{\delta_0}}, 1\right) \in (0,1], \quad j = 1,\dots,N. \tag{28}$$

The triangle inequality and (25) then show

$$\begin{aligned}\left\|\mathcal{G} - \widetilde{\mathcal{G}}\right\|_{L^2(C_r^s(\mathcal{X}),(\sigma_r^s)_\#\mathcal{P}_U;\mathcal{Y})} &\le \sup_{x\in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N^{\mathbf{M}}(x)\|_\mathcal{Y} + \left\|\mathcal{G}_N^{\mathbf{M}} - \widetilde{\mathcal{G}}\right\|_{L^2(C_r^s(\mathcal{X}),(\sigma_r^s)_\#\mathcal{P}_U;\mathcal{Y})}\\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

3.) *Overall complexity:* Let $\mathcal{N}_{para} \in \mathbb{N}$ denote the total number of parameters used to construct $\widetilde{\mathcal{G}}$. We have in total used N surrogates $\widetilde{\mathfrak{g}}_j : D_{M_j} \to \mathbb{R}$ with prescribed accuracies $\varepsilon_j \in (0,1]$ for $j = 1,\dots,N$. Each of these surrogates therefore involves (at most) $\mathcal{O}(M_j^\alpha \varepsilon_j^{-\beta}(1+\log(M_j) + |\log(\varepsilon_j)|)^\kappa)$ parameters by Lemma 5.6. Hence,

$$\mathcal{N}_{para} \le C_{dofs} \sum_{j=1}^N M_j^\alpha \varepsilon_j^{-\beta}(1+\log(M_j) + |\log(\varepsilon_j)|)^\kappa, \tag{29}$$

where the constant $C_{dofs} > 0$ is independent of ε_j, N and M_j.

For fixed $\delta_0 \in (0, s-\frac{1}{2})$ independent of ε, we recall that $M_j = M$ for all j, where M is given in (24). Substituting the choices of M and ε_j in (24) and (28), respectively, yields

$$\begin{aligned}\mathcal{N}_{para} \le C_{dofs} \sum_{j=1}^N &\left\lceil \left(\frac{\varepsilon}{4C_{L_\mathcal{G},\delta_0}C_0^{s-\frac{1}{2}-\delta_0}}\right)^{-\frac{1}{s-1/2-2\delta_0}} \frac{1}{C_0}\right\rceil^\alpha \min\left(\frac{w_j^{-(t-\frac{1}{2}-\delta_0)}\varepsilon}{2C_{L_\mathcal{G}}C_{\delta_0}}, 1\right)^{-\beta}\\ &\cdot\left(1 + \log\left(\left\lceil \left(\frac{\varepsilon}{4C_{L_\mathcal{G},\delta_0}C_0^{s-\frac{1}{2}-\delta_0}}\right)^{-\frac{1}{s-1/2-2\delta_0}} \frac{1}{C_0}\right\rceil\right) + \left|\log\left(\frac{w_j^{-(t-\frac{1}{2}-\delta_0)}\varepsilon}{2C_{L_\mathcal{G}}C_{\delta_0}}\right)\right|\right)^\kappa,\end{aligned}$$

where only the constants C_{δ_0} and $C_{L_\mathcal{G},\delta_0}$ depend on δ_0.

Further, recall that the constants $C_{L_\mathcal{G}}$ and that $C_{L_\mathcal{G},\delta_0}$ grow linearly with respect to $L_\mathcal{G}$. Hence, $t - \frac{1}{2} - \delta_0 < 0$ and N as in (23) show that for any $\varepsilon_0 \in (0,1)$ there holds

$$\begin{aligned}\mathcal{N}_{para} &\le C\sum_{j=1}^N \varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} w_j^{\beta(t-\frac{1}{2}-\delta_0)}(1 + |\log(\varepsilon)| + |\log(\varepsilon w_j)|\,|)^\kappa\\ &\le CN\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} \max_{j=1\dots,N}\left(w_j^{\beta(t-\frac{1}{2}-\delta_0)}\right)(1 + |\log(\varepsilon)| + |\log(w_j)|\,|)^\kappa\\ &\le C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} N^{1-\beta(t-\frac{1}{2}-\delta_0)(1-\varepsilon_0)}(1 + |\log(\varepsilon)| + |\log(N)|\,|)^\kappa\\ &\le C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} \varepsilon^{(-1+\beta(t-\frac{1}{2}-\delta_0)(1-\varepsilon_0))\frac{1}{t(1-\varepsilon_0)}}(1 + |\log(\varepsilon)|)^\kappa\\ &\le C\varepsilon^{-\frac{\alpha}{s-1/2}-\frac{(2+\beta)}{2t}-\delta}(1 + |\log(\varepsilon)|)^\kappa,\end{aligned}$$

where $C = C(\delta_0, \varepsilon_0, L_\mathcal{G})$ is independent of ε, and δ is given by

$$\delta = \frac{\alpha}{s-1/2-2\delta_0} + \beta + \frac{1}{t(1-\varepsilon_0)} - \frac{\beta(2t-1-2\delta_0)}{2t} - \frac{\alpha}{s-1/2} - \frac{(2+\beta)}{2t} = O(\delta_0) \qquad \text{as } \delta_0 \to 0.$$

The claim for $t \le \frac{1}{2}$ now follows for arbitrary $\delta > 0$ by choosing δ_0 and ε_0 sufficiently small.

4.) *Case* $t > \frac{1}{2}$*:* We only highlight the changes that are necessary in the proof for $t \leq \frac{1}{2}$.

4a) Observe that for arbitrary $\delta_0 \in (0, \frac{\min(s,t)}{2} - \frac{1}{4})$ and $\varepsilon_0 \in (0, \frac{\delta_0}{\max(s,t)-1/2-\delta_0})$ there holds

$$\max_{j=1,\dots,N} w_j^{t-\frac{1}{2}-\delta_0} w_{M_j+1}^{s-\frac{1}{2}-\delta_0} \leq C_0^{t+s-1-2\delta_0} \max_{j=1,\dots,N} j^{-t+\frac{1}{2}+2\delta_0} M_j^{-s+\frac{1}{2}+2\delta_0}.$$

Hence, we set

$$M_j := \left\lceil \left(\frac{\varepsilon}{4C_{L_\mathcal{G},\delta_0}} j^{t-\frac{1}{2}-2\delta_0} C_0^{t+s-1-2\delta_0} \right)^{-\frac{1}{s-1/2-2\delta_0}} \right\rceil, \quad j = 1, \dots, N. \tag{30}$$

We further observe that $M_1 \simeq M$ with M as in (24), and that M_j is decreasing with increasing j, since $\frac{t-1/2-2\delta_0}{s-1/2-2\delta_0} > 0$ as $\delta_0 < \frac{\min(s,t)}{2} - \frac{1}{4}$. Next, by Propositions 4.4 and 4.3 for any $\mathbf{M} \in \mathbb{N}^N$ there holds

$$\sup_{x \in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N^{\mathbf{M}}(x)\|_\mathcal{Y} \leq C_{L_\mathcal{G},\delta_0} \left(w_{N+1}^t + \max_{j=1,\dots,N} w_j^{t-\frac{1}{2}-\delta_0} w_{M_j+1}^{s-\frac{1}{2}-\delta_0} \right). \tag{31}$$

With $\mathbf{M} = (M_1, \dots, M_N)$ in (30) it then follows that

$$\sup_{x \in C_r^s(\mathcal{X})} \|\mathcal{G}(x) - \mathcal{G}_N^{\mathbf{M}}(x)\|_\mathcal{Y} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

4b) Let $\delta_0 \in (0, \frac{\min(s,t)}{2} - \frac{1}{4})$ and $\varepsilon_0 \in (0, \frac{\delta_0}{\max(s,t)-1/2-\delta_0})$ and use M_j as in (30) for fixed δ_0. Thus, there holds $\log(M_j) \leq C(1 + |\log(\varepsilon)| + \log(j))$ and we obtain the estimate

$$\begin{aligned}
\mathcal{N}_{para} &\leq C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} \sum_{j=1}^N j^{-\alpha\frac{t-1/2-2\delta_0}{s-1/2-2\delta_0}} w_j^{\beta(t-\frac{1}{2}-\delta_0)} (1 + |\log(\varepsilon)| + \log(j))^\kappa \\
&\leq C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} \sum_{j=1}^N j^{-(\frac{\alpha}{s-1/2-2\delta_0}+\beta(1-\varepsilon_0))(t-\frac{1}{2}-2\delta_0)} (1 + |\log(\varepsilon)| + \log(j))^\kappa \\
&\leq C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} N \left(\max_{j=1,\dots,N} j^{-(\frac{\alpha}{s-1/2-2\delta_0}+\beta(1-\varepsilon_0))(t-\frac{1}{2}-2\delta_0)} \right) (1 + |\log(\varepsilon)| + \log(N))^\kappa.
\end{aligned}$$

As $2\delta_0 < \min(s,t) - \frac{1}{2}$ and $\varepsilon_0 \in (0,1)$, it holds that $(\frac{\alpha}{s-1/2-2\delta_0} + \beta(1-\varepsilon_0))(t - \frac{1}{2} - 2\delta_0) \geq 0$ and thus with N as in (23) it follows that

$$\begin{aligned}
\mathcal{N}_{para} &\leq C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta} N(1 + |\log(\varepsilon)| + \log(N))^\kappa \\
&\leq C\varepsilon^{-\frac{\alpha}{s-1/2-2\delta_0}-\beta-\frac{1}{t(1-\varepsilon_0)}} (1 + |\log(\varepsilon)|)^\kappa.
\end{aligned}$$

The claim then follows for arbitrary small

$$\delta = \frac{\alpha}{s - 1/2 - 2\delta_0} + \frac{1}{t(1-\varepsilon_0)} - \frac{\alpha}{s - 1/2} - \frac{1}{t} > 0$$

with appropriate choices of δ_0 and ε_0, since $\delta = \mathcal{O}(\delta_0 + \varepsilon_0) = \mathcal{O}(\delta_0)$ as $\delta_0 \to 0$. □

6 Discussion and Extensions

We indicate several directions in which our main result, Theorem 5.7, can be extended. First, due to the invariance of the (metric) space of Lipschitz maps under composition, Theorem 5.7 can be applied to finite compositions of Lipschitz maps in a "component-wise" manner, with different truncation dimensions, and different encoders and decoders at each stage of composition. This allows, in particular, to leverage known expression rate bounds for the factor maps. We indicate the ideas and details in Section 6.1.

