# Stability of a weighted L2 projection in a Sobolev space

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Research Report No. 2022-33
July 2022

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# Stability of a weighted $L^2$ projection in a Sobolev space

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July 18, 2022

**Abstract**

In this paper, we prove the stability of a weighted $L^2$ projection operator onto finite-dimensional subspaces of a weighted Sobolev space. This stability property is needed for the analysis of the preconditioners introduced by Alouges and the author in "Quasi-local and frequency robust preconditioners for the Helmholtz first-kind integral equations on the disk". Namely, we consider the orthogonal projections $\pi_{N,\omega} : L^2(\mathbb{D}, 1/\omega(x)dx) \to \mathcal{X}_N$, where $\mathbb{D} \subset \mathbb{R}^2$ is the unit disk and $\omega(x) = \sqrt{1 - |x|^2}$. The spaces $\mathcal{X}_N$ are finite-dimensional subspaces of a weighted Sobolev-type space $T^1$, and consist of piecewise linear functions on a family of shape-regular and quasi-uniform triangulations of $\mathbb{D}$. We show that $\pi_{N,\omega}$ is continuous from $T^1$ to $T^1$ and prove an upper bound on the continuity constant that does not depend on $N$.

## 1 Introduction

Let $\mathbb{D}$ be the unit disk of $\mathbb{R}^2$, that is

$$\mathbb{D} = \left\{ x \in \mathbb{R}^2 \;\middle|\; |x|^2 < 1 \right\},$$

where $|x| = \sqrt{x_1^2 + x_2^2}$ stands for the Euclidean norm of $x$. In [1], we consider the Laplace equation in the domain $\mathbb{R}^3 \setminus (\overline{\mathbb{D}} \times \{0\})$, i.e. the exterior of a flat circular surface in $\mathbb{R}^3$. Some preconditioners are introduced for a boundary element discretization of this problem. When it comes to the analysis of the condition number of the preconditioned system, we are faced with the task of proving a uniform bound on the continuity constant of a weighted $L^2$ projection into a family of subspaces of a weighted Sobolev space. The purpose of this paper is to give a self-contained proof of this key stability property. Let us first state it precisely.

On $\mathbb{D}$, define the function

$$\omega(x) = \sqrt{1 - |x|^2}\,. \tag{1}$$

We introduce two Hilbert spaces. The first one is simply the weighted $L^2$ space

$$L^2_{1/\omega} := \left\{ u \in L^1_{\text{loc}}(\mathbb{D}) \;\middle|\; \|u\|^2_{1/\omega} := \int_{\mathbb{D}} \frac{|u(x)|^2}{\omega(x)} dx < +\infty \right\}. \tag{2}$$

Its inner product is denoted by $(\cdot,\cdot)_{1/\omega}$. The second space is the "weighted Sobolev space"

$$T^1 := \left\{ u \in L^2_{1/\omega} \;\middle|\; \|u\|^2_{T^1} := \|u\|^2_{1/\omega} + \int_{\mathbb{D}} \omega(x) \left|\nabla u(x)\right|^2 dx < +\infty \right\}. \tag{3}$$

For a region $U \subset \mathbb{D}$ and a function $f \in L^1_{\mathrm{loc}}(U)$, we will also use the notation

$$\|f\|^2_{U,1/\omega} := \int_U \frac{|f(x)|^2}{\omega(x)}\,dx\,, \quad \|f\|^2_{U,T^1} := \|f\|^2_{U,1/\omega} + \int_U \omega(x)\,|\nabla f(x)|^2\,dx\,. \tag{4}$$

Our aim is to establish the uniform stability of the $L^2_{1/\omega}$-orthogonal projection onto a sequence $(\mathcal{X}_N)_{N\in\mathbb{N}}$ of subspaces of $T^1$, that we define now. Let $(P_N)_{N\in\mathbb{N}}$ be a sequence of polygonal approximations of the disk. That is, $P_N \subset \overline{\mathbb{D}}$ and the vertices of $P_N$ all lie in $\partial\mathbb{D}$. Furthermore, the maximal distance between two consecutive vertices of $P_N$ is denoted by $h_N$, and we assume that

