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# ON THE CONNECTION BETWEEN UNIQUENESS FROM SAMPLES AND STABILITY IN GABOR PHASE RETRIEVAL

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Abstract. For every lattice $\Lambda$, we construct pairs of functions which are arbitrarily close to the Gaussian, do not agree up to global phase but have Gabor transform magnitudes agreeing on $\Lambda$. Additionally, we prove that the Gaussian can be uniquely recovered (up to global phase) in $L^2(\mathbb{R})$ from Gabor magnitude measurements on a sufficiently fine lattice. These two facts give evidence for the existence of functions which break uniqueness from samples without affecting stability. We prove that a uniform bound on the local Lipschitz constant of the signals is not sufficient to restore uniqueness in sampled Gabor phase retrieval and more restrictive a priori knowledge of the functions is necessary. With this, we show that there is no direct connection between uniqueness from samples and stability in Gabor phase retrieval. Finally, we provide an intuitive argument about the connection between directions of instability in phase retrieval and certain Laplacian eigenfunctions associated to small eigenvalues.

## 1. Introduction

*Gabor phase retrieval* is the problem of recovering signals $f \in L^2(\mathbb{R})$ from magnitude measurements of their Gabor transform,

$$\mathcal{G}f(x,\omega) := 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\pi(t-x)^2} \mathrm{e}^{-2\pi \mathrm{i} t\omega} \,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^2,$$

up to a constant global phase factor (cf. (1.1)). Its problem formulation is inspired by applications in audio processing (cf. [12]). In this paper, we distinguish between sampled Gabor phase retrieval problems — in which we deal with the recovery of $f$ (up to global phase) from $\mathcal{A}_\Lambda(f) := (|\mathcal{G}f(x,\omega)|)_{(x,\omega)\in\Lambda}$, where $\Lambda$ is a discrete subset of the time-frequency plane $\mathbb{R}^2$ — and (continuous) Gabor phase retrieval problems — in which we deal with the recovery of $f$ (up to global phase) from $\mathcal{A}_\Omega(f)$, where $\Omega$ is an open subset of $\mathbb{R}^2$ or $\mathbb{R}^2$ itself. While not much is known about sampled Gabor phase retrieval, we know that $f$ may be uniquely recovered (up to global phase) from $\mathcal{A}_{\mathbb{R}^2}(f)$ and that this recovery is weakly but not strongly stable, i.e. the inverse phase retrieval operator, $\mathcal{A}_{\mathbb{R}^2}^{-1}$, is continuous but not uniformly continuous [1]. This insight allows for the derivation of local Lipschitz constants $c_{\mathbb{R}^2}(f)$ for the inverse phase retrieval operator (cf. equation (3.1)).

We are interested in the interplay between stability properties of (continuous) Gabor phase retrieval problems, quantified by the local Lipschitz constant, and uniqueness properties of sampled Gabor phase retrieval problems. In connection with this, we want to mention prior work by two authors of this paper on the (non-)uniqueness of sampled Gabor phase retrieval [3]: In this work, they show that sampled Gabor phase retrieval does not enjoy uniqueness (for signals in $L^2(\mathbb{R})$), when the sampling set $\Lambda$ is any (shifted) lattice in $\mathbb{R}^2$, by constructing explicit counterexamples. It is notable that, when the sampling lattice is sufficiently fine, the counterexamples constructed strongly resemble signals proposed by P. Grohs and one of the authors to demonstrate that Gabor phase retrieval is severely ill-posed [2]. It thus appears likely that uniqueness from samples and stability in Gabor phase retrieval can be linked. We will demonstrate that it is not easy to make this link. More precisely, we will show that the class

$$\mathcal{M}_\nu(B_R) := \{f \in L^2(\mathbb{R}) \,:\, c_{B_R}(f) \leq \nu\}, \qquad \nu > 0,$$

is not a good prior for uniqueness in sampled Gabor phase retrieval, where $B_R$ denotes the open ball of radius $R > 0$ around the origin of the time-frequency plane $\mathbb{R}^2$ and $c_{B_R}(f)$ denotes the local Lipschitz constant of the inverse phase retrieval operator on $B_R$ (cf. equation (3.1)).

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**Theorem 1.1** (Main Result; cf. Theorem 3.9)**.** *There exists $\nu > 0$ such that for all $R > 1$ and all lattices $\Lambda$, there exist $f, g \in \mathcal{M}_\nu(B_R)$ such that $f$ and $g$ do* not *agree (up to a constant global phase factor) and yet*

$$|\mathcal{G}f| = |\mathcal{G}g| \text{ on } \Lambda.$$

It follows that uniqueness in sampled Gabor phase retrieval can only hold under additional assumptions on the signals under consideration. The signals whose existence is postulated in our main result, Theorem 1.1, can be constructed to be arbitrarily close to the normalized Gaussian

$$\varphi(t) = 2^{1/4} \mathrm{e}^{-\pi t^2}, \qquad t \in \mathbb{R},$$

which is generally believed to enjoy very strong stability properties. We will use a link made in [9] between stability of Gabor phase retrieval and certain weighted Poincaré constants to compare the stability properties of the signals constructed to those of the normalized Gaussian and thereby prove Theorem 1.1.

We note that the insight that the signals whose existence is postulated in Theorem 1.1 can be constructed to be arbitrarily close to the normalized Gaussian also ellicits the question whether the Gaussian itself may be uniquely recovered from sampled Gabor transform measurements. By employing a result from the theory of functions, we develop a new uniqueness guarantee in this direction.

**Theorem 1.2** (cf. Corollary 2.6)**.** *Let $0 < a < 1$ be a sampling rate and let $f \in L^2(\mathbb{R})$ be a signal satisfying*

$$|\mathcal{G}f(x, \omega)| = |\mathcal{G}\varphi(x, \omega)|, \qquad (x, \omega) \in a\mathbb{Z}^2,$$

*where $\varphi = 2^{1/4}\mathrm{e}^{-\pi(\cdot)^2}$ denotes the normalized Gaussian. Then, $f = e^{i\alpha}\varphi$ for some $\alpha \in \mathbb{R}$.*

We emphasize that this result highlights a fragility in the notion of uniqueness for sampled Gabor phase retrieval: While the Gaussian may be uniquely recovered from sampled Gabor phase retrieval measurements on a sufficiently fine lattice, it holds that for any lattice, there exist counterexamples (cf. Theorem 1.1) to uniqueness in sampled Gabor phase retrieval which are arbitrarily close to the Gaussian.

Let us finally make some remarks on our proof of Theorem 1.1 and its development. As mentioned before, we construct signals $f_\pm \in L^2(\mathbb{R})$ which are arbitrarily close to the normalized Gaussian, do not agree up to global phase but have magnitudes agreeing on a lattice. To see that the signals $f_\pm$ are elements of the class $\mathcal{M}_\nu(B_R)$, we show that the corresponding weighted Poincaré constants (cf. equation (3.4)) are finite and refer to a result from [9]. In essence, this argument works because the weighted Poincaré constant is continuous with respect to suitable variations of the weight function (cf. Lemma 3.6). The weighted Poincaré constant with non-vanishing and smooth weight function $w$ may be linked to the first non-zero eigenvalue of the Laplace operator with Dirichlet boundary conditions on the manifold $\mathbb{R}^2$ with metric

$$\left( \begin{pmatrix} w(x,\omega) & 0 \\ 0 & w(x,\omega) \end{pmatrix} \right)_{(x,\omega)\in\mathbb{R}^2}.$$

Through the results in [9], this insight links the local Lipschitz constant of the Gabor phase retrieval problem to the first non-zero eigenvalue $\lambda_1$ of a Laplace operator. We argue that the stability of Gabor phase retrieval is generally not only governed by $\lambda_1$ and a full consideration of the Laplacian eigenvalues gives a much more refined picture of the local stability in Gabor phase retrieval: More precisely, $\lambda_1$ yields a worst case estimate for the local Lipschitz constant. When $\lambda_1$ is very small, then the corresponding eigenfunction $u_1$ does determine a profile which is unstable. Higher order eigenvalues play a similar role with higher order eigenfunctions giving precise profiles of directions of instability (see Section 4). These arguments do not rely on the specific form of the Gabor transform and are more generally true for any phase retrieval problem in which the underlying transform comes with a holomorphic model space. In summary, our paper has three main contributions.

(1) *The notion of uniqueness for sampled Gabor phase retrieval is fragile.* We may obtain a positive result for the Gaussian but for every lattice, there exist signals which are arbitrarily close to the Gaussian, do not agree up to global phase and have Gabor transform magnitudes agreeing on the lattice.

(2) *Considering the class of signals for which the local Lipschitz constant satisfies a uniform bound is not enough to restore uniqueness in sampled Gabor phase retrieval.* Restoring uniqueness will necessitate a more restrictive signal class.

(3) *Local stability of (Gabor) phase retrieval may be quantified by Laplacian eigenvalues.* In particular, we suggest that small Laplacian eigenvalues correspond to unstable directions: If there are only very few small eigenvalues of the Laplacian, there are only few directions of instability. Moreover, each direction of instability corresponds to an associated Laplacian eigenfunction.

**Outline.** In Section 2, we prove one of our main contributions: The notion of uniqueness for sampled Gabor phase retrieval is fragile. In particular, we first prove Theorem 1.2 in Subsection 2.1 and then continue to modify the counterexamples from [3] in Subsection 2.2. In this way, we are able to construct a pair of functions $f_\pm \in L^2(\mathbb{R})$ (depending on $a > 0$) which are arbitrarily close to the normalized Gaussian $\varphi$, do not agree up to global phase but satisfy $|\mathcal{G} f_+| = |\mathcal{G} f_-|$ on $\mathbb{R} \times a\mathbb{Z} \supset a\mathbb{Z}^2$.

In Section 3, we formalize our second main idea: Considering the signal class $\mathcal{M}_\nu(B_R)$ is not enough to restore uniqueness for sampled Gabor phase retrieval. We do so by considering a certain uniqueness result from [9] which relates the local stability in Gabor phase retrieval to a weighted Poincaré constant. Then, we show that weighted Poincaré constants are continuous with respect to certain variations in the underlying weight and use that our pair of functions $f_\pm \in L^2(\mathbb{R})$ can be constructed to be arbitrarily close to the Gaussian.

In Section 4, we present an argument for the third main idea: The local stability of (Gabor) phase retrieval may be quantified by certain Laplacian eigenvalues. We believe that our arguments illustrate three main messages:

(1) if phase retrieval is ill-posed, then this can be seen from the presence of small eigenvalues of the Laplacian;
(2) each profile of instability corresponds to an eigenfunction of the Laplacian associated to a small eigenvalue;
(3) the vector space spanned by profiles of instabilities is finite-dimensional because the eigenvalues of the Laplacian grow and are unbounded.

