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Skeleton Integral Equations for Acoustic Transmission Problems with Varying Coefficients

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Abstract

In this paper we will derive an integral equation which transform a three-dimensional acoustic transmission problem with *variable* coefficients, non-zero absorption, and mixed boundary conditions to a non-local equation on the skeleton of the domain $\Omega \subset \mathbb{R}^3$, where "skeleton" stands for the union of the interfaces and boundaries of a Lipschitz partition of Ω. To that end, we introduce and analyze abstract layer potentials as solutions of auxiliary coercive full space variational problems and derive jump conditions across domain interfaces. This allows us to formulate the non-local skeleton equation as a *direct method* for the unknown Cauchy data of the original partial differential equation. We establish coercivity and continuity of the variational form of the skeleton equation without based on an auxiliary full space variational problem. Explicit knowledge of Green's functions is not required and our estimates are explicit in the complex wave number.

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Key Words: acoustic wave equation, transmission problem, layer potentials, Calderón operator.

1 Introduction

Setting. In this paper we consider acoustic transmission problems in Laplace domain.

$$\operatorname{div}(\mathbb{A}\nabla w) + s^2 p\, w = 0 \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{1.1}$$

We admit general essentially bounded and uniformly positive (definite) coefficient functions $\mathbb{A}$ and p and mixed boundary conditions. More precisely, the boundary conditions on $\partial\Omega$ are of Dirichlet and/or Neumann type and decay conditions are imposed at infinity if the domain is unbounded. We assume that the (complex) wave number s has positive real part so that the arising sesquilinear form in the variational formulation is coercive and well-posedness in

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$H^1(\mathbb{R}^3)$ follows by the Lax-Milgram lemma. More details are given in Section 2.2 and the following.

Goal. The goal of this paper is to develop a transformation of the partial differential equation (1.1) to an non-local equation, an "integral equation", on the *skeleton* (interfaces and the domain boundary) of a Lipschitz partition of Ω, such that coercivity is inherited from the coercivity of the PDE. We emphasize that there is a variety of ways to transform a PDE to an integral equation and we mention the *direct* and *indirect* formulation, equations of *first* and of *second* kind, *symmetric* and *non-symmetric* couplings for interface problems. Our approach is based on the direct formulation based on Green's representation formula and a symmetric formulation as a system of non-local skeleton equations. The solution is sought in *single trace spaces* where the functions on the interfaces are single valued and the transmission conditions are built into the function spaces; these trace spaces have been introduced in [8].

Main contributions. Usually, Green's representation formula contains the fundamental solution of the underlying PDE explicitly and, hence, in literature the arising boundary integral equations are mostly considered for cases where the fundamental solution is known explicitly. Our approach to transform the PDE to a non-local equation on the skeleton does not rely on the fundamental solution; neither its existence nor an explicit form is required. Instead, we define the layer potentials directly via the variational form of the PDE as solutions of certain transmission problems. We derive jump relations for these abstract potentials, Green's representation formula, and non-local skeleton operators which allow us to define the Calderón operator. We show how coercivity of the sesquilinear form on the skeleton can be derived directly from the coercivity of the PDE.

While the definition of the single layer potential as the solution of the variational form of the full space PDE for certain types of right-hand side is standard and applies also to elliptic PDEs with variable coefficients, the definition of the double layer potential is more delicate. Various (equivalent) definitions exist in literature for certain types of elliptic PDEs and we briefly review some of them:

1) If the fundamental solution, say $G(\mathbf{x}, \mathbf{y})$, of the differential operator is known the double layer potential can be defined as an integral over the skeleton of the co-normal derivative of G convoluted with a boundary density – first for sufficiently regular boundary functions and then by continuous extension as a mapping between appropriate Sobolev spaces. The analysis of the double layer potential (mapping properties/jump relations, etc.) is then derived from properties of the fundamental solution. However, if the fundamental solution is not known explicitly as, e.g., for variable L^∞ coefficients the analysis is far from trivial.

2) For problems with constant coefficients the double layer potential can be defined as the composition of the full space solution operator (acoustic Newton potential) with the dual of the co-normal derivative. However, this dual co-normal derivative maps into a space which is larger than the natural domain of the Newton potential. For PDEs with constant coefficients this problem can be solved since it is known that the Newton potential satisfies some regularity shift properties. For variable L^∞ coefficients this is a subtle issue.

3) In [10], the case of C^∞- coefficients is considered. First the double layer potential is introduced as explained in 1); then a regularity shift theorem from [18] is employed to

directly derive a Green's representation formula. This Green's formula can then be used as an alternative definition of the double layer potential.

4) The definition in [6, (4.5)] expresses the double layer potential as a composition of a trace lifting of the boundary density with the differential operator and the Newton potential and thus avoids both, the explicit knowledge of the fundamental solution and the range space of the dual co-normal derivative. Although the analysis of the double layer potential can be based on the mature theory of elliptic PDEs, it seems that our new definition allows for a much more straightforward analysis.

Our **new approach** defines the double layer potential as the solution of an *ultra-weak variational formulation* of the full space PDE with a certain type of right-hand sides. This definition allows us to derive directly the mapping properties, jump relations, and representation formula from the underlying PDE.

We derive the skeleton and Calderón operators from this idea. Our paper can be considered as a generalization of [9] and the recent paper [11] by allowing for unbounded domains (full space/half space), variable coefficients in the subdomains, and do not require the explicit knowledge of a Green's function. We also generalize the stability theory for the Calderón operator developed in [4] (see also the monograph [21]) to variable coefficients in the principal and zeroth order part of the differential equation. The estimates for the layer potentials, Calderón operators, and skeleton operators are explicit with respect to the wave number s and generalizes the known estimates for problems with piecewise constant coefficients (see, e.g., [3], [15], [11]).

Outline. The paper is structured as follows. In Section 2 we formulate the acoustic transmission problem with mixed boundary conditions. This requires the introduction of the domain partitioning along its skeleton, the definition of one-sided trace operators as well as the jumps and means of piecewise regular functions. The transmission problem is formulated in (2.23) and defines the starting point for the various steps in the derivation of the non-local skeleton equations.

In Section 3, we derive Green's representation formula in an abstract way. We consider the homogeneous PDE on a subdomain as well as on its complement domain in $\mathbb{R}^3$ (with extended coefficients) and formulate auxiliary variational full space problems which are coercive and continuous. The single layer potential is defined as the solution operator for a distribution (density) located on the interface (see (3.17)); the explicit knowledge of a fundamental solution is not required. We present a new and simple definition of the double layer potential as the solution of an ultra-weak variational full space problem for a certain type of right-hand sides. With these layer potentials at hand we prove a Green's representation formula on both subdomains (Lemma 3.15) as well as jump relations for both layer potentials.

Section 4 is devoted to the definition of the non-local skeleton operators V, K, K', W which are used to build the Calderón operator. The important projection property for the Calderón operator is derived in Lemma 4.3.

In Section 5 we define the free single trace space $\mathbb{X}^{\text{single}}$ on the skeleton and the one with incorporated boundary conditions $\mathbb{X}_0^{\text{single}}$. Then, the non-local skeleton equation is formulated in (5.4) as a variational problem with energy space $\mathbb{X}_0^{\text{single}}$. The remaining part of this section is devoted to the analysis of the skeleton equation and leads to its well-posedness, formulated in Theorem 5.5.

We summarize our main achievement in the concluding Section 6 and give comments on some straightforward extensions of this integral equation method.

In the Appendix A we give the proof of s-explicit coercivity and continuity estimates for the boundary integral operators and layer potentials. Since the arguments are very similar to those in [15, Prop. 16, 19] and [6, Lem. 5.2] we have shifted this proof to the appendix.

List of notations

In this article we prefer "verbose" notations conveying maximum information about entities. We admit that this leads to lavishly adorned symbols, but enhanced precision is worth this price.

As a convention, we denote scalar functions and spaces of scalar functions with italic letters, vectors in $\mathbb{C}^3$ (tensors of order 1) with bold letters, and matrices in $\mathbb{C}^{3\times 3}$ (tensors of order 2) by blackboard bold letters.

$\mathbb{R}_{>0}$	positive real numbers
$\mathbb{C}_{>0}$	complex numbers with positive real part
$\mathbb{R}^{3\times 3}_{\text{sym}}$	symmetric 3×3 matrices
$\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle_\omega$	bilinear form in $\mathbb{C}^3$ see §2.1 and duality pairing of a function space on a domain (or manifold) ω with its dual
$\mathbb{A}$	tensor coefficient for transmission problem, see Rem. 2.4
$\mathbb{A}_j^\sigma, p_j^\sigma$	coefficients on the subdomain Ω_j^σ, $\sigma \in \{+, -\}$, see Assumption 2.2, (2.8)
$\mathbb{A}_j^{\text{ext}}, p_j^{\text{ext}}$	extension of the coefficients $\mathbb{A}_j^{\text{ext}}, p_j^-$ to $\mathbb{R}^3$, see Assumption 2.2
$\lambda\left(\mathbb{A}_j^{\text{ext}}\right)$, $\Lambda\left(\mathbb{A}_j^{\text{ext}}\right)$	lower and upper spectral bound of the tensor coefficient $\mathbb{A}_j^{\text{ext}}$, see (2.5)
$\lambda\left(p_j^{\text{ext}}\right)$, $\Lambda\left(p_j^{\text{ext}}\right)$	lower and upper bound of the coefficient p_j^{ext}, see (2.5)
λ_j, Λ_j	$\min\left\{\lambda\left(\mathbb{A}_j^{\text{ext}}\right), \lambda\left(p_j^{\text{ext}}\right)\right\}$, $\max\left\{\Lambda\left(\mathbb{A}_j^{\text{ext}}\right), \Lambda\left(p_j^{\text{ext}}\right)\right\}$, see (3.3)
s	Laplace domain parameter ("wave number") in $\mathbb{C}_{>0}$, see (2.2)
s_0	lower bound of the modulus of s, see (2.2)
Ω	bounded or unbounded domain in $\mathbb{R}^3$, see §2.2
$\Omega_j = \Omega_j^-$,	subdomains of Ω ($1 \le j \le n_\Omega$), see §2.2
Ω_j^+	exterior complement $\mathbb{R}^3 \backslash \overline{\Omega_j^-}$, see §2.2
$\omega \subset\subset \Omega$	ω is compactly contained in Ω, i.e., $\overline{\omega} \subset \Omega$,
Γ	boundary of Ω; see §2.2
Γ_j	boundary of Ω_j; see §2.2
$\Gamma_{j,k}$	common boundary of Ω_j and Ω_k; see §2.2
Γ_{D}	part of Γ where Dirichlet boundary conditions are imposed; see §2.2
Γ_{N}	part of Γ where Neumann boundary conditions are imposed; see § 2.2
$\mathcal{P}_\Omega$	set of subdomains of Ω; see §2.2
Σ	skeleton of $\mathcal{P}_\Omega$, union of $\partial\Omega_j$, see §2.2
$\mathbf{n}_j$	outward normal vector pointing from Ω_j^- to Ω_j^+, see Prop. 2.5
$C^\infty(\omega)$, $\mathbf{C}^\infty(\omega)$	space of infinitely differentiable functions and vector valued version
$C_0^\infty(\omega)$, $\mathbf{C}_0^\infty(\omega)$	$C_0^\infty(\omega) := \{u \in C^\infty(\omega) \mid \operatorname{supp} u \subset \omega\}$ with vector valued version $\mathbf{C}_0^\infty(\omega)$
$\left(L^p(\omega), \lVert\cdot\rVert_{L^p(\omega)}\right)$	Lebesgue space for $1 \le p \le \infty$ with norm $\lVert\cdot\rVert_{L^p(\omega)}$; see §2.1
$\left(\mathbf{L}^p(\omega), \lVert\cdot\rVert_{\mathbf{L}^p(\omega)}\right)$	$\mathbf{L}^p(\omega) := L^p(\omega)^3$ with norm $\lVert\cdot\rVert_{\mathbf{L}^p(\omega)}$, see §2.1
$\left(\mathbb{L}^p(\omega), \lVert\cdot\rVert_{\mathbb{L}^p(\omega)}\right)$	$\mathbb{L}^p(\omega) := L^p(\omega)^{3\times 3}$ with norm $\lVert\cdot\rVert_{\mathbb{L}^p(\omega)}$, see 2.1

$(\cdot,\cdot)_{L^2(\omega)}$, $(\cdot,\cdot)_{\mathbf{L}^2(\omega)}$, $(\cdot,\cdot)_{\mathbb{L}^2(\omega)}$	$L^2(\omega)$ scalar product in $L^2(\Omega)$,$\mathbf{L}^2(\Omega)$, $\mathbb{L}^2(\Omega)$
$L^\infty_{>0}(\omega,\mathbb{R})$	subset of $L^\infty(\omega)$ of functions which are uniformly positive, see §2.1
$\mathbb{L}^p\left(\omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)$	subset of $\mathbb{L}^p(\omega)$ of functions which map into the set of symmetric 3×3 matrices; see §2.1
$\mathbb{L}^\infty_{>0}\left(\omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)$	subset of $\mathbb{L}^\infty\left(\omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)$ of functions which are uniformly positive definite, see Def. 2.1
$W^{k,p}(\omega)$	Sobolev space; see §2.1
$H^k(\omega)$	Sobolev space $W^{k,2}(\omega)$, see §2.1
$H^k_0(\omega), H^{-k}(\omega)$	closure of smooth functions with compact support with respect to the $\lVert\cdot\rVert_{H^k(\omega)}$ norm (see §2.1) and its dual space (see §2.1)
$H^k_{\mathrm{loc}}(\omega)$	Sobolev space of functions which locally belong to $H^k(\omega)$; see §2.1
$\lVert\cdot\rVert_{H^1(\omega);s}$, $\lVert\cdot\rVert_{H^{-1}(\mathbb{R}^3);s}$	frequency-weighted Sobolev norm and its dual norm, see (2.1), (3.13)
$\mathbf{H}(\omega,\operatorname{div})$	subspace of $\mathbf{L}^2(\omega)$ of functions $\mathbf{v}$ satisfying $\operatorname{div}\mathbf{v}\in L^2(\omega)$, see (2.2)
$\left(H^1(\omega,\mathbb{B}),\lVert u\rVert_{H^1(\omega,\mathbb{B})}\right)$	subspace of $H^1(\omega)$ of functions v such that $\operatorname{div}(\mathbb{B}\nabla v)\in L^2(\omega)$ equipped with the graph norm; see Def. 2.1
$\mathbb{H}^1(\mathcal{P}_\Omega,\mathbb{A})$	$\bigtimes_{j=1}^{n_\Omega} H^1\left(\Omega_j,\mathbb{A}_j^-\right)$
$H^\alpha(\partial\omega)$	Sobolev space on a closed manifold, see §2.1
$H^{\pm 1/2}(\Gamma_{j,k})$, $\tilde{H}^{\pm 1/2}(\Gamma_{j,k})$	Sobolev spaces on manifolds with boundary; see (2.21)
$\left(\mathbf{X}_j,\langle\cdot,\cdot\rangle_{\mathbf{X}_j},\lVert\cdot\rVert_{\mathbf{X}_j}\right)$	Sobolev space $H^{1/2}(\Gamma_j)\times H^{-1/2}(\Gamma_j)$, equipped with bilinear form $\langle\cdot,\cdot\rangle_{\mathbf{X}_j}$ and norm $\lVert\cdot\rVert_{\mathbf{X}_j}$, see Def. 2.8, (4.1a)
$(\mathbb{X}(\mathcal{P}_\Omega),\langle\cdot,\cdot\rangle_{\mathbb{X}},\lVert\cdot\rVert_{\mathbb{X}})$	Sobolev space $\mathbb{X}(\mathcal{P}_\Omega):=\bigtimes_{j=1}^{n_\Omega}\mathbf{X}_j$, with bilinear form $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ and norm $\lVert\cdot\rVert_{\mathbb{X}}$, see Def. 2.8, (4.2)
$\mathbb{X}^{\mathrm{single}}(\mathcal{P}_\Omega)$	single traces space; see (5.2)
$\mathbb{X}_0^{\mathrm{single}}(\mathcal{P}_\Omega)$	single traces space with incorporated zero boundary conditions, see (5.2)
$\gamma^\sigma_{\mathrm{D};j}$, $\gamma_{\mathrm{D};j}$, $\gamma^\sigma_{\mathrm{D};j}(s)$, $\gamma_{\mathrm{D};j}(s)$	one-sided and two-sided Dirichlet trace operators and frequency scaled versions; see Prop. 2.5, (2.18)
$\gamma^\sigma_{\mathbf{n};j}$, $\gamma_{\mathbf{n};j}$, $\gamma^\sigma_{\mathbf{n};j}(s)$, $\gamma_{\mathbf{n};j}(s)$	one-sided and two-sided normal trace operators and frequency scaled versions; see Prop. 2.5, (2.18)
$\gamma^\sigma_{\mathrm{N};j}$, $\gamma^{\mathrm{ext},\sigma}_{\mathrm{N};j}$, $\gamma_{\mathrm{N};j}$, $\gamma^{\mathrm{ext}}_{\mathrm{N};j}$, $\gamma^\sigma_{\mathrm{N};j}(s)$, $\gamma^{\mathrm{ext},\sigma}_{\mathrm{N};j}(s)$, $\gamma_{\mathrm{N};j}(s)$, $\gamma^{\mathrm{ext}}_{\mathrm{N};j}(s)$	one-sided and two-sided co-normal derivatives and frequency scaled versions, see Prop. 2.5, (2.18), Notation 2.9
$\boldsymbol{\gamma}^\sigma_{\mathrm{C};j}$, $\boldsymbol{\gamma}^{\mathrm{ext},\sigma}_{\mathrm{C};j}$, $\boldsymbol{\gamma}^\sigma_{\mathrm{C};j}(s)$, $\boldsymbol{\gamma}^{\mathrm{ext},\sigma}_{\mathrm{C};j}(s)$	one-sided and two-sided Cauchy trace operators and frequency scaled versions, see (2.15), (2.18), Notation 2.9
$\mathsf{E}_j(s)$	trace lifting operator; see Lem. 5.1
$[u]_{\mathrm{D};j}, [u]_{\mathrm{D};j}(s)$	Dirichlet jump across Γ_j and frequency scaled version; see Def. 2.7
$[\mathbf{u}]_{\mathrm{D};j,k}$	Dirichlet jump across partial boundary $\Gamma_{j,k}$; see (2.22)
$[u]_{\mathrm{N};j}, [u]^{\mathrm{ext}}_{\mathrm{N};j}$, $[u]_{\mathrm{N};j}(s), [u]^{\mathrm{ext}}_{\mathrm{N};j}(s)$	jump of co-normal derivative across Γ_j and frequency scaled version; see Def. 2.7, Notation 2.9
$[\mathbf{w}]_{\mathrm{N};j,k}$, $[\mathbf{w}]^{\mathrm{ext}}_{\mathrm{N};j,k}$	jump of co-normal derivative across partial boundary $\Gamma_{j,k}$, see (2.22)
$\{\!\{u\}\!\}_{\mathrm{D};j}, \{\!\{u\}\!\}_{\mathrm{D};j}(s)$, $\{\!\{u\}\!\}_{\mathrm{N};j}, \{\!\{u\}\!\}_{\mathrm{N};j}(s)$	mean value of Dirichlet traces and co-normal derivatives across boundary Γ_j, and their frequency scaled version; see Def. 2.7, Notation 2.9
$\ell_j(s)(\cdot,\cdot), \mathsf{L}_j(s)$	sesquilinear form associated to the full space transmission problem with coefficients $\mathbb{A}^{\mathrm{ext}}_j$, p^{ext}_j and relative operator; see Def. 3.1
$\mathsf{L}^-_j(s)$, $\mathsf{L}^+_j(s)$	differential operator on subdomains Ω^-_j, Ω^+_j, see (2.7)

$\nabla_{\text{pw};j}$	piecewise gradient; see (3.1)
$\mathsf{V}_j(s), \mathsf{K}_j(s)$, $\mathsf{K}_j'(s), \mathsf{W}_j(s)$	boundary integral operators, see Def. 4.1
$\mathsf{C}_j(s)$	Calderón operator for the subdomain Ω_j, see Def. 4.2
$\mathsf{C}(s), c(s)$	global Calderón operator and associated sesquilinear form; see Def. 4.2
Id	identity operator

2 Setting

In this section we give details about the acoustic transmission problem. First, we introduce the appropriate Sobolev spaces, standard trace operators, and co-normal derivatives. Then we specify assumptions on the coefficients of the problem and formulate boundary and decay conditions. We write $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$, and $\mathbb{C}_{>0} := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, respectively.

2.1 Function spaces

Let $\omega \subset \mathbb{R}^3$ be a bounded or unbounded Lipschitz domain with (possibly empty) boundary $\partial\omega$. For $k \geq 0$, $1 \leq p \leq \infty$, $W^{k,p}(\omega)$ denotes the classical Sobolev space of functions with norm $\|\cdot\|_{W^{k,p}(\omega)}$. As usual we write $L^p(\omega)$ instead of $W^{0,p}(\omega)$ and $H^k(\omega)$ for $W^{k,2}(\omega)$. For $k \geq 0$, we denote by $H_0^k(\omega)$ the closure of the space of infinitely smooth functions with compact support in ω with respect to the $H^k(\omega)$ norm. Its dual space is denoted by $H^{-k}(\omega) := \left(H_0^k(\omega)\right)'$. Vector- and tensor valued versions of the Lebesgue spaces are denoted by $\mathbf{L}^p(\omega) := L^p(\omega)^3$ and $\mathbb{L}^p(\omega) := L^p(\omega)^{3\times 3}$ with norm $\|\cdot\|_{\mathbf{L}^p(\omega)}$ and $\|\cdot\|_{\mathbb{L}^p(\omega)}$, respectively and we use an analogous notation for vector and tensor valued Sobolev spaces. For $p = 2$, these spaces are Hilbert spaces with scalar product $(\cdot,\cdot)_{L^2(\omega)}$, $(\cdot,\cdot)_{\mathbf{L}^2(\omega)}$, $(\cdot,\cdot)_{\mathbb{L}^2(\omega)}$. We also employ a "frequency-dependent" $H^1(\omega)$ norm and define for $s \in \mathbb{C}\backslash\{0\}$

$$\|v\|_{H^1(\omega);s} := \left(\|\nabla v\|^2_{\mathbf{L}^2(\omega)} + |s|^2 \|v\|^2_{L^2(\omega)}\right)^{1/2}. \tag{2.1}$$

The space $\mathbf{H}(\omega, \operatorname{div})$ is given by

$$\mathbf{H}(\omega, \operatorname{div}) := \left\{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{w} \in L^2(\omega)\right\}. \tag{2.2}$$

On the boundary of ω, we define the Sobolev space $H^\alpha(\partial\omega)$, $\alpha \geq 0$, in the usual way (see, e.g., [16, pp. 98]). Note that the range of α for which $H^\alpha(\partial\omega)$ is defined may be limited, depending on the global smoothness of the surface $\partial\omega$; for Lipschitz surfaces, α can be chosen in the range $[0,1]$; for $\alpha < 0$, the space $H^\alpha(\partial\omega)$ is the dual of $H^{-\alpha}(\partial\omega)$.

