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ABSTRACT. We present the mathematical theory of one-dimensional infinitely periodic chains of subwavelength resonators. We analyse both Hermitian and non-Hermitian systems. Subwavelength resonances and associated modes can be accurately predicted by a finite dimensional eigenvalue problem involving a capacitance matrix. We are able to compute the Hermitian and non-Hermitian Zak phases, showing that the former is quantised and the latter is not. Furthermore, we show the existence of localised edge modes arising from defects in the periodicity in both the Hermitian and non-Hermitian cases. In the non-Hermitian case, we provide a complete characterisation of the edge modes.

Keywords. Subwavelength resonances, one-dimensional periodic chains of subwavelength resonators, non-Hermitian topological systems, topologically protected edge modes.

AMS Subject classifications. 35B34, 35P25, 35J05, 35C20, 46T25, 78A40.

CONTENTS

1. Introduction

In the past decade, controlling and manipulating waves via interaction with objects at subwavelength scales has gained a lot of attention in both photonics and phononics [26, 27, 34]. One way to achieve subwavelength interactions is to use high-contrast metamaterials, that are media constituted by the insertion of a set of highly contrasted resonators into a background medium. Here, subwavelength means that the size of such resonators is much smaller than the operating wavelength. A typical example in acoustics of such high-contrast resonators are air bubbles in water, which give rise to Minneart resonances [7]. Examples in electromagnetics include high-contrast dielectric particles and plasmonic particles [11, 12].

High-contrast subwavelength resonators have been extensively studied in the three-dimensional case [1, 5, 7, 14, 22]. Recently, an increased interest has been dedicated to topological properties of one-dimensional resonators. Studies include one-dimensional infinite periodic media with continuous material parameters [30], simplified Su-Schrieffer-Heeger (SSH) models [13] and various physical experiments [32, 33]. A rigorous mathematical analysis of the finite one-dimensional case was recently presented [24]. The present work completes this analysis by considering the one-dimensional periodic case. Since the interactions between the subwavelength resonators only imply the nearest neighbors in one-dimension, it also connects the field of high-contrast metamaterials to condensed-matter physics.

In classical wave systems, sources of amplification and dissipation of energy can be modelled by non-real material parameters making the underlying system non-Hermitian, meaning that the left and right eigenmodes are distinct. Our work considers both Hermitian and non-Hermitian systems of subwavelength resonators. We look at these two cases separately as they present deep underlying differences. A particular case of non-Hermitian systems are those with parity-time (PT-) symmetry. Recently, non-Hermitian subwavelength resonators have been studied in three dimensions [10], but to the best of our knowledge no literature exists on the one-dimensional case, which is of interest not only for the study of one-dimensional metamaterials but also of quantum systems as the interactions in both cases are short-range.

In this work, we are able to show that, similarly to the three-dimensional case [6], also in the one-dimensional case it is possible to design subwavelength structures where certain frequencies cannot propagate and are trapped near an edge. This typically happens by introducing a defect in the geometry — in the Hermitian case — or in the material parameters — in the non-Hermitian case. Generally these localised modes are sensitive with respect to small perturbations. In order to manufacture structures presenting the said characteristics, stability with respect to perturbations is required. We take inspiration from quantum mechanics where so-called topological insulators have been extensively studied [15, 17, 18, 19]. The underlying principle of these structures is the existence of a topological invariant that captures the propagation properties of the system. In the present setup, the correct topological invariant is the *Zak phase*. The combination of two structures having different invariants will give rise to modes that are confined at the interface of the structure and that are stable with respect to imperfections. These modes are known as topologically protected edge modes. We first compute the Zak phase for both the Hermitian and non-Hermitian cases and prove that the non-Hermitian one is not quantised. Then we show the existence of the said edge modes and demonstrate their robustness. Moreover, in the non-Hermitian case, we also provide a full characterisation of these modes.

The paper is organized as follows. In Section 2, we present the mathematical setup of the problem. In Section 3, we introduce the Dirichlet-to-Neumann map and solve the exterior problem. Section 4 is dedicated to deriving an asymptotic approximation of the subwavelength resonances and their associated modes. We show that a generalised eigenvalue problem involving the capacitance matrix solves this. The exact tridiagonal structure of the capacitance matrix allows us to study the topological properties of one-dimensional systems of subwavelength resonators without dilute regime assumptions. In Section 5, we focus on the Hermitian case and show numerically the existence of edge modes in the presence of geometrical defects. Ultimately in Section 6, we first explicitly compute the Zak phase and then prove the existence of an edge mode in the case of defects in the periodicity of the material parameters. The robustness of the edge modes in both the Hermitian and non-Hermitian cases is illustrated numerically.

2. Problem statement and preliminaries

2.1. Problem formulation

We consider a one-dimensional system constituted of N periodically repeated disjoint subwavelength resonators $D_i := (x_i^{\mathsf{L}}, x_i^{\mathsf{R}})$, where $(x_i^{\mathsf{L},\mathsf{R}})_{1\leq i\leq N} \subset \mathbb{R}$ are the $2N$ extremities satisfying $x_i^{\mathsf{L}} < x_i^{\mathsf{R}} < x_{i+1}^{\mathsf{L}}$ for any $0 \leq i \leq N-1$. We assume without loss of generality that $x_1^{\mathsf{L}} = 0$. We denote by $(x_i^{\mathsf{L},\mathsf{R}})_{i\in\mathbb{N}}$ the infinite sequence obtained by setting

$$x_{i+N}^{\mathsf{L},\mathsf{R}} := x_i^{\mathsf{L},\mathsf{R}} + L,$$

for some $L > x_N^{\mathsf{R}} - x_1^{\mathsf{L}}$. Furthermore, we let $D^n = \bigcup_{i=1}^N D_i + nL$ so that $D^n = D + nL$ is just the repetition of $D^0 =: D$. We denote the entire structure by $\mathfrak{C} := \bigcup_{n\in\mathbb{Z}} D^n$. We also denote by $\ell_i = x_i^{\mathsf{R}} - x_i^{\mathsf{L}}$ the length of the i-th resonators, and by $s_i = x_{i+1}^{\mathsf{L}} - x_i^{\mathsf{R}}$ the spacing between the i-th and $(i+1)$-th resonator. With our convention, the spacing s_N is the distance that separates the last resonator of a unit cell from the first of the next one:

$$s_N := x_{N+1}^{\mathsf{L}} - x_N^{\mathsf{R}} = L - x_N^{\mathsf{R}} + x_1^{\mathsf{L}}.$$

One notices that $L = \sum_{i=1}^N \ell_i + s_i$ is the size of the unit cell, which we denote by $Y := (0, L)$. The system is illustrated on Figure 1.

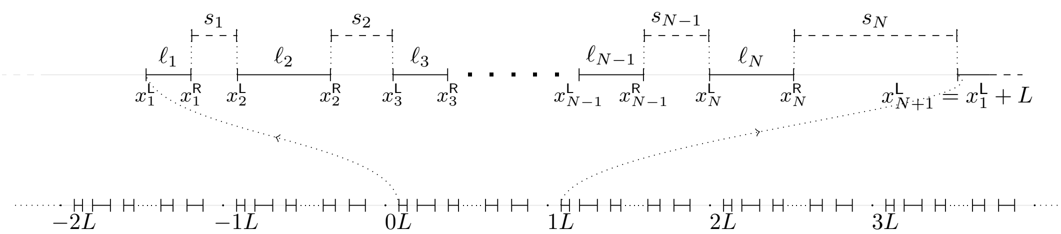


Figure 1. An infinite chain of N subwavelength resonators, with lengths $(\ell_i)_{1<i\leq N}$ and spacings $(s_i)_{1\leq i\leq N-1}$, periodically repeated with period L.

As a wave field $u(t,x)$ propagates in a heterogeneous medium, it is solution to the following one-dimensional wave equation:

$$\frac{1}{\kappa(x)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t,x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\rho(x)}\frac{\mathrm{d}}{\mathrm{d}x}u(t,x)\right) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}\,. \tag{2.1}$$

The parameters $\kappa(x)$ and $\rho(x)$ are the material parameters of the medium. We consider

$$\kappa(x) = \begin{cases}\kappa_i & x \in D_i + L\,\mathbb{Z},\\ \kappa & x \in \mathbb{R}\setminus\mathfrak{C},\end{cases}, \quad \rho(x) = \begin{cases}\rho_b & x \in D_i + L\,\mathbb{Z},\\ \rho & x \in \mathbb{R}\setminus\mathfrak{C},\end{cases} \tag{2.2}$$

where $\rho_b, \rho \in \mathbb{R}_{>0}$. We are interested in both the Hermitian $\kappa_i \in \mathbb{R}_{>0}$ and the non-Hermitian $\kappa_i \in \mathbb{C}$ (with nonzero imaginary parts) cases. In the Hermitian case we typically set for simplicity $\kappa_i = \kappa_b$ for some $\kappa_b \in \mathbb{R}$ for all $1 \leq i \leq N$. We stick with the general notation allowing different κ_i's, but think of them as equal to a positive constant in the Hermitian case.

Following the notation of [7, 22], the wave speeds inside the resonators $\mathfrak{C}$ and inside the background medium $\mathbb{R}\setminus\mathfrak{C}$, are denoted respectively by v_i and v, the wave numbers respectively by k_i and k, and the contrast between the ρ's of the resonators and the background medium by δ:

$$v_i := \sqrt{\frac{\kappa_i}{\rho_b}}, \qquad v := \sqrt{\frac{\kappa}{\rho}}, \qquad k_i := \frac{\omega}{v_i}, \qquad k := \frac{\omega}{v}, \qquad \delta := \frac{\rho_b}{\rho}. \tag{2.3}$$

Up to using a Fourier decomposition in time, we can assume that the total wave field $u(t,x)$ is time-harmonic:

$$u(t,x) = \Re(e^{-\mathbf{i}\omega t}u(x)), \tag{2.4}$$

for a function $u(x)$ which solves the one-dimensional Helmholtz equations:

$$-\frac{\omega^2}{\kappa(x)}u(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\rho(x)}\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = 0, \qquad x \in \mathbb{R}\,. \tag{2.5}$$

In these circumstances of step-wise defined material parameters, the wave problem determined by (2.5) can be rewritten as the following system of coupled one-dimensional Helmholtz equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x^2}u(x) + \dfrac{\omega^2}{v^2}u(x) = 0, & x \in \mathbb{R}\setminus\mathfrak{C}, \\ \dfrac{\mathrm{d}}{\mathrm{d}x^2}u(x) + \dfrac{\omega^2}{v_i^2}u(x) = 0, & x \in D_i + L\,\mathbb{Z}, \\ u|_{\mathsf{R}}(x_n^{\mathsf{L},\mathsf{R}}) - u|_{\mathsf{L}}(x_n^{\mathsf{L},\mathsf{R}}) = 0, & \forall n \in \mathbb{Z}, \\ \left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{R}}(x_n^{\mathsf{L}}) - \delta \left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{L}}(x_n^{\mathsf{L}}) = 0, & \forall n \in \mathbb{Z}, \\ \delta \left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{R}}(x_n^{\mathsf{R}}) - \left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{L}}(x_n^{\mathsf{R}}) = 0, & \forall n \in \mathbb{Z}, \end{cases} \tag{2.6}$$

where for a one-dimensional function w we denote by

$$w|_{\mathsf{L}}(x) := \lim_{\substack{s\to 0\\ s>0}} w(x-s) \quad \text{and} \quad w|_{\mathsf{R}}(x) := \lim_{\substack{s\to 0\\ s>0}} w(x+s)$$

if the limits exist.

2.2. Floquet-Bloch theory

To study this periodic problem we use Floquet-Bloch theory (see, for instance, [9, 25]).

Definition 2.1. Given $f(x) \in L^2(\mathbb{R})$, the Floquet transform of f with period L is defined as

$$\mathcal{F}[f](x,\alpha) := \sum_{n\in\mathbb{Z}} f(x - mL)e^{\mathbf{i}\alpha mL}.$$

The Floquet transform is an analogue of the Fourier transform in the periodic case. Also for the Floquet transform, the original function may be recovered from the collection of the transformed ones via the following Plancherel type inversion:

$$\mathcal{F}^{-1}[g](x) = \frac{L}{2\pi}\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} g(x,\alpha)\,\mathrm{d}\alpha.$$

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be α-quasiperiodic if $e^{-\mathbf{i}\alpha x}f(x)$ is periodic. One remarks that $\mathcal{F}[f](x,\alpha)$ is α-quasiperiodic in x (with period L) and periodic in α (with period $\frac{2\pi}{L}$). We will thus be interested in quasiperiodicities laying in the *first Brillouin zone* $Y^* := \mathbb{R}\,/\frac{2\pi}{L}\,\mathbb{Z} = (-\frac{\pi}{L}, \frac{\pi}{L}]$. We will denote $u^\alpha(x) := \mathcal{F}[u](x,\alpha)$ the Floquet transform of a solution to (2.6). Inserting $u^\alpha(x)$ into (2.5) and using the periodicity of the material parameters $\kappa(x)$ and $\rho(x)$, we find that $u^\alpha(x)$ solves

$$-\frac{\omega^2}{\kappa(x)}u^\alpha(x) - \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)\left[\frac{1}{\rho(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)u^\alpha(x)\right] = 0, \quad x \in \mathbb{R}, \tag{2.7}$$

where $x \mapsto u^\alpha(x)$ is L-periodic.

The following lemma describes the subwavelength resonances — that is ω^α for which (2.7) has a nontrivial solution — in the Hermitian case.

