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Sampling at twice the Nyquist rate in two frequency bins guarantees uniqueness in Gabor phase retrieval

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Abstract

We demonstrate that a fourfold redundancy in the measurements is sufficient for uniqueness in sampled Gabor phase retrieval with bandlimited signals and thereby draw a parallel between the sampled Gabor phase retrieval problem and finite-dimensional phase retrieval problems. Precisely, we show that sampling at twice the Nyquist rate in two frequency bins guarantees uniqueness in Gabor phase retrieval for signals in the Paley–Wiener space.

Keywords: Phase retrieval, Gabor transform, 4M-4 Conjecture, Nyquist–Shannon sampling, Müntz-type result

MSC Classification: 94A12 , 42B10

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1 Introduction

1.1 Foreword: an overview of uniqueness in finite-dimensional phase retrieval

In finite dimensional phase retrieval, it is known that a fourfold redundancy in the measurements is necessary and sufficient for uniqueness. In particular, if one wants to recover $\mathbf{x} \in \mathbb{C}^L$ (up to a constant global phase factor) from the measurements

$$|(\mathbf{x}, \boldsymbol{\phi}_m)|, \qquad m = 1, \ldots, M, \tag{1}$$

with measurement vectors $(\boldsymbol{\phi}_m)_{m=1}^M \in \mathbb{C}^L$, then one needs that $M \geq (4 + o(1))L$ [1]. Here, we show that a fourfold redundancy in the measurements is sufficient for uniqueness in sampled Gabor phase retrieval.

Let us give a short overview over certain uniqueness results for finite-dimensional phase retrieval problems loosely centered around the following question posed in [2].

Question 1 (Cf. Problem 8 in [2]) For any dimension L, what is the smallest number $M^*(L)$ of measurement vectors for which the phase retrieval problem with measurements (1) enjoys uniqueness and how can we design such measurement vectors?

The above question is fully answered in sign retrieval (i.e. when both the signal $\mathbf{x}$ as well as the measurements vectors $(\boldsymbol{\phi}_m)_{m=1}^M$ are assumed to be in $\mathbb{R}^L$) by [3]: indeed, there it is shown that a collection of $M = 2L - 2$ measurement vectors can never yield uniqueness in sign retrieval (cf. Proposition 2.5) while a generic set of $M = 2L - 1$ measurement vectors will always yield uniqueness (cf. Theorem 2.2).[1] This leads to $M^*(L) = 2L - 1$ and the intuition that *a twofold redundancy is necessary and sufficient for uniqueness in sign retrieval problems.*

Phase retrieval is a bit more complicated than sign retrieval: in [4], it is shown that a generic set of $M = 4L-4$ measurement vectors yields uniqueness in phase retrieval (cf. Theorem 3.3) while in [1] it is shown that $M \geq (4+o(1))L$ is necessary for uniqueness in phase retrieval (cf. Theorem 6). Combining these two insights leads to $M^*(L) = (4 + o(1))L$ and the intuition that *a fourfold redundancy is necessary and sufficient for uniqueness in phase retrieval problems.*[2]

[1] Here, generic refers to an open dense subset of the set of all M-element frames of $\mathbb{R}^L$.

[2] Interestingly, the exact expression for $M^*(L)$ does not seem to be known in the complex case. A proof of the famous $4L-4$ Conjecture in [2] would have implied that $M^*(L) = 4L-4$. However, as was shown in [5] through the explicit construction of a set of $M = 11$ measurement vectors in $\mathbb{C}^4$ which yield uniqueness in phase retrieval, the $4L - 4$ Conjecture is false for general L.

