# Exponential Convergence of Deep Operator Networks for Elliptic Partial Differential Equations

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# EXPONENTIAL CONVERGENCE OF DEEP OPERATOR NETWORKS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

CARLO MARCATI* AND CHRISTOPH SCHWAB*

**Abstract.** We construct deep operator networks (ONets) between infinite-dimensional spaces that emulate with an exponential rate of convergence the coefficient-to-solution map of elliptic second-order PDEs. In particular, we consider problems set in $d$-dimensional periodic domains, $d = 1, 2, \ldots$, and with analytic right-hand sides and coefficients. Our analysis covers diffusion-reaction problems, parametric diffusion equations, and elliptic systems such as linear isotropic elastostatics in heterogeneous materials.

We leverage the exponential convergence of spectral collocation methods for boundary value problems whose solutions are analytic. In the present periodic and analytic setting, this follows from classical elliptic regularity. Within the ONet branch and trunk construction of Chen and Chen [4] and of Lu et al. [13], we show the existence of deep ONets which emulate the coefficient-to-solution map to accuracy $\varepsilon > 0$ in the $H^1$ norm, uniformly over the coefficient set. We prove that the neural networks in the ONet have size $\mathcal{O}(|\log(\varepsilon)|^\kappa)$ for some $\kappa > 0$ depending on the physical space dimension.

**Key words.** Operator networks, deep neural networks, exponential convergence, elliptic PDEs

**AMS subject classifications.** 35J15, 65N15, 65N35, 68T07

**1. Introduction.** We construct deep operator network (ONet) emulations of coefficient-to-solution maps for boundary value problems with linear elliptic divergence-form operators. In particular, we consider operator networks with rectified linear unit (ReLU) activation and problems formulated in domains without boundary and with analytic right-hand sides and coefficients. In this setting, we construct operator networks that approximate the (nonlinear) coefficient-to-solution map with exponential accuracy in the corresponding function spaces. We bound—polylogarithmically with respect to the energy norm of the error—both the size of the approximating network and the number of sampling points where the coefficient is queried.

**1.1. Existing Results.** The use of deep neural networks (DNN) in the numerical solution of differential equations in science and engineering has received considerable attention in recent years. We refer to the survey [2] for uses and successes of DNN based numerical simulations in computational fluid mechanics, and to [22] for their use in computational finance and computational option pricing. First uses of DNNs in numerical PDE solution in engineering and the sciences focused on leveraging DNNs for "mesh-free" solution approximation and representation (see, e.g., [21, 7]), with good success explained, to some extent, by *approximation properties of DNNs in function spaces* (see, e.g., [19, 17, 18, 15, 23, 8]) in particular overcoming the so-called Curse-of-Dimensionality (CoD) in high-dimensional approximation of PDE solution manifolds [23, 9], of parametric PDEs and of PDEs on high-dimensional state spaces, as arising e.g. in computational finance (see [22, 1] and the references there).

Reference [11] addressed the expression rate of ReLU NNs for the solution maps of parametric PDEs. The analysis in that paper proceeds through the DNN emulation of *reduced bases* for the approximation of solutions of the PDEs. The expression rate bounds obtained in [11] are subject to strong hypotheses on the DNN approximability of reduced bases for the PDEs of interest. The parameter sets (i.e., the domains of the solution operator) considered in [11] are finite-dimensional; this paper mostly concerns instead the approximation of solution maps between infinite dimensional spaces. We nonetheless show how expression rates for parametric PDEs also follow from our main results, see Theorem 5.10 and Remark 5.12.

DNNs have been leveraged in [6, 13, 12] for the *DNN emulation of data-to-solution operators for partial differential equations*. See also the review [14]. Here, previous investigations have focused

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on *universality of NNs* for operator approximation. The pioneering work [4] established this for a certain type of NNs with a "branch and trunk" architecture, which will also be used in the present work. While [4] imposed strong compactness assumptions, more recently [12] extended these results to certain settings without the compactness assumptions of [4]. In these papers, focus has been on emulating nonlinear maps, such as domain-to-solution, or coefficient-to-solution maps. For well-posed PDE problems, continuous dependence on the problem data implies that these maps are continuous, in the appropriate topologies on the data and the solution space. We refer to [10, 13] and the references there. In these references, some theory explaining some of the numerically observed performances of NN emulation of nonlinear operators has been developed (see, e.g., [12, 6]).

The *convergence rate estimates* proved in these references indicate that a) DNNs are capable of parsimonious numerical representations of the nonlinear, smooth data-to-solution maps for PDEs, and b) they are not prone to the CoD in connection with the countable number of parameters due, e.g., to series representations of inputs in separable Banach spaces of possibly infinite dimension.

**1.2. Contributions.** We construct DNN approximations of data-to-solution maps, so-called "Operator Networks" for linear second order divergence-form PDEs with non-homogeneous coefficients and source terms. We establish *exponential expression rates* for these coefficient-to-solution operators for elliptic PDEs.

Our argument relies on analytic regularity for elliptic PDEs with analytic coefficients, on the a priori analysis of the spectral approximation of PDEs, and on numerical quadrature. We consider linear second order divergence-form elliptic boundary value problems with analytic coefficients, and (uniformly) analytic solutions, whose inputs and solutions can be approximated with exponential accuracy from space of high-degree polynomials. Furthermore, our results show also that neural networks can emulate accurately the (discrete) solution map of Galerkin methods for the elliptic PDEs mentioned above with numerical integration. The operator networks we construct are composed of *encoding*, *approximation*, and *reconstruction* operators. In the encoding step, the input datum is queried on collocation points in the physical domain. The reconstruction and approximation parts of the operator networks are composed of two neural networks, one that approximates a polynomial basis, while the other maps point evaluations of the diffusion coefficient to coefficients over the basis.

For the sake of clarity of exposition, we develop this strategy for model, linear second order elliptic PDEs in divergence form, with inhomogeneous coefficients. We then show, using the compositionality of NNs, how to include problems with parametric diffusion, typically arising in computational uncertainty quantification. Finally, we mention the minor modifications required for PDEs with reaction coefficients and discuss in some detail ONet emulation of the coefficient-to-solution map for linear elasticity.

**1.3. Structure of this paper.** To fix a setting for developing our results, we introduce in Section 2 a scalar, elliptic, isotropic diffusion equation. The *coefficient-to-solution operator* that will be the main target of approximation by neural networks is also introduced in this section. Then, in Section 3, we define feed forward neural networks (with ReLU activation) and operator networks with the branch and trunk architecture of [4, 13], that approximate maps between infinite dimensional spaces. We conclude the section by defining some operations on networks that will then be used for the approximation analysis. In Section 4, we gather (classical) results on the polynomial approximation of solutions to the elliptic problem. The main results of this paper are then proved in Section 5. In Theorem 5.7 we show the exponential convergence of the operator net approximation of the coefficient-to-solution map for the elliptic isotropic diffusion problem. We extend the analysis to parametric diffusion coefficients in Theorem 5.10. Finally, in Section 6 we extend our ONet approximation to further second order problems comprising reaction-diffusion with nonzero reaction coefficients and linear elastostatics.

**1.4. Notation.** We use standard notation and symbols: $\mathbb{N}$ denotes the set of positive natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We write vectors in lowercase boldface characters and matrices in uppercase boldface characters. We denote $\|\boldsymbol{a}\|_2$ the $\ell^2$-norm of a vector $\boldsymbol{a}$, while for any matrix $\mathbf{A}$, we denote $\|\mathbf{A}\|_2 = \sup_{\|\boldsymbol{x}\|_2 = 1} \|\mathbf{A}\boldsymbol{x}\|_2$ its operator norm. The spectrum of a matrix $\mathbf{A}$ is written $\sigma(\mathbf{A})$. For $n \in \mathbb{N}$, $\mathbf{Id}_n$ is the $n \times n$ identity matrix, while $\mathbf{0}_n$ is a vector of zeros of size $n$. We denote by $\otimes$ the Kronecker product: given two matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$, then $\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{mp \times nq}$, such that

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B} & \cdots & \mathbf{A}_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{m1}\mathbf{B} & \cdots & \mathbf{A}_{mn}\mathbf{B} \end{bmatrix}.$$

We denote by $[\boldsymbol{a}_1 | \dots | \boldsymbol{a}_n]$ the matrix resulting from the horizontal concatenation of $n$ equal length vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$. We indicate by $\mathbf{vec} : \mathbb{R}^{m \times n} \to \mathbb{R}^{mn}$ and $\mathbf{matr} : \mathbb{R}^{mn} \to \mathbb{R}^{m \times n}$ the vectorization and matricization operators, such that $\mathbf{matr}(\mathbf{vec}(\mathbf{A})) = \mathbf{A}$ for any matrix $\mathbf{A}$. All results are independent of the ordering of the vectorization operation; the dimensions of the matricization operation will be clear from the context. We denote by $\mathbb{R}^{n \times n}_{\mathrm{sym}}$ the space of symmetric matrices of size $n \times n$. Given two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, we write $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{ij}$.

Let $d \in \mathbb{N}$. For $k \in \mathbb{N}_0$, $p \in [1, \infty]$, and a domain $D \subset \mathbb{R}^d$, we indicate by $W^{k,p}(D)$ the classical Sobolev spaces. In the Hilbertian case $p = 2$, we write $H^k(D)$; in addition, $L^p(D) = W^{0,p}(D)$ and $L^2(D) = H^0(D)$. Given $Q = (0,1)^d$ and $\Omega = (\mathbb{R}/\mathbb{Z})^d$, we denote

$$H^k(\Omega) = H^k_{\mathrm{per}}(Q) \coloneqq \left\{ v \in H^k_{\mathrm{loc}}(\mathbb{R}^d) : v|_{Q+\boldsymbol{k}} = v|_Q, \forall \boldsymbol{k} \in \mathbb{Z}^d \right\},$$

where $H^k_{\mathrm{loc}}(\mathbb{R}^d)$ indicates functions that are in $H^k(D)$ for any bounded subset $D$ of $\mathbb{R}^d$. Furthermore, we slightly abuse notation and denote integrals as $\int_\Omega = \int_Q$ and norms as $\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{W^{k,p}(Q)}$. We denote by $(\cdot, \cdot)$ the $L^2$ scalar product in $Q$. All function spaces in $\Omega$ are understood in their periodic version.

