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Particle tracking in a live cell environment is concerned with reconstructing the trajectories, locations, or velocities of the targeting particles, which holds the promise of revealing important new biological insights. The standard approach of particle tracking consists of two steps: first reconstructing statically the source locations in each time step, and second applying tracking techniques to obtain the trajectories and velocities. In contrast to the standard approach, the dynamic reconstruction seeks to simultaneously recover the source locations and velocities from all frames, which enjoys certain advantages. In this paper, we provide a rigorous mathematical analysis for the resolution limit of reconstructing source number, locations, and velocities by general dynamical reconstruction in particle tracking problems, by which we demonstrate the possibility of achieving super-resolution for the dynamic reconstruction. We show that when the location-velocity pairs of the particles are separated beyond certain distances (the resolution limits), the number of particles and the location-velocity pair can be stably recovered. The resolution limits are related to the cut-off frequency of the imaging system, signal-to-noise ratio, and the sparsity of the source. By these estimates we also derive a stability result for a sparsity-promoting dynamic reconstruction. In addition, we further show that the reconstruction of velocities has a better resolution limit which improves constantly as the particles moving. This result is derived by a crucial observation that the inherent cut-off frequency for the velocity recovery can be viewed as the total observation time multiplies the cut-off frequency of the imaging system, which may lead to a better resolution limit as compared to the one for each diffraction-limited frame. It is anticipated that this crucial observation can inspire many new reconstruction algorithms that significantly improve the resolution of particle tracking in practice. In addition, we propose super-resolution algorithms for recovering the number and values of the velocities in the tracking problem and demonstrate theoretically or numerically their super-resolution capability.

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1 INTRODUCTION

The problem of particle tracking is of particular importance in many modern imaging experiments such as visualization of cell migration and subcellular dynamics [28, 37, 42, 54]. It is also utilized in ultrafast ultrasound localization microscopy to super-resolve structures of vascular and velocities of blood flow [17, 22]. The conventional tracking methods in above applications generally consist of two main image processing steps: 1) particle detection, and 2) particle linking. The former refers to determining the locations of particles in each frame and the later refers to linking particles between consecutive frames. However, this static reconstruction strategy suffers from two major issues. The first key issue is that the particles are not well-separated in some of the frames which results in unstable recovery of the source locations or abandon of a large amount of frames where particles are closely-spaced. The second issue is the ambiguities and heavy computational burden in linking particles with high densities and velocities [37,38,45]. To fix these issues, in [4], the authors consider a model where the particles move with constant velocities (in a short period) and propose a new dynamic reconstruction method to simultaneously recover the source locations and velocities from measurements of all the frames. However, theoretically the algorithm still requires that the point sources should be separated beyond Rayleigh limit in each frame for a stable recovery, which hampered its application in the case when sources are closely-spaced in some of the frames. In this paper, in order to understand the super-resolution capability of the tracking problem, we aim to analyze the resolution limit for super-resolving the source number, location-velocity pairs, and velocities in the tracking problem . We also provide super-resolution algorithms for reconstructing the velocities of moving particles which are able to super-resolve velocities even when static reconstruction completely fails.

1.1 OUR MODEL AND CONTRIBUTIONS

Let us first introduce our model and main contributions. We consider the same model for reconstructing dynamic point sources (or particles) as the one in [4]. We remark that we will use bold symbols for vectors, matrices and some functions, and ordinary ones for scalar values in our models and discussions throughout the paper. To be specific, we consider a cluster of point sources that are moving constantly, represented by a time-varying measures μ_t, where $t \in [0,\eta]$ with $\eta > 0$ being the observation window. Since η is expected to be small, the dynamics of each point can be linearly approximated. Thus each point is modeled as a particle moving with a constant velocity:

$$\mu_t = \sum_{j=1}^{n} a_j \delta_{\mathbf{y}_j + \mathbf{v}_j \tau t}, \quad t = 0, \cdots, T,$$

where $\mathbf{y}_j \in \mathbb{R}^d$ are the initial source locations and $\mathbf{v}_j \in \mathbb{R}^d$ are the velocities. The τt, $t = 0, \cdots, T$ is the time step in $[0,\eta]$ at which we observe the samples. We remark that in the applications of measuring the velocity of blood flow [17, 22] and particle tracking velocimetry [15], we may have a larger observation window η and our model is much more applicable. The measurement at each time step is a noisy Fourier data in a bounded domain,

$$\mathbf{Y}_t(\boldsymbol{\omega}) = \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}} + \mathbf{W}_t(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \mathbb{R}^d, ||\boldsymbol{\omega}||_2 \le \Omega,\ t = 0, \cdots, T, \tag{1.1}$$

where $\top$ denotes the transpose, Ω is the cut-off frequency of the imaging system, $\mathbf{W}_t$ is an additive noise with $|\mathbf{W}_t(\boldsymbol{\omega})| < \sigma$ and σ being the noise level. We denote

$$m_{\min} := \min_{j=1,\cdots,n} |a_j|.$$

For the tracking problem in $\mathbb{R}^d$, we are interested in reconstructing the amplitude, location, and velocity pair set

$$\left\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\right\}_{j=1}^{n},$$

from the measurements $\mathbf{Y}_t$'s.

In [4], the authors proposed a fully dynamical method to reconstruct simultaneously the source locations and velocities. They cast the inverse problem as a total variation optimization problem satisfying the measurement constraints and showed that under certain conditions, the optimization is able to reconstruct the amplitudes, locations and velocities of source from noiseless measurements with infinite precision. However, for the noisy case, theoretically, to stably recover the source, the point sources are required to be separated beyond Rayleigh limit in each frame, which is definitely not super-resolution. Thus the superiority of the dynamic reconstruction over the static recovery in the tracking problem is still unclear. Since when the spikes are separated beyond Rayleigh limit, the static reconstruction is also able to recover stably all the μ_t's.

In this paper, in contrast to [4], we consider the locations and velocities of point sources to be closely-spaced in sub-Rayleigh regimes respectively, and explore the ability of super-resolution of the dynamic reconstruction in the tracking problem. More precisely, we analyze the resolution limit for recovering the number, locations, and velocities of the source from the measurement constraint. We prove that when

$$\min_{j\neq p} \left|\left|(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top) - (\mathbf{y}_p^\top, \tau\mathbf{v}_p^\top)\right|\right|_2 \ge \frac{C_1(d,n)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}},$$

where $C_1(d,n)$ is an explicit constant only related to space dimensionality d and source number n (so do the following constants $C_2(d,n), C_3(d,n), C_4(d,n)$), then we can recover the source number n in the dynamic reconstruction. When

$$\min_{j\neq p} \left|\left|(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top) - (\mathbf{y}_p^\top, \tau\mathbf{v}_p^\top)\right|\right|_2 \ge \frac{C_2(d,n)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}},$$

then we can stably recover the source locations and velocities. The estimate demonstrates that the dynamic reconstruction could resolve the location-velocity pairs in sub-Rayleigh regimes for sufficiently large signal-to-noise ratio. It also indicates that the dynamic reconstruction could recover source locations and velocities even when static reconstruction fails in some of the frames, which demonstrates its superiority.

Moreover, by above results, we also show that any algorithms targeting at the sparsest solution satisfying measurement constraint could achieve the resolution $\frac{C_2(d,n)}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$, which demonstrates the favorable performance and the superiority of sparsity-promoting dynamic reconstruction as compared to the static reconstruction methods. This is consistent with the numerical results in [4] for comparison between reconstructions by TV minimization and static method.

In some applications, the recovery of the velocities of the particles is more interesting than the location recovery. For instance, in the ultrafast ultrasound localization microscopy based non-invasive super-resolution vascular imaging [17, 22], the velocities of the blood flow with a large dynamic range can be extracted by reconstructing the velocities of microbubbles inside. We also note that the particle tracking is a common method in velocimetry [15] which seeks to measure accurate velocities of fluid. We thus further consider the reconstruction of individual velocities. By some projection methods, we demonstrate super-resolution for resolving the number of point sources and value of velocities. More precisely, when the minimum separation distance of velocities satisfies

$$\min_{j\neq p}\left|\left|\tau\mathbf{v}_j-\tau\mathbf{v}_p\right|\right|_2\geq\frac{C_3(d,n)}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}},$$

we can stably recover the number of point sources. When

$$\min_{j\neq p}\left|\left|\tau\mathbf{v}_j-\tau\mathbf{v}_p\right|\right|_2\geq\frac{C_4(d,n)}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}},$$

we can stably reconstruct all the velocities. The results are derived by a crucial observation that the inherent cut-off frequency for recovering velocities $\tau\mathbf{v}_j$'s can be viewed as $T\Omega$, by which we obtain better resolution limits. We hope this observation could inspire new algorithms for super-resolving velocities in the tracking problem. For fixed dimensionality d and source number n, all the above theoretical results for the resolution limits are optimal in the sense that at the same separation order there are counter examples in the worst-case scenario.

We also propose super-resolution algorithms for resolving the source number and velocities respectively. They are demonstrated theoretically or numerically to achieve the same order of the resolution limits that derived above. We also demonstrate numerically their superiority over the static reconstruction.

1.2 RELATED WORKS

The main motivations of our work are ultrasound localization microscopy (ULM) [13, 14, 17, 19, 22] and the dynamic reconstruction algorithm proposed in [4]. The concept of ULM is drawing increasing research interest in recent years, due to its ability to solve the trade-off between spatial resolution, penetration depth, and acquisition time when incorporating with ultrasound contrast agents and ultrafast imaging [17, 22], by which the conventional ultrasound imaging and other non-invasive medical imaging methods for living organs are usually limited. Specifically, based on the diffraction theory, the resolution of conventional ultrasound imaging is limited by half of the wavelength of the ultrasound waves which is of the order of $300\mu m$. Thus the blood vessels separated below $300\mu m$ is unable to be distinguished by the conventional ultrasound imaging. But the ultrasound localization microscopy focus on localizing some targeting particles (such as microbubbles) inside the blood vessels, giving rise to a particle tracking problem [22] where both the locations and velocities are of interest. By this innovative method, tenfold increase in resolution is achieved as compared to conventional ultrasound imaging. Moreover, the ultrafast ultrasound imaging techniques also significantly reduce the acquisition time of ULM imaging [22].

However, the tracking approach in [22] relies on static reconstruction of point sources in each frame which suffers from some drawbacks. Among them the key issue is that a lot of data are discarded whenever static reconstruction cannot be performed as particles being too close. To solve these issues, in [4] the authors proposed a new method for the tracking problem based on a fully dynamical inversion scheme, in which the locations and velocities of the point sources are reconstructed simultaneously. To be specific, assuming the targeting particles are moving at constant velocities and considering similar measurement constraint as (1.1), they reconstruct source locations and velocities by a total variation optimization problem. They majorly demonstrate that, for the noiseless measurement, if in more than three time steps the point sources are well-separated to satisfy certain static dual certificates and the configuration does not admit any "ghost particles" defined there, then the TV optimization successfully recover the source locations and velocities. However, for recovering from the noisy measurement, the point sources are required to be separated by more than Rayleigh limit in each frame to ensure a stable recovery of the locations and velocities. Although the super-resolution ability of the algorithm isn't exhibited in the above theoretical result, but it is confirmed by the numerical experiments there. Motivated by this, in this paper we provide a rigorous analysis for the resolution limit of the dynamic reconstruction in the tracking problems and theoretically demonstrate the advantages of dynamic reconstruction over the static reconstruction methods.

On the other hand, the resolution analysis in this paper also follows the line of the authors' previous researches on exploring the super-resolution capability for different imaging configurations [32–35]. Specifically, to analyze the resolution for recovering multiple point sources from a single measurement, in [33–35] the authors defined "computational resolution limits" which characterize the minimum required distance between point sources so that their number and locations can be stably resolved under certain noise level. Based on a new approximation theory in a so-called Vandermonde space, they derived bounds for the resolution limits of one- and multi- dimensional super-resolution problems [33–35]. In particular, they showed that the computational resolution limit for number and location recovery should be respectively $\frac{C_{\mathrm{num}}(d,n)}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}}$ and $\frac{C_{\mathrm{supp}}(d,n)}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}}$ where $C_{\mathrm{num}}(d,n), C_{\mathrm{supp}(d,n)}$ are certain constants depending on space dimensionality d and source number n. In addition to the single measurement case, the stability for sparsity-promoting super-resolution algorithms in multi-illumination imaging was also derived [32].

There were also many mathematical theories for estimating the stability of super-resolution from the perceptive of minimax error estimation. In [20], Donoho considered a grid setting where a discrete measure is supported on a lattice (spacing by Δ) and regularized by a so-called "Rayleigh index" b. He demonstrated that the minimax error for the amplitude reconstruction is bounded from below and above by $SRF^{2b-1}\sigma$ and $SRF^{2b+1}\sigma$ respectively with σ being the noise level and the super-resolution factor $SRF = 1/(\Omega\Delta)$. More recently, inspired by the huge success of new super-resolution modalities [8, 25, 26, 44, 50] and the popularity of researches for super-resolution algorithms [5, 9, 18, 21, 40, 41, 49], the study of super-resolution capability of imaging problems also becomes popular in applied mathematics. In [16], the authors considered n-sparse point sources supported on a grid and obtained sharper lower and upper bounds (both $SRF^{2n-1}\sigma$) for the minimax error of amplitude recovery than those in [20]. The case of multi-clustered point sources was considered in [6, 30] and similar minimax error estimations were derived. In [3, 7], the authors considered the minimax error for recovering off-the-grid point sources. Based on an analysis of the "prony-type system", they derived bounds for both amplitude and location reconstructions of the point sources. More precisely, they showed that for $\sigma \lessapprox (SRF)^{-2p+1}$ where p is the number of point sources in a cluster, the minimax error for the amplitude and the location recoveries scale respectively as $(SRF)^{2p-1}\sigma$, $(SRF)^{2p-2}\sigma/\Omega$, while for the isolated non-clustered source, the corresponding minimax error for the amplitude and the location recoveries scale respectively as σ and σ/Ω. We also refer the readers to [10, 39] for understanding the resolution limit from the perceptive of sample complexity.

1.3 Organization of the paper

The rest of the paper is organized in the following way. In Section 2, we present the main results about the stability of location-velocity pair reconstruction in the tracking problem. More precisely, in Section 2.1, we state the results for the stability of number recovery of the point sources, and in Section 2.2 those for the stability of the recovery of the location-velocity pairs. In Section 2.3, we present the stability estimate for a sparsity-promoting algorithm in the tracking problem. In Section 3, we demonstrate better resolutions for the number and value reconstruction of velocities. In Section 4 and Section 5, we propose projection-based super-resolution algorithms for the number and value recovery of the velocities respectively. Numerical experiments are also conducted to demonstrate their efficiency. Section 6 is devoted to some concluding remarks and future works. In Section 7 and Section 8, we prove our main results in this paper.

2 Main results about the recovery of location-velocity pairs

In this section, we present the results for the stability of the dynamic reconstruction in the tracking problem. Suppose that a series of images is generated by n d-dimensional point sources located at $\mathbf{y}_j$'s with amplitudes a_j's and velocities $\mathbf{v}_j$'s. The inverse problem we are concerned with is to reconstruct the number and location-velocity pairs of the point sources. To be more specific, we consider the set of parameters $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ and denote the vectors of locations and velocities as

$$\boldsymbol{\alpha}_j = \begin{pmatrix} \mathbf{y}_j \\ \tau\mathbf{v}_j \end{pmatrix}.$$

We first consider the stability of reconstructing the number and value of the location-velocity pairs $\boldsymbol{\alpha}_j$'s. Because we are interested in the capability of super-resolution, we consider recovering point sources with close $\mathbf{y}_j$ and $\tau\mathbf{v}_j$'s. We define

$$B_\delta^m(\mathbf{x}) := \left\{\mathbf{y} \,\middle|\, \mathbf{y}\in\mathbb{R}^m,\ ||\mathbf{y}-\mathbf{x}||_2 < \delta\right\},$$

and assume that $\boldsymbol{\alpha}_j \in B_\delta^m(\mathbf{0}), j = 1,\cdots,n$, for certain $\delta > 0$. On the other hand, in order to analyze the stability of the reconstruction, below we first need to define a σ-admissible parameter set of $\mathbf{Y}_t$'s. In the following sections, we shall consider recovering the number and value of location-velocity pairs from all the σ-admissible parameter sets.

Definition 2.1. *Given measurement $\mathbf{Y}_t$'s in (1.1), we say that $\{(\hat a_j, \hat{\mathbf{y}}_j, \tau\hat{\mathbf{v}}_j)\}_{j=1}^k$ is a σ-admissible parameter set of $\mathbf{Y}_t$'s if*

$$\left|\sum_{j=1}^k \hat a_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}} - \mathbf{Y}_t(\boldsymbol{\omega})\right| < \sigma,\ ||\boldsymbol{\omega}||_2 \le \Omega,\ \textit{for all } t = 0,1,\cdots,T.$$

Definition 2.2. *We say a parameter set is a n-sparse parameter set if it contains exactly n different elements.*

2.1 Stability of number recovery in the tracking problem

In this subsection, we present our main results for recovering the number of point sources in the tracking problem. Define

$$\xi(k) = \begin{cases} \sum_{j=1}^k \frac{1}{j}, & k\ge 1,\\ 0, & k = 0.\end{cases} \tag{2.1}$$

We have the following theorem for the recovery of source number in the d-dimensional tracking problem. Its proof is given is Section 7.

Theorem 2.1. *Let $n \ge 2$ and $T \ge \frac{n(n-1)}{2}$. Let the measurement $\mathbf{Y}_t \in \mathbb{R}^d$, $t = 0,\cdots,T$, in (1.1) be generated by a n-sparse parameter set $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ with $\boldsymbol{\alpha}_j := \begin{pmatrix}\mathbf{y}_j\\ \tau\mathbf{v}_j\end{pmatrix} \in B^{2d}_{\frac{\pi}{(n+1)\Omega}}(\mathbf{0})$. Assume that the following separation condition is satisfied*

$$\min_{p\neq j, 1\le p,j\le n} \left|\left|\boldsymbol{\alpha}_p - \boldsymbol{\alpha}_j\right|\right|_2 \ge \frac{8.8e\pi^2\sqrt{(\frac{n(n-1)}{2})^2+1}(\pi/2)^{d-1}\left(\frac{n(n-1)}{\pi}\right)^{\xi(d-1)}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}, \tag{2.2}$$

where $\xi(d-1)$ is defined as in (2.1). Then there does not exist any σ-admissible parameter set of $\mathbf{Y}_t$'s with less than n elements.

Remark 2.1. *We remark that the condition $\boldsymbol{\alpha}_j \in B^{2d}_{\frac{\pi}{(n+1)\Omega}}(\mathbf{0})$ in the above theorem and similar conditions in the rest of the paper can be straightforwardly extended to $\boldsymbol{\alpha}_j \in B^{2d}_{\frac{\pi}{(n+1)\Omega}}(\mathbf{x})$ for a non-zero vector $\mathbf{x}$. Also, they can be easily extended to $\boldsymbol{\alpha}_j \in B^{2d}_\delta(\mathbf{0})$ for a larger δ with a slight modification of the results.*

Theorem 2.1 reveals that, for fixed space dimensionality d and source number n, when $\min_{j\neq p}\left|\left|\begin{pmatrix}\mathbf{y}_j\\ \tau\mathbf{v}_j\end{pmatrix}-\begin{pmatrix}\mathbf{y}_p\\ \tau\mathbf{v}_p\end{pmatrix}\right|\right|_2 \geq O\left(\frac{1}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}}\right)$, it is possible to reconstruct the exact source number in the tracking problem. We remark that since the space dimensionality of interest is usually small ($d=1,2,3$), the amplification factor in (2.2) due to the space dimensionality is not large.

We next show that in the worst-case scenario, the order $O(\frac{1}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}})$ is optimal without further information on the velocities. This is shown by Proposition 2.2. We first present a result for the d-dimensional static super-resolution problem which helps to derive Propositions 2.2 and 2.3.

Proposition 2.1. *For given* $0<\sigma<m_{\min}$ *and integer* $n\geq 2$*, there exist* $a_j\in\mathbb{C},\mathbf{y}_j\in\mathbb{R}^d, j=1,\cdots,n$*, and* $\hat a_j\in\mathbb{C},\hat{\mathbf{y}}_j\in\mathbb{R}^d, j=1,\cdots,n-1$ *such that*

$$\left|\sum_{j=1}^{n-1}\hat a_j e^{i\hat{\mathbf{y}}_j^\top\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i\mathbf{y}_j^\top\boldsymbol{\omega}}\right|<\sigma,\quad ||\boldsymbol{\omega}||_2\leq\Omega.$$

Moreover,

$$\min_{1\leq j\leq n}|a_j|=m_{\min},\quad \min_{p\neq j}\left|\left|\mathbf{y}_p-\mathbf{y}_j\right|\right|_2=\frac{0.81e^{-\frac{3}{2}}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}.$$

Proof. See Proposition 2.4 in [33]. □

Proposition 2.2. *For given* $0<\sigma<m_{\min}$ *and integer* $n\geq 2$*, there exist a* n*-sparse parameter set* $\{(a_j,\mathbf{y}_j,\tau\mathbf{v}_j)\}_{j=1}^n, \mathbf{y}_j\in\mathbb{R}^d,\mathbf{v}_j\in\mathbb{R}^d$*, and a* $(n-1)$*-sparse parameter set* $\{(\hat a_j,\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j)\}_{j=1}^{n-1}, \hat{\mathbf{y}}_j\in\mathbb{R}^d,\hat{\mathbf{v}}_j\in\mathbb{R}^d$*, such that*

$$\left|\sum_{j=1}^{n-1}\hat a_j e^{i(\hat{\mathbf{y}}_j^\top+t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i(\mathbf{y}_j^\top+t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\right|<\sigma,\quad ||\boldsymbol{\omega}||_2\leq\Omega,\ t=0,\cdots,T.$$

Moreover,

$$\min_{1\leq j\leq n}|a_j|=m_{\min},\quad \min_{p\neq j}\left|\left|\boldsymbol{\alpha}_p-\boldsymbol{\alpha}_j\right|\right|_2=\frac{0.81e^{-\frac{3}{2}}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}},$$

where $\boldsymbol{\alpha}_j=\begin{pmatrix}\mathbf{y}_j\\ \tau\mathbf{v}_j\end{pmatrix}$.

Proof. Let $\hat a_j,\hat{\mathbf{y}}_j,a_j,\mathbf{y}_j$'s be the ones in Proposition 2.1. Then we have

$$\left|\sum_{j=1}^{n-1}\hat a_j e^{i\hat{\mathbf{y}}_j^\top\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i\mathbf{y}_j^\top\boldsymbol{\omega}}\right|<\sigma,\quad ||\boldsymbol{\omega}||_2\leq\Omega,\boldsymbol{\omega}\in\mathbb{R}^d.$$

When $\hat{\mathbf{v}}_j=\mathbf{v}_j=\mathbf{v}$, we also have

$$\left|\sum_{j=1}^{n-1}\hat a_j e^{i(\hat{\mathbf{y}}_j^\top+t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i(\mathbf{y}_j^\top+t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\right|<\sigma,\quad ||\boldsymbol{\omega}||_2\leq\Omega,\ t=0,\cdots,T.$$

The other parts of the proposition are easy to verify. □

Proposition 2.2 holds for the case when all $\tau\mathbf{v}_j$'s are equal or very close to each other. If the velocities are not close to each other, we next have Proposition 2.3, which shows that the order of the resolution of number recovery in Theorem 2.1 is nearly optimal for the reconstruction with a short time period (i.e., when T is small). For a tracking problem with long time period, we expect that the resolution for recovering the number of different velocities is of order $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}})$. This will be demonstrated by results in Section 3.

Proposition 2.3. *For given $0<\sigma<m_{\min}$ and integer $n\geq 2$, there exist a n-sparse parameter set $\{(a_j,\mathbf{y}_j,\tau\mathbf{v}_j)\}_{j=1}^n$ with different $\mathbf{v}_j$'s, and a $(n-1)$-sparse parameter set $\{(\hat{a}_j,\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j)\}_{j=1}^{n-1}$, such that*

$$\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top+t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i(\mathbf{y}_j^\top+t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big|<\sigma,\ ||\boldsymbol{\omega}||_2\leq\Omega,\ t=0,\cdots,T.$$

Moreover,

$$\min_{1\leq j\leq n}|a_j|=m_{\min},\quad \min_{p\neq j}\left|\left|\boldsymbol{\alpha}_p-\boldsymbol{\alpha}_j\right|\right|_2=\frac{0.81\sqrt{2}e^{-\frac{3}{2}}}{(T+1)\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}},\tag{2.3}$$

where $\boldsymbol{\alpha}_j=\begin{pmatrix}\mathbf{y}_j\\ \tau\mathbf{v}_j\end{pmatrix}$.