Lemma 2.2. *Let $\kappa \in \mathbb{R}_{>0}$ and $\kappa_i = \kappa_b \in \mathbb{R}_{>0}$ for all i. Then there exists a family of real non-negative eigenfrequencies $(\omega_p^\alpha)_{p\in\mathbb{N}}$ such that, for any $p \in \mathbb{N}$:*

(i) *$\alpha \mapsto \omega_p^\alpha$ is an analytic, $2\pi/L$-periodic function of α for any $p \geq 1$;*

(ii) *$\alpha \mapsto \omega_0^\alpha$ is an analytic $2\pi/L$–periodic function except as $\alpha \in \frac{2\pi}{L}\mathbb{Z}$, where it has a linear behaviour:*
$$\omega_0^\alpha \sim c|\alpha| + \mathcal{O}(\alpha^2) \text{ as } |\alpha| \to 0,$$
corresponding to the crossing of the branches ω_0^α and $\omega_0^{-\alpha}$. Furthermore, a direct computation shows that
$$c = \sqrt{\frac{\int_0^L \frac{1}{\rho(x)}\,\mathrm{d}x}{\int_0^L \frac{1}{\kappa(x)}\,\mathrm{d}x}} \sim v_b \text{ as } \delta \to 0; \tag{2.8}$$

(iii) *For any $\alpha \in (-\pi/L, \pi/L)$, there exists a nontrivial L-periodic function $u_p^\alpha(x)$ solution to (2.7) with $\omega = \omega_p^\alpha$. The function $u_p^\alpha(x)$ is called a* Bloch mode *and can be chosen analytic with respect to the parameter $\alpha \in \mathbb{R}$;*

(iv) *By convention, one can choose*
$$0 = \omega_0^{\alpha=0} < \omega_1^{\alpha=0} \leq \omega_2^{\alpha=0} \leq \dots,$$
where $\omega_p^{\alpha=0}$ is the p-th eigenvalue of the symmetric eigenvalue problem (2.7) at $\alpha = 0$;

(v) *$\omega_p^\alpha = 0$ with $\alpha \in \left(-\frac{\pi}{L}, \frac{\pi}{L}\right)$ if and only if $\alpha = 0$ and $p = 0$, which is associated to the constant Bloch mode. As a consequence,*
$$\omega_p^\alpha > 0 \text{ for any } p \geq 1 \text{ or for } p = 0 \text{ with } \alpha \neq 0;$$

(vi) *$\omega_p^\alpha = \omega_p^{-\alpha}$ and $\overline{u_p^\alpha(x)}$ is a Bloch mode for the quasiperiodicity $-\alpha$.*

Proof. All these properties result from the fact that
$$\alpha \mapsto -\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)\left[\frac{1}{\rho(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)\right]$$
form a holomorphic family of Hermitian operators on the space of $H^1_{\mathrm{per}}((0, L))$, where $H^1_{\mathrm{per}}((0, L))$ is the usual Sobolev space of periodic complex-valued functions on $(0, L)$. Rellich's theorem ensures in particular that $\alpha \mapsto (\omega_p^\alpha)^2$ is analytic, and hence ω_p^α is analytic for all values of α except maybe ω_0^α at $\alpha = 0$. However, the parity property implies that $(\omega_0^\alpha)^2 = \mathcal{O}(\alpha^2)$ as $\alpha \to 0$, and then $\omega_0^\alpha/|\alpha|$ is analytic in α.

The value of c in (2.8) can be found as follows. Denoting $\lambda_0(\alpha) := (\omega_0^\alpha)^2$, we find by differentiating (2.7) with respect to α that
$$\begin{aligned}\left[-\frac{\lambda_0(\alpha)}{\kappa(x)} - \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)\left[\frac{1}{\rho(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)\right]\right]\frac{\mathrm{d}}{\mathrm{d}\alpha}u_0^\alpha(x)& \\ = \frac{\lambda_0'(\alpha)}{\kappa(x)}u_0^\alpha(x) + \mathbf{i}\frac{1}{\rho(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)u_0^\alpha(x) + \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathbf{i}\alpha\right)\left[\frac{1}{\rho(x)}\mathbf{i}\alpha u_0^\alpha(x)\right].&\end{aligned} \tag{2.9}$$
Setting $\alpha = 0$ and using that $\lambda_0(0) = 0$ and $u_0^\alpha(x) \equiv u_0$ is a constant, we obtain that $\lambda_0'(0) = 0$, and then $\frac{\mathrm{d}}{\mathrm{d}\alpha}u_0^0(x) = 0$. Then, differentiating (2.9) with respect to α and setting $\alpha = 0$, we obtain
$$\left[-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left[\frac{1}{\rho(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\right]\right]\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}u_0^0(x) = \frac{\lambda_0''(0)}{\kappa(x)}u_0 - 2\frac{1}{\rho(x)}u_0.$$
From Fredholm's alternative, this equation admits a L-periodic solution if and only if
$$\int_0^L \left(\frac{\lambda_0''(0)}{\kappa(x)}u_0 - 2\frac{1}{\rho(x)}u_0\right)\mathrm{d}x = 0,$$

which yields

$$\frac{\lambda_0''(0)}{2} = \frac{\int_0^L \frac{1}{\rho(x)}\,\mathrm{d}x}{\int_0^L \frac{1}{\kappa(x)}\,\mathrm{d}x},$$

and hence (2.8) holds. The asymptotic expansion (2.8) is obtained then from the formula

$$\frac{\int_0^L \frac{1}{\rho(x)}\,\mathrm{d}x}{\int_0^L \frac{1}{\kappa(x)}\,\mathrm{d}x} = \frac{\frac{1}{\rho_b}\sum_{i=1}^N \ell_i + \frac{1}{\rho}\left(L - \sum_{i=1}^N \ell_i\right)}{\frac{1}{\kappa_b}\sum_{i=1}^N \ell_i + \frac{1}{\kappa}\left(L - \sum_{i=1}^N \ell_i\right)} = \frac{\sum_{i=1}^N \ell_i + \delta\left(L - \sum_{i=1}^N \ell_i\right)}{\frac{1}{v_b^2}\sum_{i=1}^N \ell_i + \frac{\delta}{v^2}\left(L - \sum_{i=1}^N \ell_i\right)} = v_b^2 + O(\delta).$$

■

We recall from [9] that the *subwavelength spectrum* of the operator associated to (2.5) is given by

$$\sigma = \bigcup_{p=0}^{N-1} \bigcup_{\alpha\in Y^*} \omega_p^\alpha,$$

both in the Hermitian and the non-Hermitian cases.

This describes the band structure of the subwavelength spectrum of (2.5): for each p the spectrum traces out bands ω_p^α as α varies. In the Hermitian case, the spectrum is said to have a *subwavelength band gap* if, for some $0 \le p \le N-1$, $\max_\alpha \omega_p^\alpha < \min_\alpha \omega_{p+1}^\alpha$. In the non-Hermitian case, a subwavelength band gap is a connected component of $\mathbb{C}\setminus\sigma$. A band is said to be *non-degenerate* if it does not intersect any other band.

Consequently, we study the equivalent one-dimensional spectral problem in the unit cell Y for the function $u(x,\alpha) := u^\alpha(x)e^{\mathbf{i}\alpha x}$ for $\alpha \in Y^*$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x^2}u(x) + \frac{\omega^2}{v^2}u(x,\alpha) = 0, & x \in \mathbb{R}\setminus\mathfrak{C}, \\ \frac{\mathrm{d}}{\mathrm{d}x^2}u(x) + \frac{\omega^2}{v_i^2}u(x,\alpha) = 0, & x \in D_i + L\,\mathbb{Z}, \\ u|_{\mathsf{R}}(x_n^{\mathsf{L},\mathsf{R}},\alpha) - u|_{\mathsf{L}}(x_n^{\mathsf{L},\mathsf{R}},\alpha) = 0, & \forall n\in\mathbb{Z}, \\ \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{R}}(x_n^{\mathsf{L}},\alpha) - \delta\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{L}}(x_n^{\mathsf{L}},\alpha) = 0, & \forall n\in\mathbb{Z}, \\ \delta\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{R}}(x_n^{\mathsf{R}},\alpha) - \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{L}}(x_n^{\mathsf{R}},\alpha) = 0, & \forall n\in\mathbb{Z}, \\ u(x+L,\alpha) = u(x,\alpha)e^{\mathbf{i}\alpha L} & \text{for almost every } x\in\mathbb{R}, \end{cases} \tag{2.10}$$

and consider the subwavelength resonances for the scattering problem (2.10) by performing an asymptotic analysis in the low-frequency and high-contrast regimes

$$\omega \to 0 \quad \text{as} \quad \delta \to 0. \tag{2.11}$$

For this, we adapt the Dirichlet-to-Neumann approach of [23, 24] to the one-dimensional quasiperiodic problem (2.10).

3. Quasiperiodic Dirichlet-to-Neumann map

In this section, we characterize the Dirichlet-to-Neumann map of the Helmholtz operator on the domain Y with the quasiperiodic boundary conditions. We give a fully explicit expression of this operator in Proposition 3.3, before computing its leading-order asymptotic expansion in terms of δ in Corollary 3.4.

In all what follows, we denote by $H^1(D)$ the usual Sobolev space of complex-valued functions on D and let $H^1_{\mathrm{per}}(\mathbb{R})$ be the the usual Sobolev space of periodic complex-valued functions on $\mathbb{R}$ and $H^1_{\mathrm{per},\alpha}(\mathbb{R}) := \{u : e^{-\mathbf{i}\alpha x}u \in H^1_{\mathrm{per}}(\mathbb{R})\}$.

Throughout the paper, we also denote by $\mathbb{C}^{2N,\alpha}$ the set of quasiperiodic boundary data $f \equiv (f_i^{\mathsf{L},\mathsf{R}})_{i\in\mathbb{Z}}$ satisfying

$$f_{i+N}^{\mathsf{L},\mathsf{R}} = e^{\mathbf{i}\alpha L} f_i^{\mathsf{L},\mathsf{R}},$$

where f_i^{L} (respectively f_i^{R}) refers to the component associate to x_i^{L} (respectively to x_i^{R}). The space of such quasiperiodic sequences is clearly of dimension $2N$. The following lemma provides an explicit expression for the solution to exterior problems on $\mathbb{R}\setminus\mathfrak{C}$.

Lemma 3.1. *Assume that k is not of the form $k = n\pi/s_i$ for some nonzero integer $n \in \mathbb{Z}\setminus\{0\}$ and index $1 \le i \le N$. Then, for any quasiperiodic sequence $(f_i^{\mathsf{L},\mathsf{R}})_{1\le i\le N} \in \mathbb{C}^{2N,\alpha}$, there exists a unique solution $w_f^\alpha \in H^1_{\mathrm{per},\alpha}(\mathbb{R})$ to the exterior problem:*

$$\begin{cases} \left(\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right) w_f^\alpha(x) = 0, & x \in \mathbb{R}\setminus\mathfrak{C}, \\ w_f^\alpha(x_i^{\mathsf{L},\mathsf{R}}) = f_i^{\mathsf{L},\mathsf{R}}, & \forall\, 1 \le i \le N, \\ w_f^\alpha(x+L) = e^{\mathbf{i}\alpha L} w_f^\alpha(x), & x \in \mathbb{R}\setminus\mathfrak{C}. \end{cases} \tag{3.1}$$

Furthermore, when $k \neq 0$, the solution w_f^α reads explicitly

$$w_f^\alpha(x) = a_i e^{\mathbf{i}kx} + b_i e^{-\mathbf{i}kx} \text{ if } x \in (x_i^{\mathsf{R}}, x_{i+1}^{\mathsf{L}}), \qquad \forall i \in \mathbb{Z}, \tag{3.2}$$

where a_i and b_i are given by the matrix-vector product

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = -\frac{1}{2\mathbf{i}\sin(ks_i)} \begin{pmatrix} e^{-\mathbf{i}kx_{i+1}^{\mathsf{L}}} & -e^{-\mathbf{i}kx_i^{\mathsf{R}}} \\ -e^{\mathbf{i}kx_{i+1}^{\mathsf{L}}} & e^{\mathbf{i}kx_i^{\mathsf{R}}} \end{pmatrix} \begin{pmatrix} f_i^{\mathsf{R}} \\ f_{i+1}^{\mathsf{L}} \end{pmatrix}. \tag{3.3}$$

Proof. Identical to [24, Lemma 2.1]. ∎

Definition 3.2 (Dirichlet-to-Neumann map). For any $k \in \mathbb{C}$ which is not of the form $n\pi/s_i$ for some $n \in \mathbb{Z}\setminus\{0\}$ and $1 \le i \le N-1$, the Dirichlet-to-Neumann map with wave number k is the linear operator $\mathcal{T}^{k,\alpha} : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ defined by

$$\mathcal{T}^{k,\alpha}[(f_i^{\mathsf{L},\mathsf{R}})_{1\le i\le N}] = \left(\pm \frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_i^{\mathsf{L},\mathsf{R}})\right)_{1\le i\le N}, \tag{3.4}$$

where w_f^α is the unique solution to (3.1).

The condition that $k \in \mathbb{C}$ is not of the form $n\pi/s_i$ for some $n \in \mathbb{Z}\setminus\{0\}$ and $i \in \mathbb{Z}$ is equivalent to state that k^2 is not a (quasiperiodic) Dirichlet eigenvalue of $-\mathrm{d}^2/\mathrm{d}x^2$ on $\mathbb{R}\setminus\mathfrak{C}$. We consider a minus sign in (3.4) on the abscissa x_i^{L} because $\mathcal{T}^{k,\alpha}[(f_j^{\mathsf{L},\mathsf{R}})_{1\le j\le N}]_i^{\mathsf{L},\mathsf{R}}$ is the normal derivative of w_f^α at $x_i^{\mathsf{L},\mathsf{R}}$, with the normal pointing *outward* the segment $(x_i^{\mathsf{L}}, x_i^{\mathsf{R}})$. This convention allows us to maintain some analogy with the analysis in the three-dimensional setting considered in [23, Section 3].

In the next proposition, we compute $\mathcal{T}^{k,\alpha}$ explicitly.