6.1 Compositions of Lipschitz Maps

The preceding operator emulation bound readily applies to *composition of operators*: let $K \geq 2$ be an integer, and assume given a collection $(\mathcal{X}_k, k = 0, \ldots, K)$ of Banach spaces, with norms $\|\cdot\|_{\mathcal{X}_k}$. Further, denote for any $k = 1, ..., K$ by $\mathrm{Lip}(\mathcal{X}_{k-1}, \mathcal{X}_k)$ the set of all Lipschitz continuous mappings from $\mathcal{X}_{k-1}$ to $\mathcal{X}_k$. For nonlinear maps $\mathcal{G}_k \in \mathrm{Lip}(\mathcal{X}_{k-1}, \mathcal{X}_k)$, $k = 1, ..., K$ consider the composition

$$\mathcal{G}_{[1:k]} = \mathcal{G}_k \circ ... \circ \mathcal{G}_1, \quad k = 1, ..., K \ . \tag{32}$$

Setting $\mathcal{X} := \mathcal{X}_0$ and $\mathcal{Y} := \mathcal{X}_K$, evidently $\mathcal{G}_{[1:K]} \in \mathrm{Lip}(\mathcal{X}, \mathcal{Y})$. The following observation is elementary.

Lemma 6.1. *Assume each of the "factor" operators $\mathcal{G}_k$ constituting $\mathcal{G}_{[1:k]}$ in (32) are Lipschitz with Lipschitz constant*

$$L_k := \sup_{x, \widetilde{x} \in \mathcal{X}_{k-1}} \frac{\|\mathcal{G}_k(x) - \mathcal{G}_k(\widetilde{x})\|_{\mathcal{X}_k}}{\|x - \widetilde{x}\|_{\mathcal{X}_{k-1}}} \ , \ \ k = 1, ..., K \ .$$

Then, the composition $\mathcal{G}_{[1:k]}$ in (32) is in $\mathrm{Lip}(\mathcal{X}_0, \mathcal{X}_k)$ with Lipschitz constant $L_{[1:k]}$ bounded by

$$L_{[1:k]} \leq \prod_{j=1}^{k} L_j \ . \tag{33}$$

Operators $\mathcal{G}_{[1:K]}$ as in (32) are naturally covered by the main result, Theorem 5.7. To apply Theorem 5.7 to $\mathcal{G}_{[1:K]}$, it is in fact sufficient that only the last component $\mathcal{G}_K$ satisfies Assumption 4.1 , in the sense that there exist $s > \frac{1}{2}$, t and $r > 0$ such that

$$\|\mathcal{G}_K(x) - \mathcal{G}_K(x')\|_{\mathcal{X}_K^t} \leq L_k \|x - x'\|_{\mathcal{X}_{K-1}}, \quad x, x' \in C_r^s(\mathcal{X}_{K-1}). \tag{34}$$

For the remaining "factor" operators $\mathcal{G}_k$ we only need to assume the weaker condition $\mathcal{G}_k \in \mathrm{Lip}(\mathcal{X}_{k-1}, \mathcal{X}_k)$ for $k = 1, ..., K-1$ with Lipschitz constants $L_k > 0$. Lemma 6.1 then implies for $\mathcal{Y} = \mathcal{X}_K$ and $\mathcal{X} = \mathcal{X}_0$ the bound

$$\|\mathcal{G}_{[1:K]}(x) - \mathcal{G}_{[1:K]}(x')\|_{\mathcal{Y}^t} \leq L_{[1:K]} \|x - x'\|_{\mathcal{X}}, \quad x, x' \in C_r^s(\mathcal{X}). \tag{35}$$

6.2 Hölder Continuous Maps

The proposed methodology readily extends to the case of γ-Hölder continuous operators, with some $\gamma \in (0, 1]$ (note that, unlike Lipschitz-continuous maps, γ-Hölder-continuous maps are, for $\gamma \in (0, 1)$, not closed under composition). We briefly indicate the modifications of our main result, Theorem 5.7, to γ-Hölder continuous operators, and state the corresponding emulation rate bounds.

Suppose then that there is $\gamma \in (0, 1)$ and $L_{\mathcal{G}} > 0$ such that Assumption 4.1 holds in the weaker form

$$\|\mathcal{G}(x) - \mathcal{G}(x')\|_{\mathcal{Y}^t} \leq L_{\mathcal{G}} \|x - x'\|_{\mathcal{X}}^{\gamma}, \quad x, x' \in C_r^s(\mathcal{X}).$$

The corresponding growth bound from (11) then translates to

$$\|\mathcal{G}(x)\|_{\mathcal{Y}^t} \leq C(1 + \|x\|_{\mathcal{X}}^{\gamma}), \quad \text{with} \quad C := \max(L_{\mathcal{G}}, \|\mathcal{G}(0)\|_{\mathcal{Y}^t}). \tag{36}$$

This results in the same rate for the output truncation error from Proposition 4.3, where γ only enters the hidden constant. On the other hand, the input truncation error from Proposition 4.4 now scales worse and is of order

$$\mathcal{O}\left(\max_{j=1,\ldots,N} w_j^{t-\frac{1}{2}-\delta} w_{M_j+1}^{\gamma(s-\frac{1}{2}-\delta)} \right), \quad \text{and} \quad \mathcal{O}\left(w_{M+1}^{\gamma(s-\frac{1}{2}-\delta)} \right),$$

respectively.

In addition, the exponents $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ in Assumption 5.1 have to be scaled by $\gamma^{-1} \in (1, \infty)$ for the DNN emulation of Hölder continuous maps, as may be seen e.g. from [67, Corollary 2.2] or [58, Corollary 1.2].

In all, the estimates on the number of parameters in Theorem 5.7 change in the case of Hölder-continuous operators to

$$\mathcal{N}_{para} \lesssim \begin{cases} \varepsilon^{-\alpha/\gamma^2(s-1/2)-(2\gamma+\beta)/2\gamma t-\delta}(1+|\log(\varepsilon)|)^{\kappa/\gamma} & \text{if } t \leq \frac{1}{2}, \\ \varepsilon^{-\alpha/\gamma^2(s-1/2)-1/t-\beta/\gamma-\delta}(1+|\log(\varepsilon)|)^{\kappa/\gamma}, & \text{if } t > \frac{1}{2}. \end{cases}$$

Note that the scaling of γ^{-2} in first term of the exponent is due to the reduced rate for the input truncation error *and* deteriorating DNN emulation rates for Hölder continuous functions. The result [67, Theorem 2.1] even allows to obtain expression rate bounds when only a (possibly weak) bound on the modulus of continuity of the components $\mathfrak{g}_j$ of G holds, as e.g. in the so-called Calderón-Problem (where this modulus is logarithmic, see [33] and the references there).

7 Examples

We provide several examples to illustrate the scope of the presently obtained expression rate bounds. Naturally, all maps between function spaces which are holomorphic, as considered in [24] are in particular (locally) Lipschitz and are therefore covered by the present approximation rate bounds. In [24], under parametric holomorphy assumptions, operator surrogates were constructed which were based on strict ReLU feedforward NNs. The present constructions are considerably more involved than those in [24], so that examples from [24] are not illustrative. We opt to discuss examples of Lipschitz operators that generically do not exhibit any regularity beyond Lipschitz (or Hölder).

In Section 7.1, we show that the proposed, abstract setting naturally accommodates a broad class of (non-linear) parametric, elliptic variational inequalities. In Section 7.2, we study expression rates for certain Lipschitz maps on the space of Hilbert-Schmidt operators between two real, separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

7.1 Elliptic Variational Inequalities

We consider parameter-to-solution maps of elliptic variational inequalities (EVIs), that arise for example from obstacle problems (e.g. [18, 26]) in mechanics, optimal stopping in financial modeling or in optimal control of differential operators [19]. It is well-known that the dependence of solutions of EVIs in unilateral problems for elliptic operators on, for example, coefficient functions in the elliptic operator is Lipschitz, but not better, even for smooth obstacles. See, e.g., [16, 22] and the references there. Hence, within our presently developed, abstract framework, coefficient-to-solution maps of EVIs can be emulated by Deep Operator Nets (DONs), with the architecture as in (3). We remark that the ensuing DON emulation rate analysis is based on (classical) results on the Lipschitz stability of solutions of EVIs under perturbations of the data, as proved e.g. in [16]. An alternative approach to DON expression rate bounds is via "unrolling" known, iterative solution algorithms e.g. with recurrent DNNs, see e.g. [55].

7.1.1 Abstract Setting

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a separable Hilbert space, let $\mathcal{H}^*$ denote the dual space of $\mathcal{H}$, and let ${}_{\mathcal{H}^*}\langle\cdot,\cdot\rangle_{\mathcal{H}}$: $\mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$ be the associated dual pairing. Further, let $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{H} \to \mathcal{H}$ be (possibly nonlinear) operators, $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$ and let $f_1, f_2 \in \mathcal{H}$.

Assumption 7.1. It holds that $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$ are closed convex subsets and there exist constants $\ell, L > 0$ such that for $i = 1, 2$:

- $\mathcal{A}_i$ is strongly ℓ-monotone, i.e.

$$\langle \mathcal{A}_i u - \mathcal{A}_i v, u - v\rangle_{\mathcal{H}} \geq \ell \, \|u - v\|_{\mathcal{H}}^2 \quad \text{for all } u, v \in \mathcal{H}. \tag{37}$$

- $\mathcal{A}_i$ is L-Lipschitz continuous, i.e.

$$\|\mathcal{A}_i u - \mathcal{A}_i v\|_{\mathcal{H}} \le L \|u - v\|_{\mathcal{H}} \quad \text{for all } u, v \in \mathcal{H}. \tag{38}$$

- For all $r > 0$ exists $B(r) < \infty$ such that

$$\sup_{\|u\|_{\mathcal{H}} \le r} \|\mathcal{A}_i u\|_{\mathcal{H}} \le B(r).$$

For $i = 1, 2$ and $\mathcal{K}_i \neq \emptyset$ closed and convex, we define by

$$P_i : \mathcal{H} \to \mathcal{K}_i, \quad u \mapsto \operatorname{argmin}_{v \in \mathcal{K}_i} \|u - v\|_{\mathcal{H}} \tag{39}$$

the projection onto $\mathcal{K}_i$. We consider the variational inequality to

$$\text{find } u_i \in \mathcal{K}_i \text{ such that } \quad \langle \mathcal{A}_i u_i, v - u_i \rangle_{\mathcal{H}} \ge \langle f_i, v - u_i \rangle_{\mathcal{H}} \quad \text{for all } v \in K_i. \tag{40}$$

Under Assumption 7.1, it is well-known that (40) admits a unique solution $u_i \in \mathcal{K}_i$. In addition, u_i depends Lipschitz continuous on $\mathcal{A}_i$, f_i and $\mathcal{K}_i$ in the following sense.