For $C > 0$, define $\mathrm{Hol}(\Omega; C)$ as the set of functions $v$ that are real analytic in $\Omega$ with

$$(1.1) \qquad \|v\|_{W^{k,\infty}(\Omega)} \leq C^{k+1} k!, \qquad \forall k \in \mathbb{N}_0.$$

Define furthermore the set of all real analytic functions in $\Omega$ as $\mathrm{Hol}(\Omega) = \bigcup_{C>0} \mathrm{Hol}(\Omega; C)$. According to our convention, the functions in $\mathrm{Hol}(\Omega)$ are analytic in $\mathbb{R}^d$ and periodic with period 1 in each coordinate direction. Furthermore, by the Arzelà-Ascoli theorem, $\mathrm{Hol}(\Omega; C)$ is compact in $L^\infty(\Omega)$.

## 2. Problem formulation.

**2.1. Statement of the elliptic model problem.** We recall that $d \in \mathbb{N}$, $Q = (0,1)^d$, and $\Omega = (\mathbb{R}/\mathbb{Z})^d$. We introduce the set of admissible data $\mathcal{D}$ as follows: for each coefficient $a \in \mathcal{D}$ we assume ellipticity in the form that there exist constants $a_{\min}, a_{\max} > 0$ such that

$$(2.1) \qquad \forall \boldsymbol{x} \in \Omega, \, \forall a \in \mathcal{D} \quad a_{\min} \leq a(\boldsymbol{x}) \leq a_{\max}.$$

We also assume that all $a \in \mathcal{D}$ are real analytic, with uniform bounds on the radius of convergence of the Taylor series: there exists a constant $A_{\mathcal{D}} > 0$ such that

$$(2.2) \qquad \mathcal{D} \subset \mathrm{Hol}(\Omega; A_{\mathcal{D}}).$$

The ellipticity hypotheses (2.1) and the Poincaré inequality (2.5) below imply that for every $f \in L^2(\Omega)$ such that $\int_\Omega f = 0$, and for each $a \in \mathcal{D}$, the elliptic boundary value problem

$$(2.3) \qquad -\nabla \cdot (a\nabla u^a) = f \text{ in } \Omega$$

admits a unique solution

$$u^a \in X := \left\{ v \in H^1(\Omega) : \int_\Omega v = 0 \right\} \simeq H^1(\Omega)/\mathbb{R}.$$

It satisfies the *variational formulation*: given $a \in \mathcal{D}$, find $u \in X$ such that

$$(2.4) \qquad \mathfrak{b}^a(u, v) = (f, v) \quad \forall v \in X.$$

Here, for given $a \in \mathrm{Hol}(\Omega)$, the bilinear form $\mathfrak{b}^a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$\mathfrak{b}^a(w, v) := \int_\Omega \left( a \nabla w \cdot \nabla v \right).$$

In what follows, we assume the fixed source term $f \in \mathrm{Hol}(\Omega) \cap X$ to be given and denote, for this choice of $f$, the unique solution of (2.4) for $a \in \mathrm{Hol}(\Omega)$ by $u^a$.

We denote (still keeping the source term $f$ in (2.4) fixed) with $S$ the data-to-solution operator $a \mapsto u^a$ in (2.4). We let $\mathcal{U} = S(\mathcal{D})$ the set of solutions of (2.4) corresponding to inputs from $\mathcal{D}$. As shown in Lemma B.1 in Appendix B, the data-to-solution map $S : L^\infty(\Omega) \to H^1(\Omega)$ is Lipschitz continuous. Furthermore, standard elliptic regularity (see [16, 5] and Lemma 4.1 below) implies $S(\mathcal{D}) \subset \mathrm{Hol}(\Omega)$.

As it will be useful in the sequel, we define the Poincaré constant $C_{\mathrm{poi}} > 0$ such that

$$(2.5) \qquad \|v - \frac{1}{|\Omega|} \int_\Omega v\|_{L^2(\Omega)} \leq C_{\mathrm{poi}} \|\nabla v\|_{L^2(\Omega)}, \qquad \forall v \in H^1(\Omega).$$

## 3. Neural and operator networks.

The goal of this paper is to derive bounds for the approximation of the solution operator $S : \mathcal{D} \to X \subseteq H^1(\Omega)$ defined in Section 2 by an operator network. In order to properly define operator networks, we start by the definition of classical feed forward neural networks with ReLU activation

$$\mathrm{ReLU} : \mathbb{R} \to \mathbb{R} : x \mapsto \max\{0, x\}.$$

### 3.1. Feed forward neural network.

DEFINITION 3.1 ([19, Definition 2.1]). *Let $d, L \in \mathbb{N}$. A* neural network $\Phi$ *with input dimension $d$ and $L$ layers is a sequence of matrix-vector tuples*

$$\Phi = \left( (\mathbf{A}_1, \boldsymbol{b}_1), (\mathbf{A}_2, \boldsymbol{b}_2), \dots, (\mathbf{A}_L, \boldsymbol{b}_L) \right),$$

*where $N_0 := d$ and $N_1, \dots, N_L \in \mathbb{N}$, and where $\mathbf{A}_\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ and $\boldsymbol{b}_\ell \in \mathbb{R}^{N_\ell}$ for $\ell = 1, \dots, L$.*

*For a NN $\Phi$, we define the associated* realization *of the NN $\Phi$ as*

$$\mathrm{R}(\Phi) : \mathbb{R}^d \to \mathbb{R}^{N_L} : \boldsymbol{x} \mapsto \boldsymbol{x}_L =: \mathrm{R}(\Phi)(\boldsymbol{x}),$$

*where the output $\boldsymbol{x}_L \in \mathbb{R}^{N_L}$ results from*

$$(3.1) \qquad \begin{aligned} \boldsymbol{x}_0 &:= \boldsymbol{x}, \\ \boldsymbol{x}_\ell &:= \mathrm{ReLU}(\mathbf{A}_\ell \, \boldsymbol{x}_{\ell-1} + \boldsymbol{b}_\ell), \qquad \text{for } \ell = 1, \dots, L-1, \\ \boldsymbol{x}_L &:= \mathbf{A}_L \, \boldsymbol{x}_{L-1} + \boldsymbol{b}_L. \end{aligned}$$

*Here* ReLU *is understood to act component-wise on vector-valued inputs, i.e., for $\boldsymbol{y} = (y^1, \dots, y^m) \in \mathbb{R}^m$, $\mathrm{ReLU}(\boldsymbol{y}) := (\mathrm{ReLU}(y^1), \dots, \mathrm{ReLU}(y^m))$. We call $N(\Phi) := d + \sum_{j=1}^L N_j$ the* number of neurons *of the NN $\Phi$, $\mathrm{L}(\Phi) := L$ the* number of layers *or* depth, *$\mathrm{M}_j(\Phi) := \|A_j\|_0 + \|b_j\|_0$ the* number of nonzero weights *in the $j$-th layer, and $\mathrm{M}(\Phi) := \sum_{j=1}^L \mathrm{M}_j(\Phi)$ the* number of nonzero weights *of $\Phi$, also referred to as its* size. *We refer to $N_L$ as the* dimension of the output layer *of $\Phi$.*

$$\mathcal{D} \dashrightarrow^{S} H^1(\Omega)$$

Encoding
$a \mapsto \boldsymbol{a} = a(\boldsymbol{x}_1), \ldots, a(\boldsymbol{x}_n)$

Reconstruction
$\boldsymbol{c} \mapsto u_{\mathrm{NN}}^a = \sum_i c_i \psi_i$

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Approximation
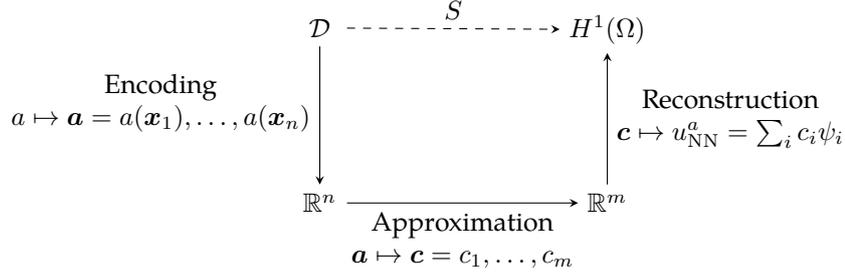$\boldsymbol{a} \mapsto \boldsymbol{c} = c_1, \ldots, c_m$

Fig. 1: Diagram of operator network between infinite dimensional spaces

**3.2. Operator networks.** The operator network approximating the solution operator $S$ can be seen as the composition $\mathcal{R} \circ \mathcal{A} \circ \mathcal{E}$ of three mappings:
- Encoding $\mathcal{E} : \mathcal{D} \to \mathbb{R}^n$, for $n \in \mathbb{N}$,
- Approximation $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, for $m \in \mathbb{N}$,
- Reconstruction $\mathcal{R} : \mathbb{R}^m \to H^1(\Omega)$,

see the diagram in Figure 1. We refer the reader to [12, 10] for a broader view on and thorough discussion of operator networks between infinite dimensional spaces. In our analysis, the encoding step will map functions $a \in \mathcal{D}$ to the vector $\boldsymbol{a} \in \mathbb{R}^n$ of their point evaluations, i.e.

$$\boldsymbol{a} = \mathcal{E}_{\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}}(a) := [a(\boldsymbol{x}_1), \ldots, a(\boldsymbol{x}_n)]^\top,$$

for suitable collection of points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \overline{Q}$. The *approximate solution operator* $\mathcal{A}$ is realized as a feed-forward NN $\Phi^{\mathrm{branch}}$:

$$\mathcal{A}_{\Phi^{\mathrm{branch}}}(\boldsymbol{a}) = \mathrm{R}(\Phi^{\mathrm{branch}})(\boldsymbol{a}).$$

For the reconstruction step $\mathcal{R}$, we introduce a neural network $\Phi^{\mathrm{trunk}}$ so that, for all $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{x} \in \overline{Q}$,

$$\mathcal{R}_{\Phi^{\mathrm{trunk}}}(\boldsymbol{c})(\boldsymbol{x}) = \left(\mathrm{R}(\Phi^{\mathrm{trunk}})(\boldsymbol{x})\right) \cdot \boldsymbol{c}.$$

This constructs the operator network mapping from $\mathcal{D}$ to $H^1(\Omega)$, defined by

$$\mathcal{R}_{\Phi^{\mathrm{trunk}}} \circ \mathcal{A}_{\Phi^{\mathrm{branch}}} \circ \mathcal{E}_{\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}} : a \mapsto u_{\mathrm{NN}}^a := \mathrm{R}(\Phi^{\mathrm{branch}})([a(\boldsymbol{x}_1), \ldots, a(\boldsymbol{x}_n)]^\top) \cdot \mathrm{R}(\Phi^{\mathrm{trunk}}),$$

see Figure 2.