Proposition 3.3. *The Dirichlet-to-Neumann map $\mathcal{T}^{k,\alpha}$ admits the following explicit matrix representation: for any $k \in \mathbb{C}\setminus\{n\pi/s_i \,|\, n \in \mathbb{Z}\setminus\{0\},\, 1 \le i \le N-1\}$, $f \equiv (f_i^{\mathsf{L},\mathsf{R}})_{1\le i\le N}$, $\mathcal{T}^{k,\alpha}[f] \equiv (\mathcal{T}^{k,\alpha}[f]_i^{\mathsf{L},\mathsf{R}})_{1\le i\le N}$ is given by*

$$\begin{pmatrix} \mathcal{T}^{k,\alpha}[f]_1^{\mathsf{L}} \\ \mathcal{T}^{k,\alpha}[f]_1^{\mathsf{R}} \\ \vdots \\ \mathcal{T}^{k,\alpha}[f]_N^{\mathsf{L}} \\ \mathcal{T}^{k,\alpha}[f]_N^{\mathsf{R}} \end{pmatrix} = T^{k,\alpha} \begin{pmatrix} f_1^{\mathsf{L}} \\ f_1^{\mathsf{R}} \\ \vdots \\ f_N^{\mathsf{L}} \\ f_N^{\mathsf{R}} \end{pmatrix}, \tag{3.5}$$

with

$$T^{k,\alpha} = \begin{pmatrix} -\frac{k\cos(ks_N)}{\sin(ks_N)} & & & & & \frac{k}{\sin(ks_N)}e^{-\mathrm{i}\alpha L} \\ & A^k(s_1) & & & & \\ & & A^k(s_2) & & & \\ & & & \ddots & & \\ & & & & A^k(s_{(N-1)}) & \\ \frac{k}{\sin(ks_N)}e^{\mathrm{i}\alpha L} & & & & & -\frac{k\cos(ks_N)}{\sin(ks_N)} \end{pmatrix}, \tag{3.6}$$

where for any real $\ell \in \mathbb{R}$, $A^k(\ell)$ denotes the 2×2 symmetric matrix given by

$$A^k(\ell) := \begin{pmatrix} -\dfrac{k\cos(k\ell)}{\sin(k\ell)} & \dfrac{k}{\sin(k\ell)} \\ \dfrac{k}{\sin(k\ell)} & -\dfrac{k\cos(k\ell)}{\sin(k\ell)} \end{pmatrix}. \tag{3.7}$$

We will thus use $T^{k,\alpha}$ and $\mathcal{T}^{k,\alpha}$ interchangeably.

Proof. Following the proof of [24, Proposition 2.1], it is easy to infer that

$$\begin{pmatrix} T^k[f]_i^{\mathsf{R}}, \\ T^k[f]_{i+1}^{\mathsf{L}} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_i^{\mathsf{R}}) \\ -\frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_{i+1}^{\mathsf{L}}) \end{pmatrix} = A^k(s_i)\begin{pmatrix} f_i^{\mathsf{R}} \\ f_{i+1}^{\mathsf{L}} \end{pmatrix} \quad \text{for all } i \in \mathbb{Z},$$

where we extend $(f_i^{\mathsf{L},\mathsf{R}})_{1\le i\le N}$ by quasiperiodicity. The values of $(\mathcal{T}^{k,\alpha}[f]_i^{\mathsf{R}}, \mathcal{T}^{k,\alpha}[f]_{i+1}^{\mathsf{L}})$ follow, with $1 \le i \le N-1$. Then, we obtain the following values at x_1^{L} and x_N^{R} by using the quasiperiodicity:

$$\begin{pmatrix} \frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_N^{\mathsf{R}}) \\ -\frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_{N+1}^{\mathsf{L}}) \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_N^{\mathsf{R}}) \\ -e^{\mathrm{i}\alpha L}\frac{\mathrm{d}w_f^\alpha}{\mathrm{d}x}(x_1^{\mathsf{L}}) \end{pmatrix} = A^k(s_i)\begin{pmatrix} f_i^{\mathsf{R}} \\ f_{i+1}^{\mathsf{L}} \end{pmatrix} = A^k(s_N)\begin{pmatrix} f_N^{\mathsf{R}} \\ f_1^{\mathsf{L}}e^{\mathrm{i}\alpha L} \end{pmatrix},$$

which yields (3.5). ■

Remarkably, the $2N\times 2N$ matrix associated to $\mathcal{T}^{k,\alpha}$ is Hermitian. It can be verified that the solution w_f^α to (3.1) with $k \neq 0$ converges as $k \to 0$ to the solution to the same equation with $k = 0$. As it can be expected from the matrix representation (3.5), the operator $\mathcal{T}^{k,\alpha}$ is analytic in a neighbourhood of $k = 0$. In all what follows, we denote by r the convergence radius

$$r := \frac{\pi}{\max_{1\le i\le N} s_i}.$$

We identify $\mathcal{T}^{k,\alpha}$ with the matrix $T^{k,\alpha}$ of (3.6).

Corollary 3.4. The Dirichlet-to-Neumann map $\mathcal{T}^{k,\alpha}$ can be extended to a holomorphic $2N \times 2N$ matrix with respect to the wave number $k \in \mathbb{C}$ on the disk $|k| < r$. Therefore, there exists a family of $2N \times 2N$ matrices $(\mathcal{T}_{2n}^\alpha)_{n\in\mathbb{N}}$ such that $\mathcal{T}^{k,\alpha}$ admits the following convergent series representation:

$$\mathcal{T}^{k,\alpha} = \sum_{n=0}^{+\infty} k^{2n}\mathcal{T}_{2n}^\alpha, \qquad \forall k \in \mathbb{C} \text{ with } |k| < r. \tag{3.8}$$

The matrices $\mathcal{T}_0^\alpha$ and $\mathcal{T}_2^\alpha$ of this series explicitly read

$$\mathcal{T}_0^\alpha = \begin{pmatrix} -\frac{1}{s_N} & & & & & & \frac{1}{s_N}e^{-\mathbf{i}\alpha L} \\ & A_0(s_1) & & & & & \\ & & A_0(s_2) & & & & \\ & & & \ddots & & & \\ & & & & A_0(s_{N-1}) & & \\ \frac{1}{s_N}e^{\mathbf{i}\alpha L} & & & & & & -\frac{1}{s_N} \end{pmatrix}, \tag{3.9}$$

$$\mathcal{T}_2^\alpha = \begin{pmatrix} \frac{1}{3}s_N & & & & \frac{1}{6}s_N e^{-\mathbf{i}\alpha L} \\ & A_2(s_1) & & & \\ & & \ddots & & \\ & & & A_2(s_{N-1}) & \\ \frac{1}{6}s_N e^{\mathbf{i}\alpha L} & & & & \frac{1}{3}s_N \end{pmatrix}, \tag{3.10}$$

where for any $\ell \in \mathbb{R}$, $A_0(\ell)$ and $A_2(\ell)$ are the 2×2 matrices

$$A_0(\ell) := \begin{pmatrix} -1/\ell & 1/\ell \\ 1/\ell & -1/\ell \end{pmatrix}, \qquad A_2(\ell) := \begin{pmatrix} \frac{\ell}{3} & \frac{\ell}{6} \\ \frac{\ell}{6} & \frac{\ell}{3} \end{pmatrix}. \tag{3.11}$$

Proof. The result is immediate by noticing that for a given $\ell > 0$, the matrix $A^k(\ell)$ of (3.7) is analytic with respect to the parameter k on the disk $|k|\ell < \pi$, and its components are even functions of k. The expressions for $\mathcal{T}_0^\alpha$ and $\mathcal{T}_2^\alpha$ follow by computing the Taylor series of $A^k(\ell)$. ∎

Remark 3.5. The expression (3.9) for $\mathcal{T}_0^\alpha$ can be more conveniently stated in terms of its action on a vector $f \equiv (f_i^{\mathsf{L},\mathsf{R}})_{1\leq i\leq N} \in \mathbb{C}^{2N}$ as

$$\begin{cases} \mathcal{T}_0^\alpha[f]_1^{\mathsf{L}} = -\dfrac{1}{s_N}(f_1^{\mathsf{L}} - e^{-\mathbf{i}\alpha L} f_N^{\mathsf{R}}), & \\ \mathcal{T}_0^\alpha[f]_i^{\mathsf{L}} = -\dfrac{1}{s_{i-1}}(f_i^{\mathsf{L}} - f_{i-1}^{\mathsf{R}}), & 2 \leq i \leq N, \\ \mathcal{T}_0^\alpha[f]_i^{\mathsf{R}} = \dfrac{1}{s_i}(f_{i+1}^{\mathsf{L}} - f_i^{\mathsf{R}}), & 1 \leq i \leq N-1, \\ \mathcal{T}_0^\alpha[f]_N^{\mathsf{R}} = \dfrac{1}{s_N}(e^{\mathbf{i}\alpha L} f_1^{\mathsf{L}} - f_N^{\mathsf{R}}), & \end{cases} \tag{3.12}$$

or even more simply

$$\begin{cases} \mathcal{T}^{k,\alpha}[f]_i^{\mathsf{L}} = -\dfrac{1}{s_{i-1}}(f_i^{\mathsf{L}} - f_{i-1}^{\mathsf{R}}) + \dfrac{k^2 s_{i-1}}{3}\left(f_i^{\mathsf{L}} + \dfrac{1}{2}f_{i-1}^{\mathsf{R}}\right) + O(k^4), \\ \mathcal{T}^{k,\alpha}[f]_i^{\mathsf{R}} = \dfrac{1}{s_i}(f_{i+1}^{\mathsf{L}} - f_i^{\mathsf{R}}) + \dfrac{k^2 s_i}{3}\left(f_i^{\mathsf{R}} + \dfrac{1}{2}f_{i+1}^{\mathsf{L}}\right) + O(k^4), \end{cases} \quad \forall i \in \mathbb{Z},$$

for any quasiperiodic sequence $(f_i^{\mathsf{L},\mathsf{R}})_{i\in\mathbb{Z}} \in \mathbb{C}^{2N,\alpha}$.

4. Subwavelength resonances

The one-dimensional problem (2.10) can be rewritten in terms of the Dirichlet-to-Neumann map as a set of coupled ordinary differential equations posed on the periodic segments $\mathfrak{C}$:

$$\begin{cases} \left(\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \dfrac{\omega^2}{v_i^2}\right) u(x,\alpha) = 0, & x \in D_i + L\,\mathbb{Z}, \\ -\dfrac{\mathrm{d}u}{\mathrm{d}x}(x_i^{\mathsf{L}}) = \delta \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{L}} & \text{for all } i \in \mathbb{Z}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(x_i^{\mathsf{R}}) = \delta \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{R}} & \text{for all } i \in \mathbb{Z}, \\ u(x+L) = u(x)e^{\mathbf{i}\alpha L} & \text{for almost every } x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where for a function $u \in H^1_{\mathrm{per},\alpha}(\mathbb{R})$, we use the notation $\mathcal{T}^{\frac{\omega}{v},\alpha}[u] \equiv \mathcal{T}^{\frac{\omega}{v}}[(u(x_i^{\mathsf{L},\mathsf{R}}))_{i\in\mathbb{Z}}]$.

Definition 4.1. Any frequency $\omega^\alpha(\delta)$ such that (4.1) admits a nontrivial solution u is called a *scattering resonance* [35]. *Subwavelength resonances* are those which in addition satisfy

$$\omega^\alpha(\delta) \to 0 \text{ as } \delta \to 0.$$

The associated nontrivial solution $u^\alpha(\omega^\alpha(\delta), \delta)$ is called a *subwavelength resonant mode.*

4.1. A first characterisation of subwavelength resonances based on an explicit representation of the solution

Let us first state a characterisation of the subwavelength resonances which relies on a finite dimensional parametrisation of the solution u.

Lemma 4.2. *The subwavelength scattering resonances ω to the wave problem* (4.1) *are the solution to the $2N \times 2N$ nonlinear eigenvalue problem*

$$\mathcal{A}^\alpha(\omega,\delta) \begin{pmatrix} a_i \\ b_i \end{pmatrix}_{1\le i\le N} = 0, \tag{4.2}$$

where $\mathcal{A}^\alpha(\omega,\delta)$ is the $2N \times 2N$ matrix given by

$$\mathcal{A}(\omega,\delta) := \mathbf{i}\,\mathrm{diag}\left(k_i \begin{pmatrix} -e^{\mathbf{i}k_i x_i^{\mathsf{L}}} & e^{-\mathbf{i}k_i x_i^{\mathsf{L}}} \\ e^{\mathbf{i}k_i x_i^{\mathsf{R}}} & -e^{-\mathbf{i}k_i x_i^{\mathsf{R}}} \end{pmatrix}\right)_{1\le i\le N} - \delta \mathcal{T}^{\frac{\omega}{v},\alpha} \times \mathrm{diag}\left(\begin{pmatrix} e^{\mathbf{i}k_i x_i^{\mathsf{L}}} & e^{-\mathbf{i}k_i x_i^{\mathsf{L}}} \\ e^{\mathbf{i}k_i x_i^{\mathsf{R}}} & e^{-\mathbf{i}k_i x_i^{\mathsf{R}}} \end{pmatrix}\right)_{1\le i\le N}, \tag{4.3}$$

and where $\mathcal{T}^{\frac{\omega}{v},\alpha}$ is the $2N \times 2N$ matrix defined by (3.5). *Furthermore, resonant modes to* (4.1) *correspond to $(a_i\ b_i)^T_{1<i\le N}$ by the formula*

$$u(x) = a_i e^{\mathbf{i}k_i x} + b_i e^{-\mathbf{i}k_i x}, \qquad \forall x \in (x_i^{\mathsf{L}}, x_i^{\mathsf{R}}). \tag{4.4}$$

Proof. Any solution u to (4.1) can be written as (4.4) and the boundary condition of (4.1) reads

$$\pm \mathbf{i}k_i(a_i e^{\mathbf{i}k_i x_i^{\mathsf{L},\mathsf{R}}} - b_i e^{-\mathbf{i}k_i x_i^{\mathsf{L},\mathsf{R}}}) - \delta \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{L},\mathsf{R}} = 0,$$

which can be rewritten as (4.2). ∎

Remark 4.3. With the characterisation of Lemma 4.2, we reduce the spectral problem (4.1) to a nonlinear finite-dimensional eigenvalue problem (4.2). We will exploit this property in the numerical computations.