Theorem 7.2. *[16, Theorem 2.1] Let Assumption 7.1 hold and define for any $r > 0$*

$$\begin{aligned}
\mathfrak{A}(r) &:= \sup_{\|u\|_{\mathcal{H}} \le r} \|\mathcal{A}_1 u - \mathcal{A}_2 u\|_{\mathcal{H}} < \infty, \\
\varrho(r) &:= \sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}} \le r} \|P_1 u - P_2 u\|_{\mathcal{H}} < \infty, \quad \textit{and} \\
\mathfrak{D} &:= \max_{i=1,2} \inf_{\varphi \in \mathcal{K}_i} \|\varphi\|_{\mathcal{H}}.
\end{aligned}$$

Let u_i be the unique solution to (40) *for $i = 1, 2$. Then, there exist constants $R = R(\ell, L, \mathfrak{D}) > 0$ and $C = C(\ell, L) > 0$ such that $\|u_i\|_{\mathcal{H}} \le R$ for $i = 1, 2$ and*

$$\|u_1 - u_2\|_{\mathcal{H}} \le C \left(\varrho \left(R + B(R) + \max \left(\|f_1\|_{\mathcal{H}}, \|f_2\|_{\mathcal{H}} \right) \right) + \|f_1 - f_2\|_{\mathcal{H}} + \mathfrak{A}(R) \right). \tag{41}$$

7.1.2 Elliptic Variational Inequalities on the Torus

For $d \in \mathbb{N}$ denote the d-dimensional torus by $\mathbb{T}^d \simeq [0,1)^d$. Let $\mathcal{H} := H^1(\mathbb{T}^d)$ and consider the closed, convex sets $\mathcal{K}_\phi \subset \mathcal{H}$, that are parameterized by *obstacles* $\phi \in \mathcal{H}$ via

$$\mathcal{K}_\phi := \{\varphi \in \mathcal{H} : \varphi \ge \phi\}.$$

We fix a (possibly non-linear) operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ satisfying Assumption 7.1 and a source term $f \in \mathcal{H}$, and identify f with an element $f^* \in \mathcal{H}^*$ by the Riesz representation theorem. For the sake of brevity, we only consider varying obstacles $\phi \in \mathcal{H}$ in the following. Then, for any $\phi \in \mathcal{H}$ and corresponding $\mathcal{K}_\phi$, there exists a unique solution $u_\phi \in \mathcal{K}_\phi$ to the variational inequality

$$\langle \mathcal{A} u_\phi, v - u_\phi \rangle_{\mathcal{H}} \ge {}_{\mathcal{H}^*}\langle f^*, v - u_\phi \rangle_{\mathcal{H}} \quad \text{for all } v \in \mathcal{K}_\phi. \tag{42}$$

We may thus define the *obstacle-to-solution operator*

$$\mathcal{G} : \mathcal{H} \to \mathcal{H}, \quad \phi \mapsto u_\phi. \tag{43}$$

Now let $\phi_1, \phi_2 \in \mathcal{H}$ and denote the respective projections by $P_i : \mathcal{H} \to \mathcal{K}_{\phi_i}$. In view of Theorem 7.2, we obtain that

$$\begin{aligned}
\|\mathcal{G}(\phi_1) - \mathcal{G}(\phi_2)\|_{\mathcal{H}} &\le C \varrho(R + B(R) + \|f^*\|_{\mathcal{H}^*}) \\
&\le C \sup_{\|u\|_{\mathcal{H}} \le R + B(R) + \|f^*\|_{H^*}} \|P_1 u - P_2 u\|_{\mathcal{H}},
\end{aligned}$$

where $C > 0$ is independent of ϕ_1, ϕ_2, but with $R > 0$ depending on $\mathfrak{D} = \max_{i=1,2} \inf_{\varphi \in \mathcal{K}_i} \|\varphi\|_{\mathcal{H}}$.

Using that $P_1 u = P_2(u - (\phi_1 - \phi_2)) + (\phi_1 - \phi_2) \in \mathcal{K}_{\phi_1}$, however, shows that there exists a $L_{\mathcal{G}} > 0$, independent of R, such that for all $\phi_1, \phi_2 \in \mathcal{H}$ it holds

$$\begin{aligned}\|\mathcal{G}(\phi_1) - \mathcal{G}(\phi_2)\|_{\mathcal{H}} &\le C \sup_{\|u\|_{\mathcal{H}} \le R + B(R) + \|f^*\|_{H^*}} \|P_2(u - (\phi_1 - \phi_2)) + (\phi_1 - \phi_2) - P_2 u\|_{\mathcal{H}} \\ &\le C \sup_{\|u\|_{\mathcal{H}} \le R + B(R) + \|f^*\|_{H^*}} \|P_2(u - (\phi_1 - \phi_2)) - P_2 u\|_{\mathcal{H}} + \|\phi_1 - \phi_2\|_{\mathcal{H}} \\ &\le L_{\mathcal{G}} \|\phi_1 - \phi_2\|_{\mathcal{H}}.\end{aligned} \tag{44}$$

7.1.3 Wavelet Encoding

Let $K_j := \{k \in \mathbb{Z}^d : 0 \le k_1, \dots, k_d < 2^j\} \subset 2^j \mathbb{T}^d$ for $j \in \mathbb{N}_0$, let $\mathcal{L}_0 := \{0,1\}^d$ and $\mathcal{L}_j := \mathcal{L}_0 \setminus \{(0, \dots, 0)\}$ for $j \in \mathbb{N}$. By [59, Proposition 1.34], there exists an $L^2(\mathbb{T}^d)$-orthonormal basis

$$\mathbf{\Psi} := \left((\psi^l_{j,k}),\ (j,k,l) \in \mathcal{I}_{\mathbf{\Psi}}\right), \quad \mathcal{I}_{\mathbf{\Psi}} := \{j \in \mathbb{N}_0,\ k \in K_j,\ l \in \mathcal{L}_j\}, \tag{45}$$

where the $\psi^l_{j,k} : \mathbb{T}^d \to \mathbb{R}$ are constructed from scaled, translated and tensorized one-periodic wavelets, such that $\lambda(\text{supp}(\psi^l_{j,k})) = \mathcal{O}(2^{-dj})$. The basis $\mathbf{\Psi}$ may be constructed from Daubechies wavelets with $m \in \mathbb{N}$ vanishing moments that have compactly supported, univariate scaling and wavelet functions. By choosing m sufficiently large, we may ensure for given $k \in \mathbb{N}$ that $\psi, \phi \in \mathrm{C}^k(\mathbb{R})$, and thus $\mathbf{\Psi} \subset \mathrm{C}^k(\mathbb{R})$. For instance, $\psi, \phi \in \mathrm{C}^1(\mathbb{R})$ holds for univariate so-called Daubechies wavelets with $m \ge 5$ vanishing moments, see [15, Section 7.1].

For $\gamma \in (0, k)$, $p \in [1, \infty]$ and $\varphi \in L^2(\mathbb{T}^d)$ recall from [59] the *Besov norms*

$$\|\varphi\|_{B^\gamma_{p,p}(\mathbb{T}^d)} := \left(\sum_{(j,k,l) \in \mathcal{I}_{\mathbf{\Psi}}} 2^{jp(\gamma + \frac{d}{2} - \frac{d}{p})} |(\varphi, \psi^l_{j,k})_{L^2(\mathbb{T}^d)}|^p \right)^{1/p}, \quad p \in [1, \infty), \tag{46}$$

and, for $p = \infty$,

$$\|\varphi\|_{B^\gamma_{\infty,\infty}(\mathbb{T}^d)} := \sup_{(j,k,l) \in \mathcal{I}_{\mathbf{\Psi}}} 2^{j(\gamma + \frac{d}{2})} |(\varphi, \psi^l_{j,k})_{L^2(\mathbb{T}^d)}| < \infty. \tag{47}$$

According to [59, Theorem 1.36], the corresponding one-periodic *Besov spaces* on $\mathbb{T}^d$ are then represented via

$$B^\gamma_{p,p}(\mathbb{T}^d) := \left\{ \varphi \in L^2(\mathbb{T}^d) : \|\varphi\|_{B^\gamma_{p,p}(\mathbb{T}^d)} < \infty \right\}. \tag{48}$$

We recall that $B^\gamma_{2,2}(\mathbb{T}^d) = H^\gamma(\mathbb{T}^d)$. Further, let $\mathcal{C}^\tau(\mathbb{T}^d)$ denote the Hölder-Zygmund space for a given exponent $\tau > 0$. There holds the embedding $B^\gamma_{p,p}(\mathbb{T}^d) \hookrightarrow B^\tau_{\infty,\infty}(\mathbb{T}^d) = \mathcal{C}^\tau(\mathbb{T}^d)$ for $\gamma - \frac{d}{p} > 0$ and $\tau \in \left(0, \gamma - \frac{d}{p}\right]$, *with embedding constant bounded by one*, see e.g. [61, Chapter 2.1].

