Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

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P. Liu and Y. He and H. Ammari

Research Report No. 2023-02
January 2023

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 200021--200307

A mathematical theory of resolution limits for super-resolution of positive sources *

Ping Liu[†] Yanchen He[†] Habib Ammari[†]

January 3, 2023

The superresolving capacity for number and location recoveries in the super-resolution of positive sources is analyzed in this work. Specifically, we introduce the computational resolution limit for respectively the number detection and location recovery in the one-dimensional super-resolution problem and quantitatively characterize their dependency on the cutoff frequency, signal-to-noise ratio, and the sparsity of the sources. As a direct consequence, we show that targeting at the sparest positive solution in the super-resolution already provides the optimal resolution order. These results are generalized to multi-dimensional spaces. Our estimates indicate that there exist phase transitions in the corresponding reconstructions, which are confirmed by numerical experiments. Our theory fills in an important puzzle towards fully understanding the super-resolution of positive sources.

Mathematics Subject Classification: 65R32, 42A10, 15A09, 94A08, 94A12

Keywords: resolution limit, super-resolution, positive sources, line spectral estimation, Vandermonde matrix, phase transition

1. Introduction

In recent years, the development of super-resolution optical microscopy led to a revolutionary improvement of resolution through the use of different technical approaches. This impressive success has generated significant interest in studying the super-resolution algorithms and the fundamental superresolving capability. In this paper, we aim to study

[*]This work was supported in part by the Swiss National Science Foundation grant number 200021–200307.

[†]Department of Mathematics, ETH Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland (ping.liu@sam.math.ethz.ch, yanchen.he@sam.math.ethz.ch, habib.ammari@math.ethz.ch).

the superresolving capacity of the number and locations recovery in the super-resolution of positive sources. To be more specific, we consider the following mathematical model. Let $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}, a_j > 0$ be a positive discrete measure, where $y_j \in \mathbb{R}, j = 1, \cdots, n$, represent the location of the point sources and $a_j > 0, j = 1, \cdots, n$, their amplitudes. Noting that y_j's are the supports of the Dirac masses in μ. In this paper we will use support recovery instead of location reconstruction. We denote by

$$m_{\min} = \min_{j=1,\cdots,n} a_j, \quad d_{\min} = \min_{p \neq j} |y_p - y_j|.$$

The measurement is the noisy Fourier data of μ in a bounded interval, that is,

$$\mathbf{Y}(\omega) = \mathscr{F}\mu(\omega) + \mathbf{W}(\omega) = \sum_{j=1}^{n} a_j e^{i y_j \omega} + \mathbf{W}(\omega), \quad \omega \in [-\Omega, \Omega], \tag{1.1}$$

with $\mathbf{W}(\omega)$ being the noise and Ω the cutoff frequency of the imaging system. We assume that

$$|\mathbf{W}(\omega)| < \sigma, \quad \omega \in [-\Omega, \Omega],$$

with σ being the noise level. The above measurement model is chosen for convenience. All the results in this paper also hold for the case when taking measurement at a sufficient number of evenly-spaced points as what was considered in [26]. Thus our results can be applied to practical situations and real-world problems.

The super-resolution problem we are interested in is to recover the positive discrete measure μ from the above noisy measurement $\mathbf{Y}$. We note that the super-resolution problem is closely related to the line spectral estimation problem [26] which is at the core of diverse fields such as wireless communications and array processing. Note also that the measurement considered in (1.1) is related to the imaging of convolution of point sources and a band-limited point spread function f.

1.1. Literature review

Fundamental limits. In [17,37,38], the authors analyzed the resolution limit in the detection of two closely-spaced point sources based on the statistical inference theory, but their theory has not been generalized to the case when there are more than two sources in the signal. The mathematical theory for analyzing the fundamental limit in superresolving multiple point sources was pioneered by Donoho [13] in 1992. In that work, he considered a grid setting where a discrete measure is supported on a lattice with spacing Δ and regularized by a so-called "Rayleigh index". The problem is to reconstruct the amplitudes of the grid points from their noisy Fourier data in $[-\Omega, \Omega]$ with Ω being the band limit. His main contribution is estimating the corresponding minimax error in the recovery, which emphasizes the importance of the sparsity of sources for the super-resolution. It was improved in recent years for the case when only n point sources are presented. In [11], the authors considered resolving n-sparse point sources supported on a grid and showed that the minimax error of amplitude recovery in the presence of noise with magnitude σ scales exactly as $SRF^{2n-1}\sigma$, where $SRF := \frac{1}{\Delta\Omega}$ is the super-resolution factor. The case of multi-clustered point sources was considered in [4,20] and similar minimax error estimates were derived. Moreover, in [2,5] the authors considered the minimax error for recovering the amplitudes and locations of off-the-grid point sources. They showed that for $\sigma \lessapprox (SRF)^{-2p+1}$, where p is the number of point sources in a cluster, the minimax error for the amplitude and the location recoveries scale respectively as $(SRF)^{2p-1}\sigma$, $(SRF)^{2p-2}\sigma/\Omega$, while for the single non-clustered source away from other sources, the corresponding minimax error for the amplitude and the location recoveries scale respectively as σ and σ/Ω. We also refer the readers to [8,28] for understanding the resolution limit from the perspective of sample complexity and to [10,42] for the resolving limit of some algorithms.

On the other hand, in order to characterize the exact resolution in resolving multiple point sources like the classical Rayleigh limit, in the earlier works [23–27] we defined the concept of "computational resolution limit" as the minimum required distance between point sources so that their number and locations can be stably resolved under certain noise level. By developing a non-linear approximation theory in a so-called Vandermonde space, we derived sharp bounds for computational resolution limits in one- and multi-dimensional super-resolution problems. In particular, we showed that the computational resolution limits for number and location recoveries should be respectively $\frac{C_{\mathrm{num}}(n,k)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}$ and $\frac{C_{\mathrm{supp}}(n,k)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$, where $C_{\mathrm{num}}(n,k), C_{\mathrm{supp}}(n,k)$ are constants depending only on source number n and space dimensionality k. In this paper, we will generalize these results to the super-resolution problem of positive sources.

Reconstruction algorithms. Due to the importance of super-resolution in applications, a number of sophisticated super-resolution algorithms have been developed over the years. Among those algorithms, a class of algorithms called subspace methods have exhibited favourable performance and have been used frequently in engineering applications. Specific examples include MUltiple SIgnal Classification (MUSIC) [35], Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) [34], and Matrix Pencil Method [18]. Note that these algorithms date back to the work of Prony [32]. Despite the appealing performance of the subspace methods in practical applications, their stability properties are not yet well-understood. The asymptotic results on the stability of MUSIC algorithm in the presence of Gaussian noise were derived just slightly after its emerging [9,40,41]. But only until recent years, some steps towards understanding the stability of MUSIC, ESPRIT and Matrix-Pencil Method in the non-asymptotic regime were taken in [22], [21] and [28], respectively. Nevertheless, the error tolerance derived in these papers are not as strong as our estimates here for the resolution limit in location reconstruction. On the other hand, it was shown numerically in [5,21,22] that these subspace methods actually achieve the optimal resolution order. Thus the theoretical demonstrations for the performance limits of subspace methods in the non-asymptotic regime are still important open problems.

In recent years, inspired by the idea of sparse modeling and compressed sensing, many sparsity promoting algorithms have been proposed for the super-resolution problem. For example, in the groundbreaking work of Candès and Fernandez-Granda [7], it was proved that off-the-grid sources can be exactly recovered from their low-frequency measurements by a TV minimization under a minimum separation condition. It invokes active researches in the off-the-grid algorithms, among them we would like to mention the BLASSO [3,15,31] and the atomic norm minimization method [43,44]. Both methods were proved to be able to stably recover the source under a minimum separation condition or a non-degeneracy condition. BLASSO (Beurling LASSO) is an off-the-grid generalization of l^1 regularization (LASSO), and exhibits excellent performance in the off-the-grid source recovery [15, 31]. The atomic norm minimization method is shown to form a nearly minimax optimal estimator when tackling line spectral estimation problems [6, 43]. Nevertheless, these convex optimization algorithms usually require a minimum separation distance of several Rayleigh limits (see, for instance, [19, 31, 42]) for the general source recovery, which may limit their applicability to superresolve closely-spaced point sources.

Super-resolution of positive sources. To the best of our knowledge, the theoretical possibility for the super-resolution of positive sources was first considered in [14]. Specifically, the authors defined

$$\omega(\sigma;\mathbf{x}) = \sup\left\{\left\|\mathbf{x}' - \mathbf{x}\right\|_1 : \left\|K\mathbf{x}' - K\mathbf{x}\right\|_2 \leqslant \sigma \text{ and } \mathbf{x}' \geqslant 0\right\},$$

where $\mathbf{x}, \mathbf{x}'$ are vectors of length M (sources are on a grid) and K consists of the first m rows of the $M \times M$ discrete Fourier transform, and said that $\mathbf{x}$ admits super-resolution if

$$\omega(\sigma;\mathbf{x}) \to 0 \quad \text{as } \sigma \to 0.$$

They demonstrated for $\mathbf{x} \geq 0$ that: (a) If $\mathbf{x}$ has $\frac{1}{2}(m-1)$ or fewer non-zero elements then $\mathbf{x}$ admits super-resolution; (b) If $\frac{1}{2}(m+1)$ divides M, there exists $\mathbf{x}$ with $\frac{1}{2}(m+1)$ non-zero elements yet does not admit super-resolution; (c) If $\mathbf{x}$ has more than m non-zero elements then $\mathbf{x}$ does not admit super-resolution. Their definition and results focused on the possibility of overcoming Rayleigh limit in the presence of sufficient small noise and hence demonstrated the possibility of super-resolution. See also [16] for a shorter exposition of the same idea.

In recent years, some researchers analyzed the stability of specific super-resolution algorithms in a non-asymptotic regime [12, 29, 30]. To be more specific, it was shown in [30] that a simple convex optimization program can already superresolve the positive sources (on a grid) to nearly optimal. The authors of [30] demonstrated that under certain conditions the deviation between the algorithm's output $\hat{\mathbf{x}}$ and the ground truth $\mathbf{x}$ obeys the following relation

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_1 \approx C \cdot \mathrm{SRF}^{2r} \cdot \sigma,$$

where r is the Rayleigh index. The theory was later generalized to the off-the-grid setting in [29] where the authors analyzed the stability of the reconstruction of high frequency information. In a different line of research, the authors studied in [12] the amplitude and support recoveries of positive discrete measures for a so-called BLASSO convex program. They demonstrated that when $\sigma/\lambda, \sigma/d_{\min}^{2n-1}$ and $\lambda/d_{\min}^{2n-1}$ are sufficiently small (with λ being the regularization parameter, σ the noise level, n the source number, $d_{\min}$ the minimum separation distance between two sources), there exists a unique solution to the BLASSO program consisting of exactly n point sources. The amplitudes and locations of the solution both converge toward those of the ground truth when the noise and the regularization parameter decay to zero faster than $d_{\min}^{2n-1}$. Note that this result is consistent with our estimates in the current paper, showing that the BLASSO achieves the optimal resolution order, which is quite impressive.

1.2. Main contribution

The main contribution of this paper is quantitative characterizations of the resolution limits to number detection and location recovery in the super-resolution of positive sources. Accurate detection of the source number (model order) is important in the super-resolution problem and many parametric estimation methods require the model order as a priori information. But there are few theoretical results which address the issue when the number of underlying sources is greater than two. In [26, 27], the first results for capacity of the number detection in the super-resolution of general sources are derived. Here we generalize the estimates to the case of positive sources, which is also the first result for understanding the capacity of superresolving positive sources. Specifically, we introduce the computational resolution limit $\mathscr{D}_{num}^+$ for the detection of n point sources (see Definition 2.2), and derive the following sharp bounds:

$$\frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}} < \mathscr{D}_{num}^+ \leq \frac{4.4\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}, \tag{1.2}$$

where $\frac{\sigma}{m_{\min}}$ is viewed as the inverse of the signal-to-noise ration (SNR). It follows that exact detection of the source number is possible when the minimum separation distance of point sources $d_{\min}$ is greater than $\frac{4.4\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}$, and impossible without additional a priori information when $d_{\min}$ is less than $\frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}$.

Following the same line of argument for the number detection problem, we also consider the support recovery in the super-resolution of positive sources. We introduce the computational resolution limit $\mathscr{D}_{supp}^+$ for the support recovery (see Definition 2.4) and derive the following bounds:

$$\frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}} < \mathscr{D}_{supp}^+ \leq \frac{5.88\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{1.3}$$

As a consequence, the resolution limit $\mathscr{D}_{supp}^+$ is of the order $O(\frac{1}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}})$. It follows that stable recovery (in certain sense) of the source locations is possible when the minimum separation distance of point sources $d_{\min}$ is greater than $\frac{5.88\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}$, and impossible without additional a priori information when $d_{\min}$ is less than $\frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}$. To further emphasize that the separation distance $O\left(\frac{1}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}\right)$ is necessary for a stable location reconstruction, we construct an example showing that if the sources are separated below the $\frac{c}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}$ for certain constant c, the recovered locations can be very unstable.

As a direct consequence of our estimates, we analyze the stability for a sparsity-promoting algorithm (l_0 minimization) in superresolving positive sources and show that it already achieves the optimal order of the resolution. These estimates for the resolution limits are also generalized to multi-dimensional spaces.

The quantitative characterizations of the resolution limits $\mathscr{D}_{num}^+$ and $\mathscr{D}_{supp}^+$ imply phase transition phenomena in the corresponding reconstructions, which are confirmed here by numerical experiments.

Also, our techniques offer a way to analyze the capability of resolving positive sources, which itself may have important applications.

1.3. Organization of the paper

The paper is organized in the following way. Section 2 presents the estimates for the resolution limit in the one-dimensional super-resolution of positive sources. Section 3 extends the estimates to multi-dimensional spaces. Section 4 proves the results in Section 2. In Sections 5 and 6, we verify the phase transition in respectively the number detection and location recovery problems. Section 7 concludes the paper. In Appendix A, we prove several auxiliary lemmas and useful inequalities.

2. Resolution limits for super-resolution in one-dimensional space

We present in this section our main results on the resolution limit for the super-resolution of one-dimensional positive sources. All the results shall be proved in Section 4. We consider the case when the point sources are tightly spaced and form a cluster. To be more specific, we define the interval

$$I(n,\Omega) := \Big[-\frac{(n-1)\pi}{2\Omega}, \quad \frac{(n-1)\pi}{2\Omega}\Big],$$

which is of length of several Rayleigh limits and assume that $y_j \in I(n,\Omega), 1\leq j\leq n$. The reconstruction process is usually targeting at some specific solutions in a so-called admissible set, which comprises of discrete measures whose Fourier data are sufficiently close to $\mathbf{Y}$. In our problem, we introduce the following concept of positive σ-admissible discrete measures. We denote in this section $||f||_{\infty} = \max_{\omega\in[-\Omega,\Omega]}|f(\omega)|$.

Definition 2.1. *Given measurement* $\mathbf{Y}$*, we say that* $\hat\mu = \sum_{j=1}^{k}\hat a_j\delta_{\hat y_j}, \hat a_j>0$ *is a positive* σ*-admissible discrete measure of* $\mathbf{Y}$ *if*

$$||\mathscr{F}[\hat\mu]-\mathbf{Y}||_{\infty} < \sigma.$$

The set of positive σ-admissible measures of $\mathbf{Y}$ characterizes all possible solutions to our super-resolution problem with the given measurement $\mathbf{Y}$. Following similar definitions in [25–27], we define the following computational resolution limit for the number detection in the super-resolution of positive sources. The reason for the definition is the fact that detecting the correct source number in μ is impossible without additional a prior information when there exists one positive σ-admissible measure with less than n supports.