Proof. Let $\Delta=\frac{0.81e^{-\frac{3}{2}}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}$. For the one-dimensional case, as in the proof in [33] or [34], Proposition 2.1 holds when the $n-1$ and n point sources are located at

$$-(n-1)\Delta,\ -(n-2)\Delta,\ \cdots,\ 0,\ \Delta,\ (n-1)\Delta,$$

with certain intensities $\hat{a}_j,a_j$'s. More specifically, let $\hat{x}_j=-j\Delta, x_j=(j-1)\Delta, j=1,\cdots,n$, we have

$$\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i\hat{x}_j\omega}-\sum_{j=1}^{n-1}a_j e^{ix_j\omega}\Big|<\sigma,\quad |\omega|\leq\Omega,\tag{2.4}$$

with certain $\hat{a}_j,a_j$'s. We consider in the proposition the above $\hat{a}_j,a_j$'s and the following $\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j,\mathbf{y}_j,\tau\mathbf{v}_j$'s,

$$(\hat{\mathbf{y}}_1^\top,\tau\hat{\mathbf{v}}_1^\top)=\frac{-\Delta}{T+1}\frac{1}{\sqrt{d}}(1,1,\cdots,1),\ \cdots,\ (\hat{\mathbf{y}}_n,\tau\hat{\mathbf{v}}_n)=\frac{-(n-1)\Delta}{T+1}\frac{1}{\sqrt{d}}(1,1,\cdots,1),$$

and

$$(\mathbf{y}_1^\top,\tau\mathbf{v}_1^\top)=(0,0,\cdots,0),\quad (\mathbf{y}_2^\top,\tau\mathbf{v}_2^\top)=\frac{\Delta}{T+1}\frac{1}{\sqrt{d}}(1,1,\cdots,1),\quad\cdots,$$
$$(\mathbf{y}_n^\top,\tau\mathbf{v}_n^\top)=\frac{(n-1)\Delta}{T+1}\frac{1}{\sqrt{d}}(1,1,\cdots,1).$$

Note that (2.3) is satisfied. On the other hand, for any $||\boldsymbol{\omega}||_2\leq\Omega,\boldsymbol{\omega}\in\mathbb{R}^d, t=0,\cdots,T$, let

$\boldsymbol{\omega}_t = \begin{pmatrix}\boldsymbol{\omega}\\ t\boldsymbol{\omega}\end{pmatrix}$, we have

$$\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big| = \Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}_t} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)\boldsymbol{\omega}_t}\Big|$$

$$=\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)(\mathbf{u}+\mathbf{v})} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)(\mathbf{u}+\mathbf{v})}\Big| \left(\mathbf{u}+\mathbf{v} = \boldsymbol{\omega}_t, \mathbf{u} = \frac{(1+t)\omega}{2\sqrt{d}}(1,\cdots,1)^\top, |\omega| \le \Omega, \mathbf{u}^\top\mathbf{v} = 0\right)$$

$$=\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\mathbf{u}} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)\mathbf{u}}\Big|$$

$$=\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i\hat{x}_j\hat{\omega}} - \sum_{j=1}^{n} a_j e^{ix_j\hat{\omega}}\Big|,$$

where

$$\hat{x}_j = (\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\frac{(1+T)}{2\sqrt{d}}(1,1,\cdots,1)^\top = -j\Delta,\ x_j = (\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)\frac{(1+T)}{2\sqrt{d}}(1,1,\cdots,1)^\top = (j-1)\Delta,$$

and

$$\left|\hat{\omega}\right| = \left|\frac{t+1}{T+1}\omega\right| \le \Omega.$$

Furthermore, by (2.4), we thus have

$$\Big|\sum_{j=1}^{n-1}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big| < \sigma,\ ||\boldsymbol{\omega}||_2 \le \Omega,\ t = 0,\cdots,T.$$

□

2.2 Stability of location and velocity recovery in the tracking problem

In this subsection, we present our main results for recovering the locations and velocities of point sources in the d-dimensional tracking problem. We have the following theorem whose proof is given in Section 7.

Theorem 2.2. *Let $n \ge 2$ and $T \ge \frac{n(n+1)}{2}$. Let the measurement $\mathbf{Y}_t, t = 0,\cdots,T$, in (1.1) be generated by a n-sparse parameter set $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ with $\boldsymbol{\alpha}_j := \begin{pmatrix}\mathbf{y}_j\\ \tau\mathbf{v}_j\end{pmatrix} \in B^{2d}_{\frac{(n-1)\pi}{n(n+2)\Omega}}(\mathbf{0})$. Assume that*

$$d_{\min} := \min_{p\neq j}\left|\left|\boldsymbol{\alpha}_p - \boldsymbol{\alpha}_j\right|\right|_2 \ge \frac{11.76e\pi^2\sqrt{(\frac{(n+1)n}{2})^2+1}4^{d-1}\Big((n+2)(n+1)/2\Big)^{\xi(d-1)}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}, \tag{2.5}$$

where $\xi(d-1)$ is defined as in (2.1). If a n-sparse parameter set $\{(\hat{a}_j, \hat{\mathbf{y}}_j, \tau\hat{\mathbf{v}}_j)\}_{j=1}^n$ with $\hat{\boldsymbol{\alpha}}_j := \begin{pmatrix}\hat{\mathbf{y}}_j\\ \tau\hat{\mathbf{v}}_j\end{pmatrix}$ supported on $B^{2d}_{\frac{(n-1)\pi}{n(n+2)\Omega}}(\mathbf{0})$ is a σ-admissible parameter set of $\mathbf{Y}_t$'s, then after reordering the $\hat{\boldsymbol{\alpha}}_j$'s, we have

$$\left|\left|\hat{\boldsymbol{\alpha}}_j - \boldsymbol{\alpha}_j\right|\right|_2 < \frac{d_{\min}}{2}.$$

Moreover, we have

$$\left|\left|\hat{\boldsymbol{\alpha}}_j - \boldsymbol{\alpha}_j\right|\right|_2 \leq \frac{C(d,n)}{\Omega} SRF^{2n-2}\frac{\sigma}{m_{\min}}, \quad 1 \leq j \leq n, \tag{2.6}$$

where SRF *is the super-resolution factor defined by* $\frac{\pi}{d_{\min}\Omega}$ *and*

$$C(d,n) = \sqrt{6\pi}(2\pi)^{2n-2}\Big((\frac{n(n+1)}{2})^2+1\Big)^{\frac{2n-1}{2}}\Big(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\Big)^{2n-1} n 2^{4n-2}e^{2n}\pi^{-\frac{1}{2}}.$$

Theorem 2.2 demonstrates that for fixed dimensionality d and source number n, when $\min_{j\neq p}\left|\left|\begin{pmatrix}\mathbf{y}_j\\ \tau\mathbf{v}_j\end{pmatrix}-\begin{pmatrix}\mathbf{y}_p\\ \tau\mathbf{v}_p\end{pmatrix}\right|\right|_2 \geq O\Big(\frac{1}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}}\Big)$, it is possible to stably reconstruct both the source locations and velocities in the tracking problem. It also indicates that even when in some of the frames, the point sources are spaced so close that are unable to be stably resolved, but the dynamic reconstruction can still stably resolve the source locations and velocities. It demonstrates the superiority of dynamic reconstruction over the static reconstruction method. In addition, by the following results, we shall see that the order $O(\frac{1}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}})$ is optimal in the worst-case scenario without further information on the velocities. We first recall the following result on the location recovery in the static super-resolution problem.

Proposition 2.4. *For given* $0<\sigma<m_{\min}$ *and integer* $n\geq 2$, *let*

$$\Delta = \frac{0.49e^{-\frac{3}{2}}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{2.7}$$

Then there exist $a_j\in\mathbb{C}, j=1,\cdots,n$ *and* $\mathbf{y}_j$*'s at* $\{(-\Delta,0,\cdots,0)^\top,(-2\Delta,0,\cdots,0)^\top,(-n\Delta,0,\cdots,0)^\top\}$ *and another* $\hat{a}_j\in\mathbb{C}, j=1,\cdots,n$ *and* $\hat{\mathbf{y}}_j$*'s at* $\{(0,0,\cdots,0)^\top,(\Delta,0,\cdots,0)^\top,((n-1)\Delta,0,\cdots,0)^\top\}$ *such that*

$$\Big|\sum_{j=1}^n \hat{a}_j e^{i\hat{\mathbf{y}}_j^\top\boldsymbol{\omega}} - \sum_{j=1}^n a_j e^{i\mathbf{y}_j^\top\boldsymbol{\omega}}\Big| < \sigma, \quad ||\boldsymbol{\omega}||_2\leq\Omega, \tag{2.8}$$

and either $\min_{1\leq j\leq n}|a_j| = m_{\min}$ *or* $\min_{1\leq j\leq n}|\hat{a}_j| = m_{\min}$.

Proof. See Proposition 2.8 in [33]. □

As a direct consequence of Proposition 2.4, we have the following result.

Proposition 2.5. *For given* $0<\sigma<m_{\min}$ *and integer* $n\geq 2$, *let*

$$\Delta = \frac{0.49e^{-\frac{3}{2}}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{2.9}$$

Then there exist n*-sparse parameter set* $\{(a_j,\mathbf{y}_j,\tau\mathbf{v}_j)\}_{j=1}^n$*'s with* $\mathbf{v}_j=\mathbf{v}, j=1,\cdots,n$ *and*

$$\mathbf{y}_1 = (-\Delta,0,\cdots,0)^\top, \mathbf{y}_2 = (-2\Delta,0,\cdots,0)^\top,\cdots,\mathbf{y}_n = (-n\Delta,\cdots,0)^\top,$$

and n*-sparse parameter set* $\{(\hat{a}_j,\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j)\}_{j=1}^n$*'s with* $\hat{\mathbf{v}}_j=\mathbf{v}, j=1,\cdots,n$ *and*

$$\hat{\mathbf{y}}_1 = (0,\cdots,0)^\top, \hat{\mathbf{y}}_2 = (\Delta,0,\cdots,0)^\top,\cdots,\hat{\mathbf{y}}_n = ((n-1)\Delta,\cdots,0)^\top$$

such that

$$\Big|\sum_{j=1}^{n}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top+t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i(\mathbf{y}_j^\top+t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big|<\sigma,\ ||\boldsymbol{\omega}||_2\le\Omega,\ t=0,\cdots,T,$$

and either $\min_{1\le j\le n}|a_j|=m_{\min}$ *or* $\min_{1\le j\le n}|\hat{a}_j|=m_{\min}$.

This result holds for the case when all $\tau\mathbf{v}_j$'s are equal or very close to each other with respect to T. Thus, theoretically the resolution of $\mathbf{y}_j$'s in the worst-case is expected to be of order $O\Big(\frac{1}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}}\Big)$.

If the velocities are not close to each other, we next have Proposition 2.6, which shows that the order of the resolution in Theorem 2.2 is nearly optimal for the reconstruction problem with short time period (i.e., when T is small). For a tracking problem with long time period, we expect the resolution for the velocities to be of order $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}})$. This will also be demonstrated by results in Section 3.

Proposition 2.6. *For given* $0<\sigma<m_{\min}$ *and integer* $n\ge 2$, *let*

$$\Delta=\frac{0.49e^{-\frac{3}{2}}}{(T+1)\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{2.10}$$

Then there exist n-sparse parameter set $\{(\hat{a}_j,\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j)\}_{j=1}^n$*'s with* $(\hat{\mathbf{y}}_1^\top,\hat{\mathbf{v}}_1^\top)=-\frac{\Delta}{\sqrt{d}}(1,\cdots,1),\ \cdots,$ $(\hat{\mathbf{y}}_n^\top,\hat{\mathbf{v}}_n^\top)=-\frac{n\Delta}{\sqrt{d}}(1,\cdots,1)$, *and n-sparse parameter set* $\{(a_j,\mathbf{y}_j,\tau\mathbf{v}_j)\}_{j=1}^n$*'s with* $(\mathbf{y}_1^\top,\mathbf{v}_1^\top)=(0,0,\cdots,0)$, $(\mathbf{y}_2^\top,\mathbf{v}_2^\top)=\frac{\Delta}{\sqrt{d}}(1,\cdots,1),\cdots,(\mathbf{y}_n^\top,\mathbf{v}_n^\top)=\frac{(n-1)\Delta}{\sqrt{d}}(1,\cdots,1)$ *such that*

$$\Big|\sum_{j=1}^{n}\hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top+t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}}-\sum_{j=1}^{n}a_j e^{i(\mathbf{y}_j^\top+t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big|<\sigma,\ ||\boldsymbol{\omega}||_2\le\Omega,\ t=0,\cdots,T,$$

and either $\min_{1\le j\le n}|a_j|=m_{\min}$ *or* $\min_{1\le j\le n}|\hat{a}_j|=m_{\min}$.

Proof. Let $\hat{\Delta}=\frac{0.49e^{-\frac{3}{2}}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}$. Let $\hat{x}_j=-j\hat{\Delta}, x_j=(j-1)\hat{\Delta}, j=1,\cdots,n$, by Proposition 2.4 for the one-dimensional case we have

$$\Big|\sum_{j=1}^{n}\hat{a}_j e^{i\hat{x}_j\omega}-\sum_{j=1}^{n-1}a_j e^{ix_j\omega}\Big|<\sigma,\quad |\omega|\le\Omega, \tag{2.11}$$

for certain $\hat{a}_j, a_j$'s. We consider the above $\hat{a}_j, a_j$'s and the following $\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j,\mathbf{y}_j,\tau\mathbf{v}_j$'s,

$$(\hat{\mathbf{y}}_1^\top,\tau\hat{\mathbf{v}}_1^\top)=\frac{-\Delta}{\sqrt{d}}(1,\cdots,1),\quad\cdots,\quad(\hat{\mathbf{y}}_n^\top,\tau\hat{\mathbf{v}}_n^\top)=\frac{-n\Delta}{\sqrt{d}}(1,\cdots,1),$$

and

$$(\mathbf{y}_1^\top,\tau\mathbf{v}_1^\top)=(0,\cdots,0),\ (\mathbf{y}_2^\top,\tau\mathbf{v}_2^\top)=\frac{\Delta}{\sqrt{d}}(1,\cdots,1),\ \cdots,\ (\mathbf{y}_n^\top,\tau\mathbf{v}_n^\top)=\frac{(n-1)\Delta}{\sqrt{d}}(1,\cdots,1).$$

For any $||\boldsymbol{\omega}||_2 \le \Omega$, $t = 0, \cdots, T$, we obtain that

$$\Big|\sum_{j=1}^{n} \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big| = \Big|\sum_{j=1}^{n} \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}_t} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)\boldsymbol{\omega}_t}\Big|$$

$$=\Big|\sum_{j=1}^{n} \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)(\mathbf{u}+\mathbf{v})} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)(\mathbf{u}+\mathbf{v})}\Big| \left(\mathbf{u}+\mathbf{v} = \boldsymbol{\omega}_t, \mathbf{u} = \frac{(1+t)\omega}{2\sqrt{d}}(1,\cdots,1)^\top, |\omega| \le \Omega, \mathbf{u}^\top \cdot \mathbf{v} = 0\right)$$

$$=\Big|\sum_{j=1}^{n} \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\mathbf{u}} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top, \tau\mathbf{v}_j^\top)\mathbf{u}}\Big|$$

$$=\Big|\sum_{j=1}^{n} \hat{a}_j e^{i\hat{x}_j\hat{\omega}} - \sum_{j=1}^{n} a_j e^{ix_j\hat{\omega}}\Big|,$$

where $\hat{x}_j = (\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\frac{(1+T)}{2\sqrt{d}}(1,\cdots,1)^\top = -j\hat{\Delta}, x_j = (\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)\frac{(1+T)}{2\sqrt{d}}(1,\cdots,1)^\top = (j-1)\hat{\Delta}$ and $\left|\hat{\omega}\right| = \left|\frac{t+1}{T+1}\omega\right| \le \Omega$. Furthermore, by (2.11), we thus have

$$\Big|\sum_{j=1}^{n} \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}} - \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\boldsymbol{\omega}}\Big| < \sigma, \quad ||\boldsymbol{\omega}||_2 \le \Omega, \ t = 0, \cdots, T.$$

□

2.3 Stability for sparsity-promoting dynamic reconstruction

Sparsity-based modeling and optimization is a common way in super-resolution that seeks to accelerate the resolving process or reduce the number of measurements. In [4], a sparsity-promoting algorithm is proposed to super-resolve source locations and velocities in the tracking problem, where the authors' aim is to find an admissible $2d$-dimensional source with minimum total variation norm. There, the theoretical stability result for the optimization requires the point sources to be separated by more than a Rayleigh length in each time step t, which is inadequate for demonstrating its super-resolution capability.

As a corollary of the results in Subsections 2.1 and 2.2, we can derive a stability result for a sparsity-promoting dynamic reconstruction in the tracking problem, which demonstrates the possibility of achieving super-resolution for the sparsity-promoting dynamic reconstructions. We consider a sparsity-promoting dynamic tracking problem which seeks to find the sparest solution satisfying the measurement constraints. The optimization problem is

$$\min_{\{(\hat{a}_j, \hat{\mathbf{y}}_j, \tau\hat{\mathbf{v}}_j)\}, (\hat{\mathbf{y}}_j^\top, \tau\hat{\mathbf{v}}_j^\top)^\top \in \mathcal{O}} \#\{(\hat{a}_j, \hat{\mathbf{y}}_j, \tau\hat{\mathbf{v}}_j)\} \quad \text{subject to } |\mathscr{F}[\rho_t](\boldsymbol{\omega}) - \mathbf{Y}_k(\boldsymbol{\omega})| < \sigma, \ t = 0, \cdots, T, \ ||\boldsymbol{\omega}||_2 \le \Omega, \tag{2.12}$$

where $\rho_t = \sum_{j=1}^n \hat{a}_j \delta_{\hat{\mathbf{y}}_j + t\tau\hat{\mathbf{v}}_j}$, and $\mathscr{F}[\rho_t](\boldsymbol{\omega}) = \sum_{j=1}^n \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\boldsymbol{\omega}}$. We have the following theorem for the stability of the minimization problem (2.12).

Theorem 2.3. *Let $n \ge 2$, $T \ge \frac{n(n+1)}{2}$, and $\sigma \le m_{\min}$. Let the measurement $\mathbf{Y}_t, t = 0, \cdots, T$, in (1.1) be generated by a n-sparse parameter set $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ with $\boldsymbol{\alpha}_j := \begin{pmatrix}\mathbf{y}_j \\ \tau\mathbf{v}_j\end{pmatrix} \in B^{2d}_{\frac{(n-1)\pi}{n(n+2)\Omega}}(\mathbf{0})$.*

Assume that

$$d_{\min} := \min_{p\neq j}\left|\left|\boldsymbol{\alpha}_p-\boldsymbol{\alpha}_j\right|\right|_2 \geq \frac{11.76e\pi^2\sqrt{(\frac{(n+1)n}{2})^2+1}4^{k-1}\Big((n+2)(n+1)/2\Big)^{\xi(k-1)}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{2.13}$$

Let $\mathcal{O}$ in (2.12) be $B^{2d}_{\frac{(n-1)\pi}{n(n+2)\Omega}}(\mathbf{0})$, then the solution to (2.12) contains exactly n elements. If a n-sparse parameter set $\{(\hat{a}_j,\hat{\mathbf{y}}_j,\tau\hat{\mathbf{v}}_j)\}_{j=1}^n$ is the solution, after reordering the $\hat{\boldsymbol{\alpha}}_j$'s, we have

$$\left|\left|\hat{\boldsymbol{\alpha}}_j-\boldsymbol{\alpha}_j\right|\right|_2 < \frac{d_{\min}}{2}.$$

Moreover, we have

$$\left|\left|\hat{\boldsymbol{\alpha}}_j-\boldsymbol{\alpha}_j\right|\right|_2 \leq \frac{C(d,n)}{\Omega}SRF^{2n-2}\frac{\sigma}{m_{\min}},\quad 1\leq j\leq n, \tag{2.14}$$

where $SRF := \frac{\pi}{d_{\min}\Omega}$ and

$$C(d,n) = \sqrt{6\pi}(2\pi)^{2n-2}\Big((\frac{n(n+1)}{2})^2+1\Big)^{\frac{2n-1}{2}}\big(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\big)^{(2n-1)}n2^{4n-2}e^{2n}\pi^{-\frac{1}{2}}.$$

Proof. Note that by $\sigma \leq m_{\min}$, (2.13) is greater than (2.2). Thus by Theorem 2.1, the solution to (2.12) contains n elements. Then by Theorem 2.2, we prove the desired result. □

Theorem 2.3 reveals that sparsity promoting over admissible solutions satisfying the measurements of all the frames could resolve the location-velocity pairs to the resolution limit order. It provides an insight that theoretically sparsity-promoting dynamic reconstruction algorithms would have favorable performances on the tracking problem. Especially, under the separation condition 2.13, any tractable sparsity-promoting algorithms (such as TV minimization) rendering the sparsest solution could stably reconstruct all the location-velocity pairs. Theorem 2.3 also indicates that even when the static reconstruction fails in some of the frames, the sparsity-promoting dynamic reconstruction could also resolve the location-velocity pairs, which is consistent with the numerical results in [4].

3 Main results for velocity recovery

In this section, we present the results for the stability of velocity recovery in the tracking problems. We shall first introduce better resolution estimates for recovering the number and values of velocities and then present a stability result for a sparsity-promoting velocity reconstruction algorithm.

3.1 Better resolution estimates for recovering the number and values of velocities

In this subsection, we shall derive better resolution estimates for recovering the number of sources and the values of velocities in the tracking problem. We first have the following theorem for the reconstruction of the number of sources.

Theorem 3.1. *Let $n \geq 2$ and $T \geq 2n-2$. Let the measurement $\mathbf{Y}_t$, $t=0,\cdots,T$, in (1.1) be generated by a n-sparse parameter set $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ with $\tau\mathbf{v}_j \in B^d_{\frac{(n-1)\pi}{T\Omega}}(0)$. Assume that*

$$\min_{p\neq j}\left|\left|\tau\mathbf{v}_p-\tau\mathbf{v}_j\right|\right|_2 \geq \frac{8.8\pi e(\pi/2)^{d-1}\Big(n(n-1)/\pi\Big)^{\xi(d-1)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}. \tag{3.1}$$

Then there does not exist any σ-admissible parameter set of $\mathbf{Y}_t$'s with less than n elements.

Theorem 3.1 reveals that when the minimum difference between velocities is greater than $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}})$, then the number of point sources can be exactly reconstructed in the tracking problem. By Proposition 2.3, we note that this order is optimal. In the following theorem, we further develop a better estimate for the resolution of the velocity reconstruction.

Theorem 3.2. *Let $n \geq 2$ and $T \geq 2n-1$. Let the measurement $\mathbf{Y}_t$, $t=0,\cdots,T$, in (1.1) be generated by a n-sparse parameter set $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ with $\tau\mathbf{v}_j \in B^d_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0})$. Assume that*

$$d_{\min} := \min_{p\neq j}\left|\left|\tau\mathbf{v}_p-\tau\mathbf{v}_j\right|\right|_2 \geq \frac{11.76e\pi 4^{d-1}\Big((n+2)(n+1)/2\Big)^{\xi(d-1)}}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}}. \tag{3.2}$$

If a n-sparse parameter set $\{(\hat{a}_j, \hat{\mathbf{y}}_j, \tau\hat{\mathbf{v}}_j)\}_{j=1}^n$ with $\tau\hat{\mathbf{v}}_j$'s supported on $B^d_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0})$ is a σ-admissible parameter set of $\mathbf{Y}_t$'s, then after reordering the $\hat{\mathbf{v}}_j$'s, we have

$$\left|\left|\tau\hat{\mathbf{v}}_j-\tau\mathbf{v}_j\right|\right|_2 < \frac{d_{\min}}{2}.$$

Moreover, we have

$$\left|\left|\tau\hat{\mathbf{v}}_j-\tau\mathbf{v}_j\right|\right|_2 \leq \frac{C(d,n)}{T\Omega}SRF^{2n-2}\frac{\sigma}{m_{\min}}, \quad 1\leq j\leq n, \tag{3.3}$$

where SRF is the super-resolution factor defined by $\frac{\pi}{d_{\min}T\Omega}$ and

$$C(d,n) = \left(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\right)^{2n-1} n 2^{6n-3} e^{2n} \pi^{-\frac{1}{2}}.$$

Theorem 3.2 reveals that when the minimum difference between the velocities is greater than $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}})$, then the velocities can be stably reconstructed in the tracking problem. By Proposition 2.6, we note that this order is optimal.