$$\lim_{N\to\infty} h_N = 0\,. \tag{5}$$

For each $N$, we consider a regular triangulation $\mathcal{T}_N$ of $P_N$, i.e. a set of pairwise disjoint open triangles, with the usual conformity assumptions (two triangles of $\mathcal{T}_N$ can only intersect along a common vertex or edge, or not at all) an such that

$$\bigcup_{\tau\in\mathcal{T}_N} \overline{\tau} = \overline{P_N}\,. \tag{6}$$

For each $\tau \in \mathcal{T}_N$, let $h_\tau$ be the diameter of $\tau$ and $\Delta_\tau$ its area. We assume that there exist constants $c_1, C_1$ and $c_2$, independent of $N$ and $\tau$, such that

$$c_1 h_N \le h_\tau \le C_1 h_N\,, \quad \text{(global quasi-uniformity)}, \tag{7}$$

$$\frac{\Delta_\tau}{h_\tau^2} \ge c_2\,, \quad \text{(uniform shape-regularity)}. \tag{8}$$

To construct piecewise linear functions in $T^1$ from $\mathcal{T}_N$, special attention must be paid to the triangles on the boundary of the triangulation. If $\tau$ has two vertices $A$ and $B$ in $\partial\mathbb{D}$, let $U_\tau$ be the open region of $\mathbb{D}$ enclosed, on the one hand, by the smallest arc of $\partial\mathbb{D}$ linking $A$ to $B$, and on the other hand, by the straight line segment $[A, B]$. Let $K_\tau$ be the open convex region resulting from the union of $\tau$ and $U_\tau$, i.e.

$$\overline{K_\tau} := \overline{\tau} \cup \overline{U_\tau}\,. \tag{9}$$

When $\tau$ has one or zero vertex in $\partial\mathbb{D}$, we simply put $K_\tau = \tau$. Then, the set $\{K_\tau\}_{\tau\in\mathcal{T}_N}$ partitions $\mathbb{D}$ in the sense that

$$\bigcup_{\tau\in\mathcal{T}_N} \overline{K_\tau} = \overline{\mathbb{D}}\,. \tag{10}$$

With these definitions, let

$$\mathcal{X}_N := \left\{ u \in C^0(\overline{\mathbb{D}}) \;\middle|\; u_{|K_\tau} \text{ is affine for all } \tau \in \mathcal{T}_N \right\}\,. \tag{11}$$

It is clear that $\mathcal{X}_N$ is a finite-dimensional subspace of $T^1$. We can now define

$$\pi_{N,\omega} : L^2_{1/\omega} \to \mathcal{X}_N\,, \tag{12}$$

the $L^2_{1/\omega}$-orthogonal projection onto $\mathcal{X}_N$. We shall prove

**Theorem 1.** *There exists a constant $C_\pi > 0$ independent of $N$ such that,*

$$\forall u \in T^1\,, \quad \|\pi_{N,\omega} u\|_{T^1} \le C_\pi \|u\|_{T^1}\,. \tag{13}$$

theorem 1 is analogous to the uniform $H^1$-stability of the $L^2$ projection onto a family of finite-dimensional supbspaces of $H^1$. In the case of piecewise linear functions on quasi-uniform meshes, such a stability property is derived easily from global inverse inequalities. Several works have been devoted to establishing this stability in more general cases [4, 5, 7], e.g. when the triangulation is not quasi-uniform. Here we restrict ourselves to the case of a quasi-uniform triangulation of $\mathbb{D}$, but the difficulty arises from the presence of the weight $\omega$ in the definitions of $L^2_{1/\omega}$ and $T^1$. In particular, notice that $\omega$ vanishes on $\partial\mathbb{D}$.

The $H^1$-stability of the $L^2$ projection is important in many aspects of Finite Elements and Boundary Elements analysis. For instance, it is useful in the study of multigrid and domain decomposition methods [10], or the quasi-optimality of Galerkin methods for parabolic problems [3]. In our case, theorem 1 is used in [1] to estimate the condition number of a preconditioned linear system arising from the Galerkin boundary element discretization of the Laplace weakly-singular integral equation on $\mathbb{D}$.