We aim for operator networks that approximate, for all $a \in \mathcal{D}$, solutions $u^a$ of (2.3) in the $H^1(\Omega)$-norm, uniformly over the input space $\mathcal{D}$, i.e., such that

$$\sup_{a \in \mathcal{D}} \|u^a - u_{\mathrm{NN}}^a\|_{H^1(\Omega)} \le \varepsilon.$$

The main result of this paper consists in proofs for upper bounds on $n$, $m$, and on the sizes of $\Phi^{\mathrm{trunk}}$ and $\Phi^{\mathrm{branch}}$ as functions of the error $\varepsilon$.

**3.3. Operations on neural networks.** We introduce and recall some operations on neural networks that will be necessary for the construction of the branch and trunk networks.
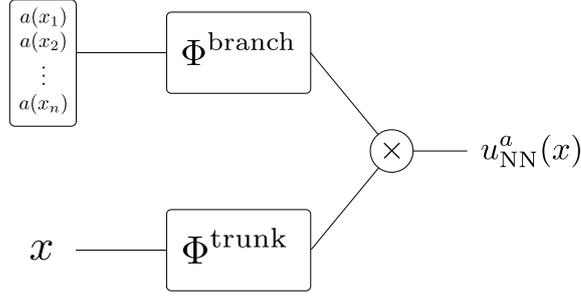
**3.3.1. Concatenation and sparse concatenation.**

Fig. 2: Structure of the branch and trunk network; $u_{\mathrm{NN}}^a(\boldsymbol{x}) := \mathrm{R}(\Phi^{\mathrm{branch}})(\boldsymbol{a}) \cdot \mathrm{R}(\Phi^{\mathrm{trunk}})(\boldsymbol{x})$.

DEFINITION 3.2 (NN concatenation, [19, Definition 2.2]). *Let* $L_1, L_2 \in \mathbb{N}$ *and let*

$$\Phi^1 = ((\mathbf{A}_1^1, \boldsymbol{b}_1^1), \ldots, (\mathbf{A}_{L_1}^1, \boldsymbol{b}_{L_1}^1)), \qquad \Phi^2 = ((\mathbf{A}_1^2, \boldsymbol{b}_1^2), \ldots, (\mathbf{A}_{L_2}^2, \boldsymbol{b}_{L_2}^2))$$

*be two neural networks such that the input layer of* $\Phi^1$ *has the same dimension as the output layer of* $\Phi^2$. *Then,* $\Phi^1 \bullet \Phi^2$ *denotes the following* $L_1 + L_2 - 1$ *layer network:*

$$\Phi^1 \bullet \Phi^2 := ((\mathbf{A}_1^2, \boldsymbol{b}_1^2), \ldots, (\mathbf{A}_{L_2-1}^2, \boldsymbol{b}_{L_2-1}^2), (\mathbf{A}_1^1 \mathbf{A}_{L_2}^2, \mathbf{A}_1^1 \boldsymbol{b}_{L_2}^2 + \boldsymbol{b}_1^1), (\mathbf{A}_2^1, \boldsymbol{b}_2^1), \ldots, (\mathbf{A}_{L_1}^1, \boldsymbol{b}_{L_1}^1)).$$

*We call* $\Phi^1 \bullet \Phi^2$ *the* concatenation *of* $\Phi^1$ *and* $\Phi^2$.

PROPOSITION 3.3 (NN sparse concatenation, [19, Remark 2.6]). *Let* $L_1, L_2 \in \mathbb{N}$, *and let* $\Phi^1, \Phi^2$ *be two NNs of respective depths* $L_1$ *and* $L_2$ *such that* $N_0^1 = N_{L_2}^2 =: d$, *i.e., the input layer of* $\Phi^1$ *has the same dimension as the output layer of* $\Phi^2$.

*Then, there exists a NN* $\Phi^1 \odot \Phi^2$, *called the* sparse concatenation *of* $\Phi^1$ *and* $\Phi^2$, *such that* $\Phi^1 \odot \Phi^2$ *has* $L_1 + L_2$ *layers,* $\mathrm{R}(\Phi^1 \odot \Phi^2) = \mathrm{R}(\Phi^1) \circ \mathrm{R}(\Phi^2)$ *and* $\mathrm{M}(\Phi^1 \odot \Phi^2) \leq 2\,\mathrm{M}(\Phi^1) + 2\,\mathrm{M}(\Phi^2)$.

**3.3.2. Emulation of matrix inversion.** Dense matrix inversion can be approximated by suitable ReLU NNs. We recall the following result from [11] where, for $Z \in \mathbb{R}_+$ and $N \in \mathbb{N}$,

$$K_N^Z := \{\mathbf{vec}(\mathbf{A}) : \mathbf{A} \in \mathbb{R}^{N \times N}, \|\mathbf{A}\|_2 \leq Z\}.$$

THEOREM 3.4. [11, Theorem 3.8] *For* $\varepsilon, \delta \in (0, 1)$ *define*

$$m(\varepsilon, \delta) := \left\lceil \frac{\log(0.5\varepsilon\delta)}{\log(1-\delta)} \right\rceil.$$

*There exists a universal constant* $C_{\mathrm{inv}} > 0$ *such that for every* $N \in \mathbb{N}$, $\varepsilon \in (0, 1/4)$ *and every* $\delta \in (0, 1)$ *there exists a NN* $\Phi_{\mathrm{inv};\varepsilon}^{1-\delta,N}$ *with* $N^2$-*dimensional input,* $N^2$-*dimensional output and the following properties:*

1. $\mathrm{L}\left(\Phi_{\mathrm{inv};\varepsilon}^{1-\delta,N}\right) \leq C_{\mathrm{inv}} \log\left(m(\varepsilon, \delta)\right) \cdot \left(\log\left(1/\varepsilon\right) + \log\left(m(\varepsilon, \delta)\right) + \log(N)\right)$,
2. $\mathrm{M}\left(\Phi_{\mathrm{inv};\varepsilon}^{1-\delta,N}\right) \leq C_{\mathrm{inv}} m(\varepsilon, \delta) \log^2(m(\varepsilon, \delta)) N^3 \cdot \left(\log\left(1/\varepsilon\right) + \log\left(m(\varepsilon, \delta)\right) + \log(N)\right)$,
3. $\sup_{\mathbf{vec}(\mathbf{A}) \in K_N^{1-\delta}} \left\| (\mathbf{Id}_N - \mathbf{A})^{-1} - \mathbf{matr}\left(\mathrm{R}\left(\Phi_{\mathrm{inv};\varepsilon}^{1-\delta,N}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 \leq \varepsilon$,
4. *for any* $\mathbf{vec}(\mathbf{A}) \in K_N^{1-\delta}$ *we have*

$$\left\| \mathbf{matr}\left(\mathrm{R}\left(\Phi_{\mathrm{inv};\varepsilon}^{1-\delta,N}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 \leq \varepsilon + \left\| (\mathbf{Id}_N - \mathbf{A})^{-1} \right\|_2 \leq \varepsilon + \frac{1}{1 - \|\mathbf{A}\|_2} \leq \varepsilon + \frac{1}{\delta}.$$

**4. Regularity and polynomial approximation.** We shall exploit the classical fact that the analyticity of the coefficient $a$ and of the source term $f$ in $\Omega$ combined with periodicity implies analyticity of the solution $u$ of (2.3). This, in turn, will imply exponential convergence of tensor product polynomial (spectral) approximations, which will be the basis of the NN approximation developed in Section 5 ahead.

**4.1. Regularity.** The following result follows from [5, Remark 1.6.5 and Theorem 1.7.1].

LEMMA 4.1. *There exists $A_{\mathcal{U}} > 0$ such that $S(\mathcal{D}) \subset \mathrm{Hol}(\Omega; A_{\mathcal{U}})$.*

*Proof.* From [5, Theorem 1.7.1], it follows that $S(\mathcal{D}) \subset \mathrm{Hol}(\Omega)$ and, for each $u \in S(\mathcal{D})$, there exists $A_u > 0$ such that

$$\frac{1}{k!}|u|_{H^k(\Omega)} \le A_u^{k+1} \left( \sum_{j=0}^{k-2} \frac{1}{j!}|f|_{H^j(\Omega)} + \|u\|_{H^1(\Omega)} \right), \qquad \forall k \in \mathbb{N}_0.$$

Furthermore, from [5, Remark 1.6.5], inspecting the proof of [5, Theorem 1.7.1], and from (2.2), the proof is completed since it follows that

$$A_{\mathcal{U}} := \sup_{u \in S(\mathcal{D})} A_u < \infty.$$

$\square$

**4.2. Polynomial basis and quadrature.** For all $p \in \mathbb{N}$, writing $n_{\mathrm{b}} = (p+1)^d - 1$, we introduce the $L^2(\Omega)$-orthonormal basis functions $\{\varphi_i\}_{i=1}^{n_{\mathrm{b}}} \subset \mathbb{Q}_p(Q)$ with

$$(4.1) \qquad \int_\Omega \varphi_i = 0, \quad \forall i \in \{1, \dots, n_{\mathrm{b}}\} \qquad \text{and} \qquad \int_\Omega \varphi_i \varphi_j = \delta_{ij}, \quad \forall (i,j) \in \{1, \dots, n_{\mathrm{b}}\}^2.$$

Those functions are $L^2(\Omega)$ normalized Legendre polynomials, excluding the constant function. In particular,

$$X_{n_{\mathrm{b}}} := \mathrm{span}(\{\varphi_1, \dots, \varphi_{n_{\mathrm{b}}}\}) \subset X.$$

For a quadrature order parameter $q \ge 2$, denoting $n_{\mathrm{q}} = q^d$, we consider the *Gauss-Lobatto quadrature rule* with weights $\{w_k^{(q)}\}_{k=1}^{n_{\mathrm{q}}}$ and points $\{\boldsymbol{x}_k^{(q)}\}_{k=1}^{n_{\mathrm{q}}}$ such that

$$\int_\Omega f = \sum_{k=1}^{n_{\mathrm{q}}} w_k^{(q)} f(\boldsymbol{x}_k^{(q)}), \qquad \forall f \in \mathbb{Q}_{2q-3}(\Omega).$$

There exist constants $c_{\mathrm{quad},1}, c_{\mathrm{quad},2} > 0$ such that

$$(4.2) \qquad c_{\mathrm{quad},1}\|v\|_{L^2(\Omega)}^2 \le \sum_{k=1}^{(p+1)^d} w_k^{(p+1)}(v(\boldsymbol{x}_k^{(p+1)}))^2 \le c_{\mathrm{quad},2}\|v\|_{L^2(\Omega)}^2, \quad \forall v \in \mathbb{Q}_p(\Omega), \forall p \in \mathbb{N},$$

see [3, Equation (6.4.52)]. We assume, for ease of notation, that $c_{\mathrm{quad},1} \le 1$ and $c_{\mathrm{quad},2} \ge 1$. We introduce furthermore the bilinear form with quadrature $\mathfrak{b}_{n_{\mathrm{q}}}^a$

$$\mathfrak{b}_{n_{\mathrm{q}}}^a(u,v) = \sum_{k=1}^{n_{\mathrm{q}}} w_k^{(q)} a(\boldsymbol{x}_k^{(q)}) \nabla u(\boldsymbol{x}_k^{(q)}) \cdot \nabla v(\boldsymbol{x}_k^{(q)}), \qquad \forall u,v \in C^1(\Omega).$$

Eventually, here $u, v$ shall be tensor product polynomials in $\Omega$.