Since $B^\gamma_{2,2}(\mathbb{T}^d) = H^\gamma(\mathbb{T}^d)$ is a Hilbert space, for representations in Fourier- or Wavelet-bases one may identify $B^\gamma_{2,2}(\mathbb{T}^d)$ with certain smoothness spaces $\mathcal{X}^s$ and $\mathcal{Y}^t$ as in Section 2.2. To relate the exponent γ with s and t, we derive an equivalent norm to (46) for $p = 2$. It is based on a weight sequence $\boldsymbol{w} = (w_i, i \in \mathbb{N})$ with a single integer index as in Section 2.2. As a first step, observe that $|K_j| = 2^{dj}$ and $|\mathcal{L}_j| \le 2^d$ for all $j \in \mathbb{N}_0$, and denote by $(i_j, j \in \mathbb{N}_0)$ an (arbitrary) collection of bijective mappings satisfying

$$\begin{aligned} &i_0 : K_0 \times \mathcal{L}_0 \to \{0, \dots, 2^d - 1\} \\ &i_j : K_j \times \mathcal{L}_j \to \left\{ \sum_{m=0}^{j-1} |K_m||\mathcal{L}_m| + 1, \dots, \sum_{m=0}^{j} |K_m||\mathcal{L}_m| \right\}, \quad j \in \mathbb{N}. \end{aligned}$$

We may then re-label all wavelet indices (j,k,l) by integers via the one-to-one mapping

$$\mathfrak{I} : \mathcal{I}_\Psi \to \mathbb{N}_0, \quad (j,k,l) \mapsto i_j(k,l).$$

Note that

$$\sum_{m=0}^{j-1} |K_m||\mathcal{L}_m| = 1 + (2^d - 1)\sum_{m=0}^{j-1} d^{dm} = 2^{dj}, \quad j \in \mathbb{N},$$

hence $\mathfrak{I}(j,k,l) \in \{2^{dj}+1, \dots, 2^{d(j+1)}\}$ for any $(j,k,l) \in \mathcal{I}_\Psi$ with $j \geq 1$. Thus, by letting $\psi_{\mathcal{I}(j,k,l)} := \psi^l_{j,k}$, there holds

$$\begin{aligned}
\|\varphi\|^p_{B^\gamma_{p,p}(\mathbb{T}^d)} &= \sum_{(j,k,l)\in\mathcal{I}_\Psi} 2^{jp(\gamma+\frac{d}{2}-\frac{d}{p})} |(\varphi, \psi^l_{j,k})_{L^2(\mathbb{T}^d)}|^p \\
&\leq \sum_{i\in\mathbb{N}_0} i^{\frac{p\gamma}{d}+\frac{p}{2}-1} |(\varphi, \psi_i)_{L^2(\mathbb{T}^d)}|^p \\
&\leq 2^d \sum_{(j,k,l)\in\mathcal{I}_\Psi} 2^{jp(\gamma+\frac{d}{2}-\frac{d}{p})} |(\varphi, \psi^l_{j,k})_{L^2(\mathbb{T}^d)}|^p \\
&= 2^d \|\varphi\|^p_{B^\gamma_{p,p}(\mathbb{T}^d)}.
\end{aligned} \tag{49}$$

Now define the Hilbert space $(\mathcal{X}, \langle\cdot,\cdot\rangle_\mathcal{X})$ with the inner product

$$\langle\varphi,\phi\rangle_\mathcal{X} := \sum_{i\in\mathbb{N}_0} \langle\varphi, i^{\gamma/2d}\psi_i\rangle_{L^2(\mathbb{T}^d)} \langle\phi, i^{\gamma/2d}\psi_i\rangle_{L^2(\mathbb{T}^d)}, \quad \varphi,\phi \in L^2(\mathbb{T}^d)$$

and by

$$\mathcal{X} := \left\{\varphi \in L^2(\mathbb{T}^d) : \ \|\varphi\|_\mathcal{X} := \sqrt{\langle\varphi,\varphi\rangle_\mathcal{X}} < \infty\right\}.$$

With $p = 2$ in (49) there holds $\mathcal{X} = H^\gamma(\mathbb{T}^d)$ with the norm equivalence

$$\|\varphi\|^2_{H^\gamma(\mathbb{T}^d)} \leq \|\varphi\|^2_\mathcal{X} \leq 2^d \|\varphi\|^2_{H^\gamma(\mathbb{T}^d)}.$$

7.1.4 Neural Network Approximation Rates of parametric EVIs

Let $w_i := i^{-1}, i \in \mathbb{N}$. Then $\boldsymbol{w} := (w_i, i \in \mathbb{N}) \in \ell^{1+\varepsilon_{\boldsymbol{w}}}(\mathbb{N})$ for all $\varepsilon_{\boldsymbol{w}} > 0$. Choose $\mathcal{X} := H^{s_0}(\mathbb{T}^d)$ for some fixed $s_0 \geq 0$, to obtain for any $s \geq 0$ and $p = 2$ that

$$\begin{aligned}
\mathcal{X}^s &= \left\{\varphi \in \mathcal{X} : \ \|\varphi\|^2_{\mathcal{X}^s} := \sum_{i\in\mathbb{N}} |(\varphi,\psi_i)_\mathcal{X}|^2 w_i^{-2s} < \infty\right\} \\
&= \left\{\varphi \in H^{s_0}(\mathbb{T}^d) : \ \|\varphi\|^2_{\mathcal{X}^s} := \sum_{i\in\mathbb{N}} |(\varphi,\psi_i)_{H^{s_0}(\mathbb{T}^d)}|^2 w_i^{-2s} < \infty\right\} \\
&= \left\{\varphi \in L^2(\mathbb{T}^d) : \ \sum_{i\in\mathbb{N}} i^{2(\frac{s_0}{d}+s)} |(\varphi,\psi_i)_{L^2(\mathbb{T}^d)}|^2 < \infty\right\} \\
&= H^{s_0+ds}(\mathbb{T}^d) \\
&= B^{s_0+ds}_{2,2}(\mathbb{T}^d).
\end{aligned}$$

Similarly, with $\mathcal{Y} := L^2(\mathbb{T}^d)$, it follows that $\mathcal{Y}^t = H^{dt}(\mathbb{T}^d) = B^{dt}_{2,2}(\mathbb{T}^d)$ for any $t \geq 0$.

Now we fix $s_0 := 1$, hence $\mathcal{X} = H^1(\mathbb{T}^d)$, and let $s > \frac{1}{2}$. For any $r > 0$ and $\phi \in C^r_s(\mathcal{X})$, there holds by (9) that $\phi \in \mathcal{X}^{s-\frac{1}{2}-\varepsilon} = H^{1+d(s-\frac{1}{2}-\varepsilon_0)}(\mathbb{T}^d)$ for any $\varepsilon_0 \in (0, s-\frac{1}{2})$. On the other hand, $\phi \in B_r(\mathcal{X}^s) =$

$B_r(H^{s_0+ds}(\mathbb{T}^d))$ is sufficient to ensure $\phi \in C_s^r(\mathcal{X})$. Assumption 4.1 is thus satisfied for any $s > \frac{1}{2}, r > 0$ and $t = \frac{1}{d}$, since for $\phi_1, \phi_2 \in C_s^r(\mathcal{X})$ we have by (44) that

$$\|\mathcal{G}(\phi_1) - \mathcal{G}(\phi_2)\|_{\mathcal{Y}^t} \leq L_{\mathcal{G}} \|\phi_1 - \phi_2\|_{\mathcal{X}}.$$

We are now in a position to bound the mean-squared error for obstacles $\phi \in \mathcal{X}$ of the form $\phi = \sigma_r^s(\mathfrak{u})$ with $\mathfrak{u} \in U = [-1,1]^{\mathbb{N}}$, given as realizations of the $\mathcal{X}$-valued random variable

$$\sigma_r^s : U \to \mathcal{X}, \quad \mathfrak{u} \mapsto r \sum_{i \in \mathbb{N}} w_i^s \mathfrak{u}_i \psi_i.$$

Provided that Assumption 5.1 holds for fixed exponents $\alpha \geq 1$ and $\overline{\alpha}, \beta, \overline{\beta} \geq 0$, Theorem 5.7 shows that for any $\varepsilon \in (0,1]$, there exists a finite-parametric neural network approximation with at most $\mathcal{N}_{para}(\varepsilon) \in \mathbb{N}$ parameters to $\mathcal{G}$ such that it holds

$$\|\mathcal{G} - \widetilde{\mathcal{G}}\|_{L^2(C_r^s(\mathcal{X}), (\sigma_r^s)_\# \mathcal{P}_U; \mathcal{Y})} \leq \varepsilon,$$

and for any $\delta > 0$ there exists a constant $C > 0$ depending on δ such that

$$\mathcal{N}_{para} \leq C \begin{cases} \varepsilon^{-\alpha/(s-1/2) - d(1+\beta/2) - \delta} (1 + |\log(\varepsilon)|)^\kappa, & d \geq 2, \\ \varepsilon^{-\alpha/(s-1/2) - 1 - \beta - \delta} (1 + |\log(\varepsilon)|)^\kappa, & d = 1. \end{cases}$$

Remark 7.3. In general $\widetilde{\mathcal{G}}(\phi) \notin \mathcal{K}_\phi$ for a given $\phi \in C_s^r(\mathcal{X})$ due to the truncation and neural network approximation of the coordinate mappings $\mathfrak{g}_j$ in (18). Further, let $\phi_M := \sum_{i=1}^M \mathcal{E}_\mathcal{X}(\phi)_i \psi_i$ denote the M-term approximation of ϕ for fixed $M \in \mathbb{N}$, and assume that M is fixed for all dimensions of the truncated output for simplicity (cf. Section 4.3). Then, in general also $\widetilde{\mathcal{G}}(\phi) \notin \mathcal{K}_{\phi_M}$, due to the bias from output truncation and the neural network surrogates.

However, a finite dimensional approximation $\widetilde{\mathcal{G}}^+(\phi) \in \mathcal{K}_{\phi_M}$ may be achieved by the following post-processing step. Let

$$\widetilde{\mathcal{G}}^+(\phi) : \mathcal{X} \to \mathcal{Y}, \quad \phi \mapsto \max \left(\widetilde{\mathcal{G}}(\phi), \sum_{i=1}^M \mathcal{E}_\mathcal{X}(\phi)_i \psi_i \right). \tag{50}$$

The maximum is understood in the point-wise sense. It is well-defined, since we assumed at hand a continuous wavelet basis of $L^2(\mathbb{T}^d)$ for de- and encoding.

