Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

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Research Report No. 2022-22
May 2022

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
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Funding SNF: 200021--200307

Topological classification of Floquet metamaterials based on Floquet theory*

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Abstract

Being driven by the goal of finding edge modes and of explaining the occurrence of edge modes in the case of time-modulated metamaterials in the high-contrast and subwavelength regime, we analyse the topological properties of Floquet-Lyapunov decompositions of periodically parameterized time-periodic ordinary differential equations $\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{T}^d}$. In fact, our main goal being the question whether an analogous principle as the bulk-boundary correspondence of solid-state physics is possible in the case of Floquet metamaterials, i.e., subwavelength high-contrast time-modulated metamaterials. This article is a first step in that direction. Since the bulk-boundary correspondence states that topological properties of the bulk materials characterize the occurrence of edge modes, we dedicate this article to the topological analysis of subwavelength band structures of Floquet metamaterials. This work should thus be considered as a basis for further investigation on whether topological properties of the bulk materials are linked to the occurrence of edge modes. The subwavelength band structures being described by a periodically parameterized time-periodic ordinary differential equation $\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{T}^d}$, we put ourselves in the general setting of periodically parameterized time-periodic ordinary differential equations and introduce a way to (topologically) classify a Floquet-Lyapunov decomposition F, P of the associated fundamental solution $\{X_\alpha(t) = P(\alpha,t)\exp(tF_\alpha)\}_{\alpha\in\mathbb{T}^d}$. This is achieved by analysing the topological properties of the Lyapunov transformation $P(\alpha,t)$ and of the Floquet exponent matrix F_α. The corresponding topological invariants can then be applied to the setting of Floquet metamaterials. In this paper these general results are considered in the case of a hexagonal structure. We show that the topological properties of a time-modulated hexagonal structure can be reduced to precisely one of the introduced topological invariants. This result is followed by two modulation examples which are topologically non-trivial showing that topologically non-trivial subwavelength band structures exist in the case of a hexagonal structure.

Mathematics Subject Classification (MSC2000): 35J05, 35C20, 35P20, 74J20

Keywords: edge mode, subwavelength quasifrequency, time-modulation, Floquet metamaterial, topological invariants, hexagonal structure

Contents

*This work was supported in part by the Swiss National Science Foundation grant number 200021–200307.
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1 Motivation and introduction

The study of topological properties in periodic physical systems is one of the most active fields in solid-state physics. The topological invariants are the building elements of topological band theory since they imply the presence of non-trivial bulk topologies, giving rise to topologically protected edge modes [8, 9, 10, 24, 30, 38, 36, 46]. Topological properties of electronic structures have been mathematically studied in the setting of the Schrödinger operator [11, 13, 15, 16, 17, 18, 27]. Topological invariants have been defined to capture the crystal's properties. Then, if part of a crystalline structure is replaced with an arrangement that is associated with a different value of this invariant, not only will certain frequencies be localized to the interface but this behaviour will be stable with respect to imperfections. These eigenmodes are known as *edge modes* and we say that they are *topologically protected* to refer to their robustness. This principle is known as the *bulk-boundary correspondence* in quantum settings [11, 12, 20, 21, 22, 30, 34]. Here, the term *bulk* is used to refer to parts of a crystal that are away from an edge.

Taking inspiration from quantum mechanics, subwavelength topological photonic and phononic crystals, based on locally resonant crystalline structures with large material contrasts, have been studied both numerically, experimentally, and mathematically in [4, 5, 14, 28, 29, 31, 33, 37, 39, 43, 44, 45, 47]. Subwavelength crystals (called also metamaterial structures) allow for the manipulation and localization of waves on very small spatial scales and are therefore very useful in physical applications, especially in situations where the operating wavelengths are very large. Recently, this field has experienced tremendous advances by exploring the novel and promising area of time-modulations. Research on *Floquet metamaterials* (or time-modulated metamaterials) aims to explore new phenomena arising from the temporal modulation of the material parameters of the structure. It has enabled to open new paradigms for the manipulation of wave-matter interactions in both spatial and temporal domains [6, 19, 40, 23, 25, 26, 32, 35, 41, 40, 42]. To the best of our knowledge, the mathematical analysis of wave propagation properties of Floquet metamaterials has just been started. In [6], a discrete characterization of the band structure of Floquet metamaterials is introduced. This characterization provides both theoretical insight and efficient numerical methods to compute the dispersion relationship of time-dependent structures. A study of exceptional points in the case of Floquet metamaterials is conducted in [7]. Furthermore, in [1, 2] the possibility of achieving non-reciprocal wave propagation and unidirectional edge modes in Floquet metamaterials is proven.

Being interested in edge modes, this article aims at being a first step in the direction of classification of edge modes in the case of periodically time-modulated metamaterials in the high-contrast and subwavelength regime. As said before, the bulk-boundary correspondence of solid-state physics states that the occurrence (and number) of edge modes (i.e., boundary modes) between two different bulk materials is given by the difference of topological invariants of the two bulk materials. Hence, in order to classify the occurrence of edge modes, one first step is to study the possible topological invariants of the bulk materials. Our main goal being the complete analogy of the bulk-boundary correspondence in the case of Floquet metamaterials, i.e., time-modulated high-contrast subwavelength metamaterials. We start our journey at the introduction of a topological classification of the Floquet-Lyapunov decomposition of the subwavelength solutions to the wave equation (with time-modulated coefficients) in Floquet metamaterials. The results of this analysis are presented in this article.

In [6], it is shown that the subwavelength band structures of periodically time-modulated metamaterials are described by a second order linear ordinary differential equation (lODE). Using this result in the case of infinite periodic crystals, it follows that the subwavelength band structures are described by a periodically parameterized lODE $\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{T}^d}$ with time-periodic coefficient matrix $A_\alpha(t)$ and parameter space $\mathbb{T}^d \cong \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ (d-times) denoting the d-dimensional torus. This motivated the study of general periodically parameterized periodic lODEs of the form

$$\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{T}^d}, \tag{1}$$

where A_α is periodic with respect to time t, say with a period T. Fixing $\alpha \in \mathbb{T}^d$ it is possible to apply the Floquet-Lyapunov theory to the lODE (1), which yields that the fundamental solution $X_\alpha(t)$ to $\frac{d}{dt}X = A_\alpha(t)X$ can be written as the product

$$X_\alpha(t) = P_\alpha(t)\exp(tF_\alpha), \tag{2}$$

where $P_\alpha(t)$ is periodic with respect to time t with period T and F_α is a matrix with the same dimensions as $A_\alpha(t)$. These results are only applicable for every $\alpha \in \mathbb{T}^d$ separately and do not tell us about the dependence of $P_\alpha(t)$ and F_α on $\alpha \in \mathbb{T}^d$. We will show that, under the condition that $A(\alpha, t) := A_\alpha(t)$ is continuous in α, it is possible to choose $P_\alpha(t)$ and F_α in a way such that P and F are also continuous and periodic with respect to α.

Based on this factorisation $P_\alpha(t)\exp(tF_\alpha)$ of $X_\alpha(t)$, we will introduce three different topological invariants. Under the assumption that $A(\alpha, t)$ will be such that $\int_0^T A(\alpha, t)dt$ is locally C^1-diagonalizable (a notion introduced in this article), it will be possible to continuously parameterize the eigenvalues $\Lambda(\alpha) = (\lambda_1(\alpha), \dots, \lambda_N(\alpha))$ and eigenspaces $\eta(\alpha) = (\eta_1(\alpha), \dots, \eta_N(\alpha))$ of the Floquet exponent matrix F_α. However, this can not always be achieved with respect to $\alpha \in \mathbb{T}^d$, but with respect to $\beta \in \tilde{\mathbb{T}}^d$ where $\beta \in \tilde{\mathbb{T}}^d \longmapsto \zeta(\beta) = \alpha \in \mathbb{T}^d$ is a covering map; see Subsection 2.2.1. The covering map ζ will give us a first topological invariant, called *Type I.a topological invariant*. Another topological invariant (*Type I.b homotopy class*) will be given by the homotopy class of the eigenspaces of F_α and a third one will be given by the homotopy class of $(\alpha, t) \longmapsto P_\alpha(t)$ and will be named *Type II homotopy class*.

Reinterpretation of the factorisation of Floquet-Lyapunov will make the naming clearer. In fact, instead of stating the results of Floquet-Lyapunov as being a factorisation of the fundamental solution X_α, one can equivalently state that a time-periodic lODE $\frac{d}{dt}X = A(t)X$ can be transformed by an invertible periodic matrix $P(t)$ (which is periodic with the same periodicity as the coefficient matrix $A(t)$), such that the resulting lODE is a constant lODE

$$\frac{d}{dt}Y(t) = FY(t).$$

This is why we understand all topological properties associated to F_α as corresponding to effects which are already present in the case of static lODEs (that is in the case of lODEs with coefficients which are constant with respect to time). Those topological effects will share the part *Type I* in their name. Topological effects associated to P are considered as not occurring in the case of static lODEs and correspond to the introduction of a (periodic) time-modulation of the system, the corresponding topological invariant will be called *Type II*. The precise definitions and a rigorous introduction of the different topological invariants and their names will be presented in Section 2.

To underline the naming of the different topological invariants, we will prove in Section 3 that the Type I.a topological invariant and the Type I.b homotopy class are the only ones relevant in the case of static (non time-dependent) periodically parameterized lODEs. Furthermore, we will see in the same section that the dimension d of the parameterization space $\mathbb{T}^d$ restricts the variety of topological effects. Indeed the Type I.b and Type II homotopy classes are only of interest in the case of a parameter space $\mathbb{T}^d$ of dimension at least 2, the Type I.a topological invariant always being relevant.

In Section 4, we will finally come back to our initial motivation: the topological classification of subwavelength band structures of time-modulated high-contrast metamaterials. High-contrast metamaterials, admissible time-modulation and subwavelength band structures as defined in [6] will be introduced. We will also present the description of subwavelength band structures of time-modulated high-contrast metamaterials which was proven in the same article. This description is given in terms of a periodically parameterized time-periodic lODE which will make our theory applicable to the setting of subwavelength band structures of high-contrast time-modulated metamaterials.

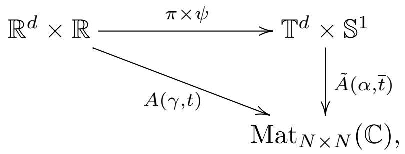
We will apply those results to the setting of a hexagonal structure (see Figure 1), similar to the crystalline structure of graphene. In the case of a hexagonal structure, the symmetry of the static system will allow to reduce from an a priori two-dimensional parameter space to a one-dimensional parameter space. The results of the former section will then imply that the topological properties of the Floquet-Lyapunov decomposition of the subwavelength band structures are uniquely described by the Type I.a topological invariant. Indeed, we can find non-trivial instances of the Type I.a topological invariant in the case of a hexagonal structure, showing that the Type I.a topological invariant fully classifies the Floquet-Lyapunov decomposition of the subwavelength solutions in the setting of periodically time-modulated high-contrast hexagonal structures in the subwavelength regime.

2 Topological classification of periodically parameterized lODEs

In this section, the topological invariants associated to a Floquet-Lyapunov decomposition of a general periodically parameterized lODE will be derived and defined. To this end, the following subsection (Subsection 2.1) specifies the setting of the analysis. It is followed by an overview subsection (Subsection 2.2) which briefly introduces the topological invariants covered in this section. It can serve as a reference to those readers not interested in the mathematical details or the derivations. Hereafter, detailed mathematical analysis will be given in Subsections 2.3 and 2.4 which are dedicated to the derivation and definition of the introduced topological invariants.

2.1 Setting

Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C}), (\gamma, t) \to A(\gamma, t)$ be a continuously differentiable function, which is periodic with respect to $(\gamma, t) \in \mathbb{R}^{d+1}$. That is, there is a maximal lattice $L \subset \mathbb{R}^d$ and a positive number $T > 0$, such that translations by lattice vectors of $L \times m\mathbb{Z}$ leave A invariant. Denoting by $\mathbb{T}^d$ the d-dimensional torus given by $\mathbb{R}^d/L$ and by $\mathbb{S}^1$ the circle $\mathbb{R}/T\mathbb{Z}$, the function A can be factored in the following way



where $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is given by $\pi(x) = x \mod L$ and $\psi : \mathbb{R} \to \mathbb{S}^1$ is given by $\psi(t) = t \mod T$. In what follows, the map $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ and the associated map $\tilde{A} : \mathbb{T}^d \times \mathbb{S}^1 \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ will both be denoted by A. It will be distinguished between A and $\tilde{A}$ by using different notation for the argument, that is, $A(\gamma, t)$ will indeed denote $A(\gamma, t)$ where A is given by $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ with $\gamma \in \mathbb{R}^d$ and $t \in \mathbb{R}$, $A(\alpha, \bar{t})$ will denote $\tilde{A}(\alpha, \bar{t})$ with $\alpha \in \mathbb{T}^d$, $\bar{t} \in \mathbb{S}^1$ and $A(\alpha, t)$ will denote $\tilde{A}(\alpha, \bar{t})$ where $\bar{t} = t \mod T$ with $\alpha \in \mathbb{T}^d$ and $t \in \mathbb{R}$.