We write $\langle\cdot,\cdot\rangle_\omega$ for the *bilinear form*

$$\langle u, v\rangle_\omega := \int_\omega uv \quad \text{so that} \quad (u,v)_{L^2(\omega)} = \langle u, \overline{v}\rangle_\omega, \tag{2.3}$$

and identify $\langle\cdot,\cdot\rangle_\omega$ with its continuous extension to the duality pairing $H^{-k}(\omega) \times H_0^k(\omega)$. For $k \geq 0$, the spaces $H^k_{\text{loc}}(\omega)$ are given by using smooth and compactly-supported cutoff functions via

$$H^k_{\text{loc}}(\omega) := \left\{v : \chi v \in H^k(\omega) \text{ for all } \chi \in C_0^\infty\left(\mathbb{R}^3\right)\right\} \tag{2.4}$$

and the subscript "loc" is used in an analogue way also for other spaces.

Let $\mathbb{R}^{3\times3}_{\text{sym}}$ denote the set of real symmetric 3×3 matrices. We denote by $\langle\cdot,\cdot\rangle : \mathbb{C}^3\times\mathbb{C}^3 \to \mathbb{C}$ the bilinear form $\langle \mathbf{a},\mathbf{b}\rangle := \sum_{\ell=1}^{3} a_\ell b_\ell$ for $\mathbf{a} = (a_\ell)_{\ell=1}^3 \in \mathbb{C}^3$ and $\mathbf{b} = (b_\ell)_{\ell=1}^3 \in \mathbb{C}^3$. Clearly, this bilinear form is the standard Euclidean scalar product if restricted to $\mathbb{R}^3\times\mathbb{R}^3$. Let $\mathbb{L}^\infty\left(\omega,\mathbb{R}^{3\times3}_{\text{sym}}\right)$ denote the space of all functions $\mathbb{B} : \omega \to \mathbb{R}^{3\times3}_{\text{sym}}$ whose components belong to the Lebesgue space $L^\infty(\omega)$. We define the *spectral bounds* for $\mathbb{B} \in \mathbb{L}^\infty\left(\omega,\mathbb{R}^{3\times3}_{\text{sym}}\right)$ and $q \in L^\infty(\omega,\mathbb{R})$ by

$$\lambda(\mathbb{B}) := \operatorname*{ess\,inf}_{\mathbf{y}\in\omega} \inf_{\mathbf{v}\in\mathbb{R}^3\backslash\{0\}} \frac{\langle \mathbb{B}(\mathbf{y})\mathbf{v},\mathbf{v}\rangle}{\langle\mathbf{v},\mathbf{v}\rangle} \leq \operatorname*{ess\,sup}_{\mathbf{y}\in\omega} \sup_{\mathbf{v}\in\mathbb{R}^3\backslash\{0\}} \frac{\langle \mathbb{B}(\mathbf{y})\mathbf{v},\mathbf{v}\rangle}{\langle\mathbf{v},\mathbf{v}\rangle} =: \Lambda(\mathbb{B}) < \infty, \tag{2.5a}$$

$$\lambda(q) := \operatorname*{ess\,inf}_{\mathbf{y}\in\omega} q(\mathbf{y}) \leq \operatorname*{ess\,sup}_{\mathbf{y}\in\omega} q(\mathbf{y}) =: \Lambda(q) < \infty. \tag{2.5b}$$

Definition 2.1 *Let*

$$L^\infty_{>0}(\omega,\mathbb{R}) := \{q \in L^\infty(\omega,\mathbb{R}) \mid \lambda(q) > 0\},$$
$$\mathbb{L}^\infty_{>0}\left(\omega,\mathbb{R}^{3\times3}_{\text{sym}}\right) := \left\{\mathbb{B} \in \mathbb{L}^\infty\left(\omega,\mathbb{R}^{3\times3}_{\text{sym}}\right) \mid \lambda(\mathbb{B}) > 0\right\}.$$

For $\mathbb{B} \in \mathbb{L}^\infty_{>0}\left(\omega,\mathbb{R}^{3\times3}_{\text{sym}}\right)$, the space $H^1(\omega,\mathbb{B})$ is given by

$$H^1(\omega,\mathbb{B}) := \left\{u \in H^1(\omega) \mid \operatorname{div}(\mathbb{B}\nabla u) \in L^2(\omega)\right\}$$

and equipped with the graph norm

$$\|u\|_{H^1(\omega,\mathbb{B})} := \left(\|u\|^2_{H^1(\omega)} + \|\operatorname{div}(\mathbb{B}\nabla u)\|^2_{L^2(\omega)}\right)^{1/2}.$$

2.2 Differential operators

Next we describe our assumptions on the computational domain and its partition. Let $\Omega \subset \mathbb{R}^3$ be a bounded or unbounded Lipschitz domain with (possibly empty) boundary $\Gamma := \partial\Omega$. We assume that there is a finite partition of Ω consisting of disjoint Lipschitz domains Ω_j, $1 \leq j \leq n_\Omega$, with closed boundary $\Gamma_j := \partial\Omega_j$, which satisfy $\overline{\Omega} = \bigcup_{j=1}^{n_\Omega} \overline{\Omega_j}$. The subdomains are collected in the partition $\mathcal{P}_\Omega = \{\Omega_j : 1 \leq j \leq n_\Omega\}$. The intersection of the boundaries $\partial\Omega_j$ and $\partial\Omega_k$ is denoted by $\Gamma_{j,k} := \partial\Omega_j \cap \partial\Omega_k$. The *skeleton* of this partition is given by $\Sigma := \bigcup_{j=1}^{n_\Omega} \partial\Omega_j$. To unify notation, we write $\Omega_j^- := \Omega_j$ and set $\Omega_j^+ := \mathbb{R}^3\backslash\overline{\Omega_j^-}$.

We consider mixed Dirichlet and Neumann boundary conditions on $\partial\Omega$. In this way, we split

$$\partial\Omega = \Gamma_{\text{D}} \cup \Gamma_{\text{N}} \tag{2.6}$$

and assume the relative interiors of these subsets are disjoint.

In the subdomains $\Omega_j \in \mathcal{P}_\Omega$, we consider partial differential equations and formulate appropriate assumptions on the coefficients next.

Assumption 2.2 *For any $1 \leq j \leq n_\Omega$, the coefficients in (2.9) satisfy*

1. $\mathbb{A}_j^- \in \mathbb{L}^\infty_{>0}\left(\Omega_j,\mathbb{R}^{3\times3}_{\text{sym}}\right)$ *and* $\mathbb{A}_j^-$ *can be extended to some* $\mathbb{A}_j^{\text{ext}} \in \mathbb{L}^\infty_{>0}\left(\mathbb{R}^3,\mathbb{R}^{3\times3}_{\text{sym}}\right)$,
2. $p_j^- \in L^\infty_{>0}(\Omega_j,\mathbb{R})$ *and* p_j^- *can be extended to some* $p_j^{\text{ext}} \in L^\infty_{>0}(\mathbb{R}^3,\mathbb{R})$,

3. $s \in \mathbb{C}_{>0}$ *and* $|s| \geq s_0$ *for some* $s_0 > 0$.

We exclude a neighborhood of 0 for the frequencies $s \in \mathbb{C}$ since our focus is on the high-frequency behavior. Note that the constants in our estimates depend continuously on s_0 and, possibly, deteriorate as $s_0 \rightarrow 0$.

For $\sigma \in \{+, -\}$, we define formally the differential operators:

$$\mathsf{L}_j^\sigma(s)\, w := -\operatorname{div}\left(\mathbb{A}_j^\sigma \nabla w\right) + s^2 p_j^\sigma w \quad \text{in } \Omega_j^\sigma, \tag{2.7}$$

where

$$\mathbb{A}_j^\sigma := \left.\mathbb{A}_j^{\text{ext}}\right|_{\Omega_j^\sigma} \quad \text{and} \quad p_j^\sigma := \left.p_j^{\text{ext}}\right|_{\Omega_j^\sigma} \quad \sigma \in \{+, -\}. \tag{2.8}$$

The differential equation on the subdomain Ω_j is given by

$$\mathsf{L}_j^-(s)\, u_j = 0 \qquad \text{in } \Omega_j. \tag{2.9}$$

Remark 2.3 *Time harmonic wave propagation* with absorption *can be described in the simplest case by a Helmholtz equation with wave number (frequency parameter) s of positive real part. Such problems arise in many applications such as, e.g., in viscoelastodynamics for materials with damping (see, e.g., [1]), in electromagnetism for wave propagation in lossy media (see, e.g., [14]) and in nonlinear optics (see, e.g., [19]). The Helmholtz equation for complex wave numbers also arises within the popular convolution quadrature method for solving time depending wave propagation problems and within some iterative algorithms for solving the linear system for the Helmholtz equation (see, e.g., [7, §2] for a more detailed description of applications).*

Remark 2.4 *Typically, the coefficients* $\mathbb{A}_j^-$, p_j^- *are the restrictions of some given global coefficients* $\mathbb{A} \in \mathbb{L}_{>0}^\infty\left(\mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3\times 3}\right)$, $p \in L_{>0}^\infty\left(\mathbb{R}^3, \mathbb{R}\right)$. *Then, the choice* $\mathbb{A}_j^{\text{ext}} := \mathbb{A}$ *is admissible and seems to be natural. In some practical applications, a different choice might be "simpler" and preferable. For instance, if the global coefficient* $\mathbb{A}$ *is constant on the subdomains* Ω_j *and given by a positive definite matrix* $\mathbb{A}_j^- \in \mathbb{R}_{\text{sym}}^{3\times 3}$ *and* p_j^- *is also constant, then, the choice of* $\mathbb{A}_j^{\text{ext}}$ *and* p_j^{ext} *as the constant extensions of* $\mathbb{A}_j^-$, p_j^- *are preferable since the Green's function is explicitly known in these cases (see, e.g., [20, (3.1.3)]). However, in our abstract setting the existence or explicit knowledge of the Green's function is not needed and hence, the concrete choice of the extension is irrelevant and gives considerable freedom depending on the application. We emphasize already at this point that the single layer and double layer operators will depend on the chosen extension; however the key point is that their combination in a Green's representation formula always represents a homogeneous solution in the corresponding subdomain as will be shown in Lemma 3.15.*

2.3 Traces and jumps

Next, we introduce *jumps* and *means* of functions across the boundaries Γ_j; the index j indicates that the two-dimensional manifold Γ_j is regarded from the domain Ω_j.

The following trace operators along their properties are well known for domains with compact boundary (see, e.g., [16, Thm. 3.37, 3.38, Lem. 4.3, Thm. 4.4], [13, Thm. 2.5]). For domains with non-compact boundary we refer to [17, Thm. 2.3, Cor. 3.14, Lem. 2.6]. We define the one-sided co-normal derivatives for an abstract diffusion coefficient $\mathbb{B} \in \mathbb{L}_{>0}^\infty\left(\mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3\times 3}\right)$; in our applications, this will be either $\mathbb{A}$ or $\mathbb{A}_j^{\text{ext}}$.

Proposition 2.5 *Let Ω, Ω_j, Ω_j^σ, $1 \le j \le n_\Omega$, $\sigma \in \{+,-\}$, be as explained above.*

1. *For $\sigma \in \{+,-\}$, there exist linear one-sided trace operators (*Dirichlet trace*)*

$$\gamma_{\mathrm{D};j}^{\sigma} : H^1\left(\Omega_j^\sigma\right) \to H^{1/2}\left(\Gamma_j\right),$$

which are the continuous extensions of the classical trace operators: for $u \in C^0\left(\overline{\Omega_j^\sigma}\right)$, it holds

$$\gamma_{\mathrm{D};j}^{\sigma} u = u|_{\Gamma_j}.$$

These operators are surjective and bounded

$$\left\|\gamma_{\mathrm{D};j}^{\sigma}\right\|_{H^{1/2}(\Gamma_j)\leftarrow H^1\left(\Omega_j^\sigma\right)} \le C_{\mathrm{D}}. \tag{2.10}$$

For $u \in H^1(\mathbb{R}^3)$, the one-sided traces coincide, i.e.,

$$\gamma_{\mathrm{D};j}^{-}\left(u|_{\Omega_j}\right) = \gamma_{\mathrm{D};j}^{+}\left(u|_{\Omega_j^+}\right) \tag{2.11}$$

and we write short $\gamma_{\mathrm{D};j} u$ for $\gamma_{\mathrm{D};j}^{\sigma}\left(u|_{\Omega_j^\sigma}\right)$, $\sigma \in \{-,+\}$, in such cases.

2. *For $\sigma \in \{+,-\}$, there exist linear one-sided normal trace operators (*normal trace*)*

$$\gamma_{\mathbf{n};j}^{\sigma} : \mathbf{H}\left(\Omega_j^\sigma, \mathrm{div}\right) \to H^{-1/2}\left(\Gamma_j\right)$$

which are continuous extensions of the classical normal trace: for $\boldsymbol{\psi}^\sigma \in \mathbf{C}^0\left(\overline{\Omega_j^\sigma}\right)$, it holds

$$\gamma_{\mathbf{n};j}^{-}\left(\boldsymbol{\psi}^-\right) = \left\langle \boldsymbol{\psi}^-|_{\Gamma_j}, \mathbf{n}_j \right\rangle \quad \text{and} \quad \gamma_{\mathbf{n};j}^{+}\left(\boldsymbol{\psi}^+\right) = \left\langle \boldsymbol{\psi}^+|_{\Gamma_j}, -\mathbf{n}_j \right\rangle,$$

where $\mathbf{n}_j$ is the unit normal vector on Γ_j pointing from Ω_j^- into Ω_j^+. These operators are bounded

$$\left\|\gamma_{\mathbf{n};j}^{\sigma}\right\|_{H^{-1/2}(\Gamma_j)\leftarrow \mathbf{H}\left(\Omega_j^\sigma,\mathrm{div}\right)} \le C_{\mathbf{n}}. \tag{2.12}$$

For $\boldsymbol{\psi} \in \mathbf{H}(\mathbb{R}^3, \mathrm{div})$ the one-sided normal traces in the fixed direction $\mathbf{n}_j$ coincide, more precisely,

$$\gamma_{\mathbf{n};j}^{-}\left(\boldsymbol{\psi}|_{\Omega_j^-}\right) = -\gamma_{\mathbf{n};j}^{+}\left(\boldsymbol{\psi}|_{\Omega_j^+}\right) \tag{2.13}$$

and we write short $\gamma_{\mathbf{n};j}\boldsymbol{\psi}$ for $\gamma_{\mathbf{n};j}^{-}\left(\boldsymbol{\psi}|_{\Omega_j^-}\right)$.

3. *Let $\mathbb{B} \in \mathbb{L}_{>0}^{\infty}\left(\mathbb{R}^3, \mathbb{R}_{\mathrm{sym}}^{3\times 3}\right)$. For $\sigma \in \{+,-\}$, $1 \le j \le n_\Omega$, set $\mathbb{B}_j^\sigma := \mathbb{B}|_{\Omega_j^\sigma}$. There exist linear one-sided co-normal derivative operators (*Neumann trace*)*

$$\gamma_{\mathrm{N};j}^{\mathbb{B},\sigma} : H^1\left(\Omega_j^\sigma, \mathbb{B}_j^\sigma\right) \to H^{-1/2}\left(\Gamma_j\right)$$

which are the continuous extensions of the classical co-normal derivatives: for $u^- \in C^1\left(\overline{\Omega_j^-}\right)$ and $u^+ \in C^1\left(\overline{\Omega_j^+}\right)$ it holds

$$\gamma_{\mathrm{N};j}^{\mathbb{B},-} u^- = \left\langle \mathbb{B}_j^- \nabla u^-, \mathbf{n}_j \right\rangle \quad \text{and} \quad \gamma_{\mathrm{N};j}^{\mathbb{B},+} u^+ = \left\langle \mathbb{B}_j^+ \nabla u^+, -\mathbf{n}_j \right\rangle.$$

These operators are bounded

$$\left\| \gamma_{\mathrm{N};j}^{\mathbb{B},\sigma} \right\|_{H^{-1/2}(\Gamma_j) \leftarrow H^1\left(\Omega_j^\sigma, \mathbb{B}_j^\sigma\right)} \leq C_{\mathrm{N}}.$$

For $u \in H^1(\mathbb{R}^3, \mathbb{B})$ the one-sided co-normal derivatives in the fixed direction $\mathbf{n}_j$ coincide, more precisely,

$$\gamma_{\mathrm{N};j}^{\mathbb{B},-}\left(u|_{\Omega_j^-}\right) = -\gamma_{\mathrm{N},j}^{\mathbb{B},+}\left(u|_{\Omega_j^+}\right) \tag{2.14}$$

and we write short $\gamma_{\mathrm{N};j}^{\mathbb{B}} u$ for $\gamma_{\mathrm{N};j}^{\mathbb{B},-}\left(u|_{\Omega_j^-}\right)$.

The one-sided Dirichlet and Neumann traces are collected in the Cauchy trace operators $\boldsymbol{\gamma}_{\mathrm{C};j}^{\mathbb{B},\sigma} : H^1\left(\Omega_j^\sigma, \mathbb{B}_j^\sigma\right) \rightarrow H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j)$ given by

$$\boldsymbol{\gamma}_{\mathrm{C};j}^{\mathbb{B},\sigma} := \left(\gamma_{\mathrm{D};j}^\sigma, \gamma_{\mathrm{N};j}^{\mathbb{B},\sigma}\right). \tag{2.15}$$

For $u \in H^1(\mathbb{R}^3, \mathbb{B})$ and $u^\sigma := u|_{\Omega_j^\sigma}$, $\sigma \in \{+,-\}$, the one-sided Cauchy traces satisfy $\left(\gamma_{\mathrm{D};j}^- u^-, \gamma_{\mathrm{N};j}^{\mathbb{B},-} u^-\right) = \left(\gamma_{\mathrm{D};j}^+ u^+, -\gamma_{\mathrm{N};j}^{\mathbb{B},+} u^+\right)$ and we write

$$\boldsymbol{\gamma}_{\mathrm{C};j}^{\mathbb{B}} : H^1\left(\mathbb{R}^3, \mathbb{B}\right) \rightarrow H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j), \quad \boldsymbol{\gamma}_{\mathrm{C};j}^{\mathbb{B}} u := \left(\gamma_{\mathrm{D};j} u, \gamma_{\mathrm{N};j}^{\mathbb{B}} u\right). \tag{2.16}$$

We will also use versions of these operators which are scaled by a frequency parameter $s \in \mathbb{C}_{>0}$ and set for $\sigma \in \{+,-\}$

$$\begin{array}{lll} \gamma_{\mathrm{D};j}^\sigma(s) := s^{1/2}\gamma_{\mathrm{D};j}^\sigma, & \gamma_{\mathbf{n};j}^\sigma(s) := s^{-1/2}\gamma_{\mathbf{n};j}^\sigma, & \gamma_{\mathrm{N};j}^{\mathbb{B},\sigma}(s) := s^{-1/2}\gamma_{\mathrm{N};j}^{\mathbb{B},\sigma}, \\ \gamma_{\mathrm{D};j}(s) := s^{1/2}\gamma_{\mathrm{D};j}, & \gamma_{\mathbf{n};j}(s) := s^{-1/2}\gamma_{\mathbf{n};j}, & \gamma_{\mathrm{N};j}^{\mathbb{B}}(s) := s^{-1/2}\gamma_{\mathrm{N};j}^{\mathbb{B}}, \end{array} \tag{2.17}$$

$$\boldsymbol{\gamma}_{\mathrm{C};j}^{\mathbb{B},\sigma}(s) := \left(s^{1/2}\gamma_{\mathrm{D};j}^\sigma, s^{-1/2}\gamma_{\mathrm{N};j}^{\mathbb{B},\sigma}\right). \tag{2.18}$$

Remark 2.6 *It will turn out that the Calderón operator (see Def. 4.2) for these scaled trace operators has a coercivity estimate which is better balanced with respect to the frequency parameter s compared to the Calderón operator for the standard trace operators (see, e.g., [4]).*

Definition 2.7 *Let $\mathbb{B} \in \mathbb{L}_{>0}^\infty\left(\mathbb{R}^3, \mathbb{R}_{\mathrm{sym}}^{3\times3}\right)$. For $\sigma \in \{+,-\}$, $1 \leq j \leq n_\Omega$, set $\mathbb{B}_j^\sigma := \mathbb{B}|_{\Omega_j^\sigma}$. For a function $u \in L^2(\Omega)$ with $u|_{\Omega_j^\sigma} \in H^1\left(\Omega_j^\sigma, \mathbb{B}_j^\sigma\right)$, the* (Dirichlet) jump *and the* jump of the co-normal derivative (Neumann jump) *of u across Γ_j are given by*

$$[u]_{\mathrm{D};j} := \gamma_{\mathrm{D};j}^+\left(u|_{\Omega_j^+}\right) - \gamma_{\mathrm{D};j}^-\left(u|_{\Omega_j^-}\right), \tag{2.19a}$$

$$[u]_{\mathrm{N};j}^{\mathbb{B}} := -\gamma_{\mathrm{N};j}^{\mathbb{B},+}\left(u|_{\Omega_j^+}\right) - \gamma_{\mathrm{N};j}^{\mathbb{B},-}\left(u|_{\Omega_j^-}\right). \tag{2.19b}$$

For $s \in \mathbb{C}_{>0}$, the frequency-scaled versions are given by $[u]_{\mathrm{D};j}(s) := s^{1/2}[u]_{\mathrm{D};j}$ and $[u]_{\mathrm{N};j}^{\mathbb{B}}(s) := s^{-1/2}[u]_{\mathrm{N};j}^{\mathbb{B}}$.