Augmenting $\widetilde{\mathcal{G}}(\phi)$ by the M-term truncation of ϕ involves $M = \mathcal{O}(\varepsilon^{-1/(s-\frac{1}{2}-\delta)})$ additional parameters (cf. (24) in the proof of Theorem 5.7), and therefore does not dominate the asymptotic complexity. Furthermore,

$$\begin{aligned} \left\|\widetilde{\mathcal{G}}^+(\phi) - \mathcal{G}(\phi)\right\|_\mathcal{Y} &\leq \left\|\widetilde{\mathcal{G}}^+(\phi) - \max(\widetilde{\mathcal{G}}(\phi), \phi)\right\|_\mathcal{Y} + \left\|\max(\widetilde{\mathcal{G}}(\phi), \phi) - \max(\mathcal{G}(\phi), \phi)\right\|_\mathcal{Y} \\ &\leq \|\phi_M - \phi\|_\mathcal{Y} + \left\|\widetilde{\mathcal{G}}(\phi) - \mathcal{G}(\phi)\right\|_\mathcal{Y}, \end{aligned}$$

and since $\mathcal{X} \hookrightarrow \mathcal{Y}$ there holds

$$\|\mathcal{G}(\phi) - \widetilde{\mathcal{G}}^+(\phi)\|_{L^2(C_r^s(\mathcal{X}), (\sigma_r^s)_\# \mathcal{P}_U; \mathcal{Y})} \leq C\varepsilon,$$

for a $C > 0$, independent of ε.

7.2 Expression Rates for Lipschitz Maps of Hilbert-Schmidt Operators

Our results also encompass the approximation of nonlinear Lipschitz maps between spaces of operators. We illustrate this for the particular class of Hilbert-Schmidt (HS) operators, $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ ("2-Schatten" class of linear operators), acting between separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. As $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ is itself a Hilbert space, the presently developed abstract framework (1) is applicable with $\mathcal{X} = \mathcal{Y} = \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$. Assuming at hand orthonormal basis $(\psi_j^{\mathcal{H}_i}, j \in \mathbb{N})$ of $\mathcal{H}_i$ for $i = 1, 2$, adopting the dyadic basis $(\psi_j^{\mathcal{H}_1} \otimes \psi_{j'}^{\mathcal{H}_2}, j, j' \in \mathbb{N})$ of $\mathcal{X} = \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$, the spaces $\mathcal{X}^s$ correspond to p-Schatten classes for suitable $p(s) < 2$.

We first present a general setting without particular structural assumptions on the map $\mathcal{G}$ and then, in Sect. 7.2.2, address a particular case of "singular value maps", through a scalar Lipschitz function, of HS operators as considered in [1].

7.2.1 Hilbert-Schmidt Operators

We denote by $(\mathcal{H}_i, (\cdot, \cdot)_{\mathcal{H}_i})$, $i = 1, 2$ separable Hilbert spaces and assume that $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a compact linear operator. Then there exists a singular value decomposition (SVD) of A, i.e., there is a sequence $\{s_j(A) : j \in \mathbb{N}\} \subset [0, \infty)$, and ONB $(v_i, i \in \mathbb{N})$ of $\mathcal{H}_1$ and another ONB $(w_j, j \in \mathbb{N})$ of $\mathcal{H}_2$ such that

$$A = \sum_{j \in \mathbb{N}} s_j(A) w_j \otimes v_j \ , \tag{51}$$

where $w_j \otimes v_j \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is defined by $(w_j \otimes v_j)\phi = \langle \phi, v_j \rangle_{\mathcal{H}_1} w_j$. The real, non-negative numbers $s_j(A) \geq 0$ in (51) are the *singular values of* A. They accumulate only at zero. For $0 < p \leq \infty$, we denote the subset of *p-Schatten class operators* as

$$S_p(\mathcal{H}_1, \mathcal{H}_2) = \{A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : (s_j(A), j \in \mathbb{N}) \in \ell^p(\mathbb{N})\} \ .$$

Of particular interest in the present context is the case $p = 2$, the so-called Hilbert-Schmidt (HS) operators, with square-summable singular values $(s_j(A), j \in \mathbb{N}) \in \ell^2(\mathbb{N})$. For $A \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$, the sum in (51) converges in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. We recall that $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ is a separable Hilbert space with basis $(w_j \otimes v_i,\ i, j \in \mathbb{N})$ and inner product given by

$$(A, B)_{HS} := \sum_{j \in \mathbb{N}} (Ae_j, Be_j)_{\mathcal{H}_2} \ ,$$

where $(e_k, k \in \mathbb{N}) \subset H_1$ denotes an orthonormal basis [3] of $\mathcal{H}_1$, and $(\cdot, \cdot)_{\mathcal{H}_2}$ denotes the $\mathcal{H}_2$ inner product. A corresponding norm in $S_2(\mathcal{H}_1, \mathcal{H}_2)$ is given by

$$\|A\|_{\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)} := (A, A)_{HS}^{1/2} \ , \quad A \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2) \ .$$

7.2.2 Singular Value Lipschitz Functional Calculus

A particular class of nonlinear maps on the space of HS operators can be constructed via *functional calculus*. For such maps, better DON emulation rates can be shown than in the general case. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ [4] be continuous with $f(0) = 0$ and consider the map

$$\mathcal{G}(f) : \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2), \quad A \mapsto \sum_{j \in \mathbb{N}} f(s_j(A))\, w_j \otimes v_j. \tag{52}$$

Note that we may have $\mathcal{G}(f)(A) \notin \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ for $A \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$, without any further assumptions on f. On the other hand, if f is Lipschitz with Lipschitz constant $L_f > 0$, [1, Theorem 4.2] shows $\mathcal{G}(f) : \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ is Lipschitz with

$$L_{\mathcal{G}(f)} := \sup_{A, B \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2), A \neq B} \frac{\|\mathcal{G}(f)(A) - \mathcal{G}(f)(B)\|_{\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)}}{\|A - B\|_{\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)}} \leq L_f < \infty. \tag{53}$$

[3] The inner product and norm are independent of the choice of orthonormal basis.

[4] It is also possible to define $\mathcal{G}_f$ in terms of a complex-valued, scalar Lipschitz function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$. For the ensuing DNN emulation of operator $\mathcal{G}_f$, one then has to rely on *complex-valued neural networks* (CVNNs), that involve complex activation functions and linear transforms, see e.g. [7]. We further note that the right hand side of (53) only yields the bound $\sqrt{2}L_f$ for complex $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$.

Now let $\boldsymbol{w} = (w_j, j \in \mathbb{N})$ be again a given sequence of positive weights, so that $\boldsymbol{w} \in \ell^{1+\varepsilon}(\mathbb{N})$ for all $\varepsilon > 0$. In view of Section 4, we encode elements $A \in \mathcal{X} = \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ using the basis $(w_j \otimes v_i, i, j \in \mathbb{N})$ and define

$$\mathcal{X}^s := \left\{ A \in \mathcal{X} :\ \|A\|^2_{\mathcal{X}^s} := \sum_{j\in\mathbb{N}} s_j(A)^2 w_j^{-2s} < \infty \right\}, \quad s > 0, \tag{54}$$

as well as the cubes

$$C_r^s(\mathcal{X}) := \left\{ A \in \mathcal{X} :\ \sup_{j\in\mathbb{N}} s_j(A)^2 w_j^{-2s} \le r \right\}, \quad r, s > 0. \tag{55}$$

In case there are $p \in (0, 2)$ and $\delta > 0$ such that $s_j(A) \le C w_j^{(1+\delta)/p}$ for all $j \in \mathbb{N}$, it follows that $A \in \mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$. In addition, for all $s \in [0, (1 + \delta)(1/p - 1/2)]$ there holds $A \in \mathcal{X}^s$, since

$$\|A\|^2_{\mathcal{X}^s} = \sum_{j\in\mathbb{N}} s_j(A)^2 w_j^{-2s} \le \left(\sup_{j\in\mathbb{N}} s_j(A)^{2-p} w_j^{-2s} \right) \sum_{j\in\mathbb{N}} s_j(A)^p \le C \left(\sup_{j\in\mathbb{N}} s_j(A)^{2-p} w_j^{-2s} \right) < \infty.$$

7.2.3 DNN Emulation Rates

We verify that operators $\mathcal{G}(f)$ of the form (52) with scalar, nonnegative Lipschitz f are a particular case of our framework. Specifically, the representation (52) implies favourable DNN expression rate bounds in terms of the number $\mathcal{N}_{para}$ of neurons. To verify this, we apply our abstract setting with the choices $\mathcal{X} = \mathcal{Y} := \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$. Our construction of a DNN surrogate starts from the N-term truncated operator $\mathcal{G}_N(f)$, which is given via

$$\mathcal{G}_N(f) : \mathcal{X} \to \mathcal{Y}, \quad A \mapsto \sum_{j=1}^{N} f(s(A_j))\, w_j \otimes v_j, \quad N \in \mathbb{N}. \tag{56}$$

Note that we have fixed input and output truncation by N terms simultaneously. As f is Lipschitz with $f(0) = 0$, it follows readily that $f(s_j(A)) \le L_f s_j(A)$ and thus

$$\|\mathcal{G}(f)(A) - \mathcal{G}_N(f)(A)\|^2_{\mathcal{Y}} = \sum_{j\ge N} f(s_j(A))^2 \le L_f^2 \sum_{j\ge N} s_j(A)^2. \tag{57}$$

Equation (57) yields for any $s > \frac{1}{2}$ and $\delta \in (0, s - \frac{1}{2})$ that there is a $C > 0$ (depending on s and on δ) such that

$$\|\mathcal{G}(f)(A) - \mathcal{G}_N(f)(A)\|_{\mathcal{Y}} \le C w_N^{s-\frac{1}{2}-\delta}, \quad A \in C_r^s(\mathcal{X}).$$

As $\lim_{j\to\infty} s_j(A) = 0$, we may assume without loss of generality that $s_j(A) \le 1$ for all $j \in \mathbb{N}$. Hence, we only need to replace the univariate Lipschitz mapping f a total of N times by a NN surrogate $\widetilde{f} : [0, 1] \to \mathbb{R}$ such that

$$\left\| f - \widetilde{f} \right\|_{L^2([0,1])} \le L_f \varepsilon, \quad \varepsilon \in (0, 1].$$

By Assumption 5.1, this may be achieved by *one common scalar surrogate* $\widetilde{f}$ with $\mathcal{O}(\varepsilon^{-\beta} |\log(\varepsilon)|^{\bar{\beta}})$ parameters. Thus, for fixed, scalar Lipschitz f and any $\varepsilon \in (0, 1]$, there exists a DNN approximation $\widetilde{\mathcal{G}}(f)$ with $\mathcal{N}_{para} \in \mathbb{N}$ parameters to $\mathcal{G}(f)$, such that

$$\|\mathcal{G}(f) - \widetilde{\mathcal{G}}(f)\|_{L^2(C_r^s(\mathcal{X}), (\sigma_r^s)_\# \mathcal{P}_U; \mathcal{Y})} \le \varepsilon,$$

and for any $\delta > 0$ there is $C_\delta > 0$ such that for all $\varepsilon \in (0, 1]$ holds

$$\mathcal{N}_{para} \le C_\delta \varepsilon^{-1/(s-1/2)-\beta-\delta} (1 + |\log(\varepsilon)|)^\kappa.$$

8 Conclusions

We obtained expression rate bounds for a class of deep operator networks (DONs) with the architecture (3) to emulate Lipschitz continuous maps between separable Hilbert spaces. It is based on linear encoder/decoder pairs $(\mathcal{E}, \mathcal{D})$ based on stable biorthogonal bases of domain $\mathcal{X}$ and target space $\mathcal{Y}$. For example, Karhunen-Loève expansions as employed in so-called PCA-Nets [36], Fourier bases as used in FNOs [41, 62], etc. Concrete encoders in this framework either access PDE inputs via point values as e.g. in ONets (e.g. [45]) (with dual bases consisting of Dirac measures), or also via "Galerkin moments" as recently promoted in discussion of transformer-encoders in [6, Section 4.1.3]. In either of these cases, *aliasing errors* due to sampling or quadrature must be accounted for, in addition to the expression error analysis performed here. To accommodate possibly low regularity of the map $\mathcal{G}$, we considered neural approximators $\tilde{G}$ in (3) from two superexpressive classes of DNNs that are not subject to the CoD: (i) NNs with superexpressive activation functions and (ii) NNs with nonstandard architecture. Specifically, the "Nest-Net" construction from [67].