4.2. Characterisation of the subwavelength resonances based on the Dirichlet-to-Neumann map

Multiplying by a test function $v \in H^1(D)$ and integrating on all the intervals $(x_i^{\mathsf{L}}, x_i^{\mathsf{R}})$, (4.1) can be rewritten in the following weak form: find a nontrivial $u \in H^1(D)$ such that for any $v \in H^1(D)$,

$$a^\alpha(u,v) = 0, \tag{4.5}$$

where a^α is the bilinear form on $H^1(D) \times H^1(D)$ defined by

$$a^\alpha(u,v) := \sum_{i=1}^N \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} \left(\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\overline{v}}{\mathrm{d}x} - \frac{\omega^2}{v_i^2} u\overline{v} \right) \mathrm{d}x - \sum_{i=1}^N \delta \left[\overline{v}(x_i^{\mathsf{R}}) \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{R}} + \overline{v}(x_i^{\mathsf{L}}) \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{L},\mathsf{R}} \right].$$

Following [24], we introduce a new bilinear form $a^\alpha_{\omega,\delta}$ on $H^1(D)\times H^1(D)$:

$$\begin{aligned} a^\alpha_{\omega,\delta}(u,v) := \sum_{i=1}^N & \left[\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\overline{v}}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u\,\mathrm{d}x \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} \overline{v}\,\mathrm{d}x \right] \\ & - \sum_{i=1}^N \left[\frac{\omega^2}{v_i^2} \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u\overline{v}\,\mathrm{d}x + \delta[\overline{v}(x_i^{\mathsf{R}}) \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{R}} + \overline{v}(x_i^{\mathsf{L}}) \mathcal{T}^{\frac{\omega}{v},\alpha}[u]_i^{\mathsf{L}}] \right]. \end{aligned} \tag{4.6}$$

The bilinear form $a^\alpha_{\omega,\delta}(u,v)$ is obtained by adding the rank-one bilinear forms $(u,v) \to \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u\,\mathrm{d}x \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} \overline{v}\,\mathrm{d}x$ to the bilinear form a^α. Clearly, $a^\alpha_{\omega,\delta}$ is an analytic perturbation in ω and δ of the bilinear form $a_{0,0}$ defined by

$$a_{0,0}(u,v) = \sum_{i=1}^N \left[\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\overline{v}}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u\,\mathrm{d}x \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} \overline{v}\,\mathrm{d}x \right],$$

which is continuous coercive on $H^1(D)$. From standard perturbation theory, $a^\alpha_{\omega,\delta}$ remains coercive for sufficiently small complex values of ω and δ.

In order to characterise the subwavelength resonant modes, it is useful to introduce $h_j^\alpha(\omega,\delta)$ the solution to the variational problems

$$a^\alpha_{\omega,\delta}(h_j^\alpha(\omega,\delta), v) = \int_{x_j^{\mathsf{L}}}^{x_j^{\mathsf{R}}} \overline{v}\,\mathrm{d}x, \quad \forall v \in H^1(D), \qquad \forall 1 \le j \le N. \tag{4.7}$$

The functions $h_j^\alpha(\omega,\delta)$ allow to reduce the $2N\times 2N$ problem (4.2) to a $N\times N$ matrix linear system, which is simpler to analyse.

Lemma 4.4. *Let $\omega \in \mathbb{C}$ and $\delta \in \mathbb{R}$ belong to a neighbourhood of zero such that $a^\alpha_{\omega,\delta}$ is coercive. The variational problem* (4.5) *admits a nontrivial solution $u \equiv u(\omega,\delta)$ if and only if the $N\times N$ nonlinear eigenvalue problem*

$$(I - \mathbf{C}^\alpha(\omega,\delta))\boldsymbol{x} = 0 \tag{4.8}$$

has a solution ω and $\boldsymbol{x} := (x_i(\omega,\delta))_{1\le i\le N}$, where $\mathbf{C}^\alpha(\omega,\delta)$ is the matrix given by

$$\mathbf{C}^\alpha(\omega,\delta) \equiv (\mathbf{C}^\alpha(\omega,\delta)_{ij})_{1\le i,j\le N} := \left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} h_j^\alpha(\omega,\delta)\,\mathrm{d}x \right)_{1\le i,j\le N}. \tag{4.9}$$

When it is the case, ω is a subwavelength resonance and an associated resonant mode $u^\alpha(\omega,\delta)$ solution to (4.5) *(equivalently, to* (2.10) *and* (4.1)*) reads*

$$u^\alpha(\omega,\delta) = \sum_{j=1}^N x_j(\omega,\delta) h_j^\alpha(\omega,\delta) \tag{4.10}$$

with $h_j^\alpha(\omega,\delta)$ being defined by (4.7).

Proof. The variational problem (4.5) reads equivalently

$$\begin{aligned} a^\alpha(u,v) = 0 \Leftrightarrow a^\alpha_{\omega,\delta}(u,v) - \sum_{i=1}^N \left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u\,\mathrm{d}x \right) a^\alpha_{\omega,\delta}(u_i^\alpha, v) = 0 \\ \Leftrightarrow u - \sum_{i=1}^N \left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u\,\mathrm{d}x \right) u_i^\alpha = 0. \end{aligned} \tag{4.11}$$

By integrating both sides of (4.11) on $(x_i^{\mathsf{L}}, x_i^{\mathsf{R}})$, we find that the vector $\boldsymbol{x} := \left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u(\omega,\delta)\,\mathrm{d}x\right)_{1\le i\le N}$ solves the linear system

$$\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} u(\omega,\delta)\,\mathrm{d}x - \sum_{j=1}^{N} \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} h_j^\alpha(\omega,\delta)\,\mathrm{d}x \int_{x_j^{\mathsf{L}}}^{x_j^{\mathsf{R}}} u(\omega,\delta)\,\mathrm{d}x = 0, \qquad 1\le i\le N,$$

which is exactly (4.8). Conversely, if (4.8) has a solution, then the second line of (4.11) shows that the solution to (4.5) is given by (4.10). ■

Subwavelength resonances are therefore the characteristic values $\omega \equiv \omega(\delta)$ for which $I - \mathbf{C}^\alpha(\omega,\delta)$ is not invertible.

4.3. Asymptotic expansions of the subwavelength resonances

We now show the existence of N subwavelength resonances for any $\alpha\in Y^*$ and we compute their leading-order asymptotic expansions in terms of δ. We start by computing explicit asymptotic expansions of the functions $h_j^\alpha(\omega,\delta)$ solutions to (4.7). Here and hereafter, the characteristic function of a set S is written as $\mathbb{1}_S$.

PROPOSITION 4.5. *Let $\omega\in\mathbb{C}$ and $\delta\in\mathbb{R}$ belong to a small enough neighbourhood of zero. The unique solution $h_j(\omega,\delta)$ with $1\le j\le N$ to the variational problem (4.7) has the following asymptotic behaviour as $\omega,\delta\to 0$:*

$$\begin{aligned} h_j(\omega,\delta) = {} & \left(\frac{1}{\ell_j} + \frac{\omega^2}{v_j^2\ell_j^2}\right)\mathbb{1}_{(x_j^{\mathsf{L}},x_j^{\mathsf{R}})} \\ & + \delta\left[\frac{\mathbb{1}_{\{2,\dots,N\}}(j)}{\ell_{j-1}^2\ell_j}\frac{1}{s_{j-1}}\mathbb{1}_{(x_{j-1}^{\mathsf{L}},x_{j-1}^{\mathsf{R}})} - \frac{1}{\ell_j^3}\left(\frac{1}{s_{j-1}}+\frac{1}{s_j}\right)\mathbb{1}_{(x_j^{\mathsf{L}},x_j^{\mathsf{R}})} + \frac{\mathbb{1}_{\{1,\dots,N-1\}}(j)}{\ell_j\ell_{j+1}^2 s_j}\mathbb{1}_{(x_{j+1}^{\mathsf{L}},x_{j+1}^{\mathsf{R}})}\right. \\ & \left. + \frac{\delta_{1j}}{\ell_N^2\ell_1}\frac{e^{-\mathrm{i}\alpha L}}{s_N}\mathbb{1}_{(x_N^{\mathsf{L}},x_N^{\mathsf{R}})} + \frac{\delta_{Nj}}{\ell_1^2\ell_N}\frac{e^{\mathrm{i}\alpha L}}{s_N}1_{(x_1^{\mathsf{L}},x_1^{\mathsf{R}})} + \widetilde{h}_{j,0,1}\right] + O((\omega^2+\delta)^2), \end{aligned} \tag{4.12}$$

where $\widetilde{h}_{j,0,1}$ is some (quadratic) functions satisfying

$$\int_{x_i^-}^{x_i^+} \widetilde{h}_{j,0,1}\,\mathrm{d}x = 0, \qquad \forall 1\le i\le N.$$

Proof. From the definition of $a_{\omega,\delta}^\alpha$, the function $h_j^\alpha \equiv h_j^\alpha(\omega,\delta)$ satisfies the following differential equation written in strong form:

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}h_j^\alpha - \dfrac{\omega^2}{v_b^2}h_j^\alpha + \displaystyle\sum_{i=1}^{N}\left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} h_j^\alpha\,\mathrm{d}x\right)\mathbb{1}_{(x_i^{\mathsf{L}},x_i^{\mathsf{R}})} = \mathbb{1}_{(x_j^{\mathsf{L}},x_j^{\mathsf{R}})} & \text{in } \displaystyle\bigsqcup_{i=1}^{N}(x_i^{\mathsf{L}},x_i^{\mathsf{R}}), \\ -\dfrac{\mathrm{d}h_j^\alpha}{\mathrm{d}x}(x_i^{\mathsf{L}}) = \delta\mathcal{T}^{\frac{\omega}{v},\alpha}[h_j^\alpha]_i^{\mathsf{L}} & \text{for all } 1\le i\le N, \\ \dfrac{\mathrm{d}h_j^\alpha}{\mathrm{d}x}(x_i^{\mathsf{R}}) = \delta\mathcal{T}^{\frac{\omega}{v},\alpha}[h_j^\alpha]_i^{\mathsf{R}} & \text{for all } 1\le i\le N. \end{cases} \tag{4.13}$$

Since $\mathcal{T}^{\frac{\omega}{v},\alpha}$ is analytic in ω^2, it follows that $h_j^\alpha(\omega,\delta)$ is analytic in ω^2 and δ: there exist functions $(h_{j,2p,k})_{p\ge0,k\ge0}$ such that $h_j^\alpha(\omega,\delta)$ can be written as the following convergent series in $H^1(D)$:

$$h_j^\alpha(\omega,\delta) = \sum_{p,k=0}^{+\infty}\omega^{2p}\delta^k h_{j,2p,k}. \tag{4.14}$$

By using Corollary 3.4 and identifying powers of ω and δ, we obtain the following equations characterizing the functions $(h_{j,2p,k})_{p\geq 0,k\geq 0}$:

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}h_{j,2p,k} + \displaystyle\sum_{i=1}^N \left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} h_{j,2p,k}\,\mathrm{d}x\right)\mathbb{1}_{(x_i^{\mathsf{L}},x_i^{\mathsf{R}})} = \dfrac{1}{v_j^2}h_{j,2p-2,k} + \mathbb{1}_{(x_j^{\mathsf{L}},x_j^{\mathsf{R}})}\delta_{0p}\delta_{0k} & \text{in } D, \\ -\dfrac{\mathrm{d}h_{j,2p,k}}{\mathrm{d}x}(x_i^{\mathsf{L}}) = \displaystyle\sum_{n=0}^{p}\dfrac{1}{v^{2n}}\mathcal{T}_{2n}^{\alpha}[h_{j,2p-2n,k-1}]_i^{\mathsf{L}}, & 1\leq i\leq N, \\ \dfrac{\mathrm{d}h_{j,2p,k}}{\mathrm{d}x}(x_i^{\mathsf{R}}) = \displaystyle\sum_{n=0}^{p}\dfrac{1}{v^{2n}}\mathcal{T}_{2n}^{\alpha}[h_{j,2p-2n,k-1}]_i^{\mathsf{R}}, & 1\leq i\leq N, \end{cases} \tag{4.15}$$

with the convention that $h_{j,2p,k} = 0$ for negative indices p and k. It can then be easily obtained by induction that

$$h_{j,2p,0} = \frac{\mathbb{1}_{(x_j^{\mathsf{L}},x_j^{\mathsf{R}})}}{v_j^{2p}\ell_j^{p+1}} \text{ for any } p\geq 0, \qquad 1\leq j\leq N.$$

Then, for $p=0$ and $k=1$, we find that $h_{j,0,1}$ satisfies

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}h_{j,0,1} + \displaystyle\sum_{i=1}^N \left(\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} h_{j,0,1}\,\mathrm{d}x\right)\mathbb{1}_{(x_i^{\mathsf{L}},x_i^{\mathsf{R}})} = 0 & \text{in } D, \\ -\dfrac{\mathrm{d}h_{j,0,1}}{\mathrm{d}x}(x_i^{\mathsf{L}}) = \mathcal{T}_0^{\alpha}[h_{j,0,0}]_i^{\mathsf{L}} & \text{for all } 1\leq i\leq N, \\ \dfrac{\mathrm{d}h_{j,0,1}}{\mathrm{d}x}(x_i^{\mathsf{R}}) = \mathcal{T}_0^{\alpha}[h_{j,0,0}]_i^{\mathsf{R}} & \text{for all } 1\leq i\leq N. \end{cases} \tag{4.16}$$

From (3.12) with $f_i^{\mathsf{L},\mathsf{R}} := h_{j,0,0}(x_i^{\mathsf{L},\mathsf{R}}) = \delta_{ij}/\ell_j$ for $1\leq i\leq N$, we obtain

$$\begin{cases} \mathcal{T}_0[h_{j,0,0}]_1^{\mathsf{L}} = -\dfrac{1}{\ell_j}\dfrac{1}{s_N}(\delta_{1j} - \delta_{Nj}e^{-\mathbf{i}\alpha L}), & \\ \mathcal{T}_0[h_{j,0,0}]_i^{\mathsf{L}} = -\dfrac{1}{\ell_j}\dfrac{1}{s_{i-1}}\left(\delta_{ij} - \delta_{(i-1)j}\right) & \text{for } 2\leq i\leq N, \\ \mathcal{T}_0[h_{j,0,0}]_i^{\mathsf{R}} = \dfrac{1}{\ell_j}\dfrac{1}{s_i}\left(\delta_{i+1} - \delta_{ij}\right) & \text{for } 1\leq i\leq N-1, \\ \mathcal{T}_0[h_{j,0,0}]_N^{\mathsf{R}} = \dfrac{1}{\ell_j}\dfrac{1}{s_N}\left(e^{\mathbf{i}\alpha L}\delta_{1j} - \delta_{Nj}\right). & \end{cases}$$