Definition 2.2. *The computational resolution limit to the number detection problem in the super-resolution of one-dimensional positive source is defined as the smallest nonnegative number* $\mathscr{D}_{num}^{+}$ *such that for all positive* n*-sparse measure* $\sum_{j=1}^{n}a_j\delta_{y_j}, a_j>0, y_j\in I(n,\Omega)$ *and the associated measurement* $\mathbf{Y}$ *in (1.1), if*

$$\min_{p\neq j}|y_j-y_p|\geq \mathscr{D}_{num}^{+},$$

then there does not exist any positive σ*-admissible measure of* $\mathbf{Y}$ *with less than* n *supports.*

The notion of "computational resolution limit" emphasizes the essential impossibility of correct number detection for very close source by any means. Also, this notion depends crucially on the signal-to-noise ratio and the sparsity of the source, which is different from all classical resolution limits [1, 33, 36, 39, 45] that depend only on the cutoff frequency. We now present sharp bounds for this computational resolution limit $\mathscr{D}_{num}^+$. The following upper bound for it is a direct consequence of [26, Theorem 1].

Theorem 2.1. *Let* $\mathbf{Y}$ *be a measurement generated by a positive measure* $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}$*, which is supported on* $I(n,\Omega)$*. Let* $n \geq 2$ *and assume that the following separation condition is satisfied*

$$\min_{p\neq j}\left|y_p - y_j\right| \geq \frac{4.4\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}. \tag{2.1}$$

Then there do not exist any positive σ*-admissible measures of* $\mathbf{Y}$ *with less than n supports.*

Theorem 2.1 gives an upper bound for the computational resolution limit $\mathscr{D}_{num}^+$. This upper bound is shown to be sharp for the super-resolution of general discrete source (not positive) by a lower bound derived in [26], but the result is unknown for the case of resolving positive sources. We next present a lower bound of $\mathscr{D}_{num}^+$ which is the main result of this paper.

Theorem 2.2. *For given* $0 < \sigma \leq m_{\min}$ *and integer* $n \geq 2$*, there exist positive measures* $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}$ *with n supports and* $\hat\mu = \sum_{j=1}^{n-1} \hat a_j \delta_{\hat y_j}$ *with* $(n-1)$ *supports such that* $||\mathscr{F}[\hat\mu] - \mathscr{F}[\mu]||_\infty < \sigma$*. Moreover,*

$$\min_{1\leq j\leq n}\left|a_j\right| = m_{\min}, \quad \min_{p\neq j}\left|y_p - y_j\right| = \frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}.$$

The above result gives a lower bound for the computational resolution limit $\mathscr{D}_{num}^+$ to the number detection problem. Combined with Theorem 2.1, it reveals that the computational resolution limit for number detection satisfies

$$\frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}} < \mathscr{D}_{num}^+ \leq \frac{4.4\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}.$$

We remark that similarly to the results of [2, 5, 26], our bounds are the worst-case bounds, and one may achieve better bounds for the case of random noise.

We now consider the location (support) recovery problem in the super-resolution of positive sources. We first introduce the following concept of δ-neighborhood of a discrete measure.

Definition 2.3. *Let* $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}$ *be a discrete measure and let* $0 < \delta$ *be such that the n intervals* $(y_k - \delta, y_k + \delta), 1 \leq k \leq n$ *are pairwise disjoint. We say that* $\hat\mu = \sum_{j=1}^{n} \hat a_j \delta_{\hat y_j}$ *is within* δ*-neighborhood of* μ *if each* $\hat y_j$ *is contained in one and only one of the n intervals* $(y_k - \delta, y_k + \delta), 1 \leq k \leq n$*.*

According to the above definition, a measure in a δ-neighbourhood preserves the inner structure of the real source. For any stable support recovery algorithm, the output should be a measure in some δ-neighborhood, otherwise it is impossible to distinguish which is the reconstructed location of some y_j's. We now introduce the computational resolution limit for stable support recoveries. For ease of exposition, we only consider measures supported in $I(n,\Omega)$, where n is the number of supports.

Definition 2.4. *The computational resolution limit to the stable support recovery problem in the super-resolution of one-dimensional positive sources is defined as the smallest nonnegative number $\mathscr{D}_{supp}^+$ such that for all positive n-sparse measures $\sum_{j=1}^n a_j\delta_{y_j}, a_j>0, y_j\in I(n,\Omega)$ and the associated measurement $\mathbf{Y}$ in (1.1), if*

$$\min_{p\neq j}|y_j-y_p|\geq \mathscr{D}_{supp}^+,$$

then there exists $\delta>0$ such that any positive σ-admissible measure for $\mathbf{Y}$ with n supports in $I(n,\Omega)$ is within a δ-neighbourhood of μ.

To state the results on the resolution limit to stable support recovery, we introduce the super-resolution factor which is defined as the ratio between Rayleigh limit $\frac{\pi}{\Omega}$ (for point spread function $sinc(x)^2$) and the minimum separation distance of sources $d_{\min}:=\min_{p\neq j}|y_p-y_j|$:

$$SRF:=\frac{\pi}{\Omega d_{\min}}.$$

As a direct consequence of [26, Theorem 2], we have the following theorem giving the upper bound of $\mathscr{D}_{supp}^+$.

Theorem 2.3. *Let $n\geq 2$, assume that the positive measure $\mu=\sum_{j=1}^n a_j\delta_{y_j}$ is supported on $I(n,\Omega)$ and that*

$$\min_{p\neq j}\left|y_p-y_j\right|\geq \frac{5.88\pi e}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{2.2}$$

If $\hat\mu=\sum_{j=1}^n \hat a_j\delta_{\hat y_j}, \hat a_j>0$ supported on $I(n,\Omega)$ is a positive σ-admissible measure for the measurement generated by μ, then $\hat\mu$ is within the $\frac{d_{\min}}{2}$-neighborhood of μ. Moreover, after reordering the $\hat y_j$'s, we have

$$\left|\hat y_j-y_j\right|\leq \frac{C(n)}{\Omega}SRF^{2n-2}\frac{\sigma}{m_{\min}},\quad 1\leq j\leq n, \tag{2.3}$$

where $C(n)=n2^{4n-2}e^{2n}\pi^{-\frac{1}{2}}$.

Theorem 2.3 gives an upper bound to the computational resolution limit $\mathscr{D}_{supp}^+$. We next show that the order of the upper bound is optimal.

Theorem 2.4. *For given $0<\sigma\leq m_{\min}$ and integer $n\geq 2$, let*

$$\tau=\frac{e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{2.4}$$

Then there exist a positive measure $\mu=\sum_{j=1}^n a_j\delta_{y_j}$ with n supports at $\{-(n-\frac{1}{2})\tau,-(n-\frac{5}{2})\tau,\cdots,(n-\frac{3}{2})\tau\}$ and a positive measure $\hat\mu=\sum_{j=1}^n \hat a_j\delta_{\hat y_j}$ with n supports at $\{-(n-\frac{3}{2})\tau,-(n-\frac{7}{2})\tau,\cdots,(n-\frac{1}{2})\tau\}$ such that

$$||\mathscr{F}[\hat\mu]-\mathscr{F}[\mu]||_\infty<\sigma,\quad \min_{1\leq j\leq n}|a_j|=m_{\min}.$$

Since the minimum distance between y_j's is $d_{\min} = 2\tau$, thus for the positive σ-admissible measure $\hat{\mu}$, it is obviously that the $\hat{y}_j$'s are not in any δ-neighborhood of y_j's for $\delta \leq \frac{d_{\min}}{2}$ (for $\delta > \frac{d_{\min}}{2}$ the intervals in Definition 2.3 are overlapped). According to Definition 2.4, Theorem 2.4 implies $\mathscr{D}^+_{supp} > \frac{2e^{-1}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$. Thus we conclude that

$$\frac{2e^{-1}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}} < \mathscr{D}^+_{supp} \leq \frac{5.88\pi e}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}.$$

To further demonstrate that the order $O\left(\frac{1}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}\right)$ is essentially optimal for stable location reconstruction, we present an example with a new distribution of the source locations as follows.

Proposition 2.1. *For given* $0 < \sigma < m_{\min}$ *and integer* $n \geq 2$, *let*

$$\tau = \frac{0.2e^{-1}}{\Omega s^{\frac{2n+1}{2n-1}}}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}. \tag{2.5}$$

Then there exist a positive measure $\mu = \sum_{j=1}^n a_j\delta_{y_j}$ *with* n *supports at* $\{t_j = -\frac{sn-2}{2}\tau + \frac{(j-2)s}{2}\tau, j = 2, 4, \cdots, 2n\}$ *and a positive measure* $\hat{\mu} = \sum_{j=1}^n \hat{a}_j\delta_{\hat{y}_j}$ *with* n *supports at* $\{t_j = t_{4\lceil\frac{j+1}{4}\rceil-2} + (-1)^{\frac{j+1}{2}}\tau, j = 1, 3, 5, \cdots, 2n-1\}$ *such that*

$$||\mathscr{F}[\hat{\mu}] - \mathscr{F}[\mu]||_\infty < \sigma, \quad \min_{1\leq j\leq n}|a_j| = m_{\min}.$$

Note that the n underlying sources in μ are spaced by

$$s\tau = \frac{0.4e^{-1}}{\Omega s^{\frac{2}{2n-1}}}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}.$$

Proposition 2.1 reveals that when the n point sources are separated by $\frac{c}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$ for some constant c, the recovered source locations from the σ-admissible measures can be very unstable; see Figure 2.1.

Remark 2.1. *Note that all of our results hold for the case when the sources are supported on a grid. Specifically, we consider the grid points* $t_j = j\Delta, j = 1, \cdots, N$, *where* N *and* Δ *are the number and spacing of grid points, respectively, and assume the sources are supported on the grid. Assume also that the grid spacing* $\Delta \leq \frac{e^{-1}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$ *for fixed* n *and* $m_{\min}$. *By Theorem 2.4, we can construct two positive measures* $\mu = \sum_{q=1}^n a_q\delta_{t_{jq}}$ *and* $\hat{\mu} = \sum_{q=1}^n \hat{a}_q\delta_{\hat{t}_{jq}}$ *supported on the grid with completely different supports such that the difference of their Fourier data is less than the noise level and the minimum separation of sources is equal or less than* $\frac{2e^{-1}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$ *with* $\min_{q=1,\cdots,n} a_q = m_{\min}$.

Remark 2.2. *Note that our estimates for both the resolution limits in the number detection and support recovery already improve the estimates in [26] for the case of general sources.*

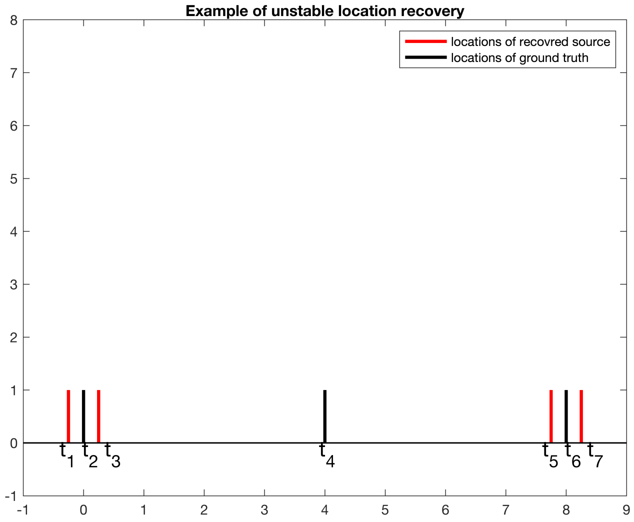


Figure 2.1: An example of unstable location recovery.

2.1. Stability analysis of sparsity-promoting algorithms

Nowadays, sparsity-promoting algorithms are popular methods in image processing, signal processing and many other fields. By our results for the resolution limits, we can derive a sharp stability result for the l_0 minimization in the super-resolution of positive sources. We consider the following l_0-minimization problem:

$$\min_{\rho \text{ supported on } \mathcal{O},\, \rho \text{ is a positive discrete measure}} ||\rho||_0 \quad \text{subject to} \quad |\mathscr{F}\rho(\omega) - \mathbf{Y}(\omega)| < \sigma, \quad \omega \in [-\Omega, \Omega], \tag{2.6}$$

where $||\rho||_0$ is the number of Dirac masses representing the discrete measure ρ. As a corollary of Theorems 2.1 and 2.3, we have the following theorem for its stability.

Theorem 2.5. *Let $n \geq 2$ and $\sigma \leq m_{\min}$. Let the measurement $\mathbf{Y}$ in (1.1) be generated by a positive n-sparse measure $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}, y_j \in I(n,\Omega)$. Assume that*

$$d_{\min} := \min_{p \neq j} \left| y_p - y_j \right| \geq \frac{5.88\pi e}{\Omega} \left(\frac{\sigma}{m_{\min}} \right)^{\frac{1}{2n-1}}. \tag{2.7}$$

Let $\mathcal{O}$ in the minimization problem (2.6) be (or be included in) $I(n,\Omega)$, then the solution to (2.6) contains exactly n point sources. For any solution $\hat{\mu} = \sum_{j=1}^{n} \hat{a}_j \delta_{\hat{y}_j}$, it is in a $\frac{d_{\min}}{2}$-neighborhood of μ. Moreover, after reordering the $\hat{y}_j$'s, we have

$$\left| \hat{y}_j - y_j \right| \leq \frac{C(n)}{\Omega} SRF^{2n-2} \frac{\sigma}{m_{\min}}, \quad 1 \leq j \leq n, \tag{2.8}$$

where $C(n) = n 2^{4n-2} e^{2n} \pi^{-\frac{1}{2}}$.

Theorem 2.5 reveals that sparsity promoting over admissible solutions can resolve the source locations to the resolution limit level. It provides an insight that theoretically sparsity-promoting algorithms would have excellent performance on the super-resolution of positive sources, which already have been corroborated by [12, 29, 30]. Especially, under the separation condition (2.7), any tractable sparsity-promoting algorithms (such as total variation minimization algorithms [7]) rendering the sparsest solution could stably reconstruct all the source locations.

3. Resolution limits for super-resolution in multi-dimensional spaces

In this section, combining the estimates in Section 2 and [24, 25], we present our main results on the resolution limits to the super-resolution of positive sources in multi-dimensional spaces. Let us first introduce the model setting. We consider the source as the n-sparse positive measure

$$\mu = \sum_{j=1}^{n} a_j \delta_{\mathbf{y}_j},$$

where δ denotes Dirac's δ-distribution in $\mathbb{R}^k$, $\mathbf{y}_j \in \mathbb{R}^k, 1 \leq j \leq n$, represent the locations of the point sources and $a_j > 0, 1 \leq j \leq n$ are their amplitudes. Denote by

$$m_{\min} = \min_{j=1,\cdots,n} |a_j|, \quad d_{\min} = \min_{p \neq j} ||\mathbf{y}_p - \mathbf{y}_j||_2. \tag{3.1}$$

The available measurement is the noisy Fourier data of μ in a bounded region, that is,

$$\mathbf{Y}(\boldsymbol{\omega}) = \mathscr{F}\mu(\boldsymbol{\omega}) + \mathbf{W}(\boldsymbol{\omega}) = \sum_{j=1}^{n} a_j e^{i\mathbf{y}_j \cdot \boldsymbol{\omega}} + \mathbf{W}(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \mathbb{R}^k, \tag{3.2}$$

where with slight abuse of notation $\mathscr{F}\mu$ denotes the Fourier transform of μ in the k-dimensional space, Ω is the cut-off frequency, and $\mathbf{W}$ is the noise. We assume that

$$||\mathbf{W}(\boldsymbol{\omega})||_\infty < \sigma,\ ||\boldsymbol{\omega}||_2 \leq \Omega,$$

where σ is the noise level and $\left|\left|f(\boldsymbol{\omega})\right|\right|_\infty := \max_{\boldsymbol{\omega} \in \mathbb{R}^k, ||\boldsymbol{\omega}||_2 \leq \Omega} |f(\boldsymbol{\omega})|$ in this section. We are interested in the resolution limit for resolving a cluster of tightly-spaced point sources. Thus, we denote by

$$B_\delta^k(\mathbf{x}) := \left\{\mathbf{y} \,\middle|\, \mathbf{y} \in \mathbb{R}^k,\ ||\mathbf{y} - \mathbf{x}||_2 < \delta\right\},$$

and assume that $\mathbf{y}_j \in B^k_{\frac{(n-1)\pi}{2\Omega}}(\mathbf{0}), j = 1, \cdots, n$, or equivalently $||\mathbf{y}_j||_2 < \frac{(n-1)\pi}{2\Omega}$.

We then define positive σ-admissible measures and computational resolution limits in the k-dimensional space analogously to those in the one-dimensional case.