**Definitions and basic notions.** The Gabor transform is a special case of the short-time Fourier transform with window $\psi \in L^2(\mathbb{R})$ of a signal $f \in L^2(\mathbb{R})$ given by

$$\mathcal{V}_\psi f(x,\omega) := \int_{\mathbb{R}} f(t)\overline{\psi(t-x)}\mathrm{e}^{-2\pi \mathrm{i} t\omega}\,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^2.$$

More precisely, the Gabor transform corresponds to the short-time Fourier transform with

$$\varphi(t) = 2^{1/4}\mathrm{e}^{-\pi t^2}, \qquad t \in \mathbb{R},$$

as window. Here, we consider the Gabor transform of elements in the modulation spaces,

$$M^p(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R})\,;\, \mathcal{G} f \in L^p(\mathbb{R}^2) \right\}, \qquad 1 \le p \le \infty,$$

where $\mathcal{S}'(\mathbb{R})$ denotes the class of tempered distributions. $M^p(\mathbb{R})$ can be equipped with the norm

$$\|f\|_{M^p(\mathbb{R})} := \|\mathcal{G} f\|_p.$$

Let us emphasize that we are exclusively interested in the modulation spaces with parameter $p \in [1,2]$. We will thus always be working with functions rather than abstract distributions. In particular, the following simple inclusion holds (cf. Proposition 1.7 on p. 408 of [13]):

$$M^p(\mathbb{R}) \subset L^r(\mathbb{R}), \qquad r \in [p,p'],$$

where $p' \in [2,\infty]$ denotes the Hölder conjugate of $p$ and where we have equality as sets if $p = 2$. We observe that the above inclusion implies that $M^p(\mathbb{R}) \subset L^2(\mathbb{R})$, for $p \in [1,2]$, such that the application of the Gabor transform to signals $f \in M^p(\mathbb{R})$ is well-defined.

As already mentioned above, in sampled Gabor phase retrieval, we ask questions about the recovery of signals $f \in M^p(\mathbb{R})$ from the measurements $\mathcal{A}_\Lambda(f) := (|\mathcal{G} f(x,\omega)|)_{(x,\omega)\in\Lambda}$, where $\Lambda$ is a discrete subset of the time-frequency plane $\mathbb{R}^2$, and in (continuous) Gabor phase retrieval, we ask questions about the recovery of signals $f \in M^p(\mathbb{R})$ from the measurements $\mathcal{A}_\Omega(f)$, where $\Omega$ is an open subset of $\mathbb{R}^2$ or $\mathbb{R}^2$ itself. There are certain trivial ambiguities in Gabor phase retrieval such that one typically only seeks to recover $f$ up to a global phase factor. To define this, we may introduce the equivalence relation

$$f \sim g :\iff \exists \alpha \in \mathbb{R} : f = \mathrm{e}^{\mathrm{i}\alpha} g \tag{1.1}$$

on $M^p(\mathbb{R})$ and consider the quotient set $X^p := M^p(\mathbb{R})/\sim$ with metric

$$d_{X^p}(f,g) := \min_{\alpha\in\mathbb{R}} \|f - \mathrm{e}^{\mathrm{i}\alpha} g\|_{M^p(\mathbb{R})}.$$

Sampled Gabor phase retrieval then amounts to inverting $\mathcal{A}_\Lambda : X^p \to [0,\infty)^\Lambda$ while (continuous) Gabor phase retrieval amounts to inverting $\mathcal{A}_\Omega : X^p \to [0,\infty)^\Omega$. Due to a classical formula of time-frequency analysis known as the ambiguity function relation, the operator $\mathcal{A}_\Omega$ is known to be injective provided that $p = 2$ and $\Omega = \mathbb{R}^2$.

## 2. Uniqueness for sampled Gabor phase retrieval is fragile

Recently, sampled Gabor phase retrieval has attracted a lot of attention [3, 4, 7, 8, 14]. This is partially due to the fact that the sampled Gabor phase retrieval problem is closer to applications than the (continuous) Gabor phase retrieval problem: Measurements may only be taken at finitely many points in the time-frequency plane in practice after all. Another reason for the increase of interest seems to be that the sampled problem has many fascinating connections to different mathematical theories. In this section, we will make use of a particularly fruitful connection to complex analysis in order to derive a new uniqueness result. Before we do so, we provide a brief sketch of what is known and what is not known about sampled Gabor phase retrieval. First, let us note that we have a complete picture of uniqueness for the recovery of bandlimited functions in the Paley–Wiener spaces,

$$\mathrm{PW}_B^p := \left\{ f : \mathbb{C} \to \mathbb{C} \,:\, \exists F \in L^p([-B,B]) : f = \int_{-B}^{B} F(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\cdot}\,\mathrm{d}\xi \right\},$$

due to [4, 7, 14]:

**Theorem 2.1** (Cf. Theorem 28 on p. 27 of [14])**.** *Let $p \in [2,\infty]$, $B > 0$ and $b \in (0, 1/4B)$. Then, the following are equivalent for $f, g \in \mathrm{PW}_B^p$:*

*(1) $f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$,*
*(2) $|\mathcal{G}f| = |\mathcal{G}g|$ on $b\mathbb{Z} \times \mathbb{N}$.*

Secondly, we remark that we know a lot about how uniqueness breaks down for the recovery of general (or real-valued) signals in $L^2(\mathbb{R})$ [3, 8]. We will discuss this in more detail in Section 2.2. For now, we want to emphasize that it is clear from the main result in [3] that there cannot be a uniqueness result for sampled Gabor phase retrieval in all of $L^2(\mathbb{R})$ when the sampling set is a subset of any collection of infinitely many equidistant parallel lines in the time-frequency plane. A question which has been interesting to us for quite some time is whether there are any natural subsets of $L^2(\mathbb{R})$ (apart from the Paley–Wiener spaces) in which uniqueness from samples can be regained. In the following, we illuminate some of the research which we have done in connection with this question.

### 2.1. Recovering the Gaussian from Gabor magnitude measurements on a lattice.

One of the most natural questions, which one may ask regarding the lack of uniqueness for sampled Gabor phase retrieval in $L^2(\mathbb{R})$, is probably whether the normalized Gaussian

$$\varphi(t) = 2^{1/4}\mathrm{e}^{-\pi t^2}, \qquad t \in \mathbb{R},$$

may be distinguished from all other functions in $L^2(\mathbb{R})$ by considering measurements of $|\mathcal{G}\varphi|$ on a sufficiently fine lattice. We recall that a lattice is a countable and discrete set $\Lambda \subset \mathbb{R}^2$ of the form $\Lambda = L\mathbb{Z}^2$, with $L \in \mathrm{GL}_2(\mathbb{R})$. In particular, we say that $\Lambda$ is a rectangular lattice if $L$ is diagonal. If it was true that for all lattices, there exist $L^2$-functions which are not equal to the Gaussian up to global phase but whose Gabor transform magnitudes on the lattice agree with those of the Gaussian, we would have to exclude these $L^2$-functions from our considerations to obtain uniqueness. The Gabor transform of the normalized Gaussian is given by

$$\mathcal{G}\varphi(x,\omega) = 2^{1/4}\int_{\mathbb{R}} \varphi(t)\mathrm{e}^{-\pi(t-x)^2}\mathrm{e}^{-2\pi\mathrm{i}t\omega}\,\mathrm{d}t = \mathrm{e}^{-\pi\mathrm{i}x\omega}\mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)}, \tag{2.1}$$

for $(x,\omega) \in \mathbb{R}^2$, which allows us to pose the above question in the following way.

**Question 2.2.** Let $0 < a < 1$ and let $f \in L^2(\mathbb{R})$ be such that

$$|\mathcal{G}f(x,\omega)|^2 = \mathrm{e}^{-\pi\left(x^2+\omega^2\right)} = |\mathcal{G}\varphi(x,\omega)|^2, \qquad (x,\omega) \in a\mathbb{Z}^2.$$

Does it follow that there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha}\varphi$?

The standard way to approach sampling problems for the Gabor transform is to relate the problem to the Fock space of entire functions via the Bargmann transform. The Bargmann transform of the Gaussian is given by

$$\mathcal{B}\varphi(z) = 2^{\frac{1}{4}} \int_{\mathbb{R}} \varphi(t)\mathrm{e}^{2\pi tz - \pi t^2 - \frac{\pi}{2}z^2}\,\mathrm{d}t = \mathrm{e}^{-\pi \mathrm{i} x\omega}\mathcal{G}\varphi(x,-\omega)\mathrm{e}^{\frac{\pi}{2}|z|^2} = 1,$$

for $z = x + \mathrm{i}\omega \in \mathbb{C}$. We note that $\mathcal{B}$ is a unitary operator from $L^2(\mathbb{R})$ onto the Fock space (see Theorem 3.4.3 on p. 56 of [6]) and recall that the Fock space $\mathcal{F}^2(\mathbb{C})$ consists of all entire functions $F$ for which the norm

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 \mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z$$

is finite. It follows that Question 2.2 is equivalent to the following question.

**Question 2.3.** Let $0 < a < 1$ and let $F \in \mathcal{F}^2(\mathbb{C})$ be such that

$$|F(z)| = 1 = |\mathcal{B}\varphi(z)|, \qquad z \in a\mathbb{Z} + \mathrm{i}a\mathbb{Z}.$$

Does it follow that there exists an $\alpha \in \mathbb{R}$ such that $F = \mathrm{e}^{\mathrm{i}\alpha}$?

Intuitively, the answer to this question seems related to the maximum modulus (or Phragmén–Lindelöf) principle since we are considering a second order entire function $F$ which is bounded on all lattice points. This intuition suggests that $F$ should be constant in the entire complex plane as long as the lattice is dense enough. This is indeed the case and will follow from a very nice result which was discovered independently by V. Ganapathy Iyer [10] and Albert Pfluger [11] in 1936.

**Theorem 2.4** (E.g. Theorem I A on p. 305 of [11])**.** *Let $h$ be an entire function such that*

$$\limsup_{r\to\infty} \frac{\log M_h(r)}{r^2} < \frac{\pi}{2},$$

*where $M_h(r) = \max_{|z|=r}|h(z)|$. If there exists a constant $\kappa > 0$ such that*

$$|h(m + \mathrm{i}n)| \leq \kappa, \qquad m,n \in \mathbb{Z},$$

*then $h$ is constant.*

We can now answer Question 2.3.

**Corollary 2.5.** Let $0 < a < 1$ and let $F \in \mathcal{F}^2(\mathbb{C})$ be such that

$$|F(z)| = 1 = |\mathcal{B}\varphi(z)|, \qquad z \in a\mathbb{Z} + \mathrm{i}a\mathbb{Z}.$$

Then, there exists an $\alpha \in \mathbb{R}$ such that $F(z) = \mathrm{e}^{\mathrm{i}\alpha}$ is constant.

*Proof.* We will consider the function $h(z) := F(az)$, for $z \in \mathbb{C}$. The Fock space is a reproducing kernel Hilbert space (see Theorem 3.4.2 on p. 54 in [6]) such that

$$|h(z)| = |F(az)| \leq \|F\|_{\mathcal{F}} \cdot \mathrm{e}^{\frac{\pi}{2}|az|^2} = \|F\|_{\mathcal{F}} \cdot \mathrm{e}^{\frac{\pi a^2}{2}|z|^2}, \qquad z \in \mathbb{C}.$$

It follows that

$$\limsup_{r\to\infty} \frac{\log M_h(r)}{r^2} \leq \limsup_{r\to\infty} \left( \frac{\log\|F\|_{\mathcal{F}}}{r^2} + \frac{\pi a^2}{2} \right) = \frac{\pi a^2}{2} < \frac{\pi}{2}.$$

By our assumptions, we have that

$$|h(m + \mathrm{i}n)| = |F(am + \mathrm{i}an)| = 1, \qquad m,n \in \mathbb{Z},$$

such that the assumptions of Theorem 2.4 are met with $\kappa = 1$ and we can conclude that $h$ is constant. As $|h(0)| = 1$, it follows that there must exist an $\alpha \in \mathbb{R}$ such that $h = \mathrm{e}^{\mathrm{i}\alpha}$. Therefore, $F(z) = \mathrm{e}^{\mathrm{i}\alpha}$. $\square$

We can now rephrase this corollary to obtain the answer to our initial question.