Multiplying (4.16) by $\mathbb{1}_{(x_i^{\mathsf{L}},x_i^{\mathsf{R}})}$ and integrating by parts, we find that

$$\begin{aligned} \int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}} h_{j,0,1}\,\mathrm{d}x &= \frac{1}{\ell_i}\left[\mathcal{T}_0^{\alpha}[h_{j,0,0}]_i^{\mathsf{L}} + \mathcal{T}_0^{\alpha}[h_{j,0,0}]_i^{\mathsf{R}}\right] \\ &= \frac{1}{\ell_i\ell_j}\frac{1}{s_{i-1}}(\delta_{(i-1)j} - \delta_{ij})\mathbb{1}_{\{2,\dots,N\}}(i) + \frac{1}{\ell_i\ell_j}\frac{1}{s_i}(\delta_{(i+1)j} - \delta_{ij})\mathbb{1}_{\{1,\dots,N-1\}}(i) \\ &\quad + \frac{1}{\ell_i\ell_j}\frac{1}{s_N}\left(\delta_{Nj}e^{-\mathbf{i}\alpha L} - \delta_{1j}\right)\delta_{i1} + \frac{1}{\ell_i\ell_j}\frac{1}{s_N}\left(e^{\mathbf{i}\alpha L}\delta_{1j} - \delta_{Nj}\right)\delta_{iN}. \end{aligned}$$

Isolating the different cases yields

$$\begin{cases} \dfrac{1}{\ell_1\ell_N}\dfrac{1}{s_N}e^{-\mathbf{i}\alpha L} & \text{if } i=1,\ j=N,\\ \dfrac{1}{\ell_{j-1}\ell_j}\dfrac{1}{s_{j-1}} & \text{if } i=j-1,\ 2\le j\le N,\\ -\dfrac{1}{\ell_j^2}\left(\dfrac{\mathbb{1}_{\{2,\dots,N\}}(j)}{s_{j-1}}+\dfrac{\mathbb{1}_{\{1,\dots,N-1\}}(j)}{s_j}+\dfrac{\delta_{1j}+\delta_{jN}}{s_N}\right) & \text{if } i=j,\\ \dfrac{1}{\ell_j\ell_{j+1}}\dfrac{1}{s_j} & \text{if } i=j+1,\ 1\le j\le N-1,\\ \dfrac{1}{\ell_1\ell_N}\dfrac{1}{s_N}e^{\mathbf{i}\alpha L} & \text{if } i=N,\ j=1. \end{cases}$$

Using Fredholm's alternative, this allows to infer that $h_{j,0,1}$ can be written as

$$\begin{aligned} h_{j,0,1} = \frac{\mathbb{1}_{\{2,\dots,N\}}(j)}{\ell_{j-1}^2\ell_j}\frac{1}{s_{j-1}}\mathbb{1}_{(x_{j-1}^{\mathsf{L}},x_{j-1}^{\mathsf{R}})}-\frac{1}{\ell_j^3}\left(\frac{1}{s_{j-1}}+\frac{1}{s_j}\right)\mathbb{1}_{(x_j^{\mathsf{L}},x_j^{\mathsf{R}})}+\frac{\mathbb{1}_{\{1,\dots,N-1\}}(j)}{\ell_j\ell_{j+1}^2}\frac{1}{s_j}\mathbb{1}_{(x_{j+1}^{\mathsf{L}},x_{j+1}^{\mathsf{R}})} \\ +\frac{\delta_{1j}}{\ell_N^2\ell_1}\frac{e^{-\mathbf{i}\alpha L}}{s_N}\mathbb{1}_{(x_N^{\mathsf{L}},x_N^{\mathsf{R}})}+\frac{\delta_{Nj}}{\ell_1^2\ell_N}\frac{e^{\mathbf{i}\alpha L}}{s_N}\mathbb{1}_{(x_1^{\mathsf{L}},x_1^{\mathsf{R}})}+\widetilde{h}_{j,0,1}, \end{aligned} \tag{4.17}$$

where $\widetilde{h}_{j,0,1}$ is a function (in fact, a second order polynomial) satisfying $\int_{x_i^{\mathsf{L}}}^{x_i^{\mathsf{R}}}\widetilde{h}_{j,0,1}\,\mathrm{d}x=0$ for any $1\le i\le N$, with the convention $s_0=s_N$. Furthermore, $\widetilde{h}_{j,0,1}$ is identically zero on $(x_i^{\mathsf{L}},x_i^{\mathsf{R}})$, where $i\notin\{j-1,j,j+1\}$. ∎

Next, we define the (quasiperiodic) capacitance matrix similar to the three-dimensional case [2, 4, 6].

Definition 4.6 (Quasiperiodic capacitance matrix). Consider the solutions $V_i^\alpha:\mathbb{R}\to\mathbb{R}$ of the problem

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}V_i^\alpha=0 & \mathbb{R}\setminus\mathfrak{C},\\ V_i^\alpha(x)=\delta_{i,j} & x\in D_j,\\ V_i^\alpha(x+mL)=e^{\mathbf{i}\alpha mL}V_i^\alpha(x) & m\in\mathbb{Z}. \end{cases} \tag{4.18}$$

Then the capacitance matrix is defined coefficient-wise by

$$\mathcal{C}_{ij}^\alpha=-\int_{\partial D_i}\frac{\partial V_j^\alpha}{\partial\nu}\,\mathrm{d}\sigma,$$

where ν is the outward-pointing normal.

Lemma 4.7. *The capacitance matrix is given by*

$$\mathcal{C}_{ij}^\alpha:=-\frac{1}{s_{j-1}}\delta_{i(j-1)}+\left(\frac{1}{s_{j-1}}+\frac{1}{s_j}\right)\delta_{ij}-\frac{1}{s_j}\delta_{i(j+1)}-\delta_{1j}\delta_{iN}\frac{e^{-\mathbf{i}\alpha L}}{s_N}-\delta_{1i}\delta_{jN}\frac{e^{\mathbf{i}\alpha L}}{s_N},$$

that is,

$$\mathcal{C}^\alpha=\begin{pmatrix} \frac{1}{s_N}+\frac{1}{s_1} & -\frac{1}{s_1} & & & -\frac{e^{-\mathbf{i}\alpha L}}{s_N}\\ -\frac{1}{s_1} & \frac{1}{s_1}+\frac{1}{s_2} & -\frac{1}{s_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{1}{s_{N-1}}\\ -\frac{e^{\mathbf{i}\alpha L}}{s_N} & & & -\frac{1}{s_{N-1}} & \frac{1}{s_{N-1}}+\frac{1}{s_N} \end{pmatrix}.$$

Proof. One notices that the solutions V_i^α to (4.18) for $1 \leq i \leq N-1$ are given by

$$V_i(x) = \begin{cases} \dfrac{1}{s_{i-1}}(x - x_i^{\mathsf{L}}), & x_{i-1}^{\mathsf{R}} \leq x \leq x_i^{\mathsf{L}}, \\ 1, & x_i^{\mathsf{L}} \leq x \leq x_i^{\mathsf{R}}, \\ -\dfrac{1}{s_i}(x - x_{i+1}^{\mathsf{L}}), & x_i^{\mathsf{R}} \leq x \leq x_{i+1}^{\mathsf{L}}, \\ 0 & \text{else}, \end{cases}$$

while for bigger and smaller i we multiply by the corresponding $e^{\mathbf{i}\alpha mL}$ factor. Derivation with respect to the outward-pointing normal and integrating on the boundary just means

$$\mathcal{C}_{ij}^\alpha = -\left(- \left.\frac{\mathrm{d}V_i^\alpha}{\mathrm{d}x}\right|_{\mathsf{L}} (x_j^{\mathsf{L}}) + \left.\frac{\mathrm{d}V_i^\alpha}{\mathrm{d}x}\right|_{\mathsf{R}} (x_j^{\mathsf{R}}) \right). \tag{4.19}$$

Evaluating (4.19) concludes the proof. ∎

Corollary 4.8. We have the following asymptotic expansion for the matrix $\mathbf{C}^\alpha(\omega,\delta)$ defined in (4.9):

$$\mathbf{C}^\alpha(\omega,\delta) = I + \omega^2 V^{-2} L^{-1} - \delta L^{-1} C^\alpha L^{-1} + O((\omega^2+\delta)^2), \tag{4.20}$$

where L is the length matrix $L \coloneqq \mathrm{diag}((\ell_i))$ and $V \coloneqq \mathrm{diag}((v_i))$ the material parameter matrix.

Proof. Integrating the asymptotic expansion (4.12) of $h_j(\omega,\delta)$ on the interval $(x_i^{\mathsf{L}}, x_i^{\mathsf{R}})$, we obtain

$$\begin{aligned} \mathcal{C}_{ij}^\alpha(\omega,\delta) = {} & \left(1 + \frac{\omega^2}{v_i^2 \ell_i}\right)\delta_{ij} \\ & + \delta \left[\frac{\mathbb{1}_{\{1,\dots,N-1\}}(i)}{\ell_i \ell_j} \frac{1}{s_{j-1}} \delta_{i(j-1)} - \frac{1}{\ell_i\ell_j}\left(\frac{1}{s_{j-1}} + \frac{1}{s_j}\right)\delta_{ij} + \frac{\mathbb{1}_{\{2,\dots,N\}}(i)}{\ell_j\ell_i}\frac{1}{s_j}\delta_{i(j+1)} \right. \\ & \left. + \frac{\delta_{1j}\delta_{iN}}{\ell_N\ell_1}\frac{e^{-\mathbf{i}\alpha L}}{s_N} + \frac{\delta_{Nj}\delta_{1N}}{\ell_1\ell_N}\frac{e^{\mathbf{i}\alpha L}}{s_N} \right] + \mathcal{O}((\omega^2+\delta)^2). \end{aligned} \tag{4.21}$$

This yields the result. ∎

It is thus useful to introduce the *generalised capacitance matrix*

$$\mathcal{C}_{\mathrm{G}}^\alpha \coloneqq V^2 L^{-1} \mathcal{C}^\alpha. \tag{4.22}$$

Proposition 4.9. *Assume that the eigenvalues of $\mathcal{C}_{\mathrm{G}}^\alpha$ are simple. Then the N subwavelength band functions $(\alpha \mapsto \omega_i^\alpha)_{1\leq i\leq N}$ satisfy to the first order*

$$\omega_i^\alpha = \pm\sqrt{\delta\lambda_i} + \mathcal{O}(\delta),$$

where $(\lambda_i^\alpha)_{1\leq i\leq N}$ are the eigenvalues of the eigenvalue problem

$$\mathcal{C}_{\mathrm{G}}^\alpha \boldsymbol{a}_i = \lambda_i^\alpha \boldsymbol{a}_i, \qquad 1 \leq i \leq N. \tag{4.23}$$

We select the N values of $\pm\sqrt{\delta\lambda_i}$ having positive real part.

Remark that in the Hermitian case it is possible to reformulate (4.23) into a symmetric eigenvalue problem so that the eigenvalues are real.

Proof. From Lemma 4.4, we know that (4.1) has a solution if and only if

$$(I - \mathbf{C}^\alpha(\omega,\delta))\boldsymbol{x} = 0$$

for some nonzero $\boldsymbol{x}$. Applying the asymptotic expansion from Corollary 4.8, we obtain that the above equation is equivalent to

$$0 = \omega^2 V^{-2} L^{-1}\boldsymbol{x} - \delta L^{-1}\,\mathcal{C}^\alpha\, \underbrace{L^{-1}\boldsymbol{x}}_{:=\boldsymbol{y}} + \mathcal{O}((\omega^2+\delta)^2)$$

$$\Leftrightarrow V^2 L^{-1}\,\mathcal{C}^\alpha\,\boldsymbol{y} = \frac{\omega^2}{\delta}\boldsymbol{y} + \mathcal{O}((\omega^2+\delta)^2),$$

meaning that $\frac{\omega^2}{\delta}$ must be approximately an eigenvalue of $V^2L^{-1}\,\mathcal{C}^\alpha = \mathcal{C}^\alpha_{\mathrm{G}}$. ∎

We refer to [22, Proposition 3.7] for a generalisation of Proposition 4.9. The capacitance matrix provides also an approximation of the eigenmodes.

Lemma 4.10. *Let u^α be a subwavelength resonant eigenmode corresponding to ω^α from Proposition 4.9. Let $\boldsymbol{a}$ be the corresponding eigenvector of the generalised capacitance matrix. Then*

$$u^\alpha(x) = \sum_j \boldsymbol{a}^{(j)} V_j^\alpha(x) + \mathcal{O}(\delta),$$

where V_j^α are the functions from (4.18) *in Definition 4.6 and $\boldsymbol{a}^{(j)}$ denotes the j-th entry of the eigenvector.*

Proof. We sketch the proof, referring to [22] for more details. We consider the case $N = 2$ as $N > 2$ is only notationally more difficult. Let $u^\alpha(x)$ be a resonant eigenmode. According to Lemma 4.4, we may represent the resonant mode (inside the resonators) as

$$u^\alpha(x) = \ell_1\boldsymbol{a}^{(1)}h_1^\alpha + \ell_2\boldsymbol{a}^{(2)}h_2^\alpha + \mathcal{O}(\delta).$$

Remark that we used the change of basis L^{-1} in Proposition 4.9, so the approximation of the $\boldsymbol{x}$ of Lemma 4.4 is $L^{-1}\boldsymbol{a}$. The asymptotic expansion of Proposition 4.5 shows that $h_i^\alpha = \frac{1}{\ell_i}\mathbb{1}_{D_i} + \mathcal{O}(\delta)$, so that we get the result inside the resonators.

In order to obtain a solution outside, we may apply Lemma 3.1. Expanding the result of Lemma 3.1 for small δ, we obtain a linear interpolation between the boundary points, that is V_j^α outside of the resonators. ∎

5. Hermitian case

In this section, we analyse in closer detail the Hermitian case. For simplicity, we consider the case when $v_i = v_b$ for all i for some $v_b \in \mathbb{R}_{>0}$. One remarks that in this case the eigenvalue problem (4.23) may be simplified by finding eigenvalues of $L^{-1}\,\mathcal{C}^\alpha$ and multiplying the eigenvalues by v_b^2.