To prove theorem 1, the main difficulty, compared to more standard Sobolev spaces, is that $L^2_{1/\omega}$ and $T^1$ do not have scaling properties like $L^2$ and $H^1$ have. Hence, many classical and convenient arguments (such as reasoning by pulling back to a reference triangle) are not directly available here. Instead, we adapt and combine two classical lines of proof. The first one is an argument, found e.g. in the proof of [2, Lemma 1], showing that inverse inequalities in combination with suitable approximation properties yield the stability of the (weighted) $L^2$ projection.

The second line of proof is aimed towards showing an approximation property of $\mathcal{X}_N$ in $T^1$. For this, we follow Clément [6] by adapting the definition of his well-known quasi-interpolant. The main new ingredient is a non-standard weighted Poincaré inequality, with careful control of the domain-dependent constant.

The remainder of this paper is organized as follows. In section 2, we state a first lemma to reduce the proof of theorem 1 to the proof of three key properties (A1)-(A3). In Sections 3 and 4, we derive weighted Poincaré and local inverse inequalities, respectively. Finally, in section 5, we define a quasi-interpolant $I_N$ and show that it meets the requirements.

In the proofs, we use the letter $C$ to denote a generic positive constant that is independent of the discretization (i.e. of the index $N \in \mathbb{N}$ of the triangulation $\mathcal{T}_N$). The value of $C$ is allowed to change from line to line. Nevertheless, we refrain from doing so in the result statements, to ensure the clarity of our discussion.

## 2 Three key properties

Our analysis of $\pi_{N,\omega}$ relies on three main ingredients.

(A1) A *quasi-interpolant* $I_N : L^2_{1/\omega} \to \mathcal{X}_N$ that is uniformly $T^1$-continuous, i.e. there exists a constant $C_I > 0$ such that

$$\forall N \in \mathbb{N}\,,\ \forall u \in T^1\,, \quad \|I_N u\|_{T^1} \leq C_I \,\|u\|_{T^1}\ , \tag{14}$$

and such that, for each $N \in \mathbb{N}$ and $\tau \in \mathcal{T}_N$, there exists a constant $C_P(K_\tau) > 0$ such that

$$\forall u \in T^1\,, \quad \|u - I_N u\|^2_{K_\tau,1/\omega} \leq C_P(K_\tau)^2 \,\|u\|^2_{\omega_\tau,T^1}\ , \tag{15}$$

where $\omega_\tau$ is the union of the domains $K_{\tau'}$ such that $\tau'$ and $\tau$ are *neighbors*, i.e. share at least one vertex $\tau$.

(A2) Some local *inverse inequalities* in $\mathcal{X}_N$: for all $\theta \in \mathcal{X}_N$ and for all $\tau \in \mathcal{T}_N$, there exists a constant $C_{\rm inv}(K_\tau) > 0$ such that

$$\|\theta\|^2_{K_\tau,T^1} \le C_{\rm inv}(K_\tau)^{-2} \, \|\theta\|^2_{K_\tau,1/\omega} \, . \tag{16}$$

(A3) A uniform estimate of the ratios $C_P(K_\tau)/C_{\rm inv}(K_\tau)$, i.e. there exists a constant $C_{\rm rat} > 0$ such that

$$\forall N \in \mathbb{N}\,,\ \forall \tau \in \mathcal{T}_N\,, \quad C_P(K_\tau)/C_{\rm inv}(K_\tau) \le C_{\rm rat}\,. \tag{17}$$

**Lemma 1.** *If (A1)-(A3) hold, then the orthogonal projection* $\pi_{N,\omega}$ *satisfies theorem 1 with*

$$C_\pi = \sqrt{2(K_\sharp C^2_{\rm rat} + C^2_I)}\,, \tag{18}$$

$K_\sharp$ *being an upper bound for all* $N$ *on the maximal number of neighbors of* $\tau \in \mathcal{T}_N$*.*