**Corollary 2.6** (Cf. Theorem 1.2)**.** Let $0 < a < 1$ and $f \in L^2(\mathbb{R})$ be such that

$$|\mathcal{G}f(x,\omega)|^2 = \mathrm{e}^{-\pi\left(x^2+\omega^2\right)} = |\mathcal{G}\varphi(x,\omega)|^2, \qquad (x,\omega) \in a\mathbb{Z}^2.$$

Then, there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha}\varphi$.

*Proof.* According to Corollary 2.5, we find that there exists an $\alpha \in \mathbb{R}$ such that $\mathcal{B}f = \mathrm{e}^{\mathrm{i}\alpha}$. By taking the inverse Bargmann transform, we obtain that $f = \mathrm{e}^{\mathrm{i}\alpha}\varphi$. □

*Remark* 2.7. A natural confusion that might arise in connection with the prior corollary is in how far it is different from the result by Grohs and Liehr [7] on the shift-invariant spaces with Gaussian generator,

$$V_\beta^p(\varphi) := \left\{ f \in L^p(\mathbb{R}) \,:\, f = \sum_{k\in\mathbb{Z}} c_k \, \mathrm{T}_{\beta k}\, \varphi,\ c \in \ell^p(\mathbb{Z}) \right\},$$

where $p \in [1,\infty]$ and $\beta \in (0,\infty)$. Here, $\mathrm{T}_x : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ denotes the translation by $x \in \mathbb{R}$ on $L^p(\mathbb{R})$, i.e.

$$\mathrm{T}_x\, f(t) := f(t-x), \qquad t \in \mathbb{R}. \tag{2.2}$$

The mentioned result applied to the Gaussian states that if $\beta > 0$ and $0 < a < \beta/2$ are such that $a\beta \notin \mathbb{Q}$, then the only functions $f \in V_\beta^1(\varphi)$ satisfying

$$|\mathcal{G}f(x,\omega)|^2 = |\mathcal{G}\varphi(x,\omega)|^2 = \mathrm{e}^{-\pi\left(x^2+\omega^2\right)}, \qquad (x,\omega) \in a\mathbb{Z}^2,$$

are of the form $f = \mathrm{e}^{\mathrm{i}\alpha}\varphi$, where $\alpha \in \mathbb{R}$. The big difference here is the assumption $f \in V_\beta^1(\varphi)$ which stands in contrast to the much weaker assumption $f \in L^2(\mathbb{R})$ in Corollary 2.6. In short, our result implies that the Gaussian can be distinguished from all other functions in $L^2(\mathbb{R})$ by looking at its sampled Gabor magnitude measurements while the result in [7] only implies that it is distinguishable from the functions in $V_\beta^1(\varphi) \subset L^2(\mathbb{R})$.

2.2. **On counterexamples for sampled Gabor phase retrieval.** The results in the prior section reveal that the Gaussian is the only function in $L^2(\mathbb{R})$ (up to global phase) which generates the measurements $|\mathcal{G}\varphi|$ when we look at the problem on a sufficiently fine rectangular lattice. In contrast, we know from [3] that sampled Gabor phase retrieval does not enjoy uniqueness in $L^2(\mathbb{R})$. If we closely consider the functions presented there, we may come up with further counterexamples which have a somewhat surprising structure: especially in light of Corollary 2.6. Let us work with

$$h_\pm(t) := \varphi(t)\left(\cosh\left(\frac{\pi t}{a}\right) \pm \mathrm{i}\sinh\left(\frac{\pi t}{a}\right)\right), \qquad t \in \mathbb{R},$$

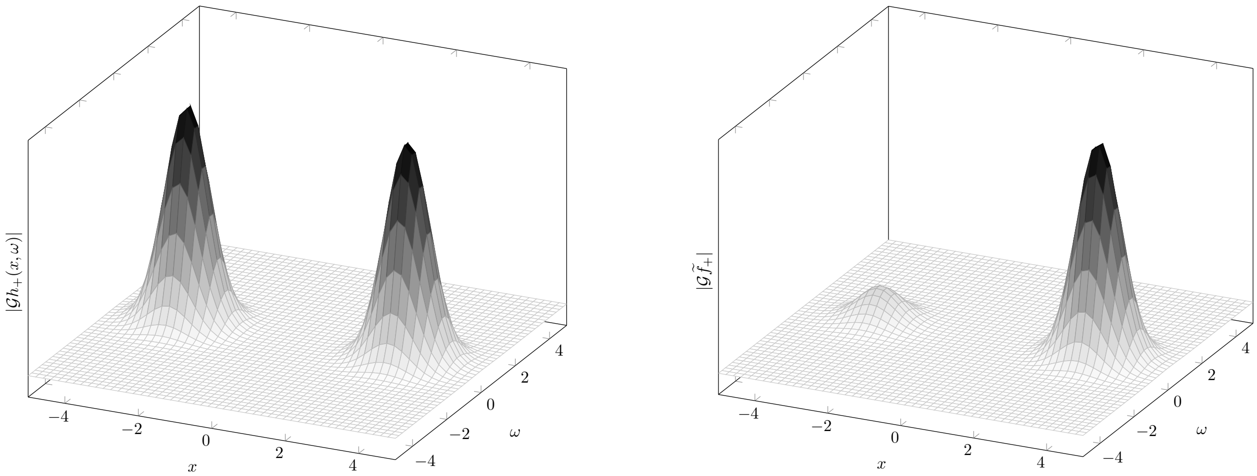
which do not agree up to global phase and still satisfy $|\mathcal{G}h_+| = |\mathcal{G}h_-|$ on $\mathbb{R} \times a\mathbb{Z}$ according to Theorem 1 on p. 6 of [3]. It is instructive to visualize the Gabor transform magnitude of these two counterexamples for a small $a > 0$. We do so in Figure 1a and note that $|\mathcal{G}f_+|$ looks remarkably similar to the spectrograms of the signals considered in [2] which are used to show the severe ill-posedness of Gabor phase retrieval. It is therefore natural to ask whether there is a connection between the uniqueness of sampled Gabor phase retrieval and the stability of (continuous) Gabor phase retrieval.

In order to investigate this question, we want to generalize the counterexamples presented in [3] slightly. To do this, we note that there is a relatively simple way of modifying the signals $h_\pm$ such that the modified signals remain counterexamples[1]: We may multiply the Bargmann transforms $\mathcal{B}h_\pm$ of the signals $h_\pm$ by an entire function of exponential type. Then, the multiplication must lie in the Fock space $\mathcal{F}^2(\mathbb{C})$ and thus give rise to new counterexamples to uniqueness in sampled Gabor phase retrieval. In particular, we will consider $\mathcal{B}h_\pm \mathrm{e}^{\pi\tau\cdot} \in \mathcal{F}^2(\mathbb{C})$, for $\tau > 0$, such that the corresponding signals

$$\widetilde{f}_\pm := \mathcal{B}^{-1}\left(\mathcal{B}h_\pm \mathrm{e}^{\pi\tau\cdot}\right) \in L^2(\mathbb{R})$$

are well-defined, do not agree up to global phase and satisfy $|\mathcal{G}\widetilde{f}_+| = |\mathcal{G}\widetilde{f}_-|$ on $\mathbb{R} \times a\mathbb{Z}$. We portray $|\mathcal{G}\widetilde{f}_+|$ in Figure 1b and note that one of the two bumps has shrunk considerably in comparison to Figure 1a. This is interesting in view of the stability results presented in [9] where stability of Gabor phase retrieval is linked

[1]Let us note that the following trick works every time because the Fock space is invariant under multiplication by first order entire functions.




(A) $|\mathcal{G}h_+|$ (B) $|\mathcal{G}\widetilde{f}_+|$, for $\tau = 1/10$.

FIGURE 1. We plot the Gabor magnitudes $|\mathcal{G}h_+|$ (Figure 1a) and $|\mathcal{G}\widetilde{f}_+|$ (Figure 1b) for $a = 1/6$. Note that we did not visualize $|\mathcal{G}h_-|$ and $|\mathcal{G}\widetilde{f}_-|$ since their plots are indistinguishable from those of $|\mathcal{G}h_+|$ and $|\mathcal{G}\widetilde{f}_+|$ upon visual inspection.

to the Cheeger constant of the spectrogram: It suggests that (continuous) Gabor phase retrieval has good stability properties at $\widetilde{f}_\pm$ while $\widetilde{f}_\pm$ cannot be recovered up to global phase from sampled Gabor measurements. We will make these two insights rigorous in the rest of this paper.

Instead of directly working with the signals $\widetilde{f}_\pm$, we want to time-shift and scale them to obtain the very nice form

$$f_\pm := \varphi \pm \mathrm{i}\gamma\, \mathrm{T}_{1/a}\, \varphi, \qquad \gamma > 0, \tag{2.3}$$

where the translation operator $\mathrm{T}_x : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is defined in equation (2.2). This has multiple benefits: First, the signals $f_\pm$ are centered in the sense of [9] such that we may directly apply the stability results proven there. Secondly, the signals $f_\pm$ converge pointwise to the normalized Gaussian $\varphi$ as $\gamma$ tends to zero. Thirdly, the functional form of $f_\pm$ is much simpler than that of $\widetilde{f}_\pm$. We may now show the following simple result.

**Lemma 2.8.** *Let $a, \gamma > 0$ and let $f_\pm$ be defined as in equation* (2.3)*. Then, $f_\pm$ do* not *agree up to global phase and yet*

$$|\mathcal{G}f_+| = |\mathcal{G}f_-| \ \text{ on } \mathbb{R} \times a\mathbb{Z}.$$

*Proof.* Let us start by computing the Gabor transforms of $f_\pm$. The lemma will then follow quite easily from there. By the linearity of the Gabor transform and the covariance property (cf. Lemma 3.1.3 on p. 41 of [6]), we find that

$$\begin{aligned}
\mathcal{G}f_\pm(x,\omega) &= \mathcal{G}\varphi(x,\omega) \pm \mathrm{i}\gamma \mathcal{G}\, \mathrm{T}_{1/a}\, \varphi(x,\omega) = \mathcal{G}\varphi(x,\omega) \pm \mathrm{i}\gamma \mathrm{e}^{-2\pi \mathrm{i} \frac{\omega}{a}} \mathcal{G}\varphi\left(x - \frac{1}{a}, \omega\right) \\
&= \mathrm{e}^{-\pi \mathrm{i} x\omega} \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)} \pm \mathrm{i}\gamma \mathrm{e}^{-2\pi\mathrm{i}\frac{\omega}{a}} \mathrm{e}^{-\pi\mathrm{i}\left(x-\frac{1}{a}\right)\omega} \mathrm{e}^{-\frac{\pi}{2}\left(\left(x-\frac{1}{a}\right)^2+\omega^2\right)} \\
&= \mathrm{e}^{-\pi \mathrm{i} x\omega} \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)} \pm \mathrm{i}\gamma \mathrm{e}^{-\pi\mathrm{i}\left(x+\frac{1}{a}\right)\omega} \mathrm{e}^{-\frac{\pi}{2}\left(\left(x-\frac{1}{a}\right)^2+\omega^2\right)},
\end{aligned}$$

for $(x,\omega) \in \mathbb{R}^2$. Therefore, we may compute

$$\begin{aligned}
|\mathcal{G}f_\pm(x,\omega)| &= \left| \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)} \pm \mathrm{i}\gamma \mathrm{e}^{-\frac{\pi\mathrm{i}\omega}{a}} \mathrm{e}^{-\frac{\pi}{2}\left(\left(x-\frac{1}{a}\right)^2+\omega^2\right)} \right| \\
&= \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)} \left| 1 \pm \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x-\mathrm{i}\omega)} \mathrm{e}^{-\frac{\pi}{2a^2}} \right|. \qquad (2.4)
\end{aligned}$$