The (Dirichlet) mean *and the* mean of the co-normal derivative (Neumann mean) *across Γ_j are given by*

$$\{\!\!\{u\}\!\!\}_{\mathrm{D};j} := \frac{1}{2}\left(\gamma_{\mathrm{D};j}^+\left(u|_{\Omega_j^+}\right) + \gamma_{\mathrm{D};j}^-\left(u|_{\Omega_j^-}\right)\right), \tag{2.20a}$$

$$\{\!\!\{u\}\!\!\}_{\mathrm{N};j}^{\mathbb{B}} := \frac{1}{2}\left(-\gamma_{\mathrm{N};j}^{\mathbb{B},+}\left(u|_{\Omega_j^+}\right) + \gamma_{\mathrm{N};j}^{\mathbb{B},-}\left(u|_{\Omega_j^-}\right)\right). \tag{2.20b}$$

For $s \in \mathbb{C}_{>0}$, the frequency-scaled versions are given by $\{\!\{u\}\!\}_{\mathrm{D};j}(s) := s^{1/2}\{\!\{u\}\!\}_{\mathrm{D};j}$ *and* $\{\!\{u\}\!\}_{\mathrm{N};j}^{\mathbb{B}}(s) := s^{-1/2}\{\!\{u\}\!\}_{\mathrm{N};j}^{\mathbb{B}}$.

We also need to formulate jump conditions on partial boundaries $\Gamma_{j,k}$ of the subdomains. For a measurable subset $M \subseteq \partial\Omega_j$ we denote by $|M|$ its two-dimensional surface measure. Let Ω_j and Ω_k be such that $\Gamma_{j,k} := \Gamma_j \cap \Gamma_j$ has positive surface measure. We define the Sobolev spaces

$$\begin{aligned} H^{1/2}(\Gamma_{j,k}) &:= \left\{ \varphi|_{\Gamma_{j,k}} : \varphi \in H^{1/2}(\Gamma_j) \right\}, \\ \tilde{H}^{-1/2}(\Gamma_{j,k}) &:= \left(H^{1/2}(\Gamma_{j,k}) \right)', \\ \tilde{H}^{1/2}(\Gamma_{j,k}) &:= \left\{ \varphi|_{\Gamma_{j,k}} : \varphi \in H^{1/2}(\Gamma_j) \wedge \varphi = 0 \text{ in } \Gamma_j \backslash \Gamma_{j,k} \right\}, \\ H^{-1/2}(\Gamma_{j,k}) &:= \left(\tilde{H}^{1/2}(\Gamma_{j,k}) \right)'. \end{aligned} \tag{2.21}$$

Definition 2.8 *The* multi trace space $\mathbb{X}(\mathcal{P}_\Omega)$ *for the partition $\mathcal{P}_\Omega$ is given by*

$$\mathbb{X}(\mathcal{P}_\Omega) := \bigtimes_{j=1}^{n_\Omega} \mathbf{X}_j \quad \textit{with} \quad \mathbf{X}_j := H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j),$$

and equipped with the norm

$$\begin{aligned} \left\|\boldsymbol{\psi}_j\right\|_{\mathbf{X}_j} &:= \left(\|\psi_{\mathrm{D};j}\|^2_{H^{1/2}(\Gamma_j)} + \|\psi_{\mathrm{N};j}\|^2_{H^{-1/2}(\Gamma_j)} \right)^{1/2} & \forall \boldsymbol{\psi}_j = (\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j}) \in \mathbf{X}_j, \\ \|\boldsymbol{\psi}\|_{\mathbb{X}} &:= \left(\sum_{j=1}^{n_\Omega} \left\|\boldsymbol{\psi}_j\right\|^2_{\mathbf{X}_j} \right)^{1/2} & \forall \boldsymbol{\psi} = \left(\boldsymbol{\psi}_j\right)_{j=1}^{n_\Omega} \in \mathbb{X}(\mathcal{P}_\Omega). \end{aligned}$$

We seek the solution of our transmission problem in the space

$$\mathbb{H}^1(\mathcal{P}_\Omega, \mathbb{A}) := \bigtimes_{j=1}^{n_\Omega} H^1\left(\Omega_j, \mathbb{A}_j^-\right)$$

(cf. Assumption 2.2, Remark 2.4).

Then, for $\mathbf{u} \in \bigtimes_{j=1}^{n_\Omega} H^1(\Omega_j)$ and $\mathbf{w} \in \mathbb{H}^1(\mathcal{P}_\Omega, \mathbb{B})$ the jump $[\mathbf{u}]_{\mathrm{D};j,k} \in H^{1/2}(\Gamma_{j,k})$ and the Neumann jump $[\mathbf{w}]_{\mathrm{N};j,k}^{\mathbb{B}} \in H^{-1/2}(\Gamma_{j,k})$ across $\Gamma_{j,k} := \Gamma_j \cap \Gamma_k$ (and frequency-scaled versions thereof) are defined by

$$[\mathbf{u}]_{\mathrm{D};j,k} := \left(\gamma_{\mathrm{D},j}^- u_j\right)\big|_{\Gamma_{j,k}} - \left(\gamma_{\mathrm{D},k}^- u_k\right)\big|_{\Gamma_{j,k}}, \qquad [\mathbf{u}]_{\mathrm{D};j,k}(s) := s^{1/2}[\mathbf{u}]_{\mathrm{D};j,k}, \tag{2.22a}$$

$$[\mathbf{w}]_{\mathrm{N};j,k}^{\mathbb{B}} := -\left(\gamma_{\mathrm{N},j}^{\mathbb{B},-} w_j\right)\Big|_{\Gamma_{j,k}} - \left(\gamma_{\mathrm{N},k}^{\mathbb{B},-} w_k\right)\Big|_{\Gamma_{j,k}}, \qquad [\mathbf{w}]_{\mathrm{N};j,k}^{\mathbb{B}}(s) := s^{-1/2}[\mathbf{w}]_{\mathrm{N};j,k}^{\mathbb{B}}. \tag{2.22b}$$

We set $[\mathbf{u}]_{\mathrm{D};j,k} := 0$ and $[\mathbf{w}]_{\mathrm{N};j,k}^{\mathbb{B}} := 0$ if $\Gamma_{j,k}$ has zero surface measure or $j = k$.

Note that for coefficients $\mathbb{B}$ and functions $\mathbf{w}$ which are piecewise sufficiently regular, the Neumann jump across $\Gamma_{j,k}$ can be written as

$$\begin{aligned} [\mathbf{w}]_{\mathrm{N};j,k}^{\mathbb{B}} &= -\left\langle \gamma_{\mathrm{D};j}^-(\mathbb{B}\nabla w_j), \mathbf{n}_j \right\rangle\big|_{\Gamma_{j,k}} - \left\langle \gamma_{\mathrm{D};k}^-(\mathbb{B}\nabla w_k), \mathbf{n}_k \right\rangle\big|_{\Gamma_{j,k}} \\ &= \left\langle \gamma_{\mathrm{D};j}^-(\mathbb{B}\nabla w_j) - \gamma_{\mathrm{D};k}^-(\mathbb{B}\nabla w_k), \mathbf{n}_k \right\rangle\big|_{\Gamma_{j,k}} = \left\langle [\mathbb{B}\nabla \mathbf{w}]_{\mathrm{D};j,k}, \mathbf{n}_k \right\rangle\Big|_{\Gamma_{j,k}}, \end{aligned}$$

where we used $\mathbf{n}_j = -\mathbf{n}_k$ on $\Gamma_{j,k}$. Clearly $[\mathbf{u}]_{\mathrm{D};j,k} = -[\mathbf{u}]_{\mathrm{D};k,j}$ depends on the ordering of the indices j, k, while the Neumann jump is independent of it.

Notation 2.9 *We have defined co-normal derivatives, Neumann jumps, and Neumann means for an abstract coefficient $\mathbb{B} \in \mathbb{L}^{\infty}_{>0}\left(\Omega_j, \mathbb{R}^{3\times 3}_{\text{sym}}\right)$ and used a superscript $\mathbb{B}$ in the notation. In our application, the choices $\mathbb{B} \leftarrow \mathbb{A}$ and $\mathbb{B} \leftarrow \mathbb{A}_j^{\text{ext}}$ will appear. To simplify notation we skip the superscript $\mathbb{B}$ if $\mathbb{B} = \mathbb{A}$ and write $\gamma^{\sigma}_{\text{N};j}$ short for $\gamma^{\mathbb{A},\sigma}_{\text{N};j}$ and similar for analogous quantities. If $\mathbb{B} = \mathbb{A}_j^{\text{ext}}$, we replace the superscript by "ext" and write $\gamma^{\text{ext},\sigma}_{\text{N};j}$ short for $\gamma^{\mathbb{A}_j^{\text{ext}},\sigma}_{\text{N};j}$ and in the same way for analogous quantities. This convention is applied verbatim also to the notation of Cauchy traces.*

2.4 Transmission problem

Now we have collected all ingredients to state the acoustic transmission problem. Let $\mathbb{A} \in \mathbb{L}^{\infty}_{>0}\left(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\text{sym}}\right)$ and $p \in L^{\infty}_{>0}\left(\mathbb{R}^3, \mathbb{R}\right)$ be given and let the coefficients in (2.9) be defined by $\mathbb{A}_j^- := \mathbb{A}|_{\Omega_j^-}$ and $p_j^- := p|_{\Omega_j^-}$ such that Assumption 2.2 is satisfied. We do not require that the extensions $\mathbb{A}_j^{\text{ext}}$, p_j^{ext} in Assumption 2.2 coincide with $\mathbb{A}$ (see Remark 2.4).

The given excitation of the acoustic transmission problem consists of given data on the skeleton as well as on the Dirichlet and Neumann parts Γ_{D} and Γ_{N} of the boundary (cf. (2.6)). Let $\boldsymbol{\beta} = \left(\boldsymbol{\beta}_j\right)_{j=1}^{n_\Omega} \in \mathbb{X}\left(\mathcal{P}_\Omega\right)$ with $\boldsymbol{\beta}_j = \left(\beta_{\text{D};j}, \beta_{\text{N};j}\right) \in \mathbf{X}_j$. For $1 \leq j,k \leq n_\Omega$, define the jumps of $\boldsymbol{\beta}$ across $\Gamma_{j,k} := \Gamma_j \cap \Gamma_k$ by

$$[\boldsymbol{\beta}]_{j,k} := \left(\beta_{\text{D};j}|_{\Gamma_{j,k}} - \beta_{\text{D};k}|_{\Gamma_{j,k}}, -\beta_{\text{N};j}|_{\Gamma_{j,k}} - \beta_{\text{N};k}|_{\Gamma_{j,k}} \right)$$

if $j \neq k$ and $\Gamma_j \cap \Gamma_k$ has positive surface measure. Otherwise, we set $[\boldsymbol{\beta}]_{j,k} := 0$.

Given data $\boldsymbol{\beta} \in \mathbb{X}\left(\mathcal{P}_\Omega\right)$, the acoustic transmission problem with mixed boundary condition seeks $\mathbf{u} = \left(u_j\right)_{j=1}^{n_\Omega} \in \mathbb{H}^1\left(\mathcal{P}_\Omega, \mathbb{A}\right)$ such that

$$\begin{aligned}
&-\operatorname{div}\left(\mathbb{A}_j \nabla u_j\right) + s^2 p_j u_j = 0 && \text{in } \Omega_j,\ 1 \leq j \leq n_\Omega,\\
&\left([\mathbf{u}]_{\text{D};j,k}(s), [\mathbf{u}]^{\mathbb{A}}_{\text{N};j,k}(s)\right) = [\boldsymbol{\beta}]_{j,k}, && 1 \leq j,k \leq n_\Omega,\\
&\left(\gamma^-_{\text{D};j}(s)\, u_j\right)\Big|_{\Gamma_j \cap \Gamma_{\text{D}}} = \beta_{\text{D};j}|_{\Gamma_j \cap \Gamma_{\text{D}}} \text{ and } \left(\gamma^-_{\text{N};j}(s)\, u_j\right)\Big|_{\Gamma_j \cap \Gamma_{\text{N}}} = \beta_{\text{N};j}|_{\Gamma_j \cap \Gamma_{\text{N}}} && 1 \leq j \leq n_\Omega.
\end{aligned} \tag{2.23}$$

Remark 2.10 *The inhomogeneity $\boldsymbol{\beta}$ in (2.23) is given in some applications via an incident wave $u_{\text{inc}} \in H^1_{\text{loc}}\left(\mathbb{R}^3, \mathbb{A}_\nu^{\text{ext}}\right)$ for some fixed $\nu \in \{1, 2, \ldots n_\Omega\}$ which satisfies $-\operatorname{div}\left(\mathbb{A}_\nu^{\text{ext}} \nabla u_{\text{inc}}\right) + s^2 p_\nu^{\text{ext}} u_{\text{inc}} = 0$ in $\mathbb{R}^3$. If Ω is unbounded, then typically, ν is chosen such that Ω_ν is unbounded. In any case, it is assumed that the Cauchy trace of u_{inc} is well defined, more precisely, (at least) one of the following two conditions is required:*

1. $\boldsymbol{\gamma}^-_{\text{C};\nu} u_{\text{inc}} \in \mathbf{X}_\nu$,
2. *the function u_{inc} belongs to $C^1\left(\mathbb{R}^3\right)$ and satisfies*
   - *(a) the traces $\gamma_{\text{D};\nu} u_{\text{inc}}$ and $\gamma_{\text{N};\nu} u_{\text{inc}}$ exist in the classical pointwise sense,*
   - *(b) the restrictions of the traces $\gamma_{\text{D};\nu} u_{\text{inc}}|_{\Gamma_{\text{D}}}$ and $\gamma_{\text{N};\nu} u_{\text{inc}}|_{\Gamma_{\text{N}}}$ have compact supports.*

We will derive the well-posedness of this problem in Section 5 via layer potentials. For this goal, we will present a general method to transform such acoustic transmission problems with mixed boundary conditions and variable coefficients to a system of non-local Calderón operators on the skeleton, without relying on the explicit knowledge of the Green's function. The resulting boundary integral operators[1] are coercive, self-dual and continuous (Thm. 5.5) so that the Lax-Milgram theorem implies well-posedness. In turn, well-posedness of the original formulation (2.23) follows.

3 Potentials and Green's formula

In the subdomains $\Omega_j \in \mathcal{P}_\Omega$, a function $u_j \in H^1(\Omega_j, \mathbb{A}_j)$ which satisfies the homogeneous partial differential equation (2.9) can be expressed in terms of its Cauchy trace via *layer potentials*. In this section, we introduce in a fairly standard way the Newton potential and the single layer potential as solutions to coercive, full space PDEs in variational form. We present a new definition for the double layer potential as a solution of an ultra-weak variational problem. This allows us to derive its mapping properties and jump relations from the theory of elliptic PDEs. Finally, we derive a *Green's representation formula* for our acoustic transmission problem based on these potentials.

3.1 Sesquilinear forms and associated operators

Throughout this section we require that Assumption 2.2 holds and employ the notation

$$\Omega_j^- := \Omega_j, \qquad \Omega_j^+ := \mathbb{R}^3 \backslash \overline{\Omega_j},$$
$$\mathbb{A}_j^+ := \mathbb{A}_j^{\mathrm{ext}}\big|_{\Omega_j^+}, \qquad p_j^+ := p_j^{\mathrm{ext}}\big|_{\Omega_j^+}.$$

We also need the piecewise gradient $\nabla_{\mathrm{pw};j}$ which is given, for a function $w \in H^1(\mathbb{R}^3 \backslash \Gamma_j)$, by

$$(\nabla_{\mathrm{pw};j} w)|_{\Omega_j^\sigma} := \nabla \left(w|_{\Omega_j^\sigma} \right), \quad \sigma \in \{-,+\} \tag{3.1}$$

and considered as a function in $\mathbf{L}^2(\mathbb{R}^3)$.

Definition 3.1 *Let Assumption 2.2 be satisfied. For $s \in \mathbb{C}_{>0}$, the sesquilinear form*

$$\ell_j(s) : H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right) \to \mathbb{C}$$

is given by

$$\ell_j(s)(u,v) := \left\langle \mathbb{A}_j^{\mathrm{ext}} \nabla u, \overline{\nabla v} \right\rangle_{\mathbb{R}^3} + s^2 \left\langle p_j^{\mathrm{ext}} u, \overline{v} \right\rangle_{\mathbb{R}^3} \qquad \forall u,v \in H^1\left(\mathbb{R}^3\right),$$

and the associated operator $\mathsf{L}_j(s) : H^1(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3)$ by

$$\langle \mathsf{L}_j(s) u, \overline{v} \rangle_{\mathbb{R}^3} := \ell_j(s)(u,v) \qquad \forall u,v \in H^1\left(\mathbb{R}^3\right). \tag{3.2}$$

Next, we prove continuity and coercivity for the sesquilinear form $\ell_j(s)(\cdot,\cdot)$ in the spirit of [3]. We take pains to elaborate the explicit dependence of the constants on s.

Lemma 3.2 *Let Assumption 2.2 be satisfied. The sesquilinear forms ℓ_j are continuous and coercive: for $\mu := s/|s|$ it holds for any $v, w \in H^1(\mathbb{R}^3)$*

$$|\ell_j(s)(v,w)| \leq \Lambda_j \|v\|_{H^1(\mathbb{R}^3);s} \|w\|_{H^1(\mathbb{R}^3);s}, \quad \operatorname{Re} \ell_j(s)(v,\mu v) \geq \lambda_j \tfrac{\operatorname{Re} s}{|s|} \|v\|^2_{H^1(\mathbb{R}^3);s},$$

with

$$\lambda_j := \min\left\{\lambda_j\left(p_j^{\mathrm{ext}}\right), \lambda_j\left(\mathbb{A}_j^{\mathrm{ext}}\right)\right\} \quad \textit{and} \quad \Lambda_j := \max\left\{\Lambda_j\left(p_j^{\mathrm{ext}}\right), \Lambda_j\left(\mathbb{A}_j^{\mathrm{ext}}\right)\right\}. \tag{3.3}$$

[1] We use here the traditional notion of *boundary* integral operators (instead of skeleton operators) since they are defined on the *subdomain* boundaries.

Proof. Fix $\mu = s/|s|$. For $v \in H^1(\mathbb{R}^3)$, it holds

$$\begin{aligned} \operatorname{Re}\ell_j(s)(v,\mu v) &= \operatorname{Re}\left\langle s^2 p_j^{\mathrm{ext}} v, \overline{\mu v}\right\rangle_{\mathbb{R}^3} + \operatorname{Re}\left\langle \mathbb{A}_j^{\mathrm{ext}}\nabla v, \overline{\mu \nabla v}\right\rangle_{\mathbb{R}^3} \\ &\geq \lambda\left(p_j^{\mathrm{ext}}\right)\operatorname{Re}\left(s^2\overline{\mu}\right)\|v\|^2_{L^2(\mathbb{R}^3)} + \lambda\left(\mathbb{A}_j^{\mathrm{ext}}\right)(\operatorname{Re}\mu)\|\nabla v\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \\ &\geq \frac{\operatorname{Re}s}{|s|}\lambda_j\|v\|^2_{H^1(\mathbb{R}^3);s}. \end{aligned} \tag{3.4}$$

To establish continuity, we use

$$\begin{aligned} |\ell_j(s)(v,w)| &= \left|s^2\left\langle p_j^{\mathrm{ext}} v, \overline{w}\right\rangle_{\mathbb{R}^3}\right| + \left|\left\langle \mathbb{A}_j^{\mathrm{ext}}\nabla v, \overline{\nabla w}\right\rangle_{\mathbb{R}^3}\right| \\ &\leq \Lambda\left(p_j^{\mathrm{ext}}\right)|s|^2\|v\|_{L^2(\mathbb{R}^3)}\|w\|_{L^2(\mathbb{R}^3)} + \Lambda\left(\mathbb{A}_j^{\mathrm{ext}}\right)\|\nabla v\|_{L^2(\mathbb{R}^3)}\|\nabla w\|_{L^2(\mathbb{R}^3)} \\ &\leq \Lambda_j\|v\|_{H^1(\mathbb{R}^3);s}\|w\|_{H^1(\mathbb{R}^3);s} \end{aligned}$$

for all $v, w \in H^1(\mathbb{R}^3)$. ■

Since the right-hand side in the first equation of (2.23) is the zero function we conclude that a solution u_j belongs to $H^1\left(\Omega_j^-, \mathbb{A}_j^-\right)$.

Lemma 3.3 (Green's identities) *Let Assumption 2.2 be satisfied and set* $\mathbb{A}_j^\sigma := \mathbb{A}_j^{\mathrm{ext}}\big|_{\Omega_j^\sigma}$, $p_j^\sigma := p_j^{\mathrm{ext}}\big|_{\Omega_j^\sigma}$ *for* $\sigma \in \{+,-\}$.

1. *For any* $\sigma \in \{+,-\}$, *assume that* $v^\sigma \in H^1\left(\Omega_j^\sigma, \mathbb{A}_j^\sigma\right)$ *satisfies*

$$\mathsf{L}_j^\sigma(s)\,v^\sigma = 0 \quad \text{in } \Omega_j^\sigma. \tag{3.5}$$

   Then, the co-normal derivative of v^σ *satisfies*

$$\left\langle \mathbb{A}_j^\sigma \nabla v^\sigma, \nabla\overline{w}\right\rangle_{\Omega_j^\sigma} + s^2\left\langle p_j^\sigma v^\sigma, \overline{w}\right\rangle_{\Omega_j^\sigma} = \left\langle \gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\,v^\sigma, \gamma_{\mathrm{D};j}^\sigma(s)\,\overline{w}\right\rangle_{\Gamma_j} \qquad \forall w \in H^1\left(\Omega_j^\sigma\right). \tag{3.6}$$

2. *For* $v \in H^1(\mathbb{R}^3)$, *set* $v^\sigma := v|_{\Omega_j^\sigma}$. *Assume that* v^σ *belongs to* $H^1\left(\Omega_j^\sigma, \mathbb{A}_j^\sigma\right)$ *and satisfies (3.5) for* $\sigma \in \{+,-\}$. *Then*

$$\ell_j(s)(v,w) = \left\langle -[v]_{\mathrm{N};j}^{\mathrm{ext}}(s), \gamma_{\mathrm{D};j}(s)\,\overline{w}\right\rangle_{\Gamma_j}, \qquad \forall w \in H^1\left(\mathbb{R}^3\right). \tag{3.7}$$

3. *For* $v \in L^2(\mathbb{R}^3)$, *set* $v^\sigma := v|_{\Omega_j^\sigma}$. *Assume* $v^\sigma \in H^1\left(\Omega_j^\sigma, \mathbb{A}_j^\sigma\right)$, $[v]_{\mathrm{N};j}^{\mathrm{ext}} = 0$, *and that* v^σ *satisfies, (3.5). Then*

$$\sum_{\sigma\in\{+,-\}}\left\langle \mathbb{A}_j^\sigma \nabla v^\sigma, \nabla\overline{w^\sigma}\right\rangle_{\Omega_j^\sigma} + s^2\left\langle p_j^\sigma v^\sigma, \overline{w^\sigma}\right\rangle_{\Omega_j^\sigma} = \left\langle \gamma_{\mathrm{N};j}^{\mathrm{ext}}(s)\,v, -[\overline{w}]_{\mathrm{D};j}(s)\right\rangle_{\Gamma_j} \tag{3.8}$$

   for any $w \in L^2(\mathbb{R}^3)$ *with* $w^\sigma := w|_{\Omega_j^\sigma} \in H^1(\Omega^\sigma)$, $\sigma \in \{+,-\}$.