In the general case, we have to solve the generalised eigenvalue problem

$$\mathcal{C}^\alpha\,\boldsymbol{a}_i = v_b^{-2}\lambda_i L\boldsymbol{a}_i. \tag{5.1}$$

After a change of basis, we recover a symmetric eigenvalue problem having the same eigenvalues as (5.1)

$$L^{-\frac{1}{2}}\,\mathcal{C}^\alpha\,L^{-\frac{1}{2}}\boldsymbol{b}_i = v_b^{-2}\lambda_i\boldsymbol{b}_i. \tag{5.2}$$

From (5.2), we see that in the Hermitian case the subwavelength resonances are real.

5.1. Dirac degeneracy and Zak phase

We first prove the following result.

Lemma 5.1. *The eigenspace associated to C^α has dimension at most two.*

Proof. This is a consequence of the tridiagonal structure of $\mathcal{C}^\alpha$: one can extract from $\mathcal{C}^\alpha - \lambda_p^\alpha L$ a full rank minor of dimension $(N-2)\times(N-2)$ which is an upper triangular matrix with diagonal $-\frac{1}{s_1}\cdots-\frac{1}{s_{N-2}}$. ∎

The following lemma concerns degeneracies of the capacitance matrix.

Lemma 5.2. *Assume that $N = 2$. The only configuration such that* (5.2) *admits a double eigenvalue is the one with $\ell_1 = \ell_2$ and $s_1 = s_2$. Moreover, this double eigenvalue occurs at $\alpha = \pm\frac{\pi}{L}$, and $\mathcal{C}^{\pm\frac{\pi}{L}} = 2s_1 I$, where I is the identity matrix.*

Proof. Problem (5.2) reduces to find the eigenvalue of

$$L^{-\frac{1}{2}}\,\mathcal{C}^\alpha\,L^{-\frac{1}{2}} = \begin{pmatrix} \left(\frac{1}{s_1}+\frac{1}{s_2}\right)\ell_1^{-1} & \left(-\frac{1}{s_1}-\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right)\ell_1^{-\frac{1}{2}}\ell_2^{-\frac{1}{2}} \\ \left(-\frac{1}{s_1}-\frac{1}{s_2}e^{-\mathbf{i}\alpha L}\right)\ell_1^{-\frac{1}{2}}\ell_2^{-\frac{1}{2}} & \left(\frac{1}{s_1}+\frac{1}{s_2}\right)\ell_2^{-1} \end{pmatrix}. \tag{5.3}$$

The characteristic polynomial of this matrix is

$$\begin{aligned} P(\lambda) &= \det\left(L^{-\frac{1}{2}}\,\mathcal{C}^\alpha\,L^{-\frac{1}{2}} - \lambda I\right) && (5.4)\\ &= \left(\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\ell_1^{-1}-\lambda\right)\left(\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\ell_2^{-1}-\lambda\right) - \ell_1^{-1}\ell_2^{-1}\left|\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right|^2 && (5.5)\\ &= \lambda^2 - \left(\frac{1}{s_1}+\frac{1}{s_2}\right)\left(\ell_1^{-1}+\ell_2^{-1}\right)\lambda + \ell_1^{-1}\ell_2^{-1}\left[\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2 - \left|\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right|\right]. && (5.6) \end{aligned}$$

Therefore, a multiple eigenvalue occurs when the discriminant of this second order polynomial vanishes, which is the case when

$$\begin{aligned} 0 &= \left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2\left(\ell_1^{-1}+\ell_2^{-1}\right)^2 - 4\ell_1^{-1}\ell_2^{-1}\left[\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2 - \left|\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right|\right] && (5.7)\\ &= \left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2\left(\ell_1^{-1}-\ell_2^{-1}\right)^2 + 4\ell_1^{-1}\ell_2^{-1}\left|\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right|^2. && (5.8) \end{aligned}$$

This readily implies $\ell_1 = \ell_2$, and then

$$\begin{cases} \dfrac{1}{s_1}+\dfrac{1}{s_2}\cos(\alpha L) = 0, \\ \dfrac{1}{s_2}\sin(\alpha L) = 0. \end{cases}$$

For this system to admit a solution with $0 < s_1, s_2$, it is necessary that $\alpha = \pm\frac{\pi}{L}$, and then we must have $s_1 = s_2$. ■

Therefore, we study the eigenvalues of $\mathcal{C}^\alpha$ for regularly spaced dimers of resonators (i.e., $N = 2$, $\ell_1 = \ell_2$ and $s_1 = s_2$). Let us rewrite (5.3) only in terms of s_1 and ℓ_1:

$$L^{-\frac{1}{2}}\,\mathcal{C}^\alpha\,L^{-\frac{1}{2}} = \frac{2}{\ell_1 s_1}\begin{pmatrix} 1 & -e^{\mathbf{i}\alpha L/2}\cos(\alpha L/2) \\ -e^{-\mathbf{i}\alpha L/2}\cos(\alpha L/2) & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\begin{aligned} \lambda_0(\alpha) &= \frac{2}{\ell_1 s_1}\left(1-\cos\left(\frac{\alpha L}{2}\right)\right) = \frac{4}{\ell_1 s_1}\sin^2\left(\frac{\alpha L}{4}\right),\\ \lambda_1(\alpha) &= \frac{2}{\ell_1 s_1}\left(1+\cos\left(\frac{\alpha L}{2}\right)\right) = \frac{4}{\ell_1 s_1}\cos^2\left(\frac{\alpha L}{4}\right). \end{aligned}$$

An associated family of eigenvectors read

$$\begin{pmatrix} 1 \\ e^{-\mathbf{i}\frac{\alpha L}{2}} \end{pmatrix}, \begin{pmatrix} 1 \\ -e^{-\mathbf{i}\frac{\alpha L}{2}} \end{pmatrix}.$$

Subwavelength resonances then read

$$\omega_0^\alpha = \frac{2}{\sqrt{\ell_1 s_1}} v_b \delta^{\frac{1}{2}}\left|\sin\left(\frac{\alpha L}{4}\right)\right| + \mathcal{O}(\delta), \qquad \omega_1^\alpha = \frac{2}{\sqrt{\ell_1 s_1}} v_b \delta^{\frac{1}{2}}\left|\cos\left(\frac{\alpha L}{4}\right)\right| + \mathcal{O}(\delta). \tag{5.9}$$

Hence, at leading order in δ, a band inversion occurs at $\alpha = \pm\frac{\pi}{L}$. Furthermore, (5.9) shows that at $\alpha = \pm\frac{\pi}{L}$ the bands form a *Dirac degeneracy.* Typically, breaking the symmetry of the structure results in the Dirac cone to open into a band gap [8, Section 4]. We show this in Section 5.2.

The following lemma gives explicit formulas for the eigenvectors of the capacitance matrix.

LEMMA 5.3. *Assume that $\ell_1 = \ell_2$. In the general case, the eigenvalues of the capacitance matrix are given by*

$$\lambda_1^\alpha = \ell_1^{-1}\left[\left(\frac{1}{s_1}+\frac{1}{s_2}\right) - \left|\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right|\right], \qquad \lambda_2^\alpha = \ell_1^{-1}\left[\left(\frac{1}{s_1}+\frac{1}{s_2}\right) + \left|\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right|\right].$$

An associated pair of eigenvector is given by

$$a_1^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{-\mathbf{i}\theta_\alpha}\end{pmatrix}, \qquad a_2^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -e^{-\mathbf{i}\theta_\alpha}\end{pmatrix},$$

where θ_α is the argument such that

$$-\left(\frac{1}{s_1}+\frac{1}{s_2}e^{\mathbf{i}\alpha L}\right) = \rho e^{\mathbf{i}\theta_\alpha}. \tag{5.10}$$

DEFINITION 5.4 (Zak phase). For a non-degenerate band ω_j^α, we let u_j^α be a family of normalised eigenmodes which depend continuously on α. Then we define the (Hermitian) Zak phase as

$$\varphi_j^{\mathrm{zak}} := \mathbf{i}\int_{Y^*}\left\langle u_j^\alpha, \frac{\partial}{\partial\alpha}u_j^\alpha\right\rangle \mathrm{d}\alpha, \tag{5.11}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product.

Using Lemma 5.3, we obtain the Zak phase of the structure.

PROPOSITION 5.5. *Let $N = 2$ and $\ell_1 = \ell_2$. Then, we have*

$$\varphi_j^{zak} = \begin{cases}\pi \text{ if } s_1 \geq s_2,\\ 0 \text{ if } s_1 < s_2.\end{cases}$$

One can prove Proposition 5.5 using a similar approach to the one in [6]. We suggest a different proof whose presentation is postponed to Section 6, where it will result as a special case of the more general Theorem 6.4.

5.2. Localised edge modes generated by geometrical defects

In this subsection, we study an infinite structure composed by two periodic parts. We consider this structures as having a geometrical defect in the periodicity, see Figure 2.

Such structures have been studied in the case of tight-binding Hamiltonian systems [16, 20, 21] and for an SSH chain of resonators in $\mathbb{R}^3$ [6].

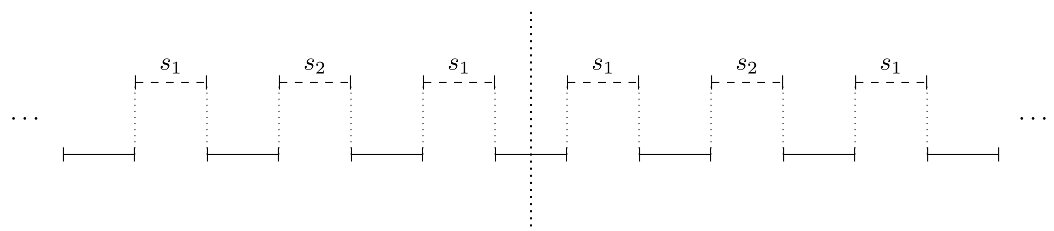


FIGURE 2. Infinite structure with a geometrical defect.

The peculiarity of such defect structures is the support for edge modes. These modes have frequencies that lay in the band gap and thus are particularly robust with respect to perturbations. Furthermore, they are spatially localised near the defect.

To show the existence of an edge mode, we compute the subwavelength resonances of a finite but large array having the same geometrical defect. In the three-dimensional case, it has been shown that this is indeed an accurate approximation [3]. Figure 3b shows the existence of edge modes. Figure 3a illustrates that the frequencies of the edge modes are well-separated from the bulk, and lay inside the band gap. Figure 3a is of particular interest as it suggests that the spectrum of the finite approximation converges exponentially to the continuous spectrum of the periodic structure. The exponential convergence is the result of the absence of long-range interactions in the system [3, 28, 29, 31].

As mentioned before, edge frequencies laying in the band gap are typically robust to perturbations. In Figure 3c, we show that these frequencies are only minimally influenced by slightly perturbing the distances between the resonators via

$$\widetilde{s_i} = s_i + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

with $\mathcal{N}(0, \sigma^2)$ being a uniform distribution with standard deviation σ and mean-value zero. In particular, they remain in the band gap. We thus call these edge modes *topologically protected*.

6. Non-Hermitian case

In the non-Hermitian case, the material parameters κ_i are complex with non vanishing imaginary parts. As we want to analyse the influence of the complex material parameters, we assume for the rest of this section that the size of the resonators is constant, i.e., $\ell_i = \ell_1$ for all $1 \leq i \leq N$.

A particular case of this non-Hermitian setup are systems with PT-symmetry. Originating from quantum mechanics, this terms defines a system where gains and losses are balanced, that is, $v_1 = \overline{v_2}$ in the case of a dimer of resonators.

6.1. Non-Hermitian Zak phase

Definition 6.1 (Non-Hermitian Zak phase). The non-Hermitian Zak phase φ_j^{zak}, for $1 \leq j \leq N$, is defined by

$$\varphi_j^{\mathrm{zak}} := \frac{\mathbf{i}}{2} \int_{Y^*} \left(\left\langle v_j^\alpha, \frac{\partial u_j^\alpha}{\partial \alpha} \right\rangle + \left\langle u_j^\alpha, \frac{\partial v_j^\alpha}{\partial \alpha} \right\rangle \right) \mathrm{d}\alpha,$$

where u_j^α and v_j^α are respectively the left and right eigenmodes.

We remark immediately that Definition 6.1 is a generalisation of Definition 5.4 as left and right eigenmodes are equal in the Hermitian case.

The following lemma is [10, Lemma 3.5].

Lemma 6.2. *Let $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$ be a bi-orthogonal system (i.e., $\langle \boldsymbol{v}_i, \boldsymbol{u}_j \rangle = \delta_{ij}$) of eigenvectors of the generalised capacitance matrix defined by* (4.22)*, so that Lemma 4.10 holds. Then, the Zak phase can be written as*

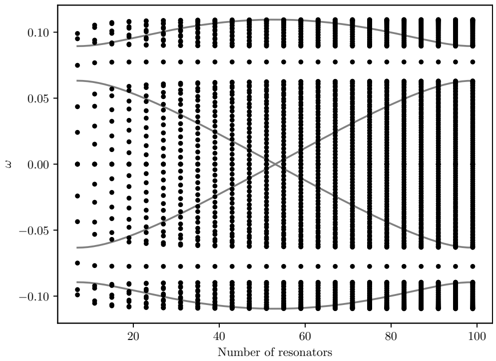
$$\varphi_j^{zak} = -\Im \left(\int_{Y^*} \left\langle \boldsymbol{v}_j, \frac{\partial \boldsymbol{u}_j}{\partial \alpha} \right\rangle \mathrm{d}\alpha \right) + \mathcal{O}(\delta). \tag{6.1}$$

We will now derive an explicit formula for the non-Hermitian Zak phase. This, as the non-Hermitian version is a generalisation of the Hermitian one, will allow us to prove Proposition 5.5.