For each $a \in \mathcal{D}$, we introduce the matrices

$$\left[\mathbf{A}_{n_\mathrm{b}}^a\right]_{ij} = \mathfrak{b}^a(\varphi_j, \varphi_i), \qquad \left[\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a\right]_{ij} = \mathfrak{b}_{n_\mathrm{q}}^a(\varphi_j, \varphi_i), \qquad (i, j) \in \{1, \ldots, n_\mathrm{b}\}^2.$$

Let $\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1$ be the matrix obtained with $a \equiv 1$ in $\Omega$. Let $q \geq p + 1$: for all nonzero $\boldsymbol{x} \in \mathbb{R}^{n_\mathrm{b}}$, there exists $v \in X_{n_\mathrm{b}} \backslash \{0\}$ such that, for all $a \in \mathcal{D}$, $\boldsymbol{x}^\top \mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a \boldsymbol{x} = \mathfrak{b}_{n_\mathrm{q}}^a(v, v) > 0$, due to the equivalence of norms (4.2) and to the Poincaré inequality (2.5). Hence, the matrices $\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a$ and $\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1$ are invertible. Denote then

$$\widetilde{\mathbf{A}}_{n_\mathrm{b}, n_\mathrm{q}}^a = (\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1)^{-1} \mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a.$$

We also introduce the right-hand side vector $\boldsymbol{c}_{f; n_\mathrm{b}} \in \mathbb{R}^{n_\mathrm{b}}$ such that

$$[\boldsymbol{c}_{f; n_\mathrm{b}}]_i = \int_\Omega f \varphi_i, \qquad i \in \{1, \ldots, n_\mathrm{b}\},$$

and the vector

$$\widetilde{\boldsymbol{c}}_{f; n_\mathrm{b}} = (\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1)^{-1} \boldsymbol{c}_{f; n_\mathrm{b}}.$$

Finally, let

(4.3) $$\boldsymbol{c}_{u; n_\mathrm{b}, n_\mathrm{q}}^a := (\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a)^{-1} \boldsymbol{c}_{f; n_\mathrm{b}}.$$

Here and in the sequel, for all $q \in \mathbb{N}$, with $n_\mathrm{q} = q^d$, we will denote $\boldsymbol{a}_{n_\mathrm{q}} \in \mathbb{R}^{n_\mathrm{q}}$ the vector with entries

$$\left[\boldsymbol{a}_{n_\mathrm{q}}\right]_i = a(\boldsymbol{x}_k^{(q)}), \qquad \forall k \in \{1, \ldots, n_\mathrm{q}\}.$$

The following two statements concern the norms of the matrices introduced, and will be necessary for later estimates.

LEMMA 4.2. *There exist $C_\mathrm{coer}, C_\mathrm{cont} > 0$ such that for all $a \in \mathcal{D}$, all $p \in \mathbb{N}$, and all integer $q \geq p + 1$,*

(4.4) $$\sigma(\widetilde{\mathbf{A}}_{n_\mathrm{b}, n_\mathrm{q}}^a) \subset [C_\mathrm{coer}, C_\mathrm{cont}], \qquad \sigma((\widetilde{\mathbf{A}}_{n_\mathrm{b}, n_\mathrm{q}}^a)^{-1}) \subset [1/C_\mathrm{cont}, 1/C_\mathrm{coer}],$$

*with $n_\mathrm{b} = (p + 1)^d - 1$ and $n_\mathrm{q} = q^d$.*

*Proof.* For all $\boldsymbol{x} \in \mathbb{R}^{n_\mathrm{b}}$,

$$a_\mathrm{min} \boldsymbol{x}^\top \mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1 \boldsymbol{x} \leq \boldsymbol{x}^\top \mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a \boldsymbol{x} \leq a_\mathrm{max} \boldsymbol{x}^\top \mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1 \boldsymbol{x}.$$

Since the matrices $\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1$ and $\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a$ are symmetric, this implies

$$\sigma((\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1)^{-1} \mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^a) \subset [a_\mathrm{min}, a_\mathrm{max}].$$

The assertion follows with $C_\mathrm{coer} = a_\mathrm{min}$ and $C_\mathrm{cont} = a_\mathrm{max}$. $\qquad \square$

LEMMA 4.3. *There exists a constant $C_\mathrm{A} > 0$ such that, for all $p \in \mathbb{N}$, and for all integer $q \geq p + 1$,*

$$\|(\mathbf{A}_{n_\mathrm{b}, n_\mathrm{q}}^1)^{-1}\|_2 \leq C_\mathrm{A},$$

*with $n_\mathrm{b} = (p + 1)^d - 1$ and $n_\mathrm{q} = q^d$.*

*Proof.* From (4.2) and (2.5) and the symmetry of the bilinear form, it follows that

$$\lambda_{\min}\left(\mathbf{A}^1_{n_b,n_q}\right) := \min_{\lambda \in \sigma(\mathbf{A}^1_{n_b,n_q})} \lambda \geq \inf_{v \in X_{n_b}} \frac{\mathfrak{b}^1_{n_q}(v,v)}{\|v\|^2_{L^2(\Omega)}} \stackrel{(4.2)}{\geq} c^2_{\text{quad},1} \inf_{v \in X_{n_b}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \stackrel{(2.5)}{\geq} \frac{c^2_{\text{quad},1}}{C^2_{\text{poi}}}.$$

This concludes the proof, since $\|(\mathbf{A}^1_{n_b,n_q})^{-1}\|_2 = 1/\lambda_{\min}\left(\mathbf{A}^1_{n_b,n_q}\right)$. $\qquad\square$

**5. NN approximation.** We detail the structure of the branch and trunk networks and state and prove their convergence rate bounds.

**5.1. Branch network.**

**5.1.1. Input layer.** For all $k \in \{1, \ldots, n_q\}$, denote $\widehat{\mathbf{D}}_{n_b}(\boldsymbol{x}^{(q)}_k)$ the matrix with entries

$$\left[\widehat{\mathbf{D}}_{n_b}(\boldsymbol{x}^{(q)}_k)\right]_{ij} = w^{(q)}_k \nabla \varphi_i(\boldsymbol{x}^{(q)}_k) \cdot \nabla \varphi_j(\boldsymbol{x}^{(q)}_k), \qquad (i,j) \in \{1,\ldots,n_b\}^2.$$

The following statement follows from this definition.

LEMMA 5.1. *For all $\alpha \in \mathbb{R}$ the one layer NN*

$$\Phi^{A,\alpha}_{n_b,n_q} := \left(\left(-\alpha\left[\mathbf{vec}(\widehat{\mathbf{D}}_{n_b}(\boldsymbol{x}^{(q)}_1))|\ldots|\mathbf{vec}(\widehat{\mathbf{D}}_{n_b}(\boldsymbol{x}^{(q)}_{n_q}))\right], \mathbf{0}_{n_b}\right)\right)$$

*is such that*

(5.1) $$\mathbf{matr}\left(\mathrm{R}(\Phi^{A,\alpha}_{n_b,n_q})(\boldsymbol{a}_{n_q})\right) = -\alpha\mathbf{A}^a_{n_b,n_q},$$

*and* $\mathrm{M}(\Phi^{A,\alpha}_{n_b,n_q}) \leq n^2_b n_q$ .

*Proof.* We have

$$\left[\mathrm{R}\left(\Phi^{A,\alpha}_{n_b,n_q}\right)(\boldsymbol{a}_{n_q})\right]_i = -\alpha\sum_{k=1}^{n_q}\left[\mathbf{vec}(\widehat{\mathbf{D}}(\boldsymbol{x}^{(q)}_k))\right]_i a(\boldsymbol{x}^{(q)}_k),$$

hence the equality after matricization. The size bound follows from the fact that

$$\|\widehat{\mathbf{D}}_{n_b}(x^{(q)}_k)\|_0 \leq n_b, \quad k \in \{1, \ldots, n_q\} .$$

$\square$

LEMMA 5.2. *For all $\alpha \in \mathbb{R}$ the two layer NN*

(5.2) $$\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_b,n_q} := \left(\left(\mathbf{Id}_{n_b} \otimes (\mathbf{A}^1_{n_b,n_q})^{-1}, \mathbf{vec}(\mathbf{Id}_{n_b})\right)\right) \odot \Phi^{A,\alpha}_{n_b,n_q}$$

*is such that*

$$\mathbf{matr}\left(\mathrm{R}(\Phi^{A,\mathrm{Id},\alpha}_{n_b,n_q})(\boldsymbol{a}_{n_q})\right) = -\alpha\widetilde{\mathbf{A}}^a_{n_b,n_q} + \mathbf{Id}_{n_b},$$

*and* $\mathrm{M}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_b,n_q}) \leq 2n^2_b n_q + 2n^3_b + 2n_b$ .

*Proof.* For all $m, n, l \in \mathbb{N}$ and all $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{X} \in \mathbb{R}^{n \times l}$, $\mathbf{vec}(\mathbf{AX}) = (\mathbf{Id}_l \otimes \mathbf{A})\mathbf{vec}(\mathbf{X})$. Hence,

$$\begin{aligned}\mathrm{R}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_b,n_q}) &\stackrel{(5.2)}{=} (\mathbf{Id}_{n_b} \otimes (\mathbf{A}^1_{n_b,n_q})^{-1})\mathrm{R}(\Phi^{A,\alpha}_{n_b,n_q}) + \mathbf{vec}(\mathbf{Id}_{n_b})\\ &\stackrel{(5.1)}{=} -\alpha(\mathbf{Id}_{n_b} \otimes (\mathbf{A}^1_{n_b,n_q})^{-1})\mathbf{vec}(\mathbf{A}^a_{n_b,n_q}) + \mathbf{vec}(\mathbf{Id}_{n_b})\\ &= -\alpha\mathbf{vec}((\mathbf{A}^1_{n_b,n_q})^{-1}\mathbf{A}^a_{n_b,n_q}) + \mathbf{vec}(\mathbf{Id}_{n_b}).\end{aligned}$$

Finally, by Proposition 3.3,

$$\mathrm{M}(\Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{\widetilde{A},\mathrm{Id},\alpha}) \leq 2\,\mathrm{M}(\Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{A,\alpha}) + 2\left(\|\mathbf{Id}_{n_{\mathrm{b}}} \otimes (\mathbf{A}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{1})^{-1}\|_0 + \|\mathbf{Id}_{n_{\mathrm{b}}}\|_0\right) \leq 2\,\mathrm{M}(\Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{A,\alpha}) + 2(n_{\mathrm{b}}^3 + n_{\mathrm{b}}). \;\square$$

The following statement is a consequence of Lemmas 4.2 and 5.2.