4. *Let* $v^\sigma, w^\sigma \in H^1\left(\Omega_j^\sigma, \mathbb{A}_j^\sigma\right)$. *Then,*

$$\begin{aligned} &\left\langle v^\sigma, \mathsf{L}_j^\sigma(s)\,\overline{w^\sigma}\right\rangle_{\Omega_j^\sigma} - \left\langle \mathsf{L}_j^\sigma(s)\,v^\sigma, \overline{w^\sigma}\right\rangle_{\Omega_j^\sigma} \\ &\qquad = \left\langle \gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\,v^\sigma, \gamma_{\mathrm{D};j}^\sigma(s)\,\overline{w^\sigma}\right\rangle_{\Gamma_j} - \left\langle \gamma_{\mathrm{D};j}^\sigma(s)\,v^\sigma, \gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\left(\overline{w^\sigma}\right)\right\rangle_{\Gamma_j}. \end{aligned} \tag{3.9}$$

Proof. @ 1. For any $v \in H^1\left(\Omega_j^\sigma, \mathbb{A}_j^\sigma\right)$, it holds

$$\left\langle \mathbb{A}_j^\sigma \nabla v, \overline{\nabla w}\right\rangle_{\Omega_j^\sigma} + \left\langle s^2 p_j^\sigma v, \overline{w}\right\rangle_{\Omega_j^\sigma} = \left\langle \mathsf{L}_j^\sigma (s) v, \overline{w}\right\rangle_{\Omega_j^\sigma} + \left\langle \gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s) v, \gamma_{\mathrm{D};j}^\sigma (s) \overline{w}\right\rangle_{\Gamma_j} \stackrel{(3.5)}{=} \left\langle \gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s) v, \gamma_{\mathrm{D};j}^\sigma (s) \overline{w}\right\rangle_{\Gamma_j} \qquad \forall w \in H^1\left(\Omega_j^\sigma\right). \tag{3.10}$$

@ 2. Let $v \in H^1(\mathbb{R}^3)$ and assume v satisfies the conditions in part 2. We conclude from part 1 that

$$\begin{aligned}\ell_j(s)(v,w) &= \sum_{\sigma\in\{+,-\}} \left\langle \mathbb{A}_j^\sigma \nabla v, \overline{\nabla w}\right\rangle_{\Omega_j^\sigma} + \left\langle s^2 p_j^\sigma v, \overline{w}\right\rangle_{\Omega_j^\sigma} \\ &= \left\langle \gamma_{\mathrm{N};j}^{\mathrm{ext},+}(s) v^+ + \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s) v^-, \gamma_{\mathrm{D};j}(s)\overline{w}\right\rangle_{\Gamma_j} = \left\langle -\left[v\right]_{\mathrm{N};j}^{\mathrm{ext}}(s), \gamma_{\mathrm{D};j}(s)\overline{w}\right\rangle_{\Gamma_j}\end{aligned}$$

holds for all $w \in H^1(\mathbb{R}^3)$.

@ 3. The relation (3.8) follows in the same fashion as (3.6).

@ 4. Relation (3.9) follows by integrating by parts the first term in (3.10). ■

3.2 Volume and layer potentials

In this section we define volume and layer potentials as solutions to certain variational formulations of elliptic partial differential equations without relying on the explicit knowledge of the Green's function.

3.2.1 The Newton potential

We will define the acoustic Newton potential as the solution of the variational formulation of a full space partial differential equation depending on a single subdomain Ω_j, corresponding to extended coefficients $\mathbb{A}_j^{\mathrm{ext}}$, p_j^{ext}, and the frequency parameter s.

Definition 3.4 *Let Assumption 2.2 be satisfied. The solution operator (acoustic Newton potential)* $\mathsf{N}_j(s) : H^{-1}(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ *is defined through*

$$\ell_j(s)(\mathsf{N}_j(s) f, w) = \langle f, \overline{w}\rangle_{\mathbb{R}^3} \qquad \forall f \in H^{-1}\left(\mathbb{R}^3\right), \quad \forall w \in H^1\left(\mathbb{R}^3\right). \tag{3.11}$$

Lemma 3.2 implies that $\ell_j(s)$ is continuous and coercive. Hence, the Lax-Milgram theorem ensures that

$$\mathsf{N}_j(s) : H^{-1}\left(\mathbb{R}^3\right) \to H^1\left(\mathbb{R}^3\right) \tag{3.12}$$

is well defined, linear, and bounded. An estimate of the operator norm in frequency dependent norms (see (2.1), (3.13)) is given by the following lemma. Note that the dual space of $\left(H^1(\mathbb{R}^3), \|\cdot\|_{H^1(\mathbb{R}^3);s}\right)$ is given by $\left(H^{-1}(\mathbb{R}^3), \|\cdot\|_{H^{-1}(\mathbb{R}^3);s}\right)$ with dual norm defined by

$$\|f\|_{H^{-1}(\mathbb{R}^3);s} := \sup_{g\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\left|\langle f, \overline{g}\rangle_{\mathbb{R}^3}\right|}{\|g\|_{H^1(\mathbb{R}^3);s}}. \tag{3.13}$$

Lemma 3.5 *Let Assumption 2.2 be satisfied. The Newton potential is an inverse of* $\mathsf{L}_j(s)$, *i.e.,*

$$v = \mathsf{N}_j(s) \circ \mathsf{L}_j(s) v \quad \forall v \in H^1\left(\mathbb{R}^3\right) \quad \text{and} \quad f = \mathsf{L}_j(s) \circ \mathsf{N}_j(s) f \quad \forall f \in H^{-1}\left(\mathbb{R}^3\right); \tag{3.14}$$

it satisfies the estimate

$$\left\| \mathsf{N}_j(s) f \right\|_{H^1(\mathbb{R}^3);s} \leq \frac{|s|}{\lambda_j \operatorname{Re} s} \|f\|_{H^{-1}(\mathbb{R}^3);s} \quad \forall f \in H^{-1}\left(\mathbb{R}^3\right), \tag{3.15}$$

with λ_j *as in (3.3).*

Proof. For $v \in H^1(\mathbb{R}^3)$, we have $\mathsf{L}_j(s) v \in H^{-1}(\mathbb{R}^3)$ and hence the Newton potential can be applied:

$$\ell_j(s)(\mathsf{N}_j(s) \circ \mathsf{L}_j(s) v, w) = \langle \mathsf{L}_j(s) v, \overline{w} \rangle_{\mathbb{R}^3} = \ell_j(s)(v, w) \qquad \forall w \in H^1\left(\mathbb{R}^3\right).$$

Since $\ell_j(s)(\cdot,\cdot)$ is coercive the first identity in (3.14) follows. The second one is a direct consequence of the definition of $\mathsf{N}_j(s)$.

To prove (3.15), we use the coercivity of $\ell_j(s)(\cdot,\cdot)$ with respect to the Hilbert space $\left(H^1(\mathbb{R}^3), \|\cdot\|_{H^1(\mathbb{R}^3);s}\right)$. The dual space with dual norm is $\left(H^{-1}(\mathbb{R}^3), \|\cdot\|_{H^{-1}(\mathbb{R}^3);s}\right)$. From the Babuška-Lax-Milgram theorem [2, Thm. 2.1] and the definition (3.13) of the dual norm the assertion follows. ■

3.2.2 The single layer potential

The single layer potential is defined by using the same sesquilinear form as for the Newton potential for a certain type of right-hand sides.

Definition 3.6 *Let Assumption 2.2 be satisfied. For* $1 \leq j \leq n_\Omega$ *and* $\varphi \in H^{-1/2}(\Gamma_j)$ *the* single layer potential $\mathsf{S}_j(s)\varphi \in H^1(\mathbb{R}^3)$ *is given as the unique solution of:*

$$\ell_j(s)(\mathsf{S}_j(s)\varphi, w) = \langle \varphi, \gamma_{\mathrm{D};j}(s)\overline{w} \rangle_{\Gamma_j} \quad \forall w \in H^1\left(\mathbb{R}^3\right). \tag{3.16}$$

This defines a continuous operator $H^{-1/2}(\Gamma_j) \to H^1(\mathbb{R}^3)$. The single layer can be represented as the composition of the Newton potential and the dual Dirichlet trace as can be seen from the next lemma, where also important properties of $\mathsf{S}_j(s)$ are collected.

Lemma 3.7 *Let Assumption 2.2 be satisfied. Then*

$$\mathsf{S}_j(s) = \mathsf{N}_j(s) \circ (\gamma_{\mathrm{D};j}(s))'. \tag{3.17}$$

For any $\varphi \in H^{-1/2}(\Gamma)$, *the single layer potential* $u := \mathsf{S}_j(s)\varphi$ *satisfies* $u \in H^1\left(\mathbb{R}^3 \backslash \Gamma_j, \mathbb{A}_j^{\mathrm{ext}}\right)$. *For the restrictions* $u^\sigma := u|_{\Omega_j^\sigma}$, $\sigma \in \{+,-\}$, *hold*

$$\mathsf{L}_j^\sigma(s) u^\sigma = 0 \quad \text{in } \Omega_j^\sigma \tag{3.18}$$

and the jump relations

$$[(\mathsf{S}_j(s)\varphi)]_{\mathrm{D};j}(s) = 0 \quad , \quad [(\mathsf{S}_j(s)\varphi)]_{\mathrm{N};j}^{\mathrm{ext}}(s) = -\varphi. \tag{3.19}$$

Proof. The representation (3.17) follows by writing (3.16) as

$$\ell_j(s)(\mathsf{S}_j(s)\varphi, w) = \left\langle (\gamma_{\mathrm{D};j}(s))' \varphi, \overline{w} \right\rangle_{\mathbb{R}^3} \quad \forall w \in H^1\left(\mathbb{R}^3\right),$$

so that $\mathsf{S}_j(s)\varphi = \mathsf{N}_j(s)(\gamma_{\mathrm{D};j}(s))'\varphi$. Indeed, the mapping properties of the dual Dirichlet trace $(\gamma_{\mathrm{D};j}(s))' : H^{1/2}(\Gamma_j) \to H^{-1}(\mathbb{R}^3)$ imply that the Newton potential can be applied in (3.17).

For $\varphi \in H^{-1/2}(\Gamma)$, let $u := \mathsf{S}_j(s)\varphi$ and $u^\sigma := u|_{\Omega_j^\sigma}$. By choosing in (3.16) test functions $v \in H^1(\mathbb{R}^3)$ with zero trace on Γ_j we obtain

$$\mathsf{L}_j^\sigma(s)\,u^\sigma = 0 \quad \text{in } \Omega_j^\sigma,\ \sigma \in \{+,-\}.$$

In particular, this implies $u \in H^1\left(\mathbb{R}^3\backslash\Gamma_j, \mathbb{A}_j^{\mathrm{ext}}\right)$. An integration by parts in (3.16) over Ω_j^- and Ω_j^+ leads to

$$-\left\langle [u]_{\mathrm{N};j}^{\mathrm{ext}}(s), \gamma_{\mathrm{D};j}(s)\,\overline{w}\right\rangle_{\Gamma_j} = \langle \varphi, \gamma_{\mathrm{D};j}(s)\,\overline{w}\rangle_{\Gamma_j} \quad \forall w \in H^1\left(\mathbb{R}^3\right).$$

Since $\gamma_{\mathrm{D};j}(s) : H^1(\mathbb{R}^3) \to H^{1/2}(\Gamma_j)$ is surjective (see, e.g., [16, Thm. 3.37], [17, Lem. 2.6]) it follows that $[u]_{\mathrm{N};j}^{\mathrm{ext}}(s) = -\varphi$. Finally, the relation $[u]_{\mathrm{D};j}(s) = 0$ follows from $u \in H^1(\mathbb{R}^3)$ (see, e.g. [16, (6.20)], [17, Lem. 2.5]). ■

3.2.3 The double layer potential

Next, we introduce the double layer potential and start by reviewing some standard definitions as already sketched in the introduction. For problems with constant coefficients as, e.g., in [20, Def. 3.1.5], the double layer potential is defined by

$$\mathsf{D}_j(s) := \mathsf{N}_j(s) \circ \left(\gamma_{\mathrm{N};j}^{\mathrm{ext}}\right)'(s). \tag{3.20}$$

The continuity of the co-normal derivative $\gamma_{\mathrm{N};j}^{\mathrm{ext}} : H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\mathrm{ext}}\right) \to H^{-1/2}(\Gamma_j)$ (see (2.14)) leads to the continuity of its dual $\left(\gamma_{\mathrm{N};j}^{\mathrm{ext}}\right)' : H^{1/2}(\Gamma_j) \to \left(H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\mathrm{ext}}\right)\right)'$. The problem with (3.20) is that the image space $\left(H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\mathrm{ext}}\right)\right)'$ in general is larger than $H^{-1}(\mathbb{R}^3)$ and hence exceeds the domain of $\mathsf{N}_j(s)$ in (3.20). The extension of the domain of $\mathsf{N}_j(s)$ for problems with varying coefficients is far from trivial. Another common definition uses explicit knowledge of the fundamental solution $G(\mathbf{x}, \mathbf{y})$ and first defines

$$(\mathsf{D}_j(s)\,\psi)(\mathbf{x}) := \int_{\Gamma_j} \left(\frac{\partial}{\partial \tilde{\mathbf{n}}_{\mathbf{y}}} G(\mathbf{x}, \mathbf{y})\right) \psi(\mathbf{y})\, d\Gamma_{\mathbf{y}} \quad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma_j$$

($\partial/\partial\tilde{\mathbf{n}}_{\mathbf{y}}$ with $\tilde{\mathbf{n}}_{\mathbf{y}} := \mathbb{A}_j^{\mathrm{ext}}\mathbf{n}_j$ denotes the co-normal derivative with respect to $\mathbf{y}$) for coefficients $\mathbb{A}_j^{\mathrm{ext}}$ and boundary densities $\psi : \Gamma_j \to \mathbb{C}$, which are sufficiently regular, and then continuously extends this definition to appropriate Sobolev spaces. However, the derivation of mapping properties of $\mathsf{D}_j(s)$ via this approach relies on properties of the unknown fundamental solution and is far from trivial for problems with L^∞ coefficients.

Instead, we present here a new definition of the double layer potential as a solution of some ultra-weak variational problem which allows us to derive properties of these potentials directly from the well-established theory of linear elliptic partial differential operators of second order.

For the definition of the double layer potential we introduce two auxiliary variational problems.

I. *Ultra-weak variational problem (UWVP)*: Given $\psi \in H^{1/2}(\Gamma_j)$, find $u \in L^2(\mathbb{R}^3)$ such that

$$\langle u, \mathsf{L}_j(s)\,\overline{v}\rangle_{\mathbb{R}^3} = \left\langle \psi, \gamma_{\mathrm{N};j}^{\mathrm{ext}}(s)\,\overline{v}\right\rangle_{\Gamma_j} \qquad \forall v \in H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\mathrm{ext}}\right). \tag{3.21}$$

II. *Mixed variational problem (MVP).* For given $\psi \in H^{1/2}(\Gamma_j)$, find $\mathbf{j} \in \mathbf{H}(\mathbb{R}^3, \operatorname{div})$ and $u \in L^2(\mathbb{R}^3)$ such that

$$\begin{aligned} -\left\langle \left(\mathbb{A}_j^{\text{ext}}\right)^{-1} \mathbf{j}, \overline{\mathbf{m}} \right\rangle_{\mathbb{R}^3} & -\langle u, \operatorname{div} \overline{\mathbf{m}}\rangle_{\mathbb{R}^3} &= \langle \psi, \gamma_{\mathbf{n};j}(s)\, \overline{\mathbf{m}}\rangle_{\Gamma_j} \quad & \forall \mathbf{m} \in \mathbf{H}(\mathbb{R}^3, \operatorname{div}), \\ -\langle \operatorname{div} \mathbf{j}, \overline{q}\rangle_{\mathbb{R}^3} & + s^2 \left\langle p_j^{\text{ext}} u, \overline{q}\right\rangle_{\mathbb{R}^3} &= 0 \quad & \forall q \in L^2(\mathbb{R}^3). \end{aligned} \tag{3.22}$$

In Lemmas 3.8 and 3.9 we will prove that the variational problems (3.9) and (3.22) are well posed.

Lemma 3.8 *Let Assumption 2.2 be satisfied. The ultra-weak variation problem (3.21) is well posed.*

Proof. We will show that there exist constants $0 < C_1, C_2, c_1 < \infty$ such that the continuity estimates

$$\forall u \in L^2(\mathbb{R}^3),\ v \in H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right) \quad \left|\langle u, \mathsf{L}_j(s)\, \overline{v}\rangle_{\mathbb{R}^3}\right| \leq C_1 \|u\|_{L^2(\mathbb{R}^3)} \|v\|_{H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right)}, \tag{3.23a}$$

$$\forall \psi \in H^{1/2}(\Gamma),\ v \in H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right) \quad \left|\left\langle \psi, \gamma_{\mathrm{N};j}^{\text{ext}}(s)\, \overline{v}\right\rangle_{\Gamma_J}\right| \leq C_2 \,|||\psi\|_{H^{1/2}(\Gamma)} \|v\|_{H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right)}. \tag{3.23b}$$

and the following inf-sup conditions hold:

$$\forall u \in L^2(\mathbb{R}^3) \quad \exists v \in H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right) \qquad \left|\langle u, \mathsf{L}_j(s)\, \overline{v}\rangle_{\mathbb{R}^3}\right| \geq c_1 \|u\|_{L^2(\mathbb{R}^3)} \|v\|_{H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right)}, \tag{3.23c}$$

$$\forall v \in H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right) \qquad \left(\sup_{u \in L^2(\mathbb{R}^3)} \left|\langle u, \mathsf{L}_j(s)\, \overline{v}\rangle_{\mathbb{R}^3}\right| = 0\right) \implies (v = 0). \tag{3.23d}$$

The Babuška-Lax-Milgram theorem (also sometimes called Banach-Nečas-Babuška theorem) (see, e.g., [2, Thm. 2.1] and, e.g., [12, Thm. 25.9] for the form we will apply it) then implies well-posedness of (3.21).

@(3.23a). The continuity of the sesquilinear form in (3.21) follows from

$$\begin{aligned} \left|\langle u, \mathsf{L}_j(s)\, \overline{v}\rangle_{\mathbb{R}^3}\right| &\leq \|u\|_{L^2(\mathbb{R}^3)} \|\mathsf{L}_j(s)\, \overline{v}\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{L^2(\mathbb{R}^3)} \left\|-\operatorname{div}\left(\mathbb{A}_j^{\text{ext}} \nabla \overline{v}\right) + s^2 p_j^{\text{ext}} \overline{v}\right\|_{L^2(\mathbb{R}^3)} \\ &\leq \sqrt{2}\, \|u\|_{L^2(\mathbb{R}^3)} \left(\left\|\operatorname{div}\left(\mathbb{A}_j^{\text{ext}} \nabla \overline{v}\right)\right\|_{L^2(\mathbb{R}^3)}^2 + |s|^4 \Lambda_j^2 \|v\|_{L^2(\mathbb{R}^3)}^2\right)^{1/2} \\ &\leq C_1 \|u\|_{L^2(\mathbb{R}^3)} \|v\|_{H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right)} \end{aligned}$$

for $C_1 = \sqrt{2} \max\left\{1, |s|^2 \Lambda_j\right\}$.

@(3.23b). It is a simple consequence of the mapping properties of the trace operator that the right-hand side in (3.21) $\left\langle \psi, \gamma_{\mathrm{N};j}^{\text{ext}}(s)\, \bar{\cdot}\right\rangle_{\Gamma_j}$ defines a continuous functional on $H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\text{ext}}\right)$ so that (3.23b) follows.