1.2 Gabor phase retrieval: prior arts and our contribution

We are interested in a phase retrieval problem which differs from the setup discussed so far. Let us introduce the *Gabor transform* of a signal $f \in L^2(\mathbb{R})$ by

$$\mathcal{G}f(x,\omega) := 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\pi(t-x)^2} \mathrm{e}^{-2\pi \mathrm{i} t\omega} \,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^2,$$

and consider *sampled Gabor phase retrieval* which refers to the problem of recovering $f \in L^2(\mathbb{R})$ from

$$|\mathcal{G}f(x,\omega)|, \qquad (x,\omega) \in S,$$

where $S \subset \mathbb{R}^2$ is a discrete subset of the time-frequency plane. The first known uniqueness result for sampled Gabor phase retrieval was the following.

Theorem 1 (Cf. Theorem 2.5 in [6]) *$f \in \mathrm{PW}_B^2$ real-valued on the real line is uniquely determined (up to a constant global phase factor) by*

$$|\mathcal{G}f(x,\omega)|, \qquad (x,\omega) \in \frac{\mathbb{Z}}{4B} \times \{0\}.$$

Notably, the above result can be interpreted as a statement about uniqueness in sign retrieval: indeed, f is assumed to be real-valued on the real line and

$$\mathcal{G}f(x,0) = 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\pi(t-x)^2} \,\mathrm{d}t \in \mathbb{R},$$

for $x \in \mathbb{R}$. Additionally, we remind the reader that any signal f in the Paley–Wiener space PW_B^2 is uniquely determined by its samples $(f(k/(2B)))_{k\in\mathbb{Z}}$ at the Nyquist sampling rate according to the famous Nyquist–Shannon sampling theorem (Theorem 6). We do therefore suggest that Theorem 1 should be interpreted as an instance of the insight that a twofold redundancy is sufficient for uniqueness in sign retrieval as it guarantees uniqueness from samples at twice the Nyquist sampling rate.

As for finite-dimensional problems, sampled Gabor phase retrieval is a bit more complicated than sampled Gabor sign retrieval. Recently, the following result has been proven in [7] based on ideas in [8].

Theorem 2 (Cf. Proposition 27 in [7]) *$f \in \mathrm{PW}_B^2$ is uniquely determined (up to a constant global phase factor) by*

$$|\mathcal{G}f(x,\omega)|, \qquad (x,\omega) \in b\mathbb{Z} \times \mathbb{N},$$

if $b \in (0, \frac{1}{4B})$.

We note that in the above result samples at twice the Nyquist sampling rate are required in time while information on infinitely many frequency bins is utilised. When compared to the intuition that a fourfold redundancy is sufficient for uniqueness in phase retrieval, the latter seems excessive and the following question comes up naturally.

Question 2 Can $f \in \mathrm{PW}_B^2$ be uniquely recovered (up to a constant global phase factor) from measurements of $|\mathcal{G}f|$ in two frequency bins sampled at twice the Nyquist sampling rate?

Here, we will use a mix of ideas from [7–9] to answer this question and prove the following result.

Theorem 3 (Main result; cf. Theorem 8) *$f \in \mathrm{PW}_B^2$ is uniquely determined (up to a constant global phase factor) by*

$$|\mathcal{G}f(x,\omega)|, \qquad (x,\omega) \in \frac{\mathbb{Z}}{4B} \times \{\omega_0, \omega_1\},$$

if $\omega_0, \omega_1 \in \mathbb{R}$ are such that $\omega_0 \neq \omega_1$.

Hence, sampling at twice the Nyquist rate in two frequency bins guarantees uniqueness in Gabor phase retrieval. We suggest that this should be interpreted as an instance of the insight that a fourfold redundancy is sufficient for uniqueness in phase retrieval.