*Proof.* We adapt a well-known argument appearing for example in the proof of [2, Lemma 1]. Given $N \in \mathbb{N}$ and $u \in T^1$, we write

$$\begin{aligned}
\|\pi_{N,\omega} u\|^2_{T^1} &\le 2(\|\pi_{N,\omega}(u - I_N u)\|^2_{T^1} + \|I_N u\|^2_{T^1}) \\
&\le 2\left(\sum_{\tau\in\mathcal{T}_N} \|\pi_{N,\omega}(u - I_N u)\|^2_{K_\tau,T^1}\right) + 2C_I^2\,\|u\|^2_{T^1} \\
&\le 2\left(\sum_{\tau\in\mathcal{T}_N} C_{\rm inv}(K_\tau)^{-2}\,\|\pi_{N,\omega}(u - I_N u)\|^2_{K_\tau,1/\omega}\right) + 2C_I^2\,\|u\|^2_{T^1} \\
&\le 2\left(\sum_{\tau\in\mathcal{T}_N} C_{\rm inv}(K_\tau)^{-2}\,\|u - I_N u\|^2_{K_\tau,1/\omega}\right) + 2C_I^2\,\|u\|^2_{T^1} \\
&\le 2\left(\sum_{\tau\in\mathcal{T}_N} C_{\rm inv}(K_\tau)^{-2} C_P(K_\tau)^2\,\|u\|^2_{U_K,T^1}\right) + 2C_I^2\,\|u\|^2_{T^1} \\
&\le 2(K_\sharp C^2_{\rm rat} + C^2_I)\,\|u\|^2_{T^1}\ .
\end{aligned}$$

We have applied, successively: the triangle inequality, the property that $\pi_{N,\omega}\theta = \theta$ for all $\theta \in \mathcal{X}_N$, the uniform continuity (i), the inverse inequalities (ii), the minimization properties of $\pi_{N,\omega}$ in $L^2_{1/\omega}$, the weighted Poincaré inequalities (i) eq.(15), the estimate of the ratio $C_P(K_\tau)/C_{\rm inv}(K_\tau)$ (iii), and the definition of $K_\sharp$. □

In the next sections, we show that (A1) - (A3) hold, see lemma 7, lemma 5 and lemma 8, respectively.

# 3 Weighted Poincaré inequalities

In what follows, for any open region $U \subset \mathbb{D}$, we write

$$\langle u\rangle_U := \left(\int_U 1/\omega(x)\right)^{-1} \int_U \frac{u(x)}{\omega(x)}\,dx\,. \tag{19}$$

It is proved in [1, Theorem 1] that

$$\forall u \in T^1\,, \quad \|u - \langle u\rangle_{\mathbb{D}}\|^2_{1/\omega} \leq \int_{\mathbb{D}} \omega(x)\,|\nabla u(x)|^2\;dx\,. \tag{20}$$

The goal of this section is to prove similar inequalities when the domain of integration is replaced by a subset of $\mathbb{D}$. We start with two technical lemmas.

**Lemma 2.** *Let $N \in \mathbb{N}$, $Q$ a vertex of the triangulation $\mathcal{T}_N$ and $\varphi_{N,Q}$ the element of $\mathcal{X}_N$ such that*

$$\varphi_{N,Q}(Q') = \begin{cases} 1 & \text{if } Q = Q', \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

*for all vertices $Q'$ of $\mathcal{T}_N$. Let $S_{N,Q} = \operatorname{supp}\varphi_{N,Q}$. Then, there exists a bilipschitz application $\kappa_{N,Q}$, mapping $\mathbb{D}$ to $S_{N,Q}$ such that*

$$\forall x, y \in \mathbb{D}\,, \quad c_3 h_N\,|x-y| \leq |\kappa_{N,Q}(x) - \kappa_{N,Q}(y)| \leq C_3 h_N\,|x-y|\;, \tag{22}$$

*where the constants $c_3$ and $C_3$ do not depend on $N$ nor on $Q$.*

The proof can be done by introducing polar coordinates in $S_{N,Q}$, centered at the vertex $Q$, and using the shape-regularity of $(\mathcal{T}_N)_{N\in\mathbb{N}}$.