@(3.23c). We choose the test function in (3.21) as $v \leftarrow \mathsf{N}_j(\overline{s})\, u$. It is easy to deduce from Definition 3.4 that $\overline{\mathsf{N}_j(\overline{s})\, u} = \mathsf{N}_j(s)\, \overline{u}$ holds so that

$$\left\langle u, \mathsf{L}_j(s)\, \overline{\mathsf{N}_j(\overline{s})\, u}\right\rangle_{\mathbb{R}^3} = \langle u, \mathsf{L}_j(s)\, \mathsf{N}_j(s)\, \overline{u}\rangle_{\mathbb{R}^3} = \|u\|_{L^2(\mathbb{R}^3)}^2.$$

Hence, the inf-sup constant for problem (3.21) can be estimated from below by

$$\inf_{u\in L^2(\mathbb{R}^3)\backslash\{0\}} \sup_{v\in H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)\backslash\{0\}} \frac{\langle u, \mathsf{L}_j(s)\,\overline{v}\rangle_{\mathbb{R}^3}}{\|u\|_{L^2(\mathbb{R}^3)}\|v\|_{H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)}} \geq \inf_{u\in L^2(\mathbb{R}^3)\backslash\{0\}} \frac{\|u\|_{L^2(\mathbb{R}^3)}}{\|\mathsf{N}_j(\overline{s})\,u\|_{H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)}}.$$

We estimate the denominator by

$$\begin{aligned}
\|\mathsf{N}_j(\overline{s})\,u\|^2_{H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)} &= \left\|\operatorname{div}\left(\mathbb{A}_j^{\mathrm{ext}}\nabla\mathsf{N}_j(\overline{s})\,u\right)\right\|^2_{L^2(\mathbb{R}^3)} + \|\mathsf{N}_j(\overline{s})\,u\|^2_{H^1(\mathbb{R}^3)} \\
&= \left\|\mathsf{L}_j(\overline{s})\,\mathsf{N}_j(\overline{s})\,u - \overline{s}^2 p_j^{\mathrm{ext}}\mathsf{N}_j(\overline{s})\,u\right\|^2_{L^2(\mathbb{R}^3)} + \|\mathsf{N}_j(\overline{s})\,u\|^2_{H^1(\mathbb{R}^3)} \\
&\leq 2\,\|\mathsf{L}_j(\overline{s})\,\mathsf{N}_j(\overline{s})\,u\|^2_{L^2(\mathbb{R}^3)} + 2\,|s|^4\,\Lambda_j^2\,\|\mathsf{N}_j(\overline{s})\,u\|^2_{L^2(\mathbb{R}^3)} + \|\mathsf{N}_j(\overline{s})\,u\|^2_{H^1(\mathbb{R}^3)} \\
&\leq 2\,\|u\|^2_{L^2(\mathbb{R}^3)} + |s|^2\,C_0^2\,\|\mathsf{N}_j(\overline{s})\,u\|^2_{H^1(\mathbb{R}^3);s}
\end{aligned}$$

for $C_0 := \max\left\{\sqrt{2\Lambda_j^2 + \frac{1}{s_0^4}}, s_0^{-1}\right\}$. From (3.15) we get

$$\begin{aligned}
\|\mathsf{N}_j(\overline{s})\,u\|_{H^1(\mathbb{R}^3);s} &\leq \frac{|s|}{\lambda_j \operatorname{Re} s}\|u\|_{H^{-1}(\mathbb{R}^3);s} \leq \frac{|s|}{\lambda_j \operatorname{Re} s}\left(\sup_{g\in H^1(\mathbb{R}^3)\backslash\{0\}} \frac{\|g\|_{L^2(\mathbb{R}^3)}}{\|g\|_{H^1(\mathbb{R}^3);s}}\right)\|u\|_{L^2(\mathbb{R}^3)} \\
&\leq \frac{1}{\lambda_j \operatorname{Re} s}\|u\|_{L^2(\mathbb{R}^3)}
\end{aligned}$$

and, in turn,

$$\|\mathsf{N}_j(\overline{s})\,u\|_{H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)} \leq \left(2 + \frac{C_0^2\,|s|^2}{\lambda_j^2\,(\operatorname{Re} s)^2}\right)^{1/2}\|u\|_{L^2(\mathbb{R}^3)}.$$

The combination of these estimates leads to the inf-sup estimate

$$\inf_{u\in L^2(\mathbb{R}^3)\backslash\{0\}} \sup_{v\in H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)\backslash\{0\}} \frac{\langle u, \mathsf{L}_j(s)\,\overline{v}\rangle_{\mathbb{R}^3}}{\|u\|_{L^2(\mathbb{R}^3)}\|v\|_{H^1\left(\mathbb{R}^3,\mathbb{A}_j^{\mathrm{ext}}\right)}} \geq c_1\frac{\operatorname{Re} s}{|s|},$$

where $c_1 > 0$ only depends on $\lambda_j, \Lambda_j, s_0$.

@(3.23d). We choose $u = \overline{\mathsf{L}_j(s)v}$ and obtain

$$\sup_{u\in L^2(\mathbb{R}^3)} \left|\langle u, \mathsf{L}_j(s)\,\overline{v}\rangle_{\mathbb{R}^3}\right| \geq \left|\left\langle\overline{\mathsf{L}_j(s)}v, \mathsf{L}_j(s)\,\overline{v}\right\rangle_{\mathbb{R}^3}\right| = \|\mathsf{L}_j(s)\,\overline{v}\|^2_{L^2(\mathbb{R}^3)}. \tag{3.24}$$

Since $\mathsf{L}_j(s) : H^1(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3)$ is an isomorphism (see (3.12)), the implication $\mathsf{L}_j(s)\,\overline{v} = 0 \implies v = 0$ holds for all $v \in H^1(\mathbb{R}^3)$. Since $H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\mathrm{ext}}\right) \subset H^1(\mathbb{R}^3)$ we conclude from (3.24) that (3.23d) holds. ■

Lemma 3.9 *Let Assumption 2.2 be satisfied. The mixed variational problem (3.22) is well posed.*

Proof. Again, we employ the Babuška-Lax-Milgram theorem and prove the relevant properties for the sesquilinear form and anti-linear form associated with (3.22). The sesquilinear form $b : (\mathbf{H}(\mathbb{R}^3, \operatorname{div}), L^2(\mathbb{R}^3)) \times (\mathbf{H}(\mathbb{R}^3, \operatorname{div}), L^2(\mathbb{R}^3)) \to \mathbb{C}$ related to the mixed variational problem (3.22) is given by

$$b((\mathbf{j}, u), (\mathbf{m}, v)) := -\left\langle\left(\mathbb{A}_j^{\mathrm{ext}}\right)^{-1}\mathbf{j}, \overline{\mathbf{m}}\right\rangle_{\mathbb{R}^3} - \langle u, \operatorname{div}\overline{\mathbf{m}}\rangle_{\mathbb{R}^3} - \langle\operatorname{div}\mathbf{j}, \overline{v}\rangle_{\mathbb{R}^3} + s^2\left\langle p_j^{\mathrm{ext}}u, \overline{v}\right\rangle_{\mathbb{R}^3}.$$

The anti-linear form associated to the right-hand side is $f : (\mathbf{H}(\mathbb{R}^3, \operatorname{div}), L^2(\mathbb{R}^3)) \to \mathbb{C}$

$$f((\mathbf{m}, v)) := \langle \psi, \gamma_{\mathbf{n};j}(s)\,\overline{\mathbf{m}} \rangle_{\Gamma_j}.$$

We will verify the four conditions for the Babuška-Lax-Milgram theorem. The continuity of b follows by straightforward Cauchy-Schwarz inequalities. For the analogue of (3.23c) we choose

$$v \leftarrow \frac{s}{|s|^3} u - \frac{s}{|s|^3} \frac{1}{p_j^{\mathrm{ext}}} \operatorname{div} \mathbf{j} \quad \text{and} \quad \mathbf{m} \leftarrow -\frac{\bar{s}}{|s|} \left(1 + \frac{1}{|s|^2}\right) \mathbf{j} \tag{3.25}$$

and obtain after some straightforward manipulations

$$\begin{aligned} b((\mathbf{j}, u), (\mathbf{m}, v)) = {} & \frac{s}{|s|}\left(1 + \frac{1}{|s|^2}\right) \left\langle \left(\mathbb{A}_j^{\mathrm{ext}}\right)^{-1} \mathbf{j}, \bar{\mathbf{j}} \right\rangle_{\mathbb{R}^3} + \frac{\bar{s}}{|s|^3} \left\langle \frac{1}{p_j^{\mathrm{ext}}} \operatorname{div} \mathbf{j}, \overline{\operatorname{div} \mathbf{j}} \right\rangle_{\mathbb{R}^3} \\ & + 2\,\mathrm{i}\,\mathrm{Im}\left(\frac{s}{|s|^3} \langle u, \operatorname{div} \bar{\mathbf{j}} \rangle_{\mathbb{R}^3}\right) + \frac{s}{|s|} \langle p_j^{\mathrm{ext}} u, \bar{u} \rangle_{\mathbb{R}^3}. \end{aligned}$$

Hence,

$$\begin{aligned} |b((\mathbf{j}, u), (\mathbf{m}, v))| & \geq \operatorname{Re} b((\mathbf{j}, u), (\mathbf{m}, v)) \\ & \geq \frac{\operatorname{Re} s}{\Lambda_j |s|}\left(1 + \frac{1}{|s|^2}\right) \|\mathbf{j}\|^2_{\mathbf{L}^2(\mathbb{R}^3)} + \frac{\operatorname{Re} s}{\Lambda_j |s|^3} \|\operatorname{div} \mathbf{j}\|^2_{L^2(\mathbb{R}^3)} + \frac{\operatorname{Re} s}{|s|} \lambda_j \|u\|^2_{L^2(\mathbb{R}^3)}. \end{aligned}$$

From this, the estimate

$$|b((\mathbf{j}, u), (\mathbf{m}, v))| \geq \frac{\operatorname{Re} s}{|s|^3} \min\left\{\frac{1}{\Lambda_j}, s_0^2 \lambda_j\right\} \left(\|\mathbf{j}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})} + \|u\|^2_{L^2(\mathbb{R}^3)}\right)$$

follows. The choice (3.25) can be bounded by

$$\begin{aligned} \|\mathbf{m}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})} + \|v\|^2_{L^2(\mathbb{R}^3)} & \leq \left(1 + \frac{1}{|s|^2}\right)^2 \|\mathbf{j}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})} + \frac{2}{|s|^4}\left(\|u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\lambda_j^2} \|\mathbf{j}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})}\right) \\ & \leq C_0 \left(\|\mathbf{j}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})} + \|u\|^2_{L^2(\mathbb{R}^3)}\right) \end{aligned}$$

for a positive constant C_0 which depends solely on s_0 and λ_j. This leads to

$$|b((\mathbf{j}, u), (\mathbf{m}, v))| \geq c_1 \left(\|\mathbf{j}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})} + \|u\|^2_{L^2(\mathbb{R}^3)}\right)^{1/2} \left(\|\mathbf{m}\|^2_{\mathbf{H}(\mathbb{R}^3, \operatorname{div})} + \|v\|^2_{L^2(\mathbb{R}^3)}\right)^{1/2}.$$

Next, we prove the analogue of (3.23d). Let $(\mathbf{m}, v) \in (\mathbf{H}(\mathbb{R}^3, \operatorname{div}), L^2(\mathbb{R}^3))$ and assume

$$\forall (\mathbf{j}, u) \in \left(\mathbf{H}\left(\mathbb{R}^3, \operatorname{div}\right), L^2\left(\mathbb{R}^3\right)\right) \qquad b((\mathbf{j}, u), (\mathbf{m}, v)) = 0. \tag{3.26}$$

The analogous choice to (3.25) for the primal variables $(\mathbf{j}, u)$ is

$$u \leftarrow \frac{\bar{s}}{|s|^3} v - \frac{\bar{s}}{|s|^3} \frac{1}{p_j^{\mathrm{ext}}} \operatorname{div} \mathbf{m} \quad \text{and} \quad \mathbf{j} = -\frac{s}{|s|}\left(1 + \frac{1}{|s|^2}\right) \mathbf{m}$$

and we obtain in the same way as before

$$b\left(\left(\mathbf{j},u\right),\left(\mathbf{m},v\right)\right)=\frac{s}{|s|}\left(1+\frac{1}{|s|^{2}}\right)\left\langle\left(\mathbb{A}_{j}^{\text{ext}}\right)^{-1}\mathbf{m},\overline{\mathbf{m}}\right\rangle_{\mathbb{R}^{3}}+\frac{\bar{s}}{|s|^{3}}\left\langle\frac{1}{p_{j}^{\text{ext}}}\operatorname{div}\mathbf{m},\operatorname{div}\overline{\mathbf{m}}\right\rangle_{\mathbb{R}^{3}}$$
$$+2\,\mathrm{i}\operatorname{Im}\left(\frac{s}{|s|^{3}}\left\langle\operatorname{div}\mathbf{m},\overline{v}\right\rangle_{\mathbb{R}^{3}}\right)$$
$$+\frac{s}{|s|}\left\langle p_{j}^{\text{ext}}v,\overline{v}\right\rangle_{\mathbb{R}^{3}}.$$

For the real part the estimate

$$\operatorname{Re}b\left(\left(\mathbf{j},u\right),\left(\mathbf{m},v\right)\right)\geq\frac{\operatorname{Re}s}{|s|^{3}}\min\left\{\frac{1}{\Lambda_{\min}},s_{0}^{2}\lambda_{\min}\right\}\left(\|\mathbf{m}\|_{\mathbf{H}(\mathbb{R}^{3},\operatorname{div})}^{2}+\|v\|_{L^{2}(\mathbb{R}^{3})}^{2}\right)$$

follows. In view of (3.26), $(\mathbf{m},v)=(\mathbf{0},0)$ follows.

The continuity of the anti-linear form f follows by combining a Cauchy-Schwarz inequality

$$\left|f\left(\left(\mathbf{m},v\right)\right)\right|\leq\|\psi\|_{H^{1/2}(\Gamma_{j})}|s|^{-1/2}\left\|\gamma_{\mathbf{n};j}(s)\left(\mathbf{m}\right)\right\|_{H^{-1/2}(\Gamma_{j})}$$

with the estimate (2.12) for the normal trace. ■

The next lemma states an equivalence of the solutions of (3.21) and (3.22).

Lemma 3.10 *Let Assumption 2.2 be satisfied. The mixed variational problem (3.22) and the ultra-weak variational problem (3.21) are equivalent:*

1. *If $(\mathbf{j},u)\in\left(\mathbf{H}\left(\mathbb{R}^{3},\operatorname{div}\right),L^{2}\left(\mathbb{R}^{3}\right)\right)$ is the solution of (3.22), then u solves (3.21).*
2. *If u is the solution of (3.21), then the pair $(\mathbf{j},u):=\left(\mathbb{A}_{j}^{\text{ext}}\nabla_{\text{pw};j}u,u\right)$ solves (3.22). In particular, it holds $\mathbf{j}\in\mathbf{H}\left(\mathbb{R}^{3},\operatorname{div}\right)$.*
3. *The solution u of the ultra-weak variational problem satisfies the jump relation*

$$[u]_{\mathrm{D};j}(s)=\psi. \tag{3.27}$$

Proof. Part 1.

Let $(\mathbf{j},u)\in\left(\mathbf{H}\left(\mathbb{R}^{3},\operatorname{div}\right),L^{2}\left(\mathbb{R}^{3}\right)\right)$ be the solution of (3.22). We test the first equation in (3.22) with $\mathbf{m}:=\mathbb{A}_{j}^{\text{ext}}\nabla v$ for $v\in H^{1}\left(\mathbb{R}^{3},\mathbb{A}_{j}^{\text{ext}}\right)$. Clearly, $\mathbf{m}\in\mathbf{H}\left(\mathbb{R}^{3},\operatorname{div}\right)$ is an admissible test function. This leads to

$$-\left\langle\mathbf{j},\nabla\overline{v}\right\rangle_{\mathbb{R}^{3}}-\left\langle u,\operatorname{div}\left(\mathbb{A}_{j}^{\text{ext}}\nabla\overline{v}\right)\right\rangle_{\mathbb{R}^{3}}=\left\langle\psi,\gamma_{\mathrm{N};j}^{\text{ext}}(s)\,\overline{v}\right\rangle_{\Gamma_{j}}\quad\forall v\in H^{1}\left(\mathbb{R}^{3},\mathbb{A}_{j}^{\text{ext}}\right).$$

Next we test the second equation in (3.22) with $q\in H^{1}\left(\mathbb{R}^{3},\mathbb{A}_{j}^{\text{ext}}\right)$ and integrate by parts

$$\left\langle\mathbf{j},\nabla\overline{q}\right\rangle_{\mathbb{R}^{3}}+s^{2}\left\langle p_{j}^{\text{ext}}u,\overline{q}\right\rangle_{\mathbb{R}^{3}}=0\quad\forall q\in H^{1}\left(\mathbb{R}^{3}.\mathbb{A}_{j}^{[ext]}\right).$$

We set $q=v$ and sum both equations, which yields

$$\left\langle u,\mathsf{L}_{j}(s)\,\overline{v}\right\rangle_{\mathbb{R}^{3}}=\left\langle\psi,\gamma_{\mathrm{N};j}^{\text{ext}}(s)\,\overline{v}\right\rangle_{\Gamma_{j}}\quad\forall v\in H^{1}\left(\mathbb{R}^{3},\mathbb{A}_{j}^{\text{ext}}\right).$$

Hence, the solution u of the mixed variational problem (3.22) solves the ultra-weak problem (3.21). Lemma 3.8 implies uniqueness of solutions of (3.22) so that u is the unique solution of (3.21).

Now, we test the first equation in (3.22) with functions $\mathbf{m} \in \mathbf{C}_0^\infty(\mathbb{R}^3)$ satisfying $\operatorname{supp}(\mathbf{m}) \subset\subset \Omega_j^\sigma$ for some $\sigma \in \{+,-\}$. This leads to $\nabla_{\mathrm{pw};j} u = \left(\mathbb{A}_j^{\mathrm{ext}}\right)^{-1} \mathbf{j} \in \mathbf{L}^2(\mathbb{R}^3)$ and, in turn, to $u \in H^1(\mathbb{R}^3 \backslash \Gamma_j)$.

Part 2.

Lemma 3.8 and 3.9 imply the existence and uniqueness of solutions for the variational problems (3.21) and (3.22). For $\psi \in H^{1/2}(\Gamma_j)$, let u_{uw} denote the solution of (3.21) and $(\mathbf{j}_{\mathrm{m}}, u_{\mathrm{m}})$ the solution of (3.22). Part 1 implies that $u_{\mathrm{m}} \in H^1(\mathbb{R}^3 \backslash \Gamma_j)$ solves the ultra-weak problem so that $u_{\mathrm{uw}} = u_{\mathrm{m}}$. Vice versa, u_{uw} equals the u_{m}-component of the solution for the mixed variational problem. We test the first equation in (3.22) with test functions $\mathbf{m} \in \mathbf{H}(\mathbb{R}^3, \operatorname{div})$ with compact support in $\Omega_j^- \cup \Omega_j^+$ and obtain by integration by parts

$$\mathbf{j}_{\mathrm{m}} = \mathbb{A}_j^{\mathrm{ext}} \nabla_{\mathrm{pw};j} u_{\mathrm{m}} = \mathbb{A}_j^{\mathrm{ext}} \nabla_{\mathrm{pw};j} u_{\mathrm{uw}}.$$

Since $\mathbf{j}_{\mathrm{m}} \in \mathbf{H}(\mathbb{R}^3, \operatorname{div})$ it follows that $\left(\mathbb{A}_j^{\mathrm{ext}} \nabla_{\mathrm{pw};j} u_{\mathrm{uw}}, u_{\mathrm{uw}}\right) \in \mathbf{H}(\mathbb{R}^3, \operatorname{div}) \times L^2(\mathbb{R}^3)$ solves the mixed variational problem.

Part 3.

We consider the first equation of the mixed problem (3.22) and employ $\mathbf{j} = \mathbb{A}_j^{\mathrm{ext}} \nabla_{\mathrm{pw};j} u$. Integration by parts in each subdomain yields

$$\begin{aligned} \langle \psi, \gamma_{\mathbf{n};j}(s)\, \overline{\mathbf{m}} \rangle_{\Gamma_j} &= -\left\langle \left(\mathbb{A}_j^{\mathrm{ext}}\right)^{-1} \mathbf{j}, \overline{\mathbf{m}} \right\rangle_{\mathbb{R}^3} - \langle u, \operatorname{div} \overline{\mathbf{m}} \rangle_{\mathbb{R}^3} = -\langle \nabla_{\mathrm{pw};j} u, \overline{\mathbf{m}} \rangle_{\mathbb{R}^3} - \langle u, \operatorname{div} \overline{\mathbf{m}} \rangle_{\mathbb{R}^3} \\ &= -\langle \nabla_{\mathrm{pw};j} u, \overline{\mathbf{m}} \rangle_{\mathbb{R}^3} + \langle \nabla u, \overline{\mathbf{m}} \rangle_{\mathbb{R}^3 \backslash \Gamma_j} + \left\langle [u]_{\mathrm{D};j}(s), \gamma_{\mathbf{n};j}(s)(\overline{\mathbf{m}}) \right\rangle_{\Gamma_j} \\ &= \left\langle [u]_{\mathrm{D};j}(s), \gamma_{\mathbf{n};j}(s)(\overline{\mathbf{m}}) \right\rangle_{\Gamma_j} \qquad \forall \mathbf{m} \in \mathbf{H}\left(\mathbb{R}^3, \operatorname{div}\right). \end{aligned} \tag{3.28}$$

The range of the normal trace is $H^{-1/2}(\Gamma_j) = \gamma_{\mathbf{n};j}(s)(\mathbf{H}(\mathbb{R}^3, \operatorname{div}))$ (cf. [13, Cor. 2.8]) so that the jump relation (3.27) follows from (3.28). ■

The well-posedness of the ultra-weak variational problem allows us to define the double layer potential as its solution.

Definition 3.11 *Let Assumption 2.2 be satisfied. For* $1 \leq j \leq n_\Omega$ *and* $\psi \in H^{1/2}(\Gamma_j)$ *the* double layer potential $\mathsf{D}_j(s)\, \psi \in L^2(\mathbb{R}^3)$ *is given as the unique solution of the ultra-weak variational problem*

$$\langle \mathsf{D}_j(s)\, \psi, \mathsf{L}_j(s)\, \overline{v} \rangle_{\mathbb{R}^3} = \left\langle \psi, \gamma_{\mathrm{N};j}^{\mathrm{ext}}(s)\, \overline{v} \right\rangle_{\Gamma_j} \qquad \forall v \in H^1\left(\mathbb{R}^3, \mathbb{A}_j^{\mathrm{ext}}\right). \tag{3.29}$$

Remark 3.12 *Note that our definition (3.29) has the same form as formula (4.7) in [10]. However, we employ this directly as the definition while, in [10] (where the coefficients are assumed to be infinitely smooth) a different definition is used and (3.29) is deduced as an intermediate step within the proof of the jump relations.*

In the following lemma, important properties of $\mathsf{D}_j(s)$ are collected which are well-known, e.g., for PDEs with piecewise constant coefficients.

Lemma 3.13 *Let Assumption 2.2 be satisfied. For $\psi \in H^{1/2}(\Gamma)$, the double layer potential $w := \mathsf{D}_j(s)\psi$ satisfies $w \in H^1(\mathbb{R}^3 \backslash \Gamma_j, \mathbb{A}_j^{\mathrm{ext}})$, the restrictions $w^\sigma := w|_{\Omega_j^\sigma}$ solve the homogeneous equations:*

$$\mathsf{L}_j^\sigma(s)\, w^\sigma = 0 \quad \text{in } \Omega_j^\sigma,\ \sigma \in \{+,-\}, \tag{3.30}$$

and the jump relations hold:

$$[(\mathsf{D}_j(s)\psi)]_{\mathrm{D};j}(s) = \psi, \quad [(\mathsf{D}_j(s)\psi)]_{\mathrm{N};j}^{\mathrm{ext}}(s) = 0. \tag{3.31}$$

In fact, the double layer potential is a continuous operator $\mathsf{D}_j : H^{1/2}(\Gamma_j) \to H^1(\mathbb{R}^3 \setminus \Gamma_j, \mathbb{A}_j^{\mathrm{ext}})$.