In Sections 4 and 5, we will propose super-resolution algorithms for reconstructing respectively the number of sources and the values of the velocities, which are demonstrated theoretically or numerically to lead to the optimal resolution orders that are shown above.

We remark that both Theorems 3.1 and 3.2 are derived by a crucial observation that the measurements at point $\boldsymbol{\omega} = \Omega\mathbf{v}$ are

$$\mathbf{Y}_t(\mathbf{v}\Omega) = \sum_{j=1}^n a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\mathbf{v}\Omega} + \mathbf{W}_t(\mathbf{v}\Omega), \quad t=0,\cdots,T. \tag{3.4}$$

Let $b_j = a_j e^{i\mathbf{y}_j^\top \mathbf{v}\Omega}$ and $y_j = \tau\mathbf{v}_j^\top \mathbf{v}$ and $\mathbf{W}(t) = \mathbf{W}_t(\mathbf{v}\Omega)$, the measurement can be written as

$$\mathbf{Y}(t) = \sum_{j=1}^{n} b_j e^{i y_j \Omega t} + \mathbf{W}(t) \quad t = 0, \cdots, T.$$

Thus the inherent cut-off frequency for recovering $\tau\mathbf{v}_j$'s can be viewed as $T\Omega$, which results in a much better resolution limit $\frac{\pi}{T\Omega}$ compared to the one $\frac{\pi}{\Omega}$ for each frame. Because recovering objects separated larger than Rayleigh limit is stable even for imaging system with low signal-to-noise ratio, our observation indicates that, by some dynamic reconstruction algorithms, we are very likely able to achieve better resolution for velocities than that of the static reconstruction. This will be confirmed by numerical algorithms in Section 5. For other examples, by applying the algorithms in [9,49] which are provable to resolve point sources with a separation distance of Rayleigh limit order to several measurements like (3.4), we are able to recover the velocities with resolution of order $O(\frac{\pi}{T\Omega})$. We anticipate the above observation can inspire many new reconstruction algorithms that enhance significantly the resolution of the tracking problem in practice.

3.2 Stability for a sparsity-promoting velocity reconstruction

As a corollary of the results in the above subsection, we can derive a stability result for velocity recovery by the sparsity-promoting dynamic reconstruction algorithm (2.12), by which we demonstrate that the sparsity-promoting dynamic reconstruction attains the optimal super-resolution capability for the velocity recovery. Specifically, we have the following theorem.

Theorem 3.3. *Let $n \geq 2$, $T \geq 2n-1$, and $\sigma \leq m_{\min}$. Let the measurement $\mathbf{Y}_t$, $t = 0, \cdots, T$, in (1.1) be generated by a n-sparse parameter set $\{(a_j, \mathbf{y}_j, \tau\mathbf{v}_j)\}_{j=1}^n$ with $\tau\mathbf{v}_j \in B^d_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0})$. Assume that*

$$d_{\min} := \min_{p\neq j} \left|\left|\tau\mathbf{v}_p - \tau\mathbf{v}_j\right|\right|_2 \geq \frac{11.76e\pi 4^{d-1}\Big((n+2)(n+1)/2\Big)^{\xi(d-1)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{3.5}$$

Let $\mathcal{O}$ in (2.12) be $B^d_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0})$, then the solution to (2.12) contains exactly n elements. If a n-sparse parameter set $\{(\hat{a}_j, \hat{\mathbf{y}}_j, \tau\hat{\mathbf{v}}_j)\}_{j=1}^n$ is the solution, after reordering the $\hat{\boldsymbol{\alpha}}_j$'s, we have

$$\left|\left|\tau\hat{\mathbf{v}}_j - \tau\mathbf{v}_j\right|\right|_2 < \frac{d_{\min}}{2}.$$

Moreover, we have

$$\left|\left|\tau\hat{\mathbf{v}}_j - \tau\mathbf{v}_j\right|\right|_2 \leq \frac{C(d,n)}{T\Omega} SRF^{2n-2}\frac{\sigma}{m_{\min}}, \quad 1 \leq j \leq n, \tag{3.6}$$

where $SRF := \frac{\pi}{d_{\min} T\Omega}$ and

$$C(d,n) = \left(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\right)^{2n-1} n 2^{6n-3} e^{2n} \pi^{-\frac{1}{2}}.$$

4 Projection-based velocity number detection algorithm

By the projection methods we use in Section 8, we propose in this section and next section new algorithms for super-resolving the velocities. The strength of our algorithms is that they are able to resolve the number of sources and reconstruct the values of the velocities at the resolution limit orders, i.e., $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}})$ and $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}})$, respectively. Therefore, as the number of time steps increases, the resolution improves as well.

In this section, we propose a projection-based sweeping singular-value-thresholding number detection algorithm for the tracking problem in two dimensions. The extension to higher dimensions is straightforward.

4.1 One-dimensional sweeping singular-value-thresholding number detection algorithm

In this subsection, we review the sweeping singular-value-thresholding number detection algorithm in dimension one [34] for the static super-resolution problem. We refer to [1, 2, 11, 23, 24, 29, 43, 47, 51, 52] and the references therein for other interesting algorithms in one dimension.

We consider a model that is tuned for our velocity recovery problem. Especially, we consider the measurement $\mathbf{Y}$ generated by $\mu = \sum_{j=1}^{n} a_j \delta_{y_j}, y_j \in \mathbb{R}$ as

$$\mathbf{Y}(t) = \sum_{j=1}^{n} a_j e^{i y_j \Omega t} + \mathbf{W}(t), \quad t = 0, \cdots, T, \tag{4.1}$$

with $|\mathbf{W}(t)| < \sigma$ and $\min_j |a_j| = m_{\min}$. We choose measurements at the sample points $qr, q = 0, \cdots, 2s$ with $s \geq n$ and r, g being integer satisfying $T + 1 = 2sr + g, 0 \leq g < 2s$:

$$\mathbf{Y}(qr) = \sum_{j=1}^{n} a_j e^{i y_j \Omega qr} + \mathbf{W}(qr), \quad 0 \leq q \leq 2s.$$

We then form the following Hankel matrix:

$$\mathbf{H}(s) = \begin{pmatrix} \mathbf{Y}(0) & \mathbf{Y}(r) & \cdots & \mathbf{Y}(sr) \\ \mathbf{Y}(r) & \mathbf{Y}(2r) & \cdots & \mathbf{Y}((s+1)r) \\ \cdots & \cdots & \ddots & \cdots \\ \mathbf{Y}(sr) & \mathbf{Y}((s+1)r) & \cdots & \mathbf{Y}(2sr) \end{pmatrix}, \tag{4.2}$$

and consider the singular value decomposition of $\mathbf{H}(s)$

$$\mathbf{H}(s) = \hat{U}\hat{\Sigma}\hat{U}^*,$$

where $\hat{\Sigma} = \mathrm{diag}(\hat{\sigma}_1, \cdots, \hat{\sigma}_n, \hat{\sigma}_{n+1}, \cdots, \hat{\sigma}_{s+1})$ with the singular values $\hat{\sigma}_j$, $1 \leq j \leq s+1$, ordered in a decreasing manner. We then determine the source number by thresholding on these singular values with a properly chosen threshold based on Theorem 4.1 below. To derive Theorem 4.1, we first introduce the notation

$$\phi_s(x) = (1, x, \cdots, x^s)^\top, \tag{4.3}$$

$$\zeta(k)=\begin{cases}(\frac{k-1}{2}!)^2, & k \text{ is odd,}\\ (\frac{k}{2})!(\frac{k-2}{2})!, & k \text{ is even,}\end{cases} \tag{4.4}$$

and the following auxiliary lemma.

Lemma 4.1. *Let $k\geq 2$ and $-\frac{\pi}{2}\leq\theta_1<\theta_2<\cdots<\theta_k\leq\frac{\pi}{2}$. Let $\theta_{\min}=\min_{p\neq j}|\theta_p-\theta_j|$ and $V_k(k-1)=\big(\phi_{k-1}(e^{i\theta_1}),\cdots,\phi_{k-1}(e^{i\theta_k})\big)$ with $\phi_{k-1}(x)$ defined as in (4.3). Then*

$$||V_k(k-1)^{-1}||_\infty\leq\frac{\pi^{k-1}}{\zeta(k)\theta_{\min}^{k-1}},$$

where $\zeta(k)$ is defined in (4.4).

Proof. See Lemma 3 in [34]. □

Theorem 4.1. *Let $s\geq n$ and $\mu=\sum_{j=1}^n a_j\delta_{y_j}$ with $y_j\in[\frac{-(n-1)\pi}{T\Omega},\frac{(n-1)\pi}{T\Omega}], 1\leq j\leq n$. We have*

$$\hat{\sigma}_j\leq(s+1)\sigma,\quad j=n+1,\cdots,s+1. \tag{4.5}$$

Moreover, if the following separation condition is satisfied

$$\min_{p\neq j}\Big|y_p-y_j\Big|>\frac{2\pi(s+1)}{T\Omega}\Big(\frac{2n(s+1)}{\zeta(n)^2}\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}, \tag{4.6}$$

where $\zeta(n)$ is defined as in (4.4), then

$$\hat{\sigma}_n>(s+1)\sigma. \tag{4.7}$$

Proof. **Step 1.** Note that $\mathbf{H}(s)$ has a decomposition that $\mathbf{H}(s)=DAD^\top+\Delta$ where $A=\mathrm{diag}(a_1,\cdots,a_n)$, $D=\big(\phi_s(e^{iy_1\Omega r}),\cdots,\phi_s(e^{iy_n\Omega r})\big)$ with $\phi_s(\omega)$ being defined in (4.3) and Δ is the matrix from the noise $\mathbf{W}$. We first consider the case when $\mathbf{W}(t)=0$, in which $\mathbf{H}(s)=DAD^\top$. Denoting σ_n as the n-th singular value of $\mathbf{H}(s)$. Note that σ_n is the minimum nonzero singular value of $DAD^\top$. Let $\ker(D^\top)$ be the kernel space of $D^\top$ and $\ker^\perp(D^\top)$ be its orthogonal complement, we have

$$\sigma_n=\min_{||x||_2=1,x\in\ker^\perp(D^\top)}||DAD^\top||_2\geq\sigma_{\min}(DA)\sigma_n(D^\top)\geq\sigma_{\min}(D)\sigma_{\min}(A)\sigma_{\min}(D).$$

Let $\theta_{\min}=\min_{p\neq j}\Big|y_pr\Omega-y_jr\Omega\Big|$. Since $s\geq n$, $y_jr\Omega\in\big[\frac{-\pi}{2},\frac{\pi}{2}\big]$ for $y_j\in\big[-\frac{(n-1)\pi}{T\Omega},\frac{(n-1)\pi}{T\Omega}\big]$. By Lemma 4.1, we have

$$\sigma_{\min}(D)\geq\frac{1}{||V_n(n-1)^{-1}||_2}\geq\frac{1}{\sqrt{n}||V_n(n-1)^{-1}||_\infty}\geq\frac{1}{\sqrt{n}}\frac{\zeta(n)\theta_{\min}^{n-1}}{\pi^{n-1}},$$

where $V_n(n-1)=\big(\phi_{n-1}(e^{iy_1\Omega r}),\cdots,\phi_{n-1}(e^{iy_n\Omega r})\big)$. It follows that

$$\sigma_n\geq\sigma_{\min}(A)\Big(\frac{1}{\sqrt{n}}\frac{\zeta(n)\theta_{\min}^{n-1}}{\pi^{n-1}}\Big)^2\geq\frac{m_{\min}\zeta(n)^2\theta_{\min}^{2n-2}}{n\pi^{2n-2}}. \tag{4.8}$$

Step 2. We now prove the theorem. Since $||\mathbf{W}||_\infty\leq\sigma$, for Δ in Step 1 we have $||\Delta||_2\leq||\Delta||_F\leq(s+1)\sigma$. By Weyl's theorem, we have $|\hat{\sigma}_j-\sigma_j|\leq||\Delta||_2, j=1,\cdots,s+1$, where σ_j is the j-th singular value of $DAD^\top$. Note that $\sigma_j = 0, n+1 \le j \le s+1$, we get $|\hat{\sigma}_j| \le ||\Delta||_2 \le (s+1)\sigma, n+1 \le j \le s+1$. This proves (4.5).

Since $y_j \in [\frac{-(n-1)\pi}{T\Omega}, \frac{(n-1)\pi}{T\Omega}]$, by the relation between r, s, T, we have $\theta_{\min} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$\theta_{\min} = r\Omega \min_{p\neq j} \left| y_p - y_j \right| > \frac{2\pi(s+1)r}{T} \Big(\frac{2n(s+1)}{\zeta(n)^2} \frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}} \ge \pi \Big(\frac{2n(s+1)}{\zeta(n)^2} \frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}},$$

where we have used the separation condition (4.6). By (4.8), we have

$$\sigma_n \ge \frac{m_{\min}\zeta(n)^2 \theta_{\min}^{2n-2}}{n\pi^{2n-2}} > 2(s+1)\sigma. \tag{4.9}$$

Similarly, by Weyl's theorem, $|\hat{\sigma}_n - \sigma_n| \le ||\Delta||_2$. Thus, $\hat{\sigma}_n > 2(s+1)\sigma - ||\Delta||_2 \ge (s+1)\sigma$. Conclusion (4.7) then follows. □

The reconstruction procedure is summarized in **Algorithm 1** below. Note that in **Algorithm 1**, it is required that the input integer s is greater than the source number n. However, a suitable s is not easy to estimate and large s may yield a deterioration of resolution. To remediate this issue, we propose a sweeping singular-value-thresholding number detection algorithm which allows us to find the minimum (or sparsest) source number from admissible measurements; see **Algorithm 2** below. We remark that since when $s = n$ the separation distance in (4.6) is near $\frac{c}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}}$ for a small constant c, the resolution of **Algorithm 2** attains the optimal resolution order derived in [34].

Algorithm 1: Singular-value-thresholding number detection algorithm

Input: Number s, noise level σ;
Input: Measurement: $\mathbf{Y} = (\mathbf{Y}(0), \cdots, \mathbf{Y}(T))^\top$;
1: $r = T+1 \mod 2s$, $\mathbf{Y}_{new} = (\mathbf{Y}(0), \mathbf{Y}(r), \cdots, \mathbf{Y}(\omega_{2sr}))^\top$;
2: Formulate the $(s+1)\times(s+1)$ Hankel matrix $\mathbf{H}(s)$ from $\mathbf{Y}_{new}$, and compute the singular values of $\mathbf{H}(s)$ as $\hat{\sigma}_1, \cdots, \hat{\sigma}_{s+1}$ distributed in a decreasing manner;
4: Determine n by $\hat{\sigma}_n > (s+1)\sigma$ and $\hat{\sigma}_j \le (s+1)\sigma, j = n+1, \cdots, s+1$;
Return: n

Algorithm 2: Sweeping singular-value-thresholding number detection algorithm

Input: Noise level σ, measurement: $\mathbf{Y} = (\mathbf{Y}(0), \cdots, \mathbf{Y}(T))^\top$;
Input: $n_{max} = 0$;
for $s = 1 : \lfloor \frac{T-1}{2} \rfloor$ **do**
  Input $s, \sigma, \mathbf{Y}$ to **Algorithm 1**, save the output of **Algorithm 1** as $n_{recover}$;
  if $n_{recover} > n_{max}$ **then**
    $n_{max} = n_{recover}$
Return n_{max}

4.2 Two-dimensional sweeping singular-value-thresholding number detection algorithms

We now derive the two-dimensional sweeping singular-value-thresholding number detection algorithm for recovering the number of sources. The strategy is considering the measurement at some proper sample points. To be specific, let the parameter set of the source be

$\{(a_j,\mathbf{y}_j,\mathbf{v}_j)\}_{j=1}^n, a_j\in\mathbb{C}, \mathbf{y}_j\in\mathbb{R}^2, \mathbf{v}_j\in\mathbb{R}^2$ and $\mathbf{Y}_t(\boldsymbol{\omega})$ in (1.1) be the associated measurement. We first choose the following $\frac{n(n+1)}{2}$ vectors

$$\mathbf{v}(\theta_q) = (\cos\theta_q, \sin\theta_q)^\top \in \mathbb{R}^2,\ q = 1,\cdots,\frac{n(n+1)}{2}, \tag{4.10}$$

where $\theta_q = \frac{q2\pi}{n(n+1)}$. For each q, the measurement at $\Omega\mathbf{v}(\theta_q)$ is

$$\mathbf{Y}_t(\Omega\mathbf{v}(\theta_q)) = \sum_{j=1}^n a_j e^{i\mathbf{y}_j^\top \mathbf{v}(\theta_q)\Omega} e^{i\tau\mathbf{v}_j^\top\mathbf{v}(\theta_q)\Omega t} + \mathbf{W}_t(\mathbf{v}(\theta_q)\Omega), \quad t = 0,\cdots,T.$$

This can be viewed as $\mathbf{Y}(t) = \sum_{j=1}^n b_j e^{i\tau\mathbf{v}_j^\top\mathbf{v}(\theta_q)\Omega t} + \mathbf{W}(t),\ t = 0,\cdots,T$ with $b_j = e^{i\mathbf{y}_j^\top\mathbf{v}(\theta_q)\Omega}$ and $\mathbf{W}(t) = \mathbf{W}_t(\mathbf{v}(\theta_q)\Omega)$. Thus we can form the Hankel matrix $\mathbf{H}_q(s)$ in the same way as in (4.2) from the above measurements. Denote $\hat\sigma_{q,j}$ the j-th singular value of $\mathbf{H}_q(s)$, we can detect the exact source number by thresholding on $\hat\sigma_{q,j}$'s under a suitable separation condition, as shown in Theorem 4.2 below. We first present a result which is needed in the proof of Theorem 4.2.

Lemma 4.2. *For $\tau\mathbf{v}_1,\cdots,\tau\mathbf{v}_n\in\mathbb{R}^2, n\geq 2$, let $d_{\min} := \min_{p\neq j}||\tau\mathbf{v}_p - \tau\mathbf{v}_j||_2$. Let $\mathbf{v}(\theta_q), q = 1,\cdots,\frac{n(n+1)}{2}$ be the ones in (4.10) and S_q's be the one-dimensional spaces spanned by $\mathbf{v}(\theta_q)$'s. There exists q^* so that*

$$\min_{p\neq j, 1\leq p,j\leq n}\left|\left|\mathscr{P}_{S_{q^*}}(\tau\mathbf{v}_p) - \mathscr{P}_{S_{q^*}}(\tau\mathbf{v}_j)\right|\right|_2 \geq \frac{2d_{\min}}{n(n+1)}.$$

Proof. It is clear there are at most $\frac{n(n-1)}{2}$ different $\mathbf{u}_{pj} = \tau\mathbf{v}_p - \tau\mathbf{v}_j, 1\leq p<j\leq n$. Denote $\mathbf{v}(\theta) = (\cos\theta,\sin\theta)^\top$ and $\Delta = \frac{\pi}{n(n+1)}$. We observe that if $|\mathbf{v}(\theta)^\top\mathbf{u}| < ||\mathbf{u}||_2\sin\Delta, \theta\in[2\Delta,\pi]$, then $|\mathbf{v}(\theta^*)^\top\mathbf{u}|\geq||\mathbf{u}||_2\sin\Delta$, for $|\theta^*-\theta|\geq 2\Delta, \theta^*\in[2\Delta,\pi]$. Define $N(\mathbf{u},\Delta) = \{\mathbf{v}|\mathbf{v}\in\mathbb{R}^2, ||\mathbf{v}||_2 = 1, |\mathbf{v}^\top\mathbf{u}| < ||\mathbf{u}||_2\sin\Delta\}$. If $\mathbf{v}(\theta_{q_0})\in N(\mathbf{u}_{p_0j_0},\Delta)$ for some $1\leq p_0, j_0\leq n$, then $\mathbf{v}(\theta_q)\notin N(\mathbf{u}_{p_0j_0},\Delta), \forall q\neq q_0, q = 1,\cdots,\frac{n(n+1)}{2}$. Since we have $\frac{n(n+1)}{2}$ different q's and only $\frac{n(n-1)}{2}$ $\mathbf{u}_{pj}$'s, there must be some $\mathbf{v}(\theta_{q^*})\notin\cup_{p<j,1\leq j,p\leq n}N(\mathbf{u}_{pj},\Delta)$. Hence,

$$\min_{p\neq j,1\leq p,j\leq n}||\mathscr{P}_{S_{q^*}}(\mathbf{v}_p) - \mathscr{P}_{S_{q^*}}(\mathbf{v}_j)||_2 \geq d_{\min}\sin\Delta \geq d_{\min}\frac{2\Delta}{\pi},$$

whence the lemma follows. □

Theorem 4.2. *Let $n\geq 2, s\geq n$ and consider the parameter set $\{(a_j,\mathbf{y}_j,\tau\mathbf{v}_j)\}_{j=1}^n$ with $\tau\mathbf{v}_j\in B^2_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0}), 1\leq j\leq n$. For the singular values of $\mathbf{H}_q(s)$, we have*

$$\hat\sigma_{q,j}\leq(s+1)\sigma, \quad j = n+1,\cdots,s+1, \quad q = 1,\cdots,\frac{n(n+1)}{2}. \tag{4.11}$$

Moreover, if the following separation condition is satisfied

$$d_{\min} := \min_{p\neq j}\left|\left|\tau\mathbf{v}_p - \tau\mathbf{v}_j\right|\right|_2 > \frac{\pi(s+1)n(n+1)}{2T\Omega}\left(\frac{n(s+1)}{\zeta(n)^2}\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}, \tag{4.12}$$

then there exists q^* so that

$$\hat\sigma_{q^*,n} > (s+1)\sigma. \tag{4.13}$$

Proof. Note that for each $\mathbf{v}(\theta_q)$ in (4.10), the $\tau\mathbf{v}_j$'s satisfy that $|\tau\mathbf{v}_j^\top\mathbf{v}(\theta_q)| \le \frac{(n-1)\pi}{T\Omega}, j = 1, \cdots, n$. By applying Theorem 4.1 on the Hankel matrix $\mathbf{H}_q(s)$ formulated from the measurements $\mathbf{Y}_t(\Omega\mathbf{v}(\theta_q))$, for $t = 0, \cdots, T$, we immediately get (4.11). Moreover, when the separation condition (4.12) holds, by Lemma 4.2, there is some q^* so that

$$\min_{p\neq j, 1\le p,j\le n} \left|\mathscr{P}_{S_{q^*}}(\mathbf{v}_p) - \mathscr{P}_{S_{q^*}}(\mathbf{v}_j)\right| \ge \frac{2d_{\min}}{n(n+1)} > \frac{\pi(s+1)}{T\Omega}\left(\frac{2n(s+1)}{\zeta(n)^2}\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}.$$

Applying Theorem 4.1 again, we get $\hat{\sigma}_{q^*,n} > (s+1)\sigma$. □

The above theorem shows that for point sources that are well-separated, we can determine the source number n by thresholding on the singular values of the Hankel matrices $\mathbf{H}_q(s)$'s. We note that the number of required unit vectors $\mathbf{v}(\theta_q)$ is not available since n is unknown. In practice, we can choose a large enough N, say $N \ge \frac{n(n+1)}{2}$. We summarize our algorithm as below.