**Lemma 3.** *Let $A$ and $B$ be two bounded open sets and $\kappa : A \to B$ such that*

$$\forall x, y \in A\,, \quad l\,\|x-y\| \leq \|\kappa(x) - \kappa(y)\| \leq L\,\|x-y\|\;. \tag{23}$$

*Then there holds*

$$\forall x \in A\,, \quad l \leq \frac{d(\kappa(x), \partial B)}{d(x, \partial A)} \leq L\,. \tag{24}$$

*Proof.* Let $x \in A$. For any $y \in \partial A$, $\kappa(y) \in \partial B$, so

$$d(\kappa(x), \partial B) \leq \|\kappa(x) - \kappa(y)\| \leq L\,\|x-y\|\;.$$

Taking the infimum over $y \in \partial A$, we deduce

$$d(\kappa(x), \partial B) \leq L d(x, \partial A)\,.$$

The left inequality is obtained by a similar reasoning. □

Let us point out that for all $x \in \mathbb{D}$,

$$1 \leq \frac{\omega(x)}{\sqrt{d(x, \partial\mathbb{D})}} \leq 2\,. \tag{25}$$

These remarks being made, we can prove the following result:

**Theorem 2.** *Let $N \in \mathbb{N}$ and $Q$ be a vertex of $\mathcal{T}_N$. Let $S = S_{N,Q}$ be defined as in lemma 2. Then*

$$\forall u \in T^1\,, \quad \|u - \langle u\rangle_S\|^2_{S,1/\omega} \leq C_4 \gamma(S) h_N\,\|u\|^2_{S,T^1} \tag{26}$$

*where $C_4 > 0$ does not depend on $N$ nor on $Q$ and where*

$$\gamma(S) := \sup_{x\in S} \frac{d(x, \partial S)}{d(x, \partial\mathbb{D})}\,. \tag{27}$$

*Proof.* To begin with, we observe that for any $\alpha \in \mathbb{C}$,

$$\|u - \langle u \rangle_S\|^2_{S,1/\omega} \leq \|u - \alpha\|^2_{S,1/\omega} \,.$$

Let $\alpha \in \mathbb{C}$ and $v = u - \alpha$. The main idea is the following estimate:

$$\int_S \frac{|v(x)|^2}{\omega(x)}\, dx \leq \int_S \frac{|v(x)|^2\, dx}{\sqrt{d(x, \partial\mathbb{D})}} \leq \sqrt{\gamma(S)} \int_S \frac{|v(x)|^2\, dx}{\sqrt{d(x, \partial S)}} \,.$$

Now, the singularity of the integrand is on $\partial S$, and by mapping $S$ to the disk, we will be able to use the Poincaré inequality (20). To see this, let us introduce the change of variables $x = \kappa(y)$, where $\kappa : \mathbb{D} \to S$ is a bilipschitz map as in lemma 2. This leads to

$$\int_S \frac{|v(x)|^2}{\omega(x)}\, dx \leq C\sqrt{\gamma(S)} h_N^2 \int_{\mathbb{D}} \frac{|v \circ \kappa(y)|^2\, dy}{\sqrt{d(\kappa(y), \partial S)}}\, dy \,.$$

By Lemmas 2 and 3, there holds

$$d(\kappa(y), \partial S) \geq C h_N d(y, \partial\mathbb{D}) \geq C h_N \omega^2 \,.$$

We deduce that

$$\int_S \frac{|v(x)|^2}{\omega(x)}\, dx \leq C\sqrt{\gamma(S)} \frac{h_N^2}{\sqrt{h_N}} \int_{\mathbb{D}} \frac{|f(y) - \alpha|^2}{\omega(y)} dy \tag{28}$$

$$\leq C\sqrt{\gamma(S)} h_N^{3/2} \int_{\mathbb{D}} \frac{|f(y) - \alpha|^2}{\omega(y)} dy \tag{29}$$

where $f(y) := u(\kappa(y))$. Taking $\alpha = \langle f \rangle_{\mathbb{D}}$, we can now apply the inequality (20) to $f$:

$$\int_{\mathbb{D}} \frac{|f(y) - \alpha|^2}{\omega(y)} dy \leq \int_{\mathbb{D}} \omega(y) \left|\nabla f(y)\right|^2 dy \,.$$

Injecting this inequality in what precedes, we obtain

$$\|u - \langle u \rangle_S\|^2_{S,1/\omega} \leq C\sqrt{\gamma(S)} h_N^{3/2} \int_{\mathbb{D}} \omega(y) \left|\nabla f(y)\right|^2 dy \,.$$