Extensions of the presently developed analysis to separable Banach spaces that admit stable, biorthogonal representation systems such as those developed for a broad range of Besov-Triebel-Lizorkin spaces in domains as constructed e.g. in [60] and the references there are conceivable, under conditions on these spaces. We refer to [35, Section 9, App. B] for conditions and techniques in such more general settings. These references considered DONs emulating maps $\mathcal{G}$ between spaces of functions in Euclidean domains. More general settings in the abstract framework of Section 7.2 may accommodate DONs that emulate maps from certain linear operators between separable Hilbert spaces to functions, a task that frequently arises in inverse problems, for example (with suitably regularizing observation functionals).

Also covered are multiresolution encoders and decoders, such as wavelets [59]. Since multiresolution bases hierarchically encode increasing spatial and temporal resolution, the presently developed results and framework comprise also so-called "multi-fidelity" operator networks as put forward in [28].

For Lipschitz continuous $\mathcal{G}$, in [38, 36] *lower expression rate bounds* for DON surrogates were obtained, for approximators $\widetilde{G}$ in (3) being standard feedforward NNs with smooth, nonpolynomial activation. Moreover, in the recent work [39], it was demonstrated that emulating Lipschitz operators (without additional structural properties) on infinite-dimensional hypercubes using a broad class of ONet architectures is subject to the CoD. See for instance the exponential lower bounds on the number of parameters in the ONet presented in [39, Theorem 2.15]. Such statements are not contradictory to our results: the analysis in [39] specifically considers *feedforward ReLU-NN surrogates for the coordinate maps and decoders*. In the present manuscript, higher rates were obtained by more general approximators $\widetilde{G}$ in (3) with either nonstandard architecture or nonstandard, "superexpressive" activations.

The encoder/decoder pairs in (3) pass between the in- and output spaces $\mathcal{X}$ and $\mathcal{Y}$ and the sequence space $\ell^2(\mathbb{N})$ via linear transformations. Approximation error bounds due to finitely truncating coefficient sequences furnished by *linear encoding* on $\mathcal{X}^s$ were obtained by N-term sequence truncation. As is well-known, however, *adaptive, nonlinear encoding* could yield the same rates for considerably larger classes of inputs, from (quasi-)Banach spaces $B^{\gamma}_{p,p}$ for some $0 < p < 2$ [11]. The extension of the presently proposed framework to such encoders will be considered elsewhere.

The expression rate bounds for DONs (3) obtained here are a consequence of *regularity and sparsity of inputs and outputs*, as expressed by weighted sequence summability, and the assumed mapping properties of $\mathcal{G} : \mathcal{X}^s \to \mathcal{Y}^t$, and super expressivity of activations in the DNN emulations $\widetilde{\mathfrak{g}}_j$ of the components $\mathfrak{g}_j$ of G in (3). They cover a large class of operator network constructions. Architectures which are essentially different such as U-Nets and transformer-based emulators to build $\widetilde{G}$ in (3), as proposed e.g. in [42, 6] and in the references there, or the branch-trunk architecture of deepONets [45, 47] could be investigated similarly.

We remark that the linear decoding used in the ONet architecture is necessarily restrictive. Generally, images of subspaces of N-term truncated encoded inputs under $\mathcal{G}$ become *Lipschitz manifolds* embedded into $\mathcal{Y}$. *Linear decoding* as assumed in Section 4.2 of the present analysis will in general poorly capture such manifolds, without any further assumptions. Here, we *assumed* $\mathcal{G}(\mathcal{X}^s)$ to be contained in $\mathcal{Y}^t$. This assumption is often satisfied in data-to-solution maps for elliptic and parabolic PDEs, when $\mathcal{X}^s$ and $\mathcal{Y}^t$ coincide with suitable function spaces of Sobolev, resp. of Besov-Triebel-Lizorkin type. In such spaces, the (linear) M-term decoding in Prop. 4.4 provides corresponding approximation rates. For the more general case of the range of $\mathcal{G}$ on N-term encoded inputs in $\mathcal{X}$, *nonlinear decoding* as in the branch-trunk architecture of [9], which allows for data-dependent representation system in the output-decoder, could result in better expressivity, closer to the *Lipschitz-width benchmark* [53].

A Proof of Universality

Lemma A.1. *Let $E \subseteq \ell^2(\mathbb{N})$ be compact. Then*

$$S := E \cup \{(c_1, \dots, c_n, 0, 0, \dots) : \boldsymbol{c} \in E,\ n \in \mathbb{N}\}$$

is a compact subset of $\ell^2(\mathbb{N})$.

Proof. Let $(O_i)_{i \in I}$ be an open cover of S. We need to show there exists a finite subcover.

For each $\boldsymbol{c} = (c_j)_{j \in \mathbb{N}} \in E$ there exists $\varepsilon_{\boldsymbol{c}} > 0$ and $i_{\boldsymbol{c}} \in I$ such that

$$B_{\varepsilon_{\boldsymbol{c}}}(\boldsymbol{c}) := \left\{\boldsymbol{d} \in \ell^2(\mathbb{N}) : \|\boldsymbol{c} - \boldsymbol{d}\|_{\ell^2} < \varepsilon_{\boldsymbol{c}}\right\} \subseteq O_{i_{\boldsymbol{c}}}. \tag{58}$$

Additionally, there exists $n_{\boldsymbol{c}} \in \mathbb{N}$ such that $\sum_{j > n_{\boldsymbol{c}}} c_j^2 < (\varepsilon_{\boldsymbol{c}}/3)^2$. Then for any $\boldsymbol{d} \in B_{\varepsilon_{\boldsymbol{c}}/3}(\boldsymbol{c})$ and any $n > n_{\boldsymbol{c}}$ there holds

$$\begin{aligned} \|\boldsymbol{c} - (d_1, \dots, d_n, 0, 0, \dots)\|_{\ell^2} &\le \|\boldsymbol{c} - \boldsymbol{d}\|_{\ell^2} + \left(\sum_{j > n_{\boldsymbol{c}}} (d_j^2 - c_j^2) + \sum_{j > n_{\boldsymbol{c}}} c_j^2 \right)^{1/2} \\ &\le \frac{\varepsilon_{\boldsymbol{c}}}{3} + \left(\frac{\varepsilon_{\boldsymbol{c}}^2}{3^2} + \frac{\varepsilon_{\boldsymbol{c}}^2}{3^2} \right)^{1/2} < \varepsilon_{\boldsymbol{c}}. \end{aligned} \tag{59}$$

Since E is compact and $E \subseteq \bigcup_{\boldsymbol{c} \in E} B_{\varepsilon_{\boldsymbol{c}}/3}(\boldsymbol{c})$, there exists $m \in \mathbb{N}$ and $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m \in E$ such that $E \subseteq \bigcup_{i=1}^m B_{\varepsilon_{\boldsymbol{c}_i}/3}(\boldsymbol{c}_i)$. Let $\boldsymbol{d} \in E$ be arbitrary. Then there exists $j \in \{1, \dots, m\}$ such that $\boldsymbol{d} \in B_{\varepsilon_{\boldsymbol{c}_j}/3}(\boldsymbol{c}_j)$. Thus by (59) and (58) it holds for all $n > N := \max_{i=1,\dots,m} n_{\boldsymbol{c}_i}$

$$(d_1, \dots, d_n, 0, 0, \dots) \in B_{\varepsilon_{\boldsymbol{c}_j}}(\boldsymbol{c}_j) \subseteq O_{i_{\boldsymbol{c}_j}}.$$

This shows

$$E \cup \{(d_1, \dots, d_n, 0, 0, \dots) : \boldsymbol{d} \in E,\ n > N\} \subseteq \bigcup_{j=1}^m O_{i_{\boldsymbol{c}_j}}.$$

Finally observe that $\boldsymbol{d} \mapsto (d_1, \dots, d_n, 0, 0, \dots) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is continuous for any fixed $n \in \mathbb{N}$. Hence compactness of E gives compactness of the set of remaining elements

$$R := \bigcup_{n=1}^N \{(d_1, \dots, d_n, 0, 0, \dots) : \boldsymbol{d} \in E\}.$$

Thus R can be covered by finitely many O_i, and therefore the same holds for S. $\square$

Proof of Theorem 3.1. It suffices to construct $\widetilde{\mathcal{G}}_n$, $n \in \mathbb{N}$, such that for any compact $K \subseteq \mathcal{X}$ holds $\lim_{n \to \infty} \widetilde{\mathcal{G}}_n|_K = \mathcal{G}|_K$ with uniform convergence. Throughout the rest of this proof fix $K \subseteq \mathcal{X}$ compact. We proceed in four steps to construct $\widetilde{\mathcal{G}}_n$ (independent of K) as claimed.