LEMMA 5.3. *Let $C_{\mathrm{cont}}, C_{\mathrm{coer}}$ be the constants introduced in Lemma 4.2. For all $a \in \mathcal{D}$, for all $p \in \mathbb{N}$, for all integer $q \geq p+1$, and for all $\alpha \in (0, 1/C_{\mathrm{cont}})$,*

$$\|\mathbf{matr}\left(\mathrm{R}(\Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{\widetilde{A},\mathrm{Id},\alpha})(\boldsymbol{a}_{n_{\mathrm{q}}})\right)\|_2 \leq 1 - \alpha C_{\mathrm{coer}} =: 1 - \delta,$$

*with $n_{\mathrm{b}} = (p+1)^d - 1$ and $n_{\mathrm{q}} = q^d$.*

*Proof.* By Lemma 5.2, $\mathbf{matr}\left(\mathrm{R}(\Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{\widetilde{A},\mathrm{Id},\alpha})(\boldsymbol{a}_{n_{\mathrm{q}}})\right) = \mathbf{Id}_{n_{\mathrm{b}}} - \alpha\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a}$. Due to Lemma 4.2 and since $\alpha \leq 1/C_{\mathrm{cont}}$, this matrix is symmetric positive definite and

$$\|\mathbf{Id}_{n_{\mathrm{b}}} - \alpha\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a}\|_2 = \sup_{\substack{\boldsymbol{x}\in\mathbb{R}^{n_{\mathrm{b}}} \\ \|\boldsymbol{x}\|_2=1}} \boldsymbol{x}^{\top}(\mathbf{Id}_{n_{\mathrm{b}}} - \alpha\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})\boldsymbol{x} = 1 - \alpha\inf_{\substack{\boldsymbol{x}\in\mathbb{R}^{n_{\mathrm{b}}} \\ \|\boldsymbol{x}\|_2=1}} \boldsymbol{x}^{\top}\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a}\boldsymbol{x} \leq 1 - \alpha C_{\mathrm{coer}},$$

where the last inequality follows from Lemma 4.2. $\qquad\square$

Thanks to Theorem 3.4 we can construct the network that approximates the inversion of the "preconditioned Galerkin-Numerical Integration matrix" $\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a}$ (more precisely, the network that emulates the map $\boldsymbol{a}_{n_{\mathrm{q}}} \mapsto (\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1}$).

PROPOSITION 5.4. *Let $C_{\mathrm{coer}}, C_{\mathrm{cont}}$ be defined as in Lemma 4.2. There exists a constant $C_{\mathrm{inv},A} > 0$ such that for all $n_{\mathrm{b}} \in \mathbb{N}$ and for all $\varepsilon_{\mathrm{inv}} \in (0,1)$, writing $\alpha = 1/(C_{\mathrm{coer}} + C_{\mathrm{cont}})$, $\delta = \alpha C_{\mathrm{coer}}$, $n_{\mathrm{q}} = n_{\mathrm{b}} + 1$, and denoting*

$$(5.3) \qquad \Phi_{\mathrm{inv};\varepsilon_{\mathrm{inv}},n_{\mathrm{b}}}^{\widetilde{A}} := ((\alpha\mathbf{Id}_{n_{\mathrm{b}}}, \mathbf{0}_{n_{\mathrm{b}}})) \bullet \Phi_{\mathrm{inv};\frac{\varepsilon_{\mathrm{inv}}}{\alpha}}^{1-\delta,n_{\mathrm{b}}} \odot \Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{\widetilde{A},\mathrm{Id},\alpha},$$

*we have*

$$\sup_{a\in\mathcal{D}} \|(\mathbf{A}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi_{\mathrm{inv};\varepsilon_{\mathrm{inv}},n_{\mathrm{b}}}^{\widetilde{A}}))(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2 \leq \varepsilon_{\mathrm{inv}},$$

*and*

$$\mathrm{L}(\Phi_{\mathrm{inv};\varepsilon_{\mathrm{inv}},n_{\mathrm{b}}}^{\widetilde{A}}) \leq C_{\mathrm{inv},A}\left[\log(1+|\log\varepsilon_{\mathrm{inv}}|) + \log(n_{\mathrm{b}})\right] \cdot \left[|\log\varepsilon_{\mathrm{inv}}| + \log(n_{\mathrm{b}}) + \log(1+|\log\varepsilon_{\mathrm{inv}}|)\right]$$

$$\mathrm{M}(\Phi_{\mathrm{inv};\varepsilon_{\mathrm{inv}},n_{\mathrm{b}}}^{\widetilde{A}}) \leq C_{\mathrm{inv},A}\,n_{\mathrm{b}}^3\left[1+|\log\varepsilon_{\mathrm{inv}}|\right]\left[\log(1+|\log\varepsilon_{\mathrm{inv}}|) + \log(n_{\mathrm{b}})\right]^2$$

$$\times \left[|\log\varepsilon_{\mathrm{inv}}| + \log(n_{\mathrm{b}}) + \log(1+|\log\varepsilon_{\mathrm{inv}}|)\right]$$

*Proof.* We start by estimating the approximation error. By Lemma 5.3,

$$\|\mathbf{matr}(\mathrm{R}(\Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{\widetilde{A},\mathrm{Id},\alpha}(\boldsymbol{a}_{n_{\mathrm{q}}})))\|_2 \leq 1 - \delta.$$

Then, we have, for all $a \in \mathcal{D}$,

$$\|(\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi_{\mathrm{inv};\varepsilon_{\mathrm{inv}},n_{\mathrm{b}}}^{\widetilde{A}}))(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2$$

$$\overset{(5.3)}{=} \alpha\|(\alpha\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi_{\mathrm{inv};\frac{\varepsilon_{\mathrm{inv}}}{\alpha}}^{1-\delta,n_{\mathrm{b}}} \odot \Phi_{n_{\mathrm{b}},n_{\mathrm{q}}}^{\widetilde{A},\mathrm{Id},\alpha}))(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2$$

$$\overset{\mathrm{L.}\,5.2}{=} \alpha\|(\alpha\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi_{\mathrm{inv};\frac{\varepsilon_{\mathrm{inv}}}{\alpha}}^{1-\delta,n_{\mathrm{b}}})(-\alpha\,\mathbf{vec}(\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a}) + \mathbf{vec}(\mathbf{Id}_{n_{\mathrm{b}}}))\|_2$$

$$\overset{\mathrm{T.}\,3.4}{\leq} \alpha\frac{\varepsilon_{\mathrm{inv}}}{\alpha} = \varepsilon_{\mathrm{inv}}.$$

We now have to bound the depth and size of $\Phi_{\text{inv};\varepsilon_{\text{inv}},n_{\text{b}}}^{\widetilde{A}}$. First, we remark that

$$m(\varepsilon_{\text{inv}}/\alpha, \delta) = \left\lceil \frac{\log\left(C_{\text{coer}}\varepsilon_{\text{inv}}/2\right)}{\log(1\delta)} \right\rceil,$$

where $m(\cdot, \cdot)$ is defined in Theorem 3.4. Now, we use the fact that there exists $C_1 > 0$ such that, for all $\varepsilon_{\text{inv}} \in (0, 1)$,

$$|\log(C_{\text{coer}}\varepsilon_{\text{inv}}/2)| \leq C_1(1 + |\log \varepsilon_{\text{inv}}|).$$

Furthermore, there exists $C_2 > 0$ such that for all $n_{\text{b}} \in \mathbb{N}, \delta \geq C_2^{-1}$. Remark then that $|\log(1 - y)| \geq y$ for all $y \in (0, 1)$, hence $|\log(1 - \delta)|^{-1} \leq C_2$. We infer that for all $\varepsilon_{\text{inv}} \in (0, 1)$ and for all $n_{\text{b}} \in \mathbb{N}$,

$$m(\varepsilon_{\text{inv}}/\alpha, \delta) \leq C_1 C_2(1 + |\log \varepsilon_{\text{inv}}|)$$

Therefore, from Theorem 3.4 we obtain that there exist constants $C_4, C_5 > 0$ dependent only on $C_{\text{coer}}, C_{\text{cont}}$, and $d$ such that

$$\text{L}\left(\Phi_{\text{inv};\frac{\varepsilon_{\text{inv}}}{\alpha}}^{1-\delta,n_{\text{b}}}\right) \leq C_4\left(\log(1 + |\log \varepsilon_{\text{inv}}|) + \log(n_{\text{b}})\right) \cdot \left(|\log \varepsilon_{\text{inv}}| + \log(n_{\text{b}}) + \log(1 + |\log \varepsilon_{\text{inv}}|)\right)$$

and

$$\text{M}\left(\Phi_{\text{inv};\frac{\varepsilon_{\text{inv}}}{\alpha}}^{1-\delta,n_{\text{b}}}\right) \leq C_5(1 + |\log \varepsilon_{\text{inv}}|)n_{\text{b}}^3\left[\log(1 + |\log \varepsilon_{\text{inv}}|) + \log(n_{\text{b}})\right]^2$$
$$\times \left[|\log \varepsilon_{\text{inv}}| + \log(n_{\text{b}}) + \log(1 + |\log \varepsilon_{\text{inv}}|)\right].$$

Since, in addition,

$$\text{L}(\Phi_{n_{\text{b}},n_{\text{q}}}^{\widetilde{A},\text{Id},\alpha}) = 2, \qquad \text{M}(\Phi_{n_{\text{b}},n_{\text{q}}}^{\widetilde{A},\text{Id},\alpha}) \leq C_6 n_{\text{b}}^3,$$

for $C_6 > 0$ independent of $n_{\text{b}}$, we obtain the bounds on the depth and size of $\Phi_{\text{inv};\varepsilon_{\text{inv}},n_{\text{b}}}^{\widetilde{A}}$.   □