2 Preliminaries

2.1 The relation between the Gabor transform and the Fock space of entire functions

As mentioned in the introduction, we are interested in phase retrieval problems involving the Gabor transform

$$\mathcal{G}f(x,\omega) = 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\pi(t-x)^2} \mathrm{e}^{-2\pi \mathrm{i} t\omega} \,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^2,$$

where $f \in L^2(\mathbb{R})$. Clearly,

$$\left|\mathcal{G}\left(\mathrm{e}^{\mathrm{i}\alpha} f\right)\right| = |\mathcal{G}f|, \qquad \alpha \in \mathbb{R},$$

such that we cannot distinguish between signals in $L^2(\mathbb{R})$ which are equivalent under the relation

$$f \sim g :\iff \exists \alpha \in \mathbb{R} : f = \mathrm{e}^{\mathrm{i}\alpha} g.$$

As is common in the literature, we call signals $f, g \in L^2(\mathbb{R})$ which satisfy $f \sim g$ *equal up to global phase.*

The definition of the Gabor transform indicates that we follow the convention

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t) \mathrm{e}^{-2\pi \mathrm{i} t\xi} \,\mathrm{d}t, \qquad \xi \in \mathbb{R},$$

for the Fourier transform of $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, which can be extended to $L^2(\mathbb{R})$ by a classical argument. We remind the reader that the Fourier transform induces a rotation of the time-frequency plane:

$$\mathcal{G}f(x,\omega) = \mathrm{e}^{-2\pi \mathrm{i} x\omega} \mathcal{G}\mathcal{F}f(\omega,-x), \qquad (x,\omega) \in \mathbb{R}^2.$$

The above is often called the *fundamental identity of time-frequency analysis*.

One of the most notable features of the Gabor transform is that it is connected to the Fock space of entire functions. We want to explain this classical relation shortly in the following. To do so, we consider the *Bargmann transform* of a signal $f \in L^2(\mathbb{R})$ given by

$$\mathcal{B}f(z) := 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} \,\mathrm{d}t, \qquad z \in \mathbb{C}.$$

Additionally, we introduce the *Fock space* $\mathcal{F}^2(\mathbb{C})$ as the Hilbert space of all entire functions F for which the norm

$$\|F\|_{\mathcal{F}} := \left(\int_{\mathbb{C}} |F(z)|^2 \,\mathrm{e}^{-\pi|z|^2} \,\mathrm{d}z \right)^{1/2}$$

is finite. It then holds that (cf. Proposition 3.4.1 in [10])

$$\mathcal{G}f(x,-\omega) = \mathrm{e}^{\pi \mathrm{i} x\omega} \mathcal{B}f(z) \mathrm{e}^{-\pi|z|^2/2}, \qquad z = x + \mathrm{i}\omega \in \mathbb{C},$$

and that $\mathcal{B}$ is an isometry from $L^2(\mathbb{R})$ into $\mathcal{F}^2(\mathbb{C})$. Moreover, one may show that the Fock space is a reproducing kernel Hilbert space (cf. Theorem 3.4.2 in [10]) and that all $F \in \mathcal{F}^2(\mathbb{C})$ satisfy

$$|F(z)| \leq \|F\|_{\mathcal{F}} \,\mathrm{e}^{\pi|z|^2/2}, \qquad z \in \mathbb{C}.$$

Therefore, any element of the Fock space must be a finite order entire function.[3]

2.2 A characterisation of entire functions whose magnitudes agree on parallel lines

We are interested in the order of elements in the Fock space since we have recently been able to give a full characterisation of finite order entire functions whose magnitudes agree on parallel lines in the complex plane [9]. We shortly explain the relevant ideas of that work here. For this purpose, we denote the set of roots (without repetitions) of an entire function F by $\mathcal{R}(F)$. Moreover, we let $\mathcal{R}_*(F) := \mathcal{R}(F) \setminus \{0\}$ and denote the multiplicity of a root $a \in \mathcal{R}(F)$

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[3] The *order* of an entire function F is defined to be

$$\rho := \limsup_{r\to\infty} \frac{\log\log \sup_{z\in\mathbb{C},\ |z|<r} |F(z)|}{\log r}.$$

by $m_F(a)$. In this way, we can interpret $m_F : \mathcal{R}(F) \to \mathbb{N}$ as a function which after trivial extension to all of $\mathbb{C}$ — let us denote this trivial extension by $M_F : \mathbb{C} \to \mathbb{N}_0$ — fully characterises the roots of F.