Algorithm 3: Two-dimensional sweeping singular-value-thresholding number detection algorithm

Input: Noise level σ, measurement: $\mathbf{Y}(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathbb{R}^2, ||\boldsymbol{\omega}||_2 \le \Omega$, and $n_{\max} = 0$
Input: A large enough N, and corresponding N unit vectors
$\mathbf{v}(\theta_q) = (\cos\theta_q, \sin\theta_q)^\top, \theta_q = \frac{\pi}{N}, \frac{2\pi}{N}, \cdots, \pi$
for $\theta_q = \frac{\pi}{N}, \frac{2\pi}{N}, \cdots, \pi$ **do**
  Input σ and $Y_t(\Omega\mathbf{v}(\theta_q)), t = 0, \cdots, T$ to **Algorithm 2**, save the output of **Algorithm 2** as $n_{recover}$;
  if $n_{recover} > n_{max}$ **then**
    $n_{max} = n_{recover}$
Return n_{max}

4.3 Numerical experiments

In this subsection, we conduct some numerical experiments to demonstrate the super-resolution ability of our number detection algorithm and the superiority of our algorithm over the static reconstruction method.

We present an example which shows that our algorithm can recover the source number even when it is impossible to recover the source number by the static reconstruction in any frame. For simplicity, we consider $\Omega = 1$, $\tau = 0.2$, $\sigma = 10^{-2}$ and the measurements at 5 time steps. We construct an example where two point sources with $O(1)$ intensities are located at $\mathbf{y}_1 = (0, 0.27), \mathbf{y}_2 = (0.20, 0.17)$ moving respectively at the velocities $\mathbf{v}_1 = (0.14, 0.51), \mathbf{v}_2 = (0.45, 0.17)$. At each of the 5 time steps, the locations of the two point sources are,

$$\begin{aligned}
&(0, 0.27), \quad (0.20, 0.17), \quad \text{at time step } t = 0,\\
&(0.028, 0.372), \quad (0.29, 0.204), \quad \text{at time step } t = 1,\\
&(0.056, 0.474), \quad (0.38, 0.238), \quad \text{at time step } t = 2,\\
&(0.084, 0.576), \quad (0.47, 0.272), \quad \text{at time step } t = 3,\\
&(0.112, 0.678), \quad (0.56, 0.306), \quad \text{at time step } t = 4.
\end{aligned}$$

Note that at each time step, the point sources are separated by a distance which is much lower than the Rayleigh limit. Thus the conventional static reconstruction cannot recover the source number and locations in each frame. However, by **Algorithm 3** we recover the number of point sources. This demonstrates the superiority of our algorithm over the static reconstructions for recovering correctly the source number.

In addition, in the above example the resolution is near $\tau||\mathbf{v}_1-\mathbf{v}_2||_2 \approx 0.1$, which is much better than the Rayleigh limit (π) of the imaging system. The reason is that we take $T=5$ and the resolution limit in Theorem 3.2 which of order $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}})$ indicates that a very good resolution for number detection can be achieved when we have multiple observations.

5 Projection-based velocity recovery algorithms

In this section, we propose a projection-based velocity recovery algorithm in two dimensions. The algorithm can be easily extended to higher dimensions.

5.1 Review of one-dimensional MUSIC algorithm

In this subsection, we review the one-dimensional MUSIC algorithm [31, 36, 46, 48]. In a standard MUSIC algorithm for solving the inverse problem (4.1), one first assembles the Hankel matrix $\mathbf{H}(s)$ as in (4.2), where

$$s = \begin{cases} \frac{T}{2}, & \text{even } T, \\ \frac{T-1}{2}, & \text{odd } T, \end{cases} \tag{5.1}$$

then performs the singular value decomposition of $\mathbf{H}(s)$,

$$\mathbf{H}(s) = \hat{U}\hat{\Sigma}\hat{U}^* = [\hat{U}_1 \quad \hat{U}_2]\mathrm{diag}(\hat{\sigma}_1, \hat{\sigma}_2, \cdots, \hat{\sigma}_n, \hat{\sigma}_{n+1}, \cdots, \hat{\sigma}_{\hat{N}+1})[\hat{U}_1 \quad \hat{U}_2]^*,$$

where $\hat{U}_1 = (\hat{U}(1), \cdots, \hat{U}(n)), \hat{U}_2 = (\hat{U}(n+1), \cdots, \hat{U}(\hat{N}+1))$ with n being the estimated source number (model order). The source number n can be detected by many algorithms such as those in [1, 11, 23, 24, 34, 47, 51, 52]. Denote the orthogonal projection to the space $\hat{U}_2$ by $\hat{P}_2 x = \hat{U}_2(\hat{U}_2^* x)$. For a test vector $\Phi(\omega) = (1, e^{i\Omega\omega}, \cdots, e^{is\Omega\omega})^\top$, one defines the MUSIC imaging functional

$$\hat{J}(\omega) = \frac{||\Phi(\omega)||_2}{||\hat{P}_2\Phi(\omega)||_2} = \frac{||\Phi(\omega)||_2}{||\hat{U}_2^*\Phi(\omega)||_2}.$$

The local maximizer of $\hat{J}(\omega)$ indicates the location of the point sources. In practice, one can test evenly spaced points in a specified region and plot the discrete imaging functional and then determine the source locations by detecting the peaks. A peak selection algorithm **Algorithm 5** is given below. Finally, we summarize the standard MUSIC algorithm in **Algorithm 4** below.

Algorithm 4: Standard MUSIC algorithm

Input: Noise level σ, source number n;
Input: Measurements: $\mathbf{Y} = (\mathbf{Y}(0), \mathbf{Y}(1), \cdots, \mathbf{Y}(T))^\top$;
Input: Region of test points $[TS, TE]$ and spacing of test points TPS;
1: Determine s by (5.1), formulate the $(s+1)\times(s+1)$ Hankel matrix $\mathbf{H}(s)$ as (4.2) from $\mathbf{Y}$;
2: Compute the singular vectors of $\mathbf{H}(s)$ as $\hat{U}(1), \hat{U}(2), \cdots, \hat{U}(s+1)$ and formulate the noise space $\hat{U}_2 = (\hat{U}(n+1), \cdots, \hat{U}(s+1))$;
3: For test points x in $[TS, TE]$ evenly spaced by TPS, construct the test vector $\Phi(\omega) = (1, e^{i\Omega\omega}, \cdots, e^{is\Omega\omega})^\top$;
4: Plot the MUSIC imaging functional $\hat{J}(\omega) = \frac{||\Phi(\omega)||_2}{||\hat{U}_2^*\Phi(\omega)||_2}$;
5: Select the peak locations $\hat{y}_j$'s in the $\hat{J}(x)$ by **Algorithm 5**;
Return $\hat{y}_j$'s.

Algorithm 5: Peak selection algorithm

Input: Image $IMG = (f(\omega_1), \cdots, f(\omega_M))$;
Input: Peak compare range PCR, differential compare range DCR, differential compare threshold DCT;
1: Initialize the local maximum points $LMP = [\,]$, peak points $PP = [\,]$;
2: Differentiate the image IMG to get the $DIMG = (f'(\omega_1), \cdots, f'(\omega_M))$;
3: **for** $j = 1 : M$ **do**
    if $f(\omega_j) = \max(f(\omega_{j-PCR}), f(\omega_{j-PCR+1}), \cdots, f(\omega_{j+PCR}))$ **then**
        LMP appends ω_j;
4: **for** ω_j *in* LMP **do**
    if $\max(|f'(\omega_{j-DCR})|, |f'(\omega_{j-DCR+1})|, \cdots, |f'(\omega_{j+DCR})|) \geq DCT$ **then**
        PP appends ω_j;
    Return: PP.

5.2 PROJECTION-BASED MUSIC ALGORITHM FOR SUPER-RESOLVING VELOCITIES

In this subsection we propose a projection-based MUSIC algorithm for super-resolving the velocities in two dimensions. As indicated by the proof of Theorem 3.2, when the velocities are well-separated in $\mathbb{R}^2$, there exist two unit vectors so that the projection of the velocities in two one-dimensional subspaces spanned by these unit vectors can be stably recovered. We can then find the original two-dimensional velocities from their projections. More precisely, let $N = \frac{(n+2)(n+1)}{2}$ and

$$\mathbf{v}(\phi) = (\cos\phi,\ \sin\phi)^\top, \quad \phi \in \left\{\frac{\pi}{N}, \frac{2\pi}{N}, \cdots, \pi\right\}. \tag{5.2}$$

For each $\mathbf{v}(\phi)$, we denote the space spanned by $\mathbf{v}(\phi)$ as $S(\phi)$ and call the $(\mathbf{v}_j^\top \mathbf{v}(\phi))\mathbf{v}(\phi)$'s the projected velocities in $S(\phi)$. We consider the measurement at $\Omega\mathbf{v}(\phi)$ that

$$\mathbf{Y}_t(\Omega\mathbf{v}(\phi)) = \sum_{j=1}^{n} a_j e^{i\mathbf{y}_j^\top \mathbf{v}(\phi)\Omega} e^{i\tau\mathbf{v}_j^\top \mathbf{v}(\phi)\Omega t} + \mathbf{W}_t(\mathbf{v}(\theta_q)\Omega), \quad t = 0, \cdots, T. \tag{5.3}$$

We then recover the projected velocities in each of those one-dimensional subspaces using the one-dimensional MUSIC algorithm fed with an estimated source number. We then only consider those subspaces $S(\phi)$'s where n peaks appear, i.e., n projected velocities are reconstructed. We further choose two vectors from the $\mathbf{v}(\phi)$'s, denoted by $\mathbf{v}(\phi_1), \mathbf{v}(\phi_2)$, so that the recovered velocities in $S(\phi_1), S(\phi_2)$ have respectively the largest and second largest minimum separation distance. We remark that, when N is large, one can require additionally that $\mathbf{v}(\phi_1)$ is not too much correlated to $\mathbf{v}(\phi_2)$, say $|\mathbf{v}(\phi_1)^\top \mathbf{v}(\phi_2)| \le c$ for some constant $0 < c < 1$, to ensure that the reconstruction of two-dimensional velocities from their projections on $\mathbf{v}(\phi_1), \mathbf{v}(\phi_2)$ is stable.

We next construct the original velocities from their projections on $S(\phi_1)$ and $S(\phi_2)$. This is usually called the pair matching in the problem of direction of arrival, where ad hoc schemes [12, 27, 53, 55] were derived to associate the estimated azimuth and elevation angles. In our paper, this can be done in the following manner. From the recovered projected velocities in $S(\phi_1)$ and $S(\phi_2)$, we first form a grid of n^2 points, say $\mathbf{z}_{1,1}, \mathbf{z}_{1,2}, \cdots, \mathbf{z}_{n,n}$. It can be shown that the original velocities are close to these grid points. These grid points reduce the off-the-grid recovery problem to an on-the-grid one. We then employ an enumeration method to recover the source velocities from these grid points. To be more specific, we define

$$G_\phi(\mathbf{z}_{1,j_1}, \cdots, \mathbf{z}_{n,j_n}) = \big(\phi_T(e^{i\mathbf{z}_{1,j_1}^\top \mathbf{v}(\phi)\Omega}),\ \phi_T(e^{i\mathbf{z}_{2,j_2}^\top \mathbf{v}(\phi)\Omega}),\ \cdots,\ \phi_T(e^{i\mathbf{z}_{n,j_n}^\top \mathbf{v}(\phi)\Omega})\big), \tag{5.4}$$

where $\phi_T(x)$ is defined as in (4.3), and solve the following optimization problem by enumeration,

$$\min_{\pi\in\gamma(n)} \Big(\sum_{\phi} \min_{\hat{\mathbf{a}}} \Big|\Big| G_\phi(\mathbf{z}_{1,\pi_1}, \cdots, \mathbf{z}_{n,\pi_n})\hat{\mathbf{a}} - \mathbf{Y}(\mathbf{v}(\phi))\Big|\Big|_2\Big), \tag{5.5}$$

where $\mathbf{Y}(\mathbf{v}(\phi)) = (\mathbf{Y}_0(\mathbf{v}(\phi)\Omega), \cdots, \mathbf{Y}_T(\mathbf{v}(\phi)\Omega))^\top$, and $\gamma(n)$ is the set of all permutations of $\{1, \cdots, n\}$. We note that the computational complexity of the enumeration is low when n is not large. We summarize the algorithm in **Algorithm 6** below.

Remark 5.1. *We remark that since in each one-dimensional space, the MUSIC algorithm attains nearly the optimal order of the resolution, say* $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}})$*, as demonstrated in [30, 31] numerically, our* ***Algorithm 6*** *achieves the optimal order of the resolution that is derived in Theorem 3.2.*

Remark 5.2. *We remark that the range of the recovered velocities is confined by the sampling rate in (5.3). Suppose we have a large range of velocities, one way to recover them is to consider all the possible projected velocities in a large range in one-dimensional space and construct a large grid. Then we recover the correct velocities in a similar way as (5.5) with more measurements in* $\mathbf{Y}_t$*'s.*

Algorithm 6: Two-dimensional projection-based velocity recovery algorithm

Input: Noise level σ, source number n;
Input: Measurement: $\mathbf{Y}_t(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathbb{R}^2, ||\boldsymbol{\omega}||_2 \leq \Omega, t = 0, \cdots, T$;
Input: Region of test points $[TS, TE]$ and spacing of test points TPS;
Input: A large enough N, and corresponding N unit vectors $\mathbf{v}(\phi)$ in (5.2), $\phi \in \left\{\frac{\pi}{N}, \cdots, \pi\right\}$

1:**for** $\phi \in \left\{\frac{\pi}{N}, \frac{2\pi}{N}, \cdots, \pi\right\}$ **do**

Input σ and $\mathbf{Y}_t(\mathbf{v}(\phi)\Omega), t = 0, \cdots, T$ to **Algorithm 2** to recover the projected source number $\hat{n}$;

Input $\sigma, \mathbf{Y}_t(\mathbf{v}(\phi)\Omega), t = 0, \cdots, T, \hat{n}, [TS, TE]$ and TPS to **Algorithm 4**, save the output as $b_1, \cdots, b_q$;

The recovered projected velocities are $\hat{\mathbf{p}}_1 = b_1\mathbf{v}(\phi), \cdots, \hat{\mathbf{p}}_q = b_q\mathbf{v}(\phi)$;

2: Choose two vectors $\mathbf{v}(\phi_1), \mathbf{v}(\phi_2)$, from those $\mathbf{v}(\phi)$'s so that n projected velocities were recovered in each of the spaces $S(\phi_1), S(\phi_2)$ and the recovered projected velocities $\hat{\mathbf{p}}_j(\mathbf{v}(\phi_1))$'s, $\hat{\mathbf{p}}_j(\mathbf{v}_2(\phi_2))$'s have respectively the largest and second largest minimum separation distance;

3:Construct the n^2 grid points $\mathbf{z}_{1,1}, \mathbf{z}_{1,2}, \cdots, \mathbf{z}_{n,n}$ by considering the intersection points of lines $\hat{\mathbf{p}}_q(\mathbf{v}(\phi_j)) + \lambda\mathbf{g}(\phi_j),\ \lambda \in \mathbb{R},\ q = 1, \cdots, n,\ j = 1, 2$, where $\mathbf{g}(\phi_j)$ is the unit vector that is perpendicular to $\mathbf{v}(\phi_j)$;

4: Solve the following optimization problem by enumeration,

$$\min_{\pi \in \gamma(n)} \left(\sum_{\phi} \min_{\hat{\mathbf{a}}} \left|\left| G_\phi(\mathbf{z}_{1,\pi_1}, \cdots, \mathbf{z}_{n,\pi_n})\hat{\mathbf{a}} - \mathbf{Y}(\mathbf{v}(\phi)) \right|\right|_2 \right),$$

where G_ϕ is defined by (5.4), $\mathbf{Y}(\mathbf{v}(\phi)) = (\mathbf{Y}_0(\mathbf{v}(\phi)\Omega), \cdots, \mathbf{Y}_T(\mathbf{v}(\phi)\Omega))^\top$, and $\gamma(n)$ is the set of all permutations of $\{1, \cdots, n\}$;

5: The minimizer $\mathbf{z}_{1,\pi_1}, \cdots, \mathbf{z}_{n,\pi_n}$'s are the recovered velocities $\tau\hat{\mathbf{v}}_1, \cdots, \tau\hat{\mathbf{v}}_n$;

Return $\tau\hat{\mathbf{v}}_1, \cdots, \tau\hat{\mathbf{v}}_n$.

5.3 Numerical experiments

In this subsection, we conduct some numerical experiments to demonstrate the super-resolution ability of our velocity recovery algorithm and the superiority of our algorithm over the static reconstruction method.

We present an example which shows that our algorithm can super-resolve the velocities even when it is impossible to recover stably any locations and velocities by the static reconstruction. For simplicity, we consider $\Omega = 1$, $\tau = 1$, $\sigma = 10^{-2}$ and the measurements at 5 time steps. We construct an example where two point sources with $O(1)$ intensities are located at $\mathbf{y}_1 = (0.22, 0.08), \mathbf{y}_2 = (0.05, 0.08)$ moving respectively at the velocities $\mathbf{v}_1 = (0.47, 0.11), \mathbf{v}_2 =$

$(0.58, 0.56)$. At each of the 5 time steps, the locations of the two point sources are,

$$\begin{aligned} &(0.22, 0.08), \quad (0.05, 0.08), \quad \text{at time step } t = 0,\\ &(0.69, 0.19), \quad (0.63, 0.64), \quad \text{at time step } t = 1,\\ &(1.16, 0.30), \quad (1.21, 1.20), \quad \text{at time step } t = 2,\\ &(1.63, 0.41), \quad (1.79, 1.76), \quad \text{at time step } t = 3,\\ &(2.10, 0.52), \quad (2.37, 2.32), \quad \text{at time step } t = 4. \end{aligned}$$

Note that at each time step, the point sources are separated by a distance below the Rayleigh limit. Thus the conventional static reconstruction cannot stably recover any of the locations or velocities. Since the point sources are so close in each frame, the super-resolution algorithms such as MUSIC algorithm even cannot resolve the locations. However, by **Algorithm 6** we recover stably the velocities: $\hat{\mathbf{v}}_1 = (0.442, 0.120), \hat{\mathbf{v}}_2 = (0.558, 0.570)$. This demonstrates the superiority of our algorithm over the static reconstructions for recovering the velocities.

In addition, in the above example the resolution for the velocities is near 0.4, which is much better than the Rayleigh limit (π) of the imaging system. The reason is that we take 5 times observations and the resolution limit in Theorem 3.2 which of order $O(\frac{1}{T\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}})$ indicates a very good resolution for velocity recovery can be achieved when we have multiple observations. On the other hand, as is indicated by measurement (5.3), the cut-off frequency for the velocity recovery can be viewed as $T\Omega$ rather than the Ω, the inherent Rayleigh limit for the velocity recovery should be $\frac{\pi}{T\Omega}$. The resolution in the example 0.4 is near the inherent Rayleigh limit $\frac{\pi}{4}$. It is also indicated that even if the signal-to-noise ratio becomes worse, we anticipate that our algorithm or any other super-resolution algorithm for velocity recovery can resolve the velocities stably when they are separated by a distance beyond $\frac{\pi}{4}$. For our algorithm, this is demonstrated by the following one-dimensional example.

We consider $\Omega = 1$, $\tau = 1$, $\sigma = 0.3$, and the measurements are taken at 5 time steps. We construct an example where two point sources with $O(1)$ intensities are located at $\mathbf{y}_1 = 0.296, \mathbf{y}_2 = 0.038$ moving respectively at the velocities $\mathbf{v}_1 = 0.2, \mathbf{v}_2 = 1.1$. Note that the signal-to-noise ratio is small. However, we can still stably recover the velocities by the one-dimensional velocity recovery algorithm (analogously to **Algorithm 6**) that yields $\hat{\mathbf{v}}_1 = 0.14, \hat{\mathbf{v}}_2 = 1.11$.

Note that we use a one-dimensional example because the resolution of our two-dimensional algorithm is also related to the projections.

6 Conclusions and future works

In this paper, we have explored the resolution limits for recovering the number and value of location-velocity pairs in the dynamic reconstruction of the tracking problem. We have also derived sharp and better resolution limits for reconstructing the number and values of velocities in the dynamic reconstruction. Also, two projection-based algorithms have been introduced to super-resolve respectively the number and values of velocities. By these results, we have demonstrated certain advantages of the dynamic reconstruction in the tracking problem over the conventional static reconstructions.

Besides our research findings, our work is also a start of many new topics. Especially, from the crucial observation of measurement (5.3), many new algorithms can be inspired to obtain much better resolution for the velocity recovery over the static reconstruction method. In practice, the point spread function may be approximated by other different functions, such as Gaussian functions, and a large amount of point sources may be clustered together in a single image [14], which hampered the application of subspace methods. In this case, the convex optimization based algorithms are great surrogates for resolving the point sources. Applying convex optimization to (5.3) or its variant may enhance significantly the resolution in the practical tracking problem. In addition, developing efficient algorithms for exploring the amplitudes and the locations of point sources when the velocities are known is also an interesting topic. Note that when the velocities are stably recovered, we can reconstruct the $\hat{a}_j, \hat{\mathbf{y}}_j$'s by solving the minimization problem

$$\min_{\hat{a}_j, \hat{\mathbf{y}}_j} \Big|\Big| \sum_{j=1}^{n} \hat{a}_j e^{i\hat{\mathbf{y}}_j^\top \boldsymbol{\omega}} e^{it\tau\hat{\mathbf{v}}_j^\top \boldsymbol{\omega}} - \mathbf{Y}_t(\boldsymbol{\omega}) \Big|\Big|_2, \quad ||\boldsymbol{\omega}||_2 \leq \Omega, t = 0, \cdots, T.$$

The aim is to develop a tractable algorithm in order to recover stably the $\hat{a}_j, \hat{\mathbf{y}}_j$'s for the above minimization problem or its variants.

7 Proofs of Theorems 2.1 and 2.2

7.1 The geometry of the problem

For each time step, the noiseless measurement $\sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j + t\tau\mathbf{v}_j)^\top \boldsymbol{\omega}}, ||\boldsymbol{\omega}||_2 \leq \Omega$, can be written as

$$\sum_{j=1}^{n} a_j e^{i\boldsymbol{\alpha}_j^\top \boldsymbol{\omega}_t}, \tag{7.1}$$

where $\boldsymbol{\alpha}_j = \begin{pmatrix} \mathbf{y}_j \\ \tau\mathbf{v}_j \end{pmatrix}, \boldsymbol{\omega}_t = \begin{pmatrix} \boldsymbol{\omega} \\ t\boldsymbol{\omega} \end{pmatrix}$, $t = 0, \cdots, T$, $||\boldsymbol{\omega}||_2 \leq \Omega$. Define the spaces

$$S_t^d := \left\{ \begin{pmatrix} \mathbf{v} \\ t\mathbf{v} \end{pmatrix} \Big| \mathbf{v} \in \mathbb{R}^d \right\}, \quad t = 0, 1, \cdots. \tag{7.2}$$

Then the measurement $\mathbf{Y}_t(\boldsymbol{\omega})$ can also be written as

$$\mathbf{Y}(\boldsymbol{\omega}_t) = \sum_{j=1}^{n} a_j e^{i\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)^\top \boldsymbol{\omega}_t} + \mathbf{W}(\boldsymbol{\omega}_t), \quad ||\boldsymbol{\omega}_t||_2 \leq \sqrt{1+t^2}\Omega, \tag{7.3}$$

with $|\mathbf{W}(\boldsymbol{\omega}_t)| < \sigma$. Note that S_t^d's are d-dimensional spaces. For each t, the above (7.3) is equivalent to a d-dimensional super-resolution problem in [33]; see Section 7.2 as well. Hence, now the tacking problem can be viewed as reconstructing the $2d$-dimensional vectors $\boldsymbol{\alpha}_j$'s from the measurements (or super-resolution problems) in several d-dimensional subspaces (S_t^d's) of $\mathbb{R}^{2d}$. By the same projection idea as the one in [33], we can estimate the stability of the recovery. We prove our main results by analyzing some geometrical properties of these subspaces S_t^d's.