Definition 3.1. *Given measurement* $\mathbf{Y}$*, we say that the positive measure* $\hat{\mu} = \sum_{j=1}^{m} \hat{a}_j \delta_{\hat{\mathbf{y}}_j}$, $\hat{\mathbf{y}}_j \in \mathbb{R}^k$*, is a positive* σ*-admissible discrete measure of* $\mathbf{Y}$ *if*

$$||\mathscr{F}\hat{\mu}(\boldsymbol{\omega}) - \mathbf{Y}(\boldsymbol{\omega})||_\infty < \sigma, \text{ for all } ||\boldsymbol{\omega}||_2 \leq \Omega,\ \boldsymbol{\omega} \in \mathbb{R}^k.$$

Definition 3.2. *The computational resolution limit to the number detection problem in k-dimensional space is defined as the smallest nonnegative number $\mathscr{D}^+_{k,num}$ such that for all positive n-sparse measures $\sum_{j=1}^n a_j\delta_{\mathbf{y}_j}, \mathbf{y}_j \in B^k_{\frac{(n-1)\pi}{2\Omega}}(\mathbf{0})$ and the associated measurement $\mathbf{Y}$ in (3.2), if*

$$\min_{p\neq j} ||\mathbf{y}_j - \mathbf{y}_p||_2 \geq \mathscr{D}^+_{k,num},$$

then there does not exist any positive σ-admissible measure with less than n supports for $\mathbf{Y}$.

As a consequence of [25, Theorem 2.3], we have the following result for the upper bound of the $\mathscr{D}^+_{k,num}$.

Theorem 3.1. *Let $n\geq 2$ and the measurement $\mathbf{Y}$ in (3.2) be generated by a positive n-sparse measure $\mu = \sum_{j=1}^n a_j\delta_{\mathbf{y}_j}, \mathbf{y}_j \in B^k_{\frac{(n-1)\pi}{2\Omega}}(\mathbf{0})$. There is a constant $C_{num}(k,n)$ which has an explicit form such that if*

$$\min_{p\neq j, 1\leq p,j\leq n} \left|\left|\mathbf{y}_p - \mathbf{y}_j\right|\right|_2 \geq \frac{C_{num}(k,n)}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}} \tag{3.3}$$

holds, then there do not exist any positive σ-admissible measures of $\mathbf{Y}$ with less than n supports.

We next show that the above upper bound is optimal in terms of the signal-to-noise ratio.

Theorem 3.2. *For given $0<\sigma\leq m_{\min}$ and integer $n\geq 2$, there exist positive measures $\mu = \sum_{j=1}^n a_j\delta_{\mathbf{y}_j}$ with n supports, and $\hat\mu = \sum_{j=1}^{n-1}\hat a_j \delta_{\hat{\mathbf{y}}_j}$ with $(n-1)$ supports such that $||\mathscr{F}[\hat\mu]-\mathscr{F}[\mu]||_\infty < \sigma$. Moreover,*

$$\min_{1\leq j\leq n}|a_j| = m_{\min}, \quad \min_{p\neq j}\left|\left|\mathbf{y}_p - \mathbf{y}_j\right|\right|_2 = \frac{2e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}.$$

Proof. Consider $\gamma = \sum_{j=1}^{2n-1} a_j\delta_{\mathbf{t}_j}$ with $\mathbf{t}_1 = (-(n-1)\tau, 0,\cdots,0), \mathbf{t}_2 = (-(n-2)\tau,0,\cdots,0),\cdots,\mathbf{t}_{2n-1} = ((n-1)\tau,0,\cdots,0)$ and $\tau = \frac{e^{-1}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}$. For every $\boldsymbol{\omega} = (\omega_1,\omega_2,\cdots,\omega_k)^\top$,

$$\mathscr{F}\gamma(\boldsymbol{\omega}) = \sum_{j=1}^{2n-1} a_j e^{i\mathbf{t}_j\cdot\boldsymbol{\omega}} = \sum_{j=1}^{2n-1} a_j e^{i(-n+j)\tau\omega_1}, |\omega_1|\leq\Omega.$$

This reduces the estimation of $\mathscr{F}\gamma(\boldsymbol{\omega})$ to the one-dimensional case. Combined with Theorem 2.2, there exist $a_{2j-1}>0, a_{2j}<0, 1\leq j\leq n, \min_{j=1,\cdots,n}|a_{2j-1}| = m_{\min}$, so that $\left|\left|\mathscr{F}\gamma(\boldsymbol{\omega})\right|\right|_\infty < \sigma$. As a consequence,

$$\mu = \sum_{j=1}^n a_{2j-1}\delta_{\mathbf{t}_{2j-1}}, \quad \hat\mu = \sum_{j=1}^{n-1} -a_{2j}\delta_{\mathbf{t}_{2j}}$$

satisfy all the conditions of the theorem. □

The above results indicate that

$$\frac{C_{1,k}(n)}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}} < \mathscr{D}^+_{k,num} \leq \frac{C_{2,k}(n)}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}},$$

with $C_{1,k}(n), C_{2,k}(n)$ being certain constants. An interesting open problem is to improve these constants. Two of the authors of this paper have made a progress in this direction [24].

To state the estimates for the resolution limits to the location recovery, we introduce the following concepts which are analogue to those in the one-dimensional case.

Definition 3.3. *Let $\mu = \sum_{j=1}^n a_j \delta_{\mathbf{y}_j}$ be a positive n-sparse discrete measure in $\mathbb{R}^k$ and let $\delta > 0$ be such that the n balls $B_\delta^k(\mathbf{y}_j), 1 \le j \le n$, are pairwise disjoint. We say that $\hat\mu = \sum_{j=1}^n \hat a_j \delta_{\hat{\mathbf{y}}_j}$ is within δ-neighborhood of μ if each $\hat{\mathbf{y}}_j$ is contained in one and only one of the n balls $B_\delta^k(\mathbf{y}_j), 1 \le j \le n$.*

Definition 3.4. *The computational resolution limit to the stable support recovery problem in k-dimensional space is defined as the smallest non-negative number $\mathscr{D}_{k,supp}^+$ such that for any positive n-sparse measure $\mu = \sum_{j=1}^n a_j \delta_{\mathbf{y}_j}, \mathbf{y}_j \in B_{\frac{(n-1)\pi}{2\Omega}}^k(\mathbf{0})$ and the associated measurement $\mathbf{Y}$ in (3.2), if*

$$\min_{p\neq j, 1\le p,j\le n} ||\mathbf{y}_p - \mathbf{y}_j||_2 \ge \mathscr{D}_{k,supp}^+,$$

then there exists $\delta > 0$ such that any σ-admissible measure of $\mathbf{Y}$ with n supports in $B_{\frac{(n-1)\pi}{2\Omega}}^k(\mathbf{0})$ is within a δ-neighbourhood of μ.

As a consequence of [25, Theorem 2.7], we have the following result on the characterization of $\mathscr{D}_{k,supp}^+$.

Theorem 3.3. *Let $n \ge 2$. Let the measurement $\mathbf{Y}$ in (3.2) be generated by a positive n-sparse measure $\mu = \sum_{j=1}^n a_j \delta_{\mathbf{y}_j}, \mathbf{y}_j \in B_{\frac{(n-1)\pi}{2\Omega}}^k(\mathbf{0})$ in the k-dimensional space. There is a constant $C_{supp}(k,n)$ which has an explicit form such that if*

$$d_{\min} := \min_{p\neq j} \left|\left|\mathbf{y}_p - \mathbf{y}_j\right|\right|_2 \ge \frac{C_{supp}(k,n)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}} \tag{3.4}$$

holds, then for any $\hat\mu = \sum_{j=1}^n \hat a_j \delta_{\hat{\mathbf{y}}_j}, \hat{\mathbf{y}}_j \in B_{\frac{(n-1)\pi}{2\Omega}}^k(\mathbf{0})$ being a positive σ-admissible measure of $\mathbf{Y}$, $\hat\mu$ is within the $\frac{d_{\min}}{2}$-neighborhood of μ. Moreover, after reordering the $\hat{\mathbf{y}}_j$'s, we have

$$\left|\left|\hat{\mathbf{y}}_j - \mathbf{y}_j\right|\right|_2 \le \frac{C(k,n)}{\Omega} SRF^{2n-2}\frac{\sigma}{m_{\min}}, \quad 1 \le j \le n, \tag{3.5}$$

where $SRF := \frac{\pi}{\Omega}$ is the super-resolution factor and $C(k,n)$ has an explicit form.

Theorem 3.3 gives an upper bound for the computational resolution limit for the stable support recovery in the k-dimensional space. This bound is optimal in terms of the order of the signal-to-noise ratio, as is shown by the theorem below.

Theorem 3.4. *For given $0 < \sigma \le m_{\min}$ and integer $n \ge 2$, let*

$$\tau = \frac{e^{-1}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}. \tag{3.6}$$

Then there exist a positive measure $\mu = \sum_{j=1}^{n} a_j \delta_{\mathbf{y}_j}, \mathbf{y}_j \in \mathbb{R}^k$, *with n supports at* $\{(-(n-\frac{1}{2})\tau, 0, \cdots, 0), (-(n-\frac{5}{2})\tau, 0, \cdots, 0), \cdots, ((n-\frac{3}{2})\tau, 0, \cdots, 0)\}$ *and a positive measure* $\hat{\mu} = \sum_{j=1}^{n} \hat{a}_j \delta_{\hat{\mathbf{y}}_j}, \hat{\mathbf{y}}_j \in \mathbb{R}^k$, *with n supports at* $\{(-(n-\frac{3}{2})\tau, 0, \cdots, 0), (-(n-\frac{7}{2})\tau, 0, \cdots, 0), \cdots, ((n-\frac{1}{2})\tau, 0, \cdots, 0)\}$ *such that*

$$\left|\left|\mathscr{F}[\hat{\mu}] - \mathscr{F}[\mu]\right|\right|_{\infty} < \sigma, \quad \min_{1 \le j \le n} |a_j| = m_{\min}.$$

Proof. Similar to the discussions in the proof of Theorem 3.2, the problem can be reduced to the one-dimensional case. Then leveraging Theorem 2.4 proves the result. □

Theorem 3.4 provides a lower bound to the computational resolution limit $\mathscr{D}_{k,supp}^{+}$. Combined with Theorem 3.3, it reveals that

$$\frac{C_{3,k}(n)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}} < \mathscr{D}_{k,supp}^{+} \le \frac{C_{4,k}(n)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$$

for certain constants $C_{3,k}(n), C_{4,k}(n)$.

Remark 3.1. *Compared to the one-dimensional case in Section 2, the upper bounds of multi-dimensional computational resolution limits for the number detection and location recovery in the super-resolution of positive sources has the same dependence on the signal-to-noise ratio and cutoff frequency. Moreover, their dependence on the dimensionality are indicated by the constant factors in the upper bound. We conjecture that the optimal constants may be independent of the source number n. Note that the constant factors in the bounds have been improved in [24] to nearly optimal for the two-dimensional case.*

4. Proofs of main results

We first introduce some notation and lemmas that are used in the following proofs. Set

$$\phi_s(t) = \left(1, t, \cdots, t^s\right)^{\top}. \tag{4.1}$$

We recall the Stirling formula that

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \le n! \le e n^{n+\frac{1}{2}} e^{-n}. \tag{4.2}$$

Lemma 4.1. *Let* $t_1, \cdots, t_k$ *be k different real numbers and let t be a real number. We have*

$$\left(D_k(k-1)^{-1}\phi_{k-1}(t)\right)_j = \Pi_{1 \le q \le k, q \ne j} \frac{t - t_q}{t_j - t_q},$$

where $D_k(k-1) := \left(\phi_{k-1}(t_1), \cdots, \phi_{k-1}(t_k)\right)$ *with* $\phi_{k-1}(\cdot)$ *defined by (4.1).*

Proof. This is [26, Lemma 5]. For the reader's convenience, we present a simple proof here. We denote $\left(D_k(k-1)^{-1}\right)_{jq} = b_{jq}$. Observe that

$$\left(D_k(k-1)^{-1}\phi_{k-1}(t)\right)_j = \sum_{q=1}^{k} b_{jq} t^{q-1}.$$

We have

$$\sum_{q=1}^{k} b_{jq}(t_p)^{q-1} = \delta_{jp},\ \forall j, p = 1, \cdots, k,$$

where δ_{jp} is the Kronecker delta function. Then the polynomial $P_j(x) = \sum_{q=1}^{k} b_{jq} x^{q-1}$ satisfies $P_j(t_1) = 0, \cdots, P_j(t_{j-1}) = 0, P_j(t_j) = 1, P_j(t_{j+1}) = 0, \cdots, P_j(t_k) = 0$. Therefore, it must be the Lagrange polynomial

$$P_j(x) = \Pi_{1\le q\le k, q\ne j} \frac{x - t_q}{t_j - t_q}.$$

It follows that

$$\left(D_k(k-1)^{-1}\phi_{k-1}(t)\right)_j = \Pi_{1\le q\le k, q\ne j} \frac{t - t_q}{t_j - t_q}.$$

□

4.1. Proof of Theorem 2.2

Proof. **Step 1.** Let

$$\tau = \frac{e^{-1}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}} \tag{4.3}$$

and $t_1 = -(n-1)\tau, t_2 = -(n-2)\tau, \cdots, t_n = 0, t_{n+1} = \tau, \cdots, t_{2n-1} = (n-1)\tau$. Consider the following system of linear equations:

$$Aa = 0, \tag{4.4}$$

where $A = \left(\phi_{2n-3}(t_1), \cdots, \phi_{2n-3}(t_{2n-1})\right)$ with $\phi_{2n-3}(\cdot)$ being defined by (4.1). Since A is underdetermined, there exists a nontrivial solution $a = (a_1, \cdots, a_{2n-1})^\top$ to (4.4). By the linear independence of the any $(2n-2)$ column vectors of A, we can show that all a_j's are nonzero. By a scaling of a, we can assume that $a_{2n-1} > 0$ and

$$\min_{1\le j\le n} |a_{2j-1}| = m_{\min}. \tag{4.5}$$

We define

$$\mu = \sum_{j=1}^{n} a_{2j-1}\delta_{t_{2j-1}}, \quad \hat\mu = \sum_{j=1}^{n-1} -a_{2j}\delta_{t_{2j}}.$$

We shall show that the intensities in $\hat\mu$ and μ are all positive and $||\mathscr{F}[\hat\mu] - \mathscr{F}[\mu]||_\infty < \sigma$ in the subsequent steps.

Step 2. We first analyze the sign of each $a_j, j = 1, \cdots, 2n-1$, based on $a_{2n-1} > 0$. The equation (4.4) implies that

$$-a_{2n-1}\phi_{2n-3}(t_{2n-1}) = \left(\phi_{2n-3}(t_1), \cdots, \phi_{2n-3}(t_{2n-2})\right)(a_1, \cdots, a_{2n-2})^\top,$$

and hence

$$-a_{2n-1}\left(\phi_{2n-3}(t_1), \cdots, \phi_{2n-3}(t_{2n-2})\right)^{-1}\phi_{2n-3}(t_{2n-1}) = (a_1, \cdots, a_{2n-2})^\top.$$

Together with Lemma 4.1, we have

$$-a_{2n-1}\Pi_{1\le q\le 2n-2, q\neq j}\frac{t_{2n-1}-t_q}{t_j-t_q}=a_j, \tag{4.6}$$

for $j=1,\cdots,2n-2$. Observe first that $\Pi_{1\le q\le 2n-2,q\neq j}(t_{2n-1}-t_q)$ is always positive for $1\le j\le 2n-2$. For $j=2n-2$, since $a_{2n-1}>0$, $-a_{2n-1}\Pi_{1\le q\le 2n-2,q\neq 2n-2}(t_{2n-2}-t_q)$ is negative in (4.6). Thus we have $a_{2n-2}<0$. In the same fashion, we see that $a_j<0$ for even j and $a_j>0$ for odd j. Hence the intensities in $\hat{\mu}$ and μ are all positive.