If we consider two entire functions F and G, then we may group the roots of F into two sets,

$$\mathcal{X} := \mathcal{R}_*(F) \cap \mathcal{R}(G), \qquad \mathcal{Y} := \{a \in \mathcal{R}_*(F) \,|\, m_F(a) > M_G(a)\},$$

with multiplicities $m_\mathcal{X} : \mathcal{X} \to \mathbb{N}$,

$$m_\mathcal{X}(a) := \min\{m_F(a), m_G(a)\}, \qquad a \in \mathcal{X},$$

and $m_\mathcal{Y} : \mathcal{Y} \to \mathbb{N}$,

$$m_\mathcal{Y}(a) := m_F(a) - M_G(a), \qquad a \in \mathcal{Y}:$$

indeed, it holds that $M_F = M_\mathcal{X} + M_\mathcal{Y}$, where $M_\mathcal{X}, M_\mathcal{Y}$ denote the trivial extensions of $m_\mathcal{X}, m_\mathcal{Y}$ to $\mathbb{C}$.[4]

If we now assume that F and G agree on the two parallel lines $\mathbb{R}$ and $\mathbb{R}+\mathrm{i}\tau$ in the complex plane, where $\tau > 0$, then we can show that $M_G = M_\mathcal{X} + (M_\mathcal{Y})^*$, where $(M_\mathcal{Y})^*(z) := M_\mathcal{Y}(\overline{z})$, for $z \in \mathbb{C}$, and that there is a translation symmetry in $\mathcal{Y}$: for all $k \in \mathbb{Z}$,

$$M_\mathcal{Y}(z + 2\mathrm{i}k\tau) = M_\mathcal{Y}(z), \qquad z \in \mathbb{C}.$$

Hence, $\mathcal{Y}$ can be fully described by knowledge of

$$\mathcal{Y}_\mathrm{u} := \{a \in \mathcal{Y} \,|\, \mathrm{Im}\, a \in [-\tau, \tau)\}.$$

More precisely, we have that $\mathcal{Y} = \mathcal{Y}_\mathrm{u} + 2\mathrm{i}\tau\mathbb{Z}$. It follows that once we know that there is a single root in $\mathcal{Y}$, we know that there must be infinitely many evenly spaced roots in $\mathcal{Y}$. This is especially interesting in light of the following simple lemma which follows directly from one of the main results (Theorem 25) in [9]:

Lemma 4 *Let $\tau > 0$ and let F and G be two finite order entire functions such that*

$$|F(x)| = |G(x)|, \qquad |F(x+\mathrm{i}\tau)| = |G(x+\mathrm{i}\tau)|, \qquad x \in \mathbb{R}.$$

Then, F and G agree (up to a constant global phase factor) if $\mathcal{Y}_\mathrm{u} = \emptyset$.

Proof If F is zero, then the equality $|F| = |G|$ on $\mathbb{R}$ along with the identity theorem of complex analysis implies that G is zero and thus F and G agree. The same is true if G is zero and thus we can assume for the rest of this proof that F and G are non-zero. If $\mathcal{Y}_\mathrm{u} = \emptyset$, then it follows from the considerations before the statement of

[4]It is instructive to have the intuition that $M_\mathcal{X}$ contains all information on the roots of F that are also roots of G while $M_\mathcal{Y}$ contains all information on the roots of F that are no roots of G.

the lemma that $\mathcal{Y} = \emptyset$. Therefore, Theorem 25 in [9] implies that there exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that

$$\sum_{\ell=1}^{q} (b_\ell - b'_\ell) \operatorname{Im}\left[(x + \mathrm{i}\tau)^\ell\right] = 0 \tag{2}$$

and

$$F(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell) z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a};\ p\right)^{m_{\mathcal{X}}(a)},$$

$$G(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell) z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a};\ p\right)^{m_{\mathcal{X}}(a)},$$

for $z \in \mathbb{C}$, where $E(z;\ p) := (1 - z)\exp(\sum_{\ell=1}^{p} z^\ell/\ell)$ denote the so-called primary factors.