We first consider the case when $d = 1$ and estimate the angle between adjacent lines in $\{(\omega, t\omega)^\top\}_{t=0}^T, \omega \in \mathbb{R}$. Note that

$$\arctan(u) \pm \arctan(v) = \arctan\Big(\frac{u \pm v}{1 \mp uv}\Big) \quad (\mod \pi), \quad uv \neq 1,$$

and

$$\arctan(x) \geq x - \frac{x^3}{3}.$$

Thus for $t \geq 1$ we have

$$\arctan(t) - \arctan(t-1) = \arctan\Big(\frac{1}{1 + t(t-1)}\Big) \geq \frac{1}{t^2 - t + 1} - \frac{1}{3(t^2 - t + 1)^3} > \frac{1}{t^2 + 1}. \tag{7.4}$$

We denote the unit vector in S_t^1's as $\mathbf{q}_t = \frac{1}{\sqrt{1+t^2}}(1, t)^\top$. Define $\angle(\mathbf{q}_t, \mathbf{q}_j)$ the angle between vectors $\mathbf{q}_t, \mathbf{q}_j$ that

$$\angle(\mathbf{q}_t, \mathbf{q}_j) = \arccos\Big(\frac{\mathbf{q}_t^\top \mathbf{q}_j}{||\mathbf{q}_t||_2 ||\mathbf{q}_j||_2}\Big).$$

By the above observation, we have

$$\angle(\mathbf{q}_t, \mathbf{q}_j) = \arctan(t) - \arctan(j) > \sum_{k=j+1}^{t} \frac{1}{1 + k^2}, \quad j < t. \tag{7.5}$$

We first estimate the projection to these one-dimensional subspaces S_t^1's.

Lemma 7.1. *For $t = 0, 1, \cdots$ and a $\mathbf{u} \in \mathbb{R}^2$, if*

$$\Big|\Big|\mathscr{P}_{S_t^1}(\mathbf{u})\Big|\Big|_2 < \frac{1}{2\pi(1 + t^2)} ||\mathbf{u}||_2,$$

then for $0 \leq j < t$, we have

$$\Big|\Big|\mathscr{P}_{S_j^1}(\mathbf{u})\Big|\Big|_2 \geq \frac{1}{2\pi(1 + j^2)} ||\mathbf{u}||_2.$$

Proof. We first prove the lemma for $j = t - 1$. By (7.5), we have

$$\frac{1}{t^2 + 1} < \angle(\mathbf{q}_t, \mathbf{q}_{t-1}) \leq \frac{\pi}{4}. \tag{7.6}$$

Since $||\mathscr{P}_{S_t^1}(\mathbf{u})||_2 < \frac{1}{2\pi}\frac{1}{1+t^2}||\mathbf{u}||_2$ and $|\mathbf{q}_t^\top \mathbf{u}| = ||\mathbf{u}||_2 |\cos(\angle(\mathbf{q}_t, \mathbf{u}))|$, we have $|\cos(\angle(\mathbf{q}_t, \mathbf{u}))| < \frac{1}{2\pi(1+t^2)}$. Thus $\frac{2}{\pi}|\frac{\pi}{2} - \angle(\mathbf{q}_t, \mathbf{u})| \leq |\sin(\frac{\pi}{2} - \angle(\mathbf{q}_t, \mathbf{u}))| = |\cos(\angle(\mathbf{q}_t, \mathbf{u}))| < \frac{1}{2\pi(1+t^2)}$, and consequently,

$$\frac{\pi}{2} - \frac{1}{4(1 + t^2)} < \angle(\mathbf{q}_t, \mathbf{u}) < \frac{\pi}{2} + \frac{1}{4(1 + t^2)}, or \ -\frac{\pi}{2} - \frac{1}{4(1 + t^2)} < \angle(\mathbf{q}_t, \mathbf{u}) < -\frac{\pi}{2} + \frac{1}{4(1 + t^2)}.$$

Without loss of generality, we only consider the first case. Together with (7.6), we have

$$0 < \angle(\mathbf{q}_{t-1}, \mathbf{u}) < \frac{\pi}{2} - \frac{3}{4(1 + t^2)}, or \ \frac{\pi}{2} + \frac{3}{4(1 + t^2)} < \angle(\mathbf{q}_{t-1}, \mathbf{u}) < \pi.$$

Thus,

$$\|\mathscr{P}_{S_{t-1}}(\mathbf{u})\|_2 \geq \sin\Big(\frac{3}{4(1+t^2)}\Big)\|\mathbf{u}\|_2.$$

By $\sin(x) \geq \frac{2x}{\pi}$ and $\frac{3}{1+t^2} \geq \frac{1}{1+(t-1)^2}$, we then show that

$$\|\mathscr{P}_{S_{t-1}}(\mathbf{u})\|_2 \geq \frac{1}{2\pi}\frac{1}{1+(t-1)^2}\|\mathbf{u}\|_2.$$

For $j < t-1$, since $\angle(\mathbf{q}_j, \mathbf{q}_t) > \sum_{k=j+1}^{t}\frac{1}{1+k^2}$, we have

$$\|\mathscr{P}_{S_j}(\mathbf{u})\|_2 \geq \sin\Big(\sum_{k=j+1}^{t-1}\frac{1}{1+k^2} + \frac{3}{4}\frac{1}{(1+t^2)}\Big)\|\mathbf{u}\|_2.$$

Thus

$$\|\mathscr{P}_{S_j}(\mathbf{u})\|_2 \geq \frac{2}{\pi}\Big(\sum_{k=j+1}^{t-1}\frac{1}{1+k^2} + \frac{3}{4}\frac{1}{(1+t^2)}\Big)\|\mathbf{u}\|_2.$$

It is clear that $\frac{2}{\pi}\Big(\sum_{k=j+1}^{t-1}\frac{1}{1+k^2} + \frac{3}{4}\frac{1}{(1+t^2)}\Big) \geq \frac{1}{2\pi}\frac{1}{1+j^2}$, whence the lemma is proved. □

We now consider the general S_t^d's.

Lemma 7.2. *For any* $\mathbf{q}_t \in S_t^d$ *and* $\mathbf{q}_j \in S_j^d$, $j < t$ *satisfying* $\mathbf{q}_t^\top \mathbf{q}_j \geq 0$, *we have*

$$\sum_{k=j+1}^{t}\frac{1}{1+k^2} < \angle(\mathbf{q}_t, \mathbf{q}_j) \leq \frac{\pi}{2}.$$

Proof. Let $\hat{\mathbf{q}}_t$ and $\hat{\mathbf{q}}_j$ be two unit vectors in $\mathbb{R}^d$ with $\hat{\mathbf{q}}_t^\top \hat{\mathbf{q}}_j > 0$. Then we have

$$\cos\Big(\angle\Big(\begin{pmatrix}\hat{\mathbf{q}}_t\\ t\hat{\mathbf{q}}_t\end{pmatrix}, \begin{pmatrix}\hat{\mathbf{q}}_j\\ j\hat{\mathbf{q}}_j\end{pmatrix}\Big)\Big) = \begin{pmatrix}\hat{\mathbf{q}}_t\\ t\hat{\mathbf{q}}_t\end{pmatrix}^\top \begin{pmatrix}\hat{\mathbf{q}}_j\\ j\hat{\mathbf{q}}_j\end{pmatrix} / \Big(\Big\|\begin{pmatrix}\hat{\mathbf{q}}_t\\ t\hat{\mathbf{q}}_t\end{pmatrix}\Big\|_2 \Big\|\begin{pmatrix}\hat{\mathbf{q}}_j\\ j\hat{\mathbf{q}}_j\end{pmatrix}\Big\|_2\Big) = \frac{(1+tj)\hat{\mathbf{q}}_t^\top \hat{\mathbf{q}}_j}{\sqrt{(1+t^2)(1+j^2)}}.$$

Thus, introducing $\mathbf{q}_t = \begin{pmatrix}\hat{\mathbf{q}}_t\\ t\hat{\mathbf{q}}_t\end{pmatrix}$ and $\mathbf{q}_j = \begin{pmatrix}\hat{\mathbf{q}}_j\\ j\hat{\mathbf{q}}_j\end{pmatrix}$ yields

$$0 \leq \cos(\angle(\mathbf{q}_t, \mathbf{q}_j)) \leq \frac{1+tj}{\sqrt{(1+t^2)(1+j^2)}}.$$

Let $\mathbf{q} = \mathbf{q}_t - \mathbf{q}_j$. Then, it follows that

$$\|\mathbf{q}\|_2 = \sqrt{\|\mathbf{q}_t\|_2^2 + \|\mathbf{q}_j\|_2^2 - 2\|\mathbf{q}_t\|_2\|\mathbf{q}_j\|_2 \cos(\angle(\mathbf{q}_t, \mathbf{q}_j))} \geq t - j.$$

Hence, by considering the two-dimensional space spanned by $\mathbf{q}_t, \mathbf{q}_j$ and using the same idea as the one in proving (7.4) and (7.5), we show that for any $\mathbf{q}_t \in S_t^d$ and $\mathbf{q}_j \in S_j^d$, $j < t$, we have

$$\sum_{k=j+1}^{t}\frac{1}{1+k^2} < \angle(\mathbf{q}_t, \mathbf{q}_j) \leq \frac{\pi}{2}.$$

□

We now extend Lemma 7.1 to the multi-dimensional case.

Lemma 7.3. *For $t = 0, 1, \cdots$ and $\mathbf{u} \in \mathbb{R}^{2d}$, if*

$$||\mathscr{P}_{S_t^d}(\mathbf{u})||_2 < \frac{1}{2\pi(1+t^2)}||\mathbf{u}||_2,$$

then for $0 \le j < t$, we have

$$||\mathscr{P}_{S_j^d}(\mathbf{u})||_2 \ge \frac{1}{2\pi(1+j^2)}||\mathbf{u}||_2.$$

Proof. For a fixed $\mathbf{u} \in \mathbb{R}^{2d}$, let

$$\mathbf{q}_t = \mathscr{P}_{S_t^d}(\mathbf{u})/||\mathscr{P}_{S_t^d}(\mathbf{u})||_2,$$

and

$$\mathbf{q}_j = \mathscr{P}_{s_j^d}(\mathbf{u})/||\mathscr{P}_{s_j^d}(\mathbf{u})||_2,$$

if $\mathbf{q}_t^\top \mathbf{q}_j \ge 0$. Otherwise, set $\mathbf{q}_j = -\mathscr{P}_{s_j^d}(\mathbf{u})/||\mathscr{P}_{s_j^d}(\mathbf{u})||_2$. Under the condition stated in the lemma, we have $|\mathbf{u}^\top \mathbf{q}_t| < \frac{1}{2\pi(1+t^2)}||\mathbf{u}||_2$. Also, by Lemma 7.2 we get

$$\frac{\pi}{2} \ge \angle(\mathbf{q}_t, \mathbf{q}_j) > \sum_{k=j+1}^{t} \frac{1}{1+k^2}.$$

Considering the two-dimensional space spanned by $\mathbf{q}_t, \mathbf{q}_j$, similarly to proof of Lemma 7.1, we obtain

$$\mathbf{u}^\top \mathbf{q}_j \ge \frac{1}{2\pi(1+j^2)}||\mathbf{u}||_2,$$

which implies that $||\mathscr{P}_{S_j^d}(\mathbf{u})||_2 \ge \frac{1}{2\pi(1+j^2)}||\mathbf{u}||_2$. □

We next present two auxiliary lemmas that are used in the proof of main results.

Lemma 7.4. *For a vector $\mathbf{u} \in \mathbb{R}^2$, and two unit vectors $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^2$ satisfying $0 \le |\mathbf{q}_1^\top \mathbf{q}_2| \le \cos\theta$, we have*

$$|\mathbf{q}_1^\top \mathbf{u}|^2 + |\mathbf{q}_2^\top \mathbf{u}|^2 \ge (1-\cos(\theta))||\mathbf{u}||_2^2. \tag{7.7}$$

Proof. We have

$$\left|\left|(\mathbf{q}_1 \cdot \mathbf{u},\ \mathbf{q}_2 \cdot \mathbf{u})^\top\right|\right|_2^2 = \left|\left|\begin{pmatrix}\mathbf{q}_1^\top \\ \mathbf{q}_2^\top\end{pmatrix} \cdot \mathbf{u}\right|\right|_2^2 \ge \sigma_{\min}^2(\begin{pmatrix}\mathbf{q}_1^\top \\ \mathbf{q}_2^\top\end{pmatrix})||\mathbf{u}||_2^2 \ge (1-\cos\theta)||\mathbf{u}||_2^2, \tag{7.8}$$

where the last inequality follows from calculating $\sigma_{\min}(\begin{pmatrix}\mathbf{q}_1^\top \\ \mathbf{q}_2^\top\end{pmatrix})$. □

Lemma 7.5. *For a vector $\mathbf{u} \in \mathbb{R}^d$, and for spaces S_t^d, S_j^d, $j < t$, we have*

$$\left|\left|\mathscr{P}_{S_t^d}(\mathbf{u})\right|\right|_2^2 + \left|\left|\mathscr{P}_{S_j^d}(\mathbf{u})\right|\right|_2^2 \ge \Big(1-\cos(\theta)\Big)||\mathbf{u}||_2^2,$$

where $\theta = \sum_{k=j+1}^{t} \frac{1}{1+k^2}$.

Proof. We first construct the basis of S_t^d and S_j^d. Let

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{e}_d := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

It is easy to verify that the vectors

$$\mathbf{e}_{1,t} := \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} \mathbf{e}_1 \\ t\mathbf{e}_1 \end{pmatrix}, \mathbf{e}_{2,t} := \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} \mathbf{e}_2 \\ t\mathbf{e}_2 \end{pmatrix}, \cdots, \mathbf{e}_{d,t} := \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} \mathbf{e}_d \\ t\mathbf{e}_d \end{pmatrix},$$

form an orthonormal basis of S_t^d. In the same manner, the vectors

$$\mathbf{e}_{1,j} := \frac{1}{\sqrt{1+j^2}} \begin{pmatrix} \mathbf{e}_1 \\ j\mathbf{e}_1 \end{pmatrix}, \mathbf{e}_{2,j} := \frac{1}{\sqrt{1+j^2}} \begin{pmatrix} \mathbf{e}_2 \\ j\mathbf{e}_2 \end{pmatrix}, \cdots, \mathbf{e}_{d,j} := \frac{1}{\sqrt{1+j^2}} \begin{pmatrix} \mathbf{e}_d \\ j\mathbf{e}_d \end{pmatrix},$$

form an orthonormal basis of S_j^d. Also, we have

$$|\angle(\mathbf{e}_{p,t}, \mathbf{e}_{q,j})| = \frac{\pi}{2}, \ p \neq q, \text{ and } \sum_{k=j+1}^{t} \frac{1}{1+k^2} < \angle(\mathbf{e}_{q,t}, \mathbf{e}_{q,j}) < \frac{\pi}{2}, \tag{7.9}$$

where the second inequality is from Lemma 7.2. Thus, if we denote by V_q the two-dimensional space spanned by $\{\mathbf{e}_{q,t}, \mathbf{e}_{q,j}\}$, then the V_q's are orthogonal to each other. Moreover, we have

$$\begin{aligned}
&\left|\left|\mathscr{P}_{S_t^d}(\mathbf{u})\right|\right|_2^2 + \left|\left|\mathscr{P}_{S_j^d}(\mathbf{u})\right|\right|_2^2 = \left|\mathbf{u}^\top \mathbf{e}_{1,t}\right|^2 + \cdots + \left|\mathbf{u}^\top \mathbf{e}_{d,t}\right|^2 + \left|\mathbf{u}^\top \mathbf{e}_{1,j}\right|^2 + \cdots + \left|\mathbf{u}^\top \mathbf{e}_{d,j}\right|^2 \\
=&\left(\left|\mathbf{u}^\top \mathbf{e}_{1,t}\right|^2 + \left|\mathbf{u}^\top \mathbf{e}_{1,j}\right|^2\right) + \left(\left|\mathbf{u}^\top \mathbf{e}_{2,t}\right|^2 + \left|\mathbf{u}^\top \mathbf{e}_{2,j}\right|^2\right) + \cdots + \left(\left|\mathbf{u}^\top \mathbf{e}_{d,t}\right|^2 + \left|\mathbf{u}^\top \mathbf{e}_{d,j}\right|^2\right) \\
\geq&\left(1 - \cos\left(\sum_{k=j+1}^{t} \frac{1}{1+k^2}\right)\right) \sum_{q=1}^{d} \left|\left|\mathscr{P}_{V_q}(\mathbf{u})\right|\right|_2^2 \quad \left(\text{by Lemma 7.4 and (7.9)}\right) \\
=&\left(1 - \cos\left(\sum_{k=j+1}^{t} \frac{1}{1+k^2}\right)\right) ||\mathbf{u}||_2^2 \quad \left(\text{since } V_q\text{'s are orthogonal to each other}\right).
\end{aligned}$$

□

7.2 Results for the static super-resolution problem

In this subsection, we review and restate the stability results in [33] for the static super-resolution problem in multi-dimensional spaces. These results are useful to prove stability results for the dynamic super-resolution problem.

In the super-resolution problem of single snapshot case, the source is the discrete measure $\mu = \sum_{j=1}^{n} a_j \delta_{\mathbf{y}_j}, \mathbf{y}_j \in \mathbb{R}^k$ with $\min_{1\leq j\leq n} |a_j| = m_{\min} > 0$. The measurement $\mathbf{Y}$ is the noisy Fourier data of μ in a bounded domain:

$$\mathbf{Y}(\boldsymbol{\omega}) = \sum_{j=1}^{n} a_j e^{i\mathbf{y}_j^\top \boldsymbol{\omega}} + \mathbf{W}(\boldsymbol{\omega}), \quad ||\boldsymbol{\omega}||_2 \leq \Omega, \boldsymbol{\omega} \in \mathbb{R}^k, \tag{7.10}$$

with $|\mathbf{W}(\boldsymbol{\omega})| < \sigma$. The inverse problem is to recover the discrete measure from the set of σ-admissible measures of $\mathbf{Y}$ defined below.

Definition 7.1. *Given the measurement* $\mathbf{Y}$*, we say that* $\mu = \sum_{j=1}^{k} \hat{a}_j \delta_{\hat{\mathbf{y}}_j}, \hat{\mathbf{y}}_j \in \mathbb{R}^k$ *is a* σ*-admissible measure of* $\mathbf{Y}$ *if*

$$\left|\sum_{j=1}^{k} \hat{a}_j e^{i\hat{\mathbf{y}}_j^{\top}\boldsymbol{\omega}} - \mathbf{Y}(\boldsymbol{\omega})\right| < \sigma, \ ||\boldsymbol{\omega}||_2 \leq \Omega.$$

From [33], we have the following stability results for the recovery of source number and locations.

Theorem 7.1. *Let the measurement* $\mathbf{Y}$ *in (7.10) be generated by a* n*-sparse measure* $\mu = \sum_{j=1}^{n} a_j \delta_{\mathbf{y}_j}, \mathbf{y}_j \in B^k_{\frac{(n-1)\pi}{2\Omega}}(\mathbf{0})$*. Let* $n \geq 2$ *and assume that the following separation condition is satisfied*

$$\min_{p\neq j, 1\leq p,j\leq n} \left|\left|\mathbf{y}_p - \mathbf{y}_j\right|\right|_2 \geq \frac{4.4\pi e\,(\pi/2)^{k-1}(n(n-1)/\pi)^{\xi(k-1)}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}, \tag{7.11}$$

where $\xi(k-1)$ *is defined by (2.1). Then there does not exist any* σ*-admissible measures of* $\mathbf{Y}$ *with less than* n *supports.*

Theorem 7.2. *Let* $n \geq 2$*. Let the measurement* $\mathbf{Y}$ *in (7.10) be generated by a* n*-sparse measure* $\mu = \sum_{j=1}^{n} a_j \delta_{\mathbf{y}_j}, \mathbf{y}_j \in B^k_{\frac{(n-1)\pi}{2\Omega}}(\mathbf{0})$ *in the* k*-dimensional space. Assume that*

$$d_{\min} := \min_{p\neq j} \left|\left|\mathbf{y}_p - \mathbf{y}_j\right|\right|_2 \geq \frac{5.88\pi e 4^{k-1}\Big((n+2)(n+1)/2\Big)^{\xi(k-1)}}{\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}, \tag{7.12}$$

where $\xi(k-1)$ *is defined as in (2.1). If* $\hat{\mu} = \sum_{j=1}^{n} \hat{a}_j \delta_{\hat{\mathbf{y}}_j}$ *supported on* $B^k_{\frac{(n-1)\pi}{2\Omega}}(\mathbf{0})$ *is a* σ*-admissible measure of* $\mathbf{Y}$*, then after reordering the* $\hat{y}_j$*'s, we have*

$$\left|\left|\hat{\mathbf{y}}_j - \mathbf{y}_j\right|\right|_2 < \frac{d_{\min}}{2}, \quad j = 1, \cdots, n.$$

Moreover, we have

$$\left|\left|\hat{\mathbf{y}}_j - \mathbf{y}_j\right|\right|_2 \leq \frac{C(k,n)}{\Omega} SRF^{2n-2}\frac{\sigma}{m_{\min}}, \quad 1 \leq j \leq n, \tag{7.13}$$

where $SRF := \frac{\pi}{d_{\min}\Omega}$ *is the super-resolution factor and*

$$C(k,n) = \big(4^{k-1}((n+2)(n+1)/2)^{\xi(k-1)}\big)^{2n-1} n 2^{4n-2} e^{2n} \pi^{-\frac{1}{2}}.$$

Remark 7.1. *Note that here we have* $(n+2)(n+1)/2$ *as a factor in the separation condition (7.12), which is different from the one in [33]. This is due to a minor error in [33] where* $(n+2)(n-1)/2$ *is used instead of* $(n+2)(n+1)/2$*.*

7.3 PROOF OF THEOREM 2.1

Proof. Note that there are at most $\frac{n(n-1)}{2}$ different vectors of the form $\mathbf{u}_{pq} = \boldsymbol{\alpha}_p - \boldsymbol{\alpha}_q, p < q$ with $\boldsymbol{\alpha}_p = \begin{pmatrix} \mathbf{y}_p \\ \tau \mathbf{v}_p \end{pmatrix}$. Because we have

$$\begin{aligned} d_{\min} :=& \min_{p\neq q} \left|\left|\mathbf{u}_{pq}\right|\right|_2 = \min_{p\neq q} \left|\left|\boldsymbol{\alpha}_p - \boldsymbol{\alpha}_q\right|\right|_2 \\ \geq& \frac{8.8e\pi^2\sqrt{(\frac{(n-1)n}{2})^2+1}(\pi/2)^{d-1}(n(n-1)/\pi)^{\xi(d-1)}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}, \end{aligned} \tag{7.14}$$

we also have,

$$\min_{p\neq q}\left|\left|\mathbf{u}_{p,q}\right|\right|_2 \geq \frac{8.8e\pi^2\sqrt{t^2+1}(\pi/2)^{d-1}(n(n-1)/\pi)^{\xi(d-1)}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}},\ \forall t \leq \frac{(n-1)n}{2}. \tag{7.15}$$

Let $\Delta = \frac{4.4e\pi(\pi/2)^{d-1}(n(n-1)/\pi)^{\xi(d-1)}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}$. We define that $N(\mathbf{u}_{pq},\Delta) := \left\{S_t^d \middle| S_t^d \in (7.2), t = 0,\cdots,\frac{(n-1)n}{2}, ||\mathscr{P}_{S_t^d}(\mathbf{u}_{pq})||_2 < \frac{1}{\sqrt{t^2+1}}\Delta\right\}$. If $S_t^d \in N(\mathbf{u}_{pq},\Delta)$, by (7.15) we have $||\mathscr{P}_{S_t^d}(\mathbf{u}_{pq})||_2 < \frac{1}{2\pi(1+t^2)}||\mathbf{u}_{pq}||_2$. By Lemma 7.3 and (7.15), we have, for $j < t$,

$$\left|\left|\mathscr{P}_{S_j^d}(\mathbf{u}_{pq})\right|\right|_2 \geq \frac{1}{2\pi(1+j^2)}||\mathbf{u}_{pq}||_2 \geq \frac{1}{\sqrt{j^2+1}}\Delta.$$