According to equation (2.4), the Gabor transform of $f_\pm$ is zero at $(x,\omega)$ if and only if

$$\mathrm{e}^{\frac{\pi}{a}(x-\mathrm{i}\omega)-\frac{\pi}{2a^2}} = \pm\frac{\mathrm{i}}{\gamma} = \mathrm{e}^{-\log\gamma \pm \frac{\pi\mathrm{i}}{2}+2\pi\mathrm{i}k},$$

for some $k\in\mathbb{Z}$, which is equivalent to

$$\frac{\pi}{a}(x-\mathrm{i}\omega) = \frac{\pi}{2a^2} - \log\gamma \pm \frac{\pi\mathrm{i}}{2} + 2\pi\mathrm{i}k.$$

Therefore, the root sets of $\mathcal{G}f_\pm$ are given by

$$\left\{ \left( \frac{1}{2a} - \frac{a\log\gamma}{\pi}, \pm\frac{a}{2} + 2ak \right) \,:\, k\in\mathbb{Z} \right\}. \tag{2.5}$$

We note here that the root sets of $\mathcal{G}f_+$ and $\mathcal{G}f_-$ are different from each other so that $\mathcal{G}f_+$ and $\mathcal{G}f_-$ do not agree up to global phase. It follows by the linearity of the Gabor transform that $f_+$ and $f_-$ cannot agree up to global phase. Finally, we consider equation (2.4) once again to see that

$$\begin{aligned}
|\mathcal{G}f_+(x,ak)| &= \mathrm{e}^{-\frac{\pi}{2}\left(x^2+a^2k^2\right)} \left| 1 + \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x-a\mathrm{i}k)} \mathrm{e}^{-\frac{\pi}{2a^2}} \right| \\
&= \mathrm{e}^{-\frac{\pi}{2}\left(x^2+a^2k^2\right)} \left| 1 + \mathrm{i}\gamma \mathrm{e}^{\frac{\pi x}{a}} \mathrm{e}^{-\pi\mathrm{i}k} \mathrm{e}^{-\frac{\pi}{2a^2}} \right| \\
&= \mathrm{e}^{-\frac{\pi}{2}\left(x^2+a^2k^2\right)} \left| 1 + \mathrm{i}(-1)^k \gamma \mathrm{e}^{\frac{\pi x}{a}} \mathrm{e}^{-\frac{\pi}{2a^2}} \right| \\
&= \mathrm{e}^{-\frac{\pi}{2}\left(x^2+a^2k^2\right)} \left| 1 - \mathrm{i}(-1)^k \gamma \mathrm{e}^{\frac{\pi x}{a}} \mathrm{e}^{-\frac{\pi}{2a^2}} \right| \\
&= \mathrm{e}^{-\frac{\pi}{2}\left(x^2+a^2k^2\right)} \left| 1 - \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x-a\mathrm{i}k)} \mathrm{e}^{-\frac{\pi}{2a^2}} \right| \\
&= |\mathcal{G}f_-(x,ak)|
\end{aligned}$$

must hold, for $x\in\mathbb{R}$ and $k\in\mathbb{Z}$. $\square$

Let us remind the reader that we have shown in the prior section that the Gaussian may be distinguished (up to global phase) from all other functions in $L^2(\mathbb{R})$ by considering its Gabor transform magnitudes sampled at $a\mathbb{Z}^2$ (cf. Corollary 2.6). At the same time, as we have proven above, there exist signals in $L^2(\mathbb{R})$ that are arbitrarily close to the Gaussian and are counterexamples to uniqueness of sampled Gabor phase retrieval with sampling set $a\mathbb{Z}^2$. Consequently, the uniqueness property of sampled Gabor phase retrieval is rather fragile.

*Remark* 2.9. We want to make three remarks on the prior lemma and its proof.

(1) First, the expression of the Gabor transform of the counterexamples is

$$|\mathcal{G}f_\pm(x,\omega)| = \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)} \left| 1 \pm \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x-\mathrm{i}\omega)} \mathrm{e}^{-\frac{\pi}{2a^2}} \right|, \quad (x,\omega)\in\mathbb{R}^2,$$

as can be seen from equation (2.4). This expression will play an important role in the next section. For now, we note that the above insight allows us to conclude that for all $a,R>0$, there exists a $\gamma_0=\gamma_0(a,R)>0$ such that for all $\gamma\in(0,\gamma_0)$, it holds that the roots of $\mathcal{G}f_\pm$ fall outside of the cube $[-R,R]^2$. Indeed, we may note that $(x,\omega)\mapsto \exp(\frac{\pi}{a}(x-\mathrm{i}\omega)-\frac{\pi}{2a^2})$ is a continuous function and will thus attain its maximum on $[-R,R]^2$. It follows that the Gabor transforms $\mathcal{G}f_\pm$ have no roots in $[-R,R]^2$ if we choose

$$\gamma < \left( \max_{x,\omega\in[-R,R]} \left| \mathrm{e}^{\frac{\pi}{a}(x-\mathrm{i}\omega)} \mathrm{e}^{-\frac{\pi}{2a^2}} \right| \right)^{-1}.$$

We will specify $\gamma_0$ precisely in the next remark.

(2) Secondly, the root sets of $\mathcal{G}f_\pm$ are

$$\left\{ \left( \frac{1}{2a} - \frac{a\log\gamma}{\pi}, \pm\frac{a}{2} + 2ak \right) \,:\, k\in\mathbb{Z} \right\},$$

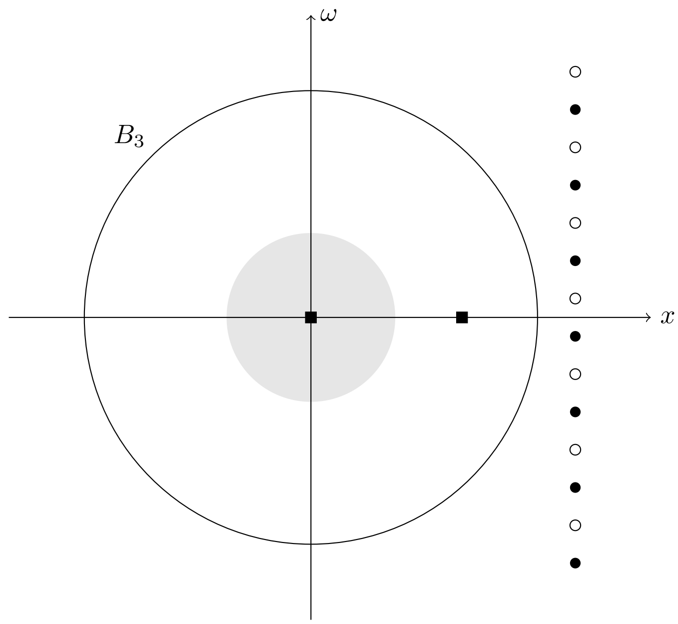



FIGURE 2. We consider $a = 1/2$ and $\gamma = \exp(-5\pi)$. The roots of $\mathcal{G}f_+$ are indicated by circles and the roots of $\mathcal{G}f_-$ are indicated by disks. We have also drawn the local maxima of $\mathcal{G}f_\pm$ as squares and indicated the region on which 99% of the $L^2$-mass of $\mathcal{G}f_\pm$ is concentrated in light gray. We highlight that we have chosen $\gamma < \gamma_0(1/2, R = 3) = \exp(-4\pi)$ such that the roots of $\mathcal{G}f_\pm$ fall outside the open ball of radius $R = 3$. We also note that there is no gray region around the local maximum at $(2, 0)$ indicating that very little mass is concentrated on the small bump.

as proven in equation (2.5). We have visualized these roots in Figure 2. If we consider arbitrary but fixed $a, R > 0$, then we may specify $\gamma_0 = \gamma_0(a, R) > 0$ such that for all $\gamma \in (0, \gamma_0]$, the roots of $\mathcal{G}f_\pm$ fall outside the strip $(-R, R) \times \mathbb{R}$ in the time-frequency plane. Precisely, we may find

$$\gamma_0 := \mathrm{e}^{-\frac{\pi}{a}\left(R - \frac{1}{2a}\right)}.$$

This insight is of great importance for our considerations in the next section.

(3) Thirdly, the above lemma may also be proven by using the results in [8]. We believe that our independent presentation is of interest because it may be used to generate counterexamples to uniqueness in sampled Gabor phase retrieval which cannot be constructed using the theory in [8]. Additionally, most of the calculations in our presentation above will be referred to at a later point in this paper.

## 3. On the stability of Gabor phase retrieval

Gabor phase retrieval deals with the reconstruction of a signal $f$ from the magnitude of its Gabor transform $|\mathcal{G}f|$. Typically, one may also be interested in recovering $\mathcal{G}f$ on a time-frequency region $\Omega$ from the magnitude measurements denoted by $|\mathcal{G}f_{|\Omega}|$. Let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $1 \leq p \leq \infty$. Given $f \in M^p(\mathbb{R})$, we denote by $c_{p,\Omega}(f)$ the smallest constant $C > 0$ such that

$$\inf_{\alpha \in \mathbb{R}} \|\mathcal{G}f - e^{i\alpha}\mathcal{G}g\|_{L^p(\Omega)} \leq C \||\mathcal{G}f_{|\Omega}| - |\mathcal{G}g_{|\Omega}|\|_{\mathfrak{B}}, \quad \forall g \in M^p(\mathbb{R}), \tag{3.1}$$

where $\|\cdot\|_{\mathfrak{B}}$ denotes a suitable norm on the space of measurements. A large local Lipschitz constant $c_{p,\Omega}(f)$ indicates that the problem of recovering $\mathcal{G}f_{|\Omega}$ from $|\mathcal{G}f_{|\Omega}|$ cannot be controlled well since there exist functions $g \in M^p(\mathbb{R})$ with $|\mathcal{G}g_{|\Omega}|$ very close to $|\mathcal{G}f_{|\Omega}|$ while the distance between $\mathcal{G}f_{|\Omega}$ and $\mathcal{G}g_{|\Omega}$ is not small.