In the following the topological invariants associated to the parameterized fundamental solution of the parameterized lODE

$$\left\{ \frac{d}{dt} X = A(\alpha, t) X \right\}_{\alpha \in \mathbb{T}^d} \tag{3}$$

will be defined and investigated. To this end, the family of fundamental solution $\{X_\alpha\}_{\alpha \in \mathbb{T}^d}$ to the parameterized lODE (3) will be understood as a function

$$\begin{aligned} X : \mathbb{T}^d \times \mathbb{R} &\longrightarrow \mathrm{GL}_N(\mathbb{C}) \\ (\alpha, t) &\longmapsto X_\alpha(t), \end{aligned}$$

where, for each $\alpha \in \mathbb{T}^d$ and each $t \in \mathbb{R}$, the matrix $X_\alpha(t)$ denotes the fundamental solution of the lODE $\frac{d}{dt} X_\alpha = A(\alpha, t) X_\alpha$ evaluated at $t \in \mathbb{R}$. More precisely, $X_\alpha(t)$ is given by

$$X_\alpha(t) = \exp\left(\int_0^t A(\alpha, s) ds \right), \tag{4}$$

which makes it clear that $X : \mathbb{T}^d \times \mathbb{R} \to \mathrm{GL}_N(\mathbb{C})$ is well defined and continuously differentiable. In the following it will also be assumed, that $X_\alpha(T)$ its diagonalizable for all $\alpha \in \mathbb{T}^d$, such that for each $\alpha \in \mathbb{T}^d$ there exists a neighborhood $W \subset \mathbb{T}^d$ of α and continuously differentiable functions $\lambda_1, \dots, \lambda_N : W \to \mathbb{C}$ and $\eta_1, \dots, \eta_N : W \to \mathbb{CP}^{N-1}$, such that for all $\alpha' \in W$ the N-tuple $(\lambda_1(\alpha'), \dots, \lambda_N(\alpha'))$ gives the eigenvalues of $X_{\alpha'}(T)$ and $(\eta_1(\alpha'), \dots, \eta_N(\alpha'))$ the respective eigenspaces. We will call this property *local C^1-diagonalizability.*[1]

2.2 Overview

In this subsection a brief overview of the topological invariants introduced in this article will be given. The starting point for the introduced topological invariants is the theory of Floquet and Lyapunov on

[1] In order to avoid pathologies in the case of eigenvalues with geometric multiplicity greater than 1, one should further assume that $\mathrm{span}_{\mathbb{C}}(\eta_1, \dots, \eta_N) = \mathbb{C}^N$, that is, that the eigenspaces $(\eta_1, \dots, \eta_N)$ span the whole space $\mathbb{C}^N$.

lODEs with time-periodic coefficients. In fact, the Floquet-Lyapunov decomposition will allow for a separate topological characterization of the time-local and the time-global behavior of the fundamental solution to a periodically parametrized lODE with periodic coefficients.

The Floquet-Lyapunov decomposition is a decomposition of the fundamental solution of lODEs $\frac{dX}{dt} = A(t)X$ with periodic coefficient matrix $A(t) \in \mathrm{Mat}_{N\times N}(\mathbb{C})$. Indeed, the fundamental solution of a periodic lODE can be decomposed into a periodic and into an exponential part, that is the fundamental solution $X(t)$ can be written as $X(t) = P(t)\exp(Ft)$, where $P(t) : \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ is periodic with the same periodicity as A and $F \in \mathrm{Mat}_{N\times N}(\mathbb{C})$ is a constant matrix. In the setting of a parameterized differential equation $\{\frac{dX}{dt} = A_\alpha(t)X\}_{\alpha\in\mathbb{T}^d}$, it is possible to choose a periodic Floquet-Lyapunov decomposition $X_\alpha(t) = P_\alpha(t)\exp(tF_\alpha)$, where $P : \mathbb{T}^d \times \mathbb{R} \to \mathrm{GL}_N(\mathbb{C})$ and $F : \mathbb{T}^d \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ are continuous (periodic) functions. The choice of a continuous, periodic Floquet-Lyapunov decomposition made in this article is given by

$$F(\alpha) := \frac{1}{T}\int_0^T A(\alpha,t)dt,$$
$$P(\alpha,t) := X_\alpha(t)\exp(-tF(\alpha)).$$

In order to analyse the topological properties of $X_\alpha(t)$, we will distinguish between two types, *Type I* and *Type II*. Topological properties associated to the Floquet exponent matrix F_α will be called Type I topological effects and topological effects associated to the Lyapunov transform $P_\alpha(t)$ will be called Type II topological effects.

From a physical point of view, the Floquet exponent matrix F_α encodes the global behavior of $X_\alpha(t)$ in the time domain and thus, Type I topological effects can be interpreted as to characterise the topological properties of the *global* behavior in the time domain of the fundamental solution $X_\alpha(t)$. From this viewpoint the *time-local* behavior of the solutions to the lODE (3) is then encoded in the *Type II* topological effects. These are topological effects and properties which are associated to the family of differential equations $\{\frac{dX}{dt} = A_\alpha(t)X\}_{\alpha\in\mathbb{T}^d}$ via the corresponding Lyapunov transformation $P_\alpha(t)$. These effects characterize the topological behavior of the fundamental solution during one (time) period and occur due to the periodic time-dependence of the coefficient matrix $A_\alpha(t)$. They are thus novel effects which do not occur in the setting of a time-independent lODE. *Type I* topological effects however, do already occur in the setting of a time-independent parametrized lODEs $\{\frac{d}{dt}X = A_\alpha X\}_{\alpha\in\mathbb{T}^d}$.

In the setting of a parameterized differential equation $\{\frac{dX}{dt} = A_\alpha(t)X\}_{\alpha\in\mathbb{T}^d}$ and under the assumption that $\int_0^T A(\alpha,t)dt$ is locally C^1-diagonalizable with respect to $\alpha \in \mathbb{T}^d$, the eigenvalues and eigenspaces of the exponential part F_α can be chosen *locally* to depend continuously differentiably on $\alpha \in \mathbb{T}^d$. However, their local representatives *need not* give rise to globally defined periodic functions on the whole torus $\mathbb{T}^d$.

Nevertheless, they will give rise to globally defined continuously differentiable functions on $\mathbb{R}^d$. Actually, one can find a lattice $\tilde{L} \subset L \subset \mathbb{R}^d$ and a covering map $\zeta : \tilde{\mathbb{T}}^d := \mathbb{R}^d/\tilde{L} \to \mathbb{T}^d, \beta \mapsto \zeta(\beta)$ such that the eigenvalues and eigenspaces of $F_{\zeta(\beta)}$ depend continuously and periodically on $\beta \in \tilde{\mathbb{T}}^d$.

The covering map $\zeta : \tilde{\mathbb{T}}^d \to \mathbb{T}^d$ will give rise to one of the topological invariants which are defined in this article. It will be called *Type I.a* topological invariant.

The following subsections will give a very brief overview on how the different topological invariants are defined, neither focusing on proofs nor on mathematical justifications.

2.2.1 Type I topological invariants

Given a time-periodic and parameterized first order lODE

$$\left\{\frac{dX}{dt} = A_\alpha(t)X\right\}_{\alpha\in\mathbb{T}^d} \tag{5}$$

with temporal period $T > 0$, let $\{X_\alpha(t)\}_{\alpha\in\mathbb{T}^d}$ denote the corresponding family of fundamental solutions. Assuming that $\int_0^T A_\alpha(t)dt \in \mathrm{Mat}_{N\times N}(\mathbb{C})$ is locally C^1-diagonalizable for all $\alpha \in \mathbb{T}^d$, *two* different kinds of Type I topological invariants can be defined.

By local C^1-diagonalizability of $\int_0^T A_\alpha(t)dt : \mathbb{T}^d \to \mathrm{Mat}_{N\times N}(\mathbb{C})$, it holds that the eigenvalues of $\int_0^T A_\alpha(t)dt \in \mathrm{Mat}_{N\times N}(\mathbb{C})$ can be continuously differentiably parameterized with respect to $\gamma \in \mathbb{R}^d$. This gives rise to the continuously differentiable function

$$\begin{array}{llll} \Lambda : & \mathbb{R}^d & \longrightarrow & \mathbb{C}^N \\ & \gamma & \longmapsto & (\lambda_1(\gamma),\ldots,\lambda_N(\gamma)), \end{array}$$

where $\lambda_1(\gamma), \dots, \lambda_N(\gamma)$ are the eigenvalues of $\int_0^T A_\gamma(t)dt$. Despite the periodic dependence of $\int_0^T A_\alpha(t)dt$ on $\alpha \in \mathbb{T}^d$, Λ might *not* correspond to a continuous function with respect to $\alpha \in \mathbb{T}^d$. However, it corresponds to a continuous function on a unique (up to isomorphism) minimal *finite* subcover $\tilde{\mathbb{T}}^d \to \mathbb{T}^d$ of the universal cover $\mathbb{R}^d \to \mathbb{T}^d$, which will be denoted by $\zeta : \tilde{\mathbb{T}}^d \to \mathbb{T}^d$, such that (the corresponding quotient map)

$$\begin{array}{llll} \Lambda : & \tilde{\mathbb{T}}^d & \longrightarrow & \mathbb{C}^N \\ & \beta & \longmapsto & (\lambda_1(\beta), \dots, \lambda_N(\beta)) \end{array}$$

is a continuously *periodic* differentiable function. The automorphism group $\mathrm{Aut}(\tilde{\mathbb{T}}^d \to \mathbb{T}^d)$ will then determine the *Type I.a topological invariant* of the periodically parameterized lODE (5).

Furthermore a parameterization of the associated eigenspaces $\eta_1, \dots, \eta_N$ will give N continuous periodic functions

$$\eta_n : \quad \tilde{\mathbb{T}}^d \longrightarrow \mathbb{CP}^{N-1},$$

where $\mathbb{CP}^{N-1}$ denotes the $(N-1)$-dimensional complex projective space. Each η_n belongs to an associated homotopy class in $[\tilde{\mathbb{T}}^d, \mathbb{CP}^{N-1}] \cong [\mathbb{T}^d, \mathbb{CP}^{N-1}]$. Grouping all N functions $\eta_1, \dots, \eta_N$ into a single function η and associating the corresponding Homotopy Class in $[\mathbb{T}^d, \mathbb{CP}^{N-1}]^N$ modulo changing the order of the eigenspaces, one can define the *Type I.b homotopy class*.

2.2.2 Type II topological invariants

Given a continuously parameterized Floquet exponent matrix $F : \mathbb{T}^d \to \mathrm{Mat}_{N\times N}(\mathbb{C})$, the Lyapunov transform $P(\alpha, t)$ is uniquely defined and given by

$$\begin{array}{llll} P : & \mathbb{T}^d \times \mathbb{S}^1 & \longrightarrow & \mathrm{GL}_N(\mathbb{C}) \\ & (\alpha, \bar{t}) & \longmapsto & X_\alpha(t) \exp(-tF_\alpha), \end{array}$$

which is a continuous and periodic function with respect to $\alpha \in \mathbb{T}^d$.

Being interested in the homotopy class of $P \in [\mathbb{T}^{d+1}, \mathrm{GL}_N(\mathbb{C})]$, one can reduce $P(\alpha, t)$ to its unitary part $U(\alpha, t)$ using the polar decomposition of invertible matrices. In fact, the homotopy class of P is uniquely identified by the homotopy class of U in $[\mathbb{T}^{d+1}, \mathrm{U}(N)]$. Using the diffeomorphism

$$\begin{array}{lll} \mathrm{U}(N) & \cong & \mathbb{S}^1 \times \mathrm{SU}(N) \\ W & \longmapsto & \left(\det(W), \begin{pmatrix} | & | & & | \\ \frac{1}{\det(W)} W^{(1)} & W^{(2)} & \dots & W^{(3)} \\ | & | & & | \end{pmatrix}\right), \end{array}$$

where $W^{(n)}$ denotes the nth column of W, we will prove that the homotopy class of

$$\det(U) \in [\mathbb{T}^{d+1}, \mathbb{S}^1]$$

is always trivial and thus obtain that the homotopy class of P is uniquely characterized by

$$\begin{pmatrix} | & | & & | \\ \frac{1}{\det(U)} U^{(1)} & U^{(2)} & \dots & U^{(3)} \\ | & | & & | \end{pmatrix} \in [\mathbb{T}^{d+1}, \mathrm{SU}(N)].$$

This homotopy class will be called *Type II homotopy class* associated to the periodically parameterized lODE (5).