It remains to show that $b_\ell = b'_\ell$, for $\ell \in \{1, \ldots, q\}$: let us note that

$$\operatorname{Im}\left[(x + \mathrm{i}\tau)^\ell\right] = \sum_{k=0}^{\ell} \binom{\ell}{k} x^k \operatorname{Im}\left[\mathrm{i}^{\ell-k}\right] \tau^{\ell-k}$$

and that therefore

$$\sum_{\ell=1}^{q} (b_\ell - b'_\ell) \operatorname{Im}\left[(x + \mathrm{i}\tau)^\ell\right] = \sum_{k=0}^{q} x^k \cdot \sum_{\ell=\min\{1,k\}}^{q} \binom{\ell}{k} \operatorname{Im}\left[\mathrm{i}^{\ell-k}\right] \tau^{\ell-k} (b_\ell - b'_\ell).$$

Comparing coefficients in equation (2) reveals that

$$\sum_{\ell=\min\{1,k\}}^{q} \binom{\ell}{k} \operatorname{Im}\left[\mathrm{i}^{\ell-k}\right] \tau^{\ell-k} (b_\ell - b'_\ell) = 0, \qquad k \in \{0, \ldots, q\}.$$

The above equation implies $b_q = b'_q$ (for $k = q$) and $b_{q-1} = b'_{q-1}$ (for $k = q - 1$) and thus (for $k = q - 2$)

$$0 = (b_{q-2} - b'_{q-2}) - q(q-1)\tau^2 (b_q - b'_q) = (b_{q-2} - b'_{q-2}).$$

Continuing these considerations for decreasing k shows that $b_\ell = b'_\ell$, for $\ell \in \{1, \ldots, q\}$, and thus that F and G agree up to a constant global phase factor. □

2.3 Uniqueness in sampled Gabor phase retrieval

Another central idea for the proof of our main result is taken from the existing uniqueness results in sampled Gabor phase retrieval [7, 8]: let us consider the *Paley–Wiener space* of bandlimited functions

$$\mathrm{PW}_B^2 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \exists F \in L^2([-B, B]) \forall z \in \mathbb{C} : f(z) = \int_{-B}^{B} F(\xi) \mathrm{e}^{2\pi\mathrm{i}\xi z} \,\mathrm{d}\xi \right\},$$

where $B > 0$. We can give meaning to $\mathcal{G}f$, for $f \in \mathrm{PW}_B^2$: indeed, the well-known Paley–Wiener theorem implies that the restriction of f to $\mathbb{R}$ — which we denote by $f|_{\mathbb{R}}$ — is in $L^2(\mathbb{R})$. Hence $\mathcal{G}f = \mathcal{G}(f|_{\mathbb{R}})$ is well-defined. Interestingly, $\mathcal{G}f$ is also bandlimited in some sense:

Lemma 5 *Let $B > 0$ and $f \in \mathrm{PW}_B^2$. Then, for all $\omega \in \mathbb{R}$, it holds that $x \mapsto |\mathcal{G}(x,\omega)|^2$ is the restriction of a function in PW_{2B}^2 to $\mathbb{R}$.*

Proof The proof of this result is very similar to that of Lemma 15 in [7]. Indeed, we only need to note that the normalised Gaussian

$$\phi(t) = 2^{1/4}\mathrm{e}^{-\pi t^2}, \qquad t \in \mathbb{R},$$

is in $L^4(\mathbb{R})$ and satisfies $\mathcal{F}\phi = \phi$ to see that

$$\xi \mapsto \mathcal{F}(f|_{\mathbb{R}})(\xi)\overline{\mathcal{F}\phi(\xi-\omega)} \in L^{4/3}(\mathbb{R}),$$

for all $\omega \in \mathbb{R}$, according to Hölder's inequality. Hence, the proof Lemma 15 implies that $x \mapsto |\mathcal{G}(x,\omega)|^2$ is the restriction of a function in PW_{2B}^2 to $\mathbb{R}$. □

Therefore, according to the famous Nyquist–Shannon sampling theorem, $x \mapsto |\mathcal{G}(x,\omega)|^2$ can be recovered from samples.