Step 3. We demonstrate that $||\mathscr{F}[\hat{\mu}]-\mathscr{F}[\mu]||_\infty<\sigma$. Observe that

$$||\mathscr{F}[\hat{\mu}]-\mathscr{F}[\mu]||_\infty=\max_{x\in[-\Omega,\Omega]}|\mathscr{F}(\gamma)(x)|, \tag{4.7}$$

where $\gamma=\sum_{j=1}^{2n-1}a_j\delta_{t_j}$ and

$$\mathscr{F}[\gamma](x)=\sum_{j=1}^{2n-1}a_je^{it_jx}=\sum_{j=1}^{2n-1}a_j\sum_{k=0}^{\infty}\frac{(it_jx)^k}{k!}=\sum_{k=0}^{\infty}Q_k(\gamma)\frac{(ix)^k}{k!}. \tag{4.8}$$

Here, $Q_k(\gamma)=\sum_{j=1}^{2n-1}a_jt_j^k$. By (4.4), we have $Q_k(\gamma)=0, k=0,\cdots,2n-3$. We next estimate $Q_k(\gamma)$ for $k>2n-3$.

Step 4. We estimate $\sum_{j=1}^{2n-1}|a_j|$ first. We begin by ordering a_j's such that

$$m_{\min}=|a_{j_1}|\le|a_{j_2}|\le\cdots\le|a_{j_{2n-1}}|.$$

Then (4.4) implies that

$$a_{j_1}\phi_{2n-3}(t_{j_1})=\big(\phi_{2n-3}(t_{j_2}),\cdots,\phi_{2n-3}(t_{j_{2n-1}})\big)(-a_{j_2},\cdots,-a_{j_{2n-1}})^\top,$$

and hence

$$a_{j_1}\left(\phi_{2n-3}(t_{j_2}),\cdots,\phi_{2n-3}(t_{j_{2n-1}})\right)^{-1}\phi_{2n-3}(t_{j_1})=(-a_{j_2},\cdots,-a_{j_{2n-1}})^\top.$$

Together with Lemma 4.1, we have

$$a_{j_1}\Pi_{2\le q\le 2n-2}\frac{t_{j_1}-t_{j_q}}{t_{j_{2n-1}}-t_{j_q}}=-a_{j_{2n-1}}.$$

Further,

$$\begin{aligned}\left|a_{j_{2n-1}}\right|&=\left|a_{j_1}\right|\Pi_{2\le q\le 2n-2}\frac{\left|t_{j_1}-t_{j_q}\right|}{\left|t_{j_{2n-1}}-t_{j_q}\right|}=\left|a_{j_1}\right|\Pi_{2\le q\le 2n-2}\frac{\left|t_{j_1}-t_{j_q}\right|}{\left|t_{j_{2n-1}}-t_{j_q}\right|}\frac{\left|t_{j_1}-t_{j_{2n-1}}\right|}{\left|t_{j_{2n-1}}-t_{j_1}\right|}\\&=\left|a_{j_1}\right|\frac{\Pi_{2\le q\le 2n-1}\left|t_{j_1}-t_{j_q}\right|}{\Pi_{1\le q\le 2n-2}\left|t_{j_{2n-1}}-t_{j_q}\right|}\le\left|a_{j_1}\right|\frac{\max_{j_1=1,\cdots,2n-1}\Pi_{2\le q\le 2n-1}\left|t_{j_1}-t_{j_q}\right|}{\min_{j_{2n-1}=1,\cdots,2n-1}\Pi_{1\le q\le 2n-2}\left|t_{j_{2n-1}}-t_{j_q}\right|}.\end{aligned} \tag{4.9}$$

Thus, based on the distribution of t_j's and (4.5), we have

$$|a_{j_{2n-1}}|\le\frac{(2n-2)!}{((n-1)!)^2}|a_{j_1}|\le\frac{(2n-2)!}{((n-1)!)^2}m_{\min},$$

and consequently,

$$\sum_{j=1}^{2n-1}|a_j| = \sum_{q=1}^{2n-1}|a_{j_q}| \le (2n-1)|a_{j_{2n-1}}| \le \frac{(2n-1)!}{((n-1)!)^2} m_{\min}. \tag{4.10}$$

It follows that for $k \ge 2n-2$,

$$|Q_k(\gamma)| = |\sum_{j=1}^{2n-1} a_j t_j^k| \le \sum_{j=1}^{2n-1}|a_j|\big((n-1)\tau\big)^k \le \frac{(2n-1)!}{((n-1)!)^2} m_{\min}\big((n-1)\tau\big)^k.$$

Step 5. Using (4.8), we have

$$\begin{aligned}
\max_{x\in[-\Omega,\Omega]}\left|\mathscr{F}[\gamma](x)\right| &\le \sum_{k\ge 2n-2}\frac{(2n-1)!}{((n-1)!)^2} m_{\min}\big((n-1)\tau\big)^k \frac{\Omega^k}{k!} \\
&= \frac{(2n-1)!m_{\min}(n-1)^{2n-2}(\tau\Omega)^{2n-2}}{((n-1)!)^2\,(2n-2)!}\sum_{k=0}^{+\infty}\frac{(\tau\Omega)^k(2n-2)!(n-1)^k}{(k+2n-2)!} \\
&< \frac{(2n-1)m_{\min}(n-1)^{2n-2}(\tau\Omega)^{2n-2}}{((n-1)!)^2}\sum_{k=0}^{+\infty}\left(\frac{\tau\Omega}{2}\right)^k \\
&\le \frac{(2n-1)m_{\min}(n-1)^{2n-2}(\tau\Omega)^{2n-2}}{((n-1)!)^2}\frac{1}{0.8} \quad \left(\text{by (4.3), } \frac{\tau\Omega}{2}\le 0.2\right) \\
&\le \frac{(2n-1)m_{\min}}{2\pi(n-1)}(e\tau\Omega)^{2n-2}\frac{1}{0.8}. \quad \big(\text{by (4.2)}\big)
\end{aligned}$$

Finally, using (4.3) and the inequality that $\frac{(2n-1)}{2\pi(n-1)}\frac{1}{0.8} < 1$, we obtain

$$\max_{x\in[-\Omega,\Omega]}\left|\mathscr{F}[\gamma](x)\right| < \sigma.$$

This completes the proof. □

4.2. Proof of Theorem 2.4

Proof. Let $t_1 = -(n-\frac{1}{2})\tau, t_2 = -(n-\frac{3}{2})\tau, \cdots, t_n = -\frac{\tau}{2}, t_{n+1} = \frac{\tau}{2}, \cdots, t_{2n} = (n-\frac{1}{2})\tau$. Consider the following system of linear equations:

$$Aa = 0, \tag{4.11}$$

where $A = \big(\phi_{2n-2}(t_1), \cdots, \phi_{2n-2}(t_{2n})\big)$ with $\phi_{2n-2}(\cdot)$ being defined in (4.1). Since A is underdetermined, there exists a nontrivial solution $a = (a_1, \cdots, a_{2n})^\top$. By the linear independence of any $(2n-1)$ column vectors of A, all a_j's are nonzero. By a scaling of a, we can assume that $a_{2n} < 0$ and

$$\min_{1\le j\le n}|a_{2j-1}| = m_{\min}. \tag{4.12}$$

We define

$$\mu = \sum_{j=1}^{n} a_{2j-1}\delta_{t_{2j-1}}, \quad \hat{\mu} = \sum_{j=1}^{n} -a_{2j}\delta_{t_{2j}}.$$

Similar to Step 2 in the proof of Theorem 2.2, we can show that $a_{2j-1} > 0, j = 1,\cdots,n$, and $a_{2j} < 0, j = 1,\cdots,n$. Thus both $\hat{\mu}$ and μ are positive measures. Similar to Step 4 in the proof of Theorem 2.2, we can show that

$$\sum_{j=1}^{2n} |a_j| \leq \frac{(2n)!}{n!(n-1)!} m_{\min}. \tag{4.13}$$

We now prove that

$$||\mathscr{F}[\hat{\mu}] - \mathscr{F}[\mu]||_{\infty} \leq \max_{x\in[-\Omega,\Omega]} |\mathscr{F}[\gamma](x)| < \sigma,$$

where $\gamma = \sum_{j=1}^{2n} a_j \delta_{t_j}$. Indeed, (4.13) implies, for $k \geq 2n-1$,

$$|\sum_{j=1}^{2n} a_j t_j^k| \leq \sum_{j=1}^{2n} |a_j|((n-1/2)\tau)^k \leq \frac{(2n)!}{n!(n-1)!} m_{\min} \left((n-1/2)\tau\right)^k.$$

On the other hand, similar to expansion (4.8), we can expand $\mathscr{F}[\gamma]$ and have

$$Q_k(\gamma) = 0,\ k = 0,\cdots,2n-2 \quad \text{and } |Q_k(\gamma)| \leq \frac{(2n)!}{n!(n-1)!} m_{\min} \left((n-1/2)\tau\right)^k, k \geq 2n-1.$$

Therefore, for $|x| \leq \Omega$,

$$\begin{aligned}
\max_{x\in[-\Omega,\Omega]} \left|\mathscr{F}[\gamma](x)\right| &\leq \sum_{k\geq 2n-1} \frac{(2n)!}{n!(n-1)!} m_{\min} \left((n-1/2)\tau\right)^k \frac{|x|^k}{k!} \leq \sum_{k\geq 2n-1} \frac{(2n)!}{n!(n-1)!} m_{\min} \left((n-1/2)\tau\right)^k \frac{\Omega^k}{k!} \\
&= \frac{(2n)! m_{\min} (n-1/2)^{2n-1} (\tau\Omega)^{2n-1}}{n!(n-1)!(2n-1)!} \sum_{k=0}^{+\infty} \frac{(\tau\Omega)^k (2n-1)!(n-1/2)^k}{(k+2n-1)!} \\
&< \frac{2n m_{\min} (n-1/2)^{2n-1} (\tau\Omega)^{2n-1}}{n!(n-1)!} \sum_{k=0}^{+\infty} \left(\frac{\tau\Omega}{2}\right)^k \\
&= \frac{2n m_{\min} (n-1/2)^{2n-1} (\tau\Omega)^{2n-1}}{n!(n-1)!} \frac{1}{0.8} \quad \left((2.4) \text{ implies } \frac{\tau\Omega}{2} \leq 0.2\right) \\
&\leq \frac{n m_{\min} (n-1/2)^{2n-1}}{\pi n^{n+\frac{1}{2}} (n-1)^{n-\frac{1}{2}}} (e\tau\Omega)^{2n-1} \frac{1}{0.8} \quad \left(\text{by } (4.2)\right) \\
&\leq \frac{n}{\pi(n-1/2)} m_{\min} (e\tau\Omega)^{2n-1} \frac{1}{0.8} \\
&< \sigma. \quad \left(\text{by } (2.4) \text{ and } \frac{n}{\pi(n-1/2)} \frac{1}{0.8} < 1\right)
\end{aligned}$$

It follows that $||\mathscr{F}[\hat{\mu}] - \mathscr{F}[\mu]||_{\infty} < \sigma$. □

4.3. Proof of Proposition 2.1

Proof. **Step 1.** For $j \in \{1,2,\cdots,2n\}$, set $t_j = -\frac{sn-2}{2}\tau + \frac{(j-2)s}{2}\tau$ if j is even and $t_j = t_{4\lceil \frac{j+1}{4}\rceil - 2} + (-1)^{\frac{j+1}{2}}\tau$ otherwise. Consider the following system of linear equations:

$$Aa = 0,$$

where $A = \left(\phi_{2n-2}(t_1), \cdots, \phi_{2n-2}(t_{2n})\right)$ with $\phi_{2n-2}(\cdot)$ defined in (4.1). Since A is underdetermined, there exists a nontrivial solution $a = (a_1, \cdots, a_{2n})^\top$. Also, by the linear independence of any $(2n-1)$ column vectors of A, we can show that all a_j's are nonzero. By a scaling of a, we can assume that $a_{2n} > 0$ and

$$\min_{1 \le j \le n} |a_{2j}| = m_{\min}. \tag{4.14}$$

We define

$$\mu = \sum_{j=1}^{n} a_{2j} \delta_{t_{2j}}, \quad \hat{\mu} = \sum_{j=1}^{n} -a_{2j-1} \delta_{t_{2j-1}}.$$

Similar to Step 2 in the proof of Theorem 2.2, we can show that $a_{2j-1} < 0, j = 1, \cdots, n$, and $a_{2j} > 0, j = 1, \cdots, n$. Thus, both $\hat{\mu}$ and μ are positive measures.

Step 2. We now estimate $\sum_{j=1}^{2n} |a_j|$. Reorder a_j such that

$$m_{\min} = |a_{j_1}| \le |a_{j_2}| \le \cdots \le |a_{j_{2n}}|.$$

Similar to Step 4 in the proof of Theorem 2.2, we have

$$a_{j_1} \Pi_{2 \le q \le 2n-1} \frac{t_{j_1} - t_{j_q}}{t_{j_{2n}} - t_{j_q}} = -a_{j_{2n}}. \tag{4.15}$$

We next estimate $\Pi_{2 \le q \le 2n-1} \left| \frac{t_{j_1} - t_{j_q}}{t_{j_{2n}} - t_{j_q}} \right|$. Note that

$$\begin{aligned} \Pi_{2 \le q \le 2n-1} \frac{\left|t_{j_1} - t_{j_q}\right|}{\left|t_{j_{2n}} - t_{j_q}\right|} &= \left(\Pi_{2 \le q \le 2n-1} \frac{\left|t_{j_1} - t_{j_q}\right|}{\left|t_{j_{2n}} - t_{j_q}\right|}\right) \cdot \frac{\left|t_{j_1} - t_{j_{2n}}\right|}{\left|t_{j_{2n}} - t_{j_1}\right|} = \frac{\Pi_{2 \le q \le 2n} \left|t_{j_1} - t_{j_q}\right|}{\Pi_{1 \le q \le 2n-1} \left|t_{j_{2n}} - t_{j_q}\right|} \\ &\le \frac{\max_{j \in \{1,2,\ldots,2n\}} \Pi_{i \in \{1,2,\ldots,2n\}, i \ne j} \left|t_i - t_j\right|}{\min_{j \in \{1,2,\ldots,2n\}} \Pi_{i \in \{1,2,\ldots,2n\}, i \ne j} \left|t_i - t_j\right|}. \end{aligned} \tag{4.16}$$

We separate $\{t_j\}_{j=1,2,\ldots,2n}$ into four classes: $C_1 = \{t_{4j-2}\}_{j=1}^{\lceil \frac{n}{2} \rceil}, C_2 = \{t_{4j}\}_{j=1}^{\lfloor \frac{n}{2} \rfloor}, C_3 = \{t_{4j-3}\}_{j=1}^{\lceil \frac{n}{2} \rceil}, C_4 = \{t_{4j-1}\}_{j=1}^{\lfloor \frac{n}{2} \rfloor}$; See Figure 4.1 for an illustration. The points in each class are evenly-spaced, by which we can estimate the right-hand side of (4.16).

Note that

$$\begin{aligned} &\max_{j \in \{1,2,\ldots,2n\}} \Pi_{p \in \{1,2,\ldots,2n\}, p \ne j} |t_p - t_j| = \max_{k=1,2,3,4} \left(\max_{x \in C_k} \Pi_{y \in \{t_1, t_2 \ldots, t_{2n}\}, y \ne x} |x - y|\right) \\ &\le \max_{k=1,2,3,4} \left(\max_{x \in C_k} \Pi_{p=1,2,3,4} \Pi_{y \in C_p, y \ne x} |x - y|\right) =: \max_{k=1,2,3,4} c_k^{\max}, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} &\min_{j \in \{1,2,\ldots,2n\}} \Pi_{p \in \{1,2,\ldots,2n\}, p \ne j} |t_p - t_j| = \min_{k=1,2,3,4} \left(\min_{x \in C_k} \Pi_{y \in \{t_1, t_2 \ldots, t_{2n}\}, y \ne x} |x - y|\right) \\ &\ge \min_{k=1,2,3,4} \left(\min_{x \in C_k} \Pi_{p=1,2,3,4} \Pi_{y \in C_p, y \ne x} |x - y|\right) =: \min_{k=1,2,3,4} c_k^{\min}. \end{aligned} \tag{4.18}$$

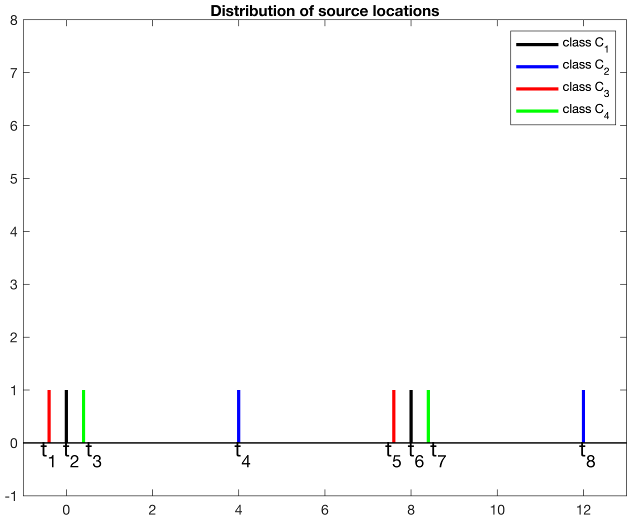


Figure 4.1: Distribution of source locations.