Proof. From Lemma 3.10 we conclude that the pair $(\mathbf{j}, w)$ with $\mathbf{j} := \mathbb{A}_j^{\mathrm{ext}} \nabla_{\mathrm{pw};j} w$ solves the mixed variational formulation (3.22). We insert this into the second equation of (3.22) and test with functions $q \in L^2(\mathbb{R}^3)$ which vanish in a neighborhood of Γ_j. From Lemma 3.10(2) it follows $w \in H^1(\mathbb{R}^3 \backslash \Gamma_j, \mathbb{A}_j^{\mathrm{ext}})$ and w satisfies (3.30). Again from Lemma 3.10 it follows $\mathbf{j} \in \mathbf{H}(\mathbb{R}^3, \mathrm{div})$ so that $[\langle \mathbf{j}, \mathbf{n}_j \rangle]_{\mathrm{D};j} = 0$. We conclude $[(\mathsf{D}_j(s)\psi)]_{\mathrm{N};j}^{\mathrm{ext}}(s) = 0$. Finally, we insert $\mathbf{j}$ into the first equation and substitute $u \leftarrow w$. Integrating by parts over Ω_j^- and Ω_j^+ leads to

$$\left\langle [w]_{\mathrm{D};j}(s), \gamma_{\mathbf{n};j}(s)\,\overline{\mathbf{m}} \right\rangle_{\mathbb{R}^3} = \langle \psi, \gamma_{\mathbf{n};j}(s)\,\overline{\mathbf{m}} \rangle_{\Gamma_j} \quad \forall \mathbf{m} \in \mathbf{H}(\mathbb{R}^3, \mathrm{div}).$$

Since the mapping $\gamma_{\mathbf{n};j} : \mathbf{H}(\mathbb{R}^3, \mathrm{div}) \to H^{-1/2}(\Gamma_j)$ is surjective (see, e.g., [13, Cor. 2.8]) it follows $[(\mathsf{D}_j(s)\psi)]_{\mathrm{D};j}(s) = \psi$. ■

3.2.4 Layer potential representation formula

The key observation for the transformation of our transmission problem to a non-local skeleton equation is the fact that solutions of the homogeneous PDE can be expressed by Green's representation formula via their Cauchy data by means of layer potentials. We start with some preliminaries. For $\varphi \in H^{-1/2}(\Gamma_j)$ and $\psi \in H^{1/2}(\Gamma_j)$ we define the potential

$$w := \mathsf{D}_j(s)\psi - \mathsf{S}_j(s)\varphi. \tag{3.32}$$

From Lemmas 3.7 and 3.13 we conclude that $w \in H^1(\mathbb{R}^3 \backslash \Gamma_j, \mathbb{A}_j^{\mathrm{ext}})$ and satisfies

$$\begin{aligned} -\operatorname{div}\left(\mathbb{A}_j^{\mathrm{ext}} \nabla w\right) + s^2 p_j^{\mathrm{ext}} w = 0 \qquad &\text{in } \mathbb{R}^3 \backslash \Gamma_j, \\ [w]_{\mathrm{D};j}(s) = \psi \quad \text{and} \quad & [w]_{\mathrm{N};j}^{\mathrm{ext}}(s) = \varphi. \end{aligned} \tag{3.33}$$

Proposition 3.14 *The transmission problem: "for given $\varphi \in H^{-1/2}(\Gamma_j)$ and $\psi \in H^{1/2}(\Gamma_j)$, find $w \in H^1(\mathbb{R}^3 \backslash \Gamma_j, \mathbb{A}_j^{\mathrm{ext}})$ such that (3.33) holds" is well posed and the unique solution is given by w in (3.32).*

Proof. Existence follows since the potential w in (3.32) defines a solution. For uniqueness, we assume that there are two solutions w_1, w_2 so that the difference $d = w_1 - w_2$ satisfies

$$\begin{aligned} -\operatorname{div}\left(\mathbb{A}_j^{\mathrm{ext}} \nabla d\right) + s^2 p_j^{\mathrm{ext}} d = 0 \quad &\text{in } \mathbb{R}^3 \backslash \Gamma_j, \\ [d]_{\mathrm{D};j}(s) = 0 \quad \text{and} \quad & [d]_{\mathrm{N};j}^{\mathrm{ext}}(s) = 0. \end{aligned}$$

We multiply the first equation by test functions $v \in H^1(\mathbb{R}^3)$ and integrate by parts over Ω_j^- and Ω_j^+. After inserting the transmission conditions we get

$$\ell_j(s)(d,v) = 0 \quad \forall v \in H^1\left(\mathbb{R}^3\right).$$

Since $\ell_j(s)(\cdot,\cdot)$ is coercive (cf. Lem. 3.2)) we conclude that $d = 0$ holds and uniqueness follows. Hence, the potential w in (3.32) defines the unique solution. Since the single and double layer operators are continuous, well-posedness follows. ■

Lemma 3.15 (Green's representation formula) *Let Assumption 2.2 be satisfied. Let* $u^- \in H^1\left(\Omega_j^-, \mathbb{A}_j^-\right)$ *and*

$$\mathsf{L}_j^-(s)\, u^- = 0 \quad \textit{in } \Omega_j^-.$$

Then, the Green's representation formulae *hold*

$$u^- = \left(\mathsf{S}_j(s)\,\gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\,u^- - \mathsf{D}_j(s)\,\gamma_{\mathrm{D};j}^-(s)\,u^-\right)\Big|_{\Omega_j^-}, \tag{3.34a}$$

$$0 = \left(\mathsf{S}_j(s)\,\gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\,u^- - \mathsf{D}_j(s)\,\gamma_{\mathrm{D};j}^-(s)\,u^-\right)\Big|_{\Omega_j^+}. \tag{3.34b}$$

Proof. Define $u \in H^1\left(\mathbb{R}^3\backslash\Gamma_j, \mathbb{A}_j^{\mathrm{ext}}\right)$ by $u|_{\Omega_j^-} := u^-$ and $u|_{\Omega_j^+} := 0$. Clearly

$$-\operatorname{div}\left(\mathbb{A}_j^{\mathrm{ext}}\nabla u\right) + s^2 p_j^{\mathrm{ext}} u = 0 \quad \text{in } \mathbb{R}^3\backslash\Gamma_j$$

and

$$[u]_{\mathrm{D};j}(s) = -\gamma_{\mathrm{D};j}^-(s)\,u^-, \qquad [u]_{\mathrm{N};j}^{\mathrm{ext}}(s) = -\gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\,u^-.$$

From Proposition 3.14 we deduce that the unique solution of this transmission problem can be written in the form

$$u = \mathsf{S}_j(s)\,\gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\,u^- - \mathsf{D}_j(s)\,\gamma_{\mathrm{D};j}^-(s)\,u^-.$$

From this and the definition of u, the representation (3.34) follows. ■

4 Calderón operators

Green's representation formula from Lemma 3.15 expresses homogeneous solutions of a linear, second order, elliptic PDE by means of their Cauchy data on the domain boundary. By applying the Cauchy trace to this formula we obtain the Calderón identity. In this way, Dirichlet and Neumann traces have to be applied to the single layer and double layer potential which give rise to non-local boundary integral operators on the subdomain boundaries.

Definition 4.1 *Let Assumption 2.2 be satisfied. For* $1 \leq j \leq n_\Omega$*, the* single layer boundary integral operator $(\mathsf{V}_j(s))$*, the* double layer boundary integral operator $(\mathsf{K}_j(s))$*, the* dual double layer boundary integral operator $(\mathsf{K}_j'(s))$*, the* hypersingular boundary integral operator $(\mathsf{W}_j(s))$ *are given by*

$$\begin{aligned}
\mathsf{V}_j(s) &: H^{-1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j), & \mathsf{V}_j(s)\,\varphi &:= \{\!\{\mathsf{S}_j(s)\,\varphi\}\!\}_{\mathrm{D};j}(s),\\
\mathsf{K}_j(s) &: H^{1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j), & \mathsf{K}_j(s)\,\psi &:= \{\!\{\mathsf{D}_j(s)\,\psi\}\!\}_{\mathrm{D};j}(s),\\
\mathsf{K}_j'(s) &: H^{-1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j), & \mathsf{K}_j'(s)\,\varphi &:= \{\!\{\mathsf{S}_j(s)\,\varphi\}\!\}_{\mathrm{N};j}^{\mathrm{ext}}(s),\\
\mathsf{W}_j(s) &: H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j), & \mathsf{W}_j(s)\,\psi &:= -\{\!\{\mathsf{D}_j(s)\,\psi\}\!\}_{\mathrm{N};j}^{\mathrm{ext}}(s),
\end{aligned}$$

for all $\varphi \in H^{-1/2}(\Gamma_j)$ *and* $\psi \in H^{1/2}(\Gamma_j)$.

In order to define the Calderón operator we introduce a bilinear form on the multi trace spaces (cf. Def. 2.8) and set, for $\boldsymbol{\phi}_j = (\phi_{\mathrm{D};j}, \phi_{\mathrm{N};j}) \in \mathbf{X}_j$ and $\boldsymbol{\psi}_j = (\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j}) \in \mathbf{X}_j$,

$$\left\langle \boldsymbol{\phi}_j, \boldsymbol{\psi}_j \right\rangle_{\mathbf{X}_j} := \langle \phi_{\mathrm{D};j}, \psi_{\mathrm{N};j} \rangle_{\Gamma_j} + \langle \psi_{\mathrm{D};j}, \phi_{\mathrm{N};j} \rangle_{\Gamma_j}, \tag{4.1a}$$

where, again, $\langle \cdot, \cdot \rangle_{\Gamma_j}$ designates the pairing between $H^{1/2}(\Gamma_j)$ and $H^{-1/2}(\Gamma_j)$. For $\boldsymbol{\phi} = \left(\boldsymbol{\phi}_j\right)_{j=1}^{n_\Omega} \in \mathbb{X}(\mathcal{P}_\Omega)$ and $\boldsymbol{\psi} = \left(\boldsymbol{\psi}_j\right)_{j=1}^{n_\Omega} \in \mathbb{X}(\mathcal{P}_\Omega)$ we define the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{X}(\mathcal{P}_\Omega) \times \mathbb{X}(\mathcal{P}_\Omega) \to \mathbb{C}$ by

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{\mathbb{X}} := \sum_{1 \leq j \leq n_\Omega} \left\langle \boldsymbol{\phi}_j, \boldsymbol{\psi}_j \right\rangle_{\mathbf{X}_j}. \tag{4.2}$$

Definition 4.2 *Let Assumption 2.2 be satisfied. The* Calderón operator $\mathsf{C}(s) : \mathbb{X}(\mathcal{P}_\Omega) \to \mathbb{X}(\mathcal{P}_\Omega)$ *is given by*

$$\mathsf{C}(s) := \operatorname{diag}\left[\mathsf{C}_j(s) : 1 \leq j \leq n_\Omega\right] \quad \text{with} \quad \mathsf{C}_j(s) := \begin{bmatrix} -\mathsf{K}_j(s) & \mathsf{V}_j(s) \\ \mathsf{W}_j(s) & \mathsf{K}_j'(s) \end{bmatrix}.$$

The sesquilinear form $c(s) : \mathbb{X}(\mathcal{P}_\Omega) \times \mathbb{X}(\mathcal{P}_\Omega) \to \mathbb{C}$ associated to the operator $\mathsf{C}(s)$ is

$$c(s)(\boldsymbol{\phi}, \boldsymbol{\psi}) := \left\langle \mathsf{C}(s)\boldsymbol{\phi}, \overline{\boldsymbol{\psi}} \right\rangle_{\mathbb{X}}.$$

Let $\mathrm{Id} : \mathbb{X}(\mathcal{P}_\Omega) \to \mathbb{X}(\mathcal{P}_\Omega)$ denote the identity. An essential property of the Calderón operator is that $\left(\mathsf{C}(s) - \frac{1}{2}\mathrm{Id}\right)$ is a projector into the space of Cauchy traces of solutions of the homogeneous PDE (2.9) as can be seen from the next Lemma. Recall the definition of the one-sided Cauchy trace $\boldsymbol{\gamma}_{\mathrm{C};j}^{\mathrm{ext},-}(s)$ from (2.16) and (2.18).

Lemma 4.3 *Let Assumption 2.2 be satisfied. Let $u^- \in H^1\left(\Omega_j^-, \mathbb{A}_j^-\right)$ and*

$$\mathsf{L}_j^-(s)\, u^- = 0 \quad \text{in } \Omega_j^-.$$

Then, for any $j \in \{1, 2, \ldots, n_\Omega\}$ it holds

$$\left(\mathsf{C}_j(s) - \frac{1}{2}\mathrm{Id}_j\right) \boldsymbol{\gamma}_{\mathrm{C};j}^{\mathrm{ext},-}(s)\, u^- = 0, \tag{4.3}$$

where $\mathrm{Id}_j : \mathbf{X}_j \to \mathbf{X}_j$ is the identity in $\mathbf{X}_j$.

Proof. Green's representation formula (3.34a) gives us

$$\begin{aligned}
\gamma_{\mathrm{D};j}^-(s)\, u^- &= \gamma_{\mathrm{D};j}^-(s)\, \mathsf{S}_j(s)\, \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- - \gamma_{\mathrm{D};j}^-(s)\, \mathsf{D}_j(s)\, \gamma_{\mathrm{D};j}^-(s)\, u^-, \\
0 &= \gamma_{\mathrm{D};j}^+(s)\, \mathsf{S}_j(s)\, \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- - \gamma_{\mathrm{D};j}^+(s)\, \mathsf{D}_j(s)\, \gamma_{\mathrm{D};j}^-(s)\, u^-, \\
\gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- &= \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, \mathsf{S}_j(s)\, \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- - \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, \mathsf{D}_j(s)\, \gamma_{\mathrm{D};j}^-(s)\, u^-, \\
0 &= -\gamma_{\mathrm{N};j}^{\mathrm{ext},+}(s)\, \mathsf{S}_j(s)\, \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- + \gamma_{\mathrm{N};j}^{\mathrm{ext},+}(s)\, \mathsf{D}_j(s)\, \gamma_{\mathrm{D};j}^-(s)\, u^-.
\end{aligned}$$

We multiply the first two relations by 1/2 and add them and do the same with the last two relations. This yields

$$\begin{aligned}
\frac{1}{2}\gamma_{\mathrm{D};j}^-(s)\, u^- &= \mathsf{V}_j(s)\, \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- - \mathsf{K}_j(s)\, \gamma_{\mathrm{D};j}^-(s)\, u^-, \\
\frac{1}{2}\gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- &= \mathsf{K}_j'(s)\, \gamma_{\mathrm{N};j}^{\mathrm{ext},-}(s)\, u^- + \mathsf{W}_j(s)\, \gamma_{\mathrm{D};j}^-(s)\, u^-
\end{aligned}$$

and after a reordering of the terms (4.3) follows. ■

5 Single-trace formulation of the transmission problem

In this section, we formulate the transmission problem (2.23) as a non-local skeleton equation for the Cauchy data of the solution. We start from a transmission problem with given jump data: We seek

$$\mathbf{u}^{\mathrm{mult}} = \left(\mathbf{u}_j^{\mathrm{mult}}\right)_{j=1}^{n\Omega} = \left(\left(u_{\mathrm{D};j}^{\mathrm{mult}}, u_{\mathrm{N};j}^{\mathrm{mult}}\right)\right)_{j=1}^{n\Omega} \in \mathbb{X}\left(\mathcal{P}_\Omega\right)$$

as the solution of

$$\begin{array}{rlr} \left(\mathsf{C}_j\left(s\right) - \frac{1}{2}\,\mathrm{Id}_j\right)\mathbf{u}_j^{\mathrm{mult}} = 0, & & 1 \leq j \leq n_\Omega, \\ \left[\mathbf{u}^{\mathrm{mult}}\right]_{j,k}\left(s\right) = \left[\boldsymbol{\beta}\right]_{j,k}, & & 1 \leq j,k \leq n_\Omega, \\ s^{1/2}\, u_{\mathrm{D};j}^{\mathrm{mult}}\big|_{\Gamma_j\cap\Gamma_\mathrm{D}} = \beta_{\mathrm{D};j}\big|_{\Gamma_j\cap\Gamma_\mathrm{D}} \text{ and } s^{-1/2}\, u_{\mathrm{N};j}^{\mathrm{mult}}\big|_{\Gamma_j\cap\Gamma_\mathrm{N}} = \beta_{\mathrm{N};j}\big|_{\Gamma_j\cap\Gamma_\mathrm{N}}, & & 1 \leq j \leq n_\Omega \end{array} \tag{5.1}$$

with $\boldsymbol{\beta}$ as in (2.23). Note that $\mathbf{u}^{\mathrm{mult}}$ is multi-valued on the inner skeleton $\Sigma \cap \Omega$. Following [9], a *single trace formulation* and single-valued functions is obtained when the transmission conditions are incorporated into the multi trace space $\mathbb{X}\left(\mathcal{P}_\Omega\right)$. We define the free single trace space $\mathbb{X}^{\mathrm{single}}\left(\mathcal{P}_\Omega\right)$ and the single trace space with incorporated homogeneous boundary conditions by

$$\mathbb{X}^{\mathrm{single}}\left(\mathcal{P}_\Omega\right) := \left\{ \left(\left(\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j}\right)\right)_{j=1}^{n\Omega} \in \mathbb{X}\left(\mathcal{P}_\Omega\right) \mid \left\{ \begin{array}{l} \exists v \in H^1\left(\Omega\right) \\ \text{s.t. } \forall 1 \leq j \leq n_\Omega \end{array} \right\} : \ \psi_{\mathrm{D};j} = \gamma_{\mathrm{D};j} v \atop \left\{ \begin{array}{l} \exists \mathbf{w} \in \mathbf{H}\left(\Omega, \mathrm{div}\right) \\ \text{s.t. } \forall 1 \leq j \leq n_\Omega \end{array} \right\} : \ \psi_{\mathrm{N};j} = \gamma_{\mathbf{n};j}\mathbf{w} \right\}, \tag{5.2}$$

$$\mathbb{X}_0^{\mathrm{single}}\left(\mathcal{P}_\Omega\right) := \left\{ \left(\left(\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j}\right)\right)_{j=1}^{n\Omega} \in \mathbb{X}^{\mathrm{single}}\left(\mathcal{P}_\Omega\right) \mid \forall 1 \leq j \leq n_\Omega : \psi_{\mathrm{D};j}\big|_{\Gamma_j\cap\Gamma_\mathrm{D}} = 0 \ \wedge \ \psi_{\mathrm{N};j}\big|_{\Gamma_j\cap\Gamma_\mathrm{N}} = 0 \right\}.$$

We set $\mathbf{u}^{\mathrm{single}} := \left(\mathbf{u}_j^{\mathrm{mult}} - \boldsymbol{\beta}_j\left(s\right)\right)_{j=1}^{n\Omega}$ and observe that $\mathbf{u}^{\mathrm{single}}$ satisfies

$$\begin{array}{lll} \left(\mathsf{C}_j\left(s\right) - \frac{1}{2}\,\mathrm{Id}_j\right)\mathbf{u}_j^{\mathrm{single}} = -\left(\mathsf{C}_j\left(s\right) - \frac{1}{2}\,\mathrm{Id}_j\right)\boldsymbol{\beta}_j & \text{in } \Omega_j, & 1 \leq j \leq n_\Omega, \\ \left[\mathbf{u}^{\mathrm{single}}\right]_{j,k}\left(s\right) = \mathbf{0}, & & 1 \leq j,k \leq n_\Omega, \\ u_{\mathrm{D};j}^{\mathrm{single}}\Big|_{\Gamma_j\cap\Gamma_\mathrm{D}} = 0 \quad \text{and} \quad u_{\mathrm{N};j}^{\mathrm{single}}\Big|_{\Gamma_j\cap\Gamma_\mathrm{N}} = 0, & & 1 \leq j \leq n_\Omega. \end{array} \tag{5.3}$$

This implies that $\mathbf{u}^{\mathrm{single}} \in \mathbb{X}_0^{\mathrm{single}}\left(\mathcal{P}_\Omega\right)$.

A reversed perspective on this derivation of the skeleton equation in the single trace space from the original transmission problem (2.23) is as follows: One solves the non-local skeleton problem in the single trace space (in variational form): find $\mathbf{u}^{\mathrm{single}} \in \mathbb{X}_0^{\mathrm{single}}\left(\mathcal{P}_\Omega\right)$ such that

$$c\left(s\right)\left(\mathbf{u}^{\mathrm{single}}, \boldsymbol{\psi}\right) - \frac{1}{2}\left\langle \mathbf{u}^{\mathrm{single}}, \overline{\boldsymbol{\psi}}\right\rangle_{\mathbb{X}} = -\left(c\left(s\right)\left(\boldsymbol{\beta}\left(s\right), \boldsymbol{\psi}\right) - \frac{1}{2}\left\langle \boldsymbol{\beta}\left(s\right), \overline{\boldsymbol{\psi}}\right\rangle_{\mathbb{X}}\right) \tag{5.4}$$

for all $\boldsymbol{\psi} \in \mathbb{X}_0^{\mathrm{single}}\left(\mathcal{P}_\Omega\right)$, and obtains $\mathbf{u}_j^{\mathrm{mult}} := \mathbf{u}^{\mathrm{single}} + \boldsymbol{\beta}\left(s\right)$. Then, we use Green's representation formula

$$u_j := \left(\mathsf{S}_j\left(s\right) u_{\mathrm{N};j}^{\mathrm{mult}} - \mathsf{D}_j\left(s\right) u_{\mathrm{D};j}^{\mathrm{mult}}\right)\big|_{\Omega_j^-}, \quad 1 \leq j \leq n_\Omega.$$

Finally, the function $\mathbf{u} = \left(u_j\right)_{j=1}^{n\Omega} \in \mathbf{H}\left(\Omega, \mathbb{A}\right)$ solves the original transmission problem (2.23).