Thus if $S_t^d \in N(\mathbf{u}_{pq},\Delta)$, for $j < t$, $S_j^d \not\in N(\mathbf{u}_{pq},\Delta)$. Now we start by considering $t = \frac{(n-1)n}{2}$. If for some $\mathbf{u}_{p_0q_0}$, $S^d_{\frac{(n-1)n}{2}}$ is in $N(\mathbf{u}_{p_0q_0},\Delta)$, then other $S_j^d, j < \frac{(n-1)n}{2}$ are all not in $N(\mathbf{u}_{p_0q_0},\Delta)$. If there is no such $\mathbf{u}_{p_0q_0}$ so that $S^d_{\frac{(n-1)n}{2}} \in N(\mathbf{u}_{p_0q_0},\Delta)$, then $\left|\left|\mathscr{P}_{S^d_{\frac{(n-1)n}{2}}}(\mathbf{u}_{pq})\right|\right|_2 \geq \frac{1}{\sqrt{(\frac{n(n-1)}{2})^2+1}}\Delta$ for all $\mathbf{u}_{pq}$'s. We can continue the process for $t = \frac{(n-1)n}{2}-1,\cdots,t=0$. Since we have $1+\frac{(n-1)n}{2}$ different S_t^d's and at most $\frac{n(n-1)}{2}$ different $\mathbf{u}_{pq}$'s, and

$$1+\frac{n(n-1)}{2}-\frac{n(n-1)}{2} = 1,$$

by the above process, we can find at least one S_t^d so that

$$\left|\left|\mathscr{P}_{S_t^d}(\mathbf{u}_{pq})\right|\right|_2 \geq \frac{1}{\sqrt{t^2+1}}\Delta \tag{7.16}$$

for all $\mathbf{u}_{pq}$'s. We consider this specific t in the following argument. In the space S_t^d, we have the noisy measurement that

$$\sum_{j=1}^n a_j e^{i\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)^\top \boldsymbol{\omega}_t} + \mathbf{W}(\boldsymbol{\omega}_t),$$

where $||\boldsymbol{\omega}_t||_2 \leq \sqrt{1+t^2}\Omega$. On the other hand, since $\boldsymbol{\alpha}_j \in B^{2d}_{\frac{\pi}{(n+1)\Omega}}(\mathbf{0})$, we have

$$||\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)||_2 \leq ||\boldsymbol{\alpha}_j||_2 \leq \frac{\pi}{(n+1)\Omega} = \frac{(n-1)\pi}{(n-1)(n+1)\Omega} \leq \frac{(n-1)\pi}{2\sqrt{1+t^2}\Omega},$$

where the last inequality is because $t \le \frac{n(n-1)}{2}$ and $\frac{1}{2\sqrt{1+t^2}} \ge \frac{1}{2\sqrt{1+(\frac{n(n-1)}{2})^2}} \ge \frac{1}{(n-1)(n+1)}$. Also, by (7.16) we have

$$\min_{p\neq q}\left\|\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_p)-\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_q)\right\|_2 \ge \frac{4.4e\pi(\pi/2)^{d-1}(n(n-1)/\pi)^{\xi(d-1)}}{\sqrt{t^2+1}\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-2}}.$$

Thus now we can apply Theorem 7.1. By Theorem 7.1, there is no $(n-1)$ vectors $\tilde{\boldsymbol{\alpha}}_j \in S_t^d$ and $(n-1)$ $\tilde{a}_j$'s so that

$$\left|\sum_{j=1}^{n-1}\tilde{a}_j e^{i\tilde{\boldsymbol{\alpha}}_j^\top \boldsymbol{\omega}_t}-\sum_{j=1}^{n} a_j e^{i\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)^\top \boldsymbol{\omega}_t}+\mathbf{W}(\boldsymbol{\omega}_t)\right|<\sigma, \quad ||\boldsymbol{\omega}_t||_2 \le \sqrt{t^2+1}\Omega.$$

Based on (7.3) for measurements $\mathbf{Y}_t$'s, the above argument proves that there is no such $(n-1)$-sparse σ-admissible parameter set $\{(\hat{a}_j,\hat{\mathbf{y}}_j,\hat{\mathbf{v}}_j)\}_{j=1}^{n-1}$ of $\mathbf{Y}_t$'s. □

7.4 Proof of Theorem 2.2

Proof. Note that there are at most $\frac{n(n-1)}{2}$ different vectors of the form $\mathbf{u}_{pq}=\boldsymbol{\alpha}_p-\boldsymbol{\alpha}_q, p<q$ with $\boldsymbol{\alpha}_p=\begin{pmatrix}\mathbf{y}_p\\ \tau\mathbf{v}_p\end{pmatrix}$. Since

$$\begin{aligned} d_{\min} &:= \min_{p\neq q}\left\|\mathbf{u}_{pq}\right\|_2 = \min_{p\neq q}\left\|\boldsymbol{\alpha}_p-\boldsymbol{\alpha}_q\right\|_2 \\ &\ge \frac{11.76e\pi^2\sqrt{(\frac{(n+1)n}{2})^2+1}4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}}, \end{aligned} \tag{7.17}$$

we also obtain that for all $t \le \frac{(n+1)n}{2}$,

$$\min_{p\neq q}\left\|\mathbf{u}_{pq}\right\|_2 = \min_{p\neq q}\left\|\boldsymbol{\alpha}_p-\boldsymbol{\alpha}_j\right\|_2 \ge \frac{11.76e\pi^2\sqrt{t^2+1}4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}{\Omega}(\frac{\sigma}{m_{\min}})^{\frac{1}{2n-1}}. \tag{7.18}$$

Let $\Delta = \frac{5.88e\pi 4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}{\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}$. We define $N(\mathbf{u}_{pq},\Delta)$ as $\left\{S_t^d | S_t^d \in (7.2), t=0,\cdots,\frac{(n+1)n}{2}, ||\mathscr{P}_{S_t^d}(\mathbf{u}_{pq})||_2 < \frac{1}{\sqrt{t^2+1}}\Delta\right\}$. If $S_t^d \in N(\mathbf{u}_{pq},\Delta)$, by (7.18) we have $||\mathscr{P}_{S_t^d}(\mathbf{u}_{pq})||_2 < \frac{1}{2\pi(1+t^2)}||\mathbf{u}_{pq}||_2$. By Lemma 7.3 and (7.18), we have, for $j<t$,

$$\left\|\mathscr{P}_{S_j^d}(\mathbf{u}_{pq})\right\|_2 \ge \frac{1}{2\pi(1+j^2)}||\mathbf{u}_{pq}||_2 \ge \frac{1}{\sqrt{j^2+1}}\Delta.$$

Thus if $S_t^d \in N(\mathbf{u}_{pq},\Delta)$, for $j<t$, $S_j^d \notin N(\mathbf{u}_{pq},\Delta)$. Again, we start by considering $t=\frac{(n+1)n}{2}$. If for some $\mathbf{u}_{pq}$, $S^d_{\frac{(n+1)n}{2}}$ is in $N(\mathbf{u}_{pq},\Delta)$, then other $S_j^d, j<\frac{n(n+1)}{2}$ are all not in $N(\mathbf{u}_{pq},\Delta)$. If there is no such $\mathbf{u}_{pq}$ so that $S^d_{\frac{(n+1)n}{2}} \in N(\mathbf{u}_{pq},\Delta)$, then $\left\|\mathscr{P}_{S^d_{\frac{(n+1)n}{2}}}(\mathbf{u}_{pq})\right\|_2 \ge \frac{1}{\sqrt{(\frac{n(n+1)}{2})^2+1}}\Delta$ for all $\mathbf{u}_{pq}$'s.

We can continue the discussion for $t = \frac{n(n+1)}{2} - 1, \cdots, t = 0$. Since we have $1 + \frac{n(n+1)}{2}$ different S_t^d's and at most $\frac{n(n-1)}{2}$ different $\mathbf{u}_{pq}$'s, and

$$1 + \frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n+1,$$

by the above process, we can find at least $n+1$ different S_t^d's so that

$$\left|\left|\mathscr{P}_{S_t^d}(\mathbf{u}_{pq})\right|\right|_2 \geq \frac{1}{\sqrt{t^2+1}}\Delta \tag{7.19}$$

for all $\mathbf{u}_{pq}$'s. We only consider these t's in the sequel.

In the space S_t^d, we have the noisy measurement given by

$$\sum_{j=1}^{n} e^{i\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)^\top \boldsymbol{\omega}_t} + \mathbf{W}(\boldsymbol{\omega}_t),$$

where $\boldsymbol{\omega}_t = \begin{pmatrix} \boldsymbol{\omega} \\ t\boldsymbol{\omega} \end{pmatrix}$,$||\boldsymbol{\omega}_t||_2 \leq \sqrt{1+t^2}\Omega$. On the other hand, since $\boldsymbol{\alpha}_j \in B^{2d}_{\frac{(n-1)\pi}{n(n+2)\Omega}}(\mathbf{0})$, we have

$$||\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)||_2 \leq ||\boldsymbol{\alpha}_j||_2 \leq \frac{(n-1)\pi}{n(n+2)\Omega} \leq \frac{(n-1)\pi}{2\sqrt{1+t^2}\Omega},$$

where the last inequality is because $t \leq \frac{n(n+1)}{2}$ and $\frac{1}{2\sqrt{1+t^2}} \geq \frac{1}{2\sqrt{1+(\frac{n(n+1)}{2})^2}} \geq \frac{1}{n(n+2)}$. Thus now we can apply Theorem 7.2. By Theorem 7.2, when (7.19) holds, i.e.,

$$\min_{p\neq q}\left|\left|\mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_p) - \mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_q)\right|\right|_2 \geq \frac{5.88e\pi 4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}{\sqrt{t^2+1}\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}} =: d_{\min,t},$$

we can conclude that for each t, we have a permutation τ_t of $\{1,\cdots,n\}$ so that

$$\begin{aligned}\left|\left|\mathscr{P}_{S_t^d}(\hat{\boldsymbol{\alpha}}_{\tau_t(j)}) - \mathscr{P}_{S_t^d}(\boldsymbol{\alpha}_j)\right|\right|_2 &\leq \frac{C(d,n)}{\sqrt{t^2+1}\Omega}\left(\frac{\pi}{d_{\min,t}\sqrt{t^2+1}\Omega}\right)^{2n-2}\frac{\sigma}{m_{\min}} \\ &= \frac{C(d,n)}{\sqrt{t^2+1}\Omega}\left(\frac{1}{5.88e4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}\right)^{2n-2}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}, \quad 1\leq j\leq n,\end{aligned} \tag{7.20}$$

where $C(d,n) = \left(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\right)^{(2n-1)} n2^{4n-2}e^{2n}\pi^{-\frac{1}{2}}$. Note that, for fixed j in (7.20), we have $n+1$ $\tau_t(j)$'s (since (7.19) holds for at least $n+1$ S_t^d's), while $\hat{\boldsymbol{\alpha}}_p$'s take at most n values. Therefore, by the pigeonhole principle, for each fixed $\boldsymbol{\alpha}_j$, we can find two different t's, say, t_1 and t_2, such that $\hat{\boldsymbol{\alpha}}_{\tau_{t_1}(j)} = \hat{\boldsymbol{\alpha}}_{\tau_{t_2}(j)} = \hat{\boldsymbol{\alpha}}_{p_j}$ for some p_j. Suppose $t_1 > t_2$, by (7.20) for t_1, t_2 and Lemma 7.5 we have

$$\left|\left|\hat{\boldsymbol{\alpha}}_{p_j} - \boldsymbol{\alpha}_j\right|\right|_2 \leq \frac{\sqrt{\frac{1}{t_1^2+1}+\frac{1}{t_2^2+1}}}{\sqrt{1-\cos\left(\sum_{k=t_2+1}^{t_1}\frac{1}{k^2+1}\right)}}\frac{C(d,n)}{\Omega}\left(\frac{1}{5.88e4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}\right)^{2n-2}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}.$$

Using the inequality $1-\cos x \geq \frac{1}{\pi}x^2$, we further obtain

$$\left|\left|\hat{\boldsymbol{\alpha}}_{p_j}-\boldsymbol{\alpha}_j\right|\right|_2 \leq \frac{\sqrt{\pi}\sqrt{\frac{1}{t_1^2+1}+\frac{1}{t_2^2+1}}}{\sum_{k=t_2+1}^{t_1}\frac{1}{k^2+1}}\frac{C(d,n)}{\Omega}\Big(\frac{1}{5.88e4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}\Big)^{2n-2}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.$$

Since

$$\frac{\sqrt{\frac{1}{t_1^2+1}+\frac{1}{t_2^2+1}}}{\sum_{k=t_2+1}^{t_1}\frac{1}{k^2+1}} < \frac{\sqrt{t_1^2+1}\sqrt{1+\frac{t_1^2+1}{t_2^2+1}}}{(t_1-t_2)} \leq \sqrt{6(t_1^2+1)},$$

and $t_1 \leq \frac{n(n+1)}{2}$, we have

$$\left|\left|\hat{\boldsymbol{\alpha}}_{p_j}-\boldsymbol{\alpha}_j\right|\right|_2 \leq \frac{\sqrt{6\pi}\sqrt{(\frac{n(n+1)}{2})^2+1}C(d,n)}{\Omega}\Big(\frac{1}{5.88e4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}\Big)^{2n-2}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}} \tag{7.21}$$

for $1 \leq j \leq n$. We next claim that

$$\left|\left|\hat{\boldsymbol{\alpha}}_{p_j}-\boldsymbol{\alpha}_j\right|\right|_2 < \frac{d_{\min}}{2}, \tag{7.22}$$

where $d_{\min}$ is defined in (7.17). Indeed, by $C(d,n) = \left(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\right)^{(2n-1)} n2^{4n-2}e^{2n}\pi^{-\frac{1}{2}}$ and a direct calculation, we can verify that

$$\begin{aligned}&\sqrt{6\pi}\sqrt{(\frac{n(n+1)}{2})^2+1}C(d,n)\Big(\frac{1}{5.88e4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}}\Big)^{2n-2}\\ <&\frac{1}{2}11.76e\pi^2\sqrt{(\frac{n(n+1)}{2})^2+1}4^{d-1}\Big((n+2)(n+1)/2\Big)^{\xi(d-1)},\end{aligned}$$

whence (7.22) follows.

So far, we have proved that for each $\boldsymbol{\alpha}_j$, there exists a point $\hat{\boldsymbol{\alpha}}_{p_j}$ satisfying $||\hat{\boldsymbol{\alpha}}_{p_j}-\boldsymbol{\alpha}_j||_2 < \frac{d_{\min}}{2}$. Thus for each $\boldsymbol{\alpha}_j$ there exists only one such $\hat{\boldsymbol{\alpha}}_{p_j} \in \{\hat{\boldsymbol{\alpha}}_1,\cdots,\hat{\boldsymbol{\alpha}}_n\}$. We can reorder the index so that

$$\left|\left|\hat{\boldsymbol{\alpha}}_j-\boldsymbol{\alpha}_j\right|\right|_2 < \frac{d_{\min}}{2},$$

and

$$\left|\left|\hat{\boldsymbol{\alpha}}_j-\boldsymbol{\alpha}_j\right|\right|_2 \leq \frac{\sqrt{6\pi}(2\pi)^{2n-2}((\frac{n(n+1)}{2})^2+1)^{\frac{2n-1}{2}}C(d,n)}{\Omega}\Big(\frac{\pi}{d_{\min}\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}}, \quad 1 \leq j \leq n,$$

which follows from (7.21) and (7.17). □

8 PROOFS OF THEOREMS 3.1 AND 3.2

8.1 LEMMAS FOR PROJECTION

In this subsection, we present some lemmas for projection in multi-dimensional spaces, which will help us to prove Theorems 3.1 and 3.2.

For a $\mathbf{v} \in \mathbb{R}^k$, we denote by $\mathbf{v}^\perp$ the $(k-1)$-dimensional orthogonal complement space of the one-dimensional space spanned by $\mathbf{v}$. We consider the unit sphere in $\mathbb{R}^k$ and the following spherical coordinates:

$$\begin{aligned} x_1(\Phi) &= \cos(\phi_1), \\ x_2(\Phi) &= \sin(\phi_1)\cos(\phi_2), \\ &\vdots \\ x_{k-1}(\Phi) &= \sin(\phi_1)\cdots\sin(\phi_{k-2})\cos(\phi_{k-1}), \\ x_k(\Phi) &= \sin(\phi_1)\cdots\sin(\phi_{k-2})\sin(\phi_{k-1}), \end{aligned} \tag{8.1}$$

where $\Phi = (\phi_1, \cdots, \phi_{k-1}) \in [0,\pi]^{k-2} \times [0, 2\pi)$. For $0 < \theta < \frac{\pi}{2}$ and $N = \lfloor \frac{\pi}{2\theta} \rfloor$, we let

$$\mathbf{v}_{\tau_1\cdots\tau_{k-1}} = \big(x_1(\Phi_{\tau_1\cdots\tau_{k-1}}), \cdots, x_k(\Phi_{\tau_1\cdots\tau_{k-1}})\big)^\top, \quad 1 \le \tau_j \le N, \tag{8.2}$$

where $\Phi_{\tau_1\cdots\tau_{k-1}} = (\tau_1\theta, \cdots, \tau_{k-1}\theta)$. It is obvious that $\Phi_{\tau_1\cdots\tau_{k-1}} \in [0, \frac{\pi}{2}]^{k-1}$ and $\mathbf{v}_{\tau_1\cdots\tau_{k-1}} \neq \mathbf{v}_{p_1\cdots p_{k-1}}$ if $(\tau_1, \cdots, \tau_{k-1}) \neq (p_1, \cdots, p_{k-1})$. There are N^{k-1} different unit vectors of the form (8.2).

Lemma 8.1. *For two different vectors* $\mathbf{v}_{\tau_1\cdots\tau_{k-1}} \neq \mathbf{v}_{p_1\cdots p_{k-1}}$ *in (8.2), we have*

$$0 \le \mathbf{v}_{\tau_1\cdots\tau_{k-1}} \cdot \mathbf{v}_{p_1\cdots p_{k-1}} \le \cos\theta. \tag{8.3}$$

Proof. See Lemma 6.3 in [33]. □

Lemma 8.2. *For a vector* $\mathbf{u} \in \mathbb{R}^k$*, suppose* $||\mathscr{P}_{\mathbf{v}^\perp_{\tau_1\cdots\tau_{k-1}}}(\mathbf{u})||_2 < \sin(\frac{\theta}{2})||\mathbf{u}||_2$ *with* $\mathbf{v}_{\tau_1\cdots\tau_{k-1}}$ *defined in (8.2), we have* $||\mathscr{P}_{\mathbf{v}^\perp_{p_1\cdots p_{k-1}}}(\mathbf{u})||_2 \ge \sin(\frac{\theta}{2})||\mathbf{u}||_2$ *for* $\mathbf{v}_{p_1\cdots p_{k-1}} \neq \mathbf{v}_{\tau_1\cdots\tau_{k-1}}$.

Proof. See Lemma 6.4 in [33]. □

Lemma 8.3. *Let* $k \ge 2$*. For a vector* $\mathbf{u} \in \mathbb{R}^k$*, and two unit vectors* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^k$ *satisfying* $0 \le \mathbf{v}_1 \cdot \mathbf{v}_2 \le \cos(\theta)$*, we have*

$$||\mathscr{P}_{\mathbf{v}_1^\perp}(\mathbf{u})||_2^2 + ||\mathscr{P}_{\mathbf{v}_2^\perp}(\mathbf{u})||_2^2 \ge (1 - \cos(\theta))||\mathbf{u}||_2^2. \tag{8.4}$$

Proof. See Lemma 3.3 in [33]. □

Lemma 8.4. *Suppose we have* $\frac{n(n-1)}{2}$ *different* $\mathbf{u}_{pq} \in \mathbb{R}^k$*, then there exists a one-dimensional space S so that for all* $\mathbf{u}_{pq}$*'s we have*

$$\Big|\Big|\mathscr{P}_S(\mathbf{u}_{pq})\Big|\Big|_2 \ge \frac{||\mathbf{u}_{pq}||_2}{(\pi/2)^{k-1}} \Big(\frac{\pi}{n(n-1)}\Big)^{\xi(k-1)},$$

where $\xi(k-1)$ *is defined as in (2.1).*

Proof. Let S_{k-1} be the unit sphere in $\mathbb{R}^k$. For a subspace S of $\mathbb{R}^k$ and $\mathbf{v} \in S$, we denote by $\mathbf{v}^\perp(S)$ the orthogonal complement space of $\mathbf{v}$ in S. For each $\mathbf{u} \in \mathbb{R}^k$, we let

$$N(\mathbf{u},\Delta) = \Big\{\mathbf{v}\Big|\mathbf{v} \in S_{k-1}, ||\mathscr{P}_{\mathbf{v}^\perp(\mathbb{R}^k)}(\mathbf{u})||_2 < ||\mathbf{u}||_2 \sin\Delta\Big\}. \tag{8.5}$$

Define area(A) as the area of set A and let $\Delta_k = \left(\frac{\pi}{n(n-1)}\right)^{\frac{1}{k-1}}$, we have

$$\text{area}\left(\cup_{pq} N(\mathbf{u}_{pq},\Delta_k)\right) \le \frac{n(n-1)}{2}\frac{2\text{area}(S_{k-2})}{k-1}\Delta_k^{k-1} = \frac{\pi\text{area}(S_{k-2})}{k-1} \le \text{area}(S_{k-1}),$$

where the first inequality is from Lemma 6.1 in [33] and the last one is from Lemma 6.2 in [33]. On the other hand, $\cup_{pq} N(\mathbf{u}_{pq},\Delta_k)$ is an open set in S_{k-1}. Thus $S_{k-1} \setminus \cup_{pq} N(\mathbf{u}_{pq},\Delta_k)$ is not empty. By the definition of $N(\mathbf{u}_{pq},\Delta_k)$, there exists a unit vector $\mathbf{v}_k \in \mathbb{R}^k$ such that

$$\left|\left|\mathscr{P}_{\mathbf{v}_k^\perp(\mathbb{R}^k)}(\mathbf{u}_{pq})\right|\right|_2 \ge ||\mathbf{u}_{pq}||_2 \sin\Delta_k \ge \frac{||\mathbf{u}_{pq}||_2}{\pi/2}\left(\frac{\pi}{n(n-1)}\right)^{\frac{1}{k-1}}$$

for all $\mathbf{u}_{pq}$'s. Note that $\mathscr{P}_{\mathbf{v}_k^\perp(\mathbb{R}^k)}(\mathbf{u}_{pq})$ are at most $n(n-1)$ different vectors in a $(k-1)$-dimensional space $\mathbf{v}_k^\perp(\mathbb{R}^k)$. Applying similar arguments as above to $\mathbf{v}_k^\perp(\mathbb{R}^k)$ and $\mathscr{P}_{\mathbf{v}_k^\perp(\mathbb{R}^k)}(\mathbf{u}_{pq})$'s, we can show that there exists a unit vector $\mathbf{v}_{k-1} \in \mathbf{v}_k^\perp(\mathbb{R}^k)$ such that for all $\mathbf{u}_{pq}$'s,

$$\left|\left|\mathscr{P}_{\mathbf{v}_{k-1}^\perp(\mathbf{v}_k^\perp(\mathbb{R}^k))}\left(\mathscr{P}_{\mathbf{v}_k^\perp(\mathbb{R}^k)}(\mathbf{u}_{pq})\right)\right|\right|_2 \ge \left|\left|\mathscr{P}_{\mathbf{v}_k^\perp(\mathbb{R}^k)}(\mathbf{u}_{pq})\right|\right|_2\left(\frac{2}{\pi}\right)\left(\frac{\pi}{n(n-1)}\right)^{\frac{1}{k-2}} \ge \frac{||\mathbf{u}_{pq}||_2}{(\pi/2)^2}\left(\frac{\pi}{n(n-1)}\right)^{\frac{1}{k-1}+\frac{1}{k-2}}.$$

On the other hand, since $\mathbf{v}_{k-1}^\perp(\mathbf{v}_k^\perp(\mathbb{R}^k))$ is a subspace of $\mathbf{v}_k^\perp(\mathbb{R}^k)$, we further obtain

$$\left|\left|\mathscr{P}_{\mathbf{v}_{k-1}^\perp(\mathbf{v}_k^\perp(\mathbb{R}^k))}(\mathbf{u}_{pq})\right|\right|_2 = \left|\left|\mathscr{P}_{\mathbf{v}_{k-1}^\perp(\mathbf{v}_k^\perp(\mathbb{R}^k))}\left(\mathscr{P}_{\mathbf{v}_k^\perp(\mathbb{R}^k)}(\mathbf{u}_{pq})\right)\right|\right|_2 \ge \frac{||\mathbf{u}_{pq}||_2}{(\pi/2)^2}\left(\frac{\pi}{n(n-1)}\right)^{\frac{1}{k-1}+\frac{1}{k-2}}.$$