The estimates of $\max_{k=1,2,3,4} c_k^{\max}$ and $\min_{k=1,2,3,4} c_k^{\min}$ are detailed in Lemmas A.3 and A.4 in Appendix A. With the aid of them we control the left-hand side of (4.16) that

$$\Pi_{2\le q\le 2n-1}\frac{|t_{j_1}-t_{j_q}|}{|t_{j_{2n}}-t_{j_q}|}\le \frac{\tau^{2n-1}s^{2n-1}\left(2\lceil\frac{n}{2}\rceil\right)!\left(2\lceil\frac{n}{2}\rceil-1\right)!}{\tau^{2n-1}s^{2n-3}\cdot\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^4}\le \frac{s^2e^{11}}{\pi^2}(n+1)^{10}2^{2n-8}, \tag{4.19}$$

where the last inequality is obtained by Lemma 4.2 in the following step.
Step 3.

Lemma 4.2. *For $n\ge 2$, we have*

$$\frac{\left(2\lceil\frac{n}{2}\rceil\right)!\left(2\lceil\frac{n}{2}\rceil-1\right)!}{\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^4}\le \frac{e^{11}}{\pi^2}(n+1)^{10}2^{2n-8}.$$

Proof. Recall the Stirling approximation of factorial, that is,

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\le n!\le en^{n+\frac{1}{2}}e^{-n}. \tag{4.20}$$

For $n \le 11$, the inequality can be checked by calculation. For $n > 11$, we have

$$
\begin{aligned}
&\frac{(2\lceil \frac{n}{2}\rceil)!\,(2\lceil \frac{n}{2}\rceil-1)!}{\left((2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^4} \le \frac{e^{2-4\lceil \frac{n}{2}\rceil+1}(2\lceil \frac{n}{2}\rceil)^{2\lceil \frac{n}{2}\rceil+\frac{1}{2}}\cdot(2\lceil \frac{n}{2}\rceil-1)^{2\lceil \frac{n}{2}\rceil-\frac{1}{2}}}{(\sqrt{2\pi})^4(2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1)^{4(2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1)+2}e^{-4(2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1)}}\\
\le&\frac{1}{4\pi^2}\frac{e^{3-2n}}{e^{-4(2\cdot\frac{n}{4}-1)}}\cdot\frac{(2(\frac{n}{2}+\frac{1}{2}))^{2n+2}}{(2(\frac{n}{4}-\frac{3}{4})-1)^{4(2(\frac{n}{4}-\frac{3}{4})-1)+2}}=\frac{1}{4e\pi^2}\frac{(n+1)^{2n+2}}{(\frac{n}{2}-\frac{5}{2})^{2n-8}}\\
=&\frac{1}{4e\pi^2}(n+1)^{10}\left(2+\frac{6}{\frac{n}{2}-\frac{5}{2}}\right)^{2n-8}=\frac{1}{4e\pi^2}(n+1)^{10}2^{2n-8}\left(1+\frac{6}{n-5}\right)^{12\cdot\frac{n-5}{6}+2}\\
\le&\frac{e^{11}}{4\pi^2}\left(1+\frac{6}{n-5}\right)^2(n+1)^{10}2^{2n-8}\le\frac{e^{11}}{\pi^2}(n+1)^{10}2^{2n-8}.
\end{aligned}
$$

□

Step 4. Thus, combined (4.15) and (4.19), we have

$$|a_{j_{2n}}| \le \frac{e^{11}s^2}{\pi^2}(n+1)^{10}2^{2n-8}|a_{j_1}|,$$

and consequently,

$$\sum_{j=1}^{2n}|a_j| \le \sum_{q=1}^{2n}|a_{j_q}| \le 2n\frac{e^{11}s^2}{\pi^2}(n+1)^{10}2^{2n-8}m_{\min}. \tag{4.21}$$

It then follows that for $k \ge 2n-1$,

$$\left|\sum_{j=1}^{2n-1}a_j t_j^k\right| \le \sum_{j=1}^{2n-1}|a_j|\left(\frac{sn}{2}\tau\right)^k \le \frac{2n(n+1)^{10}e^{11}s^2}{\pi^2}2^{2n-8}m_{\min}\left(\frac{sn}{2}\tau\right)^k.$$

Step 5. We now prove that

$$||\mathscr{F}[\hat{\mu}]-\mathscr{F}[\mu]||_\infty \le \max_{x\in[-\Omega,\Omega]}|\mathscr{F}[\gamma](x)| < \sigma,$$

where $\gamma = \sum_{j=1}^{2n} a_j\delta_{t_j}$. On the other hand, similar to expansion (4.8), we can expand $\mathscr{F}[\gamma]$ and have

$$Q_k(\gamma)=0,\ k=0,\cdots,2n-2 \quad \text{and } |Q_k(\gamma)| \le \frac{2n(n+1)^{10}e^{11}s^2 2^{2n-8}}{\pi^2}m_{\min}\left(\frac{sn}{2}\tau\right)^k, \quad k \ge 2n-1.$$

Therefore, for $|x|\leq \Omega$ and $n\geq 6$, we have

$$
\begin{aligned}
|\mathscr{F}[\gamma](x)| \leq & \sum_{k\geq 2n-1} \frac{2n(n+1)^{10}e^{11}s^2}{\pi^2} 2^{2n-8} m_{\min}\left(\frac{sn}{2}\tau\right)^k \frac{|x|^k}{k!} \\
\leq & \sum_{k\geq 2n-1} \frac{2n(n+1)^{10}e^{11}s^2}{\pi^2} 2^{2n-8} m_{\min}\left(\frac{sn}{2}\tau\right)^k \frac{\Omega^k}{k!} \\
\leq & \frac{2e^{11}}{\pi^2}\cdot \frac{s^2 n(n+1)^{10}2^{2n-8}m_{\min}\cdot (\frac{sn\tau\Omega}{2})^{2n-1}}{(2n-1)!} \sum_{k=0}^{+\infty} \frac{(\frac{s\tau\Omega}{2})^k(2n-1)!n^k}{(k+2n-1)!} \\
< & \frac{e^{11}}{2^6\pi^2}\cdot \frac{(\tau\Omega)^{2n-1}s^{2n+1}m_{\min}n(n+1)^{10}n^{2n-1}}{(2n-1)!}\cdot \sum_{k=0}^{+\infty}\left(\frac{s\tau\Omega}{2}\right)^k \\
\leq & \frac{e^{11}}{2^6\pi^2}(\tau\Omega)^{2n-1}s^{2n+1}m_{\min}\frac{(n+1)^{10}n^{2n}e^{2n-1}}{\sqrt{2\pi}(2n-1)^{2n-\frac{1}{2}}}\sum_{k=0}^{+\infty}\left(\frac{s\tau\Omega}{2}\right)^k \quad \Big(\text{by Stirling's formula (4.2)}\Big) \\
\leq & 38\cdot 0.2^{2n-1}\sigma\cdot \frac{n(n+1)^{10}}{\sqrt{2n-1}}\frac{n^{2n-1}}{(2n-1)^{2n-1}}\sum_{k=0}^{+\infty}\left(\frac{s\tau\Omega}{2}\right)^k \quad \Big(\text{by separation condition (2.5)}\Big) \\
= & 38\cdot 0.1^{2n-1}\sigma\cdot \frac{n(n+1)^{10}}{\sqrt{2n-1}}\left(\frac{n}{n-\frac{1}{2}}\right)^{2n-1}\frac{1}{1-0.04} \quad \left(\text{(2.5) implies } \frac{s\tau\Omega}{2}\leq 0.04\right) \\
\leq & 38e\cdot 0.1^{2n-1}\sigma\cdot \frac{n(n+1)^{10}}{\sqrt{2n-1}}\frac{1}{1-0.04} \\
< & \sigma \quad \left(38e\cdot 0.1^{2n-1}\cdot \frac{n(n+1)^{10}}{\sqrt{2n-1}}\frac{1}{1-0.04} < 1 \text{ for } n\geq 6\right).
\end{aligned}
$$

It then follows that $||\mathscr{F}[\hat{\mu}]-\mathscr{F}[\mu]||_\infty < \sigma$.

Now consider the case when $2\leq n\leq 5$. By (4.19), we have

$$
\Pi_{2\leq q\leq 2n-1}\frac{|t_{j_1}-t_{j_q}|}{|t_{j_{2n}}-t_{j_q}|}\leq \frac{s^2\left(2\lceil\frac{n}{2}\rceil\right)!\left(2\lceil\frac{n}{2}\rceil-1\right)!}{\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^4},
$$

and consequently,

$$
\sum_{j=1}^{2n}|a_j|\leq 2n\frac{s^2\left(2\lceil\frac{n}{2}\rceil\right)!\left(2\lceil\frac{n}{2}\rceil-1\right)!}{\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^4}.
$$

By similar arguments as those for the case when $n\geq 6$, we can show that for $2\leq n\leq 5$,

$$
|\mathscr{F}[\gamma](x)|<\sigma.
$$

□

5. Phase transition in the number detection

In this section, employing the sweeping singular-value-thresholding number detection algorithm introduced in [26], we verify the phase transition phenomenon for the number detection in the super-resolution of positive sources.

5.1. Review of the sweeping singular-value-thresholding number detection algorithm

In [26], the authors proposed a number detection algorithm called sweeping single-value-thresholding number detection algorithm. It determines the number of sources by thresholding on the singular value of a Hankel matrix formulated from the measurement data.

To be more specific, suppose the measurement is taken at M evenly-spaced points $\omega_1 = -\Omega, \omega_2, \cdots, \omega_M = \Omega$, that is,

$$\mathbf{Y}(\omega_j) = \mathscr{F}\mu(\omega_j) + \mathbf{W}(\omega_j), \quad j = 1, \cdots, M.$$

We choose a partial measurement at the sample points $z_t = \omega_{(t-1)r+1}$ for $t = 1, \cdots, 2s+1$, where $s \geq n$ and $r = (M-1) \mod 2s$. For ease of exposition, assume $r = \frac{M-1}{2s}$. Then $z_t = \omega_{(t-1)\frac{M-1}{2s}+1} = -\Omega + \frac{t-1}{s}\Omega$ (since $\omega_1 = -\Omega$, $\omega_M = \Omega$) and the partial measurement is

$$\mathbf{Y}(z_t) = \mathscr{F}\mu(z_t) + \mathbf{W}(z_t), \quad 1 \leq t \leq 2s+1.$$

Assemble the following Hankel matrix by the measurements that

$$\mathbf{H}(s) = \begin{pmatrix} \mathbf{Y}(-\Omega) & \mathbf{Y}(-\Omega + \frac{1}{s}\Omega) & \cdots & \mathbf{Y}(0) \\ \mathbf{Y}(-\Omega + \frac{1}{s}\Omega) & \mathbf{Y}(-\Omega + \frac{2}{s}\Omega) & \cdots & \mathbf{Y}(\frac{1}{s}\Omega) \\ \cdots & \cdots & \ddots & \cdots \\ \mathbf{Y}(0) & \mathbf{Y}(\frac{1}{s}\Omega) & \cdots & \mathbf{Y}(\Omega) \end{pmatrix}. \tag{5.1}$$

We observe that $\mathbf{H}(s)$ has the decomposition

$$\mathbf{H}(s) = DAD^\top + \Delta,$$

where $A = \text{diag}(e^{-iy_1\Omega}a_1, \cdots, e^{-iy_n\Omega}a_n)$ and $D = \left(\phi_s(e^{iy_1\frac{\Omega}{s}}), \cdots, \phi_s(e^{iy_n\frac{\Omega}{s}})\right)$ with $\phi_s(\omega)$ being defined as $(1, \omega, \cdots, \omega^s)^\top$ and

$$\Delta = \begin{pmatrix} \mathbf{W}(-\Omega) & \mathbf{W}(-\Omega + \frac{1}{s}\Omega) & \cdots & \mathbf{W}(0) \\ \mathbf{W}(-\Omega + \frac{1}{s}\Omega) & \mathbf{W}(-\Omega + \frac{2}{s}\Omega) & \cdots & \mathbf{W}(\frac{1}{s}\Omega) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{W}(0) & \mathbf{W}(\frac{1}{s}\Omega) & \cdots & \mathbf{W}(\Omega) \end{pmatrix}.$$

We denote the singular value decomposition of $\mathbf{H}(s)$ as

$$\mathbf{H}(s) = \hat{U}\hat{\Sigma}\hat{U}^*,$$

where $\hat{\Sigma} = \text{diag}(\hat{\sigma}_1, \cdots, \hat{\sigma}_n, \hat{\sigma}_{n+1}, \cdots, \hat{\sigma}_{s+1})$ with the singular values $\hat{\sigma}_j$, $1 \leq j \leq s+1$, ordered in a decreasing manner. From [26], we have the following theorem for the threshold to determine the source number.

Theorem 5.1. *Let $s \geq n$ and $\mu = \sum_{j=1}^n a_j \delta_{y_j}$ with $y_j \in I(n, \Omega), 1 \leq j \leq n$. We have*

$$\hat{\sigma}_j \leq (s+1)\sigma, \quad j = n+1, \cdots, s+1. \tag{5.2}$$

Moreover, if the following separation condition is satisfied

$$\min_{p\neq j}|y_p - y_j| > \frac{\pi s}{\Omega}\Big(\frac{2n(s+1)}{\zeta(n)^2}\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}, \tag{5.3}$$

where $\zeta(n) = \begin{cases} (\frac{n-1}{2}!)^2, & n \text{ is odd}, \\ (\frac{n}{2})!(\frac{k-2}{2})!, & n \text{ is even}, \end{cases}$ *then*

$$\hat{\sigma}_n > (s+1)\sigma. \tag{5.4}$$

Based on this theorem, the threshold should be $(s+1)\sigma$ and the following **Algorithm 1** was proposed to detect the source number for fixed s.

Algorithm 1: Singular-value-thresholding number detection algorithm

Input: Number s, Noise level σ;
Input: measurement: $\mathbf{Y} = (\mathbf{Y}(\omega_1), \cdots, \mathbf{Y}(\omega_M))^\top$;
1: $r = (M-1) \mod 2s$, $\mathbf{Y}_{new} = (\mathbf{Y}(\omega_1), \mathbf{Y}(\omega_{r+1}), \cdots, \mathbf{Y}(\omega_{2sr+1}))^\top$;
2: Formulate the $(s+1)\times(s+1)$ Hankel matrix $\mathbf{H}(s)$ from $\mathbf{Y}_{new}$, and compute the singular value of $\mathbf{H}(s)$ as $\hat{\sigma}_1, \cdots, \hat{\sigma}_{s+1}$ distributed in a decreasing manner;
4: Determine n by $\hat{\sigma}_n > (s+1)\sigma$ and $\hat{\sigma}_j \leq (s+1)\sigma, j = n+1, \cdots, s+1$;
Return: n.

In **Algorithm 1**, the $s \geq n$ should be properly chosen to have a good resolution. To address this issue, a sweeping strategy was utilized and the following **Algorithm 2** was proposed. It was shown in [26] that the **Algorithm 2** achieves the optimal resolution order.