Theorem 6 (Nyquist–Shannon sampling theorem) *Let $B > 0$ and $f \in \mathrm{PW}_B^2$. Then, it holds that*

$$f(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2B}\right) \operatorname{sinc}(2Bt - k), \qquad t \in \mathbb{R},$$

where the series converges uniformly on every compact subset of $\mathbb{R}$ and where the convention

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \qquad t \neq 0,$$

is used for the sinc function.

To prove a complete sampling result for Gabor phase retrieval, it remains to understand how to sample in the frequency coordinate. In [8, 9], this understanding comes from Zalik's theorem which we state in a slightly modified form.

Theorem 7 (Zalik's theorem; cf. Theorem 4 in [11]) *Let $p \in [1,\infty)$, let $a, b \in \mathbb{R}$ be such that $a < b$, let $r > 0$ and let $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}$ be a sequence of distinct numbers such that there exists a $\delta > 0$ and an $N_0 \in \mathbb{N}$ with*

$$\left|\mathrm{Re}\left[c_n - \tfrac{1}{2}\right]\right| \geq \delta \left|c_n - \tfrac{1}{2}\right|, \qquad n \geq N_0.$$

Then,

$$\left\{t \mapsto \mathrm{e}^{-r^2(t-c_n)^2};\ n \in \mathbb{N}\right\}$$

is complete in $L^p([a,b])$ if and only if

$$\sum_{n\in\mathbb{N},\ c_n\neq 0} |c_n|^{-1}$$

diverges.

Proof The theorem follows from the original proof in [11] with some small modifications. In case of confusion, the reader is also encouraged to read the proof of Theorem 20 in [7]. □

Remark 1 A combination of the Nyquist–Shannon sampling theorem and Zalik's theorem was first used in [8] to prove uniqueness in sampled Gabor phase retrieval with signals in $f \in L^4([-B,B])$. We have recently been able to generalise this result to PW_B^2 (cf. Theorem 2) [7]. In the following, we will further generalise Theorem 2 and show that it remains true if we replace the countably infinite set of frequency bins $\mathbb{N}$ by the set $\{\omega_0, \omega_1\}$. We see this result as being in line with the intuition that a fourfold redundancy is sufficient for uniqueness in phase retrieval.

3 The main result

We now have all the tools at our disposal which are needed to prove our main result.

Theorem 8 (Main result; cf. Theorem 3) *Let $B > 0$, let $\omega_0, \omega_1 \in \mathbb{R}$ be such that $\omega_0 < \omega_1$ and let $f, g \in \mathrm{PW}_B^2$. Then, $f \sim g$ if (and only if) $|\mathcal{G}f| = |\mathcal{G}g|$ on $\frac{1}{4B}\mathbb{Z} \times \{\omega_0, \omega_1\}$.*

Proof Let us suppose that $|\mathcal{G}f| = |\mathcal{G}g|$ on $\frac{1}{4B}\mathbb{Z} \times \{\omega_0, \omega_1\}$ and fix $j \in \{0,1\}$. According to Lemma 5, it holds that $x \mapsto |\mathcal{G}f(x,\omega_j)|^2$ and $x \mapsto |\mathcal{G}g(x,\omega_j)|^2$ are restrictions of functions in PW_{2B}^2 to $\mathbb{R}$. It therefore follows from the Nyquist–Shannon sampling theorem (cf. Theorem 6) that

$$\left|\mathcal{G}f(x,\omega_j)\right|^2 = \left|\mathcal{G}g(x,\omega_j)\right|^2, \qquad x \in \mathbb{R}. \tag{3}$$

In words, the Gabor transforms of f and g agree in magnitude on two parallel lines in the time-frequency plane.