Continuing the above arguments from dimension $(k-2)$ to dimension 2, we can find a unit vector $\mathbf{v}_1$ so that

$$\left|\left|\mathscr{P}_{\mathbf{v}_1^\perp(\cdots\mathbf{v}_{k-1}^\perp(\mathbf{v}_k^\perp(\mathbb{R}^k)))}(\mathbf{u}_{pq})\right|\right|_2 \ge \frac{||\mathbf{u}_{pq}||_2}{(\pi/2)^{k-1}}\left(\frac{\pi}{n(n-1)}\right)^{\xi(k-1)},$$

for all $\mathbf{u}_{pq}$'s. If we define S in the lemma as the one-dimensional space $\mathbf{v}_1^\perp(\cdots\mathbf{v}_{k-1}^\perp(\mathbf{v}_k^\perp(\mathbb{R}^k)))$, then the proof is completed. □

Lemma 8.5. *Suppose we have $\frac{n(n-1)}{2}$ different $\mathbf{u}_{pq} \in \mathbb{R}^k$, then there exist $n+1$ $(k-1)$-dimensional spaces $S_j^{k-1} = \mathbf{v}_j^\perp$, $j = 1,\cdots,n+1$ with $0 \le \mathbf{v}_p^\top \mathbf{v}_j \le \cos\left(\frac{\pi}{4}\left(\frac{2}{(n+1)(n+2)}\right)^{\frac{1}{k-1}}\right)$, $p \ne j$, so that for all $\mathbf{u}_{pq}$'s we have*

$$\left|\left|\mathscr{P}_{S_j^{k-1}}(\mathbf{u}_{pq})\right|\right|_2 \ge \frac{||\mathbf{u}_{pq}||_2}{4}\left(\frac{2}{(n+1)(n+2)}\right)^{\frac{1}{k-1}}.$$

Proof. For $k \ge 2$, let $\Delta_k = \frac{\pi}{8}\left(\frac{2}{(n+1)(n+2)}\right)^{\frac{1}{k-1}}$ and $\theta_k = 2\Delta_k$. First, we consider the vectors (8.2) in $\mathbb{R}^k$ with $\theta = \theta_k$ and $N = N_k = \lfloor\frac{\pi}{2\theta_k}\rfloor$. We denote $N(\mathbf{u},\Delta) = \Big\{\mathbf{v}\Big|\mathbf{v} \in \mathbb{R}^k, ||\mathbf{v}||_2 = 1, ||\mathscr{P}_{\mathbf{v}^\perp}(\mathbf{u})||_2 <$

$||\mathbf{u}||_2 \sin\Delta\Big\}$. Thus, by Lemma 8.2, each $N(\mathbf{u}_{pq}, \Delta_k)$ contains at most one unit vector in (8.2). Next recall that there are N_k^{k-1} different vectors in (8.2), where $N_k = \lfloor \frac{\pi}{2\theta_k} \rfloor \geq \frac{\pi}{2\theta_k} - 1$. Since

$$\theta_k = 2\Delta_k = \frac{\pi}{4}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{k-1}},$$

we have

$$N_k^{k-1} \geq \Big(\frac{\pi}{2\theta_k} - 1\Big)^{k-1} = \Big(2(\frac{(n+1)(n+2)}{2})^{\frac{1}{k-1}} - 1\Big)^{k-1} \geq \Big((\frac{(n+1)(n+2)}{2})^{\frac{1}{k-1}}\Big)^{k-1} = \frac{(n+1)(n+2)}{2}.$$

Note that $\frac{(n+1)(n+2)}{2} - \frac{n(n-1)}{2} \geq n+1$, we can at least find $(n+1)$ vectors $\mathbf{v}_j$'s in (8.2) so that $||\mathscr{P}_{\mathbf{v}_j^\perp}(\mathbf{u}_{pq})||_2 \geq ||\mathbf{u}_{pq}||_2 \sin\Delta_k$ for all $\mathbf{u}_{pq}$'s. Thus we have

$$\Big|\Big|\mathscr{P}_{\mathbf{v}_j^\perp}(\mathbf{u}_{pq})\Big|\Big|_2 \geq \frac{2||\mathbf{u}_{pq}||_2}{\pi}\frac{\pi}{8}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{k-1}} = \frac{||\mathbf{u}_{pq}||_2}{4}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{k-1}}$$

for all $\mathbf{u}_{pq}$'s. On the other hand, by Lemma 8.1 we have $0 \leq \mathbf{v}_p^\top \mathbf{v}_j \leq \cos\Big(\frac{\pi}{4}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{k-1}}\Big)$ for $p \neq j$. □

Lemma 8.6. *Define*

$$d_{\min,k} = \frac{11.76\pi e 4^{k-1}((n+2)(n+1)/2)^{\xi(k-1)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.$$

For a vector $\mathbf{u} \in \mathbb{R}^k$, *suppose we can find two* $(k-1)$*-dimensional spaces* $S_j^{k-1} = \mathbf{v}_j^\perp$, $j = 1,2$ *with* $0 \leq \mathbf{v}_1^\top \mathbf{v}_2 \leq \cos\Big(\frac{\pi}{4}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{k-1}}\Big)$ *so that*

$$\Big|\Big|\mathscr{P}_{S_j^{k-1}}(\mathbf{u})\Big|\Big|_2 < \frac{1}{2}d_{\min,k-1}, \quad j = 1,2,$$

and

$$\Big|\Big|\mathscr{P}_{S_j^{k-1}}(\mathbf{u})\Big|\Big|_2 \leq \frac{C(k-1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,k-1}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}}, \quad j = 1,2,$$

where

$$C(k-1,n) = \big(4^{k-2}((n+2)(n+1)/2)^{\xi(k-2)}\big)^{(2n-1)} n 2^{6n-3} e^{2n} \pi^{-\frac{1}{2}}.$$

Then we have

$$\Big|\Big|\mathbf{u}\Big|\Big|_2 < \frac{1}{2}d_{\min,k},$$

and

$$\Big|\Big|\mathbf{u}\Big|\Big|_2 \leq \frac{C(k,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,k}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},$$

where

$$C(k,n) = \big(4^{k-1}((n+2)(n+1)/2)^{\xi(k-1)}\big)^{(2n-1)} n 2^{6n-3} e^{2n} \pi^{-\frac{1}{2}}.$$

Proof. Let $\theta=\frac{\pi}{4}\left(\frac{2}{(n+1)(n+2)}\right)^{\frac{1}{k-1}}$. By condition for $\mathbf{v}_1,\mathbf{v}_2$ in the lemma and Lemma 8.3, we have

$$||\mathscr{P}_{\mathbf{v}_1^{\perp}}(\mathbf{u})||_2^2+||\mathscr{P}_{\mathbf{v}_2^{\perp}}(\mathbf{u})||_2^2\geq(1-\cos\theta)||\mathbf{u}||_2^2. \tag{8.6}$$

Thus

$$||\mathbf{u}||_2\leq\frac{\sqrt{2}}{\sqrt{1-\cos\theta}}\frac{C(k-1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,k-1}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}}.$$

Using the inequality $1-\cos\theta\geq\frac{2}{\pi^2}\theta^2=\frac{1}{8}\big(\frac{2}{(n+2)(n+1)}\big)^{\frac{2}{k-1}}$, we further obtain

$$\Big|\Big|\mathbf{u}\Big|\Big|_2\leq\frac{4((n+2)(n+1)/2)^{\frac{1}{k-1}}C(k-1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,k-1}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},\quad 1\leq j\leq n. \tag{8.7}$$

We next claim that

$$\Big|\Big|\mathbf{u}\Big|\Big|_2<\frac{d_{\min,k}}{2}.$$

Indeed, by a direct calculation, we can verify that

$$\begin{aligned}&4((n+2)(n+1)/2)^{\frac{1}{k-1}}C(k-1,n)\Big(\frac{1}{11.76e4^{k-2}((n+2)(n+1)/2)^{\xi(k-2)}}\Big)^{2n-2}\\<&\frac{1}{2}11.76\pi e4^{(k-1)}((n+2)(n+1)/2)^{\xi(k-1)}.\end{aligned}$$

On the other hand, we have

$$(\frac{\pi}{d_{\min,k-1}T\Omega})^{2n-2}\frac{\sigma}{m_{\min}}\leq\Big(\frac{1}{11.76e4^{k-2}((n+2)(n+1)/2)^{\xi(k-2)}}\Big)^{2n-2}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.$$

Therefore, we have

$$\begin{aligned}&\frac{4((n+2)(n+1)/2)^{\frac{1}{k-1}}C(k-1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,k-1}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}}\\<&\frac{1}{2}\frac{11.76\pi e4^{k-1}((n+2)(n+1)/2)^{\xi(k-1)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.\end{aligned}$$

This is $\Big|\Big|\mathbf{u}\Big|\Big|_2<\frac{1}{2}d_{\min,k}$. Moreover, we have

$$\Big|\Big|\mathbf{u}\Big|\Big|_2\leq\frac{(4((n+2)(n+1)/2)^{\frac{1}{k-1}})^{2n-1}C(k-1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,k}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},$$

which follows from (8.7) and the equation that $d_{\min,k-1}=\frac{d_{\min,k}}{4((n+2)(n+1)/2)^{\frac{1}{k-1}}}$. This completes the proof. □

8.2 THEOREMS ON SOURCE NUMBER AND LOCATION RECOVERY FOR THE ONE-DIMENSIONAL STATIC SUPER-RESOLUTION PROBLEM

The idea to prove Theorems 3.1 and 3.2 is to reduce the stability of a high-dimensional tracking problem to many one-dimensional super-resolution problems. Thus in this subsection we present results (Theorems 8.3 and 8.4) for the source number and location recovery in a one-dimensional static super-resolution problem that is tuned to subsequent proofs of stability of the velocity reconstruction. The results are slightly different from the ones in [34] and therefore, we present their detailed proofs here for the sake of completeness.

We first introduce some lemmas and theorems from [35]. We denote for integer $k \geq 1$,

$$\zeta(k) = \begin{cases} (\frac{k-1}{2}!)^2, & k \text{ is odd}, \\ (\frac{k}{2})!(\frac{k-2}{2})!, & k \text{ is even}, \end{cases} \quad \beta(k) = \begin{cases} \frac{1}{2}, & k = 1, \\ \frac{(\frac{k-1}{2})!(\frac{k-3}{2})!}{4}, & k \text{ is odd}, \ k \geq 3, \\ \frac{(\frac{k-2}{2}!)^2}{4}, & k \text{ is even}. \end{cases} \tag{8.8}$$

We also define for positive integers p, q, and $z_1, \cdots, z_p, \hat{z}_1, \cdots, \hat{z}_q \in \mathbb{C}$, the following vector in $\mathbb{R}^p$

$$\eta_{p,q}(z_1, \cdots, z_p, \hat{z}_1, \cdots, \hat{z}_q) = \begin{pmatrix} |(z_1 - \hat{z}_1)| \cdots |(z_1 - \hat{z}_q)| \\ |(z_2 - \hat{z}_1)| \cdots |(z_2 - \hat{z}_q)| \\ \vdots \\ |(z_p - \hat{z}_1)| \cdots |(z_p - \hat{z}_q)| \end{pmatrix}. \tag{8.9}$$

For a complex matrix A, we denote by A^* its conjugate transpose. For integer s and $z \in \mathbb{C}$, we define the complex Vandermonde-vector

$$\phi_s(z) = (1, z, \cdots, z^s)^\top. \tag{8.10}$$

Lemma 8.7. *For $-\frac{\pi}{2} \leq \theta_1 < \theta_2 < \cdots < \theta_k \leq \frac{\pi}{2}$ and $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, suppose*

$$||\eta_{k,k}(e^{i\theta_1}, \cdots, e^{i\theta_k}, e^{i\hat{\theta}_1}, \cdots, e^{i\hat{\theta}_k})||_\infty < (\frac{2}{\pi})^k \epsilon, \ \text{and}\ \theta_{\min} = \min_{q \neq j} |\theta_q - \theta_j| \geq \Big(\frac{4\epsilon}{\lambda(k)}\Big)^{\frac{1}{k}},$$

where $\eta_{k,k}$ is defined by (8.9) and

$$\lambda(k) = \begin{cases} 1, & k = 2, \\ \beta(k-2), & k \geq 3. \end{cases} \tag{8.11}$$

Then after reordering $\hat{\theta}_j$'s, we have

$$|\hat{\theta}_j - \theta_j| < \frac{\theta_{\min}}{2} \ \text{and}\ |\hat{\theta}_j - \theta_j| \leq \frac{2^{k-1}\epsilon}{(k-2)!(\theta_{\min})^{k-1}}, \quad j = 1, \cdots, k. \tag{8.12}$$

Proof. See Corollary 9 in [34]. □

Theorem 8.1. *Let $k \geq 1$. Assume that $\theta_j \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], 1 \leq j \leq k+1$ are $k+1$ distinct points, and $|a_j| \geq m_{\min}, 1 \leq j \leq k+1$. Let $\theta_{\min} = \min_{p \neq j} |\theta_p - \theta_j|$. For $q \leq k$, let $\hat{a}(q) = (\hat{a}_1, \cdots, \hat{a}_q)^\top$, $a = (a_1, \cdots, a_{k+1})^\top$ and*

$$\hat{A}(q) = \big(\phi_{2k}(e^{i\hat{\theta}_1}), \cdots, \phi_{2k}(e^{i\hat{\theta}_q})\big), \quad A = \big(\phi_{2k}(e^{i\theta_1}), \cdots, \phi_{2k}(e^{i\theta_{k+1}})\big),$$

where $\phi_{2k}(z)$ is defined as in (8.10). Then

$$\min_{\hat{a}_p\in\mathbb{C},\hat{\theta}_p\in\mathbb{R},p=1,\cdots,q}\left|\left|\hat{A}(q)\hat{a}(q)-Aa\right|\right|_2 \geq \frac{\zeta(k+1)\beta(k)m_{\min}\theta_{\min}^{2k}}{\pi^{2k}}.$$

Proof. See Theorem 4 in [34]. □

Theorem 8.2. *Let $k \geq 2$. Assume that $\theta_1,\cdots,\theta_k \in \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ are k different points and $|a_j| \geq m_{\min}, 1 \leq j \leq k$. Define $\theta_{\min} = \min_{p\neq j}|\theta_p-\theta_j|$. Let k distinct points $\hat{\theta}_1,\cdots,\hat{\theta}_k \in \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ satisfy*

$$||\hat{A}\hat{a}-Aa||_2 < \sigma,$$

where $\hat{a} = (\hat{a}_1,\cdots,\hat{a}_k)^\top$, $a = (a_1,\cdots,a_k)^\top$ and

$$\hat{A} = \left(\phi_{2k-1}(e^{i\hat{\theta}_1}),\cdots,\phi_{2k-1}(e^{i\hat{\theta}_k})\right),\ A = \left(\phi_{2k-1}(e^{i\theta_1}),\cdots,\phi_{2k-1}(e^{i\theta_k})\right).$$

Then

$$||\eta_{k,k}(e^{i\theta_1},\cdots,e^{i\theta_k},e^{i\hat{\theta}_1},\cdots,e^{i\hat{\theta}_k})||_\infty < \frac{2^k\pi^{k-1}}{\zeta(k)\theta_{\min}^{k-1}}\frac{\sigma}{m_{\min}}.$$

Proof. See Theorem 5 in [34]. □

Lemma 8.8. *Let $\zeta(n)$, $\beta(n)$ and $\lambda(n)$ be defined in (8.8) and (8.11), respectively. For $n \geq 2$, we have*

$$\left(\frac{2\sqrt{2n-1}}{\zeta(n)\beta(n-1)}\right)^{\frac{1}{2n-2}} \leq \frac{4.4e}{2n-1}, \tag{8.13}$$

$$\left(\frac{8\sqrt{2n}}{\zeta(n)\lambda(n)}\right)^{\frac{1}{2n-1}} \leq \frac{5.88e}{2n}, \tag{8.14}$$

and

$$\frac{(2n)^{2n-\frac{3}{2}}}{\zeta(n)(n-2)!} \leq 2^{3n-3}e^{2n}\pi^{-\frac{3}{2}}. \tag{8.15}$$

Proof. See the appendix in [34]. □

We define a discrete measure as $\mu = \sum_{j=1}^n a_j\delta_{y_j}$ and the vector of its Fourier transform at $0,\Omega,2\Omega,\cdots,T\Omega$ as

$$[\mu] = (\mathscr{F}[\mu](0),\mathscr{F}[\mu](\Omega),\cdots,\mathscr{F}[\mu](T\Omega))^\top,$$

where $\mathscr{F}[\mu](x) = \sum_{j=1}^n a_j e^{iy_jx}$. We have the following theorems for the source number and location recovery in a one-dimensional static super-resolution problem.

Theorem 8.3. *Let $n \geq 2$ and $T \geq 2n-2$. Suppose that the measurement is*

$$\mathbf{Y}(t) = \sum_{j=1}^n a_j e^{iy_j\Omega t} + \mathbf{W}(t), \quad t = 0,\cdots,T,$$

with $|\mathbf{W}(t)| < \sigma$, $y_j \in [-\frac{(n-1)\pi}{T\Omega}, \frac{(n-1)\pi}{T\Omega}]$ *and*

$$d_{\min} := \min_{p\neq q}\left|y_p - y_q\right| \geq \frac{8.8e\pi}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-2}}. \tag{8.16}$$

Then there is no $\hat{\mu} = \sum_{j=1}^{k} \hat{a}_j \delta_{\hat{y}_j}$ *with* $k < n$ *so that*

$$||[\hat{\mu}] - \mathbf{Y}||_{\infty} < \sigma.$$

Proof. **Step 1.** We write

$$T + 1 = (2n-1)r + q, \tag{8.17}$$

where r, q are integers with $r \geq 1$ and $0 \leq q < 2n-1$. We denote by $\theta_j = y_j r\Omega, j = 1, \cdots, n$. For $y_j \in [-\frac{(n-1)\pi}{T\Omega}, \frac{(n-1)\pi}{T\Omega}]$, in view of (8.17), it is clear that

$$\theta_j = y_j r\Omega \in \Big[\frac{-\pi}{2}, \frac{\pi}{2}\Big], \quad j = 1, \cdots, n. \tag{8.18}$$

For $\hat{\mu} = \sum_{j=1}^{k} \hat{a}_j \delta_{\hat{y}_j}$ with $k < n$, note that

$$[\hat{\mu}] - [\mu] = \big(\mathscr{F}[\hat{\mu}](0), \mathscr{F}[\hat{\mu}](\Omega), \cdots, \mathscr{F}[\hat{\mu}](T\Omega)\big)^{\top} - \big(\mathscr{F}[\mu](0), \mathscr{F}[\mu](\Omega), \cdots, \mathscr{F}[\mu](T\Omega)\big)^{\top}.$$

Using only the partial measurement at $qr\Omega, 0 \leq q \leq 2n-2$, we have

$$\big(\mathscr{F}\hat{\mu}(0), \mathscr{F}\hat{\mu}(r\Omega), \cdots, \mathscr{F}\hat{\mu}((2n-2)r\Omega)\big)^{\top} - \big(\mathscr{F}\mu(0), \mathscr{F}\mu(r\Omega), \cdots, \mathscr{F}\mu((2n-2)r\Omega)\big)^{\top} = \hat{B}\hat{a} - Ba,$$

where $\hat{a} = (\hat{a}_1, \cdots, \hat{a}_k)^{\top}$, $a = (a_1, \cdots, a_n)^{\top}$ and

$$\begin{aligned} \hat{B} &= \big(\phi_{2n-2}(e^{i\hat{\theta}_1}), \cdots, \phi_{2n-2}(e^{i\hat{\theta}_k})\big), \\ B &= \big(\phi_{2n-2}(e^{i\theta_1}), \cdots, \phi_{2n-2}(e^{i\theta_n})\big), \end{aligned} \tag{8.19}$$

with $\theta_j = y_j r\Omega, \hat{\theta}_j = \hat{y}_j r\Omega$. It is clear that

$$\begin{aligned} &\min_{\hat{a}\in\mathbb{C}^k, \hat{y}_j\in\mathbb{R}, j=1,\cdots,k} ||[\hat{\mu}] - [\mu]||_{\infty} \geq \min_{\hat{a}\in\mathbb{C}^k, \hat{y}_j\in\mathbb{R}, j=1,\cdots,k} ||\hat{B}\hat{a} - Ba||_{\infty} \\ \geq &\frac{1}{\sqrt{2n-1}} \min_{\alpha\in\mathbb{C}^k, \hat{y}_j\in\mathbb{R}, j=1,\cdots,k} ||\hat{B}\alpha - Ba||_2. \end{aligned} \tag{8.20}$$

In view of (8.18), we can apply Theorem 8.1 to get

$$\min_{\alpha\in\mathbb{C}^k, \hat{y}_j\in\mathbb{R}, j=1,\cdots,k} ||\hat{B}\alpha - Ba||_2 \geq \frac{m_{\min}\zeta(n)\beta(n-1)(\theta_{\min})^{2n-2}}{\pi^{2n-2}},$$

where $\theta_{\min} = \min_{j\neq p}|\theta_j - \theta_p|$. Combining the above estimate with (8.20), we get

$$\min_{\hat{a}\in\mathbb{C}^k, \hat{y}_j\in\mathbb{R}, j=1,\cdots,k} ||[\hat{\mu}] - [\mu]||_{\infty} \geq \frac{m_{\min}\zeta(n)\beta(n-1)(\theta_{\min})^{2n-2}}{\sqrt{2n-1}\pi^{2n-2}}. \tag{8.21}$$

Step 2. Recall that $d_{\min} = \min_{j\neq p} |y_j - y_p|$. Using the relation $\theta_j = y_j r\Omega$ and (8.17), we can show that

$$\theta_{\min} = r\Omega d_{\min} \geq \frac{rT\Omega}{(2n-1)(r+1)} d_{\min} \geq \frac{T\Omega}{2(2n-1)} d_{\min}. \quad \left(r \geq 1\right)$$

Then the separation condition (8.16) implies

$$\theta_{\min} \geq \frac{4.4\pi e}{2n-1}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}} \geq \left(\frac{2\sqrt{2n-1}}{\zeta(n)\beta(n-1)}\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}},$$

where we have used (8.13) for the last inequality above. Therefore (8.21) implies that

$$\min_{\hat{a}\in\mathbb{C}^k, \hat{y}_j\in\mathbb{R}, j=1,\cdots,k} ||[\hat{\mu}] - [\mu]||_\infty \geq 2\sigma.$$

It follows that

$$\begin{aligned} &||[\hat{\mu}] - \mathbf{Y}||_\infty = ||[\hat{\mu}] - [\mu] - \mathbf{W}||_\infty \\ \geq &||[\hat{\mu}] - [\mu]||_\infty - ||\mathbf{W}||_\infty \geq ||[\hat{\mu}] - [\mu]||_\infty - \sigma \geq \sigma, \end{aligned}$$

which shows that $||[\hat{\mu}] - \mathbf{Y}||_\infty < \sigma$ is impossible. This completes the proof. □

Theorem 8.4. *Let $n \geq 2$ and $T \geq 2n-1$. Suppose that the measurement is*

$$\mathbf{Y}(t) = \sum_{j=1}^{n} a_j e^{iy_j\Omega t} + \mathbf{W}(t), \quad t = 0, \cdots, T,$$

with $|\mathbf{W}(t)| < \sigma$, $y_j \in [-\frac{(n-1)\pi}{T\Omega}, \frac{(n-1)\pi}{T\Omega}]$ and

$$d_{\min} := \min_{p\neq q} \left|y_p - y_q\right| \geq \frac{11.76e\pi}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}. \tag{8.22}$$

For a measure $\hat{\mu} = \sum_{j=1}^{n} \hat{a}_j \delta_{\hat{y}_j}$ with $\hat{y}_j \in [-\frac{(n-1)\pi}{T\Omega}, \frac{(n-1)\pi}{T\Omega}]$ and $||[\hat{\mu}] - \mathbf{Y}||_\infty < \sigma$, we can reorder $\hat{y}_j$'s so that

$$\left|\hat{y}_j - y_j\right| < \frac{d_{\min}}{2},$$

and

$$\left|\hat{y}_j - y_j\right| \leq \frac{C(n)}{\Omega}\left(\frac{\pi}{T\Omega d_{\min}}\right)^{2n-2}\frac{\sigma}{m_{\min}},$$

where $C(n) = n2^{6n-3}e^{2n}\pi^{-\frac{1}{2}}$.