It remains to return to the domain $S$ by applying the inverse change of variables, while keeping track of the powers of $h_N$. We have, again by lemma 2, $|\nabla f(y)| \leq C h_N \left|[\nabla u](\kappa(y))\right|$, hence

$$\|u - \langle u \rangle_S\|^2_{S,1/\omega} \leq C\sqrt{\gamma(S)} h_N^{7/2} \int_{\mathbb{D}} \sqrt{d(y, \partial\mathbb{D})} \left|[\nabla u](\kappa(y))\right|^2 dy$$

We now reuse lemma 3:

$$\|u - \langle u \rangle_S\|^2_{S,1/\omega} \leq C\sqrt{\gamma(S)} h_N^3 \int_{\mathbb{D}} \sqrt{d(\kappa(y), \partial S)} \left|[\nabla u](\kappa(y))\right|^2 dy \,. \tag{30}$$

Finally, with the change of variables $x = \kappa(y)$ and using lemma 2, this leads to

$$\|u - \langle u \rangle_S\|^2_{S,1/\omega} \leq C\sqrt{\gamma(S)} \frac{h_N^3}{h_N^2} \int_S \sqrt{d(x, \partial S)} \left|\nabla u(x)\right|^2 dx \tag{31}$$

$$\leq C\sqrt{\gamma(S)} h_N \int_S \sqrt{d(x, \partial S)} \left|\nabla u(x)\right|^2 dx \,. \tag{32}$$

With the simple estimate

$$\sqrt{d(x,\partial S)} \leq \sqrt{\gamma(S)}\sqrt{d(x,\partial\mathbb{D})} \leq \sqrt{\gamma(S)}\omega\,,$$

we easily obtain the claimed inequality. □

*Remark* 1. There is a large corpus of works devoted to weighted Poincaré-type inequalities, but to the best of our knowledge, the kind of inequalities treated in other references (see e.g. [8, 9]) do not quite have the form of the one we deal with here.

# 4 Inverse inequalities

First, we have inverse inequalities *without weights*:

**Lemma 4.** *There exists a constant $C_5 > 0$ such that, for all $N \in \mathbb{N}$, $\theta \in \mathcal{X}_N$ and $\tau \in \mathcal{T}_N$, there holds*

$$\int_{K_\tau} |\nabla\theta(x)|^2\, dx \leq C_5 h_N^{-2} \int_{K_\tau} |\theta(x)|^2\, dx\,. \tag{33}$$

This is well-known when $K_\tau = \tau$ (i.e. when $K_\tau$ is a triangle). The only "difficulty" is to extend this to the case where $\tau$ has two vertices in the boundary. But in that case, we may enclose $K_\tau$ between two triangles of uniformly comparable areas, and the proof merely becomes a technical formality. We spare the readers with the details.

Corresponding weighted inverse inequalities can be deduced in the following manner:

**Lemma 5.** *Condition (A2) is satisfied with the constant*

$$C_{\text{inv}}(K_\tau)^{-2} = 1 + C_5 h_N^{-2}\rho_\omega(K_\tau)M_\omega(K_\tau)$$

*where $\rho_\omega(K_\tau)$ and $M_\omega(K_\tau)$ are the average and the maximum of $\omega$ on $K_\tau$, respectively.*

*Proof.* Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$ and $\theta \in \mathcal{X}_N$. Since $\nabla\theta$ is constant on $K_\tau$, one has

$$\int_{K_\tau} \omega(x)\,|\nabla\theta(x)|^2\, dx = \rho_\omega(K)\int_{K_\tau} |\nabla\theta|^2\,.$$

Applying the previous lemma, we get

$$\int_{K_\tau} \omega\,|\nabla\theta(x)|^2\, dx \leq C_5 h_N^{-2}\rho_\omega(K_\tau)\int_{K_\tau} |\theta(x)|^2\, dx \tag{34}$$

$$\leq C_5 h_N^{-2}\rho_\omega(K_\tau)M_\omega(K_\tau)\int_{K_\tau} \frac{|\theta(x)|^2}{\omega(x)}\, dx\,. \tag{35}$$

The result follows immediately. □

**Lemma 6.** *There exists a constant $C_6 > 0$ independent on $N$ such that for all $\tau \in \mathcal{T}_N$ and for any vertex $Q$ of $\tau$,*

$$h_N\gamma(S_{N,Q})C_{\text{inv}}(K_\tau)^{-2} \leq C_6\,,$$

*where $S_{N,Q}$ is the support of the basis function of $\mathcal{X}_N$ attached to $Q$, as defined in lemma 2.*

*Proof.* Let us rewrite $S = S_{N,Q}$. We have

$$h_N \gamma(S) C_i(K_\tau)^{-2} = h_N \gamma(S) + C_5 h_N^{-1} \gamma(S) \rho_\omega(K_\tau) M_\omega(K_\tau) =: T_1 + T_2 \,.$$

We can write $T_1 \le C$, since this term tends to 0 when $N \to \infty$. The main task is thus to estimate $T_2$.