### 5.1.2. Computation of the coefficients.

PROPOSITION 5.5. *There exists a constant $C_{c_u} > 0$ such that for all $n_{\text{b}} \in \mathbb{N}$ and for all $\varepsilon_u \in (0, 1)$, writing $n_{\text{q}} = n_{\text{b}} + 1$ and*

$$\Phi_{\varepsilon_u,n_{\text{b}}}^{c_u} := \left((\widetilde{\boldsymbol{c}}_{f;n_{\text{b}}}^{\top} \otimes \mathbf{Id}_{n_{\text{b}}}, \mathbf{0}_{n_{\text{b}}})\right) \odot \Phi_{\text{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}C_A),n_{\text{b}}}^{\widetilde{A}},$$

*where $C_A$ is the constant from Lemma 4.3, we have*

$$\sup_{a \in \mathcal{D}} \|\boldsymbol{c}_{u;n_{\text{b}},n_{\text{q}}}^a - \text{R}(\Phi_{\varepsilon_u,n_{\text{b}}}^{c_u})(\boldsymbol{a}_{n_{\text{q}}})\|_2 \leq \varepsilon_u$$

*and*

$$\text{L}(\Phi_{\varepsilon_u,n_{\text{b}}}^{c_u}) \leq C_{c_u}\left[\log(1 + |\log \varepsilon_u|) + \log(n_{\text{b}})\right] \cdot \left[|\log \varepsilon_u| + \log(n_{\text{b}}) + \log(1 + |\log \varepsilon_u|)\right]$$
$$\text{M}(\Phi_{\varepsilon_u,n_{\text{b}}}^{c_u}) \leq C_{c_u}n_{\text{b}}^3\left[1 + |\log \varepsilon_u|\right]\left[\log(1 + |\log \varepsilon_u|) + \log(n_{\text{b}})\right]^2$$
$$\times \left[|\log \varepsilon_u| + \log(n_{\text{b}}) + \log(1 + |\log \varepsilon_u|)\right].$$

*Proof.* We will use the following identity: for all $m, n, l \in \mathbb{N}$, let $\mathbf{B} \in \mathbb{R}^{m \times n}$ and $\mathbf{C} \in \mathbb{R}^{n \times l}$. Then,

$$\mathbf{vec}(\mathbf{BC}) = (\mathbf{C}^{\top} \otimes \mathbf{Id}_m)\,\mathbf{vec}(\mathbf{B}).$$

Then

$$(5.4) \quad \left( \widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}^{\top} \otimes \mathbf{Id}_{n_{\mathrm{b}}} \right) \mathrm{R}(\Phi_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}C_{\mathrm{A}}),n_{\mathrm{b}}}^{\widetilde{A}})(\boldsymbol{a}_{n_{\mathrm{q}}})$$
$$= \mathbf{matr}\left( \mathrm{R}(\Phi_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}C_{\mathrm{A}}),n_{\mathrm{b}}}^{\widetilde{A}})(\boldsymbol{a}_{n_{\mathrm{q}}}) \right) \widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}.$$

Therefore, for all $a \in \mathcal{D}$,

$$\|\boldsymbol{c}_{u;n_{\mathrm{b}},n_{\mathrm{q}}}^{a} - \mathrm{R}(\Phi_{\varepsilon_u,n_{\mathrm{b}}}^{c_u})(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2$$
$$\stackrel{(4.3)}{=} \| \left( (\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1} - \mathbf{matr}\left( \mathrm{R}(\Phi_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}C_{\mathrm{A}}),n_{\mathrm{b}}}^{\widetilde{A}})(\boldsymbol{a}_{n_{\mathrm{q}}}) \right) \right) \widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}\|_2$$
$$\leq \|(\widetilde{\mathbf{A}}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{a})^{-1} - \mathbf{matr}\left( \mathrm{R}(\Phi_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}C_{\mathrm{A}}),n_{\mathrm{b}}}^{\widetilde{A}})(\boldsymbol{a}_{n_{\mathrm{q}}}) \right) \|_2 \|\widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}\|_2$$
$$\stackrel{\mathrm{P.\,5.4}}{\leq} \frac{\varepsilon_u}{\|f\|_{L^2(\Omega)}}\|(\mathbf{A}_{n_{\mathrm{b}},n_{\mathrm{q}}}^{1})^{-1}\|_2 \|\boldsymbol{c}_{f;n_{\mathrm{b}}}\|_2$$
$$\stackrel{\mathrm{L.\,4.3,\,(4.1)}}{\leq} \frac{\varepsilon_u}{C_{\mathrm{A}}\|f\|_{L^2(\Omega)}} C_{\mathrm{A}}\|f\|_{L^2(\Omega)} = \varepsilon_u,$$

where in the last two steps we have used the orthonormality of the basis, Proposition 5.4, and Lemma 4.3. To derive the bounds on the size and depth of $\Phi_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}C_{\mathrm{A}}),n_{\mathrm{b}}}^{\widetilde{A}}$, it is sufficient to remark that

$$\|\widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}^{\top} \otimes \mathbf{Id}_{n_{\mathrm{b}}}\|_0 \leq n_{\mathrm{b}}^2,$$

that replacing $\varepsilon_{\mathrm{inv}}$ with $\varepsilon_u/(\|f\|_{L^2(\Omega)}C_{\mathrm{A}})$ can be absorbed by a change of constants in Proposition 5.4, and conclude with Proposition 5.4. $\qquad \square$

**5.2. Trunk network.** The following emulation rates for the approximation of the polynomial basis are a direct consequence of [18, Proposition 2.13].

PROPOSITION 5.6. *There exists $C_{\mathrm{b}} > 0$ such that, for all $\varepsilon_{\mathrm{b}} \in (0,1)$ and all $n_{\mathrm{b}} \in \mathbb{N}$, there exists a NN $\Phi_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}^{\mathrm{b}}$ such that $\mathrm{R}(\Phi_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}^{\mathrm{b}}) : \mathbb{R}^d \to \mathbb{R}^{n_{\mathrm{b}}}$, that*

$$\max_{i \in \{1,\dots,n_{\mathrm{b}}\}} \|\varphi_i - \left[\mathrm{R}(\Phi_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}^{\mathrm{b}})\right]_i \|_{H^1(\Omega)} \leq \varepsilon_{\mathrm{b}},$$

*and that*

$$\mathrm{L}(\Phi_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}^{\mathrm{b}}) \leq C_{\mathrm{b}}\left( |\log \varepsilon_{\mathrm{b}}| + n_{\mathrm{b}}^{1/d} \right) \log n_{\mathrm{b}}$$
$$\mathrm{M}(\Phi_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}^{\mathrm{b}}) \leq C_{\mathrm{b}}\left( n_{\mathrm{b}}^{2/d} + n_{\mathrm{b}}^{1/d}|\log \varepsilon_{\mathrm{b}}| + n_{\mathrm{b}}(1 + \log n_{\mathrm{b}} + |\log \varepsilon_{\mathrm{b}}|) \right).$$

**5.3. Operator network expression rates.** Combining the results from Sections 5.1 and 5.2, we obtain the main result on the operator network approximation of (2.3). The general structure of the operator network is represented in Figures 1 and 2.

THEOREM 5.7. *There exists $C > 0$ such that, for all $\varepsilon \in (0,1)$, for all $a \in \mathcal{D}$ with $u^a = S(a)$ as defined in Section 2, there exist*
   *(a) $n_{\mathrm{b}}, n_{\mathrm{q}} \in \mathbb{N}$,*
   *(b) a set of points $\boldsymbol{x}_{\mathrm{enc}} := \{\boldsymbol{x}_1, \dots \boldsymbol{x}_{n_{\mathrm{q}}}\} \subset \overline{Q}$,*
   *(c) two NNs $\Phi_{\varepsilon}^{\mathrm{br}}$ and $\Phi_{\varepsilon}^{\mathrm{tr}}$ with $\mathrm{R}(\Phi_{\varepsilon}^{\mathrm{br}}) : \mathbb{R}^{n_{\mathrm{q}}} \to \mathbb{R}^{n_{\mathrm{b}}}$ and $\mathrm{R}(\Phi_{\varepsilon}^{\mathrm{tr}}) : Q \to \mathbb{R}^{n_{\mathrm{b}}}$,*
*such that*

(i) $n_{\mathrm{b}}, n_{\mathrm{q}} \leq C(1 + |\log \varepsilon|^d)$,

(ii) *the following error bound holds:*

$$\sup_{a \in \mathcal{D}} \|u^a - (\mathcal{R}_{\Phi_\varepsilon^{\mathrm{tr}}} \circ \mathcal{A}_{\Phi_\varepsilon^{\mathrm{br}}} \circ \mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}})(a)\|_{H^1(\Omega)} \leq \varepsilon,$$

(iii) *as* $\varepsilon \downarrow 0$,

$$\mathrm{L}(\Phi_\varepsilon^{\mathrm{br}}) = \mathcal{O}\left(|\log \varepsilon| (\log |\log \varepsilon|)\right), \qquad \mathrm{M}(\Phi_\varepsilon^{\mathrm{br}}) = \mathcal{O}\left(|\log \varepsilon|^{3d+2} (\log |\log \varepsilon|)^2\right),$$

*and*

$$\mathrm{L}(\Phi_\varepsilon^{\mathrm{tr}}) = \mathcal{O}\left(|\log \varepsilon| (\log |\log \varepsilon|)\right), \qquad \mathrm{M}(\Phi_\varepsilon^{\mathrm{tr}}) = \mathcal{O}\left(|\log \varepsilon|^{d+1}\right).$$

*Proof.* Due to Lemma A.2, there exist constants $C_{\mathrm{G}}, b_{\mathrm{G}}, C_{\mathrm{q}} > 0$ such that for all $n_{\mathrm{b}} \in \mathbb{N}$, there exists $n_{\mathrm{q}} \leq C_{\mathrm{q}} n_{\mathrm{b}}$ such that

$$\sup_{a \in \mathcal{D}} \|u^a - u_{n_{\mathrm{b}}, n_{\mathrm{q}}}^a\|_{H^1(\Omega)} \leq C_{\mathrm{G}} \exp(-b_{\mathrm{G}} n_{\mathrm{b}}^{1/d}),$$

where $u_{n_{\mathrm{b}}, n_{\mathrm{q}}}^a = \sum_{i=1}^{n_{\mathrm{b}}} \left[\boldsymbol{c}_{u;n_{\mathrm{b}}, n_{\mathrm{q}}}^a\right]_i \varphi_i \in X_{n_{\mathrm{b}}}$ is the Galerkin-Numerical Integration projection of $u^a$, such that

$$\mathfrak{b}_{n_{\mathrm{q}}}^a(u_{n_{\mathrm{b}}, n_{\mathrm{q}}}^a, v) = (f, v), \qquad \forall v \in X_{n_{\mathrm{b}}}.$$

Fix now

(5.5)
$$n_{\mathrm{b}}(\varepsilon) = \left\lceil \frac{|\log(\varepsilon/3)| + \log C_{\mathrm{G}}}{b_{\mathrm{G}}} \right\rceil^d$$

and $n_{\mathrm{q}}(\varepsilon) = n_{\mathrm{b}}(\varepsilon) + 1$,

We observe that (5.5) implies that there exists a constant $C_{\mathrm{b}} > 0$ such that, for all $\varepsilon \in (0, 1)$, we have $n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon) \leq C_{\mathrm{b}}(1 + |\log \varepsilon|^d)$, i.e., item (i) of the statement of the theorem.