2.3 Continuous, periodic parameterization of Floquet exponents and Floquet modes

In the following subsections, the topological invariants presented in the previous subsection will be derived. To this end, the starting point is a continuous and periodic choice of a Floquet-Lyapunov decomposition and a continuous and periodic parameterization of the Floquet exponents and Floquet modes. The aim of this subsection is thus to set a coherent choice of continuous and periodic Floquet-Lyapunov decomposition and to establish a parameterization of the Floquet exponents and Floquet modes. The subsequent subsection will then build upon these results and will rigorously introduce the topological invariants which were already anticipated in the previous overview subsection.

Floquet-Lyapunov theory gives the existence of a factorization of the fundamental solution $X(t)$ to a periodic lODE $\frac{d}{dt}X = A(t)X$ which is given by

$$X(t) = P(t)\exp(tF),$$

where $P(t)$ is periodic with respect to the same period as the coefficient matrix $A(t)$ of the lODE and F is a constant matrix. Since the choice of F and P is non-unique, we make the consistent choice of

$$F := \frac{1}{T}\int_0^T A(t)dt$$

which uniquely determines P, which is then given by

$$P(t) := X(t)\exp(-tF).$$

Furthermore, one can define Floquet exponents and what we will call Floquet modes. Provided that F is diagonalizable, the *Floquet exponents* are given by the eigenvalues $\lambda_1, \dots, \lambda_N$ of F. The respective eigenvectors $v_1, \dots, v_N$ will then be called *Floquet modes.*[2]

Note that, when applying these definitions to a constant lODE with A being constant, one obtains that $F = A$ and $P(t) = \mathrm{Id}_{N\times N}$. Stated differently, F captures the constant component of $A(t)$, that is, it is given by the constant Fourier coefficient A_0 of $A(t) = \sum_{m\in\mathbb{Z}} A_m \exp(\frac{2\pi i m t}{T})$. Hence, in order to investigate the effect of the periodic time-dependence of the coefficient $A(t)$ on the topological nature of the fundamental solution $X(t)$, one needs to analyse the Lyapunov transformation $P(t)$.

Applying the above definition to the setting of the periodically parameterized lODE (5), one obtains the following lemma.

Lemma 1. *Let $T > 0$ and let $A : \mathbb{T}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be continuously differentiable and T-periodic with respect to the time variable t. Then*

$$F_\alpha := \frac{1}{T}\int_0^T A(\alpha, t)dt, \tag{6}$$

and

$$P_\alpha(t) := X_\alpha(t)\exp(-tF_\alpha), \tag{7}$$

where $X_\alpha(t)$ denotes the fundamental solution of $\frac{d}{dt}X = A(\alpha, t)X$, is a Floquet-Lyapunov decomposition of $X_\alpha(t)$ which depends continuously on $\alpha \in \mathbb{T}^d$.

Proof. From the definition, it is clear that F_α and thus also $P(\alpha, t)$ depend continuously on $\alpha \in \mathbb{T}^d$. It thus remains to prove that the choices (6) and (7) indeed give a Floquet-Lyapunov decomposition, that is, that $P(\alpha, t)$ is periodic with respect to time. To this end, let $t \in \mathbb{R}$ and $m \in \mathbb{Z}$. Then

$$P(\alpha, t + mT) = X_\alpha(t + mT)\exp(-(t + mT)F_\alpha).$$

Making a change of variables, one obtains $Y_\alpha(t) = X_\alpha(t + mT)$, where Y_α satisfies the following initial value problem:

$$\begin{cases} \frac{d}{dt}Y &= A_\alpha(t)Y, \\ Y_\alpha(0) &= X_\alpha(mT). \end{cases} \tag{8}$$

From ODE theory, it follows that the initial value problem (8) has the unique solution

$$Y_\alpha(t) = \exp\left(\int_0^t A_\alpha(s)ds\right)X_\alpha(mT),$$

which can also be written as

$$Y_\alpha(t) = X_\alpha(t)X_\alpha(mT).$$

[2]The Floquet modes usually denote the basis of solutions $\{X(t)v_n\}_{1\le n\le N}$ of a periodic lODE $\frac{d}{dt}X = A(t)X$.

Since $A(\alpha, t)$ is T-periodic with respect to t, it can be written as a Fourrier expansion $A(\alpha, t) = \sum_{m' \in \mathbb{Z}} A_{m'}(\alpha) \exp\left(\frac{2\pi i m' s}{T}\right)$. Using this notation and using the fact that

$$\int_0^{mT} \sum_{m' \in \mathbb{Z}} A_{m'}(\alpha) \exp\left(\frac{2\pi i m' t}{T}\right) dt = mTA_0(\alpha),$$

it follows that

$$X_\alpha(mT) = \exp(mTA_0(\alpha))$$

which in turn satisfies

$$\exp(mTA_0(\alpha)) = \exp(mTF_\alpha).$$

Since mTF_α and tF_α commute, we conclude that

$$\begin{aligned} P(\alpha, t + mT) &= X_\alpha(t + mT) \exp(-(t + mT)F_\alpha) \\ &= X_\alpha(t) \exp(mTF_\alpha) \exp(-(t + mT)F_\alpha) \\ &= X_\alpha(t) \exp(-tF_\alpha) \\ &= P(\alpha, t). \end{aligned}$$

□

The following definition will be necessary for the construction of a continuous differentiable parameterization of the Floquet exponents and Floquet modes.

Definition 1 (Local C^1-diagonalizability)**.** *A matrix-valued function $M : \mathbb{R}^d \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ is* locally C^1-diagonalizable *if the following holds. For all $\gamma \in \mathbb{R}^d$, there exists a neighborhood $W \subset \mathbb{R}^d$ of γ and continuously differentiable functions $\lambda_1, \ldots, \lambda_N : W \to \mathbb{C}$ and $\eta_1, \ldots, \eta_N : W \to \mathbb{CP}^{N-1}$, such that, for all $\gamma' \in W$, the N-tuple $(\lambda_1(\gamma'), \ldots, \lambda_N(\gamma'))$ gives the eigenvalues of $X_{\gamma'}(T)$ and $(\eta_1(\gamma'), \ldots, \eta_N(\gamma'))$ the respective eigenspaces of $M(\gamma')$, such that the $\mathbb{C}$-span of the vector spaces $\eta_1(\gamma'), \ldots, \eta_N(\gamma')$ is equal to* $\mathrm{span}_\mathbb{C}(\eta_1(\gamma'), \ldots, \eta_N(\gamma')) = \mathbb{C}^N$.

The above definition will not only be used for functions on $\mathbb{R}^d$, but also for functions on $\mathbb{T}^d$. In the later case the differential structure on $\mathbb{T}^d$ will be used and the definition is given by the obvious generalisation.[3]

Lemma 2. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function such that*

$$\int_0^T A(\gamma, t) dt$$

is locally C^1-diagonalizable for all $\gamma \in \mathbb{R}^d$. Denote by

$$\Lambda : \mathbb{R}^d \longrightarrow \mathbb{C}^N$$

a map that associates to each $\gamma \in \mathbb{R}^d$ the eigenvalues $\Lambda(\gamma) = (\Lambda_1(\gamma), \ldots, \Lambda_N(\gamma))$ of $\int_0^T A(\gamma, t)dt$. Furthermore, denote by

$$\eta : \mathbb{R}^d \longrightarrow (\mathbb{CP}^{N-1})^N$$

the map that associates to each $\gamma \in \mathbb{R}^d$ the eigenspaces $\eta(\gamma) = (\eta_1(\gamma), \ldots, \eta_N(\gamma))$ of $\int_0^T A(\gamma, t)dt$, such that $\eta_n(\gamma)$ is the eigenspace corresponding to the eigenvalue $\Lambda_n(\gamma)$ and $\mathbb{CP}^{N-1}$ denotes the $(N-1)$-dimensional complex projective space.

Then the following results hold.

(i) The maps $\Lambda \times \eta$ can be chosen to be continuously differentiable;

[3]In fact, in the case of $M : \mathbb{T}^d \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ it is equivalent to require local C^1-diagonalizability for the precomposed function $M \circ \pi$, where $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is a C^1-covering map.

(ii) There is a maximal full-dimensional lattice $\tilde{L} \subset L$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{R}^d & \xrightarrow{\text{mod } \tilde{L}} & \tilde{\mathbb{T}}^d & \xrightarrow{\Lambda\times\eta} & \mathbb{C}^N \times (\mathbb{CP}^{N-1})^N \\ & \searrow^{\pi} & \downarrow & & \downarrow \xi \\ & & \mathbb{T}^d & \xrightarrow{\int_0^T A(\alpha,t)dt} & \mathrm{Mat}_{N\times N}(\mathbb{C}), \end{array}$$

where

$$\begin{aligned} \xi : \mathbb{C}^n \times (\mathbb{CP}^{N-1})^N &\longrightarrow \mathrm{Mat}_{N\times N}(\mathbb{C}) \\ (l, \eta_1', \dots, \eta_N') &\longmapsto V \operatorname{diag}(l) V^{-1} \end{aligned}$$

with

$$V = \begin{pmatrix} | & & | \\ v_1 & \dots & v_N \\ | & & | \end{pmatrix},$$

and v_n being some generator of the one-dimensional vector space $\eta_n' \in \mathbb{CP}^{N-1}$.

Remark 1. *As in the above lemma, let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C}), (\gamma, t) \mapsto A(\gamma, t)$ be a $L \times T\mathbb{Z}$-periodic function. Then, one can equivalently assume that $A_0(\gamma)$ is locally C^1-diagonalizable, where $A_0(\gamma)$ denotes the constant Fourier term of $A(\gamma,t) = \sum_{m\in\mathbb{Z}} A_m(\gamma)\exp(\frac{2\pi i m}{T}t)$, instead of assuming $\int_0^T A(\gamma,t)dt$ to be locally C^1-diagonalizable.*

Proof. Let $l_1, \dots, l_d \in \mathbb{R}^d$ be $\mathbb{R}$-linearly independent vectors which generate L as a $\mathbb{Z}$-module. Define $\mathcal{F} = \{a_1 l_1 + \ldots + a_d l_d | a_m \in [0,1] \text{ for } 1 \le m \le d\}$ to be a simply connected fundamental domain of the lattice $L \subset \mathbb{R}$ with faces

$$\mathcal{F}_m = \{a_1 l_1 + \ldots + a_d l_d \,|\, a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_d \in [0,1] \text{ and } a_m = 0\}.$$

By simply connectedness of $\mathcal{F}$ and by local C^1-diagonalizability, it thus follows that there exist globally defined continuously-differentiable functions $\Lambda \times \eta : \mathcal{F} \to \mathbb{C}^N \times (\mathbb{CP}^{N-1})^N$ which satisfy for all $\gamma \in \mathcal{F}$, that $\Lambda(\gamma)$ are the eigenvalues of $\int_0^T A(\gamma,t)dt$ and $\eta(\gamma)$ the respective eigenspaces. In fact, it is possible to choose $\Lambda \times \eta$ on $\mathcal{F}$ in such a way that for each $m \in \{1, \dots, N\}$ there exists a permutation on $\{1, \dots, N\}$, denoted by $\sigma_m \in S_N$, which satisfies

$$\Lambda_{\sigma_m(n)}(\gamma) \times \eta_{\sigma_m(n)}(\gamma) = \Lambda_n(\gamma + l_m) \times \eta_n(\gamma + l_m) \text{ for all } \gamma \in \mathcal{F}_m.$$

By this way, it is possible to successively glue the respective coordinates of the functions

$$\{\Lambda(\gamma + l) \times \eta(\gamma + l)\}_{l \in L}$$

to obtain a globally defined function $\Lambda \times \eta$ on the whole of $\mathbb{R}^d$, as required in (i). It now remains to prove the existence of the desired lattice $\tilde{L}$. To this end, it suffices to prove that there exists some full-dimensional lattice $\overline{L} \subset L$ which leaves $\Lambda \times \eta$ invariant. Indeed, using the notation from above and denoting by $\mathrm{ord}(\sigma)$ the order of a permutation $\sigma \in S_N$, such a lattice is given by

$$\overline{L} := \{b_1 \operatorname{ord}(\sigma_1) l_1 + \ldots + b_d \operatorname{ord}(\sigma_d) l_d | b_1, \dots, b_d \in \mathbb{Z}\}.$$

□

2.4 Derivation of topological invariants

Given a parameterized lODE

$$\frac{d}{dt} X = A(\alpha, t) X \tag{9}$$

with coefficient matrix $A(\alpha,t)$ as in Lemma 2, let $X_\alpha(T)$ denote the associated parameterized fundamental solution. Then Lemma 2 indicates that the eigenvalues $\lambda_1(\gamma), \dots, \lambda_N(\gamma) \in \mathbb{C}$ and eigenspaces $\eta_1(\gamma), \dots, \eta_N(\gamma) \in \mathbb{CP}^{N-1}$ of $X_\gamma(T)$ can be parameterized continuously differentiably with respect to the (new) variable $\beta \in \tilde{\mathbb{T}}^d$ by finding a suitable full-dimensional maximal sublattice $\tilde{L} \subset L$. The lattice $\tilde{L}$ is unique and thus leads to the following well-defined topological invariant.