Consequently, the problem of recovering $f$ from $|\mathcal{G}f_{|\Omega}|$ is also not well controlled since

$$\inf_{\alpha\in\mathbb{R}} \|f - e^{i\alpha} g\|_{M^p(\mathbb{R})} = \inf_{\alpha\in\mathbb{R}} \|\mathcal{G}f - e^{i\alpha}\mathcal{G}g\|_{L^p(\mathbb{R}^2)} \geq \inf_{\alpha\in\mathbb{R}} \|\mathcal{G}f - e^{i\alpha}\mathcal{G}g\|_{L^p(\Omega)}.$$

On the other hand, a small $c_{p,\Omega}(f)$ translates into good stability guarantees for the recovery of $\mathcal{G}f_{|\Omega}$ from $|\mathcal{G}f_{|\Omega}|$. We observe that, if $\Omega \subsetneq \mathbb{R}^2$, this does not guarantee that the problem of recovering $f$ from $|\mathcal{G}f_{|\Omega}|$ is stable. However, if we suppose that $f$ is $\epsilon$-concentrated on $\Omega$, i.e. $f$ satisfies

$$\|\mathcal{G}f\|_{L^p(\mathbb{R}^2\setminus\Omega)} \leq \epsilon, \tag{3.2}$$

for some small $\epsilon > 0$, then we obtain a weaker notion of stability for the recovery of $f$ from $|\mathcal{G}f_{|\Omega}|$ in the sense that

$$\inf_{\alpha\in\mathbb{R}} \|f - e^{i\alpha} g\|_{M^p(\mathbb{R})} = \inf_{\alpha\in\mathbb{R}} \|\mathcal{G}f - e^{i\alpha}\mathcal{G}g\|_{L^p(\mathbb{R}^2)} \leq c_{p,\Omega}(f) \||\mathcal{G}f_{|\Omega}| - |\mathcal{G}g_{|\Omega}|\|_{\mathfrak{B}} + 2\epsilon,$$

for any $g \in M^p(\mathbb{R})$ satisfying (3.2). For quite some time, we have anticipated a direct connection between uniqueness of sampled Gabor phase retrieval and stability of full Gabor phase retrieval. From our perspective, the discovery of the counterexamples depicted in Figure 1a support such an expectation. The two bumps in Figure 1a move far apart as $a$ goes to zero, resulting in a degradation of the local stability constant. In other words, the finer the sample rate $a$ is, the larger the local stability constant of $h_\pm$ is. It thus appears likely that a uniform bound on the local Lipschitz constant could restore uniqueness from samples at a sufficiently fine scale. With this, the question we intend to address can be formulated as follows:

**Question 3.1.** Let $\nu > 0$, and let

$$\mathcal{M}_\nu(\mathbb{R}^2) = \{f \in M^p(\mathbb{R}) : c_{p,\mathbb{R}^2}(f) \leq \nu\}.$$

Is there a lattice $\Lambda \subset \mathbb{R}^2$ such that for every $f, g \in \mathcal{M}_\nu(\mathbb{R}^2)$, the following are equivalent?

(1) $f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$;
(2) $|\mathcal{G}f| = |\mathcal{G}g|$ on $\Lambda$.

Much to our surprise, the construction of the counterexamples (2.3) suggests the existence of function perturbations which break uniqueness from samples while not affecting stability, resulting in a negative answer to Question 3.1. Note that the precise statement in our Theorem 3.9 slightly differs from Question 3.1 and we discuss these technicalities in Section 3.3.

A result by Grohs & Rathmair [9] states that the stability constant $c_{p,\Omega}(f)$ can be controlled by the inverse of the Cheeger constant

$$h_{p,\Omega}(f) := \inf_{\substack{D\in\mathcal{D}, \\ \|\mathcal{G}f\|^p_{L^p(D)} \leq \frac{1}{2}\|\mathcal{G}f\|^p_{L^p(\Omega)}}} \frac{\|\mathcal{G}f\|^p_{L^p(\partial D)}}{\|\mathcal{G}f\|^p_{L^p(D)}},$$

where $\mathcal{D}$ denotes the class of open subsets $D \subseteq \Omega$ with $\partial D \cap \Omega$ smooth. Thus, a large Cheeger constant $h_{p,\Omega}(f)$ translates into good stability guarantees for the recovery of $\mathcal{G}f_{|\Omega}$ from $|\mathcal{G}f_{|\Omega}|$. Intuitively, the Cheeger constant $h_{p,\Omega}(f)$ describes the disconnectedness of the measurements $|\mathcal{G}f|$ on $\Omega$. More precisely, a small Cheeger constant indicates that $\Omega$ can be partitioned into two subsets $\Omega_1, \Omega_2 \subseteq \Omega$ such that $|\mathcal{G}f|$ is rather small along the separating boundary $\partial\Omega_1 = \partial\Omega_2$, and at the same time

$$\|\mathcal{G}f\|_{L^p(\Omega_1)} \sim \|\mathcal{G}f\|_{L^p(\Omega_2)}.$$

On the other hand, a large Cheeger constant indicates that the above situation cannot occur, resulting in the connectedness of the measurements on $\Omega$. While the meaning of the Cheeger constant is intuitively easy to understand, estimating it on concrete examples seems to be a challenging problem involving variational calculus arguments. However, the stability estimates in [9] involving the Cheeger constant originate from upper bounding $c_{p,\Omega}(f)$ by the Poincaré constant $C_{\mathrm{poinc}}(p, \Omega, |\mathcal{G}f|^p)$ for the weighted space $L^p(\Omega, |\mathcal{G}f|^p \mathrm{d}x\mathrm{d}\omega)$. Thus, to study the local stability of Gabor phase retrieval, we can work directly with weighted Poincaré constants. The relation with local stability is inversely proportional to that of the Cheeger constant: the smaller the Poincaré constant, the better the local stability of Gabor phase retrieval at $f$ (cf. Theorem 3.4). Altogether, we obtain the following picture:

$$c_{p,\Omega}(f) \lesssim C_{\mathrm{poinc}}(p, \Omega, |\mathcal{G}f|^p) \lesssim h_{p,\Omega}(f)^{-1}.$$

3.1. **The relation of stability to the weighted Poincaré constant.** We call a nonnegative function $w\colon \Omega \to \mathbb{R}_+$ a weight on a domain $\Omega \subseteq \mathbb{R}^2$ if it is locally integrable and strictly positive almost everywhere. $F_\Omega^w$ denotes the weighted average of $F$ over $\Omega$,

$$F_\Omega^w = \frac{1}{w(\Omega)} \int_\Omega F(x)w(x)\mathrm{d}x, \qquad \text{where} \quad w(\Omega) = \int_\Omega w(x)\mathrm{d}x.$$

For any $1 \leq p \leq \infty$, we denote by $L^p(\Omega, w)$ the space consisting of all measurable functions $F\colon \Omega \to \mathbb{C}$ such that

$$\|F\|_{L^p(\Omega,w)} = \left( \int_\Omega |F(x)|^p w(x)\mathrm{d}x \right)^{\frac{1}{p}} < +\infty.$$

Furthermore, for any $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, we denote by $W^{k,p}(\Omega, w)$ the space of functions $F \in L^p(\Omega, w)$ such that

$$\|F\|_{W^{k,p}(\Omega,w)} = \left( \sum_{\alpha+\beta \leq k} \left\| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} F \right\|_{L^p(\Omega,w)}^p \right)^{\frac{1}{p}} < +\infty.$$

If $w \equiv 1$, we simply write $\|F\|_{L^p(\Omega)}^p$ and $\|F\|_{W^{k,p}(\Omega)}^p$. Finally, we denote by $\mathcal{M}(\Omega)$ the field of meromorphic functions on $\Omega$.

**Definition 3.2.** Let $1 \leq p < \infty$, let $\Omega \subseteq \mathbb{R}^2$ be a domain, and let $w$ be a weight on $\Omega$. We say that a *weighted Poincaré inequality* holds if there exists a constant $C > 0$ such that for all $F \in W^{1,p}(\Omega, w) \cap \mathcal{M}(\Omega)$

$$\|F - F_\Omega^w\|_{L^p(\Omega,w)} \leq C \|\nabla F\|_{L^p(\Omega,w)}. \tag{3.3}$$

If a weighted Poincaré inequality holds, we call the smallest constant satisfying (3.3) *Poincaré constant* and we denote it by $C_{\text{poinc}}(p, \Omega, w)$. By definition, we note that the Poincaré constant is given by

$$C_{\text{poinc}}(p, \Omega, w) = \sup \left\{ \frac{\|F - F_\Omega^w\|_{L^p(\Omega,w)}}{\|\nabla F\|_{L^p(\Omega,w)}} \ : \ F \in W^{1,p}(\Omega, w) \cap \mathcal{M}(\Omega), \ F \neq \text{const.} \right\}. \tag{3.4}$$

We point out that the weighted Poincaré constant is classically defined by requiring that equation (3.3) is satisfied for every $F \in W^{1,p}(\Omega, w)$. Here, we use a slightly different definition since we apply results from [9] involving Definition 3.2. In order to state Theorem 3.4, we introduce a family of norms on the measurement space of functions.

**Definition 3.3.** Let $\Omega \subset \mathbb{R}^2$ be a domain and let $w$ be a weight on $\Omega$. For $p, q \in [1, +\infty)$, $s > 0$, $k \in \mathbb{N}$, and $F\colon \Omega \to \mathbb{C}$ smooth, we define the norms

$$\|F\|_{\mathcal{D}_{p,q}^{k,s}(\Omega)} = \|F\|_{W^{k,p}(\Omega)} + \|F\|_{L^p(\Omega)} + \|(x, \omega) \mapsto (|x| + |\omega|)^s F(x, \omega)\|_{L^p(\Omega,w)}^p.$$

Moreover, we say that a function $F\colon \mathbb{R}^2 \to \mathbb{C}$ is centered if $|F|$ has a maximum at the origin. We are now in a position to state one of the main results in [9].

**Theorem 3.4** (Theorem 5.9 in [9])**.** *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty)$. Let $\Omega \subseteq \mathbb{R}^2$ be a convex domain with boundary whose curvature is everywhere bounded by 1. Suppose that $f \in M^p(\mathbb{R})$ is such that its Gabor transform is centered. Then, there exists a constant $c > 0$ such that for every $g \in M^p(\mathbb{R})$ it holds that*

$$\inf_{\alpha \in \mathbb{R}} \|\mathcal{G}f - e^{i\alpha}\mathcal{G}g\|_{L^p(\Omega)} \leq c\ (1 + C_{\text{poinc}}(p, \Omega, |\mathcal{G}f|^p)) \||\mathcal{G}f| - |\mathcal{G}g|\|_{\mathcal{D}_{p,q}^{1,4}(\Omega)}.$$

Let us emphasize that the constant $c > 0$ in Theorem 3.4 only depends on $p, q$ and monotonically increasingly on

$$\max\{\|\mathcal{G}f\|_{L^p(\Omega)}/\|\mathcal{G}f\|_{L^\infty(\Omega)}, \|\mathcal{V}_{\varphi'}f\|_{L^\infty(\Omega)}/\|\mathcal{G}f\|_{L^\infty(\Omega)}\}.$$

3.2. **On the variation of the weighted Poincaré constant.** A natural question concerning weighted Poincaré inequalities is how the Poincaré constant $C_{\text{poinc}}(p, \Omega, w)$ changes under variations of the weight $w$. Lemma 3.6 provides a simple result in that direction. We first state a classical fact that we exploit in its proof.