We also define $C_{\mathrm{pol}} > 0$ as a constant such that, for all $p \in \mathbb{N}$,

(5.6)
$$\|\nabla q\|_{L^2(\Omega)} \leq C_{\mathrm{pol}} p^2 \|q\|_{L^2(\Omega)}, \qquad \forall q \in \mathbb{Q}_p(\Omega).$$

This inverse inequality follows straightforwardly from the classical Markov inequality in $(0, 1)$, with a tensorization argument (which yields that $C_{\mathrm{pol}} \sim \sqrt{d}$). With $n_{\mathrm{b}}(\varepsilon)$ as in (5.5) define

(5.7)
$$\varepsilon_{\mathrm{b}} := \frac{\varepsilon}{3 n_{\mathrm{b}}(\varepsilon)^{1/2}(2 + \sup_{a \in \mathcal{D}} \|u^a\|_{L^2(\Omega)})}, \quad \varepsilon_u := \frac{\varepsilon}{3(1 + C_{\mathrm{pol}}^2 n_{\mathrm{b}}(\varepsilon)^{1+4/d})^{1/2}}, \quad \varepsilon_{\mathrm{G}} := \frac{\varepsilon}{3}.$$

Due to (5.5),

(5.8)
$$\sup_{a \in \mathcal{D}} \|u^a - u_{n_{\mathrm{b}}, n_{\mathrm{q}}}^a\|_{H^1(\Omega)} \leq \varepsilon_{\mathrm{G}}.$$

Define then

$$\Phi_\varepsilon^{\mathrm{br}} = \Phi_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)}^{c_u} \qquad \text{and} \qquad \Phi_\varepsilon^{\mathrm{tr}} = \Phi_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}^{\mathrm{b}},$$

where the NNs $\Phi_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)}^{c_u}$ and $\Phi_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}^{\mathrm{b}}$ are defined in Propositions 5.5 and 5.6, respectively.

***Error estimate.*** For all $a \in \mathcal{D}$,

$$
\|u^a - \left(\mathrm{R}(\Phi^{\mathrm{br}}_{\varepsilon})(\boldsymbol{a}_{n_{\mathrm{q}}})\right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_{\varepsilon})\|_{H^1(\Omega)}
$$
$$
\leq \|u^a - u^a_{n_{\mathrm{b}},n_{\mathrm{q}}}\|_{H^1(\Omega)} + \|u^a_{n_{\mathrm{b}},n_{\mathrm{q}}} - \left(\mathrm{R}(\Phi^{\mathrm{br}}_{\varepsilon})(\boldsymbol{a}_{n_{\mathrm{q}}})\right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_{\varepsilon})\|_{H^1(\Omega)} =: (I) + (II).
$$

We have already established that $(I) \leq \varepsilon_{\mathrm{G}} = \varepsilon/3$. Let us consider $(II)$ then. We have

$$
(II) = \|\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \left(\left[\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}}\right]_i \varphi_i - \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}(\varepsilon)})\right]_i\right)\|_{H^1(\Omega)}
$$

$$
\leq \|\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \left(\left[\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}}\right]_i - \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i\right) \varphi_i\|_{H^1(\Omega)}
$$

$$
+ \|\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})\right]_i \left(\varphi_i - \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}(\varepsilon)})\right]_i\right)\|_{H^1(\Omega)}
$$

$$
=: (IIa) + (IIb).
$$

Denote, for all $i \in \{1,\dots,n_{\mathrm{b}}(\varepsilon)\}$,

$$
\eta_i := \left[\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}}\right]_i - \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i.
$$

Using the $L^2(\Omega)$-orthonormality of the basis, the Cauchy-Schwarz inequality, the polynomial inverse inequality (5.6) and Proposition 5.5, we obtain

$$
(IIa)^2 \leq \|\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \eta_i \varphi_i\|^2_{H^1(\Omega)} = \int_{\Omega} \left(\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \eta_i \varphi_i\right)^2 + \int_{\Omega} \left(\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \eta_i \nabla\varphi_i\right)^2
$$

$$
\overset{(4.1)}{=} \sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \eta_i^2 \int_{\Omega} \varphi_i^2 + \int_{\Omega} \left(\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \eta_i \nabla\varphi_i\right)^2
$$

$$
\overset{\text{C-S}}{\leq} \left(\sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \eta_i^2\right) \left(1 + \sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \int_{\Omega} |\nabla\varphi_i|^2\right)
$$

$$
\overset{(5.6)}{\leq} \|\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}} - \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\|^2_2 \left(1 + C^2_{\mathrm{pol}} n_{\mathrm{b}}(\varepsilon)^{1+4/d}\right)
$$

$$
\overset{\text{P. 5.5}}{\leq} \varepsilon_u^2 \left(1 + C^2_{\mathrm{pol}} n_{\mathrm{b}}(\varepsilon)^{1+4/d}\right)
$$

$$
\overset{(5.7)}{\leq} \left(\frac{\varepsilon}{3}\right)^2.
$$

Next, we estimate

$$
\|\mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2 \leq \|\mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}}) - \boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}}\|_2 + \|\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}}\|_2
$$

(5.9)
$$
\overset{\text{P. 5.5,(4.1)}}{\leq} 1 + \|u^a_{n_{\mathrm{b}},n_{\mathrm{q}}}\|_{L^2(\Omega)}
$$

$$
\leq 1 + \|u^a_{n_{\mathrm{b}},n_{\mathrm{q}}} - u^a\|_{L^2(\Omega)} + \|u^a\|_{L^2(\Omega)}
$$

$$
\overset{(5.8)}{\leq} 2 + \|u^a\|_{L^2(\Omega)}.
$$

Then,

$$
\begin{aligned}
(IIb)^2 &\overset{\text{C-S}}{\leq} \| \mathrm{R}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2^2 \sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \| \varphi_i - \left[ \mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}) \right]_i \|_{H^1(\Omega)}^2 \\
&\leq n_{\mathrm{b}}(\varepsilon) \| \mathrm{R}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2^2 \max_{i \in \{1, \ldots, n_{\mathrm{b}}(\varepsilon)\}} \| \varphi_i - \left[ \mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}) \right]_i \|_{H^1(\Omega)}^2 \\
&\overset{\text{P. 5.6,(5.9)}}{\leq} (2 + \|u_a\|_{L^2(\Omega)})^2 \varepsilon_{\mathrm{b}}^2 \\
&\leq \left( \frac{\varepsilon}{3} \right)^2 .
\end{aligned}
$$

We can conclude that

$$
\| u^a - \left( \mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon)(\boldsymbol{a}_{n_{\mathrm{q}}}) \right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon) \|_{H^1(\Omega)} \leq (I) + (IIa) + (IIb) \leq \varepsilon.
$$

*Depth and size bounds.* Using (5.5) and the definitions (5.7), we obtain that there exists a constant $C_1 > 0$ such that, for all $\varepsilon \in (0,1)$,

$$
1 + \max \left( |\log \varepsilon_{\mathrm{b}}|, |\log \varepsilon_{\mathrm{G}}|, |\log \varepsilon_u| \right) \leq C_1 (1 + |\log \varepsilon|).
$$

We infer then, from Proposition 5.5, that there exists $C_2 > 0$ such that, for all $\varepsilon \in (0,1)$,

$$
\mathrm{L}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)}) \leq C_2 \left( 1 + \log(1 + |\log \varepsilon|^d) \right) (1 + |\log \varepsilon|)
$$

and

$$
\mathrm{M}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)}) \leq C_2 \left( 1 + |\log \varepsilon|^d \right)^3 (1 + |\log \varepsilon|)^2 \left( 1 + \log(1 + |\log \varepsilon|^d) \right)^2 .
$$

Furthermore, from Proposition 5.6, we have that there exists $C_3 > 0$ such that for all $\varepsilon \in (0,1)$

$$
\mathrm{L}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}) \leq C_3 (1 + |\log \varepsilon|) \log(1 + |\log \varepsilon|^d)
$$

and

$$
\mathrm{M}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}) \leq C_3 (1 + |\log \varepsilon|^{d+1}).
$$

Using the definition of $\Phi^{\mathrm{tr}}_\varepsilon$ and $\Phi^{\mathrm{br}}_\varepsilon$ gives Item (iii) and concludes the proof. $\qquad\square$

**5.4. Parametric diffusion coefficient.** In many applications, for example in uncertainty quantification, one is interested in the case where the diffusion coefficient in (2.3) is parametric. This is naturally accommodated for by composition with solution operator networks and we briefly detail this here. Specifically, suppose that there exists $d_{\mathrm{p}} \in \mathbb{N}$ and a compact set $\mathcal{P} \subset \mathbb{R}^{d_{\mathrm{p}}}$ such that $\mathfrak{a} : \mathcal{P} \to \mathrm{Hol}(\Omega)$ and that there exist constants $a_{\min}, C_{\mathrm{p}}, b_{\mathrm{p}}, \alpha_{\mathrm{p}}, A_{\mathrm{p}}, A_\psi > 0$, and functions $\psi_i : \Omega \to \mathbb{R}$ and $a_i : \mathcal{P} \to \mathbb{R}$, $i \in \mathbb{N}$, such that

$$
\inf_{\boldsymbol{y} \in \mathcal{P}} \inf_{\boldsymbol{x} \in \Omega} \mathfrak{a}(\boldsymbol{y})(\boldsymbol{x}) \geq a_{\min}, \tag{5.10}
$$

that

$$
\forall n_{\mathrm{p}} \in \mathbb{N}, \qquad \sup_{\boldsymbol{y} \in \mathcal{P}} \| \mathfrak{a}(\boldsymbol{y}) - \sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y}) \psi_i \|_{L^\infty(\Omega)} \leq C_{\mathrm{p}} \exp(-b_{\mathrm{p}} n_{\mathrm{p}}^{\alpha_{\mathrm{p}}}), \tag{5.11}
$$

with

(5.12) $$\forall i \in \mathbb{N}, \qquad \psi_i \in \mathrm{Hol}(\Omega; A_\psi), \qquad a_i \in \mathrm{Hol}(\mathcal{P}; A_\mathrm{p}),$$

and that

(5.13) $$\sup_{\boldsymbol{y} \in \mathcal{P}} \sum_{i=1}^{\infty} |a_i(\boldsymbol{y})| \le A_\mathrm{p}.$$

Here, we use the same constant $A_\mathrm{p}$ in the second hypothesis in (5.12) and in (5.13) only to simplify notation. For all $\boldsymbol{y} \in \mathcal{P}$, we denote $u_{\boldsymbol{y}} \in X$ the solution to

(5.14) $$-\nabla \cdot (\mathfrak{a}(\boldsymbol{y})\nabla u_{\boldsymbol{y}}) = f, \qquad \text{in } \Omega.$$

*Remark* 5.8. Functions that can be written in Fourier series as

$$\mathfrak{a}(\boldsymbol{y})(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} a_{\boldsymbol{k}}(\boldsymbol{y}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $a_{\boldsymbol{k}} \in \mathrm{Hol}(\mathcal{P}, A_\mathrm{p})$, with exponential decrease of $\sup_{\boldsymbol{y}\in\mathcal{P}} |a_{\boldsymbol{k}}(\boldsymbol{y})|$ with respect to $|\boldsymbol{k}|$, and such that $\mathfrak{a}$ is bounded from below, fulfill conditions (5.11), (5.12) and (5.13).