Definition 2. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function, such that*

$$\int_0^T A(\gamma, t)dt$$

is locally C^1-diagonalizable for all $\gamma \in \mathbb{R}^d$. Let $\tilde{L} \subset L$ be a maximal lattice as in Lemma 2. Then

$$L/\tilde{L} \cong \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_r\mathbb{Z}$$
$$\text{with} \quad m_1|\ldots|m_r$$

and the Type I.a topological invariant *associated to the lODE (3) is defined as $(m_1, \ldots, m_r)$.*

Remark 2. *In the definition of the Type I.a topological invariant some information of the lattice $\tilde{L}$ is lost, in particular how it is embedded into L. A way to circumvent this inclarity would be to consider the whole lattice $\tilde{L} \subset L$ and how it embeds into L, instead of the r-tuple $(m_1, \ldots, m_r)$.*

Given that the eigenspaces $\eta_1(\alpha), \ldots, \eta_N(\alpha) \in \mathbb{CP}^{N-1}$ are parameterized continuously with respect to $\mathbb{R}/\tilde{L}$, it is natural to consider the associated homotopy class in the space of continuous functions $C^0(\mathbb{R}/\tilde{L}, \mathbb{CP}^{N-1})$.

Definition 3. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function, such that*

$$\int_0^T A(\gamma, t)dt$$

is locally C^1-diagonalizable for all $\gamma \in \mathbb{R}^d$ and let

$$\eta = (\eta_1, \ldots, \eta_N) : \; \mathbb{R}^d \longrightarrow (\mathbb{CP}^{N-1})^N$$

be as in Lemma 2. Then the Type I.b homotopy class *associated to the lODE (3) is defined as the unordered N-tuple of homotopy classes*

$$(\eta_n \in [\mathbb{R}^d/\tilde{L}, \mathbb{CP}^{N-1}])_{1\leq n\leq N}.$$

Up to now, both topological invariants could have been defined in the setting of parameterized linear equations, and both can be used in a more general setting than the setting of linear ordinary differential equations. That is also, why we have decided to call those invariants 'Type I', since they do not rely on the fact that we deal with an ordinary differential equation; see e.g. Lemma 5. The remainder of this section is therefore dedicated to the introduction of a topological invariant which exclusively characterises periodically parametrized *time-dependent* lODEs.

Using Lemma 1, it follows that the Lyapunov transformation $P : \mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z} \to \mathrm{GL}_N(\mathbb{C})$ is continuously differentiable and one can thus analyse its homotopy class in

$$C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{GL}_N(\mathbb{C})).$$

Using polar decomposition of matrices and the fact that the space of positive definite matrices is contractible, it suffices to examine the homotopy class of the unitary part $U(\alpha, t)$ of $P(\alpha, t)$ in

$$C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{U}(N)),$$

where $\mathrm{U}(N)$ denotes the space of $N \times N$-dimensional unitary matrices. Denoting by $\mathrm{SU}(N)$ the space of special unitary $N \times N$-dimensional matrices and using the diffeomorphism

$$\begin{array}{ccc} \mathrm{U}(N) & \longrightarrow & \mathbb{S}^1 \times \mathrm{SU}(N) \\ V & \longmapsto & \left(\det(V), \begin{pmatrix} | & | & & | \\ \frac{1}{\det(V)}V^{(1)} & V^{(2)} & \ldots & V^{(N)} \\ | & | & & | \end{pmatrix}\right), \end{array}$$

the homotopy class of $P \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{GL}_N(\mathbb{C}))$ is uniquely determined by the associated homotopy classes in $C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathbb{S}^1)$ and $C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$. More precisely, the homotopy class of P is uniquely determined by the functions

$$\frac{\det(P(\alpha, \bar{t}))}{|\det(P(\alpha, \bar{t}))|} \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathbb{S}^1)$$

and

$$S(\alpha, \bar{t}) :=$$

$$\begin{pmatrix} | & | & & | \\ \frac{|\det(P)|}{\det(P)} (P(P^*P)^{-\frac{1}{2}})^{(1)} & (P(P^*P)^{-\frac{1}{2}})^{(2)} & \dots & (P(P^*P)^{-\frac{1}{2}})^{(N)} \\ | & | & & | \end{pmatrix} \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N)). \tag{10}$$

The following two lemmas will actually prove that the homotopy class of P is uniquely given by the homotopy class of $S \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$. In fact, the first will characterise the homotopy class of $\det(P)/|\det(P)|$ and the second lemma will then make use of that result to show that the homotopy class of $\det(P)/|\det(P)|$ is trivial, thus proving that the homotopy class of P is uniquely determined by the homotopy class of S.

Lemma 3. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function. Let $P : \mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z} \to \mathrm{GL}_N(\mathbb{C})$ be the Lyapunov transform obtained in Lemma 1 and define*

$$\begin{aligned} \Delta : \quad & \mathrm{GL}_N(\mathbb{C}) && \longrightarrow && \mathbb{S}^1 \\ & M && \longmapsto && \frac{\det(M)}{|\det(M)|}, \end{aligned}$$

$$\begin{aligned} \iota : \quad & \mathbb{S}^1 && \longrightarrow && \mathbb{T}^{d+1} \\ & \bar{t} && \longmapsto && (0, \dots, 0, \bar{t}). \end{aligned}$$

Then the homotopy class of $\Delta \circ P = \frac{\det(P)}{|\det(P)|} \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathbb{S}^1)$ is uniquely determined by $\Delta \circ P \circ \iota$ in $\pi^1(\mathbb{S}^1)$.

Proof. In fact, the homotopy class of a continuous map $f : \mathbb{T}^{d+1} \to \mathbb{S}^1$ is uniquely determined by the homotopy classes of $f \circ \iota_j : \mathbb{S}^1 \to \mathbb{S}^1$, where $\iota_j : \mathbb{S}^1 \hookrightarrow \mathbb{T}^{d+1} \cong (\mathbb{S}^1)^{d+1}$ is the inclusion in the jth factor. Thus the homotopy class of $\Delta \circ P$ is also uniquely determined by the homotopy classes of the maps $\Delta \circ P \circ \iota_1, \dots, \Delta \circ P \circ \iota_{d+1} \in C^0(\mathbb{S}^1, \mathbb{S}^1)$. Since $P(\alpha, 0) = \mathrm{Id}_N$ for all $\alpha \in \mathbb{T}^d$, it follows that the maps $\Delta \circ P \circ \iota_1, \dots, \Delta \circ P \circ \iota_d$ are homotopically trivial and thus the homotopy class of $\Delta \circ P$ is uniquely determined by $\Delta \circ P \circ \iota_{d+1} = \Delta \circ P \circ \iota$. □

Using Lemma 3, one can prove that the homotopy class of $\Delta \circ P$ is actually trivial.

Lemma 4. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function. Let $P : \mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z} \to \mathrm{U}(N)$ be the Lyapunov transform obtained in Lemma 1 and define*

$$\begin{aligned} \Delta : \quad & \mathrm{GL}_N(\mathbb{C}) && \longrightarrow && \mathbb{S}^1 \\ & M && \longmapsto && \frac{\det(M)}{|\det(M)|}. \end{aligned}$$

Then the homotopy class of $\Delta \circ P = \frac{\det(P)}{|\det(P)|} \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathbb{S}^1)$ is always trivial.

Proof. Using Lemma 1, one obtains the following identity for the Lyapunov transform P:

$$P(\alpha, \bar{t}) = X_\alpha(t) \exp\left(-tF_\alpha\right),$$

where the fundamental solution is given by $X_\alpha(t) = \exp\left(\int_0^t A(\alpha, s) ds\right)$. Let

$$A(\alpha, t) = \sum_{m\in\mathbb{Z}} A_m(\alpha) \exp\left(\frac{2\pi i m}{T} t\right)$$

be the Fourier expansion of $A(\alpha, t)$ with respect to time t and period T. Then

$$F_\alpha = \frac{1}{T} \int_0^T A(\alpha, t) dt = A_0(\alpha),$$

where $A_0(\alpha)$ is the 0th Fourier coefficient of $A(\alpha, t)$. Putting everything together one obtains

$$\begin{aligned}\det(P(\alpha,t)) &= \det\left(X_\alpha(t)\exp\left(-tF_\alpha\right)\right)\\ &= \det\left(\exp\left(\int_0^t A(\alpha,s)ds\right)\right)\det\left(\exp\left(-tF_\alpha\right)\right)\\ &= \exp\left(\operatorname{Tr}\left(\int_0^t A(\alpha,s)ds\right)\right)\exp\left(-t\operatorname{Tr}(F_\alpha)\right)\\ &= \exp\left(\operatorname{Tr}\left(\int_0^t \sum_{m\in\mathbb{Z}} A_m(\alpha)\exp\left(\frac{2\pi i m}{T}s\right)ds\right)\right)\exp\left(-t\operatorname{Tr}(A_0(\alpha))\right)\\ &= \exp\left(\int_0^t \operatorname{Tr}\left(\sum_{m\in\mathbb{Z}-\{0\}} A_m(\alpha)\right)\exp\left(\frac{2\pi i m}{T}s\right)ds\right)\\ &= \exp\left(\sum_{m\in\mathbb{Z}-\{0\}} \operatorname{Tr}\left(A_m(\alpha)\right)\int_0^t \exp\left(\frac{2\pi i m}{T}s\right)ds\right)\\ &= \exp\left(\sum_{m\in\mathbb{Z}-\{0\}} \operatorname{Tr}\left(A_m(\alpha)\right)\frac{T\left(\exp\left(\frac{2\pi i m}{T}t\right)-1\right)}{2\pi i m}\right),\end{aligned}$$

where Tr denotes the trace of a matrix and the identity $\det(\exp(B)) = \exp(\operatorname{Tr}(B))$ for $B \in \mathrm{Mat}_{N\times N}(\mathbb{C})$ was used. It thus follows that

$$t \in \mathbb{R} \longmapsto \sum_{m\in\mathbb{Z}-\{0\}} \operatorname{Tr}\left(A_m(\alpha)\right)\frac{T\left(\exp\left(\frac{2\pi i m}{T}t\right)-1\right)}{2\pi i m} \in \mathbb{C}^*$$

is a T-periodic lifting of $\bar{t} \in \mathbb{S}^1 \longmapsto \det(P(\alpha, \bar{t})) \in \mathbb{C}^*$. Hence, $\Delta \circ P \circ \iota \in \pi^1(\mathbb{S}^1)$ is contractible and we conclude from Lemma 3 that the homotopy class of $\Delta \circ P$ is always trivial. □

Remark 3. *This argument (i.e., Lemmas 3 and 4) also holds in the setting where one only considers real-valued $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{R})$. In that case, $P(\alpha, t)$ takes values in $\mathrm{GL}_N(\mathbb{R})$ and one needs to replace $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ by $\mathrm{O}(N)$ and $\mathrm{SO}(N)$ (the spaces of orthogonal and special orthogonal $N \times N$-dimensional matrices), respectively.*

The above lemma implies that the homotopy class of P is uniquely determined by its special unitary part $S(\alpha, t) \in \mathrm{SU}(N)$. We thus set the following definition.

Definition 4. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function. Let P be as in Lemma 1 and let $S(\alpha, \bar{t}) \in C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$ be obtained from $P(\alpha, \bar{t})$ as indicated in equation (10). Then the* Type II homotopy class *associated to the lODE (3) is defined as the homotopy class of S in $C^0(\mathbb{R}^d/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$.*

3 Analysis of topological invariants

This section is dedicated to the analysis of the defined topological invariants under some assumptions on the dimension d of the parameter space $\mathbb{T}^d$ and on the degree of the parameterized lODE.