Step 1. We claim that for every $\delta > 0$ exists $N_1(\delta, K) \in \mathbb{N}$ such that

$$\sup_{x \in K} \left\| x - \sum_{j=1}^{m} \langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}} \psi_j \right\|_{\mathcal{X}} < \delta \qquad \forall m \geq N_1. \tag{60}$$

To prove this, for all $m \in \mathbb{N}$, define the open sets

$$\mathcal{X}_m := \operatorname{span}\{\psi_1, \dots, \psi_m\} + B_r^{\mathcal{X}},$$

where $B_r^{\mathcal{X}}$ denotes the open ball of radius $r := \frac{\delta^2}{\Lambda_{\mathcal{X}}^2} > 0$ around $0 \in \mathcal{X}$. These sets are nested and $\bigcup_{m \in \mathbb{N}} \mathcal{X}_m = \mathcal{X} \supset K$. Since K is compact there exists N_1 such that $K \subseteq \mathcal{X}_{N_1}$. Then for any $x \in K$

$$x = \sum_{j=1}^{N_1} \alpha_j \psi_j + \widetilde{x}$$

for some $\alpha_1, \dots, \alpha_{N_1} \in \mathbb{R}$ and some $\widetilde{x} \in \mathcal{X}$ with $\|\widetilde{x}\|_{\mathcal{X}}^2 < \frac{\delta^2}{\Lambda_{\mathcal{X}}^2}$ and thus $\sum_{j \in \mathbb{N}} \langle \widetilde{x}, \widetilde{\psi}_j \rangle_{\mathcal{X}}^2 < \frac{\delta^2}{\Lambda_{\mathcal{X}}}$. Then for all $m \geq N_1$

$$\left\| x - \sum_{j=1}^{m} \langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}} \psi_j \right\|_{\mathcal{X}} = \left\| \sum_{j > m} \langle \widetilde{x}, \widetilde{\psi}_j \rangle_{\mathcal{X}} \psi_j \right\|_{\mathcal{X}} < \Lambda_{\mathcal{X}} \sum_{j > N_1} \langle \widetilde{x}, \widetilde{\psi}_j \rangle_{\mathcal{X}}^2 \leq \delta^2,$$

which shows (60).

Step 2. We claim that for every $\varepsilon > 0$ exists $N_2(\varepsilon, K) \in \mathbb{N}$ such that

$$\sup_{x \in K} \left\| \mathcal{G}(x) - \mathcal{G}\left(\sum_{j=1}^{m} \langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}} \psi_j \right) \right\|_{\mathcal{Y}} \leq \varepsilon \qquad \forall m \geq N_2. \tag{61}$$

Compactness of K implies that $\mathcal{G} : K \to \mathcal{Y}$ is uniformly continuous. Hence for any $\varepsilon > 0$ exists $\delta > 0$ such that $\|x - \widetilde{x}\|_{\mathcal{X}} < \delta$ implies $\|\mathcal{G}(x) - \mathcal{G}(\widetilde{x})\|_{\mathcal{Y}} < \varepsilon$. Set $N_2(\varepsilon, K) := N_1(\delta, K)$. Then (60) gives (61).

Step 3. We claim that for every $\varepsilon > 0$ there exists $M_3(\varepsilon, K) \in \mathbb{N}$ such that[5]

$$\sup_{x \in K} \sup_{n \in \mathbb{N}} \sum_{j > M_3} \left\langle \mathcal{G}\left(\sum_{i=1}^{n} \langle x, \widetilde{\psi}_i \rangle_{\mathcal{X}} \psi_i \right), \widetilde{\eta}_j \right\rangle_{\mathcal{Y}}^2 < \varepsilon^2 . \tag{62}$$

As $x \mapsto \mathcal{E}_{\mathcal{X}}(x) : \mathcal{X} \to \ell^2(\mathbb{N})$ is a bounded and linear (thus continuous) map and $K \subseteq \mathcal{X}$ is compact, also $\mathcal{E}_{\mathcal{X}}(K)$ is compact. According to Lemma A.1 the set

$$S := \mathcal{E}_{\mathcal{X}}(K) \cup \{(c_1, \dots, c_n, 0, 0, \dots) : \boldsymbol{c} \in \mathcal{E}_{\mathcal{X}}(K),\ n \in \mathbb{N}\} \subseteq \ell^2(\mathbb{N})$$

is compact. Continuity of $\mathcal{G} \circ \mathcal{D}_{\mathcal{X}} : \ell^2(\mathbb{N}) \to \mathcal{Y} : \boldsymbol{c} \mapsto \mathcal{G}(\sum_{j \in \mathbb{N}} c_j \psi_j)$ gives that

$$\mathcal{G}(\mathcal{D}_{\mathcal{X}}(S)) = \Big\{ \mathcal{G}(x) : x \in K \Big\} \cup \Big\{ \mathcal{G}\Big(\sum_{j=1}^{n} \langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}} \psi_j \Big) : x \in K,\ n \in \mathbb{N} \Big\} \subseteq \mathcal{Y} \tag{63}$$

is compact, and (62) then follows by the statement in Step 1.

Step 4. We construct $\widetilde{\mathcal{G}}_n$ and conclude the proof.

[5] The notation N and M is chosen so that N always refers to trunction of input, M always refers to truncation of output. The subindex of N and M refers to step of proof.

For every $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ let $\widetilde{\mathfrak{g}}_j^n : [-n, n]^n \to \mathbb{R}$ be a σ-NN such that

$$\sup_{\boldsymbol{c} \in [-n,n]^n} \Bigg| \underbrace{\left\langle \mathcal{G}\Big(\sum_{i=1}^{n} c_i \psi_i\Big), \widetilde{\eta}_j \right\rangle_{\mathcal{Y}}}_{=:\mathfrak{g}_j^n(\boldsymbol{c})} - \widetilde{\mathfrak{g}}_j^n(\boldsymbol{c}) \Bigg| < 2^{-n}. \tag{64}$$

Such $\widetilde{\mathfrak{g}}_j^n$ exists according to [40, Theorem 1], since $\boldsymbol{c} \mapsto g_j^n(\boldsymbol{c}) : \mathbb{R}^n \to \mathbb{R}$ is continuous due to the continuity of $\mathcal{G}$. Now define (independent of K)

$$\widetilde{\mathcal{G}}_n(x) := \sum_{j=1}^{n} \widetilde{\mathfrak{g}}_j^n(\langle x, \widetilde{\psi}_1 \rangle_{\mathcal{X}}, \dots, \langle x, \widetilde{\psi}_n \rangle_{\mathcal{X}}) \eta_j.$$

Fix $\varepsilon > 0$ and let $n \geq \max\{N_2(\varepsilon, K), M_3(\varepsilon, K)\}$. Then for any $x \in K$ by (64) and (62)

$$\begin{aligned}
&\left\| \mathcal{G}\left(\sum_{j=1}^{n} \langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}} \psi_j \right) - \widetilde{\mathcal{G}}_n(x) \right\|_{\mathcal{Y}}^2 \\
&= \left\| \sum_{j=1}^{n} (\mathfrak{g}_j^n((\langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}})_{j=1}^n) - \widetilde{\mathfrak{g}}_j^n((\langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}})_{j=1}^n)) \eta_j + \sum_{j>n} \mathfrak{g}_j^n((\langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}})_{j=1}^n) \eta_j \right\|_{\mathcal{Y}}^2 \\
&\leq 2\Lambda_{\mathcal{Y}} \left(\sum_{j=1}^{n} 2^{-2n} + \sum_{j>n} \mathfrak{g}_j^n((\langle x, \widetilde{\psi}_j \rangle_{\mathcal{X}})_{j=1}^n)^2 \right) \\
&\leq 2\Lambda_{\mathcal{Y}}(n 2^{-2n} + \varepsilon^2).
\end{aligned}$$

Therefore using (61)

$$\lim_{n\to\infty} \sup_{x \in K} \left\| \mathcal{G}(x) - \widetilde{\mathcal{G}}_n(x) \right\|_{\mathcal{Y}} \leq \lim_{n\to\infty} \left(\varepsilon + (2\Lambda_{\mathcal{Y}}(n 2^{-2n} + \varepsilon^2))^{1/2} \right) = (1 + \sqrt{2\Lambda_{\mathcal{Y}}})\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof. □

References

[1] F. Andersson, M. Carlsson, and K.-M. Perfekt. Operator-Lipschitz estimates for the singular value functional calculus. *Proceedings of the American Mathematical Society*, 144(5):1867–1875, 2016.

[2] M. Bachmayr, A. Cohen, D. Dung, and C. Schwab. Fully discrete approximation of parametric and stochastic elliptic PDEs. *SIAM J. Numer. Anal.*, 55(5):2151–2186, 2017.

[3] A. R. Barron. Universal approximation bounds for superpositions of a sigmoidal function. *IEEE Transactions on Information Theory*, 39(3):930–945, 1993.

[4] J. A. L. Benitez, T. Furuya, F. Faucher, X. Tricoche, and M. V. de Hoop. Fine-tuning neural-operator architectures for training and generalization. *arXiv preprint arXiv:2301.11509*, 2023.

[5] S. Cai, Z. Mao, Z. Wang, M. Yin, and G. E. Karniadakis. Physics-informed neural networks (PINNs) for fluid mechanics: a review. *Acta Mech. Sin.*, 37(12):1727–1738, 2021.

[6] S. Cao. Choose a Transformer: Fourier or Galerkin. *Advances in Neural Information Processing Systems*, 2021.

[7] A. Caragea, D. G. Lee, J. Maly, G. Pfander, and F. Voigtlaender. Quantitative approximation results for complex-valued neural networks. *SIAM Journal on Mathematics of Data Science*, 4(2):553–580, 2022.

[8] N. Chegini, S. Dahlke, U. Friedrich, and R. Stevenson. Piecewise tensor product wavelet bases by extensions and approximation rates. *Math. Comp.*, 82(284):2157–2190, 2013.

[9] T. Chen and H. Chen. Approximations of continuous functionals by neural networks with application to dynamic systems. *IEEE Transactions on Neural Networks*, 4:910 – 918, 1993.

[10] O. Christensen. *An Introduction to Frames and Riesz Bases*, volume 7. Springer, 2003.

[11] A. Cohen, W. Dahmen, I. Daubechies, and R. DeVore. Tree approximation and optimal encoding. *Appl. Comput. Harmon. Anal.*, 11(2):192–226, 2001.

[12] A. Cohen, C. Schwab, and J. Zech. Shape Holomorphy of the stationary Navier-Stokes Equations. *SIAM J. Math. Analysis*, 50(2):1720–1752, 2018.