Algorithm 2: Sweeping singular-value-thresholding number detection algorithm

Input: Noise level σ, measurement: $\mathbf{Y} = (\mathbf{Y}(\omega_1), \cdots, \mathbf{Y}(\omega_M))^\top$;
Input: $n_{max} = 0$
for $s = 1 : \lfloor \frac{M-1}{2} \rfloor$ **do**
  Input $s, \sigma, \mathbf{Y}$ to **Algorithm 1**, save the output of **Algorithm 1** as $n_{recover}$;
  if $n_{recover} > n_{max}$ **then**
    $n_{max} = n_{recover}$
Return n_{max}.

5.2. Phase transition

We know from Section 2 that the resolution limit to the number detection problem in super-resolution of positive sources is bounded from below and above by $\frac{C_1}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}}$ and $\frac{C_2}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}}$, respectively for some constants C_1, C_2. This indeed implies a phase transition phenomenon in the problem. Specifically, recall that the super-resolution factor is $SRF = \frac{\pi}{d_{\min}\Omega}$ and the $\frac{m_{\min}}{\sigma}$ can be viewed as the signal-to-noise ratio SNR. Taking the logarithm of both sides of the two bounds, we can conclude that the exact number detection is guaranteed if

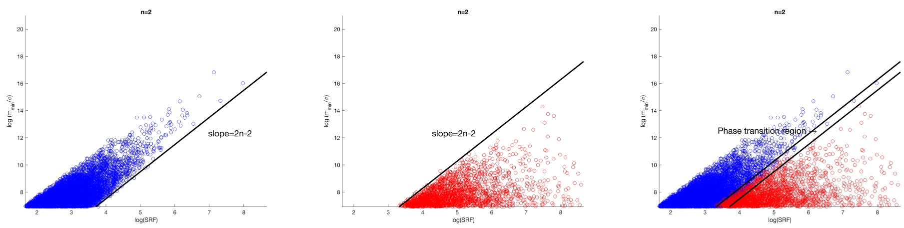
$$\log(SNR) > (2n-2)\log(SRF) + (2n-2)\log\frac{C_1}{\pi},$$

and may fail if

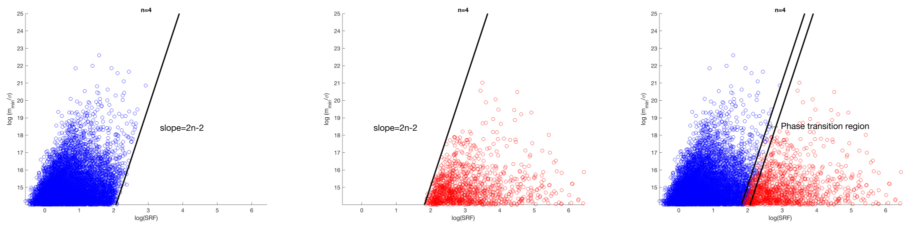
$$\log(SNR) < (2n-2)\log(SRF) + (2n-2)\log\frac{C_2}{\pi}.$$

As a consequence, we expect that in the parameter space of $\log SNR - \log SRF$, there exist two lines both with slope $2n-2$ such that the number detection is successful for cases above the first line and unsuccessful for cases below the second. In the intermediate region between the two lines, the number detection can be either successful or unsuccessful from case to case. This is clearly demonstrated in the numerical experiments below.

We fix $\Omega = 1$ and consider n point sources randomly spaced in $\left[-\frac{(n-1)\pi}{2}, \frac{(n-1)\pi}{2}\right]$ with positive amplitudes a_j's. The noise level is σ and the minimum separation distance between sources is $d_{\min}$. We perform 10000 random experiments (the randomness is in the choice of $(d_{\min}, \sigma, y_j, a_j)$) to detect the source number based on **Algorithm 2**. Figure 5.1 shows the results for $n = 2, 4$, respectively. In each case, two lines of slope $2n-2$ strictly separate the blue points (successful detection) and red points (unsuccessful detection) and in-between is the phase transition region. It clearly elucidates the phase transition phenomenon of **Algorithm 2** and is consistent with our theory.



(a) detection success (b) detection fail (c) phase transition region



(d) detection success (e) detection fail (f) phase transition region

Figure 5.1: Plots of the successful and the unsuccessful number detection by **Algorithm 2** depending on the relation between $\log(SRF)$ and $\log(\frac{m_{\min}}{\sigma})$. (a) illustrates that two positive point source can be exactly detected if $\log(\frac{m_{\min}}{\sigma})$ is above a line of slope 2 in the parameter space. Conversely, for the same case, (b) shows that the number detection fails if $\log(\frac{m_{\min}}{\sigma})$ falls below another line of slope 2. (c) highlights the phase transition region which is bounded by the black slashes in (a) and (b). (d),(e) and (f) illustrate parallel results for four positive point sources.

6. Phase transition in the location recovery

In this section, by the MUSIC algorithm we verify the phase transition phenomenon for the location recovery in the super-resolution of positive sources.

6.1. Review of the MUSIC algorithm

In this section we review the MUSIC algorithm. From the measurement $\mathbf{Y} = (\mathbf{Y}(\omega_1), \mathbf{Y}(\omega_2), \cdots, \mathbf{Y}(\omega_M))^\top$ and $\hat{M} = \lfloor \frac{M-1}{2} \rfloor$, we assemble the $(\hat{M}+1) \times (\hat{M}+1)$ Hankel matrix,

$$\mathbf{H} = \begin{pmatrix} \mathbf{Y}(\omega_1) & \mathbf{Y}(\omega_2) & \cdots & \mathbf{Y}(\omega_{\hat{M}}) \\ \mathbf{Y}(\omega_2) & \mathbf{Y}(\omega_3) & \cdots & \mathbf{Y}(\omega_{\hat{M}+1}) \\ \cdots & \cdots & \ddots & \cdots \\ \mathbf{Y}(\omega_{\hat{M}}) & \mathbf{Y}(\omega_{\hat{M}+1}) & \cdots & \mathbf{Y}(\omega_{2\hat{M}+1}) \end{pmatrix}. \tag{6.1}$$

We perform the following singular value decomposition for $\mathbf{H}$,

$$\mathbf{H} = \hat{U}\hat{\Sigma}\hat{U}^* = [\hat{U}_1 \quad \hat{U}_2]\mathrm{diag}(\hat{\sigma}_1, \hat{\sigma}_2, \cdots, \hat{\sigma}_n, \hat{\sigma}_{n+1}, \cdots, \hat{\sigma}_{\hat{M}+1})[\hat{U}_1 \quad \hat{U}_2]^*,$$

where $\hat{U}_1 = (\hat{U}(1), \cdots, \hat{U}(n)), \hat{U}_2 = (\hat{U}(n+1), \cdots, \hat{U}(\hat{M}+1))$ with n being the source number. Then we denote the orthogonal projection to the space $\hat{U}_2$ by $\hat{P}_2 x = \hat{U}_2(\hat{U}_2^* x)$. For a test vector $\Phi(x) = (1, e^{ihx}, \cdots, e^{i\hat{M}hx})^\top$ with h being the spacing parameter, we define the MUSIC imaging functional

$$\hat{J}(x) = \frac{||\Phi(x)||_2}{||\hat{P}_2\Phi(x)||_2} = \frac{||\Phi(x)||_2}{||\hat{U}_2^*\Phi(x)||_2}.$$

The local maximizers of $\hat{J}(x)$ indicate the locations of the point sources. In practice, we test evenly spaced points in a specified interval and plot the discrete imaging functional and then determine the source locations by detecting the peaks. We present the peak selection algorithm as **Algorithm 4** and summarize the MUSIC algorithm in **Algorithm 3** below.

Algorithm 3: MUSIC algorithm

Input: Measurements: $\mathbf{Y} = (\mathbf{Y}(\omega_1), \cdots, \mathbf{Y}(\omega_M))^\top$, sampling distance h, source number n;
Input: Region of test points $[TS, TE]$ and spacing of test points TPS;
1: Let $\hat{M} = \lfloor \frac{M-1}{2} \rfloor$, formulate the $(\hat{M}+1) \times (\hat{M}+1)$ Hankel matrix $\hat{X}$ from $\mathbf{Y}$;
2: Compute the singular vector of $\hat{X}$ as $\hat{U}(1), \hat{U}(2), \cdots, \hat{U}(\hat{M}+1)$ and formulate the noise space $\hat{U}_2 = (\hat{U}(n+1), \cdots, \hat{U}(\hat{M}+1))$;
3: For test points x's in $[TS, TE]$ evenly spaced by TPS, construct the test vector $\Phi(x) = (1, e^{ihx}, \cdots, e^{i\hat{M}hx})^\top$;
4: Plot the MUSIC imaging functional $\hat{J}(x) = \frac{||\Phi(x)||_2}{||\hat{U}_2^*\Phi(x)||_2}$;
5: Select the peak locations $\hat{y}_j$'s in the $\hat{J}(x)$ by **Algorithm 4**;
Return $\hat{y}_j$'s.

Algorithm 4: Peak selection algorithm

Input: Image $IMG = (f(x_1), \cdots, f(x_N))$;
Input: Peak compare range PCR, differential compare range DCR, differential compare threshold DCT;
1: Initialize the Local maximum points $LMP = [\,]$, peak points $PP = [\,]$;
2: Differentiate the image IMG to get the $DIMG = (f'(x_1), \cdots, f'(x_N))$;
3: **for** $j = 1 : N$ **do**
  if $f(x_j) = \max(f(x_{j-PCR}), f(x_{j-PCR+1}), \cdots, f(x_{j+PCR}))$ **then**
    LMP appends x_j;
4: **for** x_j *in* LMP **do**
  if $\max(|f'(x_{j-DCR})|, |f'(x_{j-DCR+1})|, \cdots, |f'(x_{j+DCR})|) \geq DCT$ **then**
    PP appends x_j;
Return: PP.

6.2. Phase transition

The derived bounds for the resolution limit $\mathscr{D}^+_{supp}$ of the location recovery in the super-resolution of positive sources implies a phase transition in the problem. Taking the logarithm of both sides of the two bounds, we can draw a conclusion that the location recovery is stable if

$$\log(SNR) > (2n-1)\log(SRF) + (2n-1)\log\frac{C_3}{\pi},$$

and may be unstable if

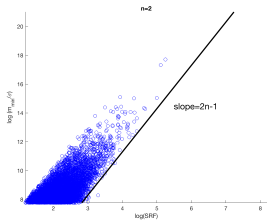
$$\log(SNR) < (2n-1)\log(SRF) + (2n-1)\log\frac{C_4}{\pi},$$

for certain constants C_3, C_4. Similar to the number detection, we expect that in the parameter space of $\log SNR - \log SRF$, there exist two lines both with slope $2n-1$ such that the location recovery is stable for cases above the first line and unstable for cases below the second. This phase transition phenomenon has been demonstrated numerically using the Matrix Pencil method, MUSIC and ESPRIT in [5, 20–22] for resolving general sparse sources.

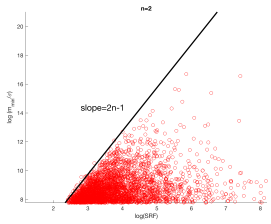
In what follows, we shall conduct numerical experiments to demonstrate the phase transition phenomenon for the MUSIC algorithm in the super-resolution of positive sources. For simplicity, we fix $\Omega = 1$ and consider $n = 2$ or 4 positive point sources separated with minimum separation $d_{\min}$. We perform 10000 random experiments (the randomness is in the choice of $(d_{\min}, \sigma, y_j, a_j)$ to recover the source locations using **Algorithm 3**. The recovery is deemed stable only if n locations $\hat{y}_j$'s are recovered and they are in a $\frac{d_{\min}}{2}$-neighborhood of the ground truth; see **Algorithm 5** for details in a single experiment. As is shown in Figure 6.1, in each case, two lines with slope $2n-1$ strictly separate the blue points (stable recoveries) and red points (unstable recoveries), and in-between is the phase transition region. This is exactly the predicted phase transition phenomenon by our theory. It also demonstrates that the MUSIC can resolve the location of positive point sources with optimal resolution order.

Algorithm 5: A single experiment

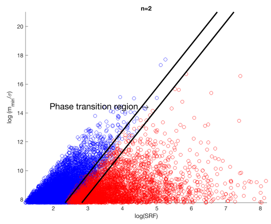
Input: Sources $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}$, source number n;
Input: Measurements: $\mathbf{Y}$
Input source number n and measurement $\mathbf{Y}$ to **Algorithm 3** and save the output as $\hat{y}_1, \cdots, \hat{y}_k$, which are ordered in an increasing manner;
if $k{=}{=}n$ **then**
  Compute the reconstruction error for the source location y_j that $e_j := |\hat{y}_j - y_j|$;
  if $\max_{j=1,\cdots,n} e_j < \frac{\min_{p \neq j} |y_p - y_j|}{2}$ **then**
    Return Stable
  else
    Return Unstable
else
  Return Unstable.



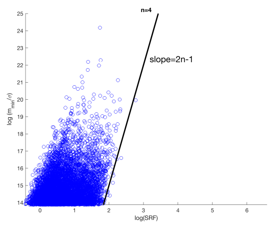
(a) recovery success



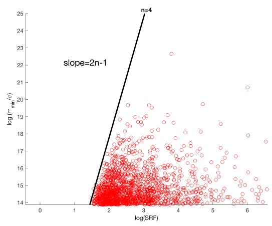
(b) recovery fail



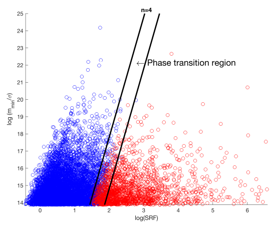
(c) phase transition region



(d) recovery success



(e) recovery fail



(f) phase transition region

Figure 6.1: Plots of the stable and the unstable location recoveries by **Algorithm 3** in view of the relation between $\log(SRF)$ and $\log(\frac{m_{\min}}{\sigma})$. (a) illustrates that the locations of two positive point sources can be stably recovered if $\log(\frac{m_{\min}}{\sigma})$ is above a line of slope 3 in the parameter space. Conversely, for the same case, (b) shows that the locations cannot be stably recovered by MUSIC if $\log(\frac{m_{\min}}{\sigma})$ falls below another line of slope 3. (c) highlights the phase transition region which is bounded by the black slashes in (a) and (b). (d),(e) and (f) illustrate parallel results for four positive point sources.

7. Conclusions and future works

In this paper, we have introduced the resolution limit for respectively the number detection and the location recovery in the super-resolution of positive sources. We have quantitatively characterized the two limits by establishing their sharp upper and lower bounds. We have also verified the phase transition phenomena that predicted by our theory in the number detection and support recovery problems.

Our new technique provides a way to analyze the resolving capability of the super-resolution of positive sources. The applications of the technique introduced here to other problems will be presented in a near future.

A. Auxiliary lemmas

The following results can be easily proved.