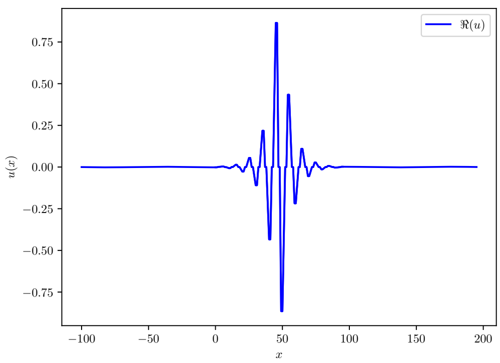
Remark 6.3. Consider an eigendecomposition

$$M = UDU^{-1}$$

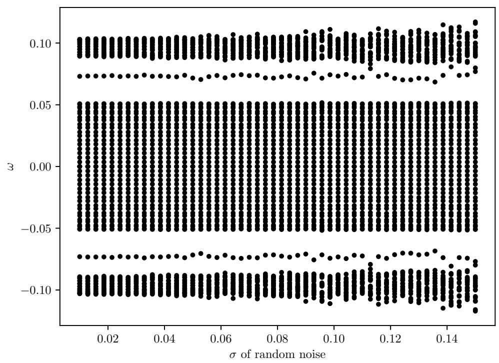
of a matrix M, where U is an invertible matrix with columns given by (right) eigenvectors and D a diagonal matrix. Then, a basis of left eigenvectors is given by the columns of the matrix $V := (U^{-1})^*$. Furthermore, the two matrices are bi-orthogonal, meaning that $V^* U = I$ so that the left and right eigenvectors satisfy $\langle v_i, u_j \rangle = \delta_{ij}$.



(A) Convergence of the subwavelength resonances of a finite structure with a geometrical defect for an increasing number of resonators. In black the subwavelength resonances while in grey underlaid the subwavelength resonant bands of the infinite right structure. (Remark that for these last frequencies, the x-axis is meaningless.)



(B) Localised edge mode for a finite but large array of $N = 39$ resonators having a geometrical defect.



(C) Stability of subwavelength edge resonances with respect to perturbations in the geometry.

FIGURE 3. Edge modes generated by geometrical defects. For the infinite structure, we use $N = 2$, $\ell_i = 1$, $s_1 = 2$, $s_2 = 1$.

Let $U(\alpha)$ be an eigenbasis of the generalised quasiperiodic capacitance matrix and $V(\alpha) = (U(\alpha))^{-1}$ be the corresponding bi-orthogonal basis according to Remark 6.3. Then, defining

$$V^*(\alpha)\frac{\partial}{\partial\alpha}U(\alpha) = U^{-1}(\alpha)\frac{\partial}{\partial\alpha}U(\alpha) =: J(\alpha),$$

the Zak phase take the following form according to Lemma 6.2:

$$\varphi_j^{\text{zak}} = -\Im\left(\int_{Y^*} J_{j,j}(\alpha)\,\mathrm{d}\alpha\right) + \mathcal{O}(\delta). \tag{6.2}$$

Let $a = \frac{1}{s_1} + \frac{1}{s_2}$ and $b(\alpha) = -\frac{1}{s_1} - \frac{e^{-\mathrm{i}L\alpha}}{s_2}$, so that the generalised capacitance matrix is given by

$$\mathcal{C}_{\mathrm{G}}^{\alpha} := \begin{pmatrix} v_1^2 a & v_1^2 b(\alpha) \\ v_2^2\overline{b(\alpha)} & v_2^2 a \end{pmatrix}$$

with eigenbasis given by the columns of

$$U(\alpha) := \begin{pmatrix} -a(v_2^2-v_1^2) - \sqrt{a^2(v_1^2-v_2^2)^2+4v_1^2v_2^2|b(\alpha)|^2} & -a(v_2^2-v_1^2) + \sqrt{a^2(v_1^2-v_2^2)^2+4v_1^2v_2^2|b(\alpha)|^2} \\ 2v_2^2\overline{b(\alpha)} & 2v_2^2\overline{b(\alpha)} \end{pmatrix} \tag{6.3}$$

$$=: \begin{pmatrix} -a(v_2^2-v_1^2) - \sqrt{f(b(\alpha))} & -a(v_2^2-v_1^2) + \sqrt{f(b(\alpha))} \\ 2v_2^2\overline{b(\alpha)} & 2v_2^2\overline{b(\alpha)} \end{pmatrix}.$$

Actually, if $b(\alpha)=0$, then this formula for $U(\alpha)$ does not work. However, as we will be later interested in integrating this quantity and the set $\{\alpha : b(\alpha)=0\}$ has zero measure, we can just work with the formula above.

In particular, for a non-degenerate $\mathcal{C}_{\mathrm{G}}^{\alpha}$, we have

$$U(\alpha)^{-1} = \frac{1}{4v_2^2\overline{b(\alpha)}f(b(\alpha))}\begin{pmatrix} 2v_2^2\overline{b(\alpha)} & +a(v_2^2-v_1^2) - \sqrt{f(b(\alpha))} \\ -2v_2^2\overline{b(\alpha)} & -a(v_2^2-v_1^2) - \sqrt{f(b(\alpha))} \end{pmatrix},$$

so that

$$J_{1,1} = \frac{2v_2^2\overline{b(\alpha)}}{4v_2^2\overline{b(\alpha)}\sqrt{f(b(\alpha))}}\frac{\partial}{\partial\alpha}(-\sqrt{f(b(\alpha))}) + \frac{a(v_2^2-v_1^2)-\sqrt{f(b(\alpha))}}{4v_2^2\overline{b(\alpha)}\sqrt{f(b(\alpha))}}\frac{\partial}{\partial\alpha}2v_2^2\overline{b(\alpha)}.$$

By periodicity, we know that $b(\alpha)$ draws a closed path in $\mathbb{C}$. Remark that $f(b(\alpha))$ is a closed curved tracing a line (or two segments), so that integrating over it always results in zero. Reformulating the above in terms of path integral we get

$$\begin{aligned}\int_{Y^*} J_{1,1}(\alpha)\,\mathrm{d}\alpha = &-\frac{1}{2\sqrt{f(b(\alpha))}}\frac{\partial}{\partial\alpha}(\sqrt{f(b(\alpha))})\,\mathrm{d}\alpha + \frac{a(v_2^2-v_1^2)}{2}\int_{Y^*}\frac{1}{\overline{b(\alpha)}\sqrt{f(b(\alpha))}}\frac{\partial}{\partial\alpha}\overline{b(\alpha)}\,\mathrm{d}\alpha \\ &-\frac{1}{2}\int_{Y^*}\frac{1}{\overline{b(\alpha)}}\frac{\partial}{\partial\alpha}\overline{b(\alpha)},\end{aligned}$$

so that (6.2) becomes

$$\varphi_j^{\mathrm{zak}} = \frac{1}{2}\Im\left(\underbrace{-\int_{\sqrt{f}}\frac{1}{z}\,\mathrm{d}z}_{=0} + (-1)^{j+1}\underbrace{a(v_2^2-v_1^2)\int_{\overline{b}}\frac{1}{z\sqrt{f(z)}}\,\mathrm{d}z}_{:=P} - \int_{\overline{b}}\frac{1}{z}\,\mathrm{d}z\right) + \mathcal{O}(\delta). \tag{6.4}$$

Thus, we have shown the following theorem.

Theorem 6.4. *Consider a geometrical structure with $N=2$ and $\ell_1=\ell_2$ with a non-degenerate corresponding band structure. Then the Zak phase has the following asymptotic expansion:*

$$\varphi_j^{zak} = (-1)^{j+1}\frac{s_1+s_2}{2s_1s_2}\Im\left((v_2^2-v_1^2)\int_\gamma\frac{1}{z\sqrt{f(z)}}\,\mathrm{d}z\right) + \pi\mathbb{1}_{\{x<s_1\}}(s_2) + \mathcal{O}(\delta), \tag{6.5}$$

where γ is the closed path

$$\gamma(t) := s_1^{-1} + s_2^{-1}e^{\mathbf{i}Lt},$$

and f is defined along γ as

$$f(z) := \left(s_1^{-1}+s_2^{-1}\right)^2(v_1^2-v_2^2)^2 + 4v_1^2v_2^2|z|^2.$$

One remarks already here that for the special case $v_1^2=v_2^2$ one obtains

$$\varphi_j^{\mathrm{zak}} = \frac{1}{2}\Im\left(\int_\gamma\frac{1}{z}\,\mathrm{d}z\right) = \begin{cases}\pi & \text{if } \dfrac{1}{s_2} > \dfrac{1}{s_1}, \\ 0 & \text{if } \dfrac{1}{s_2} \le \dfrac{1}{s_1},\end{cases} \tag{6.6}$$

as in this case the Zak phase is known to be quantised [6] so that we can drop the asymptotic factor. This proves Proposition 5.5.

In general the integrals above are tedious to evaluate because of the non-holomorphicity of the integrand but we can check numerically that the integral is not zero and not constant, showing the non-quantisation of the Zak phase in the non-Hermitian case. Some values of this integral are shown in Table 1.

s_1	s_2	v_1	v_2	$\frac{1}{2}\Im(P)$
1	2	$1+1.38\mathbf{i}$	$1-1.42\mathbf{i}$	0.408
1	1	$1+1.38\mathbf{i}$	$1-1.42\mathbf{i}$	2.420
1	1	$1-1.42\mathbf{i}$	$1+1.38\mathbf{i}$	-2.420

Table 1. Values of the perturbation factor for various geometrical and material configurations.

In the PT-symmetric case, the system is degenerate. It has twice a double eigenvalue. The following result in that case can be shown explicitly.

Lemma 6.5 (PT-symmetric Zak phase). *Assume that $N=2$ and $v_2=\overline{v_1}$. Then,*

$$\varphi_j^{zak} = \mathcal{O}(\delta).$$

Proof. For this proof, we will denote by $\varphi_j^{\text{zak}}(v)$ the Zak phase for $v_1=v$. Let also $\sigma=(1\ 2)$ be the permutation of two elements. Asymptotically, the Zak phase solely depends on the eigenvectors of the generalised capacitance matrix. We first show that $\varphi_j^{\text{zak}}(v)=\varphi_{\sigma(j)}^{\text{zak}}(\overline{v})$. To this end, we remark that using the definition of the capacitance matrix

$$\mathcal{C}_{\mathrm{G}}^{\alpha} = V^2\,\mathcal{C}^{\alpha} = \begin{pmatrix} v^2 & 0 \\ 0 & \overline{v}^2 \end{pmatrix}\begin{pmatrix} a & b(\alpha) \\ \overline{b(\alpha)} & a \end{pmatrix}$$

and the permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain the following relation:

$$\overline{V^2} = PV^2P^{-1} \quad \overline{\mathcal{C}^{\alpha}} = P\mathcal{C}^{\alpha}P = \mathcal{C}^{-\alpha}.$$

So,

$$\overline{\mathcal{C}_{\mathrm{G}}^{\alpha}} = PV^2P^{-1}P\,\mathcal{C}^{\alpha}\,P^{-1} = P\,\mathcal{C}_{\mathrm{G}}^{\alpha}\,P^{-1},$$

and $\overline{\mathcal{C}_{\mathrm{G}}^{\alpha}}$ and $\mathcal{C}_{\mathrm{G}}^{\alpha}$ are similar via a permutation matrix. However, $\overline{\mathcal{C}_{\mathrm{G}}^{\alpha}} = \overline{V^2}\mathcal{C}^{-\alpha}$ and so the eigenvectors of $\overline{V^2}\mathcal{C}^{-\alpha}$ are a permutation of the eigenvectors of $\mathcal{C}_{\mathrm{G}}^{\alpha}$. By symmetry around the origin of the Brillouin zone and Definition 6.1 of the Zak phase, we conclude that $\varphi_j^{\text{zak}}(v)=\varphi_{\sigma(j)}^{\text{zak}}(\overline{v})$.

We now show that $\varphi_j^{\text{zak}}(v)=-\varphi_{\sigma(j)}^{\text{zak}}(\overline{v})$, which will complete the proof. Remark that the eigenvectors of the capacitance matrix given by (6.4) show that complex conjugating both material parameters leads to permuted and conjugated eigenvectors. Lemma 6.2 leads to the desired conclusion. ■

6.2. Localised edge modes generated by material-parameter defects

We have shown in Section 5 that defects in the periodicity of the system can lead to edge modes. Recently, it has been shown that edge modes can also be generated in the non-Hermitian case via defects in the material parameters rather than in the geometry [10]. We follow a similar approach to the one in [10], showing that for a one dimensional chain of resonators one can explicitly identify the edge modes.

In this section, we consider the case of equally spaced dimers (i.e., with two identical resonators per cell), that is,

$$N = 2, \quad \ell_1 = \ell_2, \quad s_1 = s_2. \tag{6.7}$$

We denote by $v_i^{(m)}$ the material parameter of the i-th resonator of the m-th dimer and similarly for the resonator itself.

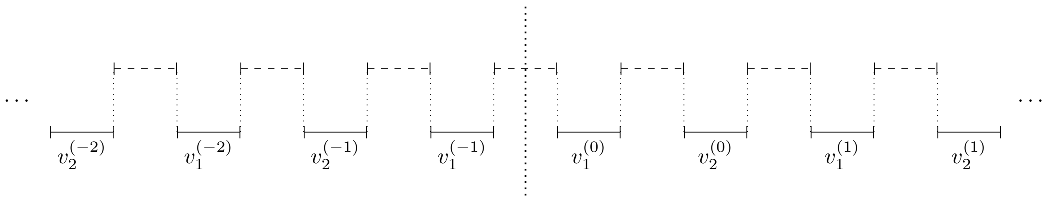


Figure 4. Infinite structure with material parameter defect.

Definition 6.6 (Localized edge mode). A solution u to (4.1) is said to be a simple eigenmode if it corresponds to a simple eigenvalue ω scaling as $\mathcal{O}(\delta)$. A solution is said to be *localised* if it is bounded in the L^2-sense, that is, $\int_{\mathbb{R}} |u(x)|^2 \,\mathrm{d}x < \infty$.

As we have seen in Proposition 4.5 and Lemma 4.10, inside the resonators a subwavelength resonant mode is almost constant

$$u(x) = u_i^m + \mathcal{O}(\delta) \text{ if } x \in D_i^{(m)}. \tag{6.8}$$

The following proposition is [10, Proposition 4.2].