	Type I.a	Type I.b	Type II
constant/static	$*$	$*$	triv.
param. space $\mathbb{S}^1$	$*$	triv.	triv.

Table 1: This table indicates which topological invariants are necessarily trivial (indicated with triv.*) and in which settings. The attribute 'constant/static' means that the coefficient matrix of the lODE is constant with respect to time $t \in \mathbb{R}$, the attribute 'param. space $\mathbb{S}^1$' means that the parameter space $\mathbb{T}^d$ of the considered lODE (3) is 1-dimensional, that is that $d = 1$ and thus $\mathbb{T}^d = \mathbb{S}^1$. The asterisk ($*$) indicates that the topological invariant might be non-trivial.*

It will turn out that depending on the setting, some of the topological invariants will always be trivial. In particular, we will show the results displayed in Table 1. As already anticipated, the Type II homotopy class is dedicated to the setting of time-dependent lODEs. That is, it is dedicated to lODEs of the form $\frac{d}{dt}X = A_\alpha(t)X$, where the coefficient matrix $A_\alpha(t)$ is not constant with respect to the time variable t. In the case of a constant coefficient matrix, the Type II homotopy class will always be trivial, which will be proven in Lemma 5. The dimension d of the parameter space $\mathbb{T}^d$ of the parameterized lODE $\{\frac{d}{dt}X = A_\alpha(t)X\}_{\alpha\in\mathbb{T}^d}$ also influences the variety of topological properties.

For example, in Lemmas 6 and 7, we will prove that the Type I.b and Type II homotopy classes are trivial whenever the dimension of the parameter space d is equal to 1. In other words, in order to have non-trivial Type I.b or Type II homotopy classes, the parameter space needs to be of dimension $d \geq 2$.

Considering the static case first, where the coefficient matrix $A(\alpha, t) = A(\alpha)$ is constant with respect to time $t \in \mathbb{R}$, the following result holds.

Lemma 5. *Let $A : \mathbb{R}^d \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a L-periodic, continuously differentiable function and let F_α and P be as in Lemma 1. Then the following identity holds:*

$$F_\alpha = A(\alpha) \quad \text{and} \quad P = \mathrm{Id}_N\,.$$

In particular, the Type II homotopy class is trivial.

Proof. Using Lemma 1, one obtains that $F_\alpha = A_0(\alpha)$, where $A_0(\alpha)$ is the constant part of $A(\alpha)$, hence $F_\alpha = A_0(\alpha) = A(\alpha)$. Since in the case of a constant lODE the fundamental solution is given by $X_\alpha(t) = \exp(tA(\alpha))$, using Lemma 1 again one obtains

$$P(\alpha, t) = X_\alpha(t)\exp(-tF_\alpha) = \exp(tA(\alpha))\exp(-tA(\alpha)) = \mathrm{Id}_N\,.$$

Hence, $P(\alpha, t) = \mathrm{Id}_N$ and the Type II homotopy class is trivial. □

The remainder of the following section will be dedicated to the analysis of the case where the lODE $\frac{d}{dt}X = A_\alpha(t)X$ is parameterized by $\alpha \in \mathbb{S}^1$, that is, where the parameter space $\mathbb{T}^d$ is of dimension $d = 1$.

We will see that a one-dimensional parameter space considerably restricts the topological properties of the Floquet-Lyapunov decomposition of the lODE

$$\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{S}^1}.$$

It will be proven that in that case only the Type I.a topological invariant remains of interest in the analysis of the topological nature of the Floquet-Lyapunov decomposition of the fundamental solution $X_\alpha(t) = P(\alpha, t)\exp(F_\alpha)$. In other words, the Type I.b and Type II homotopy classes are automatically trivial whenever the parameter space is equal to $\mathbb{S}^1$. Both results are due to the homotopical properties of $\mathbb{CP}^{N-1}$ and $\mathrm{SU}(N)$, respectively. After proving both results, the Type I.a topological invariant will be considered and its range will be determined. We will determine the precise values attained by the Type I.a topological invariant in the setting of an N-dimensional lODE and we will give constructions of lODEs for every attained value of the Type I.a topological invariant.

Considering the Type I.b homotopy class first, the following result holds.

Lemma 6. *Let $A : \mathbb{R}\times\mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function, such that*

$$\int_0^T A(\alpha, t)dt$$

is locally C^1-diagonalizable for all $\alpha \in \mathbb{S}^1$. Then the associated lODE

$$\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{S}^1\cong\mathbb{R}/L}$$

has trivial Type I.b homotopy class.

Proof. Let

$$\eta = (\eta_1, \dots, \eta_N): \quad \mathbb{R} \longrightarrow (\mathbb{CP}^{N-1})^N$$

be as in Definition 3. That is, let η be a continuous lifting of the eigenspaces of $\int_0^T A_\alpha(t)dt \in \mathrm{Mat}_{N\times N}(\mathbb{C})$. Using the notation from Lemma 1, we recall that the Type I.b homotopy class associated to the lODE $\frac{d}{dt}X = A_\alpha(t)X$ was defined as the unordered N-tuple of homotopy classes

$$\eta_n \in [\mathbb{R}/\tilde{L}, \mathbb{CP}^{N-1}] \cong [\mathbb{S}^1, \mathbb{CP}^{N-1}],$$

with $1 \leq n \leq N$. Since the complex projective space $\mathbb{CP}^{N-1}$ is simply connected for $N \geq 1$, it follows that $[\mathbb{S}^1, \mathbb{CP}^{N-1}]$ consists of precisely one element and thus the Type I.b homotopy class of $\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{S}^1}$ is always trivial. □

The argument for the Type II homotopy class is similar. It relies on the fact that $\mathrm{SU}(N)$ is 1- and 2-connected for all $N \geq 1$.

Lemma 7. *Let $A : \mathbb{R} \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function. Then the associated lODE*

$$\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{S}^1\cong\mathbb{R}/L}$$

has trivial Type II homotopy class.

Proof. Let P be as in Lemma 1 and let $S(\alpha,\bar{t}) \in C^0(\mathbb{R}/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$ be obtained from $P(\alpha,\bar{t})$ as indicated in equation (10). Using the notation from Lemma 1, recall that the Type II homotopy class associated to the lODE $\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{S}^1}$ was defined as the homotopy class of S in $C^0(\mathbb{R}/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$. In the case of a one-dimensional parameter space $\mathbb{S}^1$, the space $C^0(\mathbb{R}/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$ is homeomorphic to $C^0(\mathbb{T}^2, \mathrm{SU}(N))$. However, $\mathrm{SU}(N)$ is 1- and 2-connected for all $n \geq 1$ and thus every continuous function $\mathbb{T}^2 \to \mathrm{SU}(N)$ is homotopic to a constant function. In particular, $S \in C^0(\mathbb{R}/L \times \mathbb{R}/T\mathbb{Z}, \mathrm{SU}(N))$ is always homotopically trivial, or in other words the Type II homotopy class associated to the lODE $\left\{\frac{d}{dt}X = A_\alpha(t)X\right\}_{\alpha\in\mathbb{S}^1\cong\mathbb{R}/L}$ is trivial. □

In the setting of a one-dimensional parameter space $\mathbb{S}^1$, the Type I.a topological invariant is not necessarily trivial. The following lemma gives bounds on its values and the subsequent lemma constructs parameterized lODEs which attain all possible values for the Type I.a topological invariant.

Lemma 8. *Let $A : \mathbb{R} \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function, such that*

$$\int_0^T A(\alpha,t)dt$$

is locally C^1-diagonalizable for all $\alpha \in \mathbb{S}^1$. Let $m \geq 0$ denote the Type I.a topological invariant associated to A. Then

$$m \leq N!\,.$$

Proof. Since $\mathbb{R}$ is simply connected and since $\int_0^T A(\gamma,t)dt$ is locally C^1-diagonalizable for all $\gamma \in \mathbb{R}$, there exists a continuous parameterization of the eigenvalues $\Lambda : \mathbb{R} \to \mathbb{C}^N$. The lattice $\tilde{L}$ from Lemma 2 is then given by those elements $l \in L$ where $\Lambda(l) = \Lambda(0)$. Since at every $l \in L$, the entries of $\Lambda(l)$ are just permutations of the entries of $\Lambda(0)$, the following holds by using the periodicity of $\int_0^T A(\alpha,t)dt$. Let $l \in L$ be the smallest positive element of L. Then $\tilde{L}$ is generated by ml, where m is the order of the permutation σ, which associates the coordinates of the entries of $\Lambda(0)$ to the coordinates of the corresponding entries of $\Lambda(l)$. Since a permutation of N elements has at most order $N!$, it follows that $m \leq N!\,$. □

Remark 4. *In fact, the upper bound on m can be improved since the order of a permutation of N elements is given by the least common multiple of the lengths of the cycles of its disjoint cycle decomposition.*

Lemma 9. *For any natural number m and positive numbers $l, T > 0$, there exist a natural number N and a $l\mathbb{Z} \times T\mathbb{Z}$-periodic, continuously differentiable function $A : \mathbb{R} \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$, such that*

$$\int_0^T A(\gamma,t)dt$$

is locally C^1-diagonalizable for all $\gamma \in \mathbb{R}$, which satisfies that the Type I.a topological invariant associated to A is equal to m.

Proof. Take $N = m$. Then one just needs to construct m functions $f_1, \dots, f_m : \mathbb{R} \to \mathbb{C}$, which satisfy that they take pairwise different values at 0, and such that they fulfill $f_n(1) = f_{n+1}(0)$ and $f'_n(1) = f'_{n+1}(0)$ for all $1 \leq n \leq m$, where $m+1$ is identified with 1. Setting

$$\Lambda(\alpha) = \begin{pmatrix} f_1(\alpha) & & & \\ & f_2(\alpha) & & \\ & & \ddots & \\ & & & f_m(\alpha) \end{pmatrix}$$

and

$$V(\alpha) = \begin{pmatrix} 1+\cos(\frac{2\pi\alpha}{l}) & & & & (-1)^{m+1}(1-\cos(\frac{2\pi\alpha}{l})) \\ 1-\cos(\frac{2\pi\alpha}{l}) & 1+\cos(\frac{2\pi\alpha}{l}) & & & \\ & 1-\cos(\frac{2\pi\alpha}{l}) & \ddots & & \\ & & \ddots & 1+\cos(\frac{2\pi\alpha}{l}) & \\ & & & 1-\cos(\frac{2\pi\alpha}{l}) & 1+\cos(\frac{2\pi\alpha}{l}) \end{pmatrix},$$

then

$$M(\alpha) := V(\alpha)\Lambda(\alpha)V^{-1}(\alpha)$$

is l-periodic with respect to α. Now, choose a coefficient matrix $A(\alpha, t)$ which is $l \times T$-periodic with respect to $(\alpha, t) \in \mathbb{R} \times \mathbb{R}$, and set the constant term $A_0(\alpha) = M(\alpha)$ in its Fourier decomposition with respect to t and period T

$$A(\alpha, t) = \sum_{q \in \mathbb{Z}} A_q(\alpha) \exp\left(\frac{2\pi q}{T} t\right).$$

Then the Type I.a topological invariant of A is equal to m. □

4 Application to high-contrast hexagonal structures

In this section the derived topological invariants and their properties will be applied to the setting of a hexagonal structure, like it is present in graphene. Highly symmetric structures as honeycomb or square lattice structures allow for a reduction of the two-dimensional Brillouin zone to a one-dimensional symmetry curve (see Figure 2). This then reduces the parameterization space from a two dimensional torus $\mathbb{T}^2$ to a one-dimensional circle $\mathbb{S}^1$, which in return limits the variety of homotopic effects, as was displayed in the previous section.

In the following section, we will present the setting for which we will apply the derived topological invariants. In the subsequent section, we will briefly explain the computational procedure we use to compute the Floquet-Lyapunov decomposition, the Floquet exponents and modes. Thereafter, examples of time-modulated honeycomb structures which have non-trivial Type I.a topological invariants will be presented.

When the setting will be presented it will become apparent that we are in the case of a second order lODE which is parameterized by a one-dimensional parameter space $\mathbb{S}^1$. That is, we are in the setting of the last line of Table 1, which says that the topological nature of the Floquet-Lyapunov decomposition of an lODE which is parameterized by a circle is uniquely determined by its Type I.a topological invariant. It thus follows that the Type I.a topological invariant will determine the topological nature of the subwavelength band structure in a hexagonal structure.