Proof. **Step 1.** We first write

$$T + 1 = 2nr + q, \tag{8.23}$$

where r, q are integers with $r \geq 1$ and $0 \leq q < 2n$. Since $y_j, \hat{y}_j \in [-\frac{(n-1)\pi}{T\Omega}, \frac{(n-1)\pi}{T\Omega}]$, we have

$$\theta_j := y_j r\Omega \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \quad \hat{\theta}_p := \hat{y}_p r\Omega \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \ 1 \leq j, p \leq n. \tag{8.24}$$

Also, by (8.23),

$$\left|\theta_j-\theta_p\right| = r\Omega\left|y_j-y_p\right| \geq \frac{rT\Omega}{2n(r+1)} \geq \frac{T\Omega}{4n}\left|y_j-y_p\right|, \left(\text{since } r\geq 1\right) \tag{8.25}$$

and

$$\theta_{\min} := \min_{j\neq p}\left|\theta_j-\theta_p\right| \geq \frac{T\Omega}{4n}d_{\min}. \tag{8.26}$$

Similar to Step 1 in the proof of Theorem 8.3, we consider

$$\begin{aligned}&\left(\mathscr{F}[\hat{\mu}](0),\mathscr{F}[\hat{\mu}](r\Omega),\cdots,\mathscr{F}[\hat{\mu}]((2n-1)r\Omega)\right)^\top\\ &-\left(\mathscr{F}[\mu](0),\mathscr{F}[\mu](r\Omega),\cdots,\mathscr{F}[\mu]((2n-1)r\Omega)\right)^\top = \hat{B}\hat{a}-Ba,\end{aligned}$$

where $\hat{a}=(\hat{a}_1,\cdots,\hat{a}_n)^\top$, $a=(a_1,\cdots,a_n)^\top$, and

$$\hat{B}=\left(\phi_{2n-1}(e^{i\hat{\theta}_1}),\cdots,\phi_{2n-1}(e^{i\hat{\theta}_n})\right),\ B=\left(\phi_{2n-1}(e^{i\theta_1}),\cdots,\phi_{2n-1}(e^{i\theta_n})\right), \tag{8.27}$$

with $\theta_j = y_j r\Omega, \hat{\theta}_j=\hat{y}_j r\Omega$. It is clear that

$$||\hat{B}\hat{a}-Ba||_\infty \leq ||[\hat{\mu}]-[\mu]||_\infty.$$

On the other hand, since $||[\hat{\mu}]-\mathbf{Y}||_\infty<\sigma$, we have $||[\hat{\mu}]-[\mu]||_\infty<2\sigma$. It follows that $||\hat{B}\hat{a}-Ba||_\infty<2\sigma$, whence we get

$$||\hat{B}\hat{a}-Ba||_2 \leq \sqrt{2n}||\hat{B}\hat{a}-Ba||_\infty < 2\sqrt{2n}\sigma. \tag{8.28}$$

In view of (8.24), we can apply Theorem 8.2 to get

$$\left|\left|\eta_{n,n}(e^{i\theta_1},\cdots,e^{i\theta_n},e^{i\hat{\theta}_1},\cdots,e^{i\hat{\theta}_n})\right|\right|_\infty < \frac{\sqrt{2n}2^{n+1}\pi^{n-1}}{\zeta(n)(\theta_{\min})^{n-1}}\frac{\sigma}{m_{\min}}. \tag{8.29}$$

Step 2. We apply Lemma 8.7 to estimate $|\hat{\theta}_j-\theta_j|$'s and $|\hat{y}_j-y_j|$'s. To do so, let $\epsilon=\frac{2\sqrt{2n}\pi^{2n-1}}{\zeta(n)(\theta_{\min})^{n-1}}\frac{\sigma}{m_{\min}}$. It is clear that $||\eta_{n,n}||_\infty<(\frac{2}{\pi})^n\epsilon$ and we only need to check the following condition

$$\theta_{\min}\geq\left(\frac{4\epsilon}{\lambda(n)}\right)^{\frac{1}{n}},\quad \text{or equivalently } (\theta_{\min})^n\geq\frac{4\epsilon}{\lambda(n)}. \tag{8.30}$$

Indeed, by (8.26) and the separation condition (8.22),

$$\theta_{\min}\geq\frac{11.76\pi e}{4n}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}} \geq \pi\left(\frac{8\sqrt{2n}}{\lambda(n)\zeta(n)}\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}, \tag{8.31}$$

where we have used (8.14) in the last inequality. Then

$$(\theta_{\min})^{2n-1}\geq\frac{\pi^{2n-1}8\sqrt{2n}}{\lambda(n)\zeta(n)}\frac{\sigma}{m_{\min}},$$

whence we get (8.30). Therefore, we can apply Lemma 8.7 to get that, after reordering $\hat{\theta}_j$'s,

$$\begin{aligned}&\left|\hat{\theta}_j-\theta_j\right|<\frac{\theta_{\min}}{2},\\&\left|\hat{\theta}_j-\theta_j\right|<\frac{\sqrt{2n}2^n\pi^{2n-1}}{\zeta(n)(n-2)!(\theta_{\min})^{2n-2}}\frac{\sigma}{m_{\min}},1\leq j\leq n.\end{aligned} \tag{8.32}$$

Finally, we estimate $\left|\hat{y}_j-y_j\right|$. Since $\left|\hat{\theta}_j-\theta_j\right|<\frac{\theta_{\min}}{2}$, it is clear that $\left|\hat{y}_j-y_j\right|<\frac{d_{\min}}{2}$. On the other hand, by (8.25)

$$\left|\hat{y}_j-y_j\right|\leq\frac{4n}{T\Omega}\left|\hat{\theta}_j-\theta_j\right|.$$

Using (8.32), (8.26), and (8.15), a direct calculation shows that

$$\left|\hat{y}_j-y_j\right|\leq\frac{C(n)}{T\Omega}(\frac{\pi}{T\Omega d_{\min}})^{2n-2}\frac{\sigma}{m_{\min}},$$

where $C(n)=n2^{6n-3}e^{2n}\pi^{-\frac{1}{2}}$. This completes the proof. □

8.3 Proof of Theorem 3.1

Proof. Note that for the n different $\mathbf{v}_j$'s, we have at most $\frac{n(n-1)}{2}$ different $\mathbf{u}_{pq}=\tau\mathbf{v}_p-\tau\mathbf{v}_q, p<q$. By Lemma 8.4, we can find a one-dimensional subspace S so that

$$\left|\left|\mathscr{P}_S(\mathbf{u}_{pq})\right|\right|_2\geq\frac{||\mathbf{u}_{pq}||_2}{(\pi/2)^{d-1}}\left(\frac{\pi}{n(n-1)}\right)^{\xi(d-1)}\geq\frac{8.8e\pi}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}},$$

where the last inequality is by the separation condition in the theorem. Thus we can find a unit vector $\mathbf{v}\in\mathbb{R}^d$, so that

$$\left|\mathbf{v}^\top\mathbf{u}_{pq}\right|\geq\frac{8.8e\pi}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}. \tag{8.33}$$

We consider the measurements at $\boldsymbol{\omega}=\Omega\mathbf{v}$ that

$$\mathbf{Y}_t(\mathbf{v}\Omega)=\sum_{j=1}^n a_je^{i(\mathbf{y}_j^\top+t\tau\mathbf{v}_j^\top)\mathbf{v}\Omega}+\mathbf{W}_t(\mathbf{v}\Omega),\quad t=0,\cdots,T.$$

Let $b_j=a_je^{i\mathbf{y}_j^\top\mathbf{v}\Omega}$ and $y_j=\tau\mathbf{v}_j^\top\mathbf{v}$ and $\mathbf{W}(t)=\mathbf{W}_t(\mathbf{v}\Omega)$, the measurement can be written as

$$\mathbf{Y}(t)=\sum_{j=1}^n b_je^{iy_j\Omega t}+\mathbf{W}(t)\quad t=0,\cdots,T,$$

with $|\mathbf{W}(t)|<\sigma$. Note also that (8.33) implies $\min_{p\neq q}|y_p-y_q|\geq\frac{8.8e\pi}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-2}}$ and $\tau\mathbf{v}_j\in B^d_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0})$ implies $y_j\in[-\frac{(n-1)\pi}{T\Omega},\frac{(n-1)\pi}{T\Omega}]$. Thus by Theorem 8.3, there is no $k<n$ $\hat{y}_j$'s so that

$$|\sum_{j=1}^n\hat{b}_je^{i\hat{y}_j\Omega t}-\mathbf{Y}(t)|<\sigma,\quad t=0,\cdots,T.$$

This implies there does not exist any σ-admissible parameter set of $\mathbf{Y}_t$'s with less than n elements. □

Proof. **Step 1.** Note that we have at most $\frac{n(n-1)}{2}$ different $\mathbf{u}_{pq} = \tau\mathbf{v}_p - \tau\mathbf{v}_q, p<q$. The separation condition in this theorem means

$$\min_{p<q}\left|\left|\mathbf{u}_{pq}\right|\right|_2 \geq \frac{11.76e\pi 4^{d-1}\Big((n+2)(n+1)/2\Big)^{\xi(d-1)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.$$

For a subspace S of $\mathbb{R}^d$ and $\mathbf{v} \in S$, we denote $\mathbf{v}^{\perp}(S)$ the orthogonal complement space of $\mathbf{v}$ in S. Let

$$\theta_k = \frac{\pi}{4}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{k}},\ k = 1,\cdots,d-1,$$
$$d_{\min,k} = \frac{11.76\pi e 4^{k-1}((n+2)(n+1)/2)^{\xi(k-1)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}},\ k=1,\cdots,d-1. \tag{8.34}$$

By Lemma 8.5, in $\mathbb{R}^d$ we can construct $n+1$ $(d-1)$-dimensional spaces $S^{d-1}_{j_{d-1}} = \mathbf{v}^{\perp}_{j_{d-1}}(\mathbb{R}^d), j_{d-1} = 1,\cdots,n+1$ with $0 \leq \mathbf{v}^{\top}_{p_{d-1}}\mathbf{v}_{j_{d-1}} \leq \cos(\theta_{d-1}), p_{d-1}\neq j_{d-1}$, so that for all $\mathbf{u}_{pq}$'s we have

$$\left|\left|\mathscr{P}_{S^{d-1}_{j_{d-1}}}(\mathbf{u}_{pq})\right|\right|_2 \geq \frac{||\mathbf{u}_{pq}||_2}{4}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{d-1}} \geq \frac{11.76e\pi 4^{d-2}\Big((n+2)(n+1)/2\Big)^{\xi(d-2)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.$$

Since each $S^{d-1}_{j_{d-1}}$ is a $(d-1)$-dimensional space, applying Lemma 8.5 again for $S^{d-1}_{j_{d-1}}$ and $\mathscr{P}_{S^{d-1}_{j_{d-1}}}(\mathbf{u}_{pq})$'s, for each $S^{d-1}_{j_{d-1}}$ we can construct $n+1$ $(d-2)$-dimensional subspaces $S^{d-2}_{j_{d-2},j_{d-1}} = \mathbf{v}^{\perp}_{j_{d-2},j_{d-1}}(S^{d-1}_{j_{d-1}}), j_{d-2} = 1,\cdots,n+1$ with $0 \leq \mathbf{v}^{\top}_{p_{d-2},j_{d-1}}\mathbf{v}_{j_{d-2},j_{d-1}} \leq \cos(\theta_{d-2}), p_{d-2}\neq j_{d-2}$, so that for all $\mathbf{u}_{pq}$'s we have

$$\left|\left|\mathscr{P}_{S^{d-2}_{j_{d-2},j_{d-1}}}(\mathbf{u}_{pq})\right|\right|_2 = \left|\left|\mathscr{P}_{S^{d-2}_{j_{d-2},j_{d-1}}}\Big(\mathscr{P}_{S^{d-1}_{j_{d-1}}}(\mathbf{u}_{pq})\Big)\right|\right|_2 \geq \frac{||\mathbf{u}_{pq}||_2}{4^2}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\frac{1}{d-1}+\frac{1}{d-2}}$$
$$\geq \frac{11.76e\pi 4^{d-3}\Big((n+2)(n+1)/2\Big)^{\xi(d-3)}}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}},$$

where the first equality is because $S^{d-2}_{j_{d-2},j_{d-1}}$ is a subspace of $S^{d-1}_{j_{d-1}}$. We can continue the process and construct $n+1$ subspaces of $S^{d-2}_{j_{d-2},j_{d-1}}, S^{d-3}_{j_{d-3},j_{d-2},j_{d-1}},\cdots,S^{2}_{j_2,j_3,\cdots,j_{d-1}}$, respectively. For each two dimensional space $S^2_{j_2,j_3,\cdots,j_{d-1}}$, we can construct $n+1$ one-dimensional subspaces $S^1_{j_1,j_2,j_3,\cdots,j_{d-1}} = \mathbf{v}^{\perp}_{j_1,j_2,j_3,\cdots,j_{d-1}}(S^2_{j_2,j_3,\cdots,j_{d-1}}), j_1 = 1,\cdots,n+1$ with $0 \leq \mathbf{v}^{\top}_{p_1,j_2,\cdots,j_{d-1}}\mathbf{v}_{j_1,j_2,\cdots,j_{d-1}} \leq \cos(\theta_1), p_1 \neq j_1$, so that for all $\mathbf{u}_{pq}$'s we have

$$\left|\left|\mathscr{P}_{S^1_{j_1,j_2,j_3,\cdots,j_{d-1}}}(\mathbf{u}_{pq})\right|\right|_2 \geq \frac{||\mathbf{u}_{pq}||_2}{4^{d-1}}\Big(\frac{2}{(n+1)(n+2)}\Big)^{\xi(d-1)} \geq \frac{11.76e\pi}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}.$$

Thus for each $\{j_2,\cdots,j_{d-1}\}$, we can find $n+1$ unit vectors $\mathbf{q}_j$'s with $0 \leq |\mathbf{q}_p^{\top}\mathbf{q}_j| \leq \cos(\theta_1), p\neq j$, so that

$$\mathscr{P}_{S^1_{j_1,j_2,j_3,\cdots,j_{d-1}}}(\mathbf{u}_{pq}) = \mathbf{q}_j^{\top}\mathbf{u}_{pq},\ \text{and}\ \left|\mathbf{q}_j^{\top}\mathbf{u}_{pq}\right| \geq \frac{11.76e\pi}{T\Omega}\Big(\frac{\sigma}{m_{\min}}\Big)^{\frac{1}{2n-1}}. \tag{8.35}$$

We consider these $\mathbf{q}_j$'s in Step 2.

Step 2. Without loss of generality, we first consider $\mathbf{q}_1$. We consider the measurements at $\boldsymbol{\omega} = \Omega\mathbf{q}_1$ that

$$\mathbf{Y}_t(\mathbf{q}_1\Omega) = \sum_{j=1}^{n} a_j e^{i(\mathbf{y}_j^\top + t\tau\mathbf{v}_j^\top)\mathbf{q}_1\Omega} + \mathbf{W}_t(\mathbf{q}_1\Omega), \quad t = 0,\cdots,T.$$

Let $b_j = a_j e^{i\mathbf{y}_j^\top\mathbf{q}_1\Omega}$, $y_j = \tau\mathbf{v}_j^\top\mathbf{q}_1$ and $\mathbf{W}(t) = \mathbf{W}_t(\mathbf{q}_1\Omega)$, the measurement can be written as

$$\mathbf{Y}(t) = \sum_{j=1}^{n} b_j e^{iy_j\Omega t} + \mathbf{W}(t), \quad t = 0,\cdots,T,$$

with $|\mathbf{W}(t)| < \sigma$. By (8.35) we have

$$\min_{p\neq q}\left|y_p - y_q\right| \geq \frac{11.76e\pi}{T\Omega}\left(\frac{\sigma}{m_{\min}}\right)^{\frac{1}{2n-1}}.$$

On the other hand, the measurement constraint

$$\left|\sum_{j=1}^{n} \hat{a}_j e^{i(\hat{\mathbf{y}}_j^\top + t\tau\hat{\mathbf{v}}_j^\top)\mathbf{q}_1\Omega} - \mathbf{Y}_t(\mathbf{q}_1\Omega)\right| < \sigma,$$

can be written as

$$\left|\sum_{j=1}^{n} \hat{b}_j e^{i\hat{y}_j t} - \mathbf{Y}(t)\right| < \sigma,$$

where $\hat{b}_j = \hat{a}_j e^{i\hat{\mathbf{y}}_j^\top\mathbf{q}_1\Omega}$ and $\hat{y}_j = \tau\hat{\mathbf{v}}_j^\top\mathbf{q}_1$. Note that $|\tau\mathbf{v}_j^\top\mathbf{q}_1| \leq \frac{(n-1)\pi}{T\Omega}$ and $|\tau\hat{\mathbf{v}}_j^\top\mathbf{q}_1| \leq \frac{(n-1)\pi}{T\Omega}$ since $\tau\hat{\mathbf{v}}_j, \tau\mathbf{v}_j \in B^d_{\frac{(n-1)\pi}{T\Omega}}(\mathbf{0})$. By Theorem 8.4, we have after reordering $\hat{y}_j$'s,

$$\left|\hat{y}_j - y_j\right| < \frac{1}{2}d_{\min,1},$$

where $d_{\min,1}$ is defined as in (8.34), and

$$\left|\hat{y}_j - y_j\right| \leq \frac{C(1,n)}{T\Omega}\left(\frac{\pi}{d_{\min,1}T\Omega}\right)^{2n-2}\frac{\sigma}{m_{\min}},$$

where $C(1,n) = n2^{6n-3}e^{2n}\pi^{-\frac{1}{2}}$. Because we have $n+1$ $\mathbf{q}_j$'s, we have $n+1$ permutations τ_j's of $\{1,\cdots,n\}$ so that

$$\left|\tau\hat{\mathbf{v}}_{\tau_j(p)}^\top\mathbf{q}_j - \tau\mathbf{v}_p^\top\mathbf{q}_j\right| < \frac{1}{2}d_{\min,1}, \quad j = 1,\cdots,n+1,$$

and

$$\left|\tau\hat{\mathbf{v}}_{\tau_j(p)}^\top\mathbf{q}_j - \tau\mathbf{v}_p^\top\mathbf{q}_j\right| \leq \frac{C(1,n)}{T\Omega}\left(\frac{\pi}{d_{\min,1}T\Omega}\right)^{2n-2}\frac{\sigma}{m_{\min}}, \quad j = 1,\cdots,n+1.$$

Since we have $n+1$ $\mathbf{q}_j$'s but only n $\hat{\mathbf{v}}_p$'s, by the pigeonhole principle, for each $\mathbf{v}_p$, there exist $\hat{\mathbf{v}}_{\tau_{j_1}(p)} = \hat{\mathbf{v}}_{\tau_{j_2}(p)} = \hat{\mathbf{v}}_{p'}$ so that

$$\left|\tau\hat{\mathbf{v}}_{p'}^\top\mathbf{q}_{j_t} - \tau\mathbf{v}_p^\top\mathbf{q}_{j_t}\right| < \frac{1}{2}d_{\min,1}, \; t = 1,2,$$

and

$$\left|\tau\hat{\mathbf{v}}_{p'}^{\top}\mathbf{q}_{j_t}-\tau\mathbf{v}_p^{\top}\mathbf{q}_{j_t}\right|\le\frac{C(1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,1}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},\ t=1,2.$$

Thus we have two $S^1_{j_{1t},j_2,\cdots,j_{d-1}}$, $t=1,2$, so that

$$\left|\mathscr{P}_{S^1_{j_{1t},j_2,\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right|<\frac{1}{2}d_{\min,1},\ t=1,2,$$

and

$$\left|\mathscr{P}_{S^1_{j_{1t},j_2,\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right|\le\frac{C(1,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,1}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},\ t=1,2.$$

From the results obtained in Step 1, $S^1_{j_{1t},j_2,\cdots,j_{d-1}}=\mathbf{v}^{\perp}_{j_{1t},j_2,j_3,\cdots,j_{d-1}}(S^2_{j_2,j_3,\cdots,j_{d-1}})$, $t=1,2$, are both one-dimensional subspaces of $S^2_{j_2,\cdots,j_{d-1}}$ and $0\le\mathbf{v}^{\top}_{j_{11},j_2,\cdots,j_{d-1}}\mathbf{v}_{j_{12},j_2,\cdots,j_{d-1}}\le\cos(\theta_1)$. By Lemma 8.6, we have

$$\left\|\mathscr{P}_{S^2_{j_2,\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right\|_2<\frac{1}{2}d_{\min,2},\quad\left\|\mathscr{P}_{S^2_{j_2,\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right\|_2\le\frac{C(2,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,2}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},$$

where

$$C(2,n)=\big(4((n+2)(n+1)/2)^{\xi(1)}\big)^{(2n-1)}n2^{6n-3}e^{2n}\pi^{-\frac{1}{2}}.$$

Since we do not specify the $\{j_2,\cdots,j_{d-1}\}$, for fixed $\{j_3,\cdots,j_{d-1}\}$, the above results hold for all $\{j_2,j_3,\cdots,j_{d-1}\}$, $j_2=1,\cdots,n+1$ with that the p' is related to j_2. Again from Step 1, for fixed $\{j_3,\cdots,j_{d-1}\}$ we have $S^2_{j_2,j_3,\cdots,j_{d-1}}=\mathbf{v}^{\perp}_{j_2,j_3,\cdots,j_{d-1}}(S^3_{j_3,\cdots,j_{d-1}})$, $j_2=1,\cdots,n+1$ with $0\le\mathbf{v}^{\top}_{p_2,j_3,\cdots,j_{d-1}}\mathbf{v}_{j_2,j_3,\cdots,j_{d-1}}\le\cos(\theta_2)$, $p_2\ne j_2$. Similar to the above arguments, since we have $n+1$ j_2's while only n $\hat{\mathbf{v}}_q$'s, by the pigeonhole principle, for each $\mathbf{v}_p$ we can find $\hat{\mathbf{v}}_{p'}$ so that

$$\begin{aligned}&\left\|\mathscr{P}_{S^2_{j_{2t},\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right\|_2<\frac{1}{2}d_{\min,2},\\&\left\|\mathscr{P}_{S^2_{j_{2t},\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right\|_2\le\frac{C(2,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,2}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},\ t=1,2,\end{aligned}$$

where

$$C(2,n)=\big(4((n+2)(n+1)/2)^{\xi(1)}\big)^{(2n-1)}n2^{6n-3}e^{2n}\pi^{-\frac{1}{2}}.$$

By Lemma 8.6, we have

$$\left\|\mathscr{P}_{S^3_{j_3,\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right\|_2<\frac{1}{2}d_{\min,3},\quad\left\|\mathscr{P}_{S^3_{j_3,\cdots,j_{d-1}}}(\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p)\right\|_2\le\frac{C(3,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,3}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},$$

where

$$C(3,n)=\big(4^2((n+2)(n+1)/2)^{\xi(2)}\big)^{(2n-1)}n2^{6n-3}e^{2n}\pi^{-\frac{1}{2}}.$$

Thus, by continuing the process, there exists $\hat{\mathbf{v}}_{p'}$ so that

$$\left\|\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p\right\|_2<\frac{1}{2}d_{\min,d},\quad\left\|\tau\hat{\mathbf{v}}_{p'}-\tau\mathbf{v}_p\right\|_2\le\frac{C(d,n)}{T\Omega}\Big(\frac{\pi}{d_{\min,d}T\Omega}\Big)^{2n-2}\frac{\sigma}{m_{\min}},$$

where

$$C(d,n) = \left(4^{d-1}((n+2)(n+1)/2)^{\xi(d-1)}\right)^{(2n-1)} n 2^{6n-3} e^{2n} \pi^{-\frac{1}{2}}.$$

Since $\min_{p\neq q} ||\mathbf{v}_p - \mathbf{v}_q||_2 \geq d_{\min,d}$, for each $\mathbf{v}_p$ there exists one and only one $\hat{\mathbf{v}}_{p'}$ satisfying the above condition. Thus after reordering $\hat{\mathbf{v}}_p$'s we have

$$\left|\left|\tau\hat{\mathbf{v}}_p - \tau\mathbf{v}_p\right|\right|_2 < \frac{1}{2} d_{\min,d}, \ \left|\left|\tau\hat{\mathbf{v}}_p - \tau\mathbf{v}_p\right|\right|_2 \leq \frac{C(d,n)}{T\Omega}\left(\frac{\pi}{d_{\min,d} T\Omega}\right)^{2n-2} \frac{\sigma}{m_{\min}}, \ 1 \leq p \leq n.$$

This completes the proof. □

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