Proposition 6.7. *Any localized solution u to* (4.1) *corresponding to a subwavelength frequency ω satisfies*

$$\frac{1}{\rho}\mathcal{C}^\alpha \begin{pmatrix} \sum_{m\in\mathbb{Z}} u_1^m e^{\mathbf{i}\alpha m L} \\ \sum_{m\in\mathbb{Z}} u_2^m e^{\mathbf{i}\alpha m L} \end{pmatrix} = \omega^2 \begin{pmatrix} \sum_{m\in\mathbb{Z}} \frac{u_1^m e^{\mathbf{i}\alpha m L}}{(v_1^m)^2} \\ \sum_{m\in\mathbb{Z}} \frac{u_2^m e^{\mathbf{i}\alpha m L}}{(v_2^m)^2} \end{pmatrix}. \tag{6.9}$$

We consider the topological defect

$$v_1^{(m)} = \begin{cases} v_1 & m \le 0, \\ v_2 & m > 0, \end{cases} \quad \text{and} \quad v_2^{(m)} = \begin{cases} v_2 & m \le 0, \\ v_1 & m > 0. \end{cases} \tag{6.10}$$

The following lemma, which is [10, Lemma 4.3], exploits the symmetry in the defect to obtain a decay rate of the mode.

Lemma 6.8. *Let*

$$U_1 = \sum_{m\le 0} u_1^m e^{\mathbf{i}\alpha m L}, \quad U_2 = \sum_{m>0} u_1^m e^{\mathbf{i}\alpha m L}, \quad U_3 = \sum_{m\le 0} u_2^m e^{\mathbf{i}\alpha m L}, \quad U_4 = \sum_{m>0} u_2^m e^{\mathbf{i}\alpha m L}.$$

Then, there exists some $b \in \mathbb{C}$ independent of α satisfying $|b| < 1$ and

$$U_1 = bU_3, \quad U_4 = bU_2.$$

Using the same notation as in Lemma 6.8, we have

$$\sum_{m\in\mathbb{Z}} u_1^m e^{\mathbf{i}\alpha mL} = U_2 + U_1 = U_2 + bU_3, \quad \sum_{m\in\mathbb{Z}} u_2^m e^{\mathbf{i}\alpha mL} = U_3 + U_4 = U_3 + bU_2.$$

Furthermore, the topological defect (6.10) implies

$$\sum_{m\in\mathbb{Z}} \frac{u_1^m e^{\mathbf{i}\alpha mL}}{\left(v_1^{(m)}\right)^2} = \frac{U_2}{v_2^2} + \frac{U_1}{v_1^2} = \frac{U_2}{v_2^2} + \frac{1}{v_1^2} bU_3, \quad \sum_{m\in\mathbb{Z}} \frac{u_2^m e^{\mathbf{i}\alpha mL}}{\left(v_2^{(m)}\right)^2} = \frac{U_3}{v_2^2} + \frac{U_4}{v_1^2} = \frac{U_3}{v_2^2} + \frac{1}{v_1^2} bU_4.$$

This allows us to rewrite Proposition 6.7 as follows.

Proposition 6.9. *Assume that a structure as in* (6.7) *has a topological defect as in* (6.10). *Then, there is a localised mode in the subwavelength regime corresponding to the frequency* ω *only if* $B^{-1}\mathcal{C}^\alpha A$ *has an eigenvalue* $\mu \in \mathbb{C}$ *independent of the quasiperiodicity* α. *Here,*

$$A = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}, \quad B = \frac{1}{\delta}\begin{pmatrix} v_2^{-2} & bv_1^{-2} \\ bv_1^{-2} & v_2^{-2} \end{pmatrix}.$$

Particular of the one-dimensional case is the explicit α-dependence of the capacitance matrix, as seen in Definition 4.6, which allows us to prove the next theorem.

Remark that it has been shown in [10, Section 4.2] that the decay rate of edge modes in the case of (6.7) must be either of

$$b_\pm = \frac{1}{2}\left(3\left(1 - \frac{v_1^2}{v_2^2}\right) \pm \sqrt{9\left(1 - \frac{v_1^2}{v_2^2}\right)^2 + \frac{4v_1^2}{v_2^2}}\right), \tag{6.11}$$

whichever has magnitude smaller than 1.

Theorem 6.10. *Assume that a one-dimensional structure as in* (6.7) *has a topological defect given by* (6.10). *Then there always exists a simple eigenmode, which — if* $v_1 \neq \overline{v_2}$ *or* $v_1 = \overline{v_2} := v$ *with* $\sqrt{8}|\Im(v^2)| \leq |\Re(v^2)|$ *— is also localised.*

The frequency of the mode in the subwavelength regime satisfies

$$\omega = \pm\sqrt{\frac{\mu}{\ell_1}} + \mathcal{O}(\delta),$$

where

$$\mu = \frac{\delta}{s_1}\frac{8v_1^2(-3v_1^2 + v_2^2\sqrt{D} + 3v_2^2)}{-7v_1^2 + 3v_2^2\sqrt{D} + 9v_2^2}$$

with $D := \frac{9v_1^4}{v_2^4} - \frac{14v_1^2}{v_2^2} + 9$.

Proof. The condition about localisation arises from the form of the decay rate given in (6.11). For $v_1 \neq \overline{v_2}$ or $v_1 = \overline{v_2} := v$ with $\sqrt{8}|\Im(v^2)| \leq |\Re(v^2)|$ either b_- or b_+ must have magnitude smaller than one. However, in the $v_1 = \overline{v_2} := v$ with $\sqrt{8}|\Im(v^2)| > |\Re(v^2)|$ case both $|b_\pm| = 1$ making it impossible to have localised modes.

For the eigenvalue computations, we assume without loss of generality that $s_1 = 1$ and introduce it again in the last step.

The eigenvalues of $B^{-1}\mathcal{C}^\alpha A$ are given by

$$\mu_j = \underbrace{\frac{\delta v_1^2 v_2^2}{b^2 v_2^4 - v_1^4}}_{:=K}\Big(\mathcal{C}_{11}^\alpha(b^2v_2^2 - v_1^2) + b(v_2^2 - v_1^2)\Re(\mathcal{C}_{12}^\alpha)$$
$$+ (-1)^j\sqrt{(\mathcal{C}_{11}^\alpha(b^2v_2^2 - v_1^2) + b(v_2^2 - v_1^2)\Re(\mathcal{C}_{12}^\alpha))^2 - (b^2-1)(b^2v_2^4 - v_1^4)((\mathcal{C}_{11}^\alpha)^2 - |\mathcal{C}_{12}^\alpha|^2)}\Big)$$

and we will show that μ_0 is independent of the quasiperiodicity. We assume without loss of generality that $b = b_-$, since the case $b = b_+$ can be proved similarly. Inserting the explicit coefficients of the capacitance matrix, we obtain

$$K\Bigg(\left(2b^2v_2^2 + b\left(v_1^2 - v_2^2\right)\left(\cos\left(L\alpha\right)+1\right) - 2v_1^2\right) \\ + \sqrt{v_1^4v_2^4\left(\left(2b^2-2\right)\left(b^2v_2^4 - v_1^4\right)\left(\cos\left(L\alpha\right)-1\right) + \left(2b^2v_2^2 + b\left(v_1^2-v_2^2\right)\left(\cos\left(L\alpha\right)+1\right) - 2v_1^2\right)^2\right)}\Bigg),$$

while in order to show independence from α, it is enough to consider the term

$$b\left(v_1^2 - v_2^2\right)\left(\cos\left(L\alpha\right)+1\right) \\ + \sqrt{\left(2b^2-2\right)\left(b^2v_2^4 - v_1^4\right)\left(\cos\left(L\alpha\right)-1\right) + \left(2b^2v_2^2 + b\left(v_1^2-v_2^2\right)\left(\cos\left(L\alpha\right)+1\right) - 2v_1^2\right)^2}. \tag{6.12}$$

Inserting into (6.12) the value of b from (6.11), we obtain after some careful algebraic manipulations

$$\sqrt{2}\left(v_1^2 - v_2^2\right)\sqrt{\underbrace{\left(9v_1^4 - 3v_1^2v_2^2\sqrt{D} - 16v_1^2v_2^2 + 3v_2^4\sqrt{D} + 9v_2^4\right)}_{:=B}\left(\cos\left(L\alpha\right)+3\right)} \\ - \left(v_1^2 - v_2^2\right)\underbrace{\left(3v_1^2 - v_2^2\left(\sqrt{D}+3\right)\right)}_{:=A}\left(\cos\left(L\alpha\right)+1\right)$$

with $D := \frac{9v_1^4}{v_2^4} - \frac{14v_1^2}{v_2^2} + 9$. In order to verify independence from the quasiperiodicity, it is now enough to prove that $2B = A^2$. A direct computation shows that

$$\begin{aligned} A^2 &= Dv_2^4 + 6\sqrt{D}v_1^2v_2^2 - 6\sqrt{D}v_2^4 + 9v_1^4 - 18v_1^2v_2^2 + 9v_2^4 \\ &= 18v_1^4 - 6v_1^2v_2^2\sqrt{D} - 32v_1^2v_2^2 + 6v_2^4\sqrt{D} + 18v_2^4 \\ &= 2B. \end{aligned}$$

In particular, we have

$$\mu_0 = \frac{\delta}{s_1}\frac{8v_1^2(-3v_1^2 + v_2^2\sqrt{D} + 3v_2^2)}{-7v_1^2 + 3v_2^2\sqrt{D} + 9v_2^2}.$$

■

As in Section 5.2, we provide some numerical simulations to visualise the edge mode. We first compute the bands ω^α for the left infinite structure. These are shown in Figure 5a while Figure 5b shows their traces in $\mathbb{C}$. In these plots, we add separately the edge mode frequency predicted by Theorem 6.10.

In Figure 5c, we show the edge mode computed for a finite but large array of resonators.

As Theorem 6.10 provides an explicit formula for the edge mode frequency, it is particularly interesting to compare the subwavelength resonances of a finite structure with an increasing number of resonators with the band structure and predicted edge mode frequency — both structures having the same material parameter defect. We do this in Figure 5e.

As in the Hermitian case, we want to show that the edge mode is robust with respect to perturbations, this time in the material parameters. In Figure 5d, we compute the subwavelength resonances of a finite but large array of $N = 100$ resonators having a material parameter defect with some random perturbation given by

$$\widetilde{v_i} = v_i + (1\ \mathbf{i})\cdot\varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0,\Sigma),$$

where

$$\Sigma = \sigma\,\mathrm{diag}\left(\frac{1}{\sqrt{2}|v_i|}\right).$$

We remark that the edge mode predicted by Theorem 6.10 is stable with respect to perturbations in the material parameters. The stability is, however, less strong with respect to the Hermitian case. This is to be expected due to the non-quantisation of the Zak phase in this setup. Figure 5d shows that there is a second isolated frequency supported by this setup not predicted by Theorem 6.10. However, this frequency is not isolated from the bulk even for very small perturbations.

Acknowledgments

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Code availability

The data that support the findings of this study are openly available at https://gitlab.math.ethz.ch/silvioba/edge-modes-1d.

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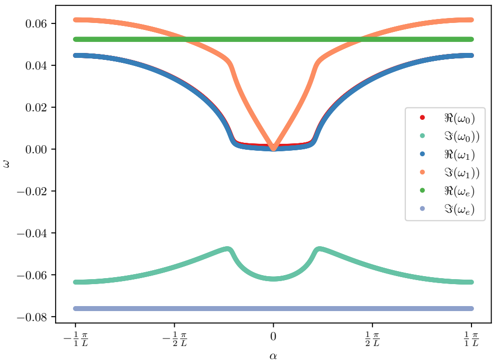
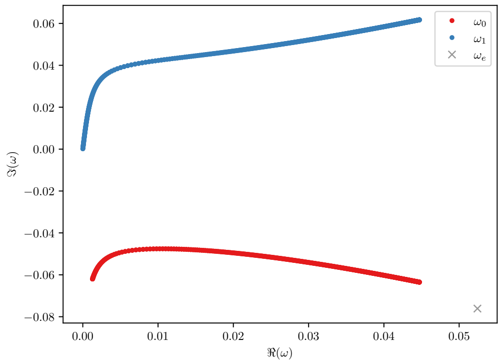
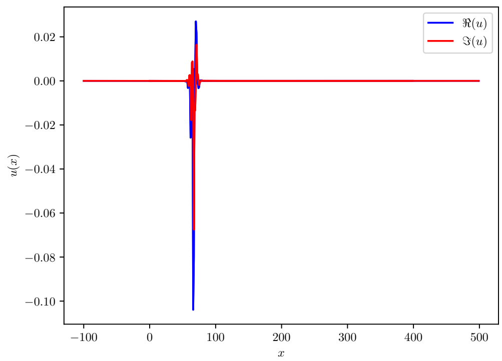


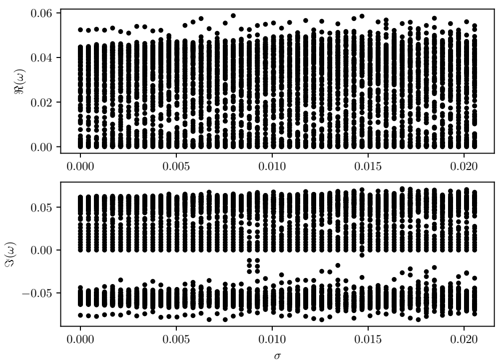
(A) Representation depending on the quasiperiodicity.



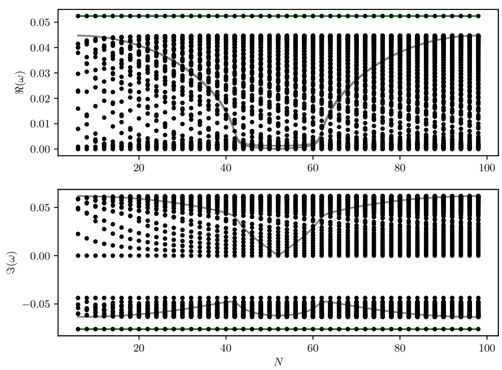
(B) Traces on the complex plane.



(C) Localised edge mode for a finite but large array of $N = 100$ resonators having a material parameter defect.



(D) Stability of the edge mode with respect to perturbations in the material parameters.



(E) Convergence of the subwavelength resonances with an increasing number of resonators. In black the subwavelength resonances while in grey underlaid the subwavelength resonant bands of the infinite right structure and in green the predicted edge mode frequency. (Remark that for these last two frequencies, the x-axis is meaningless.)

FIGURE 5. Edge modes generated by material parameter defects. For the infinite structure we used $N = 2$, $\ell_i = 1$, $s_i = 1$, $v_1 = 1 + 1.38\mathbf{i}$, $v_2 = 1 - 1.42\mathbf{i}$.

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