4.1 Setting

A metamaterial is a prototype material of the following form. It is composed of two submaterials: the *background* and the *resonator material*. While the background material fills almost the whole space $\mathbb{R}^d$, in this case we will restrict ourselves to the 2-dimensional space $\mathbb{R}^2$, the resonator material only occupies disconnected domains $D_1, \dots, D_N \subset \mathbb{R}^2$, which are repeated periodically with respect to some $\mathbb{R}$-linearly independent lattice vectors $g_1, g_2 \in \mathbb{R}^2$, which generate the lattice $G = g_1\mathbb{Z} \oplus g_2\mathbb{Z}$, such that the domain of the resonator material is given by

$$\mathcal{D} = \bigcup_{g \in G} (g + D_1 \cup \ldots \cup D_N),$$

see e.g. Figure 1 for the case of a hexagonal structure material. The *reciprocal lattice* is then defined as

$$L := \left\{ l \in \mathbb{R}^2 | < l, g > \in 2\pi\mathbb{Z} \text{ for all } g \in G \right\},$$

where $< l, g >$ denotes the standard scalar product in $\mathbb{R}^2$. Background and resonator materials are characterized by their corresponding material parameters ρ and κ which correspond to the density and the bulk modulus in the setting of scalar waves. To be precise, the density ρ and the bulk modulus κ are defined as

$$\kappa(x,t) = \begin{cases} \kappa_0, & x \in \mathbb{R}^2 \setminus \overline{\mathcal{D}}, \\ \kappa_n(t), & x \in g + D_n, \text{ with } n \in \{1, \ldots, N\}, \, g \in G, \end{cases} \tag{11}$$

$$\rho(x,t) = \begin{cases} \rho_0, & x \in \mathbb{R}^2 \setminus \overline{\mathcal{D}}, \\ \rho_n(t), & x \in g + D_n, \text{ with } n \in \{1, \ldots, N\}, \, g \in G. \end{cases} \tag{12}$$

That is, the density ρ and the bulk modulus κ are piecewise constant in space and also in time in the domain of the background material. However, the material parameters are time-dependent inside the resonators. The goal is to study the wave equation with time-dependent coefficients

$$\left(\frac{\partial}{\partial t} \frac{1}{\kappa(x,t)} \frac{\partial}{\partial t} - \nabla_x \cdot \frac{1}{\rho(x,t)} \nabla_x \right) u(x,t) = 0, \quad x \in \mathbb{R}^2, \, t \in \mathbb{R}. \tag{13}$$

To this end we will restrict ourselves to the study of its associated *subwavelength quasifrequencies*, when κ and ρ are T-periodic in time, in which case we understand the following by subwavelength quasifrequencies.

When the wave equation (13) is periodic in time and space, one can apply the Floquet transform with respect to time and space to the wave equation (13). This leads to a parameterized set of problems with restricted solution spaces. Indeed, one obtains

$$\begin{cases} \left(\frac{\partial}{\partial t} \frac{1}{\kappa(x,t)} \frac{\partial}{\partial t} - \nabla_x \cdot \frac{1}{\rho(x,t)} \nabla_x \right) u(x,t) = 0, \\ u(x,t) e^{-i\omega t} \text{ is } T\text{-periodic in } t, \\ u(x,t) e^{-i\alpha t} \text{ is } G\text{-periodic in } x, \end{cases} \tag{14}$$

where ω ranges over the elements of the *time-Brillouin zone* $Y_t^* := \mathbb{C}/(\Omega\mathbb{Z})$ with Ω being the frequency of the material parameters, which is thus given by $\Omega = 2\pi/T$, and α ranges over elements of the *Brillouin zone* $\mathbb{R}^2/L$. If a solution $u(x,t)$ to (14) exists for an $\omega \in Y_t^*$ and $\alpha \in \mathbb{R}^2/L$, then $u(x,t)$ is called a *Bloch solution* and ω its associated *(time-)quasifrequency* and α its associated *(spatial) quasifrequency*.

In order to study *subwavelength* (time-)quasifrequencies, one needs to assume that the *contrast parameter*

$$\delta := \frac{\rho_i(0)}{\rho_0}$$

is small,[4] or, to be precise, one needs to consider solutions to the wave equation (13) as $\delta \to 0$. Assuming

$$\rho(x,t) = \begin{cases} \rho_0, & x \in \mathbb{R}^2 \setminus \overline{\mathcal{D}}, \\ \rho_r \rho_n(t), & x \in g + D_n, \text{ with } n \in \{1, \ldots, N\}, \, g \in G, \end{cases} \tag{15}$$

with $\rho_n(0) = 1$ for all $n = 1, \ldots, N$, one can regard the wave equation (13) as parameterized by the contrast parameter δ and one can consider its solutions as $\delta \to 0$. This is called the *high-contrast regime*. In the setting where the frequency $\Omega = O(\sqrt{\delta})$, subwavelength frequencies are introduced as in [6] and are given by the following definition.

[4] Supposing that $\rho_n(0) = \rho_{n'}(0)$ for all $n, n' \in \{1, \ldots, N\}$.

Definition 5 (Subwavelength quasifrequency)**.** *A quasifrequency $\omega = \omega(\delta) \in Y_t^*$ of* (14) *is said to be a* subwavelength quasifrequency *if there is a corresponding Bloch solution $u(x,t)$, depending continuously on δ, which can be written as*

$$u(x,t) = e^{i\omega t} \sum_{m=-\infty}^{\infty} v_m(x) e^{im\Omega t},$$

where

$$\omega(\delta) \to 0 \in Y_t^* \text{ and } M(\delta)\Omega(\delta) \to 0 \in \mathbb{R} \text{ as } \delta \to 0,$$

for some integer-valued function $M = M(\delta)$ such that, as $\delta \to 0$, we have

$$\sum_{m=-\infty}^{\infty} \|v_m\|_{L^2(D)} = \sum_{m=-M}^{M} \|v_m\|_{L^2(D)} + o(1).$$

In [6], a capacitance matrix formulation to the subwavelength quasifrequencies as $\delta \to 0$ was proven. It describes subwavelength quasifrequencies as solutions to a (finite dimensional) linear system of equations in the setting where the density and bulk modulus are constant with respect to time. In the setting of time-dependent density and bulk modulus, the subwavelength quasifrequencies are described by a periodically parameterized lODE.

In order to state this result, we need the following definitions of the *time-dependent contrast parameters*, *wave speed* and *time-dependent wave speeds*:

$$\delta_n(t) = \frac{\rho_n(t)}{\rho_0}, \quad v_0 = \sqrt{\frac{\kappa_0}{\rho_0}}, \quad v_n(t) = \sqrt{\frac{\kappa_n(t)}{\rho_n(t)}},$$

respectively, with $n = 1, \dots, N$.

Theorem 1. *Being in the high-contrast regime of* (13)*, assume that the material parameter κ is given by* (11)*, that ρ is given by* (15) *and that they satisfy*

$$\frac{1}{\rho_n(t)} = \sum_{m=-M}^{M} r_{n,m} e^{im\Omega t}, \qquad \frac{1}{\kappa_n(t)} = \sum_{m=-M}^{M} k_{n,m} e^{im\Omega t},$$

for some $M(\delta) \in \mathbb{N}$ satisfying $M(\delta) = O\left(\delta^{-\gamma/2}\right)$ as $\delta \to 0$ for some fixed $0 < \gamma < 1$. Furthermore, suppose that the associated time-dependent contrast parameters, wave speed and time-dependent wave speeds satisfy for all $t \in \mathbb{R}$ and $n = 1, \dots, N$,

$$\delta_n(t) = O(\delta), \quad v = O(1), \quad v_n(t) = O(1) \quad \text{as} \quad \delta \to 0,$$

respectively. Then, as $\delta \to 0$, the subwavelength quasifrequencies $\omega(\delta) \in Y_t^*$ to the wave equation (13) *in the high-contrast regime are, to leading order, given by the quasifrequencies of the system of ordinary differential equations in $y(t) = (y_n(t))_n$,*

$$\sum_{m=1}^{N} C_{nm}^{\alpha} y_m(t) = -\frac{|D_n|}{\delta_n(t)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{\delta_n v_n^2} \frac{\mathrm{d}(\delta_n y_n)}{\mathrm{d}t} \right), \tag{16}$$

for $n = 1, \dots, N$, where C^α denotes the capacitance matrix associated to the infinite periodic structure of the considered metamaterial and the spatial quasifrequency $\alpha \in \mathbb{R}^2/L$.

The *capacitance matrix* is a way to encode the geometry of an infinite periodic structure into a square matrix. It has the same dimension as the total number of resonators in a fundamental domain. The capacitance matrix theory in the high-contrast regime was developed in [3]. It was first derived using *Gohberg-Sigal theory* and *layer potential techniques* for the finite structure case, where the resonators $D_1, \dots, D_N$ are not repeated periodically in space, but where the resonator domain $\mathcal{D} \subset \mathbb{R}^3$ is given by $\mathcal{D} = D_1 \cup \ldots \cup D_N$. Then, *Floquet-Bloch theory* allowed to extend the results to infinite structures.

One can rewrite (16) into the following system of Hill equations:

$$\Psi''(t) + M(t)\Psi(t) = 0, \tag{17}$$

where the vector Ψ and the matrix M are defined as

$$\Psi(t) = \left(\frac{\sqrt{\delta_n(t)}}{v_n(t)} y_n(t) \right)_{n=1}^{N}, \quad M(t) = W_1(t) C^\alpha W_2(t) + W_3(t)$$

with W_1, W_2 and W_3 being diagonal matrices with corresponding diagonal entries

$$(W_1)_{nn} = \frac{v_n \delta_n^{3/2}}{|D_n|}, \qquad (W_2)_{nn} = \frac{v_n}{\sqrt{\delta_n}}, \qquad (W_3)_{nn} = \frac{\sqrt{\delta_n} v_n}{2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{(\delta_n v_n^2)^{3/2}} \frac{\mathrm{d}(\delta_n v_n^2)}{\mathrm{d}t},$$

for $n = 1, \dots, N$.

4.2 Computation of continuous parametrization of Floquet exponents and Floquet modes

Since from a computational point of view, it is costly to compute matrix exponentials, one needs to develop another procedure to work around this problem. Since in the case of the Type I topological invariants, one needs to compute the Floquet exponents and Floquet modes, it is convenient to reuse those for the computation of $\exp(tF_\alpha)$ and, more importantly, for the computation of the Lyapunov transform $P(\alpha, t) := X_\alpha(t) \exp(-tF_\alpha)$. The details are given in the following lemma.

Lemma 10. *Let $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ be a $L \times T\mathbb{Z}$-periodic, continuously differentiable function, such that*

$$\int_0^T A(\gamma, t) dt$$

is locally C^1-diagonalizable for all $\gamma \in \mathbb{R}^d$. Let $\tilde{L} \subset L$ be a maximal lattice as in Lemma 2 and let

$$\Lambda : \ \mathbb{R}^d/\tilde{L} \longrightarrow \mathbb{C}^N$$

be a continuously differentiable map that associates to each $\gamma \in \mathbb{R}^d$ the eigenvalues $\Lambda(\gamma) = (\Lambda_1(\gamma), \dots, \Lambda_N(\gamma))$ of $\int_0^T A(\gamma, t)dt$. Furthermore, denote by

$$\eta : \ \mathbb{R}^d \longrightarrow (\mathbb{CP}^{N-1})^N$$

a continuously differentiable map that associates to each $\gamma \in \mathbb{R}^d$ the eigenspaces $\eta(\gamma) = (\eta_1(\gamma), \dots, \eta_N(\gamma))$ of $\int_0^T A(\gamma, t)dt$, such that $\eta_n(\gamma)$ is the eigenspace corresponding to the eigenvalue $\Lambda_n(\gamma)$. Then the (continuously differentiable and periodic) Floquet-Lyapunov decomposition of $X_\alpha(t)$

$$X_\alpha(t) = P(\alpha, t) \exp(tF_\alpha),$$

as in Lemma 1 is given by

$$F_\alpha = \frac{1}{T} \int_0^T A(\alpha, t) dt,$$
$$P(\alpha, t) = X_\alpha(t) \exp(-tF_\alpha)$$

and can be computed using that the Floquet exponent matrix which satisfies

$$F_\alpha = V \operatorname{diag}\left(\frac{1}{T} \Lambda(\alpha) \right) V^{-1}, \tag{18}$$

with

$$V = \begin{pmatrix} | & & | \\ v_1 & \dots & v_N \\ | & & | \end{pmatrix},$$

where v_n is some generator of the vector space $V_n = \eta_n(\alpha)$ for $1 \leq n \leq N$. The Lyapunov transformation can then be computed as

$$P(\alpha, t) = X_\alpha(t) V \operatorname{diag}\left(\exp\left(-\frac{1}{T} \Lambda(\alpha) \right) \right) V^{-1}, \tag{19}$$

where exp *denotes the 'usual' complex exponential function applied to each coordinate of $\Lambda(\alpha)$, and not the matrix analogue.*

Proof. Equation (18) is simply a diagonalization of F_α and equation (19) is due to the fact that

$$\exp(-tF_\alpha) = \exp\left(V \operatorname{diag}\left(\frac{1}{T}\Lambda(\alpha)\right) V^{-1}\right) = V \operatorname{diag}\left(\exp\left(-\frac{1}{T}\Lambda(\alpha)\right)\right) V^{-1}.$$

□

The above lemma is particularly useful, when one wants to compute the Lyapunov transform numerically. In fact, the procedure we chose for this paper is the following. We computed the fundamental solution $X_\alpha(t)$ using an appropriate numerical method, then we estimated the eigenvalues $\xi_1(\alpha), \dots, \xi_N(\alpha)$ with respective eigenvectors $v_1(\alpha), \dots, v_N(\alpha)$ of $X_\alpha(T)$ and parameterized them in a continuously differentiable manner. Since, with the notation from Subsection 2.3, $\Xi(\beta) = (\xi_1(\beta), \dots, \xi_N(\beta))$ relates to $\Lambda(\beta)$ in the following fashion:

$$\Xi(\beta) = \exp\left(\Lambda(\beta)\right), \text{ for } \beta \in \mathbb{R}^N/\tilde{L},$$

we took the appropriate logarithm branch for each $\xi_n(\beta)$ in order to obtain

$$V(\beta)\log(\Xi(\beta))V(\beta)^{-1} = \int_0^T A(\beta, t)dt.$$

Thus choosing

$$\Lambda(\beta) = \log(\Xi(\beta)),$$

one can compute $F(\alpha)$ as

$$F_\alpha = V(\beta)\frac{1}{T}\Lambda(\beta)V(\beta)^{-1}, \text{ where } \alpha = \zeta(\beta)$$

and the Lyapunov transform as

$$P(\alpha, t) = X_\alpha(t)V(\beta)\exp\left(-\frac{t}{T}\Lambda(\beta)\right)V(\beta)^{-1},$$

where it doesn't matter which $\beta \in \mathbb{R}^N/\tilde{L}$ is chosen, as long as $\zeta(\beta) = \alpha$ is satisfied.

4.3 Type I.a homotopic effects

Applying the above theory to the setting of a hexagonal structure one can observe non-trivial instances of the Type I.a topological invariant associated to the corresponding parameterized lODE (17), showing that the Type I.a invariant fully characterises the homotopy class of subwavelength band structures of high-contrast time-dependent metamaterials.

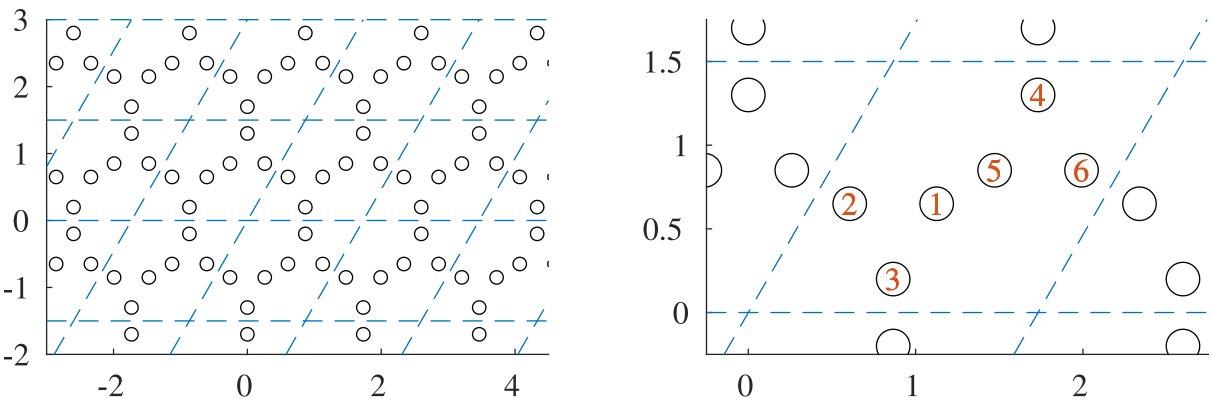


Figure 1: Displayed is the hexagonal structure used in Subsection 4.3. The lattice is displayed with blue dashed lines and the resonators are displayed as black outlined circles. In the right figure, the resonators inside the fundamental domain $\{a_1g_1 + a_2g_2 | a_1, a_2 \in [0, 1)\}$ are displayed and numbered according to the notation used in Subsection 4.3. One can distinguish the two trimers: resonators 1, 2, 3 and resonators 4, 5, 6.

Two structures of non-trivial homotopy type that are of non-trivial Type I.a will be presented in the following section. The *static* structure will in both cases be given by the same structure depicted in Figure 1. It is given by six spherical resonators $D_1, \dots, D_6$ in the fundamental domain $\{a_1 g_1 + a_2 g_2 \,|\, a_1, a_2 \in [0,1)\}$ associated to the lattice vectors

$$g_1 = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \sqrt{3}/2 \\ 3/2 \end{pmatrix}.$$

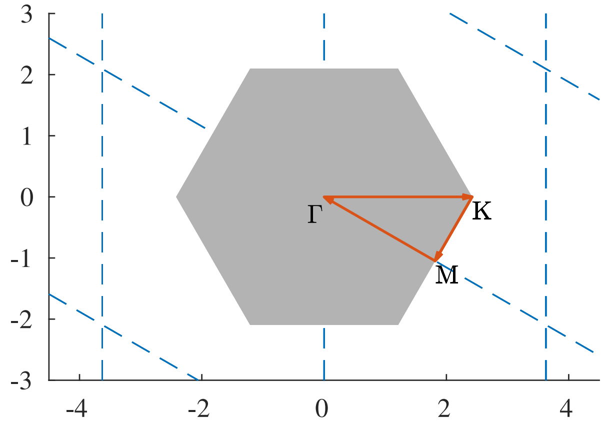


Figure 2: The first Brillouin zone associated to the hexagonal structure used in Subsection 4.3 is colored in grey. On it are displayed the high symmetry points (Γ, K and M) and the (high) symmetry curve, which is indicated with orange arrows. The reciprocal lattice is displayed using blue dashed lines.

The resonators have equal radius $R = 0.1$ and are arranged in two trimer blocks (resonators 1, 2, 3 and resonators 4, 5, 6, respectively). The positions of the centers $c_1, \dots, c_6$ of the six resonators $D_1, \dots, D_6$ in the fundamental domain are give by

$$c_1 = \frac{1}{3}(g_1+g_2) + 3R\begin{pmatrix} \cos(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) \end{pmatrix}, \; c_2 = \frac{1}{3}(g_1+g_2) + 3R\begin{pmatrix} \cos(\frac{5\pi}{6}) \\ \sin(\frac{5\pi}{6}) \end{pmatrix}, \; c_3 = \frac{1}{3}(g_1+g_2) + 3R\begin{pmatrix} \cos(\frac{9\pi}{6}) \\ \sin(\frac{9\pi}{6}) \end{pmatrix},$$
$$c_4 = \frac{2}{3}(g_1+g_2) - 3R\begin{pmatrix} \cos(\frac{9\pi}{6}) \\ \sin(\frac{9\pi}{6}) \end{pmatrix}, \; c_5 = \frac{2}{3}(g_1+g_2) - 3R\begin{pmatrix} \cos(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) \end{pmatrix}, \; c_6 = \frac{2}{3}(g_1+g_2) - 3R\begin{pmatrix} \cos(\frac{5\pi}{6}) \\ \sin(\frac{5\pi}{6}) \end{pmatrix}.$$

In the case of such a highly symmetric static structure it suffices to consider the capacitance matrix formulation on the *(high) symmetry curve* of the reciprocal lattice L; see, for instance, [1]. This reduces the a priori 2-dimensionally parameterized system to a 1-dimensionally parameterized system. The new parameterization set is then given by the piecewise linearly interpolated path through the points Γ, K and M. It is depicted in Figure 2.

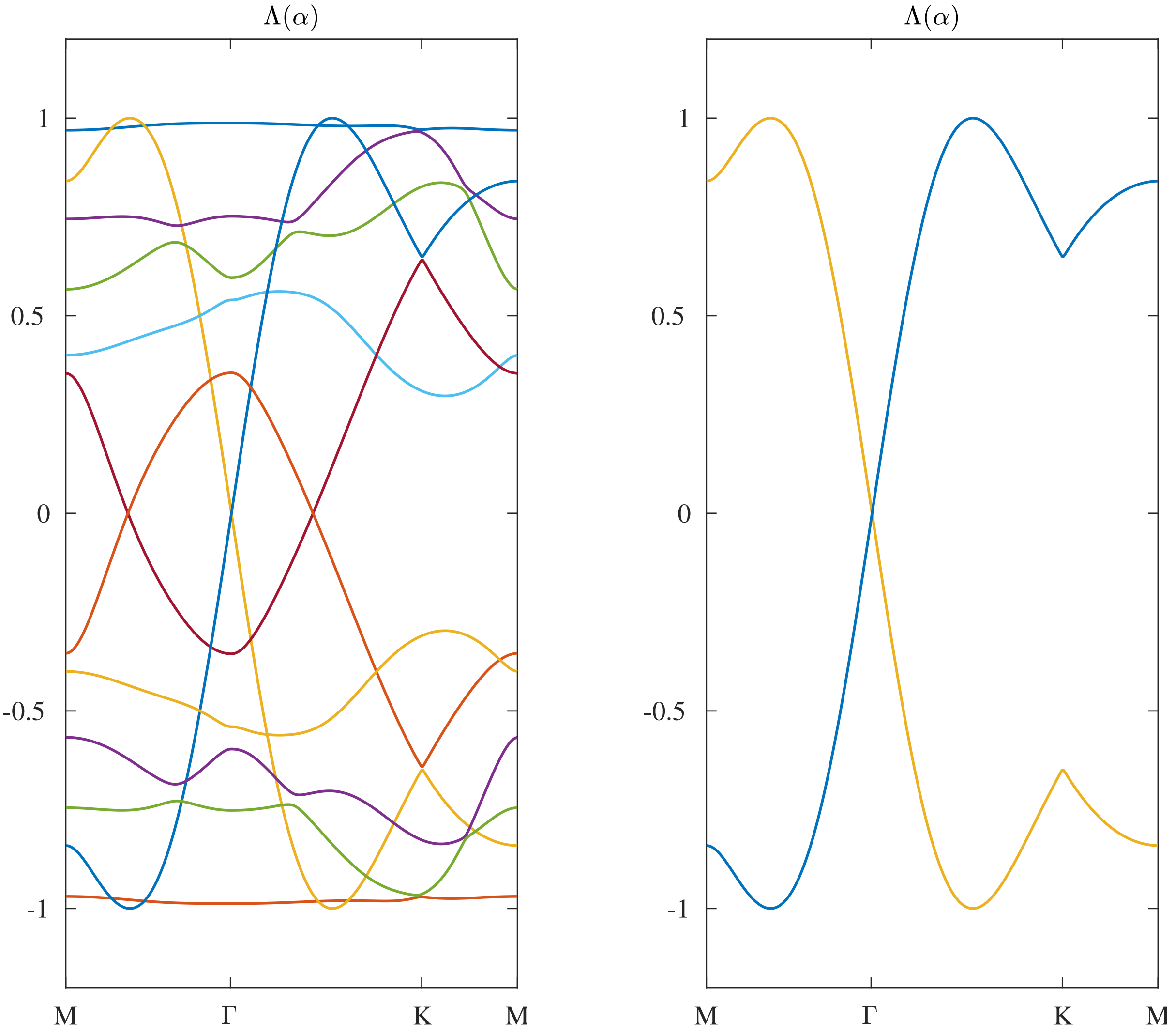
By the results of Section 3, it thus follows that the Type I.b and Type II homotopy classes are always trivial in this case. Hence, the Type I.a topological invariant fully classifies the topological nature of subwavelength band structures associated with time-modulated hexagonal structures. It remains to show that this invariant takes interesting values for some instances of time-modulated hexagonal structures. In the following we will thus present two examples of different instances of the Type I.a topological invariant.

The following modulation of the bulk modulus κ inside the resonators of the above defined material gives an example of a modulated hexagonal structure where the associated Type I.a topological invariant is *non-trivial* and *equal to 2*.

Using the notation from equation (11), the following modulation was used:

$$\kappa_1(t) = 1/(1+\varepsilon\cos(\Omega t)), \; \kappa_2(t) = 1/(1+\varepsilon\cos(\Omega t + 2\pi/3)), \; \kappa_3(t) = 1/(1+\varepsilon\cos(\Omega t + 4\pi/3)), \quad (20)$$
$$\kappa_4(t) = 1/(1+\varepsilon\cos(\Omega t)), \; \kappa_5(t) = 1/(1+\varepsilon\cos(\Omega t + 2\pi/3)), \; \kappa_6(t) = 1/(1+\varepsilon\cos(\Omega t + 4\pi/3)), \quad (21)$$

where $\Omega = 0.2$ and $\varepsilon = 0.001$. The density ρ was not modulated and set to $\rho_0 = 1$ for the background material and equal to $1/9000$ inside the resonators, giving a contrast parameter of $\delta = 1/9000$, ensuring the high-contrast regime.



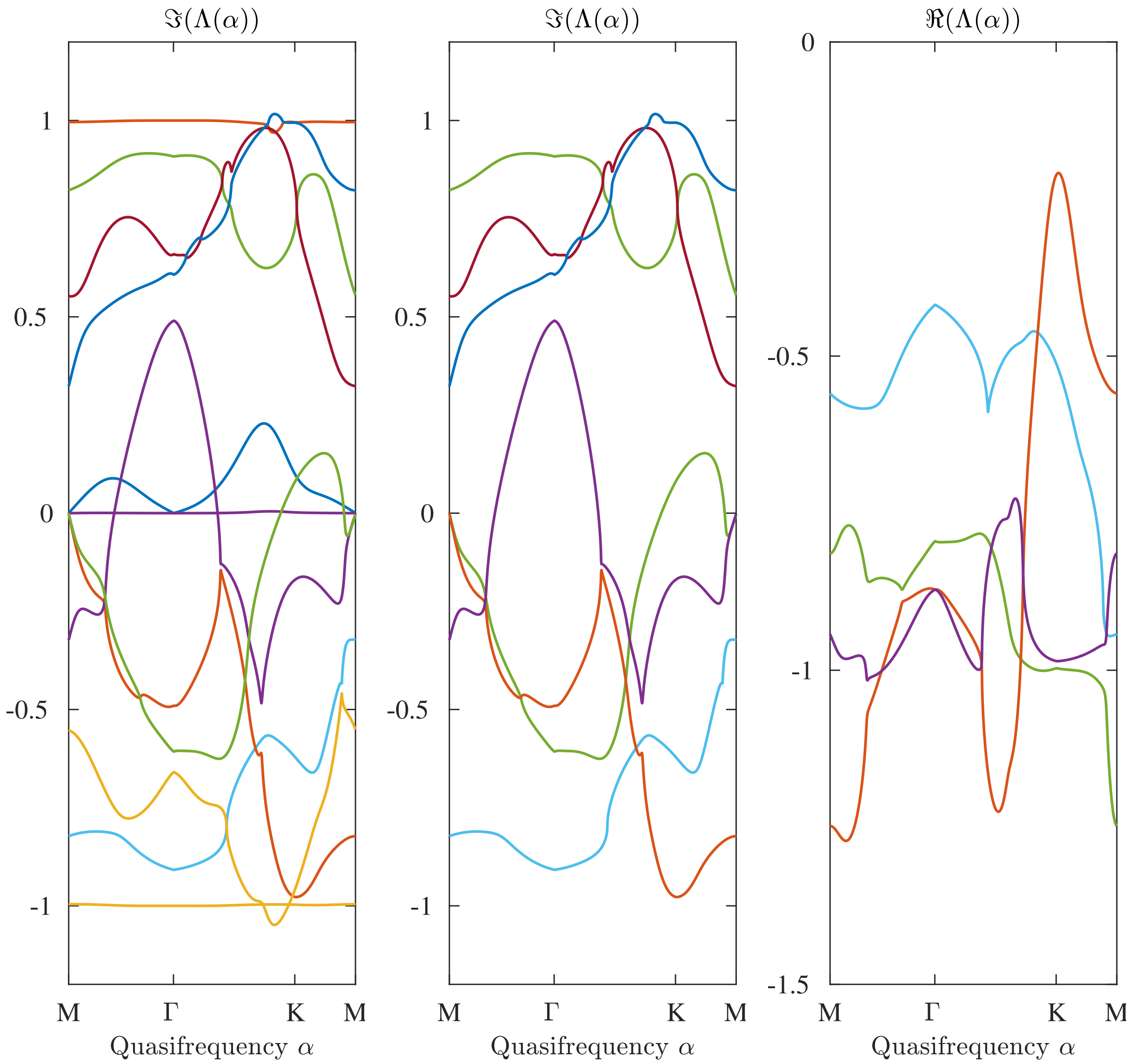
(a) Band structure on the symmetry curve.

(b) Topological non-trivial bands.

Figure 3: Example of a band structure with non-trivial Type I.a topological invariant. The band structure of the hexagonal structure presented in this section with modulation given by equation (20) and step size equal to 1.21×10^{-2}*, which means* 100 *steps between* M *and* Γ.

This setting has the associated band structure depicted in Figure 3a. Most bands can be glued continuously accross the point M. However, one band does not connect continuously (see Figure 3b), but connects with another band which then connects to the former. Thus, the Type I.a topological invariant is given by 2 and the structure is topologically non-trivial.

Next, we will present another example of a time-modulation of the hexagonal structure displayed in Figure 1. This example has the same periodicity in space as well as in time (i.e., the same frequency) as the first example. However, it has three main differences. First, not only the bulk modulus κ but also the density ρ is modulated within the resonators; second, a considerably stronger modulation is used. When in the former example a modulation amplitude of $\varepsilon = 0.001$ was used, in the following example the modulation will be 300-500 times larger, having an amplitude of $\varepsilon_\kappa = 0.5$ for the bulk modulus κ and an amplitude of $\varepsilon_\rho = 0.3$ for the density ρ. Third, the phase shifts of the modulations of the resonators are different.



(a) Imaginary part of band structure on the symmetry curve. *(b) Imaginary part of topological non-trivial bands.* *(c) Real part of lower block of topological non-trivial bands.*

Figure 4: Example of the band structure of a strong modulation with non-trivial Type I.a invariant. The band structure of the time-modulated hexagonal structure with modulation given by equations (22)–(25). The step size was set to 6.05×10^{-3}, which corresponds to 200 steps between M and Γ. On Figure 4a, the imaginary part of all bands are displayed. In Figure 4b only the imaginary part of those bands which are topologically non-trivial are depicted. This is not evident in the case of the green band. For this reason also the real part of the lower group of topologically non-trivial bands is displayed in Figure 4c.

Using the notation from equation (11), the following modulation of the bulk modulus κ was used:

$$\kappa_1(t) = \frac{1}{1+\varepsilon_\kappa \cos(\Omega t)}, \quad \kappa_2(t) = \frac{1}{1+\varepsilon_\kappa \cos\left(\Omega t + \frac{2\pi}{3}\right)}, \quad \kappa_3(t) = \frac{1}{1+\varepsilon_\kappa \cos\left(\Omega t + \frac{4\pi}{3}\right)}, \tag{22}$$

$$\kappa_4(t) = \frac{1}{1+\varepsilon_\kappa \cos\left(\Omega t + \frac{2\pi}{3}\right)}, \quad \kappa_5(t) = \frac{1}{1+\varepsilon_\kappa \cos(\Omega t)}, \quad \kappa_6(t) = \frac{1}{1+\varepsilon_\kappa \cos\left(\Omega t + \frac{4\pi}{3}\right)}, \tag{23}$$

where $\Omega = 0.2$ and $\varepsilon_\kappa = 0.5$. The shape of the modulation is different from the first example, for this example, it is *mirror-symmetric* between the two trimers.

Defining the modulation of the density, using the notation from equation (11), the following modulation of the density ρ was used:

$$\rho_1(t) = \frac{1}{1-\varepsilon_\rho \cos(\Omega t)}, \quad \rho_2(t) = \frac{1}{1-\varepsilon_\rho \cos(\Omega t + 2\pi/3)}, \quad \rho_3(t) = \frac{1}{1-\varepsilon_\rho \cos(\Omega t + 4\pi/3)}, \tag{24}$$

$$\rho_4(t) = \frac{1}{1-\varepsilon_\rho \cos(\Omega t + 2\pi/3)}, \quad \rho_5(t) = \frac{1}{1-\varepsilon_\rho \cos(\Omega t)}, \quad \rho_6(t) = \frac{1}{1-\varepsilon_\rho \cos(\Omega t + 4\pi/3)}, \tag{25}$$

where $\varepsilon_\rho = 0.3$. Note, that also ρ was chosen to be mirror-symmetric between the two trimers.

This choice of modulation leads to the band structure depicted in Figure 4a. Its instance of the Type I.a topological invariant is non-trivial, in fact, it is given by 12. In Figures 4b the two groups of non-trivial bands are displayed. The first group (see the top part of Figure 4b) being of multiplicity 3 and the second group (see the lower part of Figures 4b and 4c) is of multiplicity 4, leading to a Type I.a topological invariant of 12. In order to identify the multiplicity of the second group, it is necessary to consider the real part of the bands, which are displayed in Figure 4c. There, it becomes apparent that the purple band is continued by the green band which in turn is continued by the orange band.

5 Conclusion and remarks

In this article, a new topological classification of the time-local and time-global behaviors of the fundamental solutions of periodically parameterized periodic lODEs was introduced (see Section 2). The main starting point of the theory was given by a generalisation of the Floquet-Lyapunov decompositions of fundamental solutions of periodic lODEs. In fact, a Floquet-Lyapunov decomposition was introduced, which depends continuously and periodically on the parameter of the considered periodic lODE, allowing for the consideration of homotopy classes of the respective components of the Floquet-Lyapunov decomposition. Having introduced the respective topological invariants: Type I.a topological invariant and Type I.b homotopy class for the time-global behavior and Type II homotopy class for the time-local behavior of the associated parameterized fundamental solution, we were able to perform some analysis on the complexity of the system (e.g. degree of the lODE and the dimension of the parameter space) needed in order to obtain certain topological effects; see Section 3. Most notably, non-trivial Type I.b and Type II homotopy classes can only be obtained in the case where the parameter space $\mathbb{T}^d$ is of dimension $d \geq 2$. The Type I.a topological invariant being meaningful in the greatest amount of settings, one can use it to topologically distinguish systems with parameter spaces of any dimension $d \geq 1$ and it is even applicable in the setting of lODEs with constant coefficients.

The implications of this new topological theory for periodically time-modulated hexagonal structures in the high-contrast, subwavelength regime are the following (see Section 4). Due to the symmetry of a hexagonal structure, the topological nature of the associated subwavelength solutions to the time-modulated material is completely characterized by the Type I.a topological invariant. In fact, considering the periodically parametrized lODE, which describes the subwavelength band structures of high-contrast Floquet metamaterials, it becomes apparent that the topological nature of subwavelength band structures is determined by the constant part of the coefficient of the Hill equation (17).

We were able to provide two interesting modulation examples of a hexagonal structure which were of non-trivial Type I.a, proving that topologically non-trivial modulations of hexagonal structures exist. We even provided an example of a relatively high instance of the Type I.a topological invariant, the example provided had an associated Type I.a of 12.

To come back to the original motivation of this paper, namely the analogue of the bulk boundary correspondence in the setting of Floquet metamaterials, a next step would be to built upon the results of this paper and analyse the provided examples for the occurrence of edge modes.

Furthermore, we would like to point out that the developed theory can be refined by considering real-valued coefficient matrices $A : \mathbb{R}^d \times \mathbb{R} \to \mathrm{Mat}_{N\times N}(\mathbb{R})$ instead of complex valued coefficient matrices. This would particularly enrich the Type II topological effects. Indeed, it would mean that the Type II homotopy class would no longer be taken in the space $C^0(\mathbb{T}^{d+1}, \mathrm{SU}(N))$ but in the space $C^0(\mathbb{T}^{d+1}, \mathrm{SO}(N))$; see e.g. Remark 3. This might provide more topological effects, since $\mathrm{SO}(N)$ has richer topological properties than $\mathrm{SU}(N)$, in particular $\mathrm{SO}(N)$ is not simply connected. This would in particular imply that the *real* Type II homotopy class might be non-trivial even in the case of a one-dimensional parameter space.

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