**Lemma 3.5** ([5])**.** *Let $1 \le p < \infty$, let $\Omega \subset \mathbb{R}^2$ be a domain, and let $w$ be a weight on $\Omega$. Then, for every $F \in L^p(\Omega, w)$, it holds that*

$$\inf_{c\in\mathbb{R}} \|F - c\|_{L^p(\Omega,w)} \le \|F - F_\Omega^w\|_{L^p(\Omega,w)} \le 2 \inf_{c\in\mathbb{R}} \|F - c\|_{L^p(\Omega,w)}. \tag{3.5}$$

**Lemma 3.6.** *Let $1 \le p < \infty$, let $\Omega \subset \mathbb{R}^2$ be a domain, and let $w$ be a weight on $\Omega$. Let $w'$ be another weight on $\Omega$ which satisfies*

$$Aw(x) \le w'(x) \le Bw(x), \qquad x \in \Omega, \tag{3.6}$$

*for some constants $0 < A \le B < \infty$. Then, it holds that*

$$\frac{A^{1/p}}{2B^{1/p}}\ C_{\mathrm{poinc}}(p, \Omega, w) \le C_{\mathrm{poinc}}(p, \Omega, w') \le \frac{2B^{1/p}}{A^{1/p}}\ C_{\mathrm{poinc}}(p, \Omega, w). \tag{3.7}$$

*Proof.* By equation (3.6), it follows that for every $F \in L^p(\Omega, w)$, the integral inequality

$$A \int_\Omega |F(x)|^p w(x)\,\mathrm{d}x \le \int_\Omega |F(x)|^p w'(x)\,\mathrm{d}x \le B \int_\Omega |F(x)|^p w(x)\,\mathrm{d}x$$

holds true. Then, we have that

$$A^{1/p}\|F\|_{L^p(\Omega,w)} \le \|F\|_{L^p(\Omega,w')} \le B^{1/p}\|F\|_{L^p(\Omega,w)}$$

and consequently $L^p(\Omega, w) = L^p(\Omega, w')$ as well as $W^{1,p}(\Omega, w) = W^{1,p}(\Omega, w')$. In particular, we obtain that for every $F \in L^p(\Omega, w)$ and for every $c \in \mathbb{R}$,

$$A^{1/p}\|F - c\|_{L^p(\Omega,w)} \le \|F - c\|_{L^p(\Omega,w')} \le B^{1/p}\|F - c\|_{L^p(\Omega,w)}$$

as well as

$$A^{1/p}\|\nabla F\|_{L^p(\Omega,w)} \le \|\nabla F\|_{L^p(\Omega,w')} \le B^{1/p}\|\nabla F\|_{L^p(\Omega,w)}.$$

We can now prove the upper and lower bounds in equation 3.7. According to Lemma 3.5 and equation (3.4) along with the above inequalities, we find that

$$\begin{aligned}
C_{\mathrm{poinc}}(p, \Omega, w') &\le 2 \sup \left\{ \inf_{c\in\mathbb{R}} \frac{\|F - c\|_{L^p(\Omega,w')}}{\|\nabla F\|_{L^p(\Omega,w')}} \ :\ F \in W^{1,p}(\Omega, w') \cap \mathcal{M}(\Omega),\ F \ne \text{const.} \right\} \\
&\le \frac{2B^{1/p}}{A^{1/p}} \sup \left\{ \inf_{c\in\mathbb{R}} \frac{\|F - c\|_{L^p(\Omega,w)}}{\|\nabla F\|_{L^p(\Omega,w)}} \ :\ F \in W^{1,p}(\Omega, w) \cap \mathcal{M}(\Omega),\ F \ne \text{const.} \right\} \\
&\le \frac{2B^{1/p}}{A^{1/p}} \sup \left\{ \frac{\|F - F_\Omega^w\|_{L^p(\Omega,w)}}{\|\nabla F\|_{L^p(\Omega,w)}} \ :\ F \in W^{1,p}(\Omega, w) \cap \mathcal{M}(\Omega),\ F \ne \text{const.} \right\} \\
&= \frac{2B^{1/p}}{A^{1/p}}\ C_{\mathrm{poinc}}(p, \Omega, w).
\end{aligned}$$

We can follow essentially the same argument to show the lower bound

$$C_{\mathrm{poinc}}(p, \Omega, w') \ge \frac{A^{1/p}}{2B^{1/p}}\ C_{\mathrm{poinc}}(p, \Omega, w)$$

which concludes the proof. ☐

3.3. **Answering Question 3.1.** We apply Lemma 3.6 to the special case where the weights are given by

$$w = |\mathcal{G}\varphi|^p, \quad w'_\pm = |\mathcal{G}f_\pm|^p,$$

for some $1 \le p < \infty$, and where $f_\pm$ denote the counterexamples

$$f_\pm = \varphi \pm \mathrm{i}\gamma\, \mathrm{T}_{1/a}\, \varphi, \qquad \gamma > 0,\ a > 0,$$

constructed in Section 2.2. By equations (2.1) and (2.4), we have that for all $(x, \omega) \in \mathbb{R}^2$,

$$|\mathcal{G}f_\pm(x,\omega)|^p = |\mathcal{G}\varphi(x,\omega)|^p \left| 1 \pm \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x - \mathrm{i}\omega)} \mathrm{e}^{-\frac{\pi}{2a^2}} \right|^p.$$

We restrict to a bounded domain $\Omega \subseteq \mathbb{R}^2$ and we choose $R > 0$ such that $\Omega \subseteq (-R, R) \times \mathbb{R}$. By Remark 2.9, we know that

$$\gamma < \mathrm{e}^{-\frac{\pi}{a}\left(R - \frac{1}{2a}\right)}$$

ensures that all the roots of $|\mathcal{G}f_\pm|^p$ fall outside the domain $\Omega$. So that, by the extreme value theorem, there exist $0 < A_\gamma \le B_\gamma < \infty$ such that

$$A_\gamma \le \left|1 \pm \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x-\mathrm{i}\omega)}\mathrm{e}^{-\frac{\pi}{2a^2}}\right|^p \le B_\gamma, \qquad (x,\omega) \in \Omega.$$

More precisely, given any $0 < \delta < 1$, the stronger condition

$$\gamma < \delta \mathrm{e}^{-\frac{\pi}{a}\left(R - \frac{1}{2a}\right)}$$

implies that for all $(x,\omega) \in \Omega$,

$$(1-\delta)^p \le \left|1 \pm \mathrm{i}\gamma \mathrm{e}^{\frac{\pi}{a}(x-\mathrm{i}\omega)}\mathrm{e}^{-\frac{\pi}{2a^2}}\right|^p \le (1+\delta)^p,$$

and consequently

$$(1-\delta)^p\ |\mathcal{G}\varphi(x,\omega)|^p \le |\mathcal{G}f_\pm(x,\omega)|^p \le (1+\delta)^p |\mathcal{G}\varphi(x,\omega)|^p, \tag{3.8}$$

for all $(x,\omega) \in \Omega$.

**Corollary 3.7.** Let $1 \le p < \infty$, let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $a > 0$. Then, for any $0 < \delta < 1$, there exists a $\gamma_\delta = \gamma_\delta(a,\Omega) > 0$ such that for all $\gamma < \gamma_\delta$, it holds that

$$\frac{(1-\delta)}{2(1+\delta)}\ C_{\text{poinc}}\left(p,\Omega,|\mathcal{G}\varphi|^p\right) \le C_{\text{poinc}}\left(p,\Omega,|\mathcal{G}f_\pm|^p\right) \le \frac{2(1+\delta)}{(1-\delta)}\ C_{\text{poinc}}\left(p,\Omega,|\mathcal{G}\varphi|^p\right).$$

*Proof.* The proof follows by Lemma 3.6 along with equation (3.8). □

*Remark* 3.8. Theorem B.7 along with Theorem B.8 in [9] ensure that

$$C_{\text{poinc}}\left(p,\Omega,|\mathcal{G}\varphi|^p\right) < \infty$$

whenever $p \in [1,2]$ and $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with Lipschitz boundary.

Let $\nu > 0$ and let $B_R$ denote the ball of radius $R > 0$ centered at 0. We recall the notation $\mathcal{M}_\nu(B_R)$ introduced in Question 3.1 for the class of functions

$$\mathcal{M}_\nu(B_R) = \{f \in M^p(\mathbb{R}) : c_{p,B_R}(f) \le \nu\}.$$

The following theorem is our main result. It provides a theoretical foundation for our claim that the signal class $\mathcal{M}_\nu(\mathbb{R}^2)$ cannot serve as a prior for uniqueness in sampled Gabor phase retrieval. Precisely, it states that there is a uniform upper bound $\nu > 0$ for the local Lipschitz constant for which there exist functions in the signal class $\mathcal{M}_\nu(B_R)$, where $R > 1$, that do not agree up to global phase but whose Gabor transform magnitudes agree on a rectangular lattice $\Lambda$ — no matter how large we choose $R > 1$ and how fine we choose the lattice. Observe that every rectangular lattice $\Lambda$ is contained in a set of parallel lines; that is, there exists $a > 0$ such that $\Lambda \subseteq \mathbb{R} \times a\mathbb{Z}$.

**Theorem 3.9** (Main result; cf. Theorem 1.1)**.** *Let $p \in [1,2)$, $q \in (2p/(2-p),\infty)$. There exists $\nu > 0$ such that for all $R > 1$ and for all $a > 0$, there exist $f,g \in \mathcal{M}_\nu(B_R)$ such that $f \not\sim g$ but*

$$|\mathcal{G}f(x,\omega)| = |\mathcal{G}g(x,\omega)|, \quad (x,\omega) \in \mathbb{R} \times a\mathbb{Z}.$$

*Proof.* We show that there exists $\nu > 0$ such that for all $R > 1$ and for all $a > 0$, there exists $\gamma > 0$ such that

$$c_{p,B_R}(f_\pm) \le \nu,$$

where

$$f_\pm = \varphi \pm \mathrm{i}\gamma\,\mathrm{T}_{\frac{1}{a}}\,\varphi.$$

We already know from Section 2.2 that $f_\pm \in M^p(\mathbb{R})$, $f_+ \not\sim f_-$ and

$$|\mathcal{G}f_+(x,\omega)| = |\mathcal{G}f_-(x,\omega)|, \quad (x,\omega) \in \mathbb{R} \times a\mathbb{Z}.$$

Let $R > 1$ and $a > 0$. We choose

$$\gamma < \delta \cdot \min\left\{1, \mathrm{e}^{-\frac{\pi}{a}\left(R - \frac{1}{2a}\right)}\right\},$$

with $0 < \delta < 1$. The condition $\gamma < \delta e^{-\frac{\pi}{a}(R-\frac{1}{2a})}$ ensures that the roots of $f_\pm$ fall outside the ball $B_R$. Theorem 3.4 states that

$$c_{p,B_R}(f_\pm) \le c(1 + C_{\mathrm{poinc}}(p, B_R, |\mathcal{G}f_\pm|^p)), \tag{3.9}$$

where $c > 0$ is a constant depending on $p$, $q$ and monotonically increasingly on

$$\max\{\|\mathcal{G}f_\pm\|_{L^p(B_R)}/\|\mathcal{G}f_\pm\|_{L^\infty(B_R)}, \|\mathcal{V}_{\varphi'}f_\pm\|_{L^\infty(B_R)}/\|\mathcal{G}f_\pm\|_{L^\infty(B_R)}\}. \tag{3.10}$$

By Corollary 3.7, we know how to upper bound the weighted Poincaré constant in (3.9): We obtain

$$c_{p,B_R}(f_\pm) \le c\left(1 + \frac{2(1+\delta)}{(1-\delta)}\, C_{\mathrm{poinc}}\left(p, B_R, |\mathcal{G}\varphi|^p\right)\right). \tag{3.11}$$

By Theorem B.12 together with Theorem 5.10 in [9], there exists $\eta$ depending on $p$ but independent of $R > 0$ such that

$$C_{\mathrm{poinc}}\left(p, B_R, |\mathcal{G}\varphi|^p\right) \le \eta,$$

which yields

$$c_{p,B_R(0)}(f_\pm) \le c\left(1 + \frac{2(1+\delta)}{(1-\delta)}\eta\right).$$