On the one hand, assume that $d(S, \partial\mathbb{D}) \le h_N$. Then we use the simple estimate $\gamma(S) \le 1$. Moreover, for all $x \in K_\tau$, there holds $d(x, \partial\mathbb{D}) \le d(x, \partial S) + d(S, \partial\mathbb{D}) \le C h_N$. Using (25), we deduce $\rho_\omega(K) \le C\sqrt{h_j}$ and $M_\omega(K) \le C\sqrt{h_N}$ and thus $T_2 \le C$.

On the other hand, if $d(S, \partial\mathbb{D}) \ge h_N$, we estimate $\gamma(S)$ as follows. First, we have $d(x, \partial\mathbb{D}) \ge \omega(x)^2$ hence

$$\gamma(S) \le \frac{d(x, \partial S)}{m_\omega(S)^2} \,, \tag{36}$$

where $m_\omega(S)$ is the minimum of $\omega$ on $S$. Note that $d(S, \partial\mathbb{D}) \ge h_N$ implies that

$$h_N \le C m_\omega(S)^2 \,. \tag{37}$$

By the quasi-uniformity assumption (7) the diameter $d_S$ of $S$ satisfies

$$d_S \le C h_N \tag{38}$$

Therefore, there holds $d(x, \partial S) \le d_S \le C h_N$ . This shows that $\gamma(S) \le C \dfrac{h_N}{m_\omega(S)^2}$, which, injected in the expression of $T_2$, leads to

$$T_2 \le C \frac{\rho_\omega(K_\tau)}{m_\omega(S)} \frac{M_\omega(K_\tau)}{m_\omega(S)} \,.$$

Observing that $\nabla\omega = x/\omega$, a Taylor-Langrange inequality combined with the estimates (37) and (38) gives

$$|\rho_\omega(K_\tau) - m_\omega(S)| \le \frac{d_S}{m_\omega(S)} \le C\sqrt{h_N} \,.$$

Hence,

$$\frac{\rho_\omega(K_\tau)}{m_\omega(S)} \le 1 + \frac{|\rho_\omega(K_\tau) - m_\omega(K_\tau)|}{m_\omega(S)} \le C \,,$$

using again (37). For similar reasons, there holds $\dfrac{M_\omega(K_\tau)}{m_\omega(S)} \le C$ and so $T_2 \le C$ also in this case. This concludes the proof of the lemma. □

## 5 Clément type quasi-interpolant

Fix $N \in \mathbb{N}$ and denote by $\{Q_1, \dots, Q_n\}$ the vertices of $\mathcal{T}_N$. Let us rewrite $\varphi_{N,Q_i}$, defined in (21), as $\varphi_i$. Similarly, we write $S_i$ instead of $S_{N,Q_i}$. For the quasi-interpolant $I_N$, we put

$$\forall u \in L^2_{1/\omega} \,, \quad I_N u := \sum_{i=1}^{n} \langle u \rangle_{S_i} \varphi_i \,. \tag{39}$$

**Lemma 7.** *The quasi-interpolant* (39) *satisfies (A1) with*

$$C_P(K_\tau)^2 = C_7 h_N \sum_{i\in I(\tau)} \gamma(S_i) \tag{40}$$

*where $C_7 > 0$ is a constant independent on $N$ and $\tau$ and $I(\tau)$ is the set of indices $i$ such that $Q_i$ is a vertex of $\tau$.*