Next, we prove the well-posedness of (5.4). The essential point is to prove s-explicit continuity estimates for the layer potentials and the boundary integral operators as well as coercivity results for $\mathsf{V}\left(s\right)$, $\mathsf{W}\left(s\right)$, and $\mathsf{C}\left(s\right) - \frac{1}{2}\,\mathrm{Id}$.

We start with an estimate of the Dirichlet and Neumann trace of homogeneous solutions of the acoustic PDE.

Lemma 5.1 *Let Assumption 2.2 be satisfied and set* $\mathbb{A}_j^\sigma := \mathbb{A}_j^{\mathrm{ext}}\big|_{\Omega_j^\sigma}$, $\sigma \in \{+,-\}$. *Then there are constants* $C_D, C > 0$ *independent of* s *such that*

$$\left\|\gamma_{\mathrm{D};j}^\sigma(s)\, v\right\|_{H^{1/2}(\Gamma_j)} \leq C_{\mathrm{D}}\,|s|^{1/2}\,\|v\|_{H^1\left(\Omega_j^\sigma\right)} \leq C\,|s|^{1/2}\,\|v\|_{H^1\left(\Omega_j^\sigma\right);s} \quad \forall v \in H^1\left(\Omega_j^\sigma\right). \tag{5.5}$$

Vice versa, there exists $C > 0$ *independent of* s *and a linear bounded extension operator* $\mathsf{E}_j(s) : H^{1/2}(\Gamma_j) \to H^1(\mathbb{R}^3)$ *which satisfies for all* $\varphi \in H^{1/2}(\Gamma_j)$:

$$\gamma_{\mathrm{D},j}(s)\,\mathsf{E}_j(s)\,\varphi = \varphi \quad \text{and} \quad \|\mathsf{E}_j(s)\,\varphi\|_{H^1(\mathbb{R}^3);s} \leq C\,\|\varphi\|_{H^{1/2}(\Gamma_j)}. \tag{5.6}$$

Let $v \in H^1(\mathbb{R}^3)$ *such that* $v^\sigma := v|_{\Omega_j^\sigma}$ *belongs to* $H^1\left(\Omega_j^\sigma, \mathbb{A}_j^\sigma\right)$ *and*

$$-\operatorname{div}\left(\mathbb{A}_j^{\mathrm{ext}}\nabla v\right) + s^2 p_j^{\mathrm{ext}} v = 0 \quad \text{in } \mathbb{R}^3\backslash\Gamma_j.$$

Then,

$$\left\|\gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\, v^\sigma\right\|_{H^{-1/2}(\Gamma_j)} \leq C\Lambda_j\,\|v^\sigma\|_{H^1\left(\Omega_j^\sigma\right);s}, \tag{5.7}$$

where Λ_j *is as in Lem. 3.2 and* C *depends only on the domain* Ω_j^σ.

Proof. The estimates in (5.5) follow from the scaling of $\gamma_{\mathrm{D};j}^\sigma(s)$ with respect to s and (2.10).

The extension operator $\mathsf{E}_j(s) : H^{1/2}(\Gamma_j) \to H^1(\mathbb{R}^3)$ is defined for $\varphi \in H^{1/2}(\Gamma_j)$ piecewise in Ω_j^σ, $\sigma \in \{+,-\}$, by

$$\begin{aligned} \gamma_{\mathrm{D};j}^\sigma(s)\left(\mathsf{E}_j(s)\,\varphi\right) &= \varphi \quad \text{and} \\ \left(\nabla\left(\mathsf{E}_j(s)\,\varphi\right), \nabla w\right)_{\mathbf{L}^2\left(\Omega_j^\sigma\right)} + |s|^2\left(\mathsf{E}_j(s)\,\varphi, w\right)_{L^2\left(\Omega_j^\sigma\right)} &= 0 \quad \forall w \in H^1\left(\Omega_j^\sigma\right). \end{aligned}$$

From [21, Prop. 2.5.1] the estimate (5.6) follows.

For (5.7) we adapt the standard proof (see, e.g., [21, Prop. 2.5.2]) to our setting. For given $\psi \in H^{1/2}(\Gamma_j)$ let $w := \mathsf{E}_j(s)\,\psi$. Let $w^\sigma := w|_{\Omega_j^\sigma}$ and $v^\sigma := v|_{\Omega_j^\sigma}$, $\sigma \in \{+,-\}$. Green's first identity (3.6) gives us

$$\begin{aligned} \left|\left\langle\gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\, v^\sigma, \overline{\psi}\right\rangle_{\Gamma_j}\right| &= \left|\left(\frac{\overline{s}}{s}\right)^{1/2}\left\langle\gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\, v^\sigma, \gamma_{\mathrm{D};j}^\sigma(s)\,\overline{w^\sigma}\right\rangle_{\Gamma_j}\right| \\ &= \left|\left\langle\mathbb{A}_j^\sigma\nabla v^\sigma, \nabla\overline{w^\sigma}\right\rangle_{\Omega_j^\sigma} + s^2\left\langle p_j^\sigma v^\sigma, \overline{w^\sigma}\right\rangle_{\Omega_j^\sigma}\right| \\ &\overset{\text{Lem. 3.2}}{\leq} \Lambda_j\,\|v^\sigma\|_{H^1\left(\Omega_j^\sigma\right);s}\,\|w^\sigma\|_{H^1\left(\Omega_j^\sigma\right);s} \\ &\overset{(5.6)}{\leq} C\Lambda_j\,\|v^\sigma\|_{H^1\left(\Omega_j^\sigma\right);s}\,\|\psi\|_{H^{1/2}(\Gamma_j)}. \end{aligned}$$

Finally,

$$\left\|\gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\, v^\sigma\right\|_{H^{-1/2}(\Gamma_j)} = \sup_{\psi \in H^{1/2}(\Gamma_j)\backslash\{0\}} \frac{\left|\left\langle\gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\, v^\sigma, \overline{\psi}\right\rangle_{\Gamma_j}\right|}{\|\psi\|_{H^{1/2}(\Gamma_j)}} \leq C\Lambda_j\,\|v^\sigma\|_{H^1\left(\Omega_j^\sigma\right);s}.$$

■

Lemma 5.2 *Let Assumption 2.2 be satisfied. Then the sesquilinear form induced by the single layer boundary integral operator satisfies the coercivity and continuity estimates*

$$\operatorname{Re}\left\langle \varphi, \overline{\mathsf{V}_j(s)\varphi}\right\rangle_{\Gamma_j} \geq c\frac{\operatorname{Re}s}{|s|}\frac{\lambda_j}{\Lambda_j^2}\|\varphi\|^2_{H^{-1/2}(\Gamma_j)} \qquad \forall\varphi\in H^{-1/2}(\Gamma_j), \tag{5.8a}$$

$$\left|\left\langle \mathsf{V}_j(s)\varphi, \overline{\psi}\right\rangle_{\Gamma_j}\right| \leq C\frac{|s|^2}{\lambda_j\operatorname{Re}s}\|\varphi\|_{H^{-1/2}(\Gamma_j)}\|\psi\|_{H^{-1/2}(\Gamma_j)} \quad \forall\varphi,\psi\in H^{-1/2}(\Gamma_j). \tag{5.8b}$$

The dual double layer boundary integral operator is bounded and satisfies the estimates

$$\left\|\mathsf{K}'_j(s)\varphi\right\|_{H^{-1/2}(\Gamma_j)} \leq C\frac{\Lambda_j}{\lambda_j}\frac{|s|^{3/2}}{\operatorname{Re}s}\|\varphi\|_{H^{-1/2}(\Gamma_j)} \quad \forall\varphi\in H^{-1/2}(\Gamma_j). \tag{5.9}$$

The sesquilinear form induced by the hypersingular boundary integral operator satisfies the coercivity and continuity estimate

$$\operatorname{Re}\left\langle \mathsf{W}_j(s)\psi, \overline{\psi}\right\rangle_{\Gamma_j} \geq c\frac{\operatorname{Re}s}{|s|^2}\lambda_j\|\psi\|^2_{H^{1/2}(\Gamma_j)} \qquad \forall\psi\in H^{1/2}(\Gamma_j), \tag{5.10a}$$

$$\left|\left\langle \mathsf{W}_j(s)\psi, \overline{\varphi}\right\rangle_{\Gamma_j}\right| \leq C\frac{\Lambda_j^2}{\lambda_j}\frac{|s|}{\operatorname{Re}s}\|\psi\|_{H^{1/2}(\Gamma_j)}\|\varphi\|_{H^{1/2}(\Gamma_j)} \quad \forall\varphi,\psi\in H^{1/2}(\Gamma_j). \tag{5.10b}$$

The double layer boundary integral operator is bounded and satisfies the estimate

$$\|\mathsf{K}_j(s)\psi\|_{H^{1/2}(\Gamma_j)} \leq C\frac{\Lambda_j}{\lambda_j}\frac{|s|^{3/2}}{\operatorname{Re}s}\|\psi\|_{H^{1/2}(\Gamma_j)} \quad \forall\psi\in H^{1/2}(\Gamma_j). \tag{5.11}$$

For the single layer potential, the estimate

$$\|\mathsf{S}_j(s)\varphi\|_{H^1(\mathbb{R}^3);s} \leq C\frac{|s|^{3/2}}{\lambda_j\operatorname{Re}s}\|\varphi\|_{H^{-1/2}(\Gamma_j)} \quad \forall\varphi\in H^{-1/2}(\Gamma_j) \tag{5.12}$$

holds. The operator norm of the double layer potential is bounded by

$$\|\mathsf{D}_j(s)\psi\|_{H^1(\mathbb{R}^3\backslash\Gamma_j);s} \leq C\frac{\Lambda_j}{\lambda_j}\frac{|s|}{\operatorname{Re}s}\|\psi\|_{H^{1/2}(\Gamma_j)} \quad \forall\psi\in H^{1/2}(\Gamma_j), \tag{5.13}$$

where for $u\in L^2(\mathbb{R}^3)$ *with* $u^\sigma := u|_{\Omega_j^\sigma}\in H^1(\Omega_j^\sigma)$, $\sigma=\{+,-\}$ *the* broken H^1 *norm is given by*

$$\|u\|_{H^1(\mathbb{R}^3\backslash\Gamma_j);s} := \left(\sum_{\sigma\in\{+,-\}}\|u^\sigma\|^2_{H^1(\Omega_j^\sigma);s}\right)^{1/2}.$$

All constants $c, C > 0$ *only depend on* Ω_j *and, in particular, are independent of* s.

The proof of this lemma follows standard arguments and hence is postponed to Appendix A.

Lemma 5.3 *Let Assumption 2.2 be satisfied. The sesquilinear form* $\langle\mathsf{C}_j(s)\cdot,\bar{\cdot}\rangle_{\mathbf{X}_j}:\mathbf{X}_j\times\mathbf{X}_j\rightarrow\mathbb{C}$ *is coercive:*

$$\operatorname{Re}\left\langle \mathsf{C}_j(s)\boldsymbol{\psi}_j, \overline{\boldsymbol{\psi}_j}\right\rangle_{\mathbf{X}_j} \geq c\frac{\lambda_j}{1+\Lambda_j^2}\frac{\operatorname{Re}s}{|s|^2}\|\boldsymbol{\psi}\|^2_{\mathbf{X}_j} \quad \forall\boldsymbol{\psi}_j\in\mathbf{X}_j,$$

and continuous:

$$\left|\left\langle \mathsf{C}_j(s)\boldsymbol{\psi}_j, \overline{\boldsymbol{\phi}_j}\right\rangle_{\mathbf{X}_j}\right| \leq C\frac{1+\Lambda_j}{\lambda_j}\frac{|s|^2}{\operatorname{Re}s}\|\boldsymbol{\psi}_j\|_{\mathbf{X}_j}\|\boldsymbol{\phi}_j\|_{\mathbf{X}_j} \quad \forall\boldsymbol{\psi}_j,\boldsymbol{\phi}_j\in\mathbf{X}_j.$$

The proof follows closely the arguments in [4, Lem. 3.1] for the case of constant coefficients and we adapt it here to our general setting.

Proof. We pick some $\boldsymbol{\psi}_j := (\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j}) \in \mathbf{X}_j$ and define $u \in H^1(\mathbb{R}^3 \backslash \Gamma_j)$ by

$$u := \mathsf{S}_j(s)\,\psi_{\mathrm{N};j} - \mathsf{D}_j(s)\,\psi_{\mathrm{D};j}.$$

We set $u^\sigma := u|_{\Omega_j^\sigma}$, $\sigma \in \{-,+\}$. The jump relations (3.19), (3.31) imply

$$[u]_{\mathrm{D};j}(s) = -\psi_{\mathrm{D};j} \quad \text{and} \quad [u]_{\mathrm{N},j}^{\mathrm{ext}}(s) = -\psi_{\mathrm{N};j}$$

while the relations

$$\begin{aligned}\{\!\!\{u\}\!\!\}_{\mathrm{D};j}(s) &= \mathsf{V}_j(s)\,\psi_{\mathrm{N};j} - \mathsf{K}_j(s)\,\psi_{\mathrm{D};j},\\ \{\!\!\{u\}\!\!\}_{\mathrm{N};j}^{\mathrm{ext}}(s) &= \mathsf{K}_j'(s)\,\psi_{\mathrm{N};j} + \mathsf{W}_j(s)\,\psi_{\mathrm{D};j}\end{aligned}$$

follow directly from the definition of the boundary integral operators. A more compact formulation is

$$\mathsf{C}_j(s)\,\boldsymbol{\psi}_j = \begin{pmatrix} \{\!\!\{u\}\!\!\}_{\mathrm{D};j}(s) \\ \{\!\!\{u\}\!\!\}_{\mathrm{N};j}^{\mathrm{ext}}(s) \end{pmatrix}.$$

Since $\mathsf{S}_j(s)\,\psi_{\mathrm{N};j}$ and $\mathsf{D}_j(s)\,\psi_{\mathrm{D};j}$ satisfy the homogeneous PDE in Ω_j^- and Ω_j^+ (cf. (3.18), (3.30)) we may apply Green's identity (3.6) and the definition of the jumps and means (2.19), (2.20) to obtain by a jump-average parallelogram identity:

$$\begin{aligned}&\operatorname{Re}\left\langle \mathsf{C}_j(s)\,\boldsymbol{\psi}_j, \overline{\boldsymbol{\psi}_j}\right\rangle_{\mathbf{X}_j}\\ &\quad= -\operatorname{Re}\left(\left(\frac{s}{\overline{s}}\right)^{1/2}\left\langle \{\!\!\{u\}\!\!\}_{\mathrm{D};j}(s), [\overline{u}]_{\mathrm{N},j}^{\mathrm{ext}}(s)\right\rangle_{\Gamma_j} + \left(\frac{\overline{s}}{s}\right)^{1/2}\left\langle [\overline{u}]_{\mathrm{D};j}(s), \{\!\!\{u\}\!\!\}_{\mathrm{N};j}^{\mathrm{ext}}(s)\right\rangle_{\Gamma_j}\right)\\ &\quad= \operatorname{Re}\left(\left(\frac{\overline{s}}{s}\right)^{1/2}\left(\left\langle \mathbb{A}_j^+\nabla u^+, \nabla\overline{u^+}\right\rangle_{\Omega_j^+} + s^2\left\langle p_j^+ u^+, \overline{u^+}\right\rangle_{\Omega_j^+}\right)\right)\\ &\qquad+ \operatorname{Re}\left(\left(\frac{s}{\overline{s}}\right)^{1/2}\left(\left\langle \mathbb{A}_j^-\nabla\overline{u^-}, \nabla u^-\right\rangle_{\Omega_j^-} + s^2\left\langle p_j^-\overline{u^-}, u^-\right\rangle_{\Omega_j^-}\right)\right).\end{aligned}$$

As in the proof of Lemma 3.2 we obtain

$$\operatorname{Re}\left\langle \mathsf{C}_j(s)\,\boldsymbol{\psi}_j, \overline{\boldsymbol{\psi}_j}\right\rangle_{\mathbf{X}_j} \geq \frac{\operatorname{Re}s}{|s|}\lambda_j\,\|u\|^2_{H^1(\mathbb{R}^3\backslash\Gamma_j);s}. \tag{5.14}$$

To estimate the right-hand side we start with

$$\left\|s^{-1/2}\psi_{\mathrm{D};j}\right\|^2_{H^{1/2}(\Gamma_j)} + \|\psi_{\mathrm{N};j}\|^2_{H^{-1/2}(\Gamma_j)} = \left\|s^{-1/2}\,[u]_{\mathrm{D};j}(s)\right\|^2_{H^{1/2}(\Gamma_j)} + \left\|[u]_{\mathrm{N},j}^{\mathrm{ext}}(s)\right\|^2_{H^{-1/2}(\Gamma_j)}. \tag{5.15}$$

From (5.5) and a triangle inequality we conclude that

$$\begin{aligned}\left\|s^{-1/2}\psi_{\mathrm{D};j}\right\|^2_{H^{1/2}(\Gamma_j)} &= \left\|s^{-1/2}\,[u]_{\mathrm{D};j}(s)\right\|^2_{H^{1/2}(\Gamma_j)} \\ &\leq 2\sum_{\sigma\in\{+,-\}}|s|^{-1}\left\|\gamma_{\mathrm{D};j}^\sigma(s)\,u^\sigma\right\|^2_{H^{1/2}(\Gamma_j)} \leq C\,\|u\|^2_{H^1(\mathbb{R}^3\backslash\Gamma_j);s}.\end{aligned} \tag{5.16}$$

From (3.7) and by using the lifting $\mathsf{E}_j(s)$ as in Lemma 5.1, we obtain

$$\begin{aligned}\|\psi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)} &= \left\|[u]^{\mathrm{ext}}_{\mathrm{N},j}(s)\right\|_{H^{-1/2}(\Gamma_j)} = \sup_{\phi\in H^{1/2}(\Gamma_j)\backslash\{0\}} \frac{\left|\left\langle [u]^{\mathrm{ext}}_{\mathrm{N},j}(s), \overline{\phi}\right\rangle_{\Gamma_j}\right|}{\|\phi\|_{H^{1/2}(\Gamma_j)}} \\ &= \sup_{\phi\in H^{1/2}(\Gamma_j)\backslash\{0\}} \frac{\left|\left(\frac{\overline{s}}{s}\right)^{1/2}\left\langle [u]^{\mathrm{ext}}_{\mathrm{N},j}(s), \gamma_{\mathrm{D};j}(s)\,\overline{\mathsf{E}_j(s)\,\phi}\right\rangle_{\Gamma_j}\right|}{\|\phi\|_{H^{1/2}(\Gamma_j)}} \\ &= \sup_{\phi\in H^{1/2}(\Gamma_j)\backslash\{0\}} \frac{\left|\ell_j(s)\left(u, \mathsf{E}_j(s)\,\phi\right)\right|}{\|\phi\|_{H^{1/2}(\Gamma_j)}} \\ &\overset{\text{Lem. 3.2}}{\leq} \Lambda_j \|u\|_{H^1(\mathbb{R}^3\backslash\Gamma_j);s} \sup_{\phi\in H^{1/2}(\Gamma_j)\backslash\{0\}} \frac{\|\mathsf{E}_j(s)\,\phi\|_{H^1(\mathbb{R}^3);s}}{\|\phi\|_{H^{1/2}(\Gamma_j)}} \\ &\overset{(5.6)}{\leq} C\Lambda_j \|u\|_{H^1(\mathbb{R}^3\backslash\Gamma_j);s}.\end{aligned} \tag{5.17}$$

The combination of (5.14)-(5.17) leads to the coercivity estimate

$$\operatorname{Re}\left\langle \mathsf{C}_j(s)\,\boldsymbol{\psi}_j, \overline{\boldsymbol{\psi}_j}\right\rangle_{\mathbf{X}_j} \geq \tilde{c}\frac{\lambda_j}{1+\Lambda_j^2}\frac{\operatorname{Re} s}{|s|^2}\|\boldsymbol{\psi}_j\|^2_{\mathbf{X}_j}.$$

For the continuity estimate we obtain for any $\boldsymbol{\psi}_j = (\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j})$, $\boldsymbol{\phi} = (\varphi_{\mathrm{D};j}, \varphi_{\mathrm{N};j}) \in \mathbf{X}_j$ from Lemma 5.2

$$\begin{aligned}\left|\left\langle \mathsf{C}_j(s)\,\boldsymbol{\psi}_j, \overline{\boldsymbol{\phi}_j}\right\rangle_{\mathbf{X}_j}\right| &= \left|\left\langle \begin{pmatrix} -\mathsf{K}_j(s)\,\psi_{\mathrm{D};j} + \mathsf{V}_j(s)\,\psi_{\mathrm{N};j} \\ \mathsf{W}_j(s)\,\psi_{\mathrm{D};j} + \mathsf{K}'_j(s)\,\psi_{\mathrm{N};j}\end{pmatrix}, \overline{\begin{pmatrix}\varphi_{\mathrm{D};j} \\ \varphi_{\mathrm{N};j}\end{pmatrix}}\right\rangle_{\mathbf{X}_j}\right| \\ &= \left|\left\langle -\mathsf{K}_j(s)\,\psi_{\mathrm{D};j} + \mathsf{V}_j(s)\,\psi_{\mathrm{N};j}, \overline{\varphi_{\mathrm{N};j}}\right\rangle_{\Gamma_j} + \left\langle \overline{\varphi_{\mathrm{D};j}}, \mathsf{W}_j(s)\,\psi_{\mathrm{D};j} + \mathsf{K}'_j(s)\,\psi_{\mathrm{N};j}\right\rangle_{\Gamma_j}\right| \\ &\leq C\frac{1}{\lambda_j}\frac{|s|}{\operatorname{Re} s}\Big(\Lambda_j |s|^{1/2}\|\psi_{\mathrm{D};j}\|_{H^{1/2}(\Gamma_j)}\|\varphi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)} + |s|\,\|\psi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)}\|\varphi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)} \\ &\quad +\Lambda_j^2\|\psi_{\mathrm{D};j}\|_{H^{1/2}(\Gamma_j)}\|\varphi_{\mathrm{D};j}\|_{H^{1/2}(\Gamma_j)} + \Lambda_j|s|^{1/2}\|\psi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)}\|\varphi_{\mathrm{D};j}\|_{H^{1/2}(\Gamma_j)}\Big) \\ &\leq C\frac{1}{\lambda_j}\frac{|s|^2}{\operatorname{Re} s}\left(\Lambda_j^2\left\|s^{-1/2}\psi_{\mathrm{D};j}\right\|^2_{H^{1/2}(\Gamma_j)} + \|\psi_{\mathrm{N};j}\|^2_{H^{1/2}(\Gamma_j)}\right)^{1/2}\times \\ &\quad \times\left(\Lambda_j^2\left\|s^{-1/2}\psi_{\mathrm{D};j}\right\|^2_{H^{1/2}(\Gamma_j)} + \|\varphi_{\mathrm{N};j}\|^2_{H^{-1/2}(\Gamma_j)}\right)^{1/2} \\ &\leq C\frac{1+\Lambda_j}{\lambda_j}\frac{|s|^2}{\operatorname{Re} s}\|\boldsymbol{\psi}_j\|_{\mathbf{X}_j}\|\boldsymbol{\phi}_j\|_{\mathbf{X}_j}.\end{aligned}$$

■

A summation of the local coercivity estimates (of the local continuity estimates, resp.) over all subdomains leads to the following global coercivity (global continuity, resp.).