LEMMA 5.9. *There exists $C > 0$ such that for all $n_\mathrm{p} \in \mathbb{N}$ and for all $\varepsilon \in (0,1)$, there exists a NN $\Phi_{\varepsilon,n_\mathrm{p}}^{a,\mathrm{coef}}$ with input dimension $d_\mathrm{p}$ and output dimension $n_\mathrm{p}$ such that*

(5.15) $$\max_{i=1,\dots,n_\mathrm{p}} \|a_i - \left[\mathrm{R}(\Phi_{\varepsilon,n_\mathrm{p}}^{a,\mathrm{coef}})\right]_i\|_{L^\infty(\mathcal{P})} \le \varepsilon$$

*and that $\mathrm{L}(\Phi_{\varepsilon,n_\mathrm{p}}^{a,\mathrm{coef}}) \le C(1 + |\log\varepsilon|)(1 + \log|\log\varepsilon|)$ and $\mathrm{M}(\Phi_{\varepsilon,n_\mathrm{p}}^{a,\mathrm{coef}}) \le C(1 + |\log\varepsilon|^{d_\mathrm{p}+1})n_\mathrm{p}$.*

*Proof.* The statement follows from a parallelization of the network of [18, Theorem 3.6]  □

THEOREM 5.10. *Let $d_\mathrm{p} \in \mathbb{N}$ and let $\mathfrak{a}$ and $u_{\boldsymbol{y}}$ be defined as above. There exists $C > 0$ such that, for all $\varepsilon \in (0,1)$, there exist*
  *(a)  $n_\mathrm{b} \in \mathbb{N}$,*
  *(b)  two NNs $\Phi_\varepsilon^{\mathrm{br}}$ and $\Phi_\varepsilon^{\mathrm{tr}}$ with $\mathrm{R}(\Phi_\varepsilon^{\mathrm{br}}) : \mathbb{R}^{d_\mathrm{p}} \to \mathbb{R}^{n_\mathrm{b}}$ and $\mathrm{R}(\Phi_\varepsilon^{\mathrm{tr}}) : Q \to \mathbb{R}^{n_\mathrm{b}}$,*
*such that*
  *(i)  $n_\mathrm{b} \le C(1 + |\log\varepsilon|^d)$,*
  *(ii)  the following error estimate holds:*

$$\sup_{\boldsymbol{y} \in \mathcal{P}} \|u_{\boldsymbol{y}} - \left(\mathrm{R}(\Phi_\varepsilon^{\mathrm{br}})(\boldsymbol{y})\right) \cdot \mathrm{R}(\Phi_\varepsilon^{\mathrm{tr}})\|_{H^1(\Omega)} \le \varepsilon,$$

  *(iii)  as $\varepsilon \downarrow 0$,*

$$\mathrm{L}(\Phi_\varepsilon^{\mathrm{br}}) = \mathcal{O}\left(|\log\varepsilon| (\log|\log\varepsilon|)\right),$$
$$\mathrm{M}(\Phi_\varepsilon^{\mathrm{br}}) = \mathcal{O}\left(|\log\varepsilon|^{3d+2} (\log|\log\varepsilon|)^2 + |\log\varepsilon|^{1+d_\mathrm{p}+1/\alpha_\mathrm{p}}\right),$$

  *and*

$$\mathrm{L}(\Phi_\varepsilon^{\mathrm{tr}}) = \mathcal{O}\left(|\log\varepsilon| (\log|\log\varepsilon|)\right), \qquad \mathrm{M}(\Phi_\varepsilon^{\mathrm{tr}}) = \mathcal{O}\left(|\log\varepsilon|^{d+1}\right).$$

*Proof.* The proof proceeds in several steps. We first prove a basis consistency bound, then detail the construction of the ONet, and conclude with bounds on the depth and size of the ONet.

Let $C_L > 0$ be the constant such that, given $\mathfrak{a}_1, \mathfrak{a}_2 \in L^\infty(\Omega)$ such that

$$(5.16) \qquad 0 < \frac{a_{\min}}{4} \leq \mathfrak{a}_i \leq \max(a_{\max}, (1 + A_{\mathrm{p}})A_\psi), \qquad \text{a. e. in } \Omega \text{ and for } i = 1, 2,$$

and $u_i = S(\mathfrak{a}_i)$, $i = 1, 2$, then

$$\|u_1 - u_2\|_{H^1(\Omega)} \leq C_L \|\mathfrak{a}_1 - \mathfrak{a}_2\|_{L^\infty(\Omega)},$$

see Lemma B.1. We suppose, without loss of generality and for ease of notation, that $A_\psi \geq 1$ and $C_L \geq 1$. Let now $n_{\mathrm{p}}$ be the smallest integer such that

$$C_L C_{\mathrm{p}} \exp(-b_{\mathrm{p}} n_{\mathrm{p}}^{\alpha_{\mathrm{p}}}) \leq \min\left(\frac{\varepsilon}{3}, \frac{a_{\min}}{2}\right).$$

This implies that there exists a constant $C_1 > 0$ (depending only on $C_L, C_{\mathrm{p}}, b_{\mathrm{p}}, a_{\min}$) such that

$$n_{\mathrm{p}} \leq C_1(1 + |\log \varepsilon|^{1/\alpha_{\mathrm{p}}})$$

and that, due to (5.10), (5.11), (5.12), and (5.13),

$$(5.17) \qquad \inf_{\boldsymbol{y} \in \mathcal{P}} \inf_{x \in \Omega} \sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y})\psi_i(x) \geq \frac{a_{\min}}{2}, \qquad \sup_{\boldsymbol{y} \in \mathcal{P}} \|\sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y})\psi_i\|_{L^\infty(\Omega)} \leq A_{\mathrm{p}} A_\psi.$$

Let then

$$(5.18) \qquad \varepsilon_{\mathrm{p}} := \frac{1}{n_{\mathrm{p}} A_\psi} \min\left(\frac{\varepsilon}{3C_L}, \frac{a_{\min}}{4}\right), \qquad \widetilde{\mathfrak{a}} := \sum_{i=1}^{n_{\mathrm{p}}} \left[R(\Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}})\right]_i \psi_i,$$

where the network $\Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}}$ is defined in Lemma 5.9. We now show that $\widetilde{\mathfrak{a}}$ fulfills conditions like (2.1) and (2.2) (with updated values of the constants $a_{\min}, a_{\max}, A_{\mathcal{D}}$), uniformly with respect to $n_{\mathrm{p}}$ and $\varepsilon_{\mathrm{p}}$. From (5.15) and (5.17), it follows that, for all $k \in \mathbb{N}_0$,

$$(5.19) \qquad \sup_{\boldsymbol{y} \in \mathcal{P}} \|\widetilde{\mathfrak{a}}\|_{W^{k,\infty}(\Omega)} \leq \sup_{\boldsymbol{y} \in \mathcal{P}} \sum_{i=1}^{n_{\mathrm{p}}} \left(|a_i(\boldsymbol{y}) - \left[R(\Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}})\right]_i| + |a_i(\boldsymbol{y})|\right) \|\psi_i\|_{W^{k,\infty}}$$

$$\leq (n_{\mathrm{p}} \varepsilon_{\mathrm{p}} + A_{\mathrm{p}}) A_\psi^{k+1} k! \leq (1 + A_{\mathrm{p}}) A_\psi^{k+1} k!.$$

Furthermore, for all $\boldsymbol{y} \in \mathcal{P}$ and all $x \in \Omega$,

$$(5.20)$$
$$\widetilde{\mathfrak{a}}(\boldsymbol{y})(x) \geq \sum_{i=1}^{n_{\mathrm{p}}} \left(\left[R(\Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}})(\boldsymbol{y})\right]_i - a_i(\boldsymbol{y})\right) \psi_i(x) + \sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y})\psi_i(x)$$
$$\geq \frac{a_{\min}}{2} - n_{\mathrm{p}} \varepsilon_{\mathrm{p}} A_\psi,$$
$$\geq \frac{a_{\min}}{4}.$$

Here we have used (5.13), (5.15), (5.17), and the definition of $\varepsilon_{\mathrm{p}}$ in (5.18).

***Construction of the operator network and error estimate.*** For $q \in \mathbb{N}$, $n_{\mathrm{q}} = q^d$, we introduce the matrix $\mathbf{V}_{n_{\mathrm{q}}, n_{\mathrm{p}}} \in \mathbb{R}^{n_{\mathrm{q}} \times n_{\mathrm{p}}}$ with entries

$$(5.21) \qquad\qquad \left[\mathbf{V}_{n_{\mathrm{q}}, n_{\mathrm{p}}}\right]_{ij} = \psi_j(\boldsymbol{x}_i^{(q)}), \qquad i = 1, \ldots, n_{\mathrm{q}}, \ j = 1, \ldots, n_{\mathrm{p}},$$

where $\boldsymbol{x}_1^{(q)}, \ldots, \boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}$ are the quadrature nodes introduced in Section 4.2. Then the NN

$$(5.22) \qquad\qquad \Phi_{\varepsilon_p, n_{\mathrm{p}}, n_{\mathrm{q}}}^{\widetilde{\mathfrak{a}}} = \left(\left(\mathbf{V}_{n_{\mathrm{q}}, n_{\mathrm{p}}}, \mathbf{0}_{n_{\mathrm{q}}}\right)\right) \odot \Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}}$$

has realization such that

$$\mathrm{R}(\Phi_{\varepsilon_p, n_{\mathrm{p}}, n_{