*Proof.* We adapt the proof of [6, Theorem 1]. Let $\tau \in \mathcal{T}_N$ and fix some $j \in I(\tau)$. On $K_\tau$, we have

$$I_N u = \sum_{i\in I(\tau)} c_i\varphi_i = c_j \sum_{i\in I(\tau)} \varphi_i + \sum_{i\in I(\tau)\setminus\{j\}} (c_i - c_j)\varphi_i\,. \tag{41}$$

where $c_i = \langle u\rangle_{S_i}$. Since $\sum_{i\in I(\tau)} \varphi_i = 1$, we deduce

$$\|u - I_N u\|_{K_\tau,1/\omega} \le \|u - c_j\|_{K_\tau,1/\omega} + \sum_{i\in I(\tau)\setminus\{j\}}^{3} |c_i - c_j|\, \|\varphi_i\|_{K_\tau,1/\omega} \tag{42}$$

$$\le \|u - c_j\|_{S_j,1/\omega} + \sum_{i\in I(\tau)\setminus\{j\}} |c_i - c_j|\, \|\varphi_i\|_{K_\tau,1/\omega}\,. \tag{43}$$

By theorem 2, the first term can be estimated by

$$\|u - c_j\|_{S_j,1/\omega} \le \sqrt{C_P\gamma(S_j)h_N}\, \|u\|_{S_j,T^1}\,.$$

On the other hand for $i \in I(\tau)\setminus\{j\}$, we may write

$$|c_i - c_j|^2\, \|\varphi_i\|^2_{K_\tau,1/\omega} = \left(\int_{K_\tau} 1/\omega\right)^{-1} \|c_i - c_j\|^2_{K_\tau,1/\omega}\, \|\varphi_i\|^2_{K_\tau,1/\omega} \tag{44}$$

$$\le \|c_i - c_j\|^2_{K_\tau,1/\omega} \tag{45}$$

$$\le 2(\|u - c_i\|^2_{S_i,1/\omega} + \|u - c_j\|^2_{S_j,1/\omega})\,, \tag{46}$$

since $\varphi_p \le 1$ on $K$. Applying again theorem 2 leads to

$$\|u - I_N u\|^2_{K_\tau,1/\omega} \le C h_N \left(\sum_{i\in I(\tau)}^{3} \gamma(S_i)\right) \|u\|^2_{\omega_\tau,T^1}\,,$$

where $\omega_\tau$ is defined below Eq. (15), and we used that $S_i \subset \omega_\tau$ whenever $i \in I(\tau)$.

To show that the $T^1$-continuity (14) holds, we can write, using again (41),

$$\|u - I_N u\|_{K_\tau,T^1} \le \|u\|_{K_\tau,T^1} + \sum_{i\in I(\tau)\setminus\{j\}} |c_i - c_j|\, \|\varphi_i\|_{K_\tau,T^1}\,.$$

Using the inverse inequality shown in lemma 5 and using similar arguments as above, we find

$$\begin{aligned}\|u - I_N u\|^2_{K_\tau, T^1} &\le \left(1 + C \sum_{i \in I(\tau)\setminus\{j\}} \frac{h_N(\gamma(S_i) + \gamma(S_j))}{C_{\text{inv}}(K_\tau)^2 \displaystyle\int_{K_\tau} \frac{1}{\omega(x)} dx} \|\varphi_i\|^2_{1/\omega, K_\tau}\right) \|u\|^2_{\omega_\tau, T^1} \\ &\le \left(1 + C \sum_{i \in I(\tau)} h_N \gamma(S_i) C_{\text{inv}}(K_\tau)^{-2}\right) \|u\|^2_{\omega_\tau, T^1} \ .\end{aligned}$$

Thanks to lemma 6, we conclude that

$$\|u - I_N u\|_{K_\tau, T^1} \le C \|u\|^2_{\omega_\tau, T^1} \ .$$

The continuity (14) follows easily. □

Combining lemma 6 and lemma 7, we deduce that

**Lemma 8.** *Condition (A3) is satisfied.*

This concludes the proof of theorem 1.

# 6 Conclusions

We have shown theorem 1 by combining some inverse inequalities with a weighted Poincaré inequality. Our proof relies essentially on the fact that the constants appearing in both inequalities have a uniformly bounded ratio. Identical arguments can be used to treat quasi-uniform and shape-regular family of triangulations of more general domains, but we have restricted our attention to the disk $\mathbb{D}$ for conciseness. We do not know whether the result extends to locally refined triangulations.

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