Moreover, by equation (3.8), we have that

$$\frac{\|\mathcal{G}f_\pm\|_{L^p(B_R(0))}}{\|\mathcal{G}f_\pm\|_{L^\infty(B_R(0))}} \le \frac{(1+\delta)\|\mathcal{G}\varphi\|_{L^p(B_R(0))}}{(1-\delta)\|\mathcal{G}\varphi\|_{L^\infty(B_R(0))}} \le \frac{(1+\delta)\|\mathcal{G}\varphi\|_{L^p(\mathbb{R}^2)}}{(1-\delta)\|\mathcal{G}\varphi\|_{L^\infty(\mathbb{R}^2)}},$$

as well as

$$\frac{\|\mathcal{V}_{\varphi'}f_\pm\|_{L^\infty(B_R(0))}}{\|\mathcal{G}f_\pm\|_{L^\infty(B_R(0))}} \le \frac{\|\mathcal{V}_{\varphi'}\varphi\|_{L^\infty(B_R(0))} + \gamma\|\mathcal{V}_{\varphi'}\varphi\|_{L^\infty(\mathbb{R}^2)}}{(1-\delta)\|\mathcal{G}\varphi\|_{L^\infty(B_R(0))}} \le \frac{(1+\delta)\|\mathcal{V}_{\varphi'}\varphi\|_{L^\infty(\mathbb{R}^2)}}{(1-\delta)\|\mathcal{G}\varphi\|_{L^\infty(\mathbb{R}^2)}}.$$

Since the constant $c$ in (3.11) depends monotonically increasingly on (3.10), the above inequalities allow to upper bound the constant $c$ with a constant $c'$ independent of $R$, $a$ and $\gamma$. Hence, we conclude the proof by defining

$$\nu = c'\left(1 + \frac{2(1+\delta)}{(1-\delta)}\eta\right),$$

which is independent of $R$ and $a$. □

3.4. **Discussion of Theorem 3.9.** In this section, we give some insights and discuss possible extensions of our main theorem.

(1) The constant $\nu$ is linked to the stability constant of the Gaussian $\varphi$, which in the result in [9] is estimated by $c(1+\eta)$. The Gaussian $\varphi$ enjoys very strong stability properties for Gabor phase retrieval and the class $M_\nu(B_R)$ with our choice of $\nu$ has stability properties close to that of $\varphi$.

(2) It is worth observing that while Question 3.1 is stated for $\Omega = \mathbb{R}^2$, Theorem 3.9 is proved for arbitrary large balls $B_R$, with $R > 1$. This restriction originates from the bounds on the Poincaré constant. However, Theorem 3.9 shows that for every sampling rate $a > 0$, we can construct functions $f_\pm$ satisfying

$$c_{p,B_R}(f_\pm) \le \nu,$$

with $R > 1/a$. The condition $R > 1/a$ implies that the ball $B_R$ encloses the two bumps of $|\mathcal{G}f_\pm|^p$ and consequently all the features that may affect the local stability constants $c_{p,\mathbb{R}^2}(f_\pm)$. For this reason, it seems plausible to conjecture that $c_{p,\mathbb{R}^2}(f_\pm)$ may also be bounded by a constant $\nu'$ independent of the sampling rate $a$. A proof of this final argument would allow us to fully answer Question 3.1. We do however believe that this is a minor technicality which does not affect the value of our contribution.

(3) We can extend Theorem 3.9 to general lattices of the form $\Lambda = L\mathbb{Z}^2$, $L \in \mathrm{GL}_2(\mathbb{R})$: Given a lattice $\Lambda$, there exist $a > 0$ and $\theta \in \mathbb{R}$ such that $\Lambda \subseteq R_\theta(\mathbb{R} \times a\mathbb{Z})$, where $R_\theta$ denotes the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We can therefore adapt the proof of Theorem 3.9 to the functions

$$f_\pm^\theta(t) = \mathcal{F}_{-\theta}f_\pm(t),$$

where $\mathcal{F}_{-\theta}\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ denotes the fractional Fourier transform of order $-\theta$. It holds that $f_\pm^\theta \in M^p(\mathbb{R})$, $f_+^\theta \not\sim f_-^\theta$ and

$$|\mathcal{G} f_+^\theta(x,\omega)| = |\mathcal{G} f_-^\theta(x,\omega)|, \quad (x,\omega) \in R_\theta(\mathbb{R} \times a\mathbb{Z}) \supseteq \Lambda.$$

Furthermore, we can directly compute that

$$|\mathcal{G} f_\pm^\theta(x,\omega)| = |\mathcal{G} f_\pm(R_{-\theta}(x,\omega))|, \quad (x,\omega) \in \mathbb{R}^2. \tag{3.12}$$

The reader may consult [3, 8] for the detailed proofs of the above facts. We see from equation (3.12) that $|\mathcal{G} f_\pm^\theta|$ is the result of a rotation of $|\mathcal{G} f_\pm|$ in the time-frequency plane. Thus, it follows by equation (2.5) that the root sets of $\mathcal{G} f_\pm^\theta$ are

$$\left\{ R_\theta \left( \frac{1}{2a} - \frac{a \log \gamma}{\pi}, \pm \frac{a}{2} + 2ak \right) \,:\, k \in \mathbb{Z} \right\}.$$

Therefore, given $R > 1$ and $a > 0$, the condition

$$\gamma < \mathrm{e}^{-\frac{\pi}{a}\left(R - \frac{1}{2a}\right)}$$

ensures that all the roots of $\mathcal{G} f_\pm^\theta$ fall outside the strip $R_\theta((-R,R) \times \mathbb{R})$ in the time-frequency plane. With this, it is easy to see that analogous arguments as in the proofs of Corollary 3.7 and Theorem 3.9 apply to $|\mathcal{G} f_\pm^\theta|$.

(4) A natural question is whether a real-valuedness assumption on the signals combined with a uniform bound on the local Lipschitz constant would restore uniqueness from samples. We expect the answer to this question to be negative. In fact, we can use the results in [8] to construct the counterexamples

$$g_\pm = \varphi \pm \mathrm{i}\gamma \,\mathrm{M}_{\frac{1}{a}}\, \varphi \mp \mathrm{i}\gamma \,\mathrm{M}_{-\frac{1}{a}}\, \varphi, \quad \gamma > 0.$$

By [8, Theorem 3.13] the functions $g_\pm$ are real-valued, do not agree up to global phase and satisfy

$$|\mathcal{G} g_+(x,\omega)| = |\mathcal{G} g_-(x,\omega)|, \quad (x,\omega) \in a\mathbb{Z} \times \mathbb{R}.$$

Therefore, we see that for any given rectangular lattice $\Lambda$, we can construct real-valued functions which are arbitrarily close to the Gaussian, do not agree up to global phase but have Gabor transform magnitudes agreeing on $\Lambda$.

## 4. Directions of instability

This section is a general discussion regarding the connection between instability of phase retrieval and Laplacian eigenfunctions. For the purpose of this discussion, we assume that $\Omega \subset \mathbb{C}$ is a bounded domain and that we have two holomorphic functions $F_1, F_2 : \Omega \to \mathbb{C}$ where we assume for the sake of simplicity that $|F_1| > 0$ on all of $\Omega$. The main question to be discussed is as follows: If

$$|F_1| \approx |F_2| \text{ on most of } \Omega, \text{ does this imply that } F_1 \approx e^{i\alpha} F_2$$

on most of $\Omega$? Phrased differently: If two holomorphic functions share the same modulus over a large region, does this imply that one is a global phase-shift of the other? We observe that, throughout this section, the considerations do not invoke the short-time Fourier transform and are more generally applicable.

*Remark* 4.1. In this section, in contrast to the sections before, we will consider the classical weighted Poincaré constant, i.e. equation (3.3) is satisfied for all $F \in W^{1,p}(\Omega, w)$.

4.1. **Foreword.** This subsection may be understood as a short discussion of some of the ingredients in [9] and will set the stage for our subsequent argument. We write

$$\inf_{\alpha \in \mathbb{R}} \|F_1 - \mathrm{e}^{\mathrm{i}\alpha} F_2\|_{L^2(\Omega)}^2 = \inf_{\alpha \in \mathbb{R}} \int_\Omega \left| \frac{F_2(z)}{F_1(z)} - \mathrm{e}^{\mathrm{i}\alpha} \right|^2 |F_1(z)|^2 \,\mathrm{d}z$$

and thus by defining the measure $\mathrm{d}\mu = |F_1(z)|^2 \,\mathrm{d}z$, we have

$$\inf_{\alpha \in \mathbb{R}} \|F_1 - \mathrm{e}^{\mathrm{i}\alpha} F_2\|_{L^2(\Omega)}^2 = \inf_{\alpha \in \mathbb{R}} \int_\Omega \left| \frac{F_2(z)}{F_1(z)} - \mathrm{e}^{\mathrm{i}\alpha} \right|^2 \,\mathrm{d}\mu(z).$$

We can think of the measure $\mu$ as inducing a conformal change of the metric. Assuming $|F_1|$ is sufficiently well behaved, this allows us to interpret the quantity as the $L^2$-norm of a function on a manifold. We now define the real-valued function

$$h(z) = \left| \frac{F_2(z)}{F_1(z)} - \mathrm{e}^{\mathrm{i}\alpha} \right|, \qquad z \in \Omega.$$

At this point, we can invoke the Poincaré inequality (cf. Definition 3.2) and argue that

$$\int_\Omega h(z)^2 \,\mathrm{d}\mu(z) \le \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \,\mathrm{d}\mu(z) \right)^2 + C_{\mathrm{poinc}}(2, \Omega, w)^2 \int_\Omega |\nabla h(z)|^2 \,\mathrm{d}\mu(z),$$

where the Poincaré constant is given by

$$C_{\mathrm{poinc}}(2, \Omega, w)^2 = \frac{1}{\lambda_1}$$

and $\lambda_1$ is the first nontrivial eigenvalue of the Laplace operator on the manifold $(\Omega, \mu)$ equipped with Neumann boundary condition. Hölder's inequality immediately implies that

$$\frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \,\mathrm{d}\mu(z) \right)^2 \le \int_\Omega h(z)^2 \,\mathrm{d}\mu(z)$$

with equality if and only if $h$ is constant. In the setting of phase retrieval problems considered in this paper, we are mainly interested in the setting where the domain naturally decouples into several subdomains $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ such that on $\Omega_i$ we have $F_2(z) \sim F_1(z) e^{\alpha_i}$. We observe that in this setting $h$ is not close to a constant globally unless the $\alpha_i$ are all close to each other (corresponding, in essence, to being close to a unified global phase shift). We also note that if $f : \Omega \to \mathbb{C}$ is analytic in $z_0 \in \Omega$, then

$$|\nabla |f(z_0)|| = |f'(z_0)|$$

which follows immediately from recalling that the Cauchy–Riemann equations can be geometrically stated as saying that infinitesimal balls are mapped to infinitesimal balls. Therefore, applying this twice,

$$\int_\Omega |\nabla h(z)|^2 \,\mathrm{d}\mu(z) = \int_\Omega \left| \left( \frac{F_2(z)}{F_1(z)} \right)' \right|^2 \,\mathrm{d}\mu(z) = \int_\Omega \left| \nabla \left| \frac{F_2(z)}{F_1(z)} \right| \right|^2 \,\mathrm{d}\mu(z)$$

from which we infer

$$\inf_{\alpha \in \mathbb{R}} \| F_1 - \mathrm{e}^{\mathrm{i}\alpha} F_2 \|_{L^2(\Omega)}^2 \le \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \,\mathrm{d}\mu(z) \right)^2 + C_{\mathrm{poinc}}(2, \Omega, w)^2 \int_\Omega \left| \nabla \left| \frac{F_2(z)}{F_1(z)} \right| \right|^2 \,\mathrm{d}\mu(z).$$

If the first term on the right-hand side were to be large, then this would imply that $h$ is typically not small from which we immediately infer that $F_1 \approx \mathrm{e}^{\mathrm{i}\alpha} F_2$ cannot be true over a large region. So we may henceforth assume that the first term is small. This leaves us with the second term: If the integral were to be large, then this would be a quantitative measure indicating that $|F_1| \approx |F_2|$ is not true on most of the domain $\Omega$. However, there is one remaining possibility: It is quite conceivable that the integral is also quite small but that would then require that $C_{\mathrm{poinc}}(2, \Omega, w)$ is quite large which in turn implies that $\lambda_1$ is quite small.