Corollary 5.4 *Let Assumption 2.2 be satisfied. The sesquilinear form* $\langle \mathsf{C}(s)\,\cdot, \bar{\cdot}\rangle_{\mathbb{X}} : \mathbb{X}(\mathcal{P}_\Omega)\times\mathbb{X}(\mathcal{P}_\Omega)\to\mathbb{C}$ *is coercive: for any* $\boldsymbol{\psi}\in\mathbb{X}(\mathcal{P}_\Omega)$ *it holds*

$$\operatorname{Re}\left\langle \mathsf{C}(s)\,\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\right\rangle_{\mathbb{X}} \geq c\frac{\lambda}{1+\Lambda^2}\frac{\operatorname{Re} s}{|s|^2}\|\boldsymbol{\psi}\|^2_{\mathbb{X}}; \tag{5.18}$$

and continuous: for any $\boldsymbol{\psi}, \boldsymbol{\phi} \in \mathbb{X}(\mathcal{P}_\Omega)$ *it holds*

$$\left|\left\langle \mathsf{C}(s)\,\boldsymbol{\psi}, \overline{\boldsymbol{\phi}}\right\rangle_{\mathbb{X}}\right| \le C\frac{1+\Lambda}{\lambda}\frac{|s|^2}{\operatorname{Re} s}\,\|\boldsymbol{\psi}\|_{\mathbb{X}}\,\|\boldsymbol{\phi}\|_{\mathbb{X}} \tag{5.19}$$

with $\lambda := \min_{1\le j\le n_\Omega}\lambda_j$ *and* $\Lambda := \max_{1\le j\le n_\Omega}\Lambda_j$.

We have collected all prerequisites to prove the well-posedness of the non-local variational problem on the skeleton (5.4) in single trace spaces.

Theorem 5.5 *Let Assumption 2.2 be satisfied. The sesquilinear form* $\left(c(s)(\cdot,\cdot) - \frac{1}{2}\langle\cdot,\bar{\cdot}\rangle_{\mathbb{X}}\right) : \mathbb{X}_0^{\text{single}}(\mathcal{P}_\Omega)\times\mathbb{X}_0^{\text{single}}(\mathcal{P}_\Omega) \to \mathbb{C}$ *is coercive and continuous: for any* $\boldsymbol{\alpha} \in \mathbb{X}_0^{\text{single}}(\mathcal{P}_\Omega)$ *and* $\boldsymbol{\psi}, \boldsymbol{\phi} \in \mathbb{X}(\mathcal{P}_\Omega)$ *holds*

$$\operatorname{Re}\left(c(s)(\boldsymbol{\alpha},\boldsymbol{\alpha}) - \frac{1}{2}\langle\boldsymbol{\alpha},\overline{\boldsymbol{\alpha}}\rangle_{\mathbb{X}}\right) \ge c\frac{\lambda}{1+\Lambda^2}\frac{\operatorname{Re} s}{|s|^2}\,\|\boldsymbol{\alpha}\|_{\mathbb{X}}^2,$$

$$\left|c(s)(\boldsymbol{\psi},\boldsymbol{\phi}) - \frac{1}{2}\langle\boldsymbol{\psi},\overline{\boldsymbol{\phi}}\rangle_{\mathbb{X}}\right| \le \left(\frac{1}{2} + C\frac{1+\Lambda}{\lambda}\frac{|s|^2}{\operatorname{Re} s}\right)\|\boldsymbol{\psi}\|_{\mathbb{X}}\|\boldsymbol{\phi}\|_{\mathbb{X}}.$$

For any $\boldsymbol{\beta} \in \mathbb{X}(\mathcal{P}_\Omega)$, *the variational problem (5.4) has a solution* $\mathbf{u}^{\text{single}} \in \mathbb{X}_0^{\text{single}}(\mathcal{P}_\Omega)$ *which is unique and satisfies*

$$\left\|\mathbf{u}^{\text{single}}\right\|_{\mathbb{X}} \le C\frac{|s|^{9/2}}{(\operatorname{Re} s)^2}\,\|\boldsymbol{\beta}\|_{\mathbb{X}}, \tag{5.20}$$

where C *only depends on* λ, Λ, s_0, *and on the domain* Ω *via trace estimates.*

Proof. Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_j)_{j=1}^{n_\Omega} \in \mathbb{X}_0^{\text{single}}(\mathcal{P}_\Omega)$ with $\boldsymbol{\alpha}_j = (\alpha_{\mathrm{D};j}, \alpha_{\mathrm{N};j})$ and $\boldsymbol{\phi} = (\boldsymbol{\phi}_j)_{j=1}^{n_\Omega}$, $\boldsymbol{\psi} = (\boldsymbol{\psi}_j)_{j=1}^{n_\Omega} \in \mathbb{X}(\mathcal{P}_\Omega)$ with $\boldsymbol{\psi}_j = (\psi_{\mathrm{D};j}, \psi_{\mathrm{N};j})$ and $\boldsymbol{\phi}_j = (\varphi_{\mathrm{D};j}, \varphi_{\mathrm{N};j})$. Then

$$\operatorname{Re}\left(c(s)(\boldsymbol{\alpha},\boldsymbol{\alpha}) - \frac{1}{2}\langle\boldsymbol{\alpha},\overline{\boldsymbol{\alpha}}\rangle_{\mathbb{X}}\right) = \operatorname{Re} c(s)(\boldsymbol{\alpha},\boldsymbol{\alpha})$$

owing to the self-polarity of the single trace space, see [8, Lem. 4.1], [9, Remark 55]. Thus, the coercivity estimate follows from (5.18):

$$\operatorname{Re}\left(c(s)(\boldsymbol{\alpha},\boldsymbol{\alpha}) - \frac{1}{2}\langle\boldsymbol{\alpha},\overline{\boldsymbol{\alpha}}\rangle_{\mathbb{X}}\right) \ge c\frac{\lambda}{1+\Lambda^2}\frac{\operatorname{Re} s}{|s|^2}\,\|\boldsymbol{\alpha}\|_{\mathbb{X}}^2.$$

The continuity estimate follows by combining (5.19) with

$$\begin{aligned}\left|\left\langle\boldsymbol{\psi},\overline{\boldsymbol{\phi}}\right\rangle_{\mathbb{X}}\right| &\le \sum_{1\le j\le n_\Omega}\left|\langle\psi_{\mathrm{D};j},\overline{\varphi_{\mathrm{N};j}}\rangle_{\Gamma_j} + \langle\psi_{\mathrm{N};j},\overline{\varphi_{\mathrm{D};j}}\rangle_{\Gamma_j}\right| \\ &\le \sum_{1\le j\le n_\Omega}\left(\|\psi_{\mathrm{D};j}\|_{H^{1/2}(\Gamma_j)}\|\varphi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)} + \|\psi_{\mathrm{N};j}\|_{H^{-1/2}(\Gamma_j)}\|\varphi_{\mathrm{D};j}\|_{H^{1/2}(\Gamma_j)}\right) \\ &\le \sum_{1\le j\le n_\Omega}\left\|\boldsymbol{\psi}_j\right\|_{\mathbf{X}_j}\left\|\boldsymbol{\phi}_j\right\|_{\mathbf{X}_j} \le \|\boldsymbol{\psi}\|_{\mathbb{X}}\,\|\boldsymbol{\phi}\|_{\mathbb{X}}.\end{aligned}$$

In particular, the continuity of $\left(c(s)(\cdot,\cdot) - \frac{1}{2}\langle\cdot,\bar{\cdot}\rangle_{\mathbb{X}}\right)$ implies that for any $\boldsymbol{\beta} \in \mathbb{X}(\mathcal{P}_\Omega)$ the form $\left(c(s)(\boldsymbol{\beta}(s),\cdot) - \frac{1}{2}\langle\boldsymbol{\beta}(s),\bar{\cdot}\rangle_{\mathbb{X}}\right) : \mathbb{X}_0^{\text{single}}(\mathcal{P}_\Omega) \to \mathbb{C}$ defines an anti-linear operator with upper bound $\left(\frac{1}{2} + C\frac{1+\Lambda}{\lambda}\frac{|s|^2}{\operatorname{Re} s}\right)|s|^{1/2}\,\|\boldsymbol{\beta}\|_{\mathbb{X}}$ for its norm. By the Lax-Milgram theorem we infer well-posedness of (5.4) and the bound in (5.20). ■

6 Conclusion

In this paper, we have considered acoustic transmission problems with mixed boundary conditions and variable coefficients. We have developed a general approach to transform these equations to non-local skeleton equations in such a way that the resulting variational form is continuous and coercive so that well-posedness follows by the Lax-Milgram theorem. The transformation is based on Green's representation formula involving single and double layer potentials which are defined as solutions of some variational full space problems without relying on the explicit knowledge of the Green's function. The paper can be regarded as a generalization of [11] by allowing for unbounded domains (full space/half space) and variable coefficients in the subdomains.

In contrast to other methods such as the *indirect* method of boundary integral equations (see, e.g., [20, Chap. 3.4.1]) the well-posedness of the non-local skeleton (integral) equation follows directly from the well-posedness of the auxiliary variational problems in full space.

Another important contribution of this work is the completely s-explicit nature of all estimates. Its significance is due to the possibility to apply our boundary integral equation method to transform the space-time wave transmission problem (in analogy to (2.23)) to an integro-differential equation which may serve as a starting point for its discretization by convolution quadrature. The well-posedness of this integro-differential equation follows from the coercivity and continuity of the variational skeleton equation (5.4) via *operational calculus*; for details we refer to [11], [4], [21], [5]. We also mention that the restriction to mixed Dirichlet and Neumann boundary conditions was merely done to reduce technicalities: Dirichlet-to-Neumann boundary conditions and impedance conditions can be incorporated into the variational skeleton equation following the approach in [11].

A Proof of Lemma 5.2

The proof of Lemma 5.2 is an adaptation of the arguments in [15, Prop. 16, 19] to our setting; see also [6, Lem. 5.2]. In this appendix, we present the proof to show that the known arguments apply to our general setting.

Proof of Lemma 5.2. Let $\varphi \in H^{-1/2}(\Gamma)$ and set $u := \mathsf{S}_j(s)\varphi$. The jump relations for the single layer potential (cf. (3.19)) imply $\gamma_{\mathrm{D};j}(s)\,u = \mathsf{V}(s)\,\varphi$ and $[u]^{\mathrm{ext}}_{\mathrm{N};j}(s) = -\varphi$. Then, we have

$$\begin{aligned}\operatorname{Re}\left\langle \varphi, \overline{\mathsf{V}_j(s)\,\varphi}\right\rangle_{\Gamma_j} &= \operatorname{Re}\left\langle -[u]^{\mathrm{ext}}_{\mathrm{N};j}(s), \gamma_{\mathrm{D};j}(\overline{s})\,\overline{u}\right\rangle_{\Gamma_j} \\ &\overset{(2.18)}{=} \operatorname{Re}\left(\left(\frac{\overline{s}}{s}\right)^{1/2}\left\langle -[u]^{\mathrm{ext}}_{\mathrm{N};j}(s), \gamma_{\mathrm{D};j}(s)\,\overline{u}\right\rangle_{\Gamma_j}\right).\end{aligned}$$

We employ (3.7) with $v = w = u$ and $\lambda(\cdot)$, λ_j as in Lem. 3.2 to obtain (cf. (3.4))

$$\begin{aligned}\operatorname{Re}\left(\left(\frac{\overline{s}}{s}\right)^{1/2}\left\langle -[u]_{\mathrm{N};j}^{\mathrm{ext}}(s), \gamma_{\mathrm{D};j}(s)\,\overline{u}\right\rangle_{\Gamma_j}\right) &= \operatorname{Re}\left(\left(\frac{\overline{s}}{s}\right)^{1/2}\ell_j(s)(u,u)\right) \\ &= \frac{\operatorname{Re}s}{|s|}\left(\left\langle \mathbb{A}_j^{\mathrm{ext}}\nabla u, \nabla\overline{u}\right\rangle_{\mathbb{R}^3} + |s|^2\left\langle p_j^{\mathrm{ext}}u,\overline{u}\right\rangle_{\mathbb{R}^3}\right) \\ &\geq \frac{\operatorname{Re}s}{|s|}\left(\lambda\left(\mathbb{A}_j^{\mathrm{ext}}\right)|\nabla u|^2_{\mathbf{L}^2(\mathbb{R}^3)} + \lambda\left(p_j^{\mathrm{ext}}\right)|s|^2\|u\|^2_{L^2(\mathbb{R}^3)}\right) \\ &\geq \frac{\operatorname{Re}s}{|s|}\lambda_j\|u\|^2_{H^1(\mathbb{R}^3);s}.\end{aligned}$$

Finally, the coercivity estimate (5.8a) for $\mathsf{V}(s)$ follows from (5.7)

Next, we prove the continuity of the single layer operator. For $\varphi \in H^{-1/2}(\Gamma_j)$, let $v := \mathsf{S}_j(s)\varphi$. Then (5.12) follows from

$$\begin{aligned}\frac{\operatorname{Re}s}{|s|}\lambda_j\|v\|^2_{H^1(\mathbb{R}^3);s} \leq \operatorname{Re}\left\langle\varphi,\overline{\mathsf{V}_j(s)\varphi}\right\rangle_{\Gamma_j} &\leq \|\varphi\|_{H^{-1/2}(\Gamma_j)}\|\gamma_{\mathrm{D};j}(s)\,v\|_{H^{1/2}(\Gamma_j)} \\ &\overset{(5.5)}{\leq} C|s|^{1/2}\|\varphi\|_{H^{-1/2}(\Gamma_j)}\|v\|_{H^1(\mathbb{R}^3);s}.\end{aligned}$$

The continuity (5.8b) of $\mathsf{V}(s)$ is a direct consequence of the estimate

$$\begin{aligned}\left|\left\langle\mathsf{V}_j(s)\varphi,\overline{\psi}\right\rangle_{\Gamma_j}\right| = \left|\left\langle\gamma_{\mathrm{D};j}(s)\,\mathsf{S}_j(s)\varphi,\overline{\psi}\right\rangle_{\Gamma_j}\right| &\leq \|\gamma_{\mathrm{D};j}(s)\,\mathsf{S}_j(s)\varphi\|_{H^{1/2}(\Gamma_j)}\|\psi\|_{H^{-1/2}(\Gamma_j)} \\ \leq C|s|^{1/2}\|\mathsf{S}_j(s)\varphi\|_{H^1(\mathbb{R}^3);s}\|\psi\|_{H^{-1/2}(\Gamma_j)} &\leq C\frac{|s|^2}{\lambda_j\operatorname{Re}s}\|\varphi\|_{H^{-1/2}(\Gamma_j)}\|\psi\|_{H^{-1/2}(\Gamma_j)}.\end{aligned}$$

Finally, the dual double layer boundary integral operator $\mathsf{K}'_j(s)$ can be estimated by using the mapping properties of S_j and $\gamma_{\mathrm{N},j}^{\mathrm{ext}}$. Let $v^\sigma := \left(\mathsf{S}_j(s)\varphi\right)|_{\Omega_j^\sigma}$, $\sigma \in \{+,-\}$. Then, we have for all $\varphi \in H^{-1/2}(\Gamma_j)$:

$$\begin{aligned}\left\|\mathsf{K}'_j(s)\varphi\right\|_{H^{-1/2}(\Gamma_j)} = \left\|\{\!\{\mathsf{S}_j(s)\varphi\}\!\}_{\mathrm{N};j}^{\mathrm{ext}}(s)\right\|_{H^{-1/2}(\Gamma_j)} &\leq \sum_{\sigma\in\{+,-\}}\left\|\gamma_{\mathrm{N};j}^{\mathrm{ext},\sigma}(s)\,v^\sigma\right\|_{H^{-1/2}(\Gamma_j)} \\ \overset{(5.7)}{\leq} C\Lambda_j\sum_{\sigma\in\{+,-\}}\|v^\sigma\|_{H^1\left(\Omega_j^\sigma\right);s} &\overset{(5.12)}{\leq} C\frac{\Lambda_j}{\lambda_j}\frac{|s|^{3/2}}{\operatorname{Re}s}\|\varphi\|_{H^{-1/2}(\Gamma_j)}.\end{aligned}$$

Next, we investigate the mapping properties of the operators related to the double layer potential and start with the coercivity estimate of $\mathsf{W}_j(s)$. Let $\psi \in H^{1/2}(\Gamma_j)$ and set $u := \mathsf{D}_j(s)\psi$. The jump relations for the double layer potentials (cf. (3.31)) imply $\gamma_{\mathrm{N};j}^{\mathrm{ext}}(s)\,u = -\mathsf{W}_j(s)\psi$ and $[u]_{\mathrm{D};j}(s) = \psi$. Then, we have

$$\begin{aligned}\operatorname{Re}\left\langle\mathsf{W}_j(s)\psi,\overline{\psi}\right\rangle_{\Gamma_j} &= \operatorname{Re}\left\langle-\gamma_{\mathrm{N};j}^{\mathrm{ext}}(s)\,u,[\overline{u}]_{\mathrm{D};j}(\overline{s})\right\rangle_{\Gamma_j} \\ &\overset{(2.18)}{=} \operatorname{Re}\left(\left(\frac{\overline{s}}{s}\right)^{1/2}\left\langle-\gamma_{\mathrm{N};j}^{\mathrm{ext}}(s)\,u,[\overline{u}]_{\mathrm{D};j}(s)\right\rangle_{\Gamma_j}\right).\end{aligned}$$

We employ (3.8) with $v = w = u$ and $\lambda(\cdot)$, λ_j as in Lem. 3.2 to obtain (cf. (3.4)) with $\mathbb{A}_j^\sigma := \mathbb{A}_j^{\text{ext}}\big|_{\Omega_j^\sigma}$ and $p_j^\sigma := p_j^{\text{ext}}\big|_{\Omega_j^\sigma}$, $\sigma \in \{+,-\}$:

$$\begin{aligned}
\operatorname{Re}\left\langle \mathsf{W}_j(s)\psi, \overline{\psi}\right\rangle_{\Gamma_j} &= \operatorname{Re}\left(\left(\frac{\overline{s}}{s}\right)^{1/2} \sum_{\sigma\in\{+,-\}} \left(\left\langle \mathbb{A}_j^\sigma \nabla u^\sigma, \nabla \overline{u^\sigma}\right\rangle_{\Omega_j^\sigma} + s^2 \left\langle p_j^\sigma u^\sigma, \overline{u^\sigma}\right\rangle_{\Omega_j^\sigma}\right)\right) \\
&= \frac{\operatorname{Re} s}{|s|} \sum_{\sigma\in\{+,-\}} \left(\left\langle \mathbb{A}_j^\sigma \nabla u^\sigma, \nabla \overline{u^\sigma}\right\rangle_{\Omega_j^\sigma} + |s|^2 \left\langle p_j^\sigma u^\sigma, \overline{u^\sigma}\right\rangle_{\Omega_j^\sigma}\right) \\
&\geq \frac{\operatorname{Re} s}{|s|} \lambda_j \sum_{\sigma\in\{+,-\}} \|u^\sigma\|^2_{H^1(\Omega_j^\sigma);s} = \frac{\operatorname{Re} s}{|s|}\lambda_j \|u\|^2_{H^1(\mathbb{R}^3\backslash\Gamma_j);s}.
\end{aligned} \tag{A.1}$$

Thus, the coercivity relation (5.10a) follows from the trace estimate (cf. (5.5))

$$\begin{aligned}
\|\psi\|^2_{H^{1/2}(\Gamma_j)} &= \left\|[u]_{\mathrm{D};j}(s)\right\|^2_{H^{1/2}(\Gamma_j)} \leq \sum_{\sigma\in\{+,-\}} \left\|\gamma^\sigma_{\mathrm{D};j}(s) u^\sigma\right\|^2_{H^{1/2}(\Gamma_j)} \\
&\leq C|s| \sum_{\sigma\in\{+,-\}} \|u^\sigma\|^2_{H^1(\Omega_j^\sigma);s} = C|s| \|u\|^2_{H^1(\mathbb{R}^3\backslash\Gamma_j);s}.
\end{aligned}$$

Next, we prove the continuity of the double layer operator. For $\psi \in H^{1/2}(\Gamma_j)$, let $u := \mathsf{D}_j(s)\psi$. Then (5.13) follows from

$$\begin{aligned}
\frac{\operatorname{Re} s}{|s|}\lambda_j \|u\|^2_{H^1(\mathbb{R}^3\backslash\Gamma_j);s} &\leq \operatorname{Re}\left\langle \mathsf{W}_j(s)\psi, \overline{\psi}\right\rangle_{\Gamma_j} = \operatorname{Re}\left\langle -\gamma^{\text{ext}}_{\mathrm{N};j}(s) u, \overline{\psi}\right\rangle_{\Gamma_j} \\
&\leq \left\|\gamma^{\text{ext}}_{\mathrm{N};j}(s) u\right\|_{H^{-1/2}(\Gamma_j)} \|\psi\|_{H^{1/2}(\Gamma_j)} \\
&\overset{(5.7)}{\leq} C\Lambda_j \|u\|_{H^1(\mathbb{R}^3\backslash\Gamma_j);s} \|\psi\|_{H^{1/2}(\Gamma_j)}.
\end{aligned}$$

The continuity estimates for the operators $\mathsf{W}_j(s)$ and $\mathsf{K}_j(s)$ follow from the combination of this and the trace estimates (Lem. 5.1). ■

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