Let us now suppose, by contradiction, that f and g do not agree up to global phase and define the finite order entire functions $F(z) := \mathcal{B}f(z + \mathrm{i}\omega_0)$ and $G(z) := \mathcal{B}g(z + \mathrm{i}\omega_0)$, for $z \in \mathbb{C}$. By the linearity of the Bargmann transform, F and G do not agree up to global phase. Additionally, we can see that

$$|F(x)| = |G(x)|, \qquad |F(x + \mathrm{i}\tau)| = |G(x + \mathrm{i}\tau)|, \qquad x \in \mathbb{R},$$

for $\tau := \omega_1 - \omega_0 > 0$, according to equation (3). It therefore follows from Lemma 4 that $\mathcal{Y}_\mathrm{u} \neq \emptyset$ and thus there exists an element $a_0 = t_0 + \mathrm{i}\xi_0 \in \mathcal{Y}_\mathrm{u}$. As it holds that $\mathcal{Y}_\mathrm{u} + 2\mathrm{i}\tau\mathbb{Z} = \mathcal{Y} \subset \mathcal{R}(F)$, we conclude that $F(a_0 - 2\mathrm{i}k\tau) = 0$ such that $\mathcal{B}f(a_0 + \mathrm{i}(\omega_0 - 2k\tau)) = 0$ which implies

$$\mathcal{G}f(x_0, 2k\tau - \omega_0) = 0, \qquad k \in \mathbb{Z}, \tag{4}$$

through the relation of the Gabor and the Bargmann transform.

We note that

$$\mathcal{G}f(x,\omega) = \mathrm{e}^{-2\pi\mathrm{i}x\omega}\mathcal{G}\mathcal{F}f(\omega,-x) = \mathrm{e}^{-2\pi\mathrm{i}x\omega} \cdot \int_{-B}^{B} \mathcal{F}f(\xi)\mathrm{e}^{-\pi(\xi-\omega)^2}\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi,$$

for $(x, \omega) \in \mathbb{R}^2$, according to the fundamental identity of time-frequency analysis and $\operatorname{supp} \mathcal{F}f \subset [-B, B]$. By completing the square in the exponent, we find that

$$-\pi(\xi - \omega)^2 - 2\pi \mathrm{i} \xi x = -\pi(\xi - \omega + \mathrm{i}x)^2 - 2\pi \mathrm{i} \omega x - \pi x^2$$

and thus

$$\mathcal{G}f(x, \omega) = \mathrm{e}^{-\pi x^2} \cdot \int_{-B}^{B} \mathcal{F}f(\xi) \cdot \overline{\mathrm{e}^{-\pi(\xi - \omega + \mathrm{i}x)^2}} \, \mathrm{d}\xi.$$

It therefore follows from equation (4) that

$$\int_{-B}^{B} \mathcal{F}f(\xi) \cdot \overline{\mathrm{e}^{-\pi(\xi - 2k\tau + \omega_0 + \mathrm{i}x_0)^2}} \, \mathrm{d}\xi = 0, \qquad k \in \mathbb{Z},$$

which implies that $\mathcal{F}f$ is orthogonal to $t \mapsto \mathrm{e}^{-\pi(t - 2k\tau + \omega_0 + \mathrm{i}x_0)^2}$ in $L^2([-B, B])$, for all $k \in \mathbb{Z}$. A slight modification of Zalik's theorem (cf. Theorem 7) does therefore imply that $\mathcal{F}f = 0$ and thus $f = 0$. Hence, it holds that $F = 0$ and, as $|F| = |G|$, the identity theorem of complex analysis can be used to show that $G = 0$ such that $g = 0$. Therefore, f and g do agree up to global phase which is the desired contradiction. □

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