Lemma A.1. *Let $n \in \mathbb{N}^+$ and $\tau > 0$ and let $\mathscr{C} = \{j = 1, 2, \ldots, n\}$. Then*

(1):

$$\lceil \frac{n}{2} \rceil \in \arg\min_{z \in \mathscr{C}} \Pi_{x \in \mathscr{C}, x \neq z} |x - z|, \tag{A.1}$$

and

$$\min_{z \in \mathscr{C}} \Pi_{x \in \mathscr{C}, x \neq z} |x - z| = \left(\lceil \frac{n}{2} \rceil - 1\right)! \left(\lfloor \frac{n}{2} \rfloor\right)!; \tag{A.2}$$

(2):

$$\arg\max_{z \in \mathscr{C}} \Pi_{x \in \mathscr{C}, x \neq z} |x - z| = \{1, n\}, \tag{A.3}$$

and

$$\max_{z \in \mathscr{C}} \Pi_{x \in \mathscr{C}, x \neq z} |x - z| = (n-1)!. \tag{A.4}$$

Lemma A.2. *Let $n \in \mathbb{N}^+$ and let $p, q \in \mathbb{R}$ be such that $p > q > 0$. For the following three sets of evenly spaced points $\mathscr{C}_1 := \{x_j = (j-1)p, j = 1, \ldots, n\}$, $\mathscr{C}_2 := \{y_j = (j-1)p - q, j = 1, \ldots, n\}$, and $\mathscr{C}_3 := \{y_j = (j-1)p - q, j = 1, \ldots, n+1\}$, we have*

(1): $\arg\min_{z \in \mathscr{C}_2} \Pi_{j=1}^n |z - x_j| \subset \{y_{\lfloor \frac{n}{2} \rfloor + 1}, y_{\lfloor \frac{n}{2} \rfloor + 2}\}$, *and*

$$\min_{z \in \mathscr{C}_2} \Pi_{i=1}^n |z - x_i| = \Pi_{j=0}^{\lceil \frac{n}{2} \rceil - 1} \left(\min(p-q, q) + pj\right) \cdot \Pi_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\max(p-q, q) + pj\right); \tag{A.5}$$

(2): $\arg\min_{z \in \mathscr{C}_3} \Pi_{j=1}^n |z - x_j| \subset \{y_{\lfloor \frac{n}{2} \rfloor + 1}, y_{\lfloor \frac{n}{2} \rfloor + 2}\}$, *and*

$$\min_{z \in \mathscr{C}_3} \Pi_{j=1}^n |z - x_j| = \Pi_{j=0}^{\lceil \frac{n}{2} \rceil - 1} \left(\min(p-q, q) + pj\right) \cdot \Pi_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\max(p-q, q) + pj\right); \tag{A.6}$$

(3): $y_1 \in \arg\max_{z \in \mathscr{C}_2} \Pi_{j=1}^n |z - x_j|$, *and*

$$\max_{z \in \mathscr{C}_2} \Pi_{j=1}^n |z - x_j| = \Pi_{j=0}^{n-1} \left(q + pj\right); \tag{A.7}$$

(4): $\arg\max_{z \in \mathscr{C}_3} \Pi_{j=1}^n |z - x_j| \subset \{y_1, y_{n+1}\}$, *and*

$$\max_{z \in \mathscr{C}_3} \Pi_{i=j}^n |z - x_j| = \Pi_{j=0}^{n-1} \left(\max(p-q, q) + pj\right). \tag{A.8}$$

Proof. Cases (3) and (4) are obvious. For cases (1) and (2), we only need to prove case (2). We verify firstly that for any integer k with $0 < k < \lfloor \frac{n}{2} \rfloor + 1$,

$$\Pi_{j=1}^{n} |y_k - x_j| > \Pi_{j=1}^{n} |y_{k+1} - x_j|. \tag{A.9}$$

This holds since

$$\Pi_{j=1}^{n} |y_k - x_j| = \Pi_{j=1}^{k-1} |y_k - x_j| \cdot \Pi_{j=k}^{n-1} |y_k - x_j| \cdot |y_k - x_n|$$

and

$$\Pi_{j=1}^{n} |y_{k+1} - x_j| = |y_{k+1} - x_1| \cdot \Pi_{j=2}^{k} |y_{k+1} - x_j| \cdot \Pi_{j=k+1}^{n} |y_{k+1} - x_j|.$$

As $\Pi_{j=1}^{k-1} |y_k - x_j| = \Pi_{j=2}^{k} |y_{k+1} - x_j|$, $\Pi_{j=k}^{n-1} |y_k - x_j| = \Pi_{j=k+1}^{n} |y_{k+1} - x_j|$ and $|y_k - x_n| \geq |y_{k+1} - x_1|$ due to the geometrical structure of $\mathscr{C}_1, \mathscr{C}_3$, we have (A.9). A similar argument gives that for any integer k with $\lfloor \frac{n}{2} \rfloor + 2 < k < n + 2$,

$$\Pi_{j=1}^{n} |y_k - x_j| > \Pi_{j=1}^{n} |y_{k-1} - x_j|. \tag{A.10}$$

This proves that $\arg\min_{z \in \mathscr{C}_3} \Pi_{j=1}^{n} |z - x_j| \subset \{y_{\lfloor \frac{n}{2} \rfloor + 1}, y_{\lfloor \frac{n}{2} \rfloor + 2}\}$ and thus the minimum value can be checked directly. □

Lemma A.3. *For $c_j^{\max}$, $j = 1,2,3,4$, defined in* (4.17)*, we have*

$$\max_{j=1,2,3,4} c_j^{\max} \leq \tau^{2n-1} s^{2n-1} \left(2\lceil \frac{n}{2} \rceil\right)! \left(2\lceil \frac{n}{2} \rceil - 1\right)!.$$

Proof. We study $c_j^{\max}$, $j = 1,2,3,4$, term by term. The definition of $c_1^{\max}$ yields

$$\begin{aligned} c_1^{\max} &= \max_{x \in C_1} \Pi_{k=1,2,3,4} \Pi_{y \in C_k, y \neq x} |x - y| \\ &\leq \max_{x \in C_1} \Pi_{y \in C_1, y \neq x} |x - y| \cdot \max_{x \in C_1} \Pi_{y \in C_2} |x - y| \cdot \max_{x \in C_1} \Pi_{y \in C_3} |x - y| \cdot \max_{x \in C_1} \Pi_{y \in C_4} |x - y|. \end{aligned}$$

Note that, by Lemmas A.1 and A.2, we have

$$\max_{x\in C_1}\Pi_{y\in C_1,y\neq x}|x-y| \leq \tau^{\lceil\frac{n}{2}\rceil-1}\left(\Pi_{j=1}^{\lceil\frac{n}{2}\rceil-1}(2sj)\right) \qquad \text{(case (2) in Lemma A.1)}$$
$$\leq \tau^{\lceil\frac{n}{2}\rceil-1}s^{\lceil\frac{n}{2}\rceil-1}\left(2\lceil\frac{n}{2}\rceil-1\right)!!.$$

$$\max_{x\in C_1}\Pi_{y\in C_2}|x-y| \leq \tau^{\lfloor\frac{n}{2}\rfloor}\left(\Pi_{j=0}^{\lfloor\frac{n}{2}\rfloor-1}(2sj+s)\right) \qquad \text{(cases (3) or (4) in Lemma A.2)}$$
$$\leq \tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left(2\lfloor\frac{n}{2}\rfloor-2\right)!!.$$

$$\max_{x\in C_1}\Pi_{y\in C_3}|x-y| \leq \tau^{\lceil\frac{n}{2}\rceil}\left(\Pi_{j=0}^{\lceil\frac{n}{2}\rceil-1}(2sj+1)\right) \qquad \text{(case (3) in Lemma A.2)}$$
$$\leq \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}-1\rceil}\left(2\lceil\frac{n}{2}\rceil\right)!!.$$

$$\max_{x\in C_1}\Pi_{y\in C_4}|x-y| \leq \tau^{\lfloor\frac{n}{2}\rfloor}\left(\Pi_{j=0}^{\lfloor\frac{n}{2}\rfloor-1}(2sj+2s-1)\right) \qquad \text{(cases (3) or (4) in Lemma A.2)}$$
$$\leq \tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left(2\lfloor\frac{n}{2}\rfloor-1\right)!!.$$

Thus

$$c_1^{\max} \leq \tau^{2n-1}s^{2n-1}\left(2\lceil\frac{n}{2}\rceil-2\right)!!\left(2\lfloor\frac{n}{2}\rfloor-1\right)!!\left(2\lceil\frac{n}{2}\rceil-1\right)!!\left(2\lfloor\frac{n}{2}\rfloor\right)!!$$
$$= \tau^{2n-1}s^{2n-2}\left(2\lceil\frac{n}{2}\rceil-1\right)!\left(2\lfloor\frac{n}{2}\rfloor\right)!.$$

Regarding $c_2^{\max}$, we have

$$c_2^{\max} = \max_{x\in C_2}\Pi_{k=1,2,3,4}\Pi_{y\in C_k,y\neq x}|x-y|$$
$$\leq \max_{x\in C_2}\Pi_{y\in C_1}|x-y|\cdot\max_{x\in C_2}\Pi_{y\in C_2,y\neq x}|x-y|\cdot\max_{x\in C_2}\Pi_{y\in C_3}|x-y|\cdot\max_{x\in C_2}\Pi_{y\in C_4}|x-y|.$$

Note that, by Lemmas A.1 and A.2, we have

$$\max_{x\in C_2} \Pi_{y\in C_1}|x-y| \le \tau^{\lceil \frac{n}{2}\rceil}\left(\Pi_{j=0}^{\lceil \frac{n}{2}\rceil-1}(2sj+s)\right) \qquad \text{(cases (3) in Lemma A.2)}$$
$$\le \tau^{\lceil \frac{n}{2}\rceil} s^{\lceil \frac{n}{2}\rceil}\left(2\lceil \frac{n}{2}\rceil -1\right)!!.$$

$$\max_{x\in C_2} \Pi_{y\in C_2, y\neq x}|x-y| \le \tau^{\lfloor \frac{n}{2}\rfloor-1}\left(\Pi_{j=1}^{\lfloor \frac{n}{2}\rfloor-1}(2sj)\right) \qquad \text{(case (2) in Lemma A.1)}$$
$$\le \tau^{\lfloor \frac{n}{2}\rfloor-1} s^{\lfloor \frac{n}{2}\rfloor-1}\left(2\lfloor \frac{n}{2}\rfloor -2\right)!!.$$

$$\max_{x\in C_2} \Pi_{y\in C_3}|x-y| \le \tau^{\lceil \frac{n}{2}\rceil}\left(\Pi_{j=0}^{\lceil \frac{n}{2}\rceil-1}(2sj+s+1)\right) \qquad \text{(case (3) in Lemma A.2)}$$
$$\le \tau^{\lceil \frac{n}{2}\rceil} s^{\lceil \frac{n}{2}\rceil}\left(2\lceil \frac{n}{2}\rceil\right)!!.$$

$$\max_{x\in C_2} \Pi_{y\in C_4}|x-y| \le \tau^{\lfloor \frac{n}{2}\rfloor}\left(\Pi_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}(2sj+s-1)\right) \qquad \text{(case (3) in Lemma A.2)}$$
$$\le \tau^{\lfloor \frac{n}{2}\rfloor} s^{\lfloor \frac{n}{2}\rfloor}\left(2\lfloor \frac{n}{2}\rfloor -1\right)!!.$$

Thus

$$C_2^{\max} \le \tau^{2n-1}s^{2n-1}\left(2\lceil \frac{n}{2}\rceil-1\right)!!\left(2\lfloor \frac{n}{2}\rfloor-2\right)!!\left(2\lceil \frac{n}{2}\rceil\right)!!\left(2\lfloor \frac{n}{2}\rfloor-1\right)!!$$
$$= \tau^{2n-1}s^{2n-1}\left(2\lceil \frac{n}{2}\rceil\right)!\left(2\lfloor \frac{n}{2}\rfloor-1\right)!.$$

As for $c_3^{\max}$, we have

$$c_3^{\max} = \max_{x\in C_3} \Pi_{k=1,2,3,4}\Pi_{y\in C_k, y\neq x}|x-y|$$
$$\le \max_{x\in C_3}\Pi_{y\in C_1}|x-y|\cdot \max_{x\in C_3}\Pi_{y\in C_2}|x-y|\cdot \max_{x\in C_3, y\neq x}\Pi_{y\in C_3}|x-y|\cdot \max_{x\in C_3}\Pi_{y\in C_4}|x-y|.$$

Note that, by Lemmas A.1 and A.2, we obtain

$$\max_{x\in C_3}\Pi_{y\in C_1}|x-y| \le \tau^{\lceil \frac{n}{2}\rceil}\left(\Pi_{j=0}^{\lceil \frac{n}{2}\rceil-1}(2sj+1)\right), \qquad \text{(case (3) in Lemma A.2)}$$
$$\le \tau^{\lceil \frac{n}{2}\rceil}s^{\lceil \frac{n}{2}\rceil-1}\left(2\lceil\frac{n}{2}\rceil-1\right)!!.$$

$$\max_{x\in C_3}\Pi_{y\in C_2}|x-y| \le \tau^{\lfloor \frac{n}{2}\rfloor}\left(\Pi_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}(2sj+s+1)\right), \qquad \text{(case (3) or (4) in Lemma A.2)}$$

$$\max_{x\in C_3}\Pi_{y\in C_3,y\neq x}|x-y| \le \tau^{\lceil \frac{n}{2}\rceil-1}\left(\Pi_{j=1}^{\lceil \frac{n}{2}\rceil-1}(2sj)\right), \qquad \text{(case (2) in Lemma A.1)}$$
$$\le \tau^{\lceil \frac{n}{2}\rceil-1}s^{\lceil \frac{n}{2}\rceil-1}\left(2\lceil\frac{n}{2}\rceil-2\right)!!.$$

$$\max_{x\in C_3}\Pi_{y\in C_4}|x-y| \le \tau^{\lfloor \frac{n}{2}\rfloor}\left(\Pi_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}(2sj+2s-2)\right). \qquad \text{(case (3) or (4) in Lemma A.2)}$$

Thus

$$\begin{aligned}
C_3^{\max} &\le \tau^{2n-1}s^{2\lceil \frac{n}{2}\rceil-2}\left(2\lceil\frac{n}{2}\rceil-1\right)!\cdot\left(\Pi_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}(2sj+s+1)(2sj+2s-2)\right)\\
&\le \tau^{2n-1}s^{2\lceil \frac{n}{2}\rceil-2}\left(2\lceil\frac{n}{2}\rceil-1\right)!\cdot\left(\Pi_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}(2sj+s)(2sj+2s)\right)\\
&= \tau^{2n-1}s^{2n-3}\left(2\lceil\frac{n}{2}\rceil-1\right)!\left(2\lfloor\frac{n}{2}\rfloor\right)!.
\end{aligned}$$

Finally, we study $c_4^{\max}$. We have

$$\max_{x\in C_4}\Pi_{y\in C_1}|x-y| \le \tau^{\lceil \frac{n}{2}\rceil}(\Pi_{j=0}^{\lceil \frac{n}{2}\rceil-1}(2sj+1)), \qquad \text{(case (3) in Lemma A.2)}$$
$$\le \tau^{\lceil \frac{n}{2}\rceil}s^{\lceil \frac{n}{2}\rceil-1}\left(2\lceil\frac{n}{2}\rceil-1\right)!!$$

$$\max_{x\in C_4}\Pi_{y\in C_2}|x-y| \le \tau^{\lfloor \frac{n}{2}\rfloor}(\Pi_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}(2sj+s-1)), \qquad \text{(case (3) in Lemma A.2)}$$
$$\le \tau^{\lfloor \frac{n}{2}\rfloor}s^{\lfloor \frac{n}{2}\rfloor}\left(2\lfloor\frac{n}{2}\rfloor\right)!!$$

$$\max_{x\in C_4}\Pi_{y\in C_3}|x-y| \le \tau^{\lceil \frac{n}{2}\rceil}(\Pi_{j=0}^{\lceil \frac{n}{2}\rceil-1}(2sj+2)), \qquad \text{(case (3) in Lemma A.2)}$$
$$\le \tau^{\lceil \frac{n}{2}\rceil}s^{\lceil \frac{n}{2}\rceil}\left(2\lceil\frac{n}{2}\rceil-1\right)!!.$$

$$\max_{x\in C_4}\Pi_{y\in C_4,y\neq x}|x-y| \le \tau^{\lfloor \frac{n}{2}\rfloor-1}(\Pi_{j=1}^{\lfloor \frac{n}{2}\rfloor-1}(2sj)). \qquad \text{(case (2) in Lemma A.1)}$$
$$\le \tau^{\lfloor \frac{n}{2}\rfloor-1}s^{\lfloor \frac{n}{2}\rfloor-1}\left(2\lfloor\frac{n}{2}\rfloor-2\right)!!.$$

Thus

$$\begin{aligned} c_4^{\max} &\le \tau^{2n-1} s^{2n-1} \Big(2\lceil \frac{n}{2}\rceil - 1\Big)!! \cdot \Big(2\lfloor \frac{n}{2}\rfloor\Big)!! \cdot \Big(2\lceil \frac{n}{2}\rceil - 1\Big)!! \cdot \Big(2\lfloor \frac{n}{2}\rfloor - 2\Big)!! \\ &\le \tau^{2n-1} s^{2n-1} \Big(2\lceil \frac{n}{2}\rceil\Big)! \Big(2\lceil \frac{n}{2}\rceil - 1\Big)!. \end{aligned}$$

By concluding above discussions, we have

$$\max_{j=1,2,3,4} c_j^{\max} \le \tau^{2n-1} s^{2n-1} \Big(2\lceil \frac{n}{2}\rceil\Big)! \Big(2\lceil \frac{n}{2}\rceil - 1\Big)!.$$