4.2. **A toy example.** An example is given in Figure 3: The classical "dumbbell" example is a two-dimensional manifold comprised of two separate regions that are connected via a thin "bridge". One way of seeing that the Poincaré constant for this example is large is to show that $\lambda_1$ is small: Recall that

$$\lambda_1 = \inf_{\substack{f \in C^\infty(\Omega) \\ \int_\Omega f \,\mathrm{d}\mu = 0}} \frac{\int_\Omega |\nabla f|^2 \,\mathrm{d}\mu}{\int_\Omega |f|^2 \,\mathrm{d}\mu}.$$

By taking $f$ to be constant on the left-hand side and right-hand side of the domain and by interpolating linearly in between, we see that $|\nabla f|$ is not necessarily small but that the region over which it is actually nonzero has rather small measure. By making the "bridge" thinner, we can make $\lambda_1$ as small as possible.

The work of Cheeger then implies that the manifold $(\Omega, \mu)$ can be separated into two distinct parts. Since $\mu = |F_1|^2$, this simply means that $|F_1|$ becomes rather small in some regions and this causes the classical and familiar obstruction for phase retrieval: Indeed, when trying to do successful phase retrieval of a function whose information is stored on two separate regions and the function is close to 0 in between, it becomes very

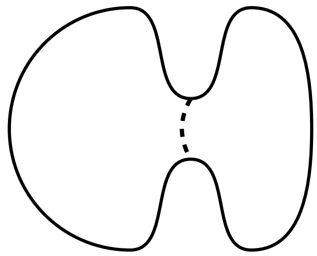


FIGURE 3. An example of a manifold $(\Omega, \mu)$ isometrically embedded into $\mathbb{R}^2$: This example corresponds to the case where $|F_1(z)|^2$ is large on two separate regions and small everywhere else (including in the area connecting the two regions). This is the prototypical example of a domain for which $C_{\text{poinc}}(2, \Omega, w)$ is large.

difficult to reconstruct the phase at once: Each of the regions may come with a different phase shift. Now, one could wonder whether this is indeed the only obstruction.

4.3. **A refinement.** Our main new idea will be to refine the inequality, valid for all real-valued $h \in W^{1,2}(\Omega, \mu)$,

$$\int_\Omega h(z)^2 \, \mathrm{d}\mu(z) \leq \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \, \mathrm{d}\mu(z) \right)^2 + C_{\text{poinc}}(2, \Omega, w)^2 \int_\Omega |\nabla h(z)|^2 \, \mathrm{d}\mu(z).$$

To this end, we introduce a sequence of Laplacian eigenfunctions: These are solutions of

$$\begin{aligned} -\Delta u_k &= \lambda_k u_k \qquad \text{inside } (\Omega, \mu), \\ \frac{\partial u_k}{\partial n} &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}$$

where

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

is a discrete sequence of eigenvalues and $n$ is the normal derivative. These eigenfunctions form an orthonormal basis of $L^2(\Omega, \mu)$ and therefore

$$\int_\Omega h(z)^2 \, \mathrm{d}\mu(z) = \sum_{k=0}^{\infty} (h, u_k)^2 = \left\| \sum_{k=0}^{\infty} (h, u_k) \, u_k \right\|_{L^2(\Omega,\mu)}^2 .$$

Since $u_0 = 1/\sqrt{\mu(\Omega)}$ is a constant function, we have

$$(h, u_0)^2 = \left( \int_\Omega \frac{h(z)}{\sqrt{\mu(\Omega)}} \, \mathrm{d}\mu(z) \right)^2 = \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \, \mathrm{d}\mu(z) \right)^2$$

which explains the first term. Moreover, assuming some minimal boundary conditions on $h$ (which do not matter if $(\Omega, \mu)$ has no boundary) and using integration by parts, we can write

$$\begin{aligned} \int_\Omega |\nabla h(z)|^2 \, \mathrm{d}\mu(z) &= \int_\Omega (\nabla h(z), \nabla h(z)) \, \mathrm{d}\mu(z) = \int_\Omega (-\Delta h(z), h(z)) \, \mathrm{d}\mu(z) \\ &= \sum_{k=1}^{\infty} \lambda_k \, (h, u_k)^2 . \end{aligned}$$

This clearly establishes the inequality

$$\int_\Omega h(z)^2 \, \mathrm{d}\mu(z) \leq \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \, \mathrm{d}\mu(z) \right)^2 + \frac{1}{\lambda_1} \int_\Omega |\nabla h(z)|^2 \, \mathrm{d}\mu(z).$$

However, it also establishes a little bit more: We see that equality in this inequality can only happen when $h$ is actually the first Laplacian eigenfunction. The same argument immediately implies the following refinement.

**Proposition 4.2.** Let $k \in \mathbb{N}$ and let

$$\pi_k : L^2(\Omega, \mu) \to \operatorname{span}\{u_j \,:\, 1 \le j \le k\}$$

denote the orthogonal projection. Then, for all real-valued $h \in W^{1,2}(\Omega, \mu)$, we have

$$\int_\Omega h(z)^2 \,\mathrm{d}\mu(z) \le \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \,\mathrm{d}\mu(z) \right)^2 + \frac{1}{\lambda_1} \|\nabla \pi_k h\|_{L^2(\Omega,\mu)}^2 + \frac{1}{\lambda_{k+1}} \int_\Omega |\nabla h(z)|^2 \,\mathrm{d}\mu(z).$$

*Proof.* Let $h \in W^{1,2}(\Omega, \mu)$ be real-valued. We can write

$$h = (h, u_0)\, u_0 + \sum_{j=1}^{k} (h, u_j)\, u_j + \sum_{j=k+1}^{\infty} (f, u_j)\, u_j.$$

We note that this is an orthogonal decomposition into three mutually orthogonal subspaces. The first term gives the constant contribution. As for the other two terms, we note that

$$\|h\|_{L^2(\Omega,\mu)}^2 = \frac{1}{\mu(\Omega)} \left( \int_\Omega h(z) \,\mathrm{d}\mu(z) \right)^2 + \sum_{j=1}^{k} (h, u_j)^2 + \sum_{j=k+1}^{\infty} (h, u_j)^2 .$$

By orthogonality, we have

$$\sum_{j=1}^{k} (h, u_j)^2 = \sum_{j=1}^{k} (\pi_k h, u_j)^2$$

and thus, arguing as above,

$$\sum_{j=1}^{k} (\pi_k h, u_j)^2 \le \frac{1}{\lambda_1} \|\nabla \pi_k h\|_{L^2(\Omega,\mu)}^2.$$

The same line of reasoning implies

$$\sum_{j=k+1}^{\infty} (h, u_j)^2 \le \frac{1}{\lambda_{k+1}} \int_\Omega |\nabla h(z)|^2 \,\mathrm{d}\mu(z).$$

□

This refinement immediately shows that the Poincaré constant describes the arising terms correctly if and only if the function $h(z)$, given by

$$h(z) = \left| \frac{F_2(z)}{F_1(z)} - \mathrm{e}^{\mathrm{i}\alpha} \right|,$$

is actually proportional to the first Laplacian eigenfunction. If that is not the case, then there is an immediate gain. Conversely, reversing the direction of the argument, Laplacian eigenfunctions give us profiles of instability.

4.4. **Applying the refinement.** The reason why this refinement is useful is that in many of the classically encountered cases (say, the dumbbell spectrogram shown in Figure 3), there are only relatively few small Laplacian eigenvalues. Indeed, returning to the dumbbell manifold, we see that it only has one arbitrarily small eigenvalue while the other eigenvalues depend on the shape of the two regions but are bounded away from 0 as the "bridge" gets thinner and thinner. This allows us to apply the refinement for $k = 1$ and to use $\lambda_2 \gtrsim 1$ to argue that phase retrieval gets more difficult but only up to one particular profile of instability *which is exactly given by the first Laplacian eigenfunction.* More generally, the refinement shows that

1. (1) if phase retrieval is ill-posed, then this can be seen from the presence of small eigenvalues of the Laplacian.
2. (2) Moreover, each eigenfunction corresponding to a small eigenvalue can be interpreted as one particular profile of instability.
3. (3) Since the sequence of eigenvalues $\lambda_k$ of the Laplacian grow and are unbounded, the vector space spanned by the profiles of instability is finite-dimensional.

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## References


1. Rima Alaifari and Philipp Grohs, *Phase retrieval in the general setting of continuous frames for Banach spaces*, SIAM Journal on Mathematical Analysis **49** (2017), no. 3, 1895–1911.
2. ———, *Gabor phase retrieval is severely ill-posed*, Applied and Computational Harmonic Analysis **50** (2021), 401–419.
3. Rima Alaifari and Matthias Wellershoff, *Phase retrieval from sampled Gabor transform magnitudes: counterexamples*, Journal of Fourier Analysis and Applications **28** (2021), no. 9, 1–8.
4. ———, *Uniqueness of STFT phase retrieval for bandlimited functions*, Applied and Computational Harmonic Analysis **50** (2021), 34–48.
5. Bartłomiej Dyda and Moritz Kassmann, *On weighted Poincaré inequalities*, Annales Academiae Scientiarum Fennicae Mathematica **38** (2013), 721–726.
6. Karlheinz Gröchenig, *Foundations of time-frequency analysis*, first ed., Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA, 2001.
7. Philipp Grohs and Lukas Liehr, *Injectivity of Gabor phase retrieval from lattice measurements*, arXiv:2008.07238 [math.FA], August 2020.
8. ———, *On foundational discretization barriers in STFT phase retrieval*, Journal of Fourier Analysis and Applications **28** (2022), no. 39, 1–21.
9. Philipp Grohs and Martin Rathmair, *Stable Gabor phase retrieval and spectral clustering*, Communications on Pure and Applied Mathematics **72** (2019), no. 5, 981–1043.
10. V. Ganapathy Iyer, *A note on integral functions of order 2 bounded at the lattice points*, Journal of the London Mathematical Society **s1-11** (1936), no. 4, 247–249.
11. A. Pfluger, *On analytic functions bounded at the lattice points*, Proceedings of the London Mathematical Society **s2-42** (1937), no. 1, 305–315.
12. Zdeněk Průša and Nicki Holighaus, *Phase vocoder done right*, 2017 25th European Signal Processing Conference (EUSIPCO) (Kos, Greece), IEEE, August 2017, pp. 976–980.
13. Joachim Toft, *Continuity properties for modulation spaces, with applications to pseudo-differential calculus — I*, Journal of Functional Analysis **207** (2004), no. 2, 399–429.
14. Matthias Wellershoff, *Injectivity of sampled Gabor phase retrieval in spaces with general integrability conditions*, arXiv:2112.10136 [math.FA], December 2021.


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