[13] W. Dahmen, A. Kunoth, and K. Urban. Biorthogonal spline wavelets on the interval—stability and moment conditions. *Appl. Comput. Harmon. Anal.*, 6(2):132–196, 1999.

[14] W. Dahmen and R. Stevenson. Element-by-element construction of wavelets satisfying stability and moment conditions. *SIAM J. Numer. Anal.*, 37(1):319–352, 1999.

[15] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.

[16] P. Doktor and M. Kučera. Perturbations of variational inequalities and rate of convergence of solutions. *Czechoslovak Mathematical Journal*, 30(3):426–437, 1980.

[17] D. Dung, V. K. Nguyen, C. Schwab, and J. Zech. Analyticity and sparsity in uncertainty quantification for PDEs with Gaussian random field inputs. *SAM Report*, 2022. (to appear in Springer Lect. Notes Math. 2023).

[18] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*, volume 219 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976.

[19] T. Führer and M. Karkulik. Least-squares finite elements for distributed optimal control problems. *arXiv preprint arXiv:2210.16377*, 2022.

[20] L. Gonon and C. Schwab. Deep ReLU neural networks overcome the curse of dimensionality for partial integrodifferential equations. *Anal. Appl. (Singap.)*, 21(1):1–47, 2023.

[21] S. Goswami, A. Bora, Y. Yu, and G. E. Karniadakis. Physics-informed neural operators. *arXiv preprint arXiv:2207.05748*, 2022.

[22] J. Gwinner, B. Jadamba, A. A. Khan, and F. Raciti. *Uncertainty quantification in variational inequalities—theory, numerics, and applications.* CRC Press, Boca Raton, FL, 2022.

[23] F. Henriquez and C. Schwab. Shape Holomorphy of the Calderón Projector for the Laplacean in $\mathbb{R}^2$. *Journ. Int. Equns. Operator Theory*, 93(4), 2021.

[24] L. Herrmann, C. Schwab, and J. Zech. Neural and spectral operator surrogates: construction and expression rate bounds. *arXiv preprint arXiv:2207.04950*, 2022.

[25] J. Hesthaven and S. Ubbiali. Non-intrusive reduced order modeling of nonlinear problems using neural networks. *Journal of Computational Physics*, 363:55–78, 2018.

[26] I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek. *Solution of variational inequalities in mechanics*, volume 66 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.

[27] V. H. Hoang and C. Schwab. Sparse Tensor Galerkin Discretizations for Parametric and Random Parabolic PDEs - Analytic Regularity and GPC Approximation. *SIAM J. Mathematical Analysis*, 45(5):3050–3083, 2013.

[28] A. A. Howard, M. Perego, G. E. Karniadakis, and P. Stinis. Multifidelity deep operator networks. *arXiv preprint arXiv:2204.09157*, 2022.

[29] M. Hutzenthaler, A. Jentzen, T. Kruse, T. Anh Nguyen, and P. von Wurstemberger. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *Proceedings of the Royal Society A*, 476(2244):20190630, 2020.

[30] A. D. Jagtap, K. Kawaguchi, and G. E. Karniadakis. Adaptive activation functions accelerate convergence in deep and physics-informed neural networks. *J. Comput. Phys.*, 404:109136, 23, 2020.

[31] C. Jerez-Hanckes, C. Schwab, and J. Zech. Electromagnetic wave scattering by random surfaces: Shape holomorphy. *Math. Mod. Meth. Appl. Sci.*, 27(12):2229–2259, 2017.

[32] M. G. Kapteyn, J. V. Pretorius, and K. E. Willcox. A probabilistic graphical model foundation for enabling predictive digital twins at scale. *Nature Computational Science*, 1(5):337–347, 2021.

[33] H. Koch, A. Rüland, and M. Salo. On instability mechanisms for inverse problems. *Ars Inveniendi Analytica*, 2021.

[34] A. N. Kolmogorov. On the representation of continuous functions of many variables by superposition of continuous functions of one variable and addition. *Dokl. Akad. Nauk SSSR*, 114:953–956, 1957.

[35] N. Kovachki, Z. Li, B. Liu, K. Azizzadenesheli, K. Bhattacharya, A. Stuart, and A. Anandkumar. Neural Operator: Learning Maps Between Function Spaces With Applications to PDEs. *Journal of Machine Learning Research*, 24(89):1–97, 2023.

[36] S. Lanthaler. Operator learning with PCA-net: upper and lower complexity bounds. *arXiv preprint arXiv:2303.16317*, 2023.

[37] S. Lanthaler, Z. Li, and A. M. Stuart. The nonlocal neural operator: Universal approximation. *arXiv preprint arXiv:2304.13221*, 2023.

[38] S. Lanthaler, S. Mishra, and G. E. Karniadakis. Error estimates for DeepONets: a deep learning framework in infinite dimensions. *Trans. Math. Appl.*, 6(1), 2022.

[39] S. Lanthaler and A. M. Stuart. The curse of dimensionality in operator learning. *arXiv preprint arXiv:2306.15924*, 2023.

[40] M. Leshno, V. Y. Lin, A. Pinkus, and S. Schocken. Multilayer feedforward networks with a nonpolynomial activation function can approximate any function. *Neural Networks*, 6(6):861–867, 1993.

[41] Z. Li, D. Z. Huang, B. Liu, and A. Anandkumar. Fourier neural operator with learned deformations for pdes on general geometries. *arXiv preprint arXiv:2207.05209*, 2022.

[42] Z. Li, K. Meidani, and A. B. Farimani. Transformer for partial differential equations' operator learning. *arXiv preprint arXiv:2205.13671*, 2022.

[43] M. Longo, J. A. A. Opschoor, N. Disch, C. Schwab, and J. Zech. De Rham compatible Deep Neural Network FEM. Technical Report 2022-03 (revised), Seminar for Applied Mathematics, ETH Zürich, 2022.

[44] G. G. Lorentz. *Approximation of Functions*. Holt, Rinehart and Winston, New York-Chicago, Ill.-Toronto, Ont., 1966.

[45] L. Lu, P. Jin, G. Pang, Z. Zhang, and G. E. Karniadakis. Learning nonlinear operators via DeepONet based on the universal approximation theorem of operators. *Nature Machine Intelligence*, 3(3):218–229, mar 2021.

[46] H. Luschgy and G. Pagès. Expansions for Gaussian processes and Parseval frames. *Electron. J. Probab.*, 14:no. 42, 1198–1221, 2009.

[47] C. Marcati and C. Schwab. Exponential convergence of deep operator networks for elliptic partial differential equations. Technical Report 2021-42, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2021. (to appear in SIAM J. Numer. Analysis 2023).

[48] R. Molinaro, Y. Yang, B. Engquist, and S. Mishra. Neural Inverse Operators for Solving PDE Inverse Problems. *arXiv preprint arXiv:2301.11167*, 2023.

[49] T. O'Leary-Roseberry, P. Chen, U. Villa, and O. Ghattas. Derivate informed neural operator: An efficient framework for high-dimensional parametric derivative learning. *arXiv preprint arXiv:2206.10745*, 2022.

[50] J. A. A. Opschoor, P. C. Petersen, and C. Schwab. Deep ReLU Networks and High-Order Finite Element Methods. *Anal. Appl. (Singap.)*, 18(5):715–770, 2020.

[51] J. A. A. Opschoor, C. Schwab, and J. Zech. Deep learning in high dimension: ReLU neural network expression for Bayesian PDE inversion. In *Optimization and control for partial differential equations—uncertainty quantification, open and closed-loop control, and shape optimization*, volume 29 of *Radon Ser. Comput. Appl. Math.*, pages 419–462. De Gruyter, Berlin, 2022.

[52] J. A. A. Opschoor, C. Schwab, and J. Zech. Exponential ReLU DNN expression of holomorphic maps in high dimension. *Constr. Approx.*, 55(1):537–582, 2022.

[53] G. Petrova and P. Wojtaszczyk. Lipschitz widths. *Constructive Approximation*, 57(2):759–805, 2023.

[54] A. Pinkus. Approximation theory of the MLP model in neural networks. In *Acta numerica, 1999*, volume 8 of *Acta Numer.*, pages 143–195. Cambridge Univ. Press, Cambridge, 1999.

[55] C. Schwab and A. Stein. Deep solution operators for variational inequalities via proximal neural networks. *Research in the Mathematical Sciences*, 9(36), 2022.

[56] C. Schwab and J. Zech. Deep learning in high dimension: neural network expression rates for generalized polynomial chaos expansions in UQ. *Anal. Appl. (Singap.)*, 17(1):19–55, 2019.

[57] C. Schwab and J. Zech. Deep Learning in High Dimension: Neural Network Expression Rates for Analytic Functions in $L^2(\mathbb{R}^d, \gamma_d)$. *SIAM/ASA J. Uncertain. Quantif.*, 11(1):199–234, 2023.

[58] Z. Shen, H. Yang, and S. Zhang. Neural network approximation: Three hidden layers are enough. *Neural Networks*, 141:160–173, 2021.

[59] H. Triebel. *Function spaces and wavelets on domains*, volume 7 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2008.

[60] H. Triebel. *Bases in function spaces, sampling, discrepancy, numerical integration*, volume 11 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2010.

[61] H. Triebel. *Theory of Function Spaces IV*, volume 107 of *Monographs in Mathematics*. Birkhäuser, 2020.

[62] G. Wen, Z. Li, K. Azizzadenesheli, A. Anandkumar, and S. M. Benson. U-FNO—An enhanced Fourier neural operator-based deep-learning model for multiphase flow. *Advances in Water Resources*, 163:104180, 2022.

[63] D. Yarotsky. Error bounds for approximations with deep ReLU networks. *Neural Networks*, 94:103–114, 2017.

[64] D. Yarotsky. Elementary superexpressive activations. In *International Conference on Machine Learning*, pages 11932–11940. PMLR, 2021.

[65] D. Yarotsky and A. Zhevnerchuk. The phase diagram of approximation rates for deep neural networks. *Advances in neural information processing systems*, 33:13005–13015, 2020.

[66] J. Zech, D. Dung, and C. Schwab. Multilevel approximation of parametric and stochastic pdes. *M3AS*, 29(9):1753–1817, 2019.

[67] S. Zhang, Z. Shen, and H. Yang. Neural network architecture beyond width and depth. *Advances in Neural Information Processing Systems*, 35:5669–5681, 2022.