□

Lemma A.4. *For* $c_j^{\min}$, $j = 1,2,3,4$, *defined in* (4.18), *we have*

$$\min_{j=1,2,3,4} c_j^{\min} \ge \tau^{2n-1} s^{2n-3} \cdot \Big((2\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \rfloor - 1)! \Big)^4.$$

Proof. We evaluate $c_k^{\min}, k = 1,2,3,4$. For $c_1^{\min}$,

$$\begin{aligned} c_1^{\min} &= \min_{x\in C_1} \Pi_{k=1,2,3,4} \Pi_{y\in C_k, y\neq x} |x-y| \\ &\ge \min_{x\in C_1} \Pi_{y\in C_1, y\neq x} |x-y| \cdot \min_{x\in C_1} \Pi_{y\in C_2} |x-y| \cdot \min_{x\in C_1} \Pi_{y\in C_3} |x-y| \cdot \min_{x\in C_1} \Pi_{y\in C_4} |x-y|. \end{aligned}$$

By using Lemmas A.1 and A.2, we have

$$\begin{aligned} \min_{x\in C_1} \Pi_{y\in C_1} |x-y| &= \tau^{\lceil \frac{n}{2}\rceil - 1} \Pi_{j=1}^{\lceil \frac{\lceil \frac{n}{2}\rceil}{2}\rceil - 1} (j \cdot 2s) \cdot \Pi_{j=1}^{\lfloor \frac{\lceil \frac{n}{2}\rceil}{2}\rfloor} (j \cdot 2s), \quad \text{(case (1) in Lemma A.1)} \\ &= \tau^{\lceil \frac{n}{2}\rceil - 1} s^{\lceil \frac{n}{2}\rceil - 1} \Big((2\lceil \frac{\lceil \frac{n}{2}\rceil}{2} \rceil - 2)!! \cdot (2\lfloor \frac{\lceil \frac{n}{2}\rceil}{2} \rfloor)!! \Big) \\ &\ge \tau^{\lceil \frac{n}{2}\rceil - 1} s^{\lceil \frac{n}{2}\rceil - 1} \Big((2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2} \rfloor - 2)!! \cdot (2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2} \rfloor)!! \Big). \end{aligned}$$

$$\begin{aligned} \min_{x\in C_1} \Pi_{y\in C_2, y\neq x} |x-y| &= \tau^{\lfloor \frac{n}{2}\rfloor} \Pi_{j=0}^{\lceil \frac{\lfloor \frac{n}{2}\rfloor}{2}\rceil - 1} \big(s + 2sj\big) \cdot \Pi_{j=0}^{\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor - 1} \big(s + 2sj\big), \quad \text{(case (1) or (2) in Lemma A.2)} \\ &= \tau^{\lfloor \frac{n}{2}\rfloor} s^{\lfloor \frac{n}{2}\rfloor} \Big((2\lceil \frac{\lfloor \frac{n}{2}\rfloor}{2} \rceil - 1)!! \cdot (2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2} \rfloor - 1)!! \Big) \\ &\ge \tau^{\lfloor \frac{n}{2}\rfloor} s^{\lfloor \frac{n}{2}\rfloor} \Big((2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2} \rfloor - 1)!! \cdot (2\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2} \rfloor - 1)!! \Big). \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\min_{x\in C_1}\Pi_{y\in C_3}|x-y| &= \tau^{\lceil \frac{n}{2}\rceil}\Pi_{j=0}^{\lceil \frac{\lceil \frac{n}{2}\rceil}{2}\rceil-1}\left(1+2sj\right)\cdot\Pi_{k=0}^{\lfloor \frac{\lceil \frac{n}{2}\rceil}{2}\rfloor-1}(2s-1+2sk) \quad \text{(case (1) in Lemma A.2)}\\
&\geq \tau^{\lceil \frac{n}{2}\rceil}\Pi_{j=1}^{\lceil \frac{\lceil \frac{n}{2}\rceil}{2}\rceil-1}\left(2sj\right)\cdot\Pi_{k=0}^{\lfloor \frac{\lceil \frac{n}{2}\rceil}{2}\rfloor-1}(s+2sk)\\
&= \tau^{\lceil \frac{n}{2}\rceil}s^{\lceil \frac{n}{2}\rceil-1}\left((2\lceil\frac{\lceil \frac{n}{2}\rceil}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lceil \frac{n}{2}\rceil}{2}\rfloor-1)!!\right)\\
&\geq \tau^{\lceil \frac{n}{2}\rceil}s^{\lceil \frac{n}{2}\rceil-1}\left((2\lfloor\frac{\lceil \frac{n}{2}\rceil}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lceil \frac{n}{2}\rceil}{2}\rfloor-1)!!\right)\\
&\geq \tau^{\lfloor \frac{n}{2}\rfloor}s^{\lfloor \frac{n}{2}\rfloor-1}\cdot\left(2\lfloor\frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1\right)!.
\end{aligned}
$$

$$
\begin{aligned}
\min_{x\in C_1}\Pi_{y\in C_4}|x-y| &= \tau^{\lfloor \frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil \frac{\lfloor \frac{n}{2}\rfloor}{2}\rceil-1}\left(1+2sj\right)\cdot\Pi_{k=0}^{\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1}(2s-1+2sk) \quad \text{(case (1) or (2) in Lemma A.2)}\\
&\geq \tau^{\lfloor \frac{n}{2}\rfloor}\Pi_{j=1}^{\lceil \frac{\lfloor \frac{n}{2}\rfloor}{2}\rceil-1}\left(2sj\right)\cdot\Pi_{k=0}^{\lfloor \frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1}(s+2sk)\\
&= \tau^{\lfloor \frac{n}{2}\rfloor}s^{\lfloor \frac{n}{2}\rfloor-1}\left((2\lceil\frac{\lfloor \frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)\\
&\geq \tau^{\lfloor \frac{n}{2}\rfloor}s^{\lfloor \frac{n}{2}\rfloor-1}\left(2\lfloor\frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1\right)!.
\end{aligned}
$$

By combining estimates in the above four cases, it follows that

$$
c_1^{\min} \geq \tau^{2n-1}s^{2n-3}\left(2\lfloor\frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor\right)!\left(\left(2\lfloor\frac{\lfloor \frac{n}{2}\rfloor}{2}\rfloor-1\right)!\right)^3.
$$

Regarding $c_2^{\min}$, we have

$$
\begin{aligned}
c_2^{\min} &= \min_{x\in C_2}\Pi_{k=1,2,3,4}\Pi_{y\in C_k,y\neq x}|x-y|\\
&\geq \min_{x\in C_2}\Pi_{y\in C_1}|x-y|\cdot\min_{x\in C_2}\Pi_{y\in C_2,y\neq x}|x-y|\cdot\min_{x\in C_2}\Pi_{y\in C_3}|x-y|\cdot\min_{x\in C_2}\Pi_{y\in C_4}|x-y|.
\end{aligned}
$$

Note that by Lemmas A.1 and A.2 we have

$$\min_{x\in C_2}\Pi_{y\in C_1}|x-y| = \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=0}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(s+2sj\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}\left(s+2sj\right) \qquad \text{(case (1) or (2) in Lemma A.2)}$$

$$= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil}\left((2\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\right)$$

$$\geq \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right).$$

$$\min_{x\in C_2}\Pi_{y\in C_2,y\neq x}|x-y| = \tau^{\lfloor\frac{n}{2}\rfloor-1}\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}(j\cdot 2s)\cdot\Pi_{j=1}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor}(j\cdot 2s) \qquad \text{(case (1) in Lemma A.1)}$$

$$= \tau^{\lfloor\frac{n}{2}\rfloor-1}s^{\lfloor\frac{n}{2}\rfloor-1}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor)!!\right)$$

$$\geq \tau^{\lfloor\frac{n}{2}\rfloor-1}s^{\lfloor\frac{n}{2}\rfloor-1}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor)!!\right).$$

$$\min_{x\in C_2}\Pi_{y\in C_3}|x-y| = \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=0}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(s-1+2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}(s+1+2sk) \qquad \text{(case (1) or (2) in Lemma A.2)}$$

$$\geq \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=1}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}(s+2sk)$$

$$= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\left((2\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\right)$$

$$\geq \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\left((2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\right)$$

$$= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!.$$

$$\min_{x\in C_2}\Pi_{y\in C_4}|x-y| = \tau^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(s-1+2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}(s+1+2sk) \qquad \text{(case (1) in Lemma A.2)}$$

$$\geq \tau^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}(s+2sk)$$

$$= \tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor-1}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)$$

$$\geq \tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor-1}\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)$$

$$= \tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor-1}(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!.$$

Thus,

$$c_2^{\min} \geq \tau^{2n-1}s^{2n-3}\cdot\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)\cdot\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor)!!\right)\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!$$
$$= \tau^{2n-1}s^{2n-3}\cdot(2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor)!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!$$
$$\geq \tau^{2n-1}s^{2n-3}\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor)!\cdot\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^3.$$

We then turn to estimate $c_3^{\min}$. We have that

$$\begin{aligned} c_3^{\min} &= \min_{x\in C_3}\Pi_{k=1,2,3,4}\Pi_{y\in C_k, y\neq x}|x-y| \\ &\geq \min_{x\in C_3}\Pi_{y\in C_1}|x-y|\cdot\min_{x\in C_3}\Pi_{y\in C_2}|x-y|\cdot\min_{x\in C_3}\Pi_{y\in C_3,y\neq x}|x-y|\cdot\min_{x\in C_3}\Pi_{y\in C_4}|x-y|. \end{aligned}$$

By Lemmas A.1 and A.2, it follows that we

$$\begin{aligned} \min_{x\in C_3}\Pi_{y\in C_1}|x-y| &= \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=0}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(1+2sj\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}\left(2s-1+2sj\right) \qquad \text{(case (1) or (2) in Lemma A.1)} \\ &\geq \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=1}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(2sj\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}\left(s+2sj\right) \\ &= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\left((2\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\right) \\ &\geq \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right) \\ &= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!. \\ \min_{x\in C_3}\Pi_{y\in C_2}|x-y| &= \tau^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(s-1+2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}\left(s+1+2sk\right) \qquad \text{(case (1) in Lemma A.1)} \\ &= \tau^{\lfloor\frac{n}{2}\rfloor}\left(s^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(\frac{s-1}{s}+2j\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}\left(\frac{s+1}{s}+2j\right)\right) \\ &\geq \tau^{\lfloor\frac{n}{2}\rfloor}\left(s^{\lfloor\frac{n}{2}\rfloor}\cdot\frac{1}{2}\cdot\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(2j\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}\left(1+2j\right)\right) \\ &= \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left(\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(2j\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}\left(1+2j\right)\right) \\ &= \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right) \\ &\geq \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right) \\ &\geq \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!. \end{aligned}$$

$$\min_{x\in C_3}\Pi_{y\in C_3,y\neq x}|x-y| = \tau^{\lceil\frac{n}{2}\rceil-1}\Pi_{j=1}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}(j\cdot 2s)\cdot\Pi_{j=1}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor}(j\cdot 2s) \qquad \text{(case (1) in Lemma A.2)}$$

$$= \tau^{\lceil\frac{n}{2}\rceil-1}s^{\lceil\frac{n}{2}\rceil-1}\Big((2\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor)!!\Big)$$

$$\geq \tau^{\lceil\frac{n}{2}\rceil-1}s^{\lceil\frac{n}{2}\rceil-1}\Big((2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\Big)$$

$$= \tau^{\lceil\frac{n}{2}\rceil-1}s^{\lceil\frac{n}{2}\rceil-1}(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!.$$

$$\min_{x\in C_3}\Pi_{y\in C_4}|x-y| = \tau^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\big(2+2sj\big)\cdot\Pi_{k=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}(2s-2+2sk) \qquad \text{(case (1) or (2) in Lemma A.2)}$$

$$\geq 2\tau^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\big(2sj\big)\cdot\Pi_{k=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}(s+2sk)$$

$$= 2\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor-1}\Big((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\Big)$$

$$\geq 2\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor-1}\Big((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\Big)$$

$$= 2\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor-1}(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!.$$

Thus,

$$c_3^{\min} \geq \cdot\tau^{2n-1}s^{2n-3}\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!$$

$$\geq \tau^{2n-1}s^{2n-3}\cdot\Big((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\Big)^4$$

Finally, regarding $c_4^{\min}$, we write

$$c_4^{\min} = \min_{x\in C_4}\Pi_{k=1,2,3,4}\Pi_{y\in C_k,y\neq x}|x-y|$$

$$\geq \min_{x\in C_4}\Pi_{y\in C_1}|x-y|\cdot\min_{x\in C_4}\Pi_{y\in C_2}|x-y|\cdot\min_{x\in C_4}\Pi_{y\in C_3}|x-y|\cdot\min_{x\in C_4}\Pi_{y\in C_4,y\neq x}|x-y|.$$

Note that, by Lemmas A.1 and A.2, we obtain

$$\min_{x\in C_4}\Pi_{y\in C_1}|x-y| = \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=0}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\big(1+2sj\big)\cdot\Pi_{j=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}\big(2s-1+2sj\big) \qquad \text{(case (1) in Lemma A.1)}$$

$$\geq \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=1}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\big(2sj\big)\cdot\Pi_{j=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}\big(s+2sj\big)$$

$$= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\Big((2\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\Big)$$

$$\geq \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\Big((2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\Big)$$

$$= \tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!.$$

$$\min_{x\in C_4}\Pi_{y\in C_2}|x-y| = \tau^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(s-1+2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}(s+1+2sk) \qquad \text{(case (1) in Lemma A.1)}$$

$$= \tau^{\lfloor\frac{n}{2}\rfloor}\left(s^{\lfloor\frac{n}{2}\rfloor}\Pi_{j=0}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(\frac{s-1}{s}+2j\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}\left(\frac{s+1}{s}+2j\right)\right)$$

$$\geq \tau^{\lfloor\frac{n}{2}\rfloor}\left(s^{\lfloor\frac{n}{2}\rfloor}\cdot\frac{1}{2}\cdot\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}\left(2j\right)\cdot\Pi_{j=0}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1}\left(1+2j\right)\right)$$

$$= \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)$$

$$\geq \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)$$

$$= \frac{1}{2}\tau^{\lfloor\frac{n}{2}\rfloor}s^{\lfloor\frac{n}{2}\rfloor}(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!.$$

$$\min_{x\in C_4}\Pi_{y\in C_3}|x-y| = \tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=0}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(2+2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}(2s-2+2sk) \qquad \text{(case (1) in Lemma A.1)}$$

$$\geq 2\tau^{\lceil\frac{n}{2}\rceil}\Pi_{j=1}^{\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-1}\left(2sj\right)\cdot\Pi_{k=0}^{\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1}(s+2sk)$$

$$= 2\tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\left((2\lceil\frac{\lceil\frac{n}{2}\rceil}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\right)$$

$$\geq 2\tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}\left((2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!!\right)$$

$$= 2\tau^{\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil-1}(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!.$$

$$\min_{x\in C_4}\Pi_{y\in C_4,y\neq x}|x-y| = \tau^{\lfloor\frac{n}{2}\rfloor-1}\Pi_{j=1}^{\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-1}(j\cdot 2s)\cdot\Pi_{j=1}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor}(j\cdot 2s) \qquad \text{(case (1) in Lemma A.2)}$$

$$= \tau^{\lfloor\frac{n}{2}\rfloor-1}s^{\lfloor\frac{n}{2}\rfloor-1}\left((2\lceil\frac{\lfloor\frac{n}{2}\rfloor}{2}\rceil-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor)!!\right)$$

$$\geq \tau^{\lfloor\frac{n}{2}\rfloor-1}s^{\lfloor\frac{n}{2}\rfloor-1}\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-2)!!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!!\right)$$

$$= \tau^{\lfloor\frac{n}{2}\rfloor-1}s^{\lfloor\frac{n}{2}\rfloor-1}(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!.$$

Thus,

$$c_4^{\min} \geq \tau^{2n-1}s^{2n-3}\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lceil\frac{n}{2}\rceil}{2}\rfloor-1)!\cdot(2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!$$

$$\geq \tau^{2n-1}s^{2n-3}\cdot\left((2\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{2}\rfloor-1)!\right)^4.$$

Summarizing all claims above finishes the proof. □

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