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SUBWAVELENGTH RESONANT ACOUSTIC SCATTERING IN FAST TIME-MODULATED MEDIA

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Abstract. This article provides a rigorous mathematical analysis of acoustic wave scattering induced by a high-contrast subwavelength resonator whose material density is periodically modulated in time, and with a modulation frequency that is much larger than the one of the incident wave. We find that in general, the effect of the fast modulation is averaged over time and that the system behaves as an unmodulated resonator with an apparent effective density. However, under a suitable tuning of the modulation, which achieves a matching between temporal Sturm-Liouville and spatial Neumann eigenvalues, the low frequency incident wave becomes suddenly able to excite high frequency modes in the resonator. This phenomenon leads to the generation of scattered waves carrying high frequency components in the far field, and to the existence of exponentially growing outgoing modes. From these findings, it is expected that such time-modulated system could serve as a spontaneously radiating device, or as a high harmonic generator.

Keywords. Acoustic scattering, time-modulated metamaterial, subwavelength resonances, high-contrast media, Sturm-Liouville eigenmodes.

AMS Subject classifications. 35B34, 35B40, 45M05, 35L05, 35B10.

Contents

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1. Introduction

Time-modulated metamaterials [94, 28, 27] have been receiving a lot of attention for the promising applications they offer in the design of wave devices. Modulating in time the physical parameters of a medium allows indeed to perform a variety of wave operations, such as temporal dispersion or frequency conversion [92, 85, 84, 81, 53], signal amplification [49, 76], signal compression [35], spacetime cloaking [70], and non-reciprocal propagation [34, 78]. On the other hand, subwavelength resonators are known to be ideal constituents of spatial metamaterials or metasurfaces [68, 66, 69]: these can be achieved through high contrast inclusions which have the property of resonating with incident wavelengths much larger than the size of the resonators, thereby allowing to manipulate waves at subwavelength scales [88, 69, 37, 89]. High-contrast metamaterials obtained by filling a homogeneous medium with many of such inclusions enable to achieve a variety of applications in photonics and phononics, such as negative index scattering [13], spatial cloaking [5, 60], superfocusing [65, 10], guiding [75, 16, 9], sensing [35, 7] or superresolution imaging [56, 57, 20].

Quite many works have been proposed for modelling and understanding time-modulated media from physical or numerical studies, e.g. [29, 28, 30, 41, 50, 49, 55, 62, 76, 77, 82, 85, 83, 94, 80, 93]. There exist, however and to the best of our knowledge, very few works proposing a systematic mathematical analysis of wave propagation in temporal media. We can mention the work of Koutserimpas and Fleury [61, 62] who propose an approach for understanding wave amplification, in periodic media only. Quite recently, Ammari and Hiltunen [15] proposed an analysis for predicting the arising of subwavelength resonances in high contrast time-modulated phononic crystals, with the motivation of achieving non-reciprocal wave propagation [3]. The authors assumed the frequency of the modulation to be small–of the same order of the propagating wave frequency–leading to band gaps asymmetric with respect to the origin, ω–gaps and folding effects. Still, the rigorous characterization of the resonances in this regime is delicate, and the derivation proposed in this work remains formal. In this case also, the authors assume that the medium is extended periodically and infinitely in the three space directions, which allows for simplifications thanks to the Floquet transform on the coordinate variables. Apart from these contributions, we are not aware of any work investigating wave scattering in an open medium due to a time-modulated obstacle.

In this work, we propose a rigorous mathematical analysis of wave scattering induced by a subwavelength resonator subjected to a *fast* periodic time-modulation of one of the physical parameters. By *fast*, we mean that the time modulation has a significantly larger frequency compared to the incident frequency. Although a fast time-modulation may be challenging to realize in a practical physical system, this setting is mathematically more amenable to a rigorous analysis than to the case where the time modulation has a frequency of the same order of the incident wave. Furthermore, it leads to interesting physical effects. More specifically, we consider the following acoustic scattering problem in a three-dimensional homogeneous medium, with a time-modulated and highly contrasted resonator D:

$$\begin{cases} \dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R}\times\mathbb{R}^3\backslash\overline{D}, \\ \dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho(t)\rho_r}\Delta u = 0 \text{ in } \mathbb{R}\times D, \\ \dfrac{1}{\rho_0}\left.\dfrac{\partial u}{\partial n}\right|_+ = \dfrac{1}{\rho_r\rho(t)}\left.\dfrac{\partial u}{\partial n}\right|_- \text{ on } \mathbb{R}\times\partial D, \quad 1\le i\le N, \\ u|_+ = u|_- \text{ on } \mathbb{R}\times\partial D, \\ u - u_{\text{in}} \text{ is outgoing}, \end{cases} \tag{1.1}$$

where n is the outward normal to D. The parameters ρ_0 and κ_0 denote respectively the density and bulk modulus of the homogeneous background medium. The resonator D is a simply connected, smooth bounded domain; it is characterized by its bulk modulus κ_r and a time-modulated density $\rho_r\rho(t)$. The modulation $t\mapsto\rho(t)$ is a T–periodic function whose frequency is denoted by

$$\Omega = \frac{2\pi}{T}.$$

We assume that u_{in} is an incident plane wave, which is a time-harmonic solution to the wave equation in the free-space $\mathbb{R}^3$ propagating with a given frequency $\omega > 0$:

$$u_{\text{in}}(t,x) = \hat{u}_{\text{in}}(x)e^{-\mathrm{i}\omega t}, \tag{1.2}$$

where $\hat{u}_{\text{in}}$ is solution to the Helmholtz equation:

$$\left(\Delta + \frac{\omega^2}{v_0^2}\right)\hat{u}_{\text{in}} = 0 \text{ in } \mathbb{R}^3. \tag{1.3}$$

In what follows, we denote by v_0 and v_r the wave speeds

$$v_0 := \sqrt{\frac{\kappa_0}{\rho_0}}, \qquad v_r := \sqrt{\frac{\kappa_r}{\rho_r}},$$

and by δ the ratio between the densities of both media:

$$\delta := \frac{\rho_r}{\rho_0}.$$

We consider the scattering problem (1.1) in the subwavelength and high contrast regimes:

$$\omega \to 0 \text{ and } \delta \to 0. \tag{1.4}$$

On the other hand, the frequency Ω of the modulation $\rho(t)$ is kept constant, so that Ω is large compared to the incident frequency ($\omega \ll \Omega$), which corresponds to the *fast* modulation setting. In Section 4.3, we show that the solution $u(t,x)$ to (1.1) is fully determined by considering a suitable outgoing radiation condition, where "outgoing" means that $u - u_{\text{in}}$ should be a function of $t - |x|/v_0$ at infinity.

When there is no time-modulation, i.e. $\rho(t) = 1$, the system (1.1) models the scattering of sound waves by a small bubble in water. It is then known that the high contrast regime $\delta \to 0$ leads to the arising of subwavelength or the so-called Minnaert resonances [73, 11], corresponding to a strong amplification of the scattered field for incident frequencies close to the resonances. In the present study, we consider, for the simplicity of the derivation, only the case where only the density of the medium is modulated. The modulation of the parameter κ_r could be taken into account up to a small adaptation described in Remark 4.1.

Given this context, the purpose of this article is to show the arising of a special kind of subwavelength resonances under some exceptional–yet feasible–conditions on the time modulation $\rho(t)$. In general, the effect of the modulation is averaged over time and the system behaves as if the time-dependent physical parameter $1/\rho(t)$ were replaced with its average $\frac{1}{T}\int_0^T \frac{1}{\rho(t)}\,\mathrm{d}t$ in (1.1), thereby yielding no particularly visible effect on the scattered field at first order (Section 5.1.3). However, when a set of explicit conditions are fulfilled, a strong coupling between the low frequency incident field and the high frequency modulation arises, leading the resonator D to generate pulsed scattered waves with high frequency components (we illustrate this phenomenon on Figure 1). Furthermore, we find the existence of modes exponentially growing in time, which suggests that such a tuned system could serve as a spontaneously radiating device or for high harmonic generation [91, 79, 33]. One possible application of such metamaterial could be the design of acoustic insulators, due to its potentiality in converting large (audible) wavelength into tiny (inaudible) ones, which could then be more easily absorbed by more classical materials adapted to small wavelengths. If these properties are expected to arise in time-modulated metamaterials [76, 49, 61], our work is the first, to the best of our knowledge, to propose a rigorous mathematical analysis of such phenomena in open systems, and high-frequency time-modulated metamaterials. A significant advance is also to account for modulations ρ which can be arbitrarily rough (we assume only $1/\rho \in L^\infty_{\text{per}}((0,T))$), while [15] assumed that ρ has a finite number of Fourier modes.

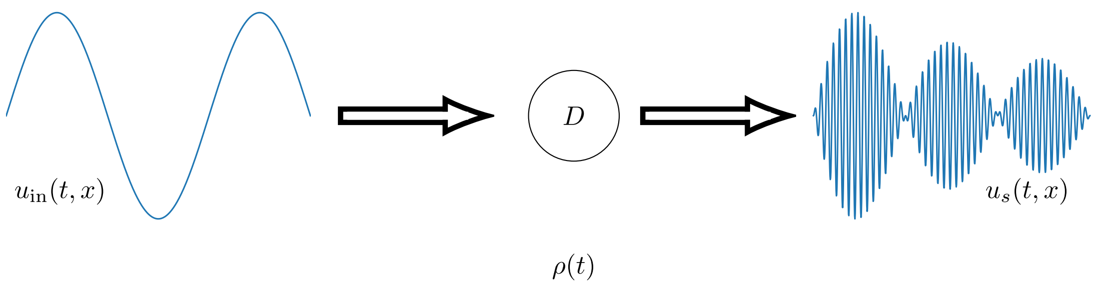

FIGURE 1. High frequency coupling between a low frequency incident field and a subwavelength resonator D modulated by a high-frequency time modulation $\rho(t)$. Upon a strong coupling between the resonator and the modulation (namely when $\Lambda \neq \{(0,0)\}$, where Λ is given by (1.7)), the scattered field $u_s = u - u_{\text{in}}$ carries high frequency components in the far field.

The conditions under which the exceptional resonant coupling arise can be formulated in terms of two eigenvalue problems naturally associated to (1.1). First, let us denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and by $(\phi_l)_{l\in\mathbb{N}}$ the eigenvalues and eigenvectors of the Laplace operator with Neumann boundary conditions on ∂D:

$$\begin{cases} -\Delta \phi_l = \lambda_l \phi_l \text{ in } D, \\ \dfrac{\partial \phi_l}{\partial n} = 0 \text{ on } \partial D, \end{cases} \qquad l \in \mathbb{N}. \tag{1.5}$$

Second, consider the Sturm-Liouville eigenvalue problem associated to the modulation $1/\rho(t)$ with eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$ and eigenvectors $(p_m)_{m\in\mathbb{N}}$:

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} p_m(t) = \dfrac{\mu_m}{\rho(t)} p_m(t), \\ \qquad\qquad p_m \text{ is } T\text{–periodic,} \end{cases} \qquad m \in \mathbb{N}. \tag{1.6}$$

Let us finally denote by Λ the set of indices associated to the eigenvalues λ_l and μ_m which are proportional by a factor $1/v_r^2$:

$$\Lambda := \left\{ (m,l) \in \mathbb{N} \times \mathbb{N} \,|\, \lambda_l = \frac{\mu_m}{v_r^2} \right\}. \tag{1.7}$$

Since $\lambda_0 = \mu_0/v_r^2 = 0$ (these eigenvalues are associated to constant functions in D and on $(0,T)$), the set Λ is not empty and contains the tuple $(0,0)$. It is reasonable to believe that generically, the coincidence of further eigenvalues is exceptional and that in most situations, $\Lambda = \{(0,0)\}$. In that case, we show that the tuple $(0,0)$ is associated to the existence of a single subwavelength resonant mode that is approximately constant in time and space in the resonator D, which generates scattered waves devoid of high frequency components at first order.

However, under an appropriate tuning of the modulation $\rho(t)$, the set Λ *may* contain a non-trivial pair of indices $(m,l) \neq (0,0)$. This can be achieved, for instance, upon a magnification of $\rho(t)$ by a suitable constant multiplicative factor, since this would result in the same magnification for the eigenvalues $(\mu_m)_{m\in\mathbb{N}}$. Then, this configuration leads to the arising of as many additional subwavelength resonant modes as the size of the set Λ, which are approximately equal to linear combinations of the functions $(p_m(t)\phi_l(x))_{(l,m)\in\Lambda}$ inside D. More precisely, the result of Proposition 5.3 shows that there are $2\#\Lambda$ resonant frequencies whose leading order asymptotics are given by

$$\omega_i^{\pm}(\delta) \sim \pm v_r \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}}, \qquad 1 \leq i \leq \#\Lambda, \tag{1.8}$$

where $\#\Lambda$ denotes the number of elements of the set Λ, and where $(\lambda_i)_{1\leq i\leq \#\Lambda}$ are the eigenvalues of the following generalized eigenvalue problem with eigenvectors $(\boldsymbol{a}_i)_{1\leq i\leq\#\Lambda}$:

$$T\boldsymbol{a}_i + \lambda_i G \boldsymbol{a}_i = 0.$$

The matrix $T \simeq (T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda}$ is determined by the Dirichlet-to-Neumann operator of the (time-modulated) acoustic scattering problem (defined in (5.11)). The matrix $G = \mathrm{diag}(\gamma_m)_{1\leq m\leq\#\Lambda}$ is a diagonal matrix of coefficients $(\gamma_m)_{1\leq m\leq\#\Lambda}$ (equation (5.10)). In the exceptional coupling case $\Lambda \neq \{(0,0)\}$, both matrices have no distinguished signs, so that the eigenvalues λ_i are complex in all generality. Then, we find in Proposition 5.5 that at least in the case where D is the disk, one of the complex square roots $\pm\lambda_i^{\frac{1}{2}}$ in (1.8) must have a strictly positive imaginary part, which is associated to the existence of some exponentially growing outgoing mode (i.e. a non-trivial solution to (1.1) with $u_{\mathrm{in}} = 0$).

Besides the analysis of these resonant phenomena, we identify in Corollary 5.1 a leading order approximation for the far field pattern of the scattered wave. After suitable rescalings and a Foldy-Lax approximation argument inspired from [21, 43], this allows us to discuss, at least at a formal level, the arising of an effective medium for the temporal metamaterial which would be constituted of many tiny copies of such time-modulated resonator D.

The paper outlines as follows. After describing our notation conventions in Section 2, our work starts in Section 3 with the introduction of a novel mathematical method for the study of subwavelength resonances in open media. Our approach relies on a variational formulation of (1.1) and the Dirichlet-to-Neumann operator associated to the Helmholtz equation in exterior domains. We expose our method on the unmodulated version of (1.1) in which $\rho(t) = 1$, which allows us to verify that we retrieve all the results of [11, 8, 45, 21, 43] concerned with this setting, namely the leading asymptotic of the subwavelength resonant frequencies, point scatterer approximations, and a formal derivation of an effective medium theory for a system constituted of many small subwavelength resonators. Our novel approach leads to slightly simpler derivations because we do not rely on a layer potential representation, but is also more flexible in the sense that it could easily be applied in other dimensions ($d = 1$ or $d = 2$, left for a future work), and to the time-modulated case which is the object of the remainder of the paper.

We then focus on the scattering problem (1.1) with the time-dependent coefficient $\rho(t)$. The formulation of an associated outgoing radiation condition for such time-dependent problem is non-standard and requires a particular study, which is the object of Section 4. We formulate an outgoing radiation condition which ensures the existence and uniqueness of a quasi-periodic, outgoing solution to the scattering problem (1.1). We define the Dirichlet-to-Neumann operator associated with such radiating solutions and we describe its main properties. We mention the existence of an associated time-modulated fundamental solution and of an associated layer potential theory which has resemblances with retarded potentials [87], and which enables us to compute far field asymptotic expansions for time-modulated outgoing waves.

In Section 5, we establish the arising of subwavelength resonances and we identify the leading order asymptotics of resonant frequencies for the time-modulated system (1.1), using the Dirichlet-to-Neumann approach introduced in Section 3. We show that there are exactly as many (complex) subwavelength resonant frequencies as twice the size of the set Λ. Assuming that the modulation is tuned in such a way that $\Lambda = \{(0,0),(l,m)\}$ for some $(l,m) \neq (0,0)$, we show how the nature of these resonances can be determined by the sign of the parameter γ_m and a matrix $(T_{lm,l'm'})_{(l,m),(l',m')\in\Lambda}$ of coefficients determined by the Dirichlet-to-Neumann operator. Then, we derive a far field pattern approximation of the scattered field in the strong coupling tuning for which $\Lambda = \{(0,0),(l,m)\}$, and we discuss the arising of an effective medium theory for a temporal metamaterial that would be constituted of many small identical time-modulated resonators. We find that under some assumptions on the size and number of the resonators, the effective medium is governed by an integral equation of the second-kind with a direction and time-dependent kernel. In general, let us mention that the arising of high frequency scattered waves makes difficult the identification of a homogenized equation for the complete scattered field, but such homogenization can be achieved for the low frequency part of the scattered field; in that case we retrieve the possibility of achieving negative index refraction or strong absorption as in the unmodulated setting. However, we believe that the true originality brought by the exceptional coupling lies in the generation of high frequency scattered waves.

A final Appendix A gathers the definition and properties of the Bloch and Floquet transforms of tempered distributions, which would be the appropriate setting for studying time-modulated wave scattering induced by not necessarily time harmonic incident fields. These transforms are usually defined on L^2 spaces and are applied in the spatial domain to solution fields decaying at infinity. This setting is too restrictive for treating periodicity in the time-domain, hence our motivation for extending the Bloch and Floquet transform on the larger space of tempered distributions.

2. General setting and notation conventions

In all what follows, $D \subset \mathbb{R}^3$ is a simply connected, smooth bounded domain. Its characteristic function is written 1_D, whose values are defined by

$$1_D(x) = \begin{cases} 1 \text{ if } x \in D, \\ 0 \text{ if } x \in \mathbb{R}^3 \backslash D. \end{cases} \tag{2.1}$$

2.1. **Functional spaces**

We write $H^s(D)$ and $H^s(\partial D)$ the usual space of complex-valued functions with Sobolev exponent $s \in \mathbb{R}$. For a given Hilbert space W, we denote by $H^s_{\text{per}}((0,T),W)$ the space of functions $(t,x) \mapsto u(t,x)$ which are T–periodic with respect to the variable t ($u(t+T,\cdot) = u(t,\cdot)$) with $u(t,\cdot) \in W$ for almost every $t \in \mathbb{R}$, and such that

$$\sum_{n\in\mathbb{Z}} (1+n^2)^{\frac{s}{2}} ||\hat{u}_n||^2_W < +\infty,$$

where $||\cdot||_W$ is the norm of W and $(\hat{u}_n)_{n\in\mathbb{Z}}$ denote the Fourier trigonometric coefficients of u:

$$u(t,x) = \sum_{n\in\mathbb{Z}} \hat{u}_n e^{-\mathrm{i}n\Omega t} \text{ with } \hat{u}_n := \frac{1}{T}\int_0^T u(t,x) e^{\mathrm{i}n\Omega t}\,\mathrm{d}t.$$

In a similar manner, we consider $L^\infty_{\text{per}}((0,T),W)$ (resp. $\mathcal{C}^\infty_{\text{per}}((0,T),W)$) the space of bounded (resp. smooth) periodic functions with values in W.

Throughout the paper, we consider the Hilbert space $H := L^2_{\text{per}}((0,T),L^2(D))$, equipped with the inner product induced by the modulation ρ:

$$\langle u,v\rangle_H := \frac{1}{T}\int_0^T \int_D \frac{1}{\rho(t)} u(t,x)\overline{v}(t,x)\,\mathrm{d}x\,\mathrm{d}t, \tag{2.2}$$

and the Hilbert space

$$V := H^1_{\text{per}}((0,T),L^2(D)) \cap L^2_{\text{per}}((0,T),H^1(D)) \tag{2.3}$$

equipped with the inner product

$$\langle u,v\rangle_V := \frac{1}{T}\left(\int_0^T \int_D u\,\mathrm{d}x\,\mathrm{d}t\right)\left(\int_0^T \int_D \overline{v}\,\mathrm{d}x\,\mathrm{d}t\right) + \int_0^T \int_D \left[\nabla u \cdot \nabla \overline{v} + \frac{1}{\rho(t)}\partial_t u \partial_t \overline{v}\right]\mathrm{d}x\,\mathrm{d}t. \tag{2.4}$$

We denote by $||u||_H := \langle u,u\rangle_H^{\frac{1}{2}}$ and $||u|_V := \langle u,u\rangle_V^{\frac{1}{2}}$ the associated norms.

2.2. **Layer potentials**

For a given real number $k \in \mathbb{R}$, $\mathcal{S}_D^k$ and $\mathcal{K}_D^{k*}$ denote respectively the single layer potential and the adjoint of the Neumann-Poincaré operator on D: for any $\phi \in H^{-\frac{1}{2}}(\partial D)$,

$$\mathcal{S}_D^k[\phi](x) := \int_{\partial D} \Gamma^k(x-y)\phi(y)\,\mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^3, \tag{2.5}$$

$$\mathcal{K}_D^{k*}[\phi](x) := \int_{\partial D} \nabla_x \Gamma^k(x-y)\cdot n(x)\phi(y)\,\mathrm{d}\sigma(y), \qquad x \in \partial D, \tag{2.6}$$

where

$$\Gamma^k(x) := -\frac{e^{\mathrm{i}k|x|}}{4\pi|x|} \tag{2.7}$$

is the fundamental solution to the Helmholtz equation and $\mathrm{d}\sigma$ is the surface measure of ∂D. We recall that when k is not a Dirichlet eigenvalue of D, $\mathcal{S}_D^k$ is an invertible operator from $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$, whose inverse is denoted by $(\mathcal{S}_D^k)^{-1} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ [18]. The purely imaginary number is denoted by (a straight) i.

Finally, we denote by Φ the solution to the exterior problem

$$\begin{cases} -\Delta\Phi = 0 \text{ in } \mathbb{R}^3\backslash D, \\ \Phi = 1 \text{ on } \partial D, \\ \Phi(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty, \end{cases} \tag{2.8}$$

and we recall the definition of the capacity of D:

$$\mathrm{cap}(D) := -\int_{\partial D} \frac{\partial \Phi}{\partial n}\,\mathrm{d}\sigma,$$

where n is the outward normal to D. We denote by $|D|$ the volume of D.

3. A Dirichlet-to-Neumann approach to subwavelength resonances in the unmodulated case

In this section, we introduce a novel approach for analyzing subwavelength resonances based on the weak formulation of the wave scattering problem and the Dirichlet-to-Neumann operator. We illustrate this method by retrieving well-established results for the static version of (1.1) in which the modulation is kept constant; namely $\rho(t) = 1$ for all $t \in \mathbb{R}$. This procedure will then be extended to the time-modulated system (1.1) in the subsequent Sections 4 and 5.

Since we assume the time harmonic regime (1.2) for the incident field, the solution to (1.1) is itself time-harmonic and is given by

$$u(t,x) = \hat{u}(x)e^{-\mathrm{i}\omega t}, \tag{3.1}$$

where $\hat{u}$ solves the following system of coupled Helmholtz equations:

$$\begin{cases} \Delta\hat{u} + \dfrac{\omega^2}{v_0^2}\hat{u} = 0 \text{ in } \mathbb{R}^3\backslash D, \\ \Delta\hat{u} + \dfrac{\omega^2}{v_r^2}\hat{u} = 0 \text{ in } D, \\ u|_+ = u|_- \text{ on } \partial D, \\ \dfrac{\partial \hat{u}}{\partial n}\Big|_- = \delta\,\dfrac{\partial \hat{u}}{\partial n}\Big|_+ \text{ on } \partial D, \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}\omega}{v_0}\right)(\hat{u} - \hat{u}_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{3.2}$$

The last equality is the outgoing Sommerfeld radiation condition, which ensures the uniqueness of the solution and that the scattered wave is outgoing [71]. The analysis of the amplitude response to (3.2) in the high-contrast regime $\delta \to 0$ by using integral representations of the solution is now quite established [11, 6, 45]. The object of this section is to propose a slightly simpler characterization of the subwavelength resonances, by reformulating (3.2) in the domain D in terms of the Dirichlet-to-Neumann map associated to the Helmholtz equation in $\mathbb{R}^3\backslash\overline{D}$. This approach turns out to be quite flexible, and it can be generalized naturally for the analysis of the time-modulated system (1.1).

This section is organized in four parts. We first recall the definition of the Dirichlet-to-Neumann map in Section 3.1 and we rewrite the scattering problem (3.2) in terms of this operator. We then provide an explicit characterization of the subwavelength resonances in Section 3.2 based on a variational formulation posed in the bounded domain D, and we compute their leading order asymptotic. Then, we explain how to retrieve a modal decomposition and a point scatterer approximation formula in Section 3.3. Finally, we outline in Section 3.4

the Foldy-Lax approximation argument which allows to derive an effective medium theory for a heterogeneous medium filled with many rescaled copies of such resonators.

3.1. Formulation of the scattering problem in terms of the Dirichlet-to-Neumann map

We start by recalling the definition of the Dirichlet-to-Neumann map [36, 63, 74, 71].

Definition 3.1. The Dirichlet-to-Neumann map with wave number $k \in \mathbb{R}$ is the operator denoted by $\mathcal{T}^k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ and defined by

$$\mathcal{T}^k f := \left.\frac{\partial w_f}{\partial n}\right|_+ \quad \text{where} \quad \begin{cases} \Delta w_f + k^2 w_f = 0 \text{ in } \mathbb{R}^d \backslash D, \\ w_f = f \text{ on } \partial D, \\ \left(\partial_{|x|} - \mathrm{i}k\right) w_f(x) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{3.3}$$

Throughout this section, we assume that $k = \omega / v_0$ with $\omega \to 0$. We have the following well-known result regarding the analyticity of $\mathcal{T}^{\frac{\omega}{v_0}}$ with respect to ω in a neighborhood of zero, in three dimensions (see e.g. [19, 12, 17], and also the proof of Proposition 4.7 below for the analyticity on the whole real line).

Proposition 3.1. *The operator $\mathcal{T}^{\frac{\omega}{v_0}} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is analytic with respect to $\omega \in \mathbb{C}$ in a neighborhood of zero. In other words, there exist operators $(\mathcal{T}_n)_{n\in\mathbb{N}}$ such that $\mathcal{T}^{\frac{\omega}{v_0}}$ can be decomposed as the following convergent power series in the space of operators $H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$:*

$$\mathcal{T}^{\frac{\omega}{v_0}} = \sum_{n=0}^{+\infty} \frac{\omega^n}{v_0^n} \mathcal{T}_n. \tag{3.4}$$

Furthermore, the leading order asymptotic of $\mathcal{T}^{\frac{\omega}{v_0}}$ reads, as $\omega \to 0$:

$$\mathcal{T}^{\frac{\omega}{v_0}}[\phi] = \mathcal{T}_0[\phi] + \frac{\mathrm{i}\omega}{4\pi v_0} \left(\int_{\partial D} \frac{\partial \Phi}{\partial n} \phi \,\mathrm{d}\sigma \right) \frac{\partial \Phi}{\partial n} + O(\omega^2),$$

where Φ is the solution to the exterior problem (2.8).

Proof. It is well-known that the Dirichlet-to-Neumann map can be expressed in terms of the single layer potential and the Neumann-Poincaré operator (see [12, 17]):

$$\mathcal{T}^{\frac{\omega}{v_0}} = \left(\frac{1}{2} I + \mathcal{K}_D^{\frac{\omega}{v_0}*} \right) (\mathcal{S}_D^{\frac{\omega}{v_0}})^{-1}. \tag{3.5}$$

The analyticity of $\mathcal{T}^{\frac{\omega}{v_0}}$ follows because the operators $\mathcal{S}_D^{\frac{\omega}{v_0}}$ and $\mathcal{K}_D^{\frac{\omega}{v_0}*}$ are analytic in ω and $\mathcal{S}_D^{\frac{\omega}{v_0}}$ is invertible for small values of $\omega \in \mathbb{C}$. We can then compute the expansion of the Dirichlet-to-Neumann operator thanks to a Neumann series:

$$\begin{aligned} \mathcal{T}^{\frac{\omega}{v_0}} &= \left(\frac{1}{2} I + \mathcal{K}_D^* + O(\omega^2) \right) \left(\mathcal{S}_D^{-1} - \frac{\omega}{v_0} \mathcal{S}_D^{-1} \mathcal{S}_{D,1} \mathcal{S}_D^{-1} + O(\omega^2) \right) \\ &= \mathcal{T}_0 - \frac{\omega}{v_0} \mathcal{T}_0 \mathcal{S}_{D,1} \mathcal{S}_D^{-1} + O(\omega^2), \end{aligned}$$

where $\mathcal{S}_{D,1}$ is the operator $\mathcal{S}_{D,1}[\phi] := -\frac{\mathrm{i}}{4\pi} \int_{\partial D} \phi \,\mathrm{d}\sigma$ (see e.g [45, Proposition 3.1]). Therefore, we find that

$$\mathcal{T}^{\frac{\omega}{v_0}}[\phi] = \mathcal{T}_0[\phi] + \frac{\mathrm{i}\omega}{4\pi v_0} \left(\int_{\partial D} \mathcal{S}_D^{-1}[\phi] \,\mathrm{d}\sigma \right) \mathcal{T}_0[1_D] + O(\omega^2).$$

The result follows since $\mathcal{S}_D^{-1}$ is self-adjoint and $\mathcal{S}_D^{-1}[1_{\partial D}] = \frac{\partial \Phi}{\partial n}$. □

Classically, the scattering problem (3.2) can be rewritten in terms of $\mathcal{T}^{\frac{\omega}{v_0}}$ as a PDE posed on the bounded domain D:

$$\begin{cases} \Delta \hat{u} + \dfrac{\omega^2}{v_r^2} \hat{u} = 0 \text{ in } D, \\ \dfrac{\partial \hat{u}}{\partial n} = \delta \mathcal{T}^{\frac{\omega}{v_0}}[\hat{u} - \hat{u}_{\mathrm{in}}] + \delta \dfrac{\partial \hat{u}_{\mathrm{in}}}{\partial n} \text{ on } \partial D. \end{cases} \tag{3.6}$$

It can equivalently be rewritten as the following variational formulation: find $\hat{u} \in H^1(D)$ such that for any $v \in H^1(D)$,

$$\int_D \left(\nabla \hat{u} \cdot \nabla \overline{v} - \frac{\omega^2}{v_r^2} \hat{u} \overline{v} \right) \mathrm{d}x - \delta \int_{\partial D} \mathcal{T}^{\frac{\omega}{v_0}}[\hat{u}] \overline{v} \,\mathrm{d}\sigma = \delta \int_{\partial D} \left(\frac{\partial \hat{u}_{\mathrm{in}}}{\partial n} - \mathcal{T}^{\frac{\omega}{v_0}}[\hat{u}_{\mathrm{in}}] \right) \overline{v} \,\mathrm{d}\sigma. \tag{3.7}$$

The system (3.6) fully determines $\hat{u}$ inside D, while the value of $\hat{u}$ outside D can be obtained by solving an exterior Helmholtz problem. In the regime $\omega \to 0$, we have for instance the representation

$$\hat{u} = \hat{u}_{\mathrm{in}} + \mathcal{S}_D^{\frac{\omega}{v_0}}[(\mathcal{S}_D^{\frac{\omega}{v_0}})^{-1}[\hat{u}_{|\partial D} - \hat{u}_{\mathrm{in}|\partial D}]] \text{ in } \mathbb{R}^3 \backslash \overline{D}, \tag{3.8}$$

where the trace $\hat{u}_{|\partial D}$ is determined by (3.6).

The goal of this section is to analyse the arising of the resonances of (3.6), which are defined as the poles of the scattering operator.

Definition 3.2. We call subwavelength "resonance" a complex frequency $\omega \equiv \omega(\delta) \in \mathbb{C}$ satisfying $\omega(\delta) \to 0$ as $\delta \to 0$ and such that (3.6) admits a non-zero solution $u(\omega, \delta) \in H^1(D)$ for a zero right-hand side $\hat{u}_{\text{in}} = 0$:

$$\begin{cases} \Delta u(\omega, \delta) + \dfrac{\omega^2}{v_r^2} u(\omega, \delta) = 0 \text{ in } D, \\ \qquad\qquad \dfrac{\partial u(\omega, \delta)}{\partial n} = \delta \mathcal{T}^{\frac{\omega}{v_0}}[u(\omega, \delta)]. \end{cases} \tag{3.9}$$

Equation (3.9) is an instance of a nonlinear eigenvalue problem [72] in $\omega \in \mathbb{C}$. Setting $\delta = 0$ in (3.6), we see that $\omega = 0$ is a resonant frequency associated to the constant function $u(0,0) := 1_D$, where 1_D is the characteristic function (2.1) of D. By using a continuation argument as in [45] or Gohberg-Sigal theory, [12], we find that it is possible to construct two resonant frequencies $\omega^+(\delta)$ and $\omega^-(\delta)$ of order $O(\delta^{\frac{1}{2}})$ upon a suitable perturbation $u(\omega, \delta)$ of $u(0,0)$. The problem of characterizing the resonances is therefore reduced to the analysis of the perturbation and the splitting of the nonlinear eigenvalue $\omega = 0$ with respect to the parameter $\delta = 0$.

It is possible thanks to a simple calculation to predict the leading order of the resonance and its connexion with the capacity of D (assumed here to have a single connected component for simplicity), based on an argument inspired from [88]. Setting $v = 1_D$ in (3.7) for a zero right-hand side $\hat{u}_{\text{in}} \equiv 0$ and $\hat{u} \equiv u(\omega, \delta) \simeq u(0,0) = 1_D$, we obtain in the regime $\delta, \omega \to 0$:

$$0 \simeq -\frac{\omega^2}{v_r^2}|D| - \delta \int_{\partial D} \mathcal{T}^{\frac{\omega}{v_0}}[1_{\partial D}] \, \mathrm{d}\sigma \simeq -\frac{\omega^2}{v_r^2}|D| - \delta \int_{\partial D} \mathcal{T}^0[1_{\partial D}] \, \mathrm{d}\sigma. \tag{3.10}$$

Since by the definition (3.3) of $\mathcal{T}^0$, it holds

$$\operatorname{cap}(D) = -\int_{\partial D} \mathcal{T}^0[1_{\partial D}] \, \mathrm{d}\sigma,$$

we can predict from (3.10) that there are two resonant frequencies $\omega^+(\delta)$ and $\omega^-(\delta)$ satisfying

$$\omega^{\pm}(\delta) \simeq \pm v_r \sqrt{\frac{\operatorname{cap}(D)}{|D|}} \delta^{\frac{1}{2}},$$

which is the result derived in [11, 6, 45]. In the next subsections, we make this argument more systematic and we express the leading order asymptotic of the solution $\hat{u}$ to (3.6) for a positive real frequency ω close to the resonant value $\omega^+(\delta)$.

3.2. Characterization of the resonances

In what follows, we introduce the bilinear form $a_{\omega,\delta}$ defined for $u, v \in H^1(D)$ by

$$a_{\omega,\delta}(u, v) := \int_D \nabla u \cdot \nabla \overline{v} \, \mathrm{d}x + \int_D u \, \mathrm{d}x \int_D \overline{v} \, \mathrm{d}x - \frac{\omega^2}{v_r^2} \int_D u \overline{v} \, \mathrm{d}x - \delta \int_{\partial D} \mathcal{T}^{\frac{\omega}{v_0}}[u] \overline{v} \, \mathrm{d}\sigma.$$

The bilinear form $a_{\omega,\delta}$ is obtained by adding the rank-one bilinear form $(u, v) \mapsto \int_D u \, \mathrm{d}x \int_D \overline{v} \, \mathrm{d}x$ to the left-hand side of (3.7). It is an analytic perturbation in ω and δ of the bilinear form $a_{0,0}$ defined by

$$a_{0,0}(u, v) := \int_D \nabla u \cdot \nabla \overline{v} \, \mathrm{d}x + \int_D u \, \mathrm{d}x \int_D \overline{v} \, \mathrm{d}x,$$

which is continuous coercive on $H^1(D)$ owing to the Poincaré-Wirtinger inequality.

From standard perturbation theory, it is clear that $a_{\omega,\delta}$ remains coercive for sufficiently small complex values of δ, ω. Hence, for any right hand-side $f \in H^{-1}(D)$, there exists a unique Lax-Milgram solution $u_f(\omega, \delta)$ to the problem

$$a_{\omega,\delta}(u_f(\omega, \delta), v) = \langle f, v \rangle_{H^{-1}(D), H^1(D)}, \tag{3.11}$$

which is analytic in ω and δ (see e.g. [42]). In the context of (3.7), we shall consider

$$\langle f, v \rangle_{H^1(D), H^{-1}(D)} := \delta \int_{\partial D} \left(\frac{\partial \hat{u}_{\text{in}}}{\partial n} - \mathcal{T}^{\frac{\omega}{v_0}}[\hat{u}_{\text{in}}] \right) \overline{v} \, \mathrm{d}\sigma.$$

We now show that subwavelength resonant frequencies $\omega^{\pm}(\delta)$ are the root of a single well-determined scalar equation.

Lemma 3.1. *Let $\omega \in \mathbb{C}$ and $\delta \in \mathbb{R}$ belong to a neighborhood of zero. For any $f \in H^{-1}(D)$, the variational problem*

find $u \in H^1(D)$ such that $\forall v \in H^1(D)$,

$$\int_D \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{v_r^2} u \overline{v} \right) \mathrm{d}x - \delta \int_{\partial D} \mathcal{T}^{\frac{\omega}{v_0}}[u] \overline{v} \, \mathrm{d}\sigma = \langle f, v \rangle_{H^{-1}(D), H^1(D)}, \tag{3.12}$$

admits a unique solution u if and only if

$$\int_D u_1(\omega, \delta) \, \mathrm{d}x \neq 1, \tag{3.13}$$

where $u_1(\omega, \delta) \in H^1(D)$ is the unique solution to the variational problem

$$a_{\omega,\delta}(u_1(\omega, \delta)) = \langle 1_D, v \rangle_{H^{-1}(D), H^1(D)} = \int_D \overline{v} \, \mathrm{d}x. \tag{3.14}$$

When (3.13) is satisfied, the solution to (3.12) reads

$$u \equiv u(\omega, \delta) = u_f(\omega, \delta) + \frac{\int_D u_f(\omega, \delta) \, \mathrm{d}x}{1 - \int_D u_1(\omega, \delta) \, \mathrm{d}x} u_1(\omega, \delta), \tag{3.15}$$

where $u_f(\omega, \delta)$ is the solution to (3.11). When (3.13) is not satisfied, then $u_1(\omega, \delta)$ is a non-zero solution to (3.12) with $f = 0$ and $\omega \equiv \omega(\delta)$ is a subwavelength resonant frequency.

Proof. Clearly, (3.12) is equivalent to

$$\begin{aligned} & a_{\omega,\delta}(u, v) - \int_D u \, \mathrm{d}x \int_D \overline{v} \, \mathrm{d}x = \langle f, v \rangle_{H^{-1}(\Omega), H^1(\Omega)} \\ \Leftrightarrow \ & a_{\omega,\delta}(u, v) = a_{\omega,\delta}(u_f(\omega, \delta), v) + \left(\int_D u \, \mathrm{d}x \right) a_{\omega,\delta}(u_1(\omega, \delta), v), \end{aligned}$$

which implies

$$u = u_f(\omega, \delta) + \left(\int_D u \, \mathrm{d}x \right) u_1(\omega, \delta).$$

Integrating on D, this equation has a solution given by (3.15) if and only if (3.13) is satisfied. $\square$

In other words, subwavelength resonances $\omega \equiv \omega(\delta)$ are characterized by the equation

$$\int_D u_1(\omega, \delta) \, \mathrm{d}x = 1, \tag{3.16}$$

where $u_1(\omega, \delta)$ is the solution to (3.14), which also reads in strong form

$$\begin{cases} -\Delta u_1(\omega, \delta) - \dfrac{\omega^2}{v_r^2} u_1(\omega, \delta) + \left(\displaystyle\int_D u_1(\omega, \delta) \, \mathrm{d}x \right) 1_D = 1_D \text{ in } D, \\ \dfrac{\partial u_1(\omega, \delta)}{\partial n} = \delta \mathcal{T}^{\frac{\omega}{v_0}}[u_1(\omega, \delta)] \text{ on } \partial D. \end{cases} \tag{3.17}$$

The existence of the complex resonant frequencies $\omega \equiv \omega(\delta)$ such that (3.16) is then guaranteed by applying the analytic implicit function theorem (see [59, Chapter 0]) to (3.16) as in [45]. In order to obtain an asymptotic expansion of $\omega(\delta)$, we start by computing the leading order term in the asymptotic expansion of $u_1(\omega, \delta)$ as $\omega \to 0$ and $\delta \to 0$.

Proposition 3.2. *The solution $u_1(\omega, \delta)$ to (3.17) has the following asymptotic behavior as $\omega, \delta \to 0$:*

$$u_1(\omega, \delta) = \left(\frac{1}{|D|} + \frac{\omega^2}{v_r^2 |D|^2} - \frac{\mathrm{cap}(D)}{|D|^3} \delta + \frac{\mathrm{i}\omega\delta \, \mathrm{cap}(D)^2}{4\pi v_0 |D|^3} \right) 1_D + \left(1 - \frac{\mathrm{i}\omega \, \mathrm{cap}(D)}{4\pi v_0} \right) \delta \widetilde{u}_{0,1} + O(\omega^2 + \delta)^2, \tag{3.18}$$

where the remainder is estimated with the $H^1(D)$–norm. The function $\widetilde{u}_{0,1}$ is the solution to the following Laplace problem:

$$\begin{cases} -\Delta \widetilde{u}_{0,1} = \dfrac{\mathrm{cap}(D)}{|D|^2} \text{ in } D, \\ \dfrac{\partial \widetilde{u}_{0,1}}{\partial n} = \dfrac{1}{|D|} \dfrac{\partial \Phi}{\partial n} \text{ on } \partial D, \\ \displaystyle\int_D \widetilde{u}_{0,1} = 0, \end{cases} \tag{3.19}$$

where Φ is the solution to (2.8).

Proof. Since $u_1(\omega,\delta)$ is analytic in ω and δ, there exist functions $(u_{p,k})_{p,k\geq 0}$ such that the following series is convergent in $H^1(D)$:

$$u_1(\omega,\delta) = \sum_{p,k=0}^{+\infty} \omega^p \delta^k u_{p,k}. \tag{3.20}$$

Inserting (3.4) into (3.14) and identifying powers of δ and ω, we obtain the following cascade of equations characterizing the functions $(u_{p,k})_{p,k\geq 0}$:

$$\begin{cases} -\Delta u_{p,k} + \left(\int_D u_{p,k}\,\mathrm{d}x\right) 1_D = \dfrac{1}{v_r^2} u_{p-2,k} + 1_D \delta_{p=0}\delta_{k=0} \text{ in } D, \\ \dfrac{\partial u_{p,k}}{\partial n} = \displaystyle\sum_{n=0}^{p} \frac{1}{v_0^n} \mathcal{T}_n[u_{p-n,k-1}], \end{cases}$$

where we assume that $u_{p,k}=0$ for negative indices p and k. It is easily obtained by induction that

$$u_{2p,0} = \frac{1_D}{v_r^{2p}|D|^{p+1}} \text{ and } u_{2p+1,0} = 0, \qquad \text{for any } p \geq 0.$$

Then, for $p=0$ and $k=1$, we find that $u_{0,1}$ satisfies, for any $v \in H^1(D)$:

$$\int_D \nabla u_{0,1}\cdot\nabla \overline{v}\,\mathrm{d}x + \int_D u_{0,1}\,\mathrm{d}x \int_D \overline{v}\,\mathrm{d}x = \int_{\partial D} \mathcal{T}_0\left[\frac{1_D}{|D|}\right]\overline{v}\,\mathrm{d}\sigma = \frac{1}{|D|}\int_{\partial D} \frac{\partial \Phi}{\partial n}\overline{v}\,\mathrm{d}\sigma.$$

Setting $v = 1_D$, we find that $\int_D u_{0,1}\,\mathrm{d}x = -\frac{\operatorname{cap}(D)}{|D|^2}$ and

$$u_{0,1} = -\frac{\operatorname{cap}(D)}{|D|^3} 1_D + \widetilde{u}_{0,1},$$

where $\widetilde{u}_{0,1}$ is the solution to the Laplace problem (3.19). Let us finally compute $u_{1,1}$, that is the solution to

$$\begin{cases} -\Delta u_{1,1} + \left(\int_D u_{1,1}\,\mathrm{d}x\right) 1_D = 0 \text{ in } D, \\ \dfrac{\partial u_{1,1}}{\partial n} = \mathcal{T}_0[u_{1,0}] + \dfrac{1}{v_0}\mathcal{T}_1[u_{0,0}] = \dfrac{1}{v_0|D|}\mathcal{T}_1[1_D]. \end{cases}$$

Using the result of Proposition 3.1, we find $\mathcal{T}_1[1_D] = \frac{-\mathrm{i}\operatorname{cap}(D)}{4\pi}\frac{\partial\Phi}{\partial n}$, and then

$$u_{1,1} = -\frac{\mathrm{i}\operatorname{cap}(D)}{4\pi v_0} u_{0,1} = \frac{\mathrm{i}\operatorname{cap}(D)^2}{4\pi v_0 |D|^3} 1_D - \frac{\mathrm{i}\operatorname{cap}(D)}{4\pi v_0}\widetilde{u}_{0,1}.$$

Substituting these results in (3.20) yields the expansion (3.18). □

Corollary 3.1. *We have the following asymptotic expansion for the equation (3.16) characterizing the resonant frequencies:*

$$0 = \int_D u_1(\omega,\delta)\,\mathrm{d}x - 1 = \frac{\omega^2}{v_r^2|D|} - \frac{\operatorname{cap}(D)}{|D|^2}\delta + \frac{\mathrm{i}\omega\delta\operatorname{cap}(D)^2}{4\pi v_0|D|^2} + O(\omega^2+\delta)^2.$$

Consequently, there exist two subwavelength resonant frequencies $\omega^+(\delta)$ and $\omega^-(\delta)$ which are analytic functions of $\delta^{\frac{1}{2}}$ and whose leading asymptotic expansions are given by

$$\omega^{\pm}(\delta) = \pm\frac{v_r\operatorname{cap}(D)^{\frac{1}{2}}}{|D|^{\frac{1}{2}}}\delta^{\frac{1}{2}} - \frac{\mathrm{i}v_r^2\operatorname{cap}(D)^2}{8\pi v_0|D|}\delta + O(\delta^{\frac{3}{2}}). \tag{3.21}$$

The asymptotic (3.21) coincides with the result derived from layer potential representations in [11, 45].

3.3. Modal decomposition and point scatterer approximation

We now show how a modal decomposition and a point-scatterer approximation can be derived for the solution $\hat{u}$ to (3.2) based on this Dirichlet-to-Neumann approach, which we shall reproduce later in Section 5.2 on the time-modulated system (1.1). In this part, we assume that the resonator D is centered around the origin.

We first need an asymptotic expansion of the solution $u_f(\omega,\delta)$ to the variational problem (3.11), which reads in strong form

$$\begin{cases} -\Delta u_f(\omega,\delta) + \left(\int_D u_f(\omega,\delta)\,\mathrm{d}x\right) 1_D - \dfrac{\omega^2}{v_r^2} u_f(\omega,\delta) = 0 \text{ in } D, \\ \dfrac{\partial u_f(\omega,\delta)}{\partial n} - \delta\mathcal{T}^{\frac{\omega}{v_0}}[u_f(\omega,\delta)] = \delta\dfrac{\partial \hat{u}_{\mathrm{in}}}{\partial n} - \delta\mathcal{T}^{\frac{\omega}{v_0}}[\hat{u}_{\mathrm{in}}] \text{ on } \partial D. \end{cases} \tag{3.22}$$

Proposition 3.3. *The function $u_f(\omega,\delta)$ solution to (3.22) satisfies at the leading order:*

$$u_f(\omega,\delta) = \delta\frac{\mathrm{cap}(D)}{|D|}\hat{u}_{\mathrm{in}}(0)1_D - \delta\hat{u}_{\mathrm{in}}(0)|D|\widetilde{u}_{0,1} + O(\omega\delta), \tag{3.23}$$

where $\widetilde{u}_{0,1}$ is the solution to (3.19).

Proof. Since the incident field solves the homogeneous Helmholtz equation (1.3) in $\mathbb{R}^3$, its Fourier transform is supported in the sphere of radius ω/v_0. Therefore, there exists a function $\alpha : \mathbb{S}^2 \to \mathbb{C}$ such that

$$\hat{u}_{\mathrm{in}}(x) = \int_{\mathbb{S}^2} \alpha(\theta) e^{\mathrm{i}\frac{\omega}{v_0}\theta\cdot x}\,\mathrm{d}\sigma(\theta).$$

From this expression, it is clear that we can expand the incident field in powers of ω as follows:

$$\hat{u}_{\mathrm{in}}(x) = \sum_{p=0}^{+\infty}\int_{\mathbb{S}^2}\alpha(\theta)\frac{\mathrm{i}^p\omega^p}{p!v_0^p}(\theta\cdot x)^p\,\mathrm{d}\sigma(\theta) \equiv \sum_{p=0}^{+\infty}\omega^p u_{\mathrm{in},p}(x),$$

where we remark that the functions $u_{\mathrm{in},p}$ satisfy

$$\omega^p u_{\mathrm{in},p}(x) = \frac{1}{p!}\nabla^p\hat{u}_{\mathrm{in}}(0)\cdot x^p, \quad \nabla\hat{u}_{\mathrm{in}}(x) = \sum_{p\geq 1}\omega^p\nabla u_{\mathrm{in},p}(x).$$

Substituting $\hat{u}_{\mathrm{in}}$ into (3.22), we write an ansatz $u_f(\omega,\delta) = \sum_{p\geq 0,k\geq 1}\omega^p\delta^k v_{p,k}$ which yields the following cascade of equations for $(v_{p,k})_{p\geq 0,k\geq 1}$:

$$\begin{cases} -\Delta v_{p,k} + \left(\int_D v_{p,k}\,\mathrm{d}x\right)1_D = \dfrac{1}{v_r^2}v_{p-2,k} \text{ in } D, \\ \qquad\qquad \dfrac{\partial v_{p,k}}{\partial n} = \displaystyle\sum_{n=0}^{p}\frac{1}{v_0^n}\mathcal{T}_n[v_{p-n,k-1}] + \left(\nabla u_{\mathrm{in},p}\cdot n - \sum_{n=0}^{p}\frac{1}{v_0^n}\mathcal{T}_n[u_{\mathrm{in},p-n}]\right)\delta_{k=1} \text{ on } \partial D, \end{cases}$$

where we assume that $v_{p,k} = 0$ for $k \leq 0$ and $p < 0$ by convention. Therefore, we find by using an integration by parts on D for $k = 1$ and $p = 0$ that

$$\int_D v_{0,1}\,\mathrm{d}x = -\int_{\partial D}\hat{u}_{\mathrm{in}}(0)\frac{\partial\Phi}{\partial n}\,\mathrm{d}\sigma = \mathrm{cap}(D)\hat{u}_{\mathrm{in}}(0),$$

and hence $v_{0,1}$ can be decomposed as

$$v_{0,1} = \frac{\mathrm{cap}(D)}{|D|}\hat{u}_{\mathrm{in}}(0)1_D + \hat{u}_{\mathrm{in}}(0)\widetilde{v}_{0,1},$$

with $\widetilde{v}_{0,1}$ being the unique solution to

$$\begin{cases} -\Delta\widetilde{v}_{0,1} = -\dfrac{\mathrm{cap}(D)}{|D|}1_D \text{ in } D, \\ \dfrac{\partial\widetilde{v}_{0,1}}{\partial n} = -\dfrac{\partial\Phi}{\partial n} \text{ on } \partial D, \\ \displaystyle\int_D \widetilde{v}_{0,1} = 0. \end{cases}$$

Remembering the definition (3.19), we find that $\widetilde{v}_{0,1} = -|D|\widetilde{u}_{0,1}$ and the result follows. □

Coming back to the formula (3.15), we obtain the following modal approximation of the solution $\hat{u}$ inside the domain D.

Corollary 3.2. *For real values of the frequency ω satisfying $\omega = O(\delta^{\frac{1}{2}})$, the solution $\hat{u}$ to (3.12) has the following asymptotic expansion in $H^1(D)$:*

$$\hat{u} \equiv \hat{u}(\omega,\delta) = -\frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega\,\mathrm{cap}(D)}{4\pi v_0}}(1 + O(\delta^{\frac{1}{2}}))1_D,$$

where

$$\omega_M := v_r\sqrt{\frac{\mathrm{cap}(D)}{|D|}}\delta^{\frac{1}{2}}. \tag{3.24}$$

Proof. Inserting (3.18) and (3.23) into (3.15), we obtain:

$$\hat{u} = O(\delta) - \frac{\delta u_{\mathrm{in}}(0)\,\mathrm{cap}(D) + O(\delta\omega)}{\frac{\omega^2}{v_r^2} - \frac{\mathrm{cap}(D)}{|D|}\delta + \frac{\mathrm{i}\omega\delta\,\mathrm{cap}(D)^2}{4\pi v_0|D|} + O((\omega^2+\delta)^2)}\left(1_D + O(\omega^2+\delta)\right),$$

which yields the result after using [45, Lemma 4.3] for bounding the error induced by approximating the denominator. □

Coming back to the representation (3.8) outside the resonator D, we obtain that D behaves in the far field as a point source with a resonant amplification coefficient. The following result was obtained in [45, Proposition 5.5] and in a slightly different form in [11].

Corollary 3.3. *The following far field approximation holds for the scattered field in the regime where ω is real and $\omega = O(\delta^{\frac{1}{2}})$:*

$$\hat{u}(x) - \hat{u}_{\mathrm{in}}(x) = \frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega\,\mathrm{cap}(D)}{4\pi v_0}} \mathrm{cap}(D)(1 + O(\delta^{\frac{1}{2}}) + O(|x|^{-1}))\Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty, \tag{3.25}$$

where we recall that $\Gamma^{\frac{\omega}{v_0}}$ is the fundamental solution (2.7) to the Helmholtz equation in the background medium with wave number ω/v_0.

Proof. Let us recall the following far field approximation for the single layer potential in the low frequency regime:

$$\mathcal{S}_D^{\frac{\omega}{v_0}}[\phi](x) = \left(\int_{\partial D} \phi \, \mathrm{d}\sigma \right) (1 + O(\omega) + O(|x|^{-1}))\Gamma^{\frac{\omega}{v_0}}(x).$$

The result follows from (3.8) with

$$\phi = (\mathcal{S}_D^{\frac{\omega}{v_0}})^{-1}[\hat{u}_{|\partial D} - \hat{u}_{\mathrm{in}|\partial D}] = -\frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega\,\mathrm{cap}(D)}{4\pi v_0}}(1 + O(\delta^{\frac{1}{2}}))(\mathcal{S}_D)^{-1}[1_D] + O(1).$$

□

3.4. Dilute regime and effective medium theory for high-contrast metamaterial

The point scatterer approximation (3.25) is a key ingredient for deriving an effective medium theory for a medium constituted of many small copies of the resonator D. In this part, we sketch the derivation of

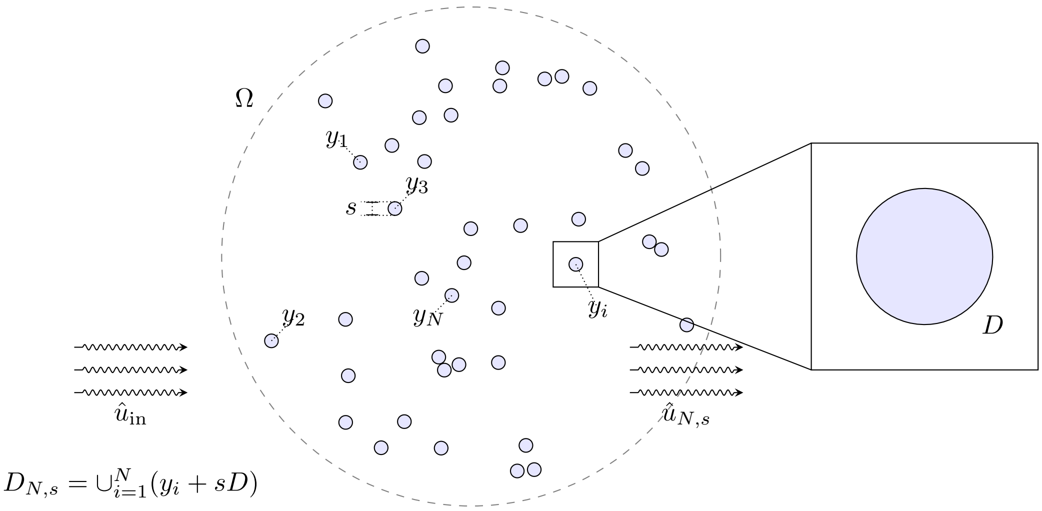

FIGURE 2. Scattering of an incident wave $\hat{u}_{\mathrm{in}}$ by a cloud of N copies of a high-contrast subwavelength resonator D rescaled by a small size factor $s > 0$ and located around centers $(y_i)_{1 \le i \le N}$. The highly-contrasted medium is denoted by $D_{N,s}$ and is filling a bounded domain Ω.

the homogenized equation for a medium containing N resonators rescaled by a factor s, based on [21]. Our motivation is later to adapt this procedure in Section 5.2 in order to obtain, under some conditions, an effective medium theory for the time-modulated system. The procedure outlined in this article remains formal and is based on the well-established Foldy-Lax approximation of the effective medium [48, 47, 21]; we refer the reader to our recent work [43] for a rigorous justification with quantitative error estimates.

The setting is depicted on Figure 2: we consider a set $D_{N,s}$ of N copies of the inclusions D, rescaled by a small factor $s > 0$ and translated in the vicinity of centers $(y_i)_{1 \le i \le N}$:

$$D_{N,s} = \bigcup_{1 \le i \le N} (y_i + sD).$$

The centers $(y_i)_{1 \le i \le N}$ are randomly and independently distributed in a bounded domain $\Omega \subset \mathbb{R}^3$ according to a probability density $V \,\mathrm{d}x$ (this density satisfies $V \in L^\infty(\Omega)$, $V \ge 0$ and $\int_\Omega V \,\mathrm{d}x = 1$). In particular, the law of large number asserts that we have the convergence of the empirical measure in the sense of distributions:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{y_i} \to V\,\mathrm{d}x \text{ as } N \to +\infty.$$

We denote by $\hat{u}_{N,s}$ the solution to the scattering problem in the medium $D_{N,s}$:

$$\begin{cases} \Delta \hat{u}_{N,s} + \dfrac{\omega^2}{v_0^2}\hat{u}_{N,s} = 0 \text{ in } \mathbb{R}^3 \backslash D_{N,s}, \\ \Delta \hat{u}_{N,s} + \dfrac{\omega^2}{v_r^2}\hat{u}_{N,s} = 0 \text{ in } D_{N,s}, \\ \hat{u}_{N,s}|_+ = u_{N,s}|_- \text{ on } \partial D_{N,s}, \\ \left.\dfrac{\partial \hat{u}_{N,s}}{\partial n}\right|_- = \delta \left.\dfrac{\partial \hat{u}_{N,s}}{\partial n}\right|_+ \text{ on } \partial D_{N,s}, \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}\omega}{v_0}\right)(\hat{u}_{N,s} - \hat{u}_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{3.26}$$

Recalling that

$$\mathrm{cap}(sD) = s\,\mathrm{cap}(D), \qquad |sD| = s^3|D|, \tag{3.27}$$

we find that the subwavelength resonant frequency ω_M (equation (3.24)) scales as

$$\omega_M = v_r\sqrt{\frac{\mathrm{cap}(D)}{|D|}}\frac{\delta^{\frac{1}{2}}}{s}.$$

Hence, we assume that there exists some constant $\alpha > 0$ such that the size factor s scales as

$$s \sim \alpha\delta^{\frac{1}{2}},$$

so that ω_M converges to a determined frequency that is of order one. Then, the Foldy-Lax approximation states that in the medium $D_{N,s}$, if the centers y_i are "sufficiently far" from one another, then the resonators behave as a system of N distant point sources. Furthermore, the field scattered at the point y_i should match the contribution of the other point sources according to the law given by point scatterer approximation formula (3.25). Using the rescaling (3.27), we expect therefore $u_{N,s}(y_i) \simeq z_{N,i}$ where $(z_{N,i})_{1\le i\le N}$ satisfies

$$z_{N,i} - \hat{u}_{\mathrm{in}}(y_i) = \sum_{j\ne i}\frac{z_{N,j}}{\frac{\omega^2}{\omega_M^2}-1}s\,\mathrm{cap}(D)\Gamma^{\frac{\omega}{v_0}}(y_i - y_j), \qquad 1 \le i \le N.$$

The latter equation is an algebraic system of N equations for the N unknowns $(z_{N,i})_{1\le i\le N}$, called Foldy-Lax system [46, 4]. Using a law of large number result from [44], we can show the convergence of the solution $(z_{N,i})_{1\le i\le N}$ as $N \to +\infty$ to the values of the solution $(\hat{u}(y_i))_{1\le i\le N}$ of the integral equation

$$\hat{u}(y) - sN\frac{\mathrm{cap}(D)}{\frac{\omega^2}{\omega_M^2}-1}\int_\Omega \Gamma^{\frac{\omega}{v_0}}(y-y')\hat{u}(y')V(y')\,\mathrm{d}y' = \hat{u}_{\mathrm{in}}(y).$$

Left-multiplying this equation by the operator $\Delta + \omega^2/v_0^2$, we finally obtain the following equation for the homogenized wave field $\hat{u}$ in the effective medium Ω:

$$\begin{cases} \Delta\hat{u} + \left(\dfrac{\omega^2}{v_0^2} - sN\dfrac{\mathrm{cap}(D)}{\frac{\omega^2}{\omega_M^2}-1}V1_\Omega\right)u = 0 \text{ in } \mathbb{R}^3, \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}\omega}{v_0}\right)(\hat{u} - \hat{u}_{\mathrm{in}})(x) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{3.28}$$

Observe that the resonant denominator $\omega^2/\omega_M^2 - 1$ can take negative or positive values depending on whether $\omega < \omega_M$ or $\omega > \omega_M$. The heterogeneous medium behaves then respectively as a highly dispersive or a highly dissipative medium. We refer the reader to [43] for a formal analysis and a statement of assumptions on $s \to 0$, $N \to +\infty$ and $\delta \to 0$ for which there is a rigorous convergence result $\hat{u}_{N,s} \to \hat{u}$.

4. Outgoing waves in a periodically time-modulated medium

In the remainder of this paper, we consider the time-modulated scattering problem (1.1) with a non-constant time-modulation $\rho \in L^\infty_{\mathrm{per}}((0,T),\mathbb{R})$. The goal of this section is to give a precise sense to "outgoing" for the scattered wave generated by the time-modulated resonator D. Inspired from the Floquet-Bloch decomposition [64, 1, 15], it is natural to seek a solution to (1.1) of the form

$$u(t,x) = e^{-\mathrm{i}\omega t}\hat{u}(t,x), \tag{4.1}$$

where the function $(t,x) \mapsto \hat{u}(t,x)$ is T–periodic in time. The ansatz (4.1) generalizes the time-harmonic regime (3.1) to the periodically modulated case; this form is peculiar to the time-harmonic assumption (1.2) on the incident field but it would generalize to arbitrary incident waves by using an appropriate Floquet-Bloch decomposition in time (see the Appendix A for a mathematical setting and Appendix A.4 for a formal justification of the ansatz (4.1) based on the Bloch transform). Inserting (4.1) into (1.1) leads us to consider the following time-dependent Helmholtz equation for $\hat{u}(t,x)$:

$$\begin{cases} \dfrac{1}{v_0^2}\left(-\mathrm{i}\omega+\partial_t\right)^2\hat{u}(t,x)-\Delta\hat{u}(t,x)=0, & (t,x)\in\mathbb{R}\times\mathbb{R}^3\backslash\overline{D}, \\ \dfrac{1}{v_r^2}\left(-\mathrm{i}\omega+\partial_t\right)^2\hat{u}(t,x)-\dfrac{1}{\rho(t)}\Delta\hat{u}(t,x)=0, & (t,x)\in\mathbb{R}\times D, \\ \dfrac{1}{\rho(t)}\left.\dfrac{\partial\hat{u}(t,x)}{\partial n}\right|_{-}=\delta\left.\dfrac{\partial\hat{u}(t,x)}{\partial n}\right|_{+}, & (t,x)\in\mathbb{R}\times\partial D, \\ \hat{u}_{|+}(t,x)=\hat{u}_{|-}(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto\hat{u}(t,x)\text{ is }T\text{–periodic}, & \\ e^{-\mathrm{i}\omega t}(\hat{u}(t,x)-\hat{u}_{\mathrm{in}}(t,x))\text{ is outgoing}, & \end{cases} \tag{4.2}$$

where we now need to clarify the meaning of the word "outgoing" in the last line of (4.2).

This section is organized as follows. Section 4.1 starts by stating the set of assumptions on the geometry of the resonator D and on the modulation $\rho(t)$ which are considered in our analysis. In Section 4.2, we show that a suitable definition of "outgoing" for (4.2) is to require that all the Fourier modes of the T-periodic scattered wave be outgoing (in the usual sense), which translates into the following radiation condition for the modulated amplitude $\hat{u}(t,x)$:

$$\left(\partial_{|x|}-\frac{\mathrm{i}\omega}{v_0}+\frac{1}{v_0}\partial_t\right)u(t,x)=O(|x|^{-2})\text{ as }|x|\to+\infty. \tag{4.3}$$

This outgoing radiation condition is associated to a time-perodic Dirichlet-to-Neumann operator $\mathcal{T}^{\omega}_{\mathrm{per}}$, for which we state the main properties. Relying on Fredholm's theory, we propose a well-posedness theory for the time-modulated wave equation (4.2) subjected to the radiation condition (4.3) in Section 4.3, in the regime where both the incident frequency ω and the contrast parameter $\delta>0$ are small. Finally, the last Section 4.4 defines periodic layer potentials $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ and $\mathcal{K}^{\omega *}_{D,\mathrm{per}}$ from a suitable T–periodic Green function $\Gamma^{\omega}_{\mathrm{per}}(t,x)$ satisfying the outgoing radiation condition (4.3). We prove a number of properties which we use in the subsequent Section 5.2.

4.1. Assumptions on the geometry of the resonator D and on the modulation ρ

In order to establish the well-posedness of the problem (4.2), we assume a set of conditions on the geometry of the resonator D and on the modulation $\rho(t)$.

4.1.1. *Nontrapping assumption on D and explicit high frequency bounds*

The remainder of our analysis requires frequency explicit bounds on the exterior Dirichlet-to-Neumann map $\mathcal{T}^k$ of the Helmholtz equation on the domain $\mathbb{R}^3\backslash D$ (Definition 3.1). We therefore assume the following hypothesis which brings substantial simplifications.

(H1) The resonator D is a smooth *nontrapping* set, in the sense of [54, Definition 1.1].

The "nontrapping" condition physically states that incident rays illuminating and reflecting on D according to the law of geometric optics exit any bounded set in finite time, which is the case for instance if D is convex. This property implies the following convenient result obtained in [52].

Proposition 4.1. *For any sufficiently large $R>0$ and $f\in H^{\frac{1}{2}}(\partial D)$, the solution $w_{f,k}$ to the problem*

$$\begin{cases} \left(-k^2-\Delta\right)w_{f,k}=0 \text{ in } \mathbb{R}^3\backslash\overline{D}, \\ w_{f,k}=f \text{ on } \partial D, \\ \left(\partial_{|x|}-\mathrm{i}k\right)w_{f,k}=O(|x|^{-2}) \text{ as } |x|\to+\infty, \end{cases} \tag{4.4}$$

satisfies the bound

$$||\nabla w_{f,k}||_{L^2(B(0,R)\backslash D)}+|k|\,||w_{f,k}||_{L^2(B(0,R)\backslash D)}\leq C||f||_{H^{\frac{1}{2}}(\partial D)}, \tag{4.5}$$

with a constant C independent of $k\in\mathbb{R}$ and f.

This result also entails that the Dirichlet-to-Neumann map $\mathcal{T}^k$ of (3.3) is linearly bounded with respect to the frequency $k\in\mathbb{R}$ (see [32, Lemma 4.2]).

Proposition 4.2. *Assume (H1). The Dirichlet-to-Neumann map $\mathcal{T}^k$ of Definition 3.1 satisfies the following frequency dependent bound:*

$$||\mathcal{T}^k[f]||_{H^{-\frac{1}{2}}(\partial D)} \leq C(|k|\,||f||_{L^2(\partial D)} + ||f||_{H^{\frac{1}{2}}(\partial D)}), \tag{4.6}$$

for a constant $C > 0$ independent of $f \in H^{\frac{1}{2}}(\partial D)$ and k.

4.1.2. *Non-coincidence of Dirichlet eigenvalues with the harmonic frequencies $n\Omega/v_0$*

We consider the following non-degeneracy condition in Section 4.4 when examining the invertibility of a periodic single layer potential:

(H2) $\inf_{\lambda\in\sigma_{\mathrm{dir}}(-\Delta),n\in\mathbb{Z}} \left|\frac{n\Omega}{v_0} - \sqrt{\lambda}\right| > 0$ where $\sigma_{\mathrm{dir}}(-\Delta)$ is the Dirichlet spectrum of $-\Delta$.

This assumption implies in particular that the single layer potential $\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is an invertible operator for any $n \in \mathbb{Z}$ and ω sufficiently small.

4.1.3. *Assumptions on the modulation ρ: coincidence of a finite number of Sturm-Liouville and Neumann eigenvalues*

We now list the assumptions on the modulation ρ which we consider in all the remainder of this paper, and in particular in our well-posedness analysis of the time-modulated reduced wave equation (4.2). First, we assume that the modulation is T–periodic ($\rho(t+T) = \rho(t)$ for almost any $t \in \mathbb{R}$), and is such that $\frac{1}{\rho}$ is a bounded function, positively bounded from below:

$$\frac{1}{\rho} \in L^\infty_{\mathrm{per}}((0,T),\mathbb{R}) \text{ and } \frac{1}{\rho(t)} > c \text{ for a.e. } t \in \mathbb{R}, \text{ for a constant } c > 0.$$

In addition, we make some hypothesis on the kernel of the time-modulated wave operator with periodic and Neumann boundary conditions on $(0,T) \times \partial D$. Let us first introduce an analytic family of relevant Sturm-Liouville eigenvalues and eigenvectors which extend those of the problem (1.6).

Proposition 4.3. *There exist a sequence of eigenfunctions $(p_m(\cdot;\omega))_{m\in\mathbb{N}}$ and eigenvalues $(\mu_m(\omega))_{m\in\mathbb{N}}$ parameterized by $\omega \in (-\Omega/2, \Omega/2)$ solving the eigenvalue problem*

$$\begin{cases} -\left(-\mathrm{i}\omega + \dfrac{\mathrm{d}}{\mathrm{d}t}\right)^2 p_m(t;\omega) = \dfrac{\mu_m(\omega)}{\rho(t)} p_m(t;\omega), \\ p_m(\cdot;\omega) \text{ is } T\text{–periodic}, \end{cases} \tag{4.7}$$

and which satisfy the following properties:

(i) the eigenvalues $(\mu_m(\omega))$ are non-negative and ordered increasingly at $\omega = 0$:

$$0 = \mu_0(0) < \mu_1(0) \leq \mu_2(0) \ldots;$$

(ii) the sequence $(p_m(\cdot;\omega))_{m\in\mathbb{N}}$ is an orthonormal basis of $H^1_{\mathrm{per}}((0,T))$ with respect to the weighted inner product induced by $1/\rho$:

$$\frac{1}{T}\int_0^T \frac{1}{\rho(t)} p_m(t;\omega)\overline{p_{m'}(t;\omega)}\,\mathrm{d}t = \delta_{mm'} \text{ for any } m, m' \in \mathbb{N}; \tag{4.8}$$

(iii) the eigenvalues $(\mu_m(\omega))$ and the orthonormal sequence $(p_m(\cdot;\omega))$ are analytic functions of ω.

Proof. This proposition is the consequence of classical perturbation theory for the analytic self-adjoint operator $\omega \mapsto -(-\mathrm{i}\omega + \mathrm{d}/\mathrm{d}t)$, see [58, 25]. □

By convention, we assume that $p_m(t) \equiv p_m(t;0)$ and $\mu_m \equiv \mu_m(0)$ where we recall the definition (1.6) for the Sturm-Liouville eigenfunctions $(p_m(t))_{m\in\mathbb{N}}$ and eigenvalues $(\mu_m)_{m\in\mathbb{N}}$. Let us also recall the definition (1.5) of the Neumann eigenfunctions $(\phi_l)_{l\in\mathbb{N}}$ and eigenvalues $(\lambda_l(x))_{l\in\mathbb{N}}$. The following proposition motivates the consideration of the set Λ introduced in (1.7).

Proposition 4.4. *The non-zero solutions $v \in H^1_{\mathrm{per}}((0,T), H^1(D))$ to the problem*

$$\begin{cases} \left(\dfrac{1}{v_r^2}\partial_{tt} - \dfrac{1}{\rho(t)}\Delta\right) v = 0, & (t,x) \in \mathbb{R}\times D, \\ \dfrac{1}{\rho(t)}\dfrac{\partial v}{\partial n} = 0, & (t,x) \in \mathbb{R}\times\partial D, \\ t \mapsto v(t,x) \text{ is } T\text{–periodic}, \end{cases}$$

are the functions of the form

$$v(t,x) = p_m(t)\phi_l(x), \text{ with } (m,l) \in \Lambda,$$

where Λ is the set defined by (1.7), that is the set of tuples (m,l) associated to a Sturm-Liouville eigenvalue μ_m and a Neumann eigenvalue satisfying $\lambda_l = \mu_m/v_r^2$.

Proof. Decomposing $v(t,x) := \sum_{m\in\mathbb{N}} p_m(t)u_m(x)$, we find that

$$\begin{cases} -\dfrac{\mu_m}{v_r^2}u_m - \Delta u_m = 0, \\ \qquad\qquad\dfrac{\partial u_m}{\partial n} = 0 \text{ on } \partial D, \end{cases}$$

which shows that u_m is a Neumann eigenvector of $-\Delta$ with eigenvalue μ_m/v_r^2. □

Throughout our analysis of the reduced T–periodic wave equation (4.2), we assume the following assumptions on the modulation ρ.

(H3) The set Λ defined by (1.7) is finite and all the eigenvalues μ_m or λ_l with $(m,l)\in\Lambda$ are simple. Furthermore, there exists a constant $c>0$ such that

$$\inf_{\substack{(m,l)\in\mathbb{N}\times\mathbb{N}\backslash\Lambda,\\ |\omega|\le c}} \left|1-\frac{\mu_m(\omega)}{\lambda_l v_r^2}\right| > 0, \tag{4.9}$$

where $\mu_m(\omega)$ is the Sturm-Liouville eigenvalue of (4.7).

The bound (4.9) requires that the Sturm-Liouville eigenvalues $(\mu_m(\omega))$ and the Neumann eigenvalues $(\lambda_l(\omega))_{l\in\mathbb{N}}$ remain well separated for $(m,l)\notin\Lambda$ and for $|\omega|<c$. For instance, there are no subsequences $(\mu_{m_n})_{n\in\mathbb{N}}$ and $(\lambda_{l_n})_{n\in\mathbb{N}}$ such that the difference $|\mu_{m_n}/v_r^2-\lambda_{l_n}|$ becomes arbitrarily small for subsequences of integers l_n, $m_n\to+\infty$. We expect that this assumption of non coincidence of more than a finite number of eigenvalues is generically satisfied: in fact, in general, we expect that Λ is reduced to the singleton $\{(0,0)\}$.

Remark 4.1. It is possible to adapt the above hypotheses in order to take into account a time-modulated bulk-modulus $\kappa_r \equiv \kappa_r\kappa(t)$ with $\kappa(t+T)=\kappa(t)$. The Sturm-Liouville eigenvalue problem (4.7) would need to be changed into

$$\begin{cases} -\left(-\mathrm{i}\omega+\dfrac{\mathrm{d}}{\mathrm{d}t}\right)\left[\dfrac{1}{\kappa(t)}\left(-\mathrm{i}\omega+\dfrac{\mathrm{d}}{\mathrm{d}t}\right)\right]p_m(t;\omega) = \dfrac{\mu_m(\omega)}{\rho(t)}p_m(t;\omega), \\ \qquad\qquad p_m(\cdot;\omega) \text{ is } T\text{–periodic.} \end{cases}$$

4.2. **Time-dependent Dirichlet-to-Neumann map and outgoing periodic waves**

We now discriminate a suitable outgoing radiation condition for the reduced T–periodic wave equation (4.2), before introducing an associated Dirichlet-to-Neumann map which characterize time-modulated waves in the background medium $\mathbb{R}^3\backslash D$.

4.2.1. *A time-periodic outgoing radiation condition*

For a T–periodic boundary datum $f\in L^2((0,T),H^{\frac12}(\partial D))$ and $\omega\in(-\Omega/2,\Omega/2)$, we consider the following time-dependent Helmholtz problem in the exterior domain $\mathbb{R}^3\backslash\overline{D}$:

$$\begin{cases} \dfrac{1}{v_0^2}\left(-\mathrm{i}\omega+\partial_t\right)^2 w_f(t,x) - \Delta w_f(t,x) = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}^3\backslash\overline{D}, \\ w_f(t,x;\omega) = f(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto w_f(t,x;\omega) \text{ is } T\text{–periodic.} \end{cases} \tag{4.10}$$

We denote by $(\hat f_n)_{n\in\mathbb{Z}}$ the Fourier coefficients of f, namely

$$f(t,x) = \sum_{n\in\mathbb{Z}} \hat f_n(x)e^{-\mathrm{i}n\Omega t}, \quad \text{with } ||f||^2_{L^2_{\mathrm{per}}((0,T),H^{\frac12}(\partial D))} = \sum_{n\in\mathbb{Z}} ||\hat f_n||^2_{H^{\frac12}(\partial D)} < +\infty.$$

The following lemma shows that one can discriminate an outgoing solution to (4.15) by requiring all the Fourier modes of w_f to be outgoing.

Lemma 4.1. *Assume (H1). For any $f\in L^2_{\mathrm{per}}((0,T),H^{\frac12}(\partial D))$, there exists a unique solution*

$$w_f \in H^1_{\mathrm{per}}((0,T),L^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D}))\cap L^2_{\mathrm{per}}((0,T),H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D}))$$

to (4.15) whose Fourier expansion reads

$$w_f(t,x) = \sum_{n\in\mathbb{Z}} \hat w_n(x)e^{-\mathrm{i}n\Omega t}, \tag{4.11}$$

and which is determined by the following cascade of Helmholtz equations for the coefficients $\hat{w}_n$:

$$\begin{cases} -\dfrac{1}{v_0^2}(\omega+n\Omega)^2\hat{w}_n - \Delta\hat{w}_n = 0 \ \text{in } \mathbb{R}^3\backslash\overline{D}, \\ \hat{w}_n = \hat{f}_n \ \text{on } \partial D, \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}(\omega+n\Omega)}{v_0}\right)\hat{w}_n = O(|x|^{-2}) \ \text{as } |x|\to+\infty, \ \text{for any } n\in\mathbb{Z}. \end{cases} \tag{4.12}$$

Owing to the Sommerfeld radiation condition, w_f is the unique solution such that $\hat{w}_n(x)e^{-\mathrm{i}n\Omega t}e^{-\mathrm{i}\omega t}$ is outgoing for any $n\in\mathbb{Z}$, and it further satisfies the following continuity estimate on any ball $B(0,R)$ of radius $R>0$:

$$||w_f||_{H^1_{\mathrm{per}}((0,T),L^2(B(0,R)))} + ||w_f||_{L^2_{\mathrm{per}}((0,T),H^1(B(0,R)))} \leq C||f||_{L^2_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))}, \tag{4.13}$$

for a constant $C>0$ independent of f. Going back to the temporal domain, the cascade of radiation conditions of (4.12) can be rewritten in the following radiation condition for the solution $w_f(t,x)$ to (4.15):

$$\left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} + \frac{1}{v_0}\partial_t\right)w_f(t,x) = O(|x|^{-2}) \ \text{as } |x|\to+\infty. \tag{4.14}$$

Proof. The fact that w_f is a solution to (4.15) is clear by decomposing w_f on its Fourier modes. The continuity estimate (4.13) is a consequence of the assumption (H1) and the bound (4.5), which implies that for any $R>0$, there exists a constant $C>0$ such that

$$||w||^2_{H^1_{\mathrm{per}}((0,T),L^2(B(0,R)))} = \sum_{n\in\mathbb{Z}}(1+n^2)||\hat{w}_n||^2_{L^2(B(0,R))} \leq C\sum_{n\in\mathbb{Z}}||\hat{f}_n||^2_{H^{\frac{1}{2}}(\partial D)} \leq C||f||^2_{L^2_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))},$$

and similarly,

$$||w||^2_{L^2_{\mathrm{per}}((0,T),H^1(B(0,R)))} = \sum_{n\in\mathbb{Z}}||\hat{w}_n||^2_{H^1(B(0,R))} \leq C\sum_{n\in\mathbb{Z}}||\hat{f}_n||^2_{H^{\frac{1}{2}}(\partial D)} \leq C||f||^2_{L^2_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))}.$$

□

4.2.2. *Time-periodic Dirichlet-to-Neumann map and its main properties*

Definition 4.1 (Time-periodic Dirichlet-to-Neumann map)**.** Assume (H1). The Dirichlet-to-Neumann map associated to the time-dependent Helmholtz equation (4.15) is the operator $\mathcal{T}^{\omega}_{\mathrm{per}} : L^2_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D)) \to L^2_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ defined by

$$\mathcal{T}^{\omega}_{\mathrm{per}}[f] = \frac{\partial w_f}{\partial n}, \qquad \text{for any } f\in L^2_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D)),$$

where w_f is the unique solution to (4.15) equipped with the radiation condition (4.14):

$$\begin{cases} \dfrac{1}{v_0^2}\left(-\mathrm{i}\omega+\partial_t\right)^2 w_f(t,x) - \Delta w_f(t,x) = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}^3\backslash\overline{D}, \\ w_f(t,x;\omega) = f(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto w_f(t,x;\omega) \text{ is } T\text{–periodic}, & \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}\omega}{v_0} + \dfrac{1}{v_0}\partial_t\right)w_f(x) = O(|x|^{-2}) \text{ as } |x|\to+\infty. & \end{cases} \tag{4.15}$$

In the next propositions, we establish a number of useful properties of the T–periodic Dirichlet-to-Neumann map $\mathcal{T}^{\omega}_{\mathrm{per}}$.

Proposition 4.5. *Assume (H1). The periodic Dirichlet-to-Neumann map $\mathcal{T}^{\omega}_{\mathrm{per}}$ admits the following trigonometric series expansion for any $f\in L^2_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))$:*

$$\mathcal{T}^{\omega}_{\mathrm{per}}[f] = \sum_{n\in\mathbb{Z}}\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}[\hat{f}_n]e^{-\mathrm{i}n\Omega t}, \tag{4.16}$$

where $(\hat{f}_n)_{n\in\mathbb{Z}}$ are the Fourier coefficients of f, and $\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}$ is the exterior Dirichlet-to-Neumann operator of the Helmholtz equation with wave number $(\omega+n\Omega)/v_0$ defined in (3.3).

Proof. This is a direct consequence of the Fourier expansion (4.11). □

The following property plays an important role in the well-posedness analysis of (4.2) in Section 4.3 below.

Proposition 4.6. *The periodic Dirichlet-to-Neumann $\mathcal{T}^\omega_{\text{per}}$ is a continuous operator from $L^2((0,T),H^{\frac12}(\partial T))\cap H^{\frac12}((0,T),L^2(\partial D))$ into its dual: there exists a uniform constant $C>0$ such that for any $f\in L^2((0,T),H^{\frac12}(\partial T))\cap H^{\frac12}((0,T),L^2(\partial D))$,*

$$\left|\int_0^T\int_{\partial D}\mathcal{T}^\omega_{\text{per}}[f]\overline{g}\,\mathrm{d}\sigma\,\mathrm{d}t\right|\leq C||f||_{L^2((0,T),H^{\frac12}(\partial D))\cap H^{\frac12}_{\text{per}}((0,T),L^2(\partial D))}||g||_{H^{\frac12}_{\text{per}}((0,T),L^2(\partial D))}.$$

Proof. Using the Plancherel identity and the bound (4.6), we obtain:

$$\begin{aligned}\left|\int_0^T\int_{\partial D}\mathcal{T}^\omega_{\text{per}}[f]\overline{g}\,\mathrm{d}\sigma\,\mathrm{d}t\right| &= \left|T\sum_{n\in\mathbb{Z}}\int_{\partial D}\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}[\hat{f}_n]\overline{g}_n\,\mathrm{d}\sigma\right|\\ &\leq TC\sum_{n\in\mathbb{Z}}(|n|\,||\hat{f}_n||_{L^2(\partial D)}+||\hat{f}_n||_{H^{\frac12}(\partial D)})||\hat{g}_n||_{L^2(\partial D)}\\ &\leq C(||f||_{H^{\frac12}_{\text{per}}((0,T),L^2(\partial D))}||g||_{H^{\frac12}_{\text{per}}((0,T),L^2(\partial D))}+||f||_{L^2_{\text{per}}((0,T),H^{\frac12}(\partial D)}||g||_{L^2((0,T),L^2(\partial D))}).\end{aligned}$$

□

Remark 4.2. If (H2) holds, then $\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}$ can be rewritten for any $\omega\in\mathbb{R}$ sufficiently small and $n\in\mathbb{Z}$ as

$$\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}=\left(\frac12+\mathcal{K}_D^{\frac{\omega+n\Omega}{v_0}*}\right)(\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}})^{-1}.$$

The next proposition will be needed in the analysis of resonances in Section 5.

Proposition 4.7. *Assume (H1). $\mathcal{T}^\omega_{\text{per}}$ is an analytic operator for ω in a neighborhood of zero: there exist operators $(\mathcal{T}_k)_{k\in\mathbb{N}}$ of $L^2_{\text{per}}((0,T),H^{\frac12}(\partial D))\to H^{-1}_{\text{per}}((0,T),H^{-\frac12}(\partial D))$ such that*

$$\mathcal{T}^\omega_{\text{per}}=\sum_{k=0}^{+\infty}\frac{\omega^k}{v_0^k}\mathcal{T}_{\text{per},k}. \tag{4.17}$$

In fact, in view of (4.16), the operators $\mathcal{T}_{\text{per},k}$ are given by

$$\mathcal{T}_{\text{per},k}[f]=\sum_{n\in\mathbb{Z}}\mathcal{T}_k^{\frac{n\Omega}{v_0}}[f_n]e^{-\mathrm{i}n\Omega t},$$

where $(\mathcal{T}_k^{\frac{n\Omega}{v_0}})_{k\in\mathbb{N}}$ denote the coefficients of the asymptotic expansion $\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}=\sum_{k\in\mathbb{N}}\frac{\omega^k}{v_0^k}\mathcal{T}_k^{\frac{n\Omega}{v_0}}$ of the Dirichlet-to-Neumann operator of the Helmholtz equation with wave number $(\omega+n\Omega)/v_0$.

Proof. It is a classical fact that the exterior Dirichlet-to-Neumann map $\mathcal{T}^k$ is analytic with respect to any wave number $k\in\mathbb{R}$ in dimension 3; for instance, $\mathcal{T}^k$ can be expressed in terms of the inverse of the combined field layer potential, see [31, Equation (2.85)]. This implies the analyticity of $\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}$, hence of $\mathcal{T}^\omega_{\text{per}}$. □

The first term $\mathcal{T}_{\text{per},0}\equiv\mathcal{T}^0_{\text{per}}$ of the asymptotic expansion (4.17) of $\mathcal{T}^\omega_{\text{per}}$ exhibits a symmetry property upon time-reversal.

Proposition 4.8. *Assume (H1) and (H2). Let $f,g\in L^2_{\text{per}}((0,T),H^{\frac12}(\partial D))$. It holds*

$$\int_0^T\int_{\partial D}\mathcal{T}^0_{\text{per}}[f]\overline{g}\,\mathrm{d}\sigma\,\mathrm{d}t=\int_0^T\int_{\partial D}f\circ\tau\overline{\mathcal{T}^0_{\text{per}}[g\circ\tau]}\,\mathrm{d}\sigma\,\mathrm{d}t, \tag{4.18}$$

where $\tau(t)=-t$ is the time-reversal operation. In particular, $\mathcal{T}^0_{\text{per}}$ is a hermitian operator when restricted to T–periodic functions which are invariant by time-reversal.

Proof. Denote by $(\hat{f}_n)_{n\in\mathbb{Z}}$ and $(\hat{g}_n)_{n\in\mathbb{Z}}$ the Fourier coefficients of f and g. From (4.16) and (4.17), the following trigonometric expansion holds for $\mathcal{T}^0_{\text{per}}[f]\equiv\mathcal{T}_{\text{per},0}[f]$:

$$\mathcal{T}^0_{\text{per}}[f](x,t)=\sum_{n\in\mathbb{Z}}\left(\frac12 I+\mathcal{K}_D^{\frac{n\Omega}{v_0}*}\right)(\mathcal{S}^{\frac{n\Omega}{v_0}})^{-1}[\hat{f}_n](x)e^{-\mathrm{i}n\Omega t},\qquad t\in\mathbb{R},\ x\in\partial D.$$

Therefore, the Plancherel and the Calderón identity imply

$$\begin{aligned}\int_0^T\int_{\partial D}\mathcal{T}_{\text{per},0}[f]\overline{g}\,\mathrm{d}\sigma\,\mathrm{d}t &= T\sum_{n\in\mathbb{Z}}\int_{\partial D}\left(\frac12+\mathcal{K}_D^{\frac{n\Omega}{v_0}*}\right)(\mathcal{S}_D^{\frac{n\Omega}{v_0}})^{-1}[\widehat{f}_n]\overline{\widehat{g}_n}\,\mathrm{d}\sigma=T\sum_{n\in\mathbb{Z}}\int_{\partial D}\overline{\widehat{f}_n(\mathcal{S}_D^{\frac{-n\Omega}{v_0}})^{-1}\left(\frac12+\mathcal{K}_D^{\frac{-n\Omega}{v_0}}\right)[\widehat{g}_n]}\,\mathrm{d}\sigma\\ &= T\sum_{n\in\mathbb{Z}}\int_{\partial D}\widehat{f}_n\overline{\left(\frac12+\mathcal{K}_D^{\frac{-n\Omega}{v_0}*}\right)(\mathcal{S}_D^{\frac{-n\Omega}{v_0}})^{-1}[\widehat{g}_n]}\,\mathrm{d}\sigma=T\sum_{n\in\mathbb{Z}}\int_{\partial D}\widehat{f}_{-n}\overline{\left(\frac12+\mathcal{K}_D^{\frac{n\Omega}{v_0}*}\right)(\mathcal{S}_D^{\frac{n\Omega}{v_0}})^{-1}[\widehat{g}_{-n}]}\,\mathrm{d}\sigma\\ &= \int_0^T\int_{\partial D}f\circ\tau\overline{\mathcal{T}_{\text{per},0}[g\circ\tau]}\,\mathrm{d}\sigma\,\mathrm{d}t,\end{aligned}$$

where we have used that $(\hat{f}_{-n})_{n\in\mathbb{Z}}$ and $(\hat{g}_{-n})_{n\in\mathbb{Z}}$ are the Fourier coefficients of $f\circ\tau$ and $g\circ\tau$. $\square$

We conclude this part by stating a positivity property for $\mathcal{T}^0_{\mathrm{per}}$ in the case of a spherical resonator D.

Proposition 4.9. *Assume (H1) and that D is a ball: $D = B(0,R)$ for some $R > 0$. Then there exists a constant $C > 0$ such that for any $f \in L^2_{\mathrm{per}}((0,T), H^{\frac{1}{2}}(\partial D)) \cap H^{\frac{1}{2}}_{\mathrm{per}}((0,T), H^{\frac{1}{2}}(\partial D))$,*

$$-\Re\left(\int_0^T\int_{\partial D}\mathcal{T}^0_{\mathrm{per}}[f]\overline{f}\,\mathrm{d}\sigma\,\mathrm{d}t\right) \geq C\|f\|^2_{L^2((0,T),L^2(\partial D))}. \tag{4.19}$$

Proof. Let us recall the uniform negativity of the real part of the exterior Dirichlet-to-Neumann map $\mathcal{T}^k$ with wave number $k\in\mathbb{R}$ on the ball, also known as the capacity operator: [74, Theorem 2.6.4] implies the existence of a constant $C>0$ independent of k such that

$$-\Re\left(\int_0^T\int_{\partial D}\mathcal{T}^k[\hat{f}]\overline{\hat{f}}\,\mathrm{d}\sigma\,\mathrm{d}t\right) \geq C\|\hat{f}\|^2_{L^2((0,T),L^2(\partial D))}. \tag{4.20}$$

Using the Plancherel identity, we therefore obtain

$$\begin{aligned}-\Re\left(\int_0^T\int_{\partial D}\mathcal{T}^0_{\mathrm{per}}[f]\overline{f}\,\mathrm{d}\sigma\,\mathrm{d}t\right) &= -T\sum_{n\in\mathbb{Z}}\Re\left(\int_{\partial D}\mathcal{T}_{0,n}[\hat{f}_n]\overline{\hat{f}_n}\,\mathrm{d}\sigma\right) \geq TC\sum_{n\in\mathbb{Z}}\|\hat{f}_n\|^2_{L^2(\partial D)}\\ &\geq C\int_0^T\int_{\partial D}|f(t,x)|^2\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned}$$

$\square$

4.3. Well-posedness of the time-periodic reduced wave equation

Based on the properties of the time-periodic Dirichlet-to-Neumann operator $\mathcal{T}^\omega_{\mathrm{per}}$ established in the previous Section 4.2.2, we now provide an existence and uniqueness theory for the time-periodic reduced wave equation (4.2). Following Section 3, the system (4.2) can be rewritten as a partial differential equation posed on the bounded domain $(0,T)\times D$:

$$\begin{cases}\dfrac{1}{v_r^2}\left(-\mathrm{i}\omega+\partial_t\right)^2\hat{u}(t,x) - \dfrac{1}{\rho(t)}\Delta\hat{u}(t,x) = 0, & (t,x)\in\mathbb{R}\times D,\\ \dfrac{1}{\rho(t)}\dfrac{\partial\hat{u}(t,x)}{\partial n} - \delta\mathcal{T}_\omega[\hat{u}(t,x)] = \delta\left(\dfrac{\partial\hat{u}_{\mathrm{in}}}{\partial n} - \mathcal{T}^\omega_{\mathrm{per}}[\hat{u}_{\mathrm{in}}]\right), & (t,x)\in\mathbb{R}\times\partial D,\\ t\mapsto\hat{u}(t,x)\text{ is }T\text{–periodic.}\end{cases} \tag{4.21}$$

We recall the definition of the space $V = H^1_{\mathrm{per}}((0,T),L^2(D))\cap L^2_{\mathrm{per}}((0,T),H^1(D))$ (see Section 2.1) and we consider the following variational formulation for (4.21):

$$\text{find } u\in V \text{ such that } \forall v\in V,\quad a(u,v) = \delta\int_0^T\int_{\partial D}\left(\frac{\partial\hat{u}_{\mathrm{in}}}{\partial n} - \mathcal{T}^\omega_{\mathrm{per}}[\hat{u}_{\mathrm{in}}]\right)\overline{v}\,\mathrm{d}\sigma\,\mathrm{d}t, \tag{4.22}$$

where $a(u,v)$ is defined for any $u,v\in V$ by

$$a(u,v) := \int_0^T\int_D\left[\frac{1}{\rho(t)}\nabla u\cdot\nabla\overline{v} - \frac{1}{v_r^2}\left[(-\mathrm{i}\omega+\partial_t)u\right]\left[\overline{(-\mathrm{i}\omega+\partial_t)v}\right]\right]\mathrm{d}x\,\mathrm{d}t - \delta\int_0^T\int_{\partial D}\mathcal{T}^\omega_{\mathrm{per}}[u]\overline{v}\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{4.23}$$

In order to prove that (4.21) is well-posed, we resort to Fredholm's theory [71]. For this purpose, we introduce $a_{0,0}$ the bilinear form

$$a_{0,0}(u,v) := \int_0^T\int_D\left(\frac{1}{\rho(t)}\nabla u\cdot\nabla v - \frac{1}{v_r^2}\partial_t u\partial_t\overline{v}\right)\mathrm{d}x\,\mathrm{d}t + \sum_{(m,l)\in\Lambda}\langle u, p_m\phi_l\rangle_H\overline{\langle v,p_m\phi_l\rangle_H}, \tag{4.24}$$

where we recall the definition (2.2) of the inner product on $H = L^2_{\mathrm{per}}((0,T),L^2(D))$, and where the assumption (H3) ensures that the set Λ is finite.

Remark that any $u\in V$ can be decomposed as the convergent sum

$$u(t,x) = \sum_{m,l\in\mathbb{N}}u_{ml}p_m(t)\phi_l(x) \text{ with } \|u\|^2_V = |u_{00}|^2 + \sum_{m,l\in\mathbb{N}}(\mu_m+\lambda_l)|u_{ml}|^2, \tag{4.25}$$

where we have denoted $u_{ml} := \langle u, p_m\phi_l\rangle_H$. In what follows, we denote by $^\dagger : H\to H$ the operator defined by

$$\forall u\in H,\ u^\dagger := \sum_{m,l\in\mathbb{N}}\mathrm{sign}\left(\lambda_l - \frac{\mu_m}{v_r^2}\right)\langle u,p_m\phi_l\rangle_H p_m\phi_l \text{ with } \mathrm{sign}(t) := \begin{cases}1 \text{ if } t\geq 0,\\ -1 \text{ if } t<0.\end{cases}$$

Obviously, the operator † is an isometry of H and V, and it holds $(u^\dagger)^\dagger = u$ for any $u\in H$.

Proposition 4.10. *Assume (H3). There exists a uniform constant $c > 0$ such that the following coercivity inequality holds:*

$$\forall u \in V, \quad a_{0,0}(u, u^\dagger) \geq c||u||_V^2. \tag{4.26}$$

Proof. By the definition of $\dagger$, it holds

$$\begin{aligned} a_{0,0}(u, u^\dagger) &= \sum_{(m,l)\in\mathbb{N}\times\mathbb{N}} \left|\lambda_l - \frac{\mu_m}{v_r^2}\right| |u_{ml}|^2 + \sum_{(m,l)\in\Lambda} |u_{ml}|^2 \\ &= \sum_{(m,l)\in\mathbb{N}\times\mathbb{N}\setminus\Lambda} \frac{1}{\lambda_l + \mu_m} \left|\lambda_l - \frac{\mu_m}{v_r^2}\right| (\lambda_l + \mu_m)|u_{ml}|^2 + |u_{0,0}|^2 + \sum_{(m,l)\in\Lambda\setminus\{(0,0)\}} \frac{1}{\lambda_l + \mu_m}(\lambda_l + \mu_m)|u_{ml}|^2. \end{aligned}$$

Since

$$\frac{1}{\lambda_l + \mu_m} \geq \frac{1}{2\max(\lambda_l, \mu_m)} \geq \min\left(\frac{1}{2\lambda_l}, \frac{1}{2\mu_m}\right),$$

we have the lower bound

$$\frac{1}{\lambda_l + \mu_m}\left|\lambda_l - \frac{\mu_m}{v_r^2}\right| \geq \frac{1}{2} \inf_{(m,l)\in\mathbb{N}\times\mathbb{N}\setminus\Lambda} \min\left(\left|1 - \frac{\mu_m}{\lambda_l v_r^2}\right|, \frac{1}{v_r^2}\left|\frac{\lambda_l v_r^2}{\mu_m} - 1\right|\right) \geq c,$$

for a constant $c > 0$ which is inferred from (H3). Then, we obtain

$$a_{0,0}(u, u^\dagger) \geq \min\left(c, \min_{(m,l)\in\Lambda\setminus\{(0,0)\}} \frac{1}{\lambda_l + \mu_m}, 1\right)\left(|u_{0,0}|^2 + \sum_{(m,l)\in\mathbb{N}\times\mathbb{N}} (\lambda_l + \mu_m)|u_{ml}|^2\right),$$

and the result follows from (4.25). □

The property (4.26) entails that the bilinear form a satisfies a Garding's inequality for $\delta > 0$.

Lemma 4.2. *Assume (H1) and (H3). For any $\delta \in \mathbb{R}$ in small neighborhood of zero, there exist constants $\alpha, \beta > 0$ such that the bilinear form a of (4.23) satisfies the following Garding's inequality:*

$$\forall u \in V, \ |a(u, u^\dagger)| \geq \alpha||u||_V^2 - \beta||u||_H^2. \tag{4.27}$$

Proof. First, it is a classical fact that the trace of a function $u \in V = H^1_{\text{per}}((0,T), L^2(D)) \cap L^2_{\text{per}}((0,T), H^1(D))$ is an element of $H^{\frac{1}{2}}_{\text{per}}((0,T), L^2(\partial D)) \cap L^2_{\text{per}}((0,T), H^{\frac{1}{2}}(\partial D))$ (see [67, Theorem 2.1]). Therefore, owing to the Proposition 4.6, the bilinear form a is continuous on V. Then, one can rewrite $a(u, u^\dagger)$ in terms of $a_{0,0}(u, u^\dagger)$ as follows:

$$\begin{aligned} \Re\left(a(u, u^\dagger)\right) &= a_{0,0}(u, u^\dagger) - \sum_{(m,l)\in\Lambda} |\langle u, p_m\phi_l\rangle_H|^2 \\ &\quad - \frac{1}{v_r^2}\int_0^T \int_D \left(\omega^2 u\overline{u}^\dagger - \mathrm{i}\omega u\partial_t\overline{u}^\dagger + \mathrm{i}\omega\partial_t u\overline{u}^\dagger\right) \mathrm{d}x\,\mathrm{d}t - \delta\int_0^T \int_{\partial D} \mathcal{T}^\omega_{\text{per}}[u]\overline{u}^\dagger \,\mathrm{d}\sigma\,\mathrm{d}t. \end{aligned} \tag{4.28}$$

This yields (4.27) by using the Haussdorff-Young inequality ($|ab| \leq (\epsilon a^2 + \epsilon^{-1}b^2)/2$) and the continuity property of $\mathcal{T}^\omega_{\text{per}}$ (Proposition 4.6). □

Relying on Fredholm's theory, we obtain the following existence and uniqueness result.

Proposition 4.11. *Assume (H1) and (H3) and that $\delta \in \mathbb{R}$ is sufficiently small for (4.27) to hold. For any $\delta > 0$, the wave problem (4.2), admits a unique solution $u \in H^1_{\text{per}}((0,T), L^2(D)) \cap L^2_{\text{per}}((0,T), H^1(D))$ for any real ω satisfying $|\omega| < c$ with c being the constant of (H3).*

Proof. Due to the Garding's inequality (4.27), we infer that the operator $A : V \mapsto V'$ defined by

$$\langle Au, v\rangle_{V',V} := a(u, v)$$

is Fredholm. Therefore, (4.23) admits a unique solution in V if and only if A is injective. In order to obtain the injectivity of A, it is enough to show that $a(u, v) = 0$ for any $v \in V$ implies $u = 0$. Assuming therefore $a(u, v) = 0$ for any $v \in V$, let us denote by

$$u(t, x) = \sum_{n\in\mathbb{Z}} \hat{u}_n(x)e^{-\mathrm{i}n\Omega t}, \qquad t \in \mathbb{R},\ x \in D,$$

the Fourier decomposition of u in $L^2_{\text{per}}((0,T), H^1(D))$. Taking the imaginary part of $a(u, u_n e^{-\mathrm{i}n\cdot}) = 0$ for a given $n \in \mathbb{Z}$ yields

$$0 = -\delta T \int_{\partial D} \mathcal{T}^{\frac{\omega + n\Omega}{v_0}}[\hat{u}_n]\overline{\hat{u}_n}\,\mathrm{d}\sigma.$$

This equality implies from the Rellich's uniqueness theorem [36] that $\hat{u}_n = 0$ on ∂D for any $n \in \mathbb{N}$, and hence $u(t,x) = 0$ for any $t \in \mathbb{R}$ and $x \in \partial D$. Decompose now $u(t,x)$ on the Sturm-Liouville and Neumann eigenvectors $p_m(t;\omega)$ and $\phi_l(x)$ defined in (4.7):

$$u(t,x) = \sum_{m,l\in\mathbb{Z}} u_{ml} p_m(t;\omega)\phi_l(x), \tag{4.29}$$

where this series is convergent in V. Computing $a(u(t,x), p_m(\cdot;\omega)\phi_l) = 0$, we find that for any $(m,l) \in \mathbb{N}\times\mathbb{N}$,

$$u_{ml}\left(\lambda_l - \frac{\mu_m(\omega)}{v_r^2}\right) = 0.$$

Therefore, either $u_{ml} = 0$ or $\lambda_l = \mu_m(\omega)/v_r^2$, however (H3) implies that the latter equality is possible only for $(m,l) \in \Lambda$. Consequently, the sum (4.29) is finite and involves indices $(m,l) \in \Lambda$. Then, $u(t,x) = 0$ for any $t \in \mathbb{R}$ and $x \in \partial D$, and if $u(t,x)$ is not identically zero on D, then one of the Neumann eigenfunctions ϕ_l must satisfy $\phi_l = 0$ on ∂D. However, this is not possible, because the only function solving the overdetermined problem

$$\begin{cases} -\Delta\phi = \lambda_l \phi \text{ in } D, \\ \phi = 0 \text{ on } \partial D, \\ \dfrac{\partial \phi}{\partial n} = 0 \text{ on } \partial D, \end{cases}$$

is $\phi = 0$ (this can be seen using the layer potential representation $\phi = \mathcal{D}_D^{\sqrt{\lambda_l}}[[u]] - \mathcal{S}_D^{\sqrt{\lambda_l}}\left[[\frac{\partial\phi}{\partial n}]\right]$). Consequently, $u(t,x) = 0$ for any $t \in \mathbb{R}$ and $x \in D$, which completes the proof. □

4.4. **Time-periodic layer potentials**

In this last part, we give the expression of the outgoing fundamental solution to the exterior T–periodic Helmholtz equation (4.15). We then introduce associated periodic layer potentials and we state their main properties.

4.4.1. *Outgoing fundamental solution*

The radiation condition (4.14) is related to the choice of a particular Green function for (4.15). We define the periodic outgoing fundamental solution $\Gamma^{\omega}_{\text{per}}(t,x)$ to be the unique T–periodic distribution solution to the time-dependent Helmholtz equation

$$\left[\frac{1}{v_0^2}(-\mathrm{i}\omega + \partial_t)^2 - \Delta\right]\Gamma^{\omega}_{\text{per}}(t,x) = \sum_{n\in\mathbb{Z}} \delta(t-nT)\delta(x), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \tag{4.30}$$

satisfying the radiation condition of (4.15).

Proposition 4.12. *The distribution*

$$\Gamma^{\omega}_{\text{per}}(t,x) := -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi|x|}\sum_{n\in\mathbb{Z}}\delta\left(\frac{|x|}{v_0} - t - nT\right) = \sum_{n\in\mathbb{Z}} \Gamma^{\frac{\omega+n\Omega}{v_0}}(x)e^{-\mathrm{i}n\Omega t}, \tag{4.31}$$

is the unique fundamental solution to (4.30) *satisfying the outgoing radiation condition*

$$\left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} + \frac{1}{v_0}\partial_t\right)\Gamma^{\omega}_{\text{per}}(t,x) = O(|x|^{-2}) \text{ as } |x| \to +\infty.$$

Proof. Let us solve (4.30) using Fourier series: we represent $\Gamma^{\omega}_{\text{perper}}$ as

$$\Gamma^{\omega}_{\text{per}}(t,x) = \sum_{n\in\mathbb{Z}} \widehat{\Gamma}^{\omega}_{\text{per},n}(x)e^{-\mathrm{i}n\Omega t}.$$

Using the Poisson summation formula, we find

$$\sum_{n\in\mathbb{Z}}\delta(t-nT) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\delta\left(\frac{t}{T} - n\right) = \frac{1}{T}\sum_{n\in\mathbb{Z}} e^{\frac{2\mathrm{i}\pi}{T}nt} = \frac{1}{T}\sum_{n\in\mathbb{Z}} e^{-\mathrm{i}n\Omega t}.$$

Inserting into (4.30), we find that $\widehat{\Gamma}^{\omega}_{\text{per},n}$ is solution to the Helmholtz equation

$$-\frac{1}{v_0^2}(\omega+n\Omega)^2\widehat{\Gamma}^{\omega}_{\text{per},n} - \Delta\widehat{\Gamma}^{\omega}_{\text{per},n} = \frac{1}{T}\delta \text{ in } \mathbb{R}^3.$$

The only solution $\widehat{\Gamma}^{\omega}_{\mathrm{per},n}(x)e^{-\mathrm{i}(\omega+n\Omega)t}$ that is outgoing is $\hat{G}^{\omega}_n(x) = -\frac{e^{\mathrm{i}\frac{\omega+n\Omega}{v_0}|x|}}{4\pi T|x|}$. Consequently, using the Poisson summation formula, we find

$$
\begin{aligned}
\Gamma^{\omega}_{\mathrm{per}}(t,x) &= -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi T|x|}\sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\Omega(\frac{1}{v_0}|x|-t)} = -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi T|x|}\sum_{n\in\mathbb{Z}}\delta\left(\frac{1}{T}\left(\frac{|x|}{v_0}-t\right)-n\right)\\
&= -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi|x|}\sum_{n\in\mathbb{Z}}\delta\left(\frac{|x|}{v_0}-t-nT\right).
\end{aligned}
$$

□

4.4.2. *Periodic retarded layer potentials*

We now define the single and double layer potential associated to the periodic Green function (4.31). We first define these potentials on functions smooth with respect to time, before extending them to T–periodic distributions. In what follows, let us denote by $\mathcal{D}'_{\mathrm{per}}((0,T),W')$ the space of T–periodic distributions with values in the dual W' of a Hilbert space W. We recall that any $T\in\mathcal{D}'_{\mathrm{per}}((0,T),W')$ can be decomposed as a trigonometric Fourier expansion

$$
T = \sum_{n\in\mathbb{Z}}\hat{T}_n e^{-\mathrm{i}n\Omega t},\ \text{ with } \hat{T}_n\in W' \text{ for any } n\in\mathbb{Z},
$$

where the Fourier coefficients grow at most polynomially: $||\hat{T}_n||_{W'}\leq C|n|^p$ for some constant $C>0$ and a polynomial exponent $p\in\mathbb{N}$ (see [40]). Then, $T\in\mathcal{D}'_{\mathrm{per}}(W')$ defines a distribution of $\mathcal{C}^{\infty}_{\mathrm{per}}(W)$ from the duality pairing

$$
\langle T,\phi\rangle_{\mathcal{D}'_{\mathrm{per}}((0,T),W')} := \sum_{n\in\mathbb{Z}}\langle\hat{T}_n,\overline{\hat{\phi}_n}\rangle_{W',W},\ \text{ for any } \phi\in\mathcal{C}^{\infty}_{\mathrm{per}}((0,T),W) \text{ with } \phi(t,x)=\sum_{n\in\mathbb{Z}}\hat{\phi}_n e^{-\mathrm{i}n\Omega t}. \tag{4.32}
$$

In particular, if $T\in L^2_{\mathrm{per}}((0,T),W)$, it holds

$$
\langle T,\phi\rangle_{\mathcal{D}'_{\mathrm{per}}((0,T),W)} = \frac{1}{T}\int_0^T\langle T,\overline{\phi}\rangle_W\,\mathrm{d}t. \tag{4.33}
$$

Accordingly, for a given function $\phi(t,x)\in\mathcal{C}^{\infty}_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ which is T–periodic in t and for $x\in\partial D$, we pose

$$
\mathcal{S}^{\omega}_{D,\mathrm{per}}[\phi](t,x) := \langle\Gamma^{\omega}_{\mathrm{per}}(t-\cdot,x-\cdot)\phi(\cdot,\cdot)\rangle_{\mathcal{D}'_{\mathrm{per}}(L^2(\partial D))}, \tag{4.34}
$$

which, from the identification (4.32) of the duality pairing and (4.31), leads to the following definition.

Definition 4.2. We denote by $\mathcal{S}^{\omega}_{D,\mathrm{per}}:\mathcal{C}^{\infty}_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))\to\mathcal{C}^{\infty}_{\mathrm{per}}((0,T),H^1_{\mathrm{loc}}(\mathbb{R}^3))$ the single layer potential defined for $\phi\in\mathcal{C}^{\infty}_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ by

$$
\mathcal{S}^{\omega}_{D,\mathrm{per}}[\phi](t,x) := \int_{\partial D}\Gamma^{\frac{\omega}{v_0}}(x-y)\phi\left(t-\frac{|x-y|}{v_0},y\right)\mathrm{d}\sigma(y), \tag{4.35}
$$

where we recall the definition (2.7) for the fundamental solution $\Gamma^{\frac{\omega}{v_0}}$ of the (static) Helmholtz equation.

Remark 4.3. The operator $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ can be called a "retarded" potential, following the terminology of the works [38, 90, 23, 22, 87] considering scattering problems for the wave equation in the time-domain.

One could think from the definition (4.35) that ϕ requires some smoothness in the variable t for (4.35) to be well-defined. In fact, this is not the case as the periodic layer potential $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ can be extended as an operator

$$
\mathcal{S}^{\omega}_{D,\mathrm{per}}:\mathcal{D}'_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))\to\mathcal{D}'_{\mathrm{per}}((0,T),H^1_{\mathrm{loc}}(\mathbb{R}^3)).
$$

Indeed, we have the following characterization.

Lemma 4.3. *The periodic single layer potential $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ can be extended to periodic distributions by setting*

$$
\mathcal{S}^{\omega}_{D,\mathrm{per}}[\phi](t,x) := \sum_{n\in\mathbb{Z}}\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}[\hat{\phi}_n]e^{-\mathrm{i}n\Omega t}, \tag{4.36}
$$

for any $\phi\in\mathcal{D}'_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ with Fourier coefficients $(\hat{\phi}_n)_{n\in\mathbb{Z}}$, where for $k\in\mathbb{R}$, $\mathcal{S}^k_D$ is the "usual" single layer potential as defined in (2.5). The operator $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ thus defined is a mapping from $\mathcal{D}'_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ into $\mathcal{D}'_{\mathrm{per}}((0,T),H^1_{\mathrm{loc}}(\mathbb{R}^3))$.

Proof. Expanding $\phi \in \mathcal{C}^\infty_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ in Fourier series, we obtain

$$\begin{aligned}\mathcal{S}^\omega_{D,\text{per}}[\phi](t,x) &= \sum_{n\in\mathbb{Z}} \int_{\partial D} \Gamma^{\frac{\omega}{v_0}}(x-y) e^{\mathrm{i}n\Omega\left(\frac{|x-y|}{v_0}-t\right)} \widehat{\phi}_n(y)\,\mathrm{d}\sigma(y)\\ &= \sum_{n\in\mathbb{Z}} e^{-\mathrm{i}n\Omega t} \int_{\partial D} \Gamma^{\frac{\omega+n\Omega}{v_0}}(x-y)\widehat{\phi}_n(y)\,\mathrm{d}\sigma(y) = \sum_{n\in\mathbb{Z}} \mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}[\widehat{\phi}_n](x) e^{-\mathrm{i}n\Omega t},\end{aligned}$$

which shows that the identity (4.36) holds for functions $\phi \in \mathcal{C}^\infty_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$. Now, let us assume that $\phi \in \mathcal{D}'_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$. In order to show that (4.36) defines a periodic distribution of $\mathcal{D}'_{\text{per}}((0,T), H^{\frac{1}{2}}(\partial D))$, it is sufficient to show that the Fourier coefficients of $\mathcal{S}^\omega_{D,\text{per}}[\phi]$ are polynomially bounded with respect to the norm of $H^{\frac{1}{2}}(\partial D)$. This holds because there exists a constant $C>0$ independent of n such that

$$|||\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}|||_{H^{-\frac{1}{2}}(\partial D)\to H^{\frac{1}{2}}(\partial D)} \leq C n^{\frac{1}{2}} \log n, \tag{4.37}$$

for a smooth domain D (see [51]), and the Fourier coefficients $(\hat{\phi}_n)_{n\in\mathbb{Z}})$ grow also polynomially. Finally, the uniform estimate of (4.5) implies that $\mathcal{S}^\omega_{D,\text{per}}[\phi]$ defines also a distribution of $\mathcal{D}'_{\text{per}}((0,T), H^1_{\text{loc}}(\mathbb{R}^3))$. ☐

Remark 4.4. The bound (4.37) implies that $\mathcal{S}^\omega_{D,\text{per}}$ maps $H^s_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ into $H^{s-\frac{1}{2}-\epsilon}_{\text{per}}((0,T), H^{\frac{1}{2}}(\partial D))$ for any $\epsilon > 0$.

4.4.3. *Neumann-Poincaré operator and jump relations*

We define the adjoint of the Neumann-Poincaré operator by a definition similar to (4.34); we pose for any $\phi \in \mathcal{C}^\infty_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$:

$$\mathcal{K}^{\omega *}_{D,\text{per}}[\phi](t,x) := \langle \boldsymbol{n}(x)\cdot\nabla_x \Gamma^\omega_{\text{per}}(t-\cdot, x-\cdot)\phi(\cdot,\cdot)\rangle_{\mathcal{D}'_{\text{per}}((0,T),L^2(\partial D))},$$

which leads to the following definition.

Definition 4.3. We denote by $\mathcal{K}^{\omega *}_{D,\text{per}} : \mathcal{C}^\infty_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D)) \to \mathcal{C}^\infty_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ the operator defined by

$$\begin{aligned}\mathcal{K}^{\omega *}_{D,\text{per}}[\phi](t,x) := \int_{\partial D} \boldsymbol{n}(x)\cdot \Bigg[&\nabla_x \Gamma^{\frac{\omega}{v_0}}(x-y)\phi\left(t-\frac{|x-y|}{v_0}, y\right)\\ &-\frac{1}{v_0}\frac{x-y}{|x-y|}\Gamma^{\frac{\omega}{v_0}}(x-y)\partial_t\phi\left(t-\frac{|x-y|}{v_0},y\right)\Bigg]\,\mathrm{d}\sigma(y), \qquad x\in\partial D.\end{aligned} \tag{4.38}$$

The first term of (4.38) is analogous to the standard Neumann-Poincaré operator and features a singular kernel. The second part of the integral is non-singular, but features partial derivative with respect to time.

Similarly, $\mathcal{K}^{\omega *}_{D,\text{per}}$ can be extended into an operator

$$\mathcal{K}^{\omega *}_{D,\text{per}} : \mathcal{D}'_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D)) \to \mathcal{D}'_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D)).$$

Lemma 4.4. *The operator $\mathcal{K}^{\omega *}_{D,\text{per}}$ can be extended to periodic distributions by setting*

$$\mathcal{K}^{\omega *}_{D,\text{per}}[\phi](t,x) := \sum_{n\in\mathbb{Z}} \mathcal{K}_D^{\frac{\omega+n\Omega}{v_0}*}[\widehat{\phi}_n] e^{-\mathrm{i}n\Omega t},$$

for any T–periodic distribution $\phi \in \mathcal{D}'_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ with Fourier coefficients $(\hat{\phi}_n)_{n\in\mathbb{Z}}$, where for $k\in\mathbb{R}$, $\mathcal{K}^{k*}_D$ is the "usual" adjoint of the Neumann-Poincaré operator defined by (2.6). The operator $\mathcal{K}^{\omega *}_{D,\text{per}}$ defines a mapping from $\mathcal{D}'_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ into itself.

Proof. The result is obtained similarly as in the proof of Lemma 4.3, using the wave number explicit bound

$$|||\mathcal{K}_D^{\frac{\omega+n\Omega}{v_0}*}|||_{H^{-\frac{1}{2}}(\partial D)\to H^{-\frac{1}{2}}(\partial D)} \leq C n^{\frac{1}{4}} \log n, \tag{4.39}$$

which holds for smooth domains D (see the appendix of [54]). ☐

Remark 4.5. The bound (4.39) implies that $\mathcal{K}^{\omega *}_{D,\text{per}}$ maps $H^s_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ into $H^{s-\frac{1}{4}-\epsilon}_{\text{per}}((0,T), H^{-\frac{1}{2}}(\partial D))$ for any $\epsilon > 0$.

Due to standard properties of singular kernels [71], we have the following result.

Proposition 4.13. *The time-periodic single layer potential $\mathcal{S}^\omega_{D,\text{per}}$ satisfies the following jump relation:*

$$\frac{\partial \mathcal{S}^\omega_{D,\text{per}}[\phi]}{\partial n}\bigg|_{\pm} = \left(\pm\frac{1}{2}I + \mathcal{K}^{\omega *}_{D,\text{per}}\right)[\phi], \qquad \textit{where } I \textit{ is the identity mapping.}$$

4.4.4. *Invertibility of the single layer potential*

It seems difficult, if not unfeasible, to show that $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ is invertible as a mapping $H^s_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ into $H^{s-\frac{1}{2}-\epsilon}_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))$ for some $s>0$ and $\epsilon>0$. However, it is fairly easy to state an invertibility result when considering $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ as an operator acting on T–periodic distributions.

Proposition 4.14. *Assume (H1) and (H2). For any $\omega\in\mathbb{R}$ in a small neighborhood of zero, the single layer potential $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ is an invertible operator from $\mathcal{D}'_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))\to\mathcal{D}'_{\mathrm{per}}((0,T),H^{\frac{1}{2}}((0,T),\partial D))$, whose inverse reads:*

$$(\mathcal{S}^{\omega}_{D,\mathrm{per}})^{-1}[f]=\sum_{n\in\mathbb{Z}}(\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}})^{-1}[\hat{f}_n]e^{-\mathrm{i}n\Omega t}, \tag{4.40}$$

where $(\hat{f}_n)_{n\in\mathbb{Z}}$ denote the Fourier coefficients of a given $f\in\mathcal{D}'_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))$.

Proof. Assumption (H2) ensures that $\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}$ is invertible for any $n\in\mathbb{N}$ and ω sufficiently small. Owing to the Fourier series expansion (4.36), it is necessary that (4.40) be the inverse of $\mathcal{S}^{\omega}_{D,\mathrm{per}}$, if this operator is invertible. Therefore, the only point to verify is that (4.40) defines a periodic distribution of $\mathcal{D}'_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$, i.e. that the coefficients of the trigonometric series (4.40) grow polynomially with respect to n. For this, we need a wave number explicit bound on the operator norm $(\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}})^{-1}$. From the jump relation, we infer that

$$(\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}})^{-1}=\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}-\mathcal{T}_-^{\frac{\omega+n\Omega}{v_0}}, \tag{4.41}$$

where $\mathcal{T}_-^{\frac{\omega+n\Omega}{v_0}}$ denotes the *interior* Dirichlet-to-Neumann map of the Helmholtz operator $\Delta+(\omega+n\Omega)^2/v_0^2$, and $\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}$ is the exterior one (Definition 3.1). By standard variational estimates, the mapping $\mathcal{T}_-^{\frac{\omega+n\Omega}{v_0}}$ satisfies the following bound for any $\phi\in H^{\frac{1}{2}}(\partial D)$:

$$||\mathcal{T}_-^{\frac{\omega+n\Omega}{v_0}}[\phi]||_{H^{-\frac{1}{2}}(\partial D)}\leq\frac{C}{\inf_{\lambda\in\sigma_{\mathrm{dir}}(-\Delta),n\in\mathbb{Z}}\left|\left(\frac{n\Omega+\omega}{v_0}\right)^2-\lambda\right|}||\phi||_{H^{\frac{1}{2}}(\partial D)}, \tag{4.42}$$

for an independent constant $C>0$. Since for $\omega\in\mathbb{R}$ sufficiently small,

$$\inf_{\lambda\in\sigma_{\mathrm{dir}},n\in\mathbb{Z}}\left|\left(\frac{n\Omega+\omega}{v_0}\right)^2-\lambda\right|=\left|\left|\frac{n\Omega+\omega}{v_0}\right|-\lambda\right|\times\left|\frac{n\Omega+\omega}{v_0}+\lambda\right|\geq C\inf_{\lambda\in\sigma_{\mathrm{dir}}(-\Delta),n\in\mathbb{Z}}\left|\frac{n\Omega}{v_0}-\sqrt{\lambda}\right|,$$

the bound (4.42) together with (H2) imply that for ω sufficiently small, the operator norm of $\mathcal{T}_-^{\omega+n\Omega}$ is uniformly bounded in n and ω. Therefore, (4.41) and the asumption (H1) imply the existence of a constant $C>0$ independent of n such that

$$|||(\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}})^{-1}|||_{H^{-\frac{1}{2}}(\partial D)\to H^{\frac{1}{2}}(\partial D)}\leq C|n|, \tag{4.43}$$

so that (4.40) defines an element of $\mathcal{D}'_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))$. □

Remark 4.6. The wave number explicit bound (4.43) implies that $(\mathcal{S}^{\omega}_{D,\mathrm{per}})^{-1}$ maps $H^s_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$ into $H^{s-1}_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))$.

Finally, it is possible to express the periodic Dirichlet-to-Neumann operator $\mathcal{T}^{\omega}_{\mathrm{per}}$ in terms of the potentials $\mathcal{S}^{\omega}_{D,\mathrm{per}}$ and $\mathcal{K}^{\omega*}_{D,\mathrm{per}}$, generalizing the identity (3.5).

Proposition 4.15. *Assume (H1) and (H2). The periodic Dirichlet-to-Neumann map $\mathcal{T}^{\omega}_{\mathrm{per}}$ of (4.15) can be extended to periodic distributions by setting*

$$\mathcal{T}^{\omega}_{\mathrm{per}}[\phi]:=\sum_{n\in\mathbb{Z}}\mathcal{T}^{\frac{\omega+n\Omega}{v_0}}[\hat{\phi}_n]e^{-\mathrm{i}n\Omega t},$$

for any $\phi\in\mathcal{D}'_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))$. Then, the operator

$$\mathcal{T}^{\omega}_{\mathrm{per}}:\mathcal{D}'_{\mathrm{per}}((0,T),H^{\frac{1}{2}}(\partial D))\to\mathcal{D}'_{\mathrm{per}}((0,T),H^{-\frac{1}{2}}(\partial D))$$

thus defined admits the following representation:

$$\mathcal{T}^{\omega}_{\mathrm{per}}=\left(\frac{1}{2}I+\mathcal{K}^{\omega*}_{D,\mathrm{per}}\right)(\mathcal{S}^{\omega}_{D,\mathrm{per}})^{-1}.$$

5. Subwavelength resonances in fast time-modulated media

This last section is dedicated to the study of scattering resonances in the fast time-modulated medium (1.1), for time-harmonic incident fields (1.2) and in the subwavelength and high-contrast regimes (1.4). Our analysis is based on the Dirichlet-to-Neumann approach introduced in Section 3, where the considered Dirichlet-to-Neumann map is the one associated to outgoing time-periodic outgoing waves studied in Section 4. Throughout this part, we assume that the assumptions (H1) to (H3) are fulfilled.

We start in Section 5.1 by characterizing the behavior of the solution operator of the reduced time-modulated wave equation (4.2) for small ω and δ. This allows us to determine the leading order asymptotics of the subwavelength resonant frequencies which arise due to the scattering with the time-modulated resonator D. We find that in the exceptional situation where $\Lambda \neq \{(0,0)\}$, resonant frequencies with positive imaginary parts may exist, leading to the arising of exponentially growing outgoing modes.

Assuming that the modulation is tuned in order to ensure that the exceptional coupling occurs, we determine a modal decomposition for the scattered field in (5.2) which is shown to oscillate like $p_m(t)\phi_l(x)$ inside the resonator D. We then identify a far field expansion for the scattered wave.

Finally, we consider a metamaterial constituted of N time-modulated resonators rescaled by a factor s in Section 5.3. We show how to formally identify an effective medium theory for such an heterogeneous medium in the subcritical regime $sN \to 0$, and we find that effective scattered waves carry high-frequency components. In the critical regime where sN converges to a positive constant, higher order reflections involve high frequency scattered waves which cannot be easily described by an effective medium theory. Following [14], we content ourselves to provide a homogenized equation for the low frequency truncation of the total wave field.

5.1. **Asymptotics of the subwavelength resonant frequencies**

In what follows, we consider the reformulation of (4.2) in terms of the Dirichlet-to-Neumann map $\mathcal{T}^{\omega}_{\mathrm{per}}$ given by (4.21), or equivalently by the variational formulation (4.22). Following the Definition 3.2, we call "subwavelength resonance" a complex frequency $\omega \equiv \omega(\delta) \in \mathbb{C}$ satisfying $\omega(\delta) \to 0$ as $\delta \to 0$ and such that (4.21) admits a non-zero solution $u(\omega,\delta)$ for a zero right-hand side $\hat{u}_{\mathrm{in}} = 0$. According to the well-posedness result of Proposition 4.11, such frequencies must necessarily have a non-zero imaginary part.

5.1.1. *Characterization of the inverse of the scattering operator*

In this part, we characterize the inverse of the solution operator to (4.21), whose poles are the resonant frequencies.

Following Section 3.2, we introduce the bilinear form $a_{\omega,\delta}$ defined for $u, v \in V$ by

$$a_{\omega,\delta}(u,v) := a_{0,0}(u,v) - \frac{1}{v_r^2}\int_0^T \int_D (2\mathrm{i}\omega\partial_t + \omega^2)u\overline{v}\,\mathrm{d}x - \delta \int_0^T \int_D \mathcal{T}^{\omega}_{\mathrm{per}}[u]\overline{v}\,\mathrm{d}\sigma,$$

where we recall the definition (4.24) of the bilinear form $a_{0,0}$ and the definition (2.3) of the space V. Since $a_{0,0}$ satisfies the coercivity property (4.26), the perturbed bilinear form $a_{\omega,\delta}$ is also coercive for small ω and $\delta > 0$. By using Fredholm's theory as in Proposition 4.11, we obtain for any $f \in V'$ the existence and uniqueness of a solution $u_f(\omega,\delta)$ to the variational problem

$$a_{\omega,\delta}(u_f(\omega,\delta), v) = \langle f, v\rangle_{V',V}, \qquad \forall v \in V. \tag{5.1}$$

Let us, in particular, denote by $u_{ml}(\omega,\delta)$ the unique solution to the problem

$$a_{\omega,\delta}(u_{ml}(\omega,\delta), v) = \langle p_m\phi_l, v\rangle_H, \tag{5.2}$$

where we recall the definition (2.2) of the inner product $\langle\cdot,\cdot\rangle_H$.

Let us finally recall that the solution $\hat{u}(t,x)$ to the time-periodic reduced wave problem (4.22) satisfies

$$a(\hat{u}, v) = \langle f, v\rangle_{V',V}, \qquad \forall v \in V,$$

with f being the linear form

$$\langle f, v\rangle_{V',V} := \delta \int_0^T \int_{\partial D} \left(\frac{\partial \hat{u}_{\mathrm{in}}}{\partial n} - \mathcal{T}^{\omega}_{\mathrm{per}}[\hat{u}_{\mathrm{in}}]\right)\mathrm{d}\sigma\,\mathrm{d}t. \tag{5.3}$$

The following proposition is the counterpart of Lemma 3.1 in the time-modulated setting.

Proposition 5.1. *For a given right-hand side $f \in V'$, the variational problem*

$$a(u,v) = \langle f, v\rangle_{V',V}, \text{ for any } v \in V, \tag{5.4}$$

admits a unique solution $u \in V$ if and only if the linear system

$$(I - C(\omega,\delta))\boldsymbol{x} = \boldsymbol{F} \tag{5.5}$$

has a unique solution $\boldsymbol{x} \equiv \boldsymbol{x}(\omega, \delta) \equiv (x_{ml}(\omega, \delta))_{(m,l)\in\Lambda}$, where $C(\omega, \delta)$ and $\boldsymbol{F}$ are the matrix and column vectors given by

$$C(\omega, \delta) := (\langle u_{m',l'}(\omega, \delta), p_m\phi_l\rangle_H)_{(m,l),(m',l')\in\Lambda\times\Lambda}, \qquad \boldsymbol{F} := (\langle u_f(\omega, \delta), p_m\phi_l\rangle_H)_{(m,l)\in\Lambda}. \tag{5.6}$$

When it is the case, the solution $u \equiv u(\omega, \delta)$ to (5.4) reads

$$u(\omega, \delta) = u_f(\omega, \delta) + \sum_{(m,l)\in\Lambda} x_{ml}(\omega, \delta) u_{ml}(\omega, \delta), \tag{5.7}$$

with $u_f(\omega, \delta)$ and $u_{ml}(\omega, \delta)$ defined by (5.1) and (5.2).

Proof. The variational problem (5.4) can be equivalently written as

$$\begin{aligned}
&a(u, v) = a_{\omega,\delta}(u(\omega, \delta), v) - \sum_{(m,l)\in\Lambda} \langle u(\omega, \delta), p_m\phi_l\rangle_H \langle v, p_m\phi_l\rangle_H = \langle f, v\rangle_{V',V} \\
\Leftrightarrow \quad & a_{\omega,\delta}(u(\omega, \delta), v) - \sum_{(m,l)\in\Lambda} \langle u(\omega, \delta), p_m\phi_l\rangle_H a_{\omega,\delta}(u_{ml}(\omega, \delta), v) = a_{\omega,\delta}(u_f(\omega, \delta), v) \\
\Leftrightarrow \quad & u(\omega, \delta) - \sum_{(m,l)\in\Lambda} \langle u(\omega, \delta), p_m\phi_l\rangle_H u_{ml}(\omega, \delta) = u_f(\omega, \delta).
\end{aligned}$$

Integrating against $p_m\phi_l/\rho$, we find that this linear system has a solution if and only if

$$\langle u(\omega, \delta), p_m\phi_l\rangle_H - \sum_{(m',l')\in\Lambda} \langle u_{m',l'}(\omega, \delta), p_m\phi_l\rangle_H \langle u(\omega, \delta), p_{m'}\phi'_l\rangle_H = \langle u_f(\omega, \delta), p_m\phi_l\rangle_H.$$

This system is equivalent to (5.5) with $x_{ml}(\omega, \delta) := \langle u(\omega, \delta), p_m\phi_l\rangle_H$ and admits (5.7) as a solution. ☐

5.1.2. *Asymptotic expansion of the subwavelength resonances*

The Proposition 5.1 reduces the study of the resonances of the infinite dimensional problem (5.4) to the one of the finite-dimensional problem (5.5): the subwavelength resonant frequencies of (5.4) are the complex numbers $(\omega(\delta))$ for which the matrix $I - C(\omega(\delta), \delta)$ is not invertible. To characterize these numbers, we write an asymptotic expansion for $I - C(\omega, \delta)$, which is obtained from an asymptotic expansion of $u_{ml}(\omega, \delta)$.

Proposition 5.2. *The solution $u_{ml}(\omega, \delta)$ to (5.2) has the following asymptotic expansion as $\omega, \delta \to 0$:*

$$\begin{aligned}
u_{ml}(\omega, \delta)(t, x) = {} & p_m(t)\phi_l(x) + \mathrm{i}\omega p_m^1(t)\phi_l(x) + \omega^2 \left(\frac{\gamma_m}{v_r^2} p_m(t) + p_m^2(t)\right) \phi_l(x) \\
& + \delta \sum_{(m',l')\in\Lambda} T_{m'l',ml} p_{m'}(t)\phi_{l'}(x) + \delta\widetilde{w}_{ml}(t, x) + O(\omega^3 + \delta\omega + \delta^2),
\end{aligned} \tag{5.8}$$

where for any $(m, l) \in \Lambda$:

- *p_m is the eigenvector of the Sturm-Liouville problem (1.6), and p_m^1 and p_m^2 are the unique T–periodic solutions to*

$$\begin{cases} -\dfrac{\mathrm{d}^2 p_m^1}{\mathrm{d}t^2} - \dfrac{\mu_m}{\rho(t)} p_m^1 = -2\dfrac{\mathrm{d}p_m}{\mathrm{d}t}, \\ p_m^1 \text{ is } T\text{–periodic}, \\ \displaystyle\int_0^T \frac{1}{\rho(t)} p_m^1(t) p_m(t)\,\mathrm{d}t = 0, \end{cases} \qquad \begin{cases} -\dfrac{\mathrm{d}^2 p_m^2}{\mathrm{d}t^2} - \dfrac{\mu_m}{\rho(t)} p_m^2 = 2\dfrac{\mathrm{d}p_m^1}{\mathrm{d}t} - p_m + \dfrac{\gamma_m}{T}\dfrac{p_m}{\rho(t)}, \\ p_m^2 \text{ is } T\text{–periodic}, \\ \displaystyle\int_0^T \frac{1}{\rho(t)} p_m^2(t) p_m(t)\,\mathrm{d}t = 0, \end{cases} \tag{5.9}$$

- *$\widetilde{w}_{ml}(t, x)$ is a function satisfying $\langle \widetilde{w}_{ml}, p_m\phi_l\rangle_H = 0$ for all $(m, l) \in \Lambda$,*
- *γ_m is the real number*

$$\gamma_m := \int_0^T \left(-2\frac{\mathrm{d}}{\mathrm{d}t} p_m^1 p_m + |p_m(t)|^2\right) \mathrm{d}t = \int_0^T \left(-\left|\frac{\mathrm{d}}{\mathrm{d}t} p_m^1\right|^2 + \frac{\mu_m}{\rho(t)} \left|p_m^1\right|^2 + |p_m(t)|^2\right) \mathrm{d}t, \tag{5.10}$$

- *$T \equiv (T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$ is the real matrix*

$$T_{ml,m'l'} := \int_0^T \int_{\partial D} \mathcal{T}_{\mathrm{per}}^0 [p_{m'}\phi_{l'}] p_m(t)\phi_l(x)\,\mathrm{d}t\,\mathrm{d}\sigma(x), \tag{5.11}$$

where $\mathcal{T}_{\mathrm{per}}^0$ is the time-periodic Dirichlet-to-Neumann map of Definition 4.1 with frequency $\omega = 0$.

Proof. To simplify the notation, let us write $w(\omega,\delta) := u_{ml}(\omega,\delta)$ in this proof. The strong form of (5.2) reads

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt} w - \dfrac{1}{\rho(t)}\Delta w + \displaystyle\sum_{(m',l')\in\Lambda} \langle w, p_{m'}\phi_{l'}\rangle_H \frac{1}{T}\frac{1}{\rho(t)} p_{m'}(t)\phi_{l'}(x) - \frac{1}{v_r^2}(2\mathrm{i}\omega\partial_t + \omega^2) w = \frac{1}{T}\frac{1}{\rho(t)} p_m(t)\phi_l(x) \\ \dfrac{1}{\rho(t)}\dfrac{\partial w}{\partial n} - \delta \mathcal{T}^{\omega}_{\mathrm{per}}[w] = 0. \end{cases}$$

Since this system is invertible for any values of ω and δ small enough, $w \equiv w(\omega,\delta)$ is analytic in ω and δ and we can write the asymptotic expansion

$$w = \sum_{p,k=0}^{+\infty} \omega^p \delta^k w_{p,k}.$$

Inserting this ansatz in the above equation yields the following cascade of equations determining $(w_{p,k})_{p,k\geq 0}$:

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt} w_{p,k} - \dfrac{1}{\rho(t)}\Delta w_{p,k} + \displaystyle\sum_{(m',l')\in\Lambda} \langle w_{p,k}, p_{m'}\phi_{l'}\rangle_H \frac{1}{T}\frac{1}{\rho(t)} p_{m'}\phi_{l'} \\ \qquad = \dfrac{1}{T}\dfrac{1}{\rho(t)} p_m(t)\phi_l(x)\delta_{p=0}\delta_{k=0} + \dfrac{1}{v_r^2}(2\mathrm{i}\partial_t w_{p-1,k} + w_{p-2,k}), \\ \dfrac{1}{\rho(t)}\dfrac{\partial w_{p,k}}{\partial n} = \displaystyle\sum_{n=0}^{p} \frac{1}{v_0^n}\mathcal{T}_{\mathrm{per},n}[w_{p-n,k-1}], \end{cases} \tag{5.12}$$

where we define $w_{p,k} = 0$ for negative $p,k < 0$ by convention. Let us now compute the first terms of this cascade of equation. In what follows, we consider two indices $(m',l') \in \Lambda$.

- $p = 0$, $k = 0$ yields $w_{0,0} = p_m\phi_l$;
- $p = 1$, $k = 0$. Integrating (5.12) against $v = p_{m'}\phi_{l'}$ yields

$$\langle w_{1,0}, p_{m'}\phi_{l'}\rangle_H = \frac{2\mathrm{i}}{v_r^2}\int_0^T\int_D \frac{\mathrm{d}}{\mathrm{d}t}p_m(t)\phi_l(x)p_{m'}(t)\phi_{l'}(x)\,\mathrm{d}x\,\mathrm{d}t = \delta_{l=l'}\delta_{m=m'}\frac{2\mathrm{i}}{v_r^2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}p_m(t)p_m(t)\,\mathrm{d}t = 0.$$

  Therefore, $w_{1,0}$ is the solution to

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt} w_{1,0} - \dfrac{1}{\rho(t)}\Delta w_{1,0} = \dfrac{2\mathrm{i}}{v_r^2}\dfrac{\mathrm{d}}{\mathrm{d}t}p_m(t)\phi_l(x), \\ \dfrac{1}{\rho(t)}\dfrac{\partial w_{1,0}}{\partial n} = 0. \end{cases}$$

  We deduce that $w_{1,0}(t,x) = \mathrm{i}p_m^1(t)\phi_l(x)$ where $p_m^1(t)$ satisfies

$$\frac{1}{v_r^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}p_m^1 + \frac{\lambda_l}{\rho(t)}p_m^1 = \frac{2}{v_r^2}\frac{\mathrm{d}}{\mathrm{d}t}p_m \quad \Leftrightarrow \quad -\frac{\mathrm{d}^2}{\mathrm{d}t^2}p_m^1 - \frac{\mu_m}{\rho(t)}p_m^1 = -2\frac{\mathrm{d}}{\mathrm{d}t}p_m.$$

  According to the Fredholm's alternative, this equation has a unique T–periodic solution p_m^1 which satisfies

$$\int_0^T \frac{1}{\rho(t)}p_m^1(t)p_m(t)\,\mathrm{d}t = 0.$$

- $p = 2$, $k = 0$. Integrating (5.12) against $v = p_{m'}\phi_{l'}$ yields

$$\begin{aligned} \langle w_{2,0}, p_{m'}\phi_{l'}\rangle_H &= \frac{1}{v_r^2}\left(2\mathrm{i}\times\mathrm{i}\int_0^T\int_D \frac{\mathrm{d}}{\mathrm{d}t}p_m^1(t)\phi_l(x)p_{m'}(t)\phi_{l'}(x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_D p_m(t)\phi_l(x)p_{m'}(t)\phi_{l'}(x)\,\mathrm{d}x\,\mathrm{d}t\right) \\ &= \frac{\delta_{m=m'}\delta_{l=l'}}{v_r^2}\left(-2\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}p_m^1 p_m\,\mathrm{d}t + \int_0^T |p_m(t)|^2\,\mathrm{d}t\right). \end{aligned}$$

  Therefore, we obtain $w_{2,0} = \frac{\gamma_m}{v_r^2}p_m(t)\phi_l(x) + \widetilde{w}_{2,0}(t,x)$, where $\widetilde{w}_{2,0}(t,x)$ is the unique solution to

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt}\widetilde{w}_{2,0} - \dfrac{1}{\rho(t)}\Delta\widetilde{w}_{2,0} = \dfrac{1}{v_r^2}\left(-2\partial_t p_m^1 + p_m - \dfrac{\gamma_m}{T}\dfrac{1}{\rho(t)}p_m\right)\phi_l, \\ \dfrac{1}{\rho(t)}\dfrac{\partial\widetilde{w}_{2,0}}{\partial n} = 0, \\ \langle\widetilde{w}_{2,0}, p_{m'}\phi_{l'}\rangle_H = 0 \text{ for all } (m',l')\in\Lambda. \end{cases}$$

  Similarly, we find $\widetilde{w}_{2,0}(t,x) = p_m^2(t)\phi_l(x)$ with $p_m^2(t)$ being the unique solution to (5.9).

- $p=0$, $k=1$. Integrating (5.12) against $v=p_{m'}\phi_{l'}$ yields

$$\langle w_{0,1}, p_{m'}\phi_{l'}\rangle_H = \int_0^T \int_{\partial D} \mathcal{T}^0_{\mathrm{per}}[p_m\phi_l] p_{m'}(t)\phi_{l'}(x)\,\mathrm{d}t\,\mathrm{d}\sigma(x) = T_{m'l',ml}.$$

Therefore, $w_{0,1} = \sum_{(m',l')\in\Lambda} T_{m'l',ml} p_{m'}\phi_{l'} + \widetilde{w}_{ml}$, where $\widetilde{w}_{ml}$ is the unique solution to

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt}\widetilde{w}_{ml} - \dfrac{1}{\rho(t)}\Delta\widetilde{w}_{ml} = -\displaystyle\sum_{(m',l')\in\Lambda} T_{ml,m'l'}\frac{1}{T}\frac{1}{\rho(t)} p_{m'}\phi_{l'}, \\ \dfrac{1}{\rho(t)}\dfrac{\partial\widetilde{w_{ml}}}{\partial n} = \mathcal{T}^0_{\mathrm{per}}[p_m\phi_l], \\ \langle \widetilde{w}_{ml}, p_{m'}\phi_{l'}\rangle_H = 0 \qquad \text{for all } (m',l')\in\Lambda. \end{cases} \tag{5.13}$$

The asymptotic (5.8) follows. □

Inserting this expansion into the definition (5.6) of $C(\omega,\delta)$ yields the following characterization of the resonances.

Proposition 5.3. *The asymptotic expansion of the matrix $C(\omega,\delta)$ defined in (5.6) reads at the leading order:*

$$C(\omega,\delta) = I + \frac{\omega^2}{v_r^2}\operatorname{diag}(\gamma_m)_{(m,l)\in\Lambda} + \delta T + O(\omega^3+\delta\omega+\delta^2), \tag{5.14}$$

where $(\gamma_m)_{(m,l)\in\Lambda}$ and the matrix T are given by (5.10) and (5.11). Consequently, the problem (4.2) admits exactly $2\#\Lambda$ subwavelength resonances $(\omega_i^{\pm}(\delta))_{1\le i\le\#\Lambda}$ whose leading asymptotic expansions are given by:

$$\omega_i^{\pm}(\delta) \sim \pm v_r\delta^{\frac12}\lambda_i^{\frac12}, \tag{5.15}$$

where $(\lambda_i)_{1\le i\le\#\Lambda}$ are the (complex) eigenvalues of the generalized eigenvalue problem

$$T\boldsymbol{a}_i + \lambda_i G\boldsymbol{a}_i = 0, \qquad G = \operatorname{diag}(\gamma_m)_{(m,l)\in\Lambda}. \tag{5.16}$$

Proof. Integrating (5.8) against the function $p_m(t)\phi_l(x)/\rho(t)$ yields the following asymptotic expansion for $C(\omega,\delta)$:

$$C(\omega,\delta)_{ml,m'l'} = \delta_{m=m'}\delta_{l=l'} + \frac{\omega^2}{v_r^2}\gamma_m\delta_{m=m'}\delta_{l=l'} + \delta T_{ml,m'l'} + O(\omega^3+\delta\omega+\delta^2),$$

which coincides with (5.14). Resorting to the generalized Rouché theorem [12] or to the implicit function theorem as in [45] (this would require the eigenvalues of T to be simple), we obtain the existence of exactly $2\#\Lambda$ subwavelength resonances $(\omega_i^{\pm}(\delta))_{1\le i\le\#\Lambda}$ for which $I - C(\omega_i^{\pm}(\delta),\delta)$ is not invertible, and these can be approximated at first order by those of the following nonlinear eigenvalue problem in ω:

$$\frac{\omega^2}{v_r^2}G\boldsymbol{x} + \delta T\boldsymbol{x} = 0,$$

which yields the expansion (5.15). □

The matrix T plays a role analogous to the capacity $\operatorname{cap}(D)$ involved in (3.18), or to the capacitance matrix in the context of subwavelength resonances induced by several simply connected resonators (see [45]). The next proposition lists a few properties of the matrix T.

Proposition 5.4. *Let us denote by $(\hat{p}_{m,n})_{n\in\mathbb{Z}}$ the Fourier coefficients of the Sturm-Liouville eigenvector p_m:*

$$p_m(t) = \sum_{n\in\mathbb{Z}} \hat{p}_{m,n} e^{-\mathrm{i}n\Omega t}.$$

(i) The coefficients $(T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$ are real and are also given by

$$T_{ml,m'l'} = T\sum_{n\in\mathbb{Z}} \hat{p}_{m',n}\overline{\hat{p}_{m,n}} \int_{\partial D} \mathcal{T}^{\frac{n\Omega}{v_0}}[\phi_{l'}]\phi_l\,\mathrm{d}\sigma.$$

(ii) The matrix T is symmetric if ρ is invariant by time reversal: $\rho(t)=\rho(-t)$. If further, D is a ball, then T is symmetric negative definite.

(iii) $T_{00,00} = -Tp_0^2\dfrac{\operatorname{cap}(D)}{|D|}$ and $\gamma_0 = Tp_0^2$ where

$$p_0 := \left(\frac{1}{T}\int_0^T \frac{1}{\rho(t)}\,\mathrm{d}t\right)^{-\frac12}. \tag{5.17}$$

(iv) For any $(m,l)\in\Lambda$,

$$T_{00,ml} = T_{ml,00} = T|D|^{-\frac12}p_0\hat{p}_{m,0}\int_{\partial D}\frac{\partial\Phi}{\partial n}\phi_l\,\mathrm{d}\sigma,$$

where Φ is the solution to the exterior problem (2.8).

Proof. (i) Decomposing $p_m\phi_l$ into Fourier modes, we find

$$\mathcal{T}^0_{\mathrm{per}}[p_{m'}\phi_{l'}] = \sum_{n\in\mathbb{Z}} \hat{p}_{m',n}\mathcal{T}^{\frac{n\Omega}{v_0}}[\phi_{l'}]e^{-\mathrm{i}n\Omega t},$$

and consequently,

$$T_{ml,m'l'} = \int_0^T\int_{\partial D} \mathcal{T}_0[p_{m'}\phi_{l'}]p_m\phi_l \,\mathrm{d}\sigma\,\mathrm{d}t = T\sum_{n\in\mathbb{Z}} \hat{p}_{m',n}\overline{\hat{p}_{m,n}}\int_{\partial D}\mathcal{T}^{\frac{n\Omega}{v_0}}[\phi_{l'}]\phi_l\,\mathrm{d}\sigma.$$

Using the property $\overline{\mathcal{T}^{\frac{n\Omega}{v_0}}[\phi_l]} = \mathcal{T}^{-\frac{n\Omega}{v_0}}[\phi_l]$, and the fact that p_m is real, we find that $T_{ml,m'l'}$ is real.

(ii) If $\rho(t)=\rho(-t)$, then the simplicity assumption (H3) of the eigenvector p_m implies that $p_m(t)=p_m(-t)$. The symmetry of T is then a consequence of Proposition 4.8. If further D is the sphere, then the bound of Proposition 4.9 implies the negative definiteness of the matrix T.

(iii)-(iv) It is easy to compute $T_{00,ml}$ for $(m,l)\in\Lambda$, because it is associated with the constant, normalized Sturm-Liouville and Neumann eigenmodes p_0 and ϕ_0 given by

$$p_0(t) = \left(\frac{1}{T}\int_0^T\frac{1}{\rho(t)}\,\mathrm{d}t\right)^{-\frac{1}{2}}, \qquad \phi_0(x) = \frac{1_D(x)}{|D|^{\frac{1}{2}}}.$$

Hence, $\hat{p}_{0,n} = \left(\frac{1}{T}\int_0^T\frac{1}{\rho(t)}\,\mathrm{d}t\right)^{-\frac{1}{2}}\delta_{0n}$ and the result follows.

□

In the next subsections, we discuss the nature of the subwavelength resonances depending on the set Λ introduced in (1.7), the constant γ_m and the matrix T of (5.10) and (5.11).

5.1.3. *Absence of exceptional subwavelength resonances in the generic case* $\Lambda=\{(0,0)\}$

In the generic case where the modulation $\rho(t)$ and the resonator D are not "tuned", it can be expected that $\Lambda=\{(0,0)\}$, i.e. only the zero eigenvalues μ_0/v_r^2 and λ_0 coincide. In this situation, we find that the matrix $C(\omega,\delta)$ of (5.14) is reduced to a constant, which has the following asymptotic expansion:

$$C(\omega,\delta) = 1 + T\left(\frac{1}{T}\int_0^T\frac{1}{\rho(t)}\,\mathrm{d}t\right)^{-1}\frac{\omega^2}{v_r^2} - T\left(\frac{1}{T}\int_0^T\frac{1}{\rho(t)}\,\mathrm{d}t\right)^{-1}\frac{\mathrm{cap}(D)}{|D|}\delta + O(\omega^3+\delta\omega+\delta^2).$$

Computing the zeros of $C(\omega,\delta)-I$, we retrieve two subwavelength resonances which coincide at first order with the classical "Minnaert resonance" $\omega^{\pm}(\delta)$ of (3.21):

$$\omega_1^{\pm}(\delta)\sim v_r\sqrt{\frac{\mathrm{cap}(D)}{|D|}}\delta^{\frac{1}{2}}.$$

This resonant frequency is associated to a resonant mode that is approximately constant inside the resonator D in both time and space. Owing to the properties of the time-periodic Dirichlet-to-Neumann map $\mathcal{T}^{\omega}_{\mathrm{per}}$, the associated scattered field does not propagate high-frequency components in the exterior domain $\mathbb{R}^3\backslash\overline{D}$. One can then verify that everything happens at first order as in the unmodulated case, with the modulation $\rho(t)$ being replaced with its harmonic average $\left(\frac{1}{T}\int_0^T\frac{1}{\rho(t)}\,\mathrm{d}t\right)^{-1}$.

5.1.4. *Subwavelength resonances in the exceptional coupling situation* $\Lambda=\{(0,0),(m,l)\}$

We now consider the more interesting case where the modulation $\rho(t)$ is tuned in such a way there exists another pair of coinciding eigenvalues $\mu_m/v_r^2=\lambda_l\neq 0$ with $(m,l)\neq(0,0)$:

$$\Lambda=\{(0,0),(m,l)\}.$$

From the point (iii) of Proposition 5.4, the constant γ_0 is positive, but the constant γ_m defined in (5.10) does not have a clear sign. In fact, we conjecture from numerical experiments that the modulation $\rho(t)$ can be tuned in such a way both cases $\gamma_m>0$ and $\gamma_m<0$ are possible. The following proposition shows that complex resonances with positive imaginary parts generally arise.

Proposition 5.5. *Assume that D is a disk and that the modulation is invariant by time-reversal: $\rho(t)=\rho(-t)$. Let us denote by $P(\xi)$ the second order polynomial*

$$P(\xi) := \gamma_0\gamma_m\xi^2 + (T_{00,00}\gamma_m + T_{ml,ml}\gamma_0)\,\xi + (T_{00,00}T_{ml,ml} - T_{ml,00}^2), \tag{5.18}$$

and by Δ its discriminant:

$$\Delta = (T_{00,00}\gamma_m - T_{ml,ml}\gamma_0)^2 + 4\gamma_0\gamma_m T_{ml,00}^2. \tag{5.19}$$

Then, the three following situations are possible depending on the sign of γ_m and Δ:

- *if $\gamma_m > 0$, then the four subwavelength resonances of (4.2) are real at the leading order and satisfy*

$$\omega_i^\pm(\delta) \sim \pm \xi_i^{\frac{1}{2}} v_r \delta^{\frac{1}{2}}, \qquad 1 \le i \le 2,$$

where the eigenvalues $(\xi_i)_{i=1,2}$ of (5.16) are the real positive numbers given by

$$\xi_1 = \frac{-T_{00,00}\gamma_m - T_{ml,ml}\gamma_0 - \sqrt{\Delta}}{2\gamma_0\gamma_m}, \qquad \xi_2 = \frac{-T_{00,00}\gamma_m - T_{ml,ml}\gamma_0 + \sqrt{\Delta}}{2\gamma_0\gamma_m}; \tag{5.20}$$

- *if $\gamma_m < 0$ and $\Delta > 0$, then the reduced wave problem (4.2) admits exactly two resonances $\omega_1^\pm(\delta)$ which are purely imaginary at first order, and two resonances $\omega_2^\pm(\delta)$ which are purely real at first order:*

$$\omega_1^\pm(\delta) \sim \pm \mathrm{i}\sqrt{-\xi_1} v_r \delta^{\frac{1}{2}}, \qquad \omega_2^\pm(\delta) \sim \pm \xi_2^{\frac{1}{2}} v_r \delta^{\frac{1}{2}},$$

where the eigenvalues ξ_1 and ξ_2 of (5.20) are respectively negative and positive;

- *if $\gamma_m < 0$ and $\Delta < 0$, then the four resonances of (4.2) are complex at the order in $O(\delta^{\frac{1}{2}})$, and their leading asymptotics are given by*

$$\omega_1^\pm(\delta) = \pm v_r \delta^{\frac{1}{2}} \sqrt{a + \mathrm{i}b} \text{ and } \omega_2^\pm(\delta) = \pm v_r \delta^{\frac{1}{2}} \sqrt{a - \mathrm{i}b}, \tag{5.21}$$

where a and b are the positive numbers

$$a = \frac{-T_{00,00}\gamma_m - T_{ml,ml}\gamma_0}{2\gamma_0\gamma_m}, \qquad b = \frac{\sqrt{-\Delta}}{2\gamma_0\gamma_m},$$

and where $\sqrt{\cdot}$ is a determination of the square root in $\mathbb{C}$.

Proof. In this context, the matrix of the eigenvalue problem (5.16) reads, for a given $\xi \in \mathbb{C}$:

$$T + \xi G = \begin{pmatrix} \xi\gamma_0 + T_{00,00} & T_{00,ml} \\ T_{ml,00} & \xi\gamma_m + T_{ml,ml}. \end{pmatrix}.$$

This matrix is not invertible when $P(\xi) = 0$. Recalling that T is a symmetric negative definite matrix due to the symmetry assumption on D and ρ, which implies that $T_{00,00}T_{ml,ml} - T_{ml,00}^2 \ge 0$. The result follows by discussing the complex nature of the root of the polynomial (5.18) depending on the sign of γ_m and Δ. $\square$

This proposition shows that at least if $\gamma_m < 0$ and in the most symmetric case where D is a ball and ρ is symmetric upon time-reversal, there exist outgoing nonzero solutions to the time-modulated scattering problem (1.1), which grow exponentially in time and with a growth rate of order $O(\delta^{\frac{1}{2}})$.

5.2. Modal decomposition and point scatterer approximation of the scattered field

We now follow the lines of Section 3.3 to obtain a modal decomposition and a far field approximation for the scattered solution $\hat{u} - \hat{u}_{\text{in}}$ based on the formula (5.7). In the remainder of this section, we consider the "exceptional" coupling situation where there are two pairs of coinciding Sturm-Liouville and Neumann eigenvalues:

$$\Lambda = \{(0,0),(m,l)\} \text{ for some } (m,l) \in \mathbb{N} \times \mathbb{N}\backslash\{(0,0)\}. \tag{5.22}$$

5.2.1. *Modal decomposition of the scattered field in the exceptional coupling*

In order to obtain a modal decomposition of $u_f(\omega,\delta)$ based on (5.7), we first need an approximation of the function $u_f(\omega,\delta)$ solution to (5.1) where $f \in V'$ is the linear form (5.3).

Proposition 5.6. *The function $u_f(\omega,\delta)$ solution to (5.1) with f given by (5.3) reads at first order*

$$u_f(\omega,\delta) = -\delta \hat{u}_{\text{in}}(0) p_0^{-1} |D|^{\frac{1}{2}} \left(\sum_{(m,l)\in\Lambda} T_{00,ml} p_m \phi_l + \widetilde{w}_{00} \right) + O(\omega\delta), \tag{5.23}$$

where $\widetilde{w}_{00}$ is the function of the Proposition 5.2 satisfying

$$\langle \widetilde{w}_{00}, p_m\phi_l \rangle_H = 0 \text{ for all } (m,l) \in \Lambda.$$

Proof. The problem (5.1) reads in strong form

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt} u_f - \dfrac{1}{\rho(t)}\Delta u_f + \displaystyle\sum_{(m,l)\in\Lambda} \langle u_f, p_m\phi_l\rangle_H \frac{1}{T}\frac{1}{\rho(t)} p_m(t)\phi_l(x) - \frac{1}{v_r^2}(2\mathrm{i}\omega\partial_t + \omega^2)u_f = 0, & (t,x) \in \mathbb{R}\times D, \\ \dfrac{1}{\rho(t)}\dfrac{\partial u_f}{\partial n} = \delta\left(\dfrac{\partial \widehat{u_{\text{in}}}}{\partial n} - \mathcal{T}_{\text{per}}^\omega[\widehat{u_{\text{in}}}]\right), & (t,x)\in\mathbb{R}\times\partial D, \\ u_f \text{ is } T\text{–periodic.} \end{cases}$$

Following the proof of Proposition 3.3, we find that

$$\frac{\partial \hat{u}_{\text{in}}}{\partial n} - \mathcal{T}^{\omega}_{\text{per}}[\hat{u}_{\text{in}}] = -\hat{u}_{\text{in}}(0)\mathcal{T}^{0}_{\text{per}}[1_D] + O(\omega) = -\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}\mathcal{T}^{0}_{\text{per}}[p_0\phi_0] + O(\omega).$$

Hence, $u_f(\omega,\delta) = \delta(u_{0,1} + \widetilde{u}_{0,1}) + O(\omega\delta)$, where

$$\begin{aligned} u_{0,1}(t,x) &= -\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}} \sum_{(m,l)\in\Lambda} \left(\int_0^T \int_{\partial D} \mathcal{T}^{0}_{\text{per}}[p_0\phi_0]p_m\phi_l \,\mathrm{d}\sigma(x)\,\mathrm{d}t\right) p_m(t)\phi_l(x) \\ &= -\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}} \sum_{(m,l)\in\Lambda} T_{00,ml}p_m(t)\phi_l(x), \end{aligned}$$

and $\widetilde{u}_{0,1}$ is the unique solution to

$$\begin{cases} \dfrac{1}{v_r^2}\partial_{tt}\widetilde{u}_{0,1} - \dfrac{1}{\rho(t)}\Delta\widetilde{u}_{0,1} = u_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}} \displaystyle\sum_{(m',l')\in\Lambda} T_{00,m'l'}\frac{1}{T}\frac{1}{\rho(t)}p_{m'}(t)\phi_{l'}(x), \\ \dfrac{1}{\rho(t)}\dfrac{\partial \widetilde{u}_{0,1}}{\partial n} = -\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}\mathcal{T}^{0}_{\text{per}}[p_0\phi_0], \\ \langle \widetilde{u}_{0,1}, p_m\phi_l\rangle_H = 0 \text{ for all } (m,l)\in\Lambda, \\ \widetilde{u}_{0,1} \text{ is } T\text{–periodic.} \end{cases}$$

Comparing with (5.13), we obtain the result with $\widetilde{u}_{0,1} = -\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}\widetilde{w}_{00}$. □

The next proposition establishes a "modal decomposition"in the exceptional coupling situation (5.22). For simplicity, we consider only the "least favourable" case where both resonant frequencies are complex at first order, that is the situation of (5.21). This case is the "least favourable" in the sense that the resonances are damped and there is no amplification of the input wave field (a resonance which is purely real at first order would result in an amplification of the incident field by a factor $O(\delta^{-\frac{1}{2}})$). However, we find that the scattered field has an amplitude of the same order of the incident wave field and contains high frequency components generated by the mode $p_m(t)\phi_l(x)$. In the other cases, one observes an amplification of the scattered field near the real part of the resonant frequency.

Proposition 5.7. *Assume that the determinant Δ of (5.19) is negative and $\Lambda = \{(0,0),(m,l)\}$. Then, for a real subwavelength frequency $\omega = O(\delta^{\frac{1}{2}})$, we have the following modal decomposition for the wave field $\hat{u}(t,x)$ solution to (4.2) for $x \in D$:*

$$\hat{u}(t,x) = \frac{\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}}{P\left(\frac{\omega^2}{v_r^2\delta}\right)}\left[\left(\frac{\omega^2}{v_r^2\delta}\gamma_m T_{00,00} + T_{ml,ml}T_{00,00} - T_{00,ml}^2\right)p_0(t)\phi_0(x) + \frac{\omega^2}{v_r^2\delta}\gamma_0 T_{00,ml}p_m(t)\phi_l(x)\right] + O(\delta^{\frac{1}{2}}), \tag{5.24}$$

where $P(\xi)$ is the second order polynomial of (5.18), which is negative and bounded from below by a constant independent of ω and δ.

Proof. We use the exact formula (5.7). In order to have an approximation of the coefficients $x_{ml}(\omega,\delta)$, we need to compute the right-hand side $\boldsymbol{F} = (F_{00}, F_{ml})$. By integrating (5.23) against $p_{m'}\phi_{l'}/\rho$, we obtain

$$F_{m'l'} = -\delta\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}T_{00,m'l'} + O(\omega\delta) \text{ for } (m',l') \in \{(0,0),(m,l)\}.$$

Then, computing explicitly the inverse of the matrix $(I - C(\omega,\delta))$ yields

$$(I - C(\omega,\delta))^{-1} = -\frac{1}{\det((I - C(\omega,\delta)))}\begin{pmatrix} \frac{\omega^2}{v_r^2}\gamma_m + \delta T_{ml,ml} & -\delta T_{00,ml} \\ -\delta T_{ml,00} & \frac{\omega^2}{v_r^2}\gamma_0 + \delta T_{00,00} \end{pmatrix} + O\left(\frac{\omega^3 + \delta\omega + \delta^2}{\det((I - C(\omega,\delta)))}\right).$$

For a real frequency $\omega = O(\delta^{\frac{1}{2}})$ in the regime where the polynomial P of (5.18) has no real roots ($\Delta < 0$), the determinant can be approximated by

$$\det((I - C(\omega,\delta))) = \delta^2 P\left(\frac{\omega^2}{v_r^2\delta}\right) + O(\omega^5 + \delta\omega^3 + \delta^2\omega) = \delta^2 P\left(\frac{\omega^2}{v_r^2\delta}\right)\left(1 + O(\delta^{\frac{1}{2}})\right).$$

Therefore, the coefficients $x_{ml}(\omega,\delta)$ of (5.7) read

$$\begin{aligned} x_{00} &= \frac{\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}}{P\left(\frac{\omega^2}{v_r^2\delta}\right)}\left[\left(\frac{\omega^2}{v_r^2\delta}\gamma_m + T_{ml,ml}\right)T_{00,00} - T_{00,ml}^2\right] + O(\delta^{\frac{1}{2}}), \\ x_{ml} &= \frac{\hat{u}_{\text{in}}(0)p_0^{-1}|D|^{\frac{1}{2}}}{P\left(\frac{\omega^2}{v_r^2\delta}\right)}\left[\left(\frac{\omega^2}{v_r^2\delta}\gamma_0 + T_{00,00}\right)T_{00,ml} - T_{00,ml}T_{00,00}\right] + O(\delta^{\frac{1}{2}}), \end{aligned}$$

which yields the result. □

Remark 5.1. The expansion (5.24) still holds the other situations in which $\Delta > 0$ and where $P(\lambda)$ has a real root up to a correction of the approximation error $O(\delta^{\frac{1}{2}})$: in that case $P\left(\frac{\omega^2}{v_r^2\delta}\right)$ becomes a resonant denominator which vanishes as ω becomes closer to the resonant frequency. A more precise approximation would be obtained by using a higher order expansion of $\det((I - C(\omega,\delta)))$, which would capture the leading order of the imaginary parts of the resonances, expected to be of order $O(\delta)$ (see e.g. [45, Section 4.3]).

Remark 5.2. In this situation where the resonances are complex at the leading order in $O(\delta^{\frac{1}{2}})$, (5.24) shows that these "resonances" are damped to the point where there is no significant amplification of the incident field in the regime $\omega = O(\delta^{\frac{1}{2}})$. However, we still find an order one high-frequency response which periodically oscillates in time at the "fast" frequency Ω.

5.2.2. *Far field expansion of the scattered field in the exceptional coupling situation*

We now determine a far field approximation for the scattered field. The scattered field $\hat{u}-\hat{u}_{\text{in}}$ can be rewritten thanks to the periodic single layer potential $\mathcal{S}^{\omega}_{D,\text{per}}$. Using the inversion property of Proposition 4.14, we can indeed write the following representation formula for $\hat{u}$ in $\mathbb{R}^3\backslash D$, assuming (H3):

$$\hat{u} - \hat{u}_{\text{in}} = \mathcal{S}^{\omega}_{D,\text{per}}[(\mathcal{S}^{\omega}_{D,\text{per}})^{-1}[\hat{u}_{|\partial D} - \hat{u}_{\text{in}}]] \text{ in } \mathbb{R}^3\backslash D. \tag{5.25}$$

In what follows, we denote by $\hat{c}_{l,n} : \mathbb{S}^2 \to \mathbb{R}$ the functions defined for any θ in the unit sphere $\mathbb{S}^2$ by

$$\hat{c}_{l,n}(\theta) := \int_{\partial D} e^{-\mathrm{i}\frac{n\Omega}{v_0}y\cdot\theta}(\mathcal{S}_D^{\frac{n\Omega}{v_0}})^{-1}[\phi_l](y)\,\mathrm{d}\sigma(y), \tag{5.26}$$

where ϕ_l is the Neumann eigenmode of (1.5). From the functions $(\hat{c}_{l,n})_{n\in\mathbb{Z}}$, we construct for any $(m,l) \in \mathbb{N}\times\mathbb{N}$ a T–periodic distribution $G_{ml} : \mathbb{R}\times\mathbb{S}^2$ by the formula

$$G_{ml}(t,\theta) := \sum_{n\in\mathbb{Z}} \hat{p}_{m,n}\hat{c}_{l,n}(\theta)e^{-\mathrm{i}n\Omega t}, \tag{5.27}$$

where we recall that $(\hat{p}_{m,n})_{n\in\mathbb{N}}$ denote the Fourier coefficients of the Sturm-Liouville eigenmode $p_m(t)$. The functions G_{ml} allow to obtain the far field patterns generated by the modes $p_m(t)\phi_l(x)$ as shown in the following lemma.

Lemma 5.1. *The following far field expansions hold in the regimes $\omega \to 0$ and $|x| \to +\infty$ and for $(m,l) \in \mathbb{N}\times\mathbb{N}$:*

$$\mathcal{S}^{\omega}_{D,\text{per}}[(\mathcal{S}^{\omega}_{D,\text{per}})^{-1}[p_m\phi_l]](x) = G_{ml}\left(t - \frac{|x|}{v_0}, \frac{x}{|x|}\right)(1 + O(|x|^{-1}) + O(\omega))\Gamma^{\frac{\omega}{v_0}}(x).$$

Proof. For a given $\phi \in H^{-\frac{1}{2}}(\partial D)$, the far-field expansion of the single layer potential $\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}[\phi](x)$ reads

$$\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}[\phi](x) = \left(\int_{\partial D} e^{-\mathrm{i}\frac{n\Omega}{v_0}y\cdot\frac{x}{|x|}}\phi(y)\,\mathrm{d}\sigma(y)\right)(1 + O(|x|^{-1}) + O(\omega))\Gamma^{\frac{\omega+n\Omega}{v_0}}(x).$$

By using (H3), we have the asymptotic expansion

$$(\mathcal{S}^{\frac{\omega+n\Omega}{v_0}})^{-1} = (\mathcal{S}_D^{\frac{n\Omega}{v_0}})^{-1} + O(\omega).$$

Consequently, we obtain that for any $(m,l) \in \mathbb{N}\times\mathbb{N}$,

$$\begin{aligned}\mathcal{S}^{\omega}_{D,\text{per}}[(\mathcal{S}^{\omega}_{D,\text{per}})^{-1}[p_m(t)\phi_l(x)]] &= \sum_{n\in\mathbb{Z}} \hat{p}_{m,n}\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}}[(\mathcal{S}_D^{\frac{\omega+n\Omega}{v_0}})^{-1}[\phi_l]]e^{-\mathrm{i}n\Omega t} \\ &= \sum_{n\in\mathbb{Z}} \hat{p}_{m,n}\left(\int_{\partial D} e^{-\mathrm{i}\frac{n\Omega}{v_0}y\cdot\frac{x}{|x|}}(\mathcal{S}_D^{\frac{n\Omega}{v_0}})^{-1}[\phi_l](y)\,\mathrm{d}\sigma(y)\right)(1 + O(\omega) + O(|x|^{-1}))\Gamma^{\frac{\omega+n\Omega}{v_0}}(x)e^{-\mathrm{i}n\Omega t}.\end{aligned} \tag{5.28}$$

Using $\Gamma^{\frac{\omega+n\Omega}{v_0}}(x)e^{-\mathrm{i}n\Omega t} = \Gamma^{\frac{\omega}{v_0}}(x)e^{-\mathrm{i}n\Omega\left(t-\frac{|x|}{v_0}\right)}$, we obtain the result. □

Remark 5.3. It is easy to verify that G_{00} is the constant function given by

$$G_{00}(t,\theta) \equiv G_{00} = -p_0|D|^{-\frac{1}{2}}\,\text{cap}(D), \qquad \forall(t,\theta) \in \mathbb{R}\times\mathbb{S}^2. \tag{5.29}$$

Remark 5.4. Far field expansions of retarded potentials featuring such a function G_{ml} depending on the propagation direction $x/|x|$ are classical, see e.g. [26, Lemma 4.3].

Using this result and the modal decomposition (5.24), we obtain the far field expansion of the scattered field, assuming the exceptional coupling situation (5.22).

Corollary 5.1. *Assume that the determinant Δ of (5.19) is negative, and that the exceptional coupling situation (5.22) is satisfied. The scattered field generated by the time modulated resonator admits the following far field expansion as $|x| \to +\infty$ and $\omega = O(\delta^{\frac{1}{2}})$:*

$$\hat{u}(x) - \hat{u}_{\rm in}(x) = \hat{u}_{\rm in}(0) \left(A\left(\frac{\omega^2}{v_r^2\delta}\right) + B\left(\frac{\omega^2}{v_r\delta}\right) G_{ml}\left(t - \frac{|x|}{v_0}, \frac{x}{|x|}\right)\right) (1 + O(\delta^{\frac{1}{2}}) + O(|x|^{-1}))\Gamma^{\frac{\omega}{v_0}}(x), \quad (5.30)$$

where the constant coefficients $A(\xi)$ and $B(\xi)$ are given by

$$A(\xi) := \operatorname{cap}(D)\left(1 - \frac{\xi\gamma_m T_{00,00} + T_{ml,ml}T_{00,00} - T_{00,ml}^2}{P(\xi)}\right), \qquad B(\xi) := \frac{p_0^{-1}|D|^{\frac{1}{2}}}{P(\xi)}\gamma_0 T_{00,ml}\xi, \quad (5.31)$$

where P is the second order polynomial of (5.18).

Proof. This result is obtained by inserting (5.24) into the layer potential representation formula (5.25). Using Lemma 5.1 and remembering that $\hat{u}_{\rm in} = u_{\rm in}(0)p_0^{-1}|D|^{\frac{1}{2}} + O(\omega)$ on ∂D yields:

$$\hat{u}(x) - \hat{u}_{\rm in}(x) = \frac{\hat{u}_{\rm in}(0)p_0^{-1}|D|^{\frac{1}{2}}}{P\left(\frac{\omega^2}{v_r^2\delta}\right)}\left[\left(\frac{\omega^2}{v_r^2\delta}\gamma_m T_{00,00} + T_{ml,ml}T_{00,00} - T_{00,ml}^2\right)G_{00}\right.$$
$$\left. + \frac{\omega^2}{v_r^2\delta}\gamma_0 T_{00,ml}G_{ml}\left(t - \frac{|x|}{v_0}, \frac{x}{|x|}\right)\right]\Gamma^{\frac{\omega}{v_0}}(x) - \hat{u}_{\rm in}(0)p_0^{-1}|D|^{\frac{1}{2}}G_{00}(1 + O(|x|^{-1}) + O(\omega))\Gamma^{\frac{\omega}{v_0}}(x). \quad (5.32)$$

The result follows by substituting G_{00} with (5.30). $\square$

Remark 5.5. Using the Fourier expansion of G_{ml}, we can also write $\hat{u}(x) - \hat{u}_{\rm in}(x)$ as a superposition of spherical waves with wave numbers $(\omega + n\Omega)/v_0$ for all $n \in \mathbb{Z}$:

$$\hat{u}(x) - \hat{u}_{\rm in}(x) = \hat{u}_{\rm in}(0)(1 + O(\delta^{\frac{1}{2}}) + O(|x|^{-1}))\left(A\left(\frac{\omega^2}{v_r^2\delta}\right)\Gamma^{\frac{\omega}{v_0}}(x) + B\left(\frac{\omega^2}{v_r^2\delta}\right)\sum_{n\in\mathbb{Z}}\hat{p}_{m,n}\hat{c}_{l,n}\left(\frac{x}{|x|}\right)\Gamma^{\frac{\omega+n\Omega}{v_0}}(x)e^{-\mathrm{i}n\Omega t}\right).$$

Remark 5.6. The point scatterer approximation (5.30) appeals to two remarks: first, the scattered field propagates outgoing waves with high frequency components, more precisely with wave numbers $\frac{\omega+n\Omega}{v_0}$ for any $n \in \mathbb{Z}$. Second, these high frequency waves propagate with an amplitude which depends on the space direction $\frac{x}{|x|}$. This suggests that a suitable tuning of the geometry of the resonator D could be used for confining the energy of the scattered high frequency pulse to a desired spatial direction.

5.3. Effective medium theory for fast time-modulated high-contrast metamaterials

In this conclusive part, we consider a metamaterial obtained by filling a bounded domain Ω with N time-modulated high-contrast resonators D around centers $(y_n)_{1\le n\le N}$ and rescaled by a factor s: we assume the homogenization setting of Section 3.4 which is also illustrated on Figure 2. In this dilute regime where $s \to 0$ and $N \to +\infty$, the incident frequency is of order one ($\omega = O(1)$) while the modulation frequency is rescaled by a factor $1/s$; we assume that the modulation is sT–periodic and is of the form $\rho(\cdot/s)$ for a T–periodic modulation ρ. These rescalings can be summarized as follows:

$$D \to sD, \qquad T \to sT, \qquad \Omega \to \frac{\Omega}{s}, \qquad \rho \to \rho(\cdot/s). \quad (5.33)$$

Therefore, the modulation frequency is still much larger than the incident frequency: $\Omega \gg \omega$. Then, we assume that the size factor is of order $s = O(\delta^{\frac{1}{2}})$. Finally, we still assume that the modulation ρ is tuned to the "exceptional" coupling situation (5.22) featuring two pairs of coinciding Sturm-Liouville and Neumann eigenvalues.

In this part, we denote by $\hat{u}_{N,s}$ the solution to the scattering problem in the time-modulated, heterogeneous medium $D_{N,s}$:

$$\begin{cases} \frac{1}{v_0^2}(-\mathrm{i}\omega + \partial_t)\hat{u}_{N,s}(t,x) - \Delta\hat{u}_{N,s} = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^3\backslash\overline{D_{N,s}}, \\ \frac{1}{v_r^2}(-\mathrm{i}\omega + \partial_t)\hat{u}_{N,s}(t,x) - \frac{1}{\rho(t/s)}\Delta\hat{u}_{N,s} = 0, & (t,x) \in \mathbb{R}\times D_{N,s}, \\ \frac{1}{\rho(t/s)}\left.\frac{\partial\hat{u}_{N,s}}{\partial n}\right|_- = \delta\left.\frac{\partial\hat{u}_{N,s}}{\partial n}\right|_+ \text{ on } \partial D_{N,s}, \\ \hat{u}_{N,s}|_+ = \hat{u}_{N,s}|_+ \text{ on } \partial D_{N,s}, \\ t \mapsto \hat{u}_{N,s}(t,x) \text{ is } sT\text{–periodic}, \\ \left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} - \frac{1}{v_0}\partial_t\right)(\hat{u}_{N,s} - \hat{u}_{\rm in}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \quad (5.34)$$

5.3.1. *Rescalings of the point-scatterer approximation formula*

In what follows, we reproduce the Foldy-Lax approximation argument of Section 3.4 in order to formally derive a homogenized equation for the heterogeneous system (5.34). To do so, we rewrite the far field approximation formula (5.30) in this dilute setting.

First, the Neumann eigenvectors and eigenvalues of (1.7), and the Sturm-Liouville eigenvectors and eigenvalues of (1.6) must be replaced according to the dilute regime (5.33) as follows:

$$\phi_l \to s^{-\frac{3}{2}}\phi_l(\cdot/s), \qquad \lambda_l \to \frac{\lambda_l}{s^2}, \qquad p_m \to p_m\left(\cdot/s\right), \qquad \mu_m \to \frac{\mu_m}{s^2}.$$

We verify that the obtained eigenvectors are normalized since

$$\frac{1}{sT}\int_0^{sT}\frac{1}{\rho(t/s)}p_m(t/s)^2\,\mathrm{d}t = \frac{1}{T}\int_0^{T}\frac{1}{\rho(t)}p_m(t)^2\,\mathrm{d}t = 1 \text{ and } \int_{sD} s^{-3}|\phi_l(x/s)|^2\,\mathrm{d}x = \int_D |\phi_l|^2\,\mathrm{d}x = 1.$$

Now, we observe that the set Λ of (1.7) remains invariant by the rescaling (5.33) since

$$\Lambda = \left\{(m,l)\in\mathbb{N}\times\mathbb{N}\,|\,\frac{\lambda_l}{s^2} = \frac{\mu_m}{s^2v_r^2}\right\} = \left\{(m,l)\in\mathbb{N}\times\mathbb{N}\,|\,\lambda_l = \frac{\mu_m}{v_r^2}\right\}.$$

The next lemma summarizes how the different quantities occuring in the point-scatterer approximation formula (5.30) are affected by the rescaling.

Lemma 5.2. *The coefficients $(\gamma_m)_{(m,l)\in\Lambda}$, $(T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$, the polynomial P of (5.24), the coefficients $\hat{c}_{l,n}(\theta)$ of (5.26) and the functions $(G_{ml})_{(m,l)\in\Lambda}$ of (5.27) are changed as follows after the rescaling (5.33):*

(i) $\gamma_m \to s\gamma_m$,
(ii) $T_{ml,m'l'} \to s^{-1}T_{ml,m'l'}$,
(iii) $P(\xi) \to s^{-2}P(s^2\xi)$,
(iv) $\hat{c}_{l,n}(\theta) \to s^{-\frac{1}{2}}\hat{c}_{l,n}(\theta)$,
(v) $G_{ml}(t,\theta) \to s^{-\frac{1}{2}}G_{ml}(t,\theta)$,
(vi) $A\left(\frac{\omega^2}{v_r^2\delta}\right) \to sA\left(\frac{s^2\omega^2}{v_r^2\delta}\right)$ *and* $B\left(\frac{\omega^2}{v_r^2\delta}\right) \to s^{\frac{3}{2}}B\left(\frac{s^2\omega^2}{v_r^2\delta}\right)$.

Proof. Changing $\rho(t)$ in $\rho(t/s)$, we find that the function p_m^1 of (5.9) needs to be changed according to the transformation $p_m^1 \to sp_m^1(\cdot/s)$, from where the point (i) follows. Then one can verify that the sT–periodic Dirichlet-to-Neumann map $\mathcal{T}^0_{sD,\mathrm{per}}$ associated to the dilute domain $D_{N,s}$ verifies

$$\mathcal{T}^0_{sD,\mathrm{per}}[p_m(\cdot/s)\phi_l(\cdot/s)] = s^{-1}\mathcal{T}_{0,D}[p_m\phi_l](\cdot/s,\cdot/s),$$

which implies that the transformation (ii) for the matrix $(T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$. The point (iii) follows by inserting the rescalings (i) and (ii) in (5.18). Then, the rescaling of the coefficients $\hat{c}_{l,n}(\theta)$ of (5.26) can be computed from the integral

$$\int_{s\partial D} e^{-\mathrm{i}\frac{n\Omega}{sv_0}y\cdot\theta}\left(\mathcal{S}_{sD}^{\frac{n\Omega}{sv_0}}\right)^{-1}[\phi_l\left(\cdot/s\right)](y)\,\mathrm{d}\sigma(y) = s^2\int_{\partial D} e^{-\mathrm{i}\frac{n\Omega}{v_0}y\cdot\theta}s^{-1}\left(\mathcal{S}_{D}^{\frac{n\Omega}{v_0}}\right)^{-1}[s^{-\frac{3}{2}}\phi_l](y)\,\mathrm{d}\sigma(y),$$

which yields the updates (iv)-(v). The result (vi) is a consequence of (i)-(iii). □

The rescaling rules of Lemma 5.2 yields the following point-scatterer approximation formula for a single rescaled resonator sD centered at zero and modulated by the sT–periodic function $\rho(\cdot/s)$:

$$\hat{u}(t,x) - \hat{u}_{\mathrm{in}}(x) \simeq s\hat{u}_{\mathrm{in}}(0)\left(A\left(\frac{\omega^2s^2}{v_r^2\delta}\right) + B\left(\frac{\omega^2s^2}{v_r^2\delta}\right)G_{ml}\left(\frac{t}{s} - \frac{|x|}{v_0s},\frac{x}{|x|}\right)\right)\Gamma^{\frac{\omega}{v_0}}(x) \text{ as } \frac{|x|}{s}\to+\infty. \tag{5.35}$$

Assuming now the setting of Figure 2 where we consider N time-modulated resonators located at the centers $(y_i)_{1\le i\le N}$, the resonator located at y_i receives the incident source field and $N-1$ scattered waves coming from the other centers $(y_j)_{1\le j\ne i\le N}$. If s is sufficiently small in the sense that the scatterers located at $(y_j)_{1\le j\ne i\le N}$ behave as point sources for the center y_i, we can expect that the total field $\hat{u}_{N,s}$ of (5.34) can be approximated by the contribution of $\hat{u}_{\mathrm{in}}$ and these scattered fields:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\mathrm{in}}(y_i) \simeq s\sum_{1\le j\ne i\le N}\left[A\left(\frac{\omega^2s^2}{v_r^2\delta}\right) + B\left(\frac{\omega^2v_r^2}{\delta}\right)G_{ml}\left(\frac{t}{s} - \frac{|y_i-y_j|}{sv_0},\frac{y_i-y_j}{|y_i-y_j|}\right)\right]\Gamma^{\frac{\omega}{v_0}}(y_i-y_j)\hat{u}_{\mathrm{in}}(y_j). \tag{5.36}$$

Based on this formula, we can discuss the formulation of an effective medium theory following the Foldy-Lax argument of Section 3.4.

5.3.2. *Effective medium theory for the time-modulated medium in a subcritical regime*

Using the law of large numbers, we deduce that $\hat{u}_{N,s}$ can be approximated by

$$\hat{u}_{N,s}(x) \simeq \hat{u}_{\text{in}}(x) + sN \int_{\Omega} \left[K\left(t - \frac{|y-y'|}{v_0}, \frac{y-y'}{|y-y'|} \right) \right] \Gamma^{\frac{\omega}{v_0}}(y-y') \hat{u}_{\text{in}}(t,y') V(y') \, \mathrm{d}y',$$

where $K(t,\theta)$ is the sT–periodic integral kernel defined for any $t \in \mathbb{R}$ and $\theta \in \mathbb{S}^2$ by

$$K(t,\theta) := A\left(\frac{\omega^2 s^2}{v_r^2 \delta} \right) + B\left(\frac{\omega^2 s^2}{v_r^2 \delta} \right) G_{ml}\left(\frac{t}{s}, \frac{y-y'}{|y-y'|} \right),$$

and where A and B are the coefficients defined by (5.31). When $sN \to 0$, this equation can be seen as the first "Born" approximation of the solution $\hat{u}_{\text{eff}}$ to an effective integral equation.

Corollary 5.2. *Assume that the Foldy-Lax approximation (5.36) is valid and the "subcritical" regime $sN \to 0$. Then we can expect the convergence of the total wave field $\hat{u}_{N,s}$ to an effective wave field $\hat{u}_{\text{eff}}$ solution to the integral equation*

$$\hat{u}_{\text{eff}}(t,y) - sN \int_{\Omega} \left[K\left(t - \frac{|y-y'|}{v_0}, \frac{y-y'}{|y-y'|} \right) \right] \Gamma^{\frac{\omega}{v_0}}(y-y') \hat{u}_{\text{eff}}(t,y') V(y') \, \mathrm{d}y' = \hat{u}_{\text{in}}(y), \quad y \in \Omega, \tag{5.37}$$

where $V \, \mathrm{d}x$ is the probability distribution of the centers $(y_i)_{1 \le i \le N}$.

Therefore, in the regime where $sN \to 0$, the effective medium is governed by the integral equation (5.37), which further emphasizes that the output signal carries a high frequency component in the far field. This "subcritical" regime corresponds to the one in which higher-order reflections, which correspond to those coming from subsequent interactions between the high-frequency components of (5.36) and the resonators themselves, can be neglected: indeed, (5.36) shows that these reflections are of order $O((sN)^2)$. Nevertheless, the scattered field remains small in this situation (of order $O(sN)$), and it seems that it would be more relevant to consider the case where sN converges to a constant.

Unfortunately, it does not seem clear that one can write a homogenization theory for (5.34) in the "critical" regime $sN \to \alpha$ for some $\alpha > 0$. Indeed, the higher order reflections induce scattered waves with arbitrarily small wavelengths (equal to $\frac{2\pi v_0}{\omega + n\Omega/s}$ for any $n \in \mathbb{Z}$), which can resolve the lower order reflections entirely on a "small" resonator $y_i + sD$. Hence, these rescaled resonators $y_i + sD$ can neither be considered "small" by the higher order reflecting waves nor be approximated by point sources. Hence, a different analysis (for instance, a high frequency homogenization approach in the spirit of [39]) is needed to derive a homogenized medium in the regime $sN = O(1)$, left for a future work.

However, it is still possible to write a homogenized equation for the low frequency component of the scattered field in the regime $sN = O(1)$, following the ideas of [14]. Let us introduce the time-averaging operator P defined by

$$P\hat{v} := \frac{1}{T} \int_0^T \hat{v}(t,\cdot) \, \mathrm{d}t \text{ for any } \hat{v} \in L^2_{\text{per}}((0,T), H^1_{\text{loc}}(\mathbb{R}^3)) \cap H^1_{\text{per}}((0,T), L^2_{\text{loc}}(\mathbb{R}^3)).$$

Then, (5.35) and (5.27) imply that the far field of $P\hat{u}$ generated by a single rescaled resonator sD is

$$P\hat{u}(t,x) - \hat{u}_{\text{in}}(x) \simeq s\hat{u}_{\text{in}}(0) \left(A\left(\frac{\omega^2 s^2}{v_r^2 \delta} \right) - B\left(\frac{\omega^2 s^2}{v_r^2 \delta} \right) \hat{p}_{m,0} |D|^{-\frac{1}{2}} \operatorname{cap}(D) \right) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } \frac{|x|}{s} \to +\infty.$$

Repeating the Foldy-Lax approximation argument, one can expect that $P\hat{u}(t,x)$ converges to the solution $\hat{u}^*_{\text{eff}}$ to the following Lippmann-Schwinger equation with varying refractive index:

$$\left[\Delta + \left(\frac{\omega^2}{v_0^2} - sNC\left(\frac{\omega^2 s^2}{v_r^2 \delta} \right) V 1_{\Omega} \right) \right] \hat{u}^*_{\text{eff}} = 0,$$

where $C(\xi) := \Big(A(\xi) - B(\xi) \hat{p}_{m,0} |D|^{-\frac{1}{2}} \operatorname{cap}(D) \Big)$, with $A(\xi)$ and $B(\xi)$ defined in (5.31). This equation has properties similar to those of the effective equation (3.28) obtained in the unmodulated case, namely the possibility to achieve negative index wave propagation and amplification close to the resonance frequency. Therefore, the true interest of the time-modulated medium lies in the generation of high frequency scattered waves.

Appendix A. Bloch transform of tempered distributions

The Bloch transform (or its variant, the Floquet transform), is a classical mathematical tool in nanophotonics and condensed matter theory; it allows to study the properties of periodic crystals, which are lattice structures featuring a periodically repeated pattern in the three directions of space $\mathbb{R}^3$. Its most importance property is the invariance with respect to the multiplication by a periodic function (see Appendix A.3), which allows to reduce a spectral posed in $H^1(\mathbb{R}^3)$ into a parameterized family of elliptic problems posed on the unit-cell of the periodic lattice with periodic boundary conditions. The Bloch transform is in most textbooks [64, 2] taken to be $L^2(\mathbb{R}^3)$, because wave packets which propagate in such structures decay at infinity. We are only aware of [24] regarding the definition of the Bloch transform of L^p functions with a general $p \geq 1$.

In the context of scattering due to a time-modulated media, where the periodicity affects the time variable rather than the space variable, it is not physically relevant to assume that the wave fields decay at infinite times, due to the energy conservation. Hence, there is a need for introducing a Bloch transform on a larger space of functions, which do not necessary decay at infinity.

In this appendix, we propose a definition of the Bloch transform of tempered distributions. Our construction is quite classical: we transpose the ideas that are considered for defining the Fourier transform of tempered distribution. Namely, we start by defining the Bloch transform $\mathcal{B}$ in the Schwartz space of smooth rapidly decaying functions in Appendix A.1. We also define the Floquet transform $\mathcal{U}$, which is its counterpart in the Fourier space. Then, in Appendix A.2, we extend $\mathcal{B}$ to tempered distributions by duality. We show that the main properties of this extension of the Bloch transform remain true, namely the fact that the Bloch transform of a distribution is equal, up to a multiplicative constant, to the Floquet transform of its Fourier transform. In Appendix A.3, we show an extension of the invariance property of the Bloch transform, namely $\mathcal{B}[\rho f] = \rho \mathcal{B} f$ for a periodic smooth function ρ and a tempered distribution f. In fact, this invariance property holds for periodic distributions ρ, whenever the multiplication ρf makes sense (see Definition A.3). Finally, we compute the Bloch transform of a time-harmonic function $t \mapsto e^{-\mathrm{i}\omega t}$ for $\omega \in \mathbb{R}$ in Appendix A.4, and we highlight how we can retrieve from this formalism the time harmonic ansatz (4.1) for the solution to the time-modulated wave equation (1.1).

In what follows, we denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decaying functions (see e.g. [86]):

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}) \,|\, \sup_{x \in \mathbb{R}} \left| x^p \frac{\mathrm{d}^q f}{\mathrm{d}x^q} \right| < +\infty, \qquad \text{for any } p, q \in \mathbb{N} \right\}.$$

We denote by $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions T, which are linear forms on $\mathcal{S}(\mathbb{R})$ satisfying the continuity property

$$\langle T, f \rangle_{\mathcal{S}', \mathcal{S}} \leq C \sup_{\substack{x \in \mathbb{R} \\ 0 \leq i, j \leq p}} \left| x^i \frac{\mathrm{d}^j f}{\mathrm{d}x^j} \right|,$$

for some integer $p \in \mathbb{N}$ and independent constant $C > 0$.

A.1. The Bloch and Floquet transforms on the Schwartz space

We start by introducing a variant of the Floquet transform for functions defined in the Fourier space. For any $\hat{f} \in \mathcal{S}(\mathbb{R})$, we consider the transformed function $\mathcal{U}\hat{f}$ defined by the formula

$$\mathcal{U}\hat{f}(t, \alpha) := \sum_{p \in \mathbb{Z}} \hat{f}(\alpha + p\Omega) e^{-\mathrm{i}p\Omega t}, \qquad \forall (t, \alpha) \in \mathbb{R} \times \mathbb{R}. \tag{A.1}$$

Since $\hat{f} \in \mathcal{S}(\mathbb{R})$, this series and all its derivatives converge normally, which implies that $\mathcal{U}\hat{f} \in \mathcal{V}(\mathbb{R} \times \mathbb{R})$ where $\mathcal{V}(\mathbb{R} \times \mathbb{R})$ is the space of $\mathcal{C}^\infty$ functions $\phi(t, \alpha)$ which are T–periodic in the first variable and Ω quasi-periodic in the second variable:

$$\mathcal{V}(\mathbb{R} \times \mathbb{R}) := \{ \phi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}) \,|\, \forall t \in \mathbb{R}, \forall \alpha \in \mathbb{R}, \phi(t + T, \alpha) = \phi(t, \alpha) \text{ and } \phi(t, \alpha + \Omega) = \phi(t, \alpha) e^{\mathrm{i}\Omega t} \}.$$

Lemma A.1. *The transform $\mathcal{U}$ is an isomorphism from $\mathcal{S}(\mathbb{R})$ to $\mathcal{V}(\mathbb{R} \times \mathbb{R})$, whose inverse is the mapping $\mathcal{U}^{-1} : \mathcal{V}(\mathbb{R} \times \mathbb{R}) \to \mathcal{S}(\mathbb{R})$ given by*

$$\mathcal{U}^{-1}\phi(\alpha) := \frac{1}{T} \int_0^T \phi(t, \alpha) \,\mathrm{d}t. \tag{A.2}$$

Proof. It is straightforward to see that $\hat{f} = \mathcal{U}^{-1}(\mathcal{U}\hat{f})$ with $\mathcal{U}^{-1}$ given by (A.2), which proves that $\mathcal{U}$ is injective. In order to prove that $\mathcal{U}$ is also surjective, it is enough to show that $\mathcal{U}^{-1}\phi \in \mathcal{S}(\mathbb{R})$ for any $\phi \in \mathcal{V}(\mathbb{R} \times \mathbb{R})$, because the Fourier inversion formula for trigonometric series yields $\mathcal{U}(\mathcal{U}^{-1}\phi) = \phi$. For an arbitrary $n \in \mathbb{N}$, we have:

$$\begin{aligned} \alpha^p \frac{\mathrm{d}^q}{\mathrm{d}\alpha^q} \mathcal{U}^{-1}\phi(\alpha) &= \frac{\alpha^p}{T} \int_0^T \partial_\alpha^q \phi(t, \alpha) \,\mathrm{d}t = \frac{\alpha^p}{T} \int_0^T \partial_\alpha^q \phi(t, \alpha - n\Omega + n\Omega) \,\mathrm{d}t \\ &= \frac{\alpha^p}{T} \int_0^T \partial_\alpha^q \phi(t, \alpha - n\Omega) e^{\mathrm{i}n\Omega t} \,\mathrm{d}t, \end{aligned}$$

where we use the fact that for any $q \in \mathbb{N}$, $\partial_\alpha^q \phi$ is Ω–quasi-periodic in the second variable. Using $p+1$ integration by parts with $n := \lfloor \alpha/\Omega \rfloor$, we obtain

$$\begin{aligned} \left| \alpha^p \frac{\mathrm{d}^q}{\mathrm{d}\alpha^q} [\mathcal{U}^{-1}\phi(\alpha)] \right| &= \left| \frac{\alpha^p (-1)^{p+1}}{T (\mathrm{i}n\Omega)^{p+1}} \int_0^T \partial_\alpha^q \partial_t^{p+1} \phi(t, \alpha - n\Omega) e^{\mathrm{i}n\Omega t} \,\mathrm{d}t \right| \\ &\leq C \frac{|\alpha|^p}{|\alpha|^{p+1}} \sup_{t \in \mathbb{R}, \alpha' \in \mathbb{R}} |\partial_\alpha^q \partial_t^{p+1} \phi(t, \alpha')| \end{aligned} \tag{A.3}$$

for some constant $C > 0$ independent of α. This shows that $\mathcal{U}^{-1}\phi \in \mathcal{S}(\mathbb{R})$. $\square$

For a given function $f \in \mathcal{S}(\mathbb{R})$ defined in the time domain, the Bloch transform of f is the $\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R})$ function $\mathcal{B}f$ defined by

$$\mathcal{B}f(t,\alpha) = \sum_{n\in\mathbb{Z}} f(t-nT)e^{\mathrm{i}\alpha(t-nT)}, \qquad (t,\alpha) \in \mathbb{R}\times\mathbb{R}. \tag{A.4}$$

The following lemma relates the Bloch transform $\mathcal{B}$ in the time-domain and the Floquet transform $\mathcal{U}$ in the Fourier space.

Lemma A.2. *We have*

$$\mathcal{B}f(t,\alpha) = \frac{1}{T}\mathcal{U}\hat{f}(t,\alpha) = \frac{1}{T}\sum_{p\in\mathbb{Z}} \hat{f}(\alpha+p\Omega)e^{-\mathrm{i}p\Omega t},$$

where $\hat{f}$ is the Fourier transform of f:

$$\hat{f}(\alpha) := \int_{-\infty}^{\infty} f(t)e^{\mathrm{i}\alpha t}\,\mathrm{d}t.$$

Proof. Let us compute the trigonometric coefficients of $\mathcal{B}(f,\alpha)$: for any $p \in \mathbb{N}$, we find

$$\begin{aligned}\frac{1}{T}\int_0^T \mathcal{B}f(t,\alpha)e^{\mathrm{i}p\Omega t}\,\mathrm{d}t &= \sum_{n\in\mathbb{Z}}\frac{1}{T}\int_0^T f(t-nT)e^{\mathrm{i}\alpha(t-nT)}e^{\mathrm{i}p\Omega t}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}}\frac{1}{T}\int_{-nT}^{-(n-1)T} f(t)e^{\mathrm{i}\alpha t}e^{\mathrm{i}p\Omega(t+nT)}\,\mathrm{d}t\\ &= \frac{1}{T}\int_{-\infty}^{+\infty} f(t)e^{\mathrm{i}(\alpha+p\Omega)t}\,\mathrm{d}t = \frac{1}{T}\hat{f}(\alpha+p\Omega).\end{aligned}$$

$\square$

By using the properties of the Floquet transform $\mathcal{U}$, we obtain the following result.

Corollary A.1. *The Bloch transform is an isomorphism from $\mathcal{S}(\mathbb{R})$ to $\mathcal{V}(\mathbb{R}\times\mathbb{R})$, and the following inversion formula holds true:*

$$f(t) = \frac{1}{\Omega}\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathcal{B}f(t,\alpha)e^{-\mathrm{i}\alpha t}\,\mathrm{d}\alpha, \qquad \text{for any } f \in \mathcal{S}(\mathbb{R}). \tag{A.5}$$

We have also the following inversion formula for the Fourier transform of $f \in \mathcal{S}(\mathbb{R})$:

$$\hat{f}(\alpha) = \int_0^T \mathcal{B}f(t,\alpha)\,\mathrm{d}t.$$

We now state the well-known Plancherel identities which hold for the Floquet and Bloch transforms.

Proposition A.1. *The following Plancherel identities hold for any $f, g \in \mathcal{S}(\mathbb{R})$:*

$$\frac{1}{T}\int_{\mathbb{R}} f(t)\overline{g}(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^T\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathcal{B}f(t,\alpha)\overline{\mathcal{B}g}(t,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}t, \tag{A.6}$$

$$\frac{1}{\Omega}\int_{\mathbb{R}} \hat{f}(\alpha)\overline{\hat{g}(\alpha)}\,\mathrm{d}\alpha = \frac{1}{2\pi}\int_0^T\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \mathcal{U}\hat{f}(t,\alpha)\overline{\mathcal{U}\hat{g}(t,\alpha)}\,\mathrm{d}\alpha\,\mathrm{d}t. \tag{A.7}$$

Proof. This classical property is an easy consequence of the result of Lemma A.2. Using the usual Plancherel identity for the Fourier transforms $\hat{f}$ and $\hat{g}$ of f and g, we write

$$\begin{aligned}\int_{\mathbb{R}} f(t)\overline{g}(t)\,\mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\alpha)\overline{\hat{g}(\alpha)}\,\mathrm{d}\alpha = \frac{1}{2\pi}\sum_{p\in\mathbb{Z}}\int_{(-\frac{1}{2}+p)\Omega}^{(\frac{1}{2}+p)\Omega}\hat{f}(\alpha)\overline{\hat{g}(\alpha)}\,\mathrm{d}\alpha\\ &= \frac{1}{2\pi}\sum_{p\in\mathbb{Z}}\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}}\hat{f}(\alpha+p\Omega)\overline{\hat{g}(\alpha+p\Omega)}\,\mathrm{d}\alpha = \frac{1}{2\pi}\int_0^T\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}}\mathcal{U}\hat{f}(t,\alpha)\overline{\mathcal{U}\hat{g}(t,\alpha)}\,\mathrm{d}\alpha\,\mathrm{d}t\\ &= \frac{1}{2\pi}\frac{1}{T}T^2\int_0^T\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}}\mathcal{B}f(t,\alpha)\overline{\mathcal{B}g(t,\alpha)}\,\mathrm{d}t\,\mathrm{d}\alpha = \frac{1}{\Omega}\int_0^T\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}}\mathcal{B}f(t,\alpha)\overline{\mathcal{B}g(t,\alpha)}\,\mathrm{d}\alpha\,\mathrm{d}t.\end{aligned}$$

$\square$

A.2. **Bloch transform of tempered distributions**

We now use the Plancherel identities of Proposition A.1 to extend $\mathcal{B}$ to tempered distributions by duality. Let us introduce the space of distributions $\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ on $\mathcal{V}(\mathbb{R}\times\mathbb{R})$.

Definition A.1. We call Φ a distribution on $\mathcal{V}(\mathbb{R}\times\mathbb{R})$ a linear form on $\mathcal{V}(\mathbb{R}\times\mathbb{R})$ for which there exist an integer $p\in\mathbb{N}$ and an independent constant $C>0$ such that

$$|\langle\Phi,\phi\rangle_{\mathcal{V}',\mathcal{V}}|\leq C\sup_{\substack{t,\alpha\in\mathbb{R}\\0\leq i,j\leq p}}|\partial_t^i\partial_\alpha^j\phi(\alpha,t)|\text{ for all }\phi\in\mathcal{V}(\mathbb{R}\times\mathbb{R}).$$

We denote by $\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ the corresponding space of such distributions.

Any function $(t,\alpha)\mapsto\phi(t,\alpha)$ integrable on $(0,T)\times(-\frac{\Omega}{2},\frac{\Omega}{2})$ can be identified to a distribution $\Phi_\phi\in\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ by defining

$$\langle\Phi_\phi,\psi\rangle_{\mathcal{V}',\mathcal{V}}:=\frac{1}{2\pi}\int_0^T\int_{-\Omega/2}^{\frac{\Omega}{2}}\phi(t,\alpha)\overline{\psi(t,\alpha)}\,\mathrm{d}t\,\mathrm{d}\alpha,\qquad\text{for all }\psi\in\mathcal{V}(\mathbb{R}\times\mathbb{R}),$$

where we choose $2\pi=\Omega T$ as the normalisation constant. We can now define the Bloch transform of a tempered distribution by mimicking the duality identity (A.4).

Definition A.2. For a given tempered distribution $f\in\mathcal{S}'(\mathbb{R})$, the Bloch transform of f is the distribution $\mathcal{B}f\in\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ defined by the formula

$$\langle\mathcal{B}f,\phi\rangle_{\mathcal{V}',\mathcal{V}}:=\frac{1}{T}\langle f,\mathcal{B}^{-1}\phi\rangle_{\mathcal{S}',\mathcal{S}},\qquad\text{for any }\phi\in\mathcal{V}(\mathbb{R}\times\mathbb{R}).\tag{A.8}$$

For a given tempered distribution $\hat{f}\in\mathcal{S}(\mathbb{R})$, the Floquet transform of $\hat{f}$ is the distribution $\mathcal{U}\hat{f}\in\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ defined by the formula

$$\langle\mathcal{U}\hat{f},\phi\rangle_{\mathcal{V}',\mathcal{V}}:=\frac{1}{\Omega}\langle\hat{f},\mathcal{U}^{-1}\phi\rangle_{\mathcal{S}',\mathcal{S}},\qquad\text{for any }\phi\in\mathcal{V}(\mathbb{R}\times\mathbb{R}).\tag{A.9}$$

Remark A.1. The fact that $\mathcal{U}$ maps $\mathcal{S}'(\mathbb{R})$ to $\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ is a direct consequence of the inequality (A.3), and similarly for $\mathcal{B}$.

In what follows, we still write (A.1) and (A.4) for $\mathcal{B}f$ and $\mathcal{U}\hat{f}$, remembering that these mean (A.8) and (A.9) in the distributional sense. Clearly, the property of Lemma A.2 remains true for $f\in\mathcal{S}'(\mathbb{R})$.

Proposition A.2. *The following properties hold:*

(i) the Bloch transform $\mathcal{B}$ and Floquet transform $\mathcal{U}$ of (A.8) and (A.9) are bijective mappings from $\mathcal{S}'(\mathbb{R})$ onto $\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ whose reciprocal map are the transform denoted by $\mathcal{B}^{-1}:\mathcal{V}'(\mathbb{R}\times\mathbb{R})\to\mathcal{S}'(\mathbb{R})$ and $\mathcal{U}^{-1}:\mathcal{V}'(\mathbb{R}\times\mathbb{R})\to\mathcal{S}'(\mathbb{R})$ defined for any $\phi\in\mathcal{V}'(\mathbb{R}\times\mathbb{R})$ by

$$\langle\mathcal{B}^{-1}\phi,f\rangle_{\mathcal{S}',\mathcal{S}}=T\langle\phi,\mathcal{B}f\rangle_{\mathcal{V}',\mathcal{V}},\qquad\forall f\in\mathcal{S}(\mathbb{R}),$$
$$\langle\mathcal{U}^{-1}\phi,\hat{f}\rangle_{\mathcal{S}',\mathcal{S}}=\Omega\langle\phi,\mathcal{U}\hat{f}\rangle_{\mathcal{V}',\mathcal{V}},\qquad\forall\hat{f}\in\mathcal{S}(\mathbb{R});$$

(ii) for any $f\in\mathcal{S}'(\mathbb{R})$, it holds $\mathcal{B}f=\frac{1}{T}\mathcal{U}\hat{f}$, where $\hat{f}$ is the Fourier transform of the tempered distribution f.

Proof. (i) is a straightforward consequence of the definition and the inverse mapping properties of $\mathcal{B}$ and $\mathcal{U}$ defined on the Schwartz class. The point (ii) is obtained as follows: using the Definition A.2, we find, for any $\phi\in\mathcal{V}(\mathbb{R}\times\mathbb{R})$:

$$\langle\mathcal{B}f,\phi\rangle=\frac{1}{T}\langle f,\mathcal{B}^{-1}\phi\rangle=\frac{1}{2\pi}\langle\hat{f},\mathcal{U}^{-1}\phi\rangle=\frac{\Omega}{2\pi}\langle\mathcal{U}\hat{f},\phi\rangle=\langle\frac{1}{T}\mathcal{U}\hat{f},\phi\rangle.$$

□

A.3. **Invariance property by multiplication with a periodic distribution**

We now extend one the main properties of the Bloch transform, which is its invariance by multiplication by a smooth periodic function (or by a L^∞ periodic function for the Bloch transform in L^2).

An important observation is to remark that, although the multiplication of two distributions is in general not well determined, the product of a tempered distribution by a T–periodic distribution ρ can be defined upon a certain condition. We recall that a T–periodic distribution can be written in the form of an arbitrary trigonometric series $\sum_{p\in\mathbb{Z}}\rho_p e^{-\mathrm{i}p\Omega\cdot}$ with coefficients $(\rho_p)_{p\in\mathbb{Z}}$ growing at most polynomially as $p\to\infty$, see e.g. [86, 40].

Definition A.3. Let ρ be a T–periodic distribution and denote by $(\rho_p)_{p\in\mathbb{N}}$ the Fourier coefficients of ρ:

$$\rho(t)=\sum_{p\in\mathbb{Z}}\rho_p e^{-\mathrm{i}p\Omega t}.\tag{A.10}$$

Let $f \in \mathcal{S}'(\mathbb{R})$ be a tempered distribution. Whenever the series $\sum_{p\in\mathbb{Z}} \rho_p(f e^{-\mathrm{i}p\Omega\cdot})$ is convergent in $\mathcal{S}'$, we say that the "product" of ρ and f is well-defined and we write

$$\rho f := \sum_{p\in\mathbb{Z}} \rho_p f e^{-\mathrm{i}p\Omega\cdot} \in \mathcal{S}'(\mathbb{R}). \tag{A.11}$$

Remark A.2. The definition (A.11) makes sense thanks to two remarks. First, the product of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ with the smooth bounded function $t \mapsto e^{-\mathrm{i}p\Omega t}$ is a well-defined tempered distribution. This series converges in $\mathcal{S}'(\mathbb{R})$ if and only if the series $\sum_{p\in\mathbb{Z}} \rho_p \langle f e^{-\mathrm{i}p\Omega\cdot}, \phi\rangle_{\mathcal{S}',\mathcal{S}}$ converges for any $\phi \in \mathcal{S}(\mathbb{R})$. When this is the case, the Banach-Steinhaus theorem implies that the linear form ρf defined by

$$\langle \rho f, \phi\rangle := \lim_{n\to+\infty} \sum_{|p|\le n} \rho_p \langle f e^{-\mathrm{i}p\Omega\cdot}, \phi\rangle_{\mathcal{S}',\mathcal{S}}$$

belongs automatically to $\mathcal{S}'(\mathbb{R})$.

Similarly, we can define the product of a T–periodic distribution with a distribution $\phi \in \mathcal{V}'(\mathbb{R}\times\mathbb{R})$.

Definition A.4. Let ρ be a T–periodic distribution whose Fourier series can be written as (A.10). Whenever the series $\sum_{p\in\mathbb{Z}} \rho_p(\Phi e^{-\mathrm{i}\Omega p\cdot})$ converges in $\mathcal{V}'(\mathbb{R}\times\mathbb{R})$, we say that the product $\rho\Phi$ is well-defined and we write

$$\rho\Phi = \sum_{p\in\mathbb{Z}} \rho_p \Phi e^{-\mathrm{i}\Omega p\cdot}.$$

We can now extend the classical invariance result $\mathcal{B}(\rho f) = \rho\mathcal{B}f$, which is obvious for $f \in \mathcal{S}$ and $\rho \in \mathcal{C}^\infty_{\mathrm{per}}((0,T),\mathbb{R})$ a smooth T–periodic function.

Proposition A.3. *Let $f \in \mathcal{S}'(\mathbb{R})$ and ρ be a T–periodic distribution. If ρf is well-defined (in the sense of Definition A.3), then the product $\rho\mathcal{B}f$ is also well-defined (in the sense of Definition A.4) and it holds*

$$\mathcal{B}(\rho f) = \rho\mathcal{B}f.$$

Proof. For any $\phi \in \mathcal{V}(\mathbb{R}\times\mathbb{R})$, we have by using the Definition A.2:

$$\langle \mathcal{B}(\rho f), \phi\rangle_{\mathcal{V}',\mathcal{V}} = \frac{1}{T}\langle \rho f, \mathcal{B}^{-1}\phi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{T}\sum_{p\in\mathbb{Z}} \rho_p \langle f, e^{-\mathrm{i}\Omega p\cdot}\mathcal{B}^{-1}\phi\rangle_{\mathcal{S}',\mathcal{S}}.$$

It is clear from Corollary A.1 that for $\phi \in \mathcal{V}(\mathbb{R}\times\mathbb{R})$, $e^{-\mathrm{i}\Omega p\cdot}\mathcal{B}^{-1}\phi = \mathcal{B}^{-1}(e^{-\mathrm{i}\Omega p\cdot}\phi)$. Therefore, we arrive at

$$\langle \mathcal{B}(\rho f), \phi\rangle_{\mathcal{V}',\mathcal{V}} = \frac{1}{T}\sum_{p\in\mathbb{Z}} \rho_p \langle f, \mathcal{B}^{-1}(e^{-\mathrm{i}\Omega p\cdot}\phi)\rangle_{\mathcal{S}',\mathcal{S}} = \sum_{p\in\mathbb{Z}} \langle \mathcal{B}f, e^{-\mathrm{i}\Omega p\cdot}\phi\rangle_{\mathcal{S}',\mathcal{S}}.$$

From the Definition A.4, we obtain that the product $\rho\mathcal{B}f$ is well-defined, and that

$$\langle \mathcal{B}(\rho f), \phi\rangle_{\mathcal{V}',\mathcal{V}} = \langle \rho\mathcal{B}f, \phi\rangle_{\mathcal{V}',\mathcal{V}}.$$

□

Remark A.3. One can verify that if $\rho \in L^\infty_{\mathrm{per}}((0,T),\mathbb{R})$ and $f \in \mathcal{L}^2(\mathbb{R})$, then the tempered distribution ρf of (A.11) is well-defined and coincides with the "usual" product ρf. Therefore, we retrieve the well-known invariance property $\mathcal{B}(\rho f) = \rho\mathcal{B}f$ which holds on these spaces.

A.4. The Bloch transform of time-harmonic functions

We conclude this appendix by computing the Bloch transform of time-harmonic functions $t \mapsto e^{-\mathrm{i}\omega t}$, $\omega \in \mathbb{R}$. These functions do not belong to $L^2(\mathbb{R})$ but they are tempered distributions.

Lemma A.3. *Let $\omega \in \mathbb{R}$. The Bloch transform of $t \mapsto e^{-\mathrm{i}\omega t}$ is the distribution*

$$\langle \mathcal{B}e^{-\mathrm{i}\omega\cdot}, \phi\rangle_{\mathcal{V}',\mathcal{V}} = \frac{1}{T}\int_0^T \phi(t,\omega)\,\mathrm{d}t, \qquad \phi \in \mathcal{V}, \tag{A.12}$$

which can also formally be written as

$$\mathcal{B}(e^{-\mathrm{i}\omega\ \cdot})(t,\alpha) = \Omega\sum_{p\in\mathbb{Z}} \delta_0(\omega-\alpha-p\Omega)e^{-\mathrm{i}p\Omega t}. \tag{A.13}$$

Proof. Using Proposition A.2 and the fact that the Fourier transform of $e^{-\mathrm{i}\omega\cdot}$ is $\alpha \mapsto 2\pi\delta_0(\omega-\alpha)$, we find

$$\mathcal{B}e^{-\mathrm{i}\omega\cdot}(t,\alpha) = \frac{2\pi}{T}[\mathcal{U}\delta(\omega-\cdot)](t,\alpha) = \frac{2\pi}{T}\sum_{p\in\mathbb{Z}} \delta(\omega-\alpha-p\Omega)e^{-\mathrm{i}p\Omega t}, \tag{A.14}$$

which is the result (A.13). The distributional sense (A.12) is a consequence of (A.9), which states that

$$\frac{2\pi}{T}\langle[\mathcal{U}\delta(\omega-\cdot)], \phi\rangle_{\mathcal{V},\mathcal{V}'} = \frac{2\pi}{T\Omega}\langle \delta(\omega-\cdot), \mathcal{U}^{-1}\phi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{T}\int_0^T \phi(t,\omega)\,\mathrm{d}t.$$

□

As an application, we can use Lemma A.3 to formally deduce that the solution $u(t,x)$ to (1.1) must be in the time-harmonic form (4.1). Due to Lemma A.3, the Bloch transform of the incident field $u_{\text{in}}(t,x)$ of (1.2) is equal to

$$\mathcal{B}u_{\text{in}}(t,x;\alpha) = \Omega \sum_{p\in\mathbb{Z}} \hat{u}_{\text{in}}(x)\delta(\omega-\alpha-p\Omega)e^{-\mathrm{i}p\Omega t}, \tag{A.15}$$

where we observe some separability between the variable α and ω. Then, if $1/\rho(t)u$ is well-defined, the Bloch transform $\mathcal{B}u$ solves the following set of partial differential equations:

$$\begin{cases} \dfrac{1}{\kappa_0}\left(-\mathrm{i}\alpha+\dfrac{\partial}{\partial t}\right)^2 \mathcal{B}u(t,x;\alpha) - \dfrac{1}{\rho_0}\Delta\mathcal{B}u(t,x;\alpha) = 0, \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^3\setminus\overline{D}, \\ \dfrac{1}{\kappa_r}\left(-\mathrm{i}\alpha+\dfrac{\partial}{\partial t}\right)^2 \mathcal{B}u(t,x;\alpha) - \dfrac{1}{\rho(t)\rho_r}\Delta\mathcal{B}u(t,x;\alpha) = 0, \qquad (t,x)\in\mathbb{R}\times D, \qquad 1\le i\le N, \\ \dfrac{1}{\rho_0}\left.\dfrac{\partial\mathcal{B}u(t,x;\alpha)}{\partial n}\right|_{+} = \dfrac{1}{\rho_r\rho(t)}\left.\dfrac{\partial\mathcal{B}u(t,x;\alpha)}{\partial n}\right|_{+}, \qquad (t,x)\in\mathbb{R}\times\partial D, \quad 1\le i\le N, \\ \mathcal{B}u_{|+}(t,x;\alpha) = \mathcal{B}u_{|-}(t,x;\alpha), \qquad (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto \mathcal{B}u(t,x;\alpha) \text{ is } T\text{--periodic}, \\ e^{-\mathrm{i}\alpha t}(\mathcal{B}u(t,x;\alpha)-\mathcal{B}u_{\text{in}}(t,x;\alpha)) \text{ is outgoing}. \end{cases} \tag{A.16}$$

Using the decomposition (A.15), we expect that

$$\mathcal{B}u(t,x;\alpha) = \Omega\sum_{p\in\mathbb{Z}} \delta(\omega-\alpha-p\Omega)\hat{u}(t,x)e^{-\mathrm{i}p\Omega t},$$

where $\hat{u}(t,x)$ is the solution to (4.2), because the solution to the problem (4.2) with right-hand side $\hat{u}_{\text{in}}(x)\delta(\omega-\alpha-p\Omega)e^{-\mathrm{i}p\Omega t}$ is $\hat{u}(t,x)\delta(\omega-\alpha-p\Omega)e^{-\mathrm{i}p\Omega t}$. Then, the inverse Bloch transform yields

$$u(t,x;\omega) = \mathcal{B}^{-1}\left[\Omega\sum_{p\in\mathbb{Z}}\delta(\omega-\alpha-p\Omega)\hat{u}(t,x)e^{-\mathrm{i}p\Omega t}\right] = \hat{u}(t,x)\mathcal{B}^{-1}\left[\Omega\sum_{p\in\mathbb{Z}}\delta(\omega-\alpha-p\Omega)e^{-\mathrm{i}p\Omega t}\right] = e^{-\mathrm{i}\omega t}\hat{u}(t,x),$$

which is the time-harmonic decomposition (4.1).

References

[1] Allaire, G., Arnold, A., Degond, P., and Hou, T. Y. *Quantum Transport*, vol. 1946 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[2] Allaire, G., Conca, C., and Vanninathan, M. The Bloch transform and applications. *ESAIM: Proceedings 3* (1998), 65–84.

[3] Ammari, H., Cao, J., and Hiltunen, E. O. Non-reciprocal wave propagation in space-time modulated media. *arXiv:2109.07220 [cond-mat, physics:math-ph]* (Sept. 2021). arXiv: 2109.07220.

[4] Ammari, H., Challa, D. P., Choudhury, A. P., and Sini, M. The equivalent media generated by bubbles of high contrasts: Volumetric metamaterials and metasurfaces. *Multiscale Modeling and Simulation 18*, 1 (2020), 240–293.

[5] Ammari, H., Ciraolo, G., Kang, H., Lee, H., and Milton, G. W. Spectral theory of a Neumann–Poincaré-type operator and analysis of cloaking due to anomalous localized resonance. *Archive for Rational Mechanics and Analysis 208*, 2 (2013), 667–692.

[6] Ammari, H., and Davies, B. Mimicking the active cochlea with a fluid-coupled array of subwavelength Hopf resonators. *Proceedings of the Royal Society A 476*, 2234 (2020), 20190870.

[7] Ammari, H., Davies, B., Hiltunen, E. O., Lee, H., and Yu, S. High-order exceptional points and enhanced sensing in subwavelength resonator arrays. *Studies in Applied Mathematics 146*, 2 (nov 2020), 440–462.

[8] Ammari, H., Davies, B., Hiltunen, E. O., Lee, H., and Yu, S. Wave interaction with subwavelength resonators. 1–44.

[9] Ammari, H., Davies, B., Hiltunen, E. O., and Yu, S. Topologically protected edge modes in one-dimensional chains of subwavelength resonators. *Journal de Mathématiques Pures et Appliquées 144* (2020), 17–49.

[10] Ammari, H., Fitzpatrick, B., Gontier, D., Lee, H., and Zhang, H. Sub-wavelength focusing of acoustic waves in bubbly media. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 473*, 2208 (2017), 20170469.

[11] Ammari, H., Fitzpatrick, B., Gontier, D., Lee, H., and Zhang, H. Minnaert resonances for acoustic waves in bubbly media. *Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire 35*, 7 (2018), 1975–1998.

[12] Ammari, H., Fitzpatrick, B., Kang, H., Ruiz, M., Yu, S., and Zhang, H. *Mathematical and Computational Methods in Photonics and Phononics*, vol. 235 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, oct 2018.

[13] Ammari, H., Fitzpatrick, B., Lee, H., Yu, S., and Zhang, H. Double-negative acoustic metamaterials. *Quarterly of Applied Mathematics 77*, 4 (2019), 767–791.

[14] Ammari, H., Garapon, P., Bustos, L. G., and Kang, H. Transient anomaly imaging by the acoustic radiation force. *Journal of Differential Equations 249*, 7 (2010), 1579–1595.

[15] Ammari, H., and Hiltunen, E. O. Time-dependent high-contrast subwavelength resonators. *Journal of Computational Physics 445* (2021), 110594.

[16] Ammari, H., Hiltunen, E. O., and Yu, S. A High-Frequency Homogenization Approach Near the Dirac Points in Bubbly Honeycomb Crystals. *Archive for Rational Mechanics and Analysis 238*, 3 (2020), 1559–1583.

[17] Ammari, H., and Kang, H. *Polarization and moment tensors: with applications to inverse problems and effective medium theory*, vol. 162. Springer Science & Business Media, 2007.

[18] Ammari, H., Kang, H., and Lee, H. Layer Potential Techniques in Spectral Analysis. *Mathematical Surveys and Monographs, Vol. 153, 153* (2009).

[19] Ammari, H., and Nédélec, J. C. Full low-frequency asymptotics for the reduced wave equation. *Applied Mathematics Letters 12*, 1 (1999), 127–131.

[20] Ammari, H., and Zhang, H. Super-resolution in high-contrast media. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 471*, 2178 (2015), 20140946.

[21] Ammari, H., and Zhang, H. Effective medium theory for acoustic waves in bubbly fluids near minnaert resonant frequency. *SIAM Journal on Mathematical Analysis 49*, 4 (2017), 3252–3276.

[22] Bamberger, A., and Ha Duong, T. Formulation variationnelle espace-temps pour le calcul par potentiel retardé de la diffraction d'une onde acoustique (I). *Mathematical Methods in the Applied Sciences 8*, 1 (1986), 405–435.

[23] Bamberger, A., and Ha Duong, T. Formulation variationnelle pour le calcul de la diffraction d'une onde acoustique par une surface rigide. *Mathematical Methods in the Applied Sciences 8*, 1 (1986), 598–608.

[24] Barth, B. *The Bloch Transform on Lp-Spaces.* PhD thesis, 2013.

[25] Baumgärtel, H. *Analytic perturbation theory for matrices and operators*, vol. 15. Birkhauser Verlag AG, 1985.

[26] Cakoni, F., Haddar, H., and Lechleiter, A. On the factorization method for a far field inverse scattering problem in the time domain. *SIAM Journal on Mathematical Analysis 51*, 2 (2019), 854–872.

[27] Caloz, C. Ten applications of metamaterials. In *2016 IEEE International Symposium on Antennas and Propagation (APSURSI)* (2016), IEEE, pp. 1299–1300.

[28] Caloz, C., and Deck-Léger, Z.-L. Spacetime metamaterials—part i: general concepts. *IEEE Transactions on Antennas and Propagation 68*, 3 (2019), 1569–1582.

[29] Caloz, C., and Deck-Leger, Z.-L. Spacetime metamaterials—part ii: theory and applications. *IEEE Transactions on Antennas and Propagation 68*, 3 (2019), 1583–1598.

[30] Caloz, C., Deck-Léger, Z.-L., and Chamanara, N. Towards space-time metamaterials. In *2017 11th International Congress on Engineered Materials Platforms for Novel Wave Phenomena (Metamaterials)* (2017), IEEE, pp. 40–42.

[31] Chandler-Wilde, S. N., Graham, I. G., Langdon, S., and Spence, E. A. Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering. *Acta numerica 21* (2012), 89–305.

[32] Chandler-Wilde, S. N., Spence, E. A., Gibbs, A., and Smyshlyaev, V. P. High-frequency bounds for the helmholtz equation under parabolic trapping and applications in numerical analysis. *SIAM Journal on Mathematical Analysis 52*, 1 (2020), 845–893.

[33] Chen, H.-T., Padilla, W. J., Cich, M. J., Azad, A. K., Averitt, R. D., and Taylor, A. J. A metamaterial solid-state terahertz phase modulator. *Nature photonics 3*, 3 (2009), 148–151.

[34] Chen, Y., Li, X., Nassar, H., Norris, A. N., Daraio, C., and Huang, G. Nonreciprocal wave propagation in a continuum-based metamaterial with space-time modulated resonators. *Physical Review Applied 11*, 6 (2019), 064052.

[35] Chen, Y., Liu, H., Reilly, M., Bae, H., and Yu, M. Enhanced acoustic sensing through wave compression and pressure amplification in anisotropic metamaterials. *Nature Communications 5*, 1 (oct 2014), 1–9.

[36] Colton, D., and Kress, R. *Integral Equation Methods in Scattering Theory.* Society for Industrial and Applied Mathematics, Philadelphia, PA, nov 2013.

[37] Cosmo, F. d., Laudato, M., and Spagnuolo, M. Acoustic metamaterials based on local resonances: homogenization, optimization and applications. In *Generalized Models and Non-classical Approaches in Complex Materials 1.* Springer, 2018, pp. 247–274.

[38] Costabel, M. Time-Dependent Problems with the Boundary Integral Equation Method. *Encyclopedia of Computational Mechanics* (2004), 1–28.

[39] Craster, R. V., Kaplunov, J., and Pichugin, A. V. High-frequency homogenization for periodic media. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 466*, 2120 (2010), 2341–2362.

[40] Donoghue, W. F. *Distributions and Fourier transforms.* Academic Press, 2014.

[41] Engheta, N. Metamaterials with high degrees of freedom: space, time, and more. *Nanophotonics 10*, 1 (2021), 639–642.

[42] Ern, A., and Guermond, J.-L. *Theory and Practice of Finite Elements*, vol. 159. Springer, 2004.

[43] Feppon, F., and Ammari, H. Homogenization of sound-absorbing and high-contrast acoustic metamaterials in subcritical regimes. *HAL preprint hal-03372593* (Oct. 2021). working paper or preprint.

[44] Feppon, F., and Ammari, H. Analysis of a Monte-Carlo Nystrom method. *SIAM Journal of Numerical Analysis* (2022).

[45] Feppon, F., and Ammari, H. Modal decompositions and point scatterer approximations near the Minnaert resonance frequencies. *Studies in Applied Mathematics* (2022).

[46] Figari, R., Papanicolaou, G., and Rubinstein, J. Remarks on the point interaction approximation. *Hydrodynamic behavior and interacting particle systems 9* (1987), 45–55.

[47] Figari, R., Papanicolaou, G., and Rubinstein, J. The point interaction approximation for diffusion in regions with many small holes. 202–209.

[48] Foldy, L. L. The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers. *Physical review 67*, 3-4 (1945), 107.

[49] Galiffi, E., Huidobro, P., and Pendry, J. B. Broadband nonreciprocal amplification in luminal metamaterials. *Physical review letters 123*, 20 (2019), 206101.

[50] Galiffi, E., Tirole, R., Yin, S., Li, H., Vezzoli, S., Huidobro, P. A., Silveirinha, M. G., Sapienza, R., Alù, A., and Pendry, J. Photonics of time-varying media. *arXiv preprint arXiv:2111.08640* (2021).

[51] Galkowski, J., and Spence, E. A. Wavenumber-explicit regularity estimates on the acoustic single-and double-layer operators. *Integral Equations and Operator Theory 91*, 1 (2019), 1–35.

[52] Galkowski, J., Spence, E. A., and Wunsch, J. Optimal constants in nontrapping resolvent estimates and applications in numerical analysis. *Pure and Applied Analysis 2*, 1 (2019), 157–202.

[53] Gao, H., Hu, S., Du, C., and Zhang, Y. Theoretical model for manipulating light distribution in the time domain by using a spatially homogenous dynamic medium. *JOSA B 30*, 3 (2013), 622–625.

[54] Han, X., and Tacy, M. Sharp norm estimates of layer potentials and operators at high frequency. *Journal of Functional Analysis 269*, 9 (2015), 2890–2926.

[55] Huang, X., Zhou, L., and Chan, C. T. Modulating image oscillations in focusing by a metamaterial lens: Time-dependent green's function approach. *Physical Review B 74*, 4 (2006), 045123.

[56] J. B. Pendry. Negative Refraction Makes a Perfect Lens. *Physical Review Letters 85*, 18 (2000), 3966–3969.

[57] Kaina, N., Lemoult, F., Fink, M., and Lerosey, G. Negative refractive index and acoustic superlens from multiple scattering in single negative metamaterials. *Nature 525*, 7567 (2015), 77–81.
[58] Kato, T. *Perturbation Theory for Linear Operators*, vol. 132 of *Classics in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1995.
[59] Kaup, L., and Kaup, B. *Holomorphic functions of several variables: an introduction to the fundamental theory*, vol. 3. Walter de Gruyter, 2011.
[60] Kohn, R. V., Lu, J., Schweizer, B., and Weinstein, M. I. A variational perspective on cloaking by anomalous localized resonance. *Communications in Mathematical Physics 328*, 1 (2014), 1–27.
[61] Koutserimpas, T. T., and Fleury, R. Electromagnetic Waves in a Time Periodic Medium With Step-Varying Refractive Index. *IEEE Transactions on Antennas and Propagation 66*, 10 (Oct. 2018), 5300–5307.
[62] Koutserimpas, T. T., and Fleury, R. Zero refractive index in time-floquet acoustic metamaterials. *Journal of Applied Physics 123*, 9 (2018), 091709.
[63] Kress, R. *Linear integral equations*, vol. 82. Springer New York, 2014.
[64] Kuchment, P. *Floquet Theory for Partial Differential Equations*. Birkhäuser Basel, Basel, 1993.
[65] Lanoy, M., Pierrat, R., Lemoult, F., Fink, M., Leroy, V., and Tourin, A. Subwavelength focusing in bubbly media using broadband time reversal. *Physical Review B 91*, 22 (2015), 224202.
[66] Lemoult, F., Kaina, N., Fink, M., and Lerosey, G. Wave propagation control at the deep subwavelength scale in metamaterials. *Nature Physics 9*, 1 (nov 2012), 55–60.
[67] Lions, J. L., and Magenes, E. *Non-Homogeneous Boundary Value Problems and Applications, Vol II*. Springer-Verlag Berlin Heidelberg, 1972.
[68] Liu, Z., Zhang, X., Mao, Y., Zhu, Y., Yang, Z., Chan, C. T., and Sheng, P. Locally resonant sonic materials. *science 289*, 5485 (2000), 1734–1736.
[69] Ma, G., and Sheng, P. Acoustic metamaterials: From local resonances to broad horizons. *Science Advances 2*, 2 (feb 2016), e1501595.
[70] McCall, M. W., Favaro, A., Kinsler, P., and Boardman, A. A spacetime cloak, or a history editor. *Journal of optics 13*, 2 (2010), 024003.
[71] McLean, W. C. H. *Strongly elliptic systems and boundary integral equations*, vol. 86. Cambridge university press, 2000.
[72] Mennicken, R., and Möller, M. Root functions, eigenvectors, associated vectors and the inverse of a holomorphic operator function. *Archiv der Mathematik 42*, 5 (1984), 455–463.
[73] Minnaert, M. XVI. On musical air-bubbles and the sounds of running water. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 16*, 104 (1933), 235–248.
[74] Nédélec, J.-C. *Acoustic and electromagnetic equations: integral representations for harmonic problems*, vol. 144. Springer Science \& Business Media, 2013.
[75] Orazbayev, B., and Fleury, R. Quantitative robustness analysis of topological edge modes in C6 and valley-Hall metamaterial waveguides. *Nanophotonics 8*, 8 (aug 2019), 1433–1441.
[76] Pacheco-Peña, V., and Engheta, N. Temporal metamaterials with gain and loss. *arXiv preprint arXiv:2108.01007* (2021).
[77] Pacheco-Peña, V., and Engheta, N. Effective medium concept in temporal metamaterials. *Nanophotonics 9*, 2 (Feb. 2020), 379–391.
[78] Palermo, A., Celli, P., Yousefzadeh, B., Daraio, C., and Marzani, A. Surface wave non-reciprocity via time-modulated metamaterials. *Journal of the Mechanics and Physics of Solids 145* (2020), 104181.
[79] Paul, O., Imhof, C., Lägel, B., Wolff, S., Heinrich, J., Höfling, S., Forchel, A., Zengerle, R., Beigang, R., and Rahm, M. Polarization-independent active metamaterial for high-frequency terahertz modulation. *Optics express 17*, 2 (2009), 819–827.
[80] Preble, S. F., and Lipson, M. Conversion of a signal wavelength in a dynamically tuned resonator. In *Integrated Photonics Research and Applications* (2006), Optical Society of America, p. IMC5.
[81] Preble, S. F., Xu, Q., and Lipson, M. Changing the colour of light in a silicon resonator. *Nature Photonics 1*, 5 (2007), 293–296.
[82] Ptitcyn, G., Mirmoosa, M. S., and Tretyakov, S. A. Time-modulated meta-atoms. *Physical Review Research 1*, 2 (2019), 023014.
[83] Ramaccia, D., Alù, A., Toscano, A., and Bilotti, F. Propagation and scattering effects in metastructures based on temporal metamaterials. In *2021 Fifteenth International Congress on Artificial Materials for Novel Wave Phenomena (Metamaterials)* (2021), IEEE, pp. 356–358.
[84] Ramaccia, D., Alù, A., Toscano, A., and Bilotti, F. Propagation and scattering effects in temporal metastructures. In *Journal of Physics: Conference Series* (2021), vol. 2015, IOP Publishing, p. 012120.
[85] Ramaccia, D., Alù, A., Toscano, A., and Bilotti, F. Temporal multilayer structures for designing higher-order transfer functions using time-varying metamaterials. *Applied Physics Letters 118*, 10 (2021), 101901.
[86] Rudin, W. *Functional analysis*, second ed. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
[87] Sayas, F.-J. *Retarded Potentials and Time Domain Boundary Integral Equations*, vol. 50. 2016.
[88] Schweizer, B. Resonance Meets Homogenization. *Jahresbericht der Deutschen Mathematiker-Vereinigung 119*, 1 (2017), 31–51.
[89] Sheng, P., Zhang, X. X., Liu, Z., and Chan, C. T. Locally resonant sonic materials. *Physica B: Condensed Matter 338*, 1-4 (2003), 201–205.
[90] Steinbach, O., and Urzúa-Torres, C. A new approach to space-time boundary integral equations for the wave equation. *Arxiv* (2021), 1–24.
[91] Steinmeyer, G. A review of ultrafast optics and optoelectronics. *Journal of Optics A: Pure and Applied Optics 5*, 1 (2002), R1.
[92] Wu, Z., and Grbic, A. Serrodyne frequency translation using time-modulated metasurfaces. *IEEE Transactions on Antennas and Propagation 68*, 3 (2019), 1599–1606.
[93] Yanik, M. F., and Fan, S. Dynamic photonic structures: stopping, storage, and time reversal of light. *Studies in Applied Mathematics: Special issue: Nonlinear optics 115*, 2 (2005), 233–253.
[94] Zangeneh-Nejad, F., and Fleury, R. Active times for acoustic metamaterials. *Reviews in Physics 4* (Nov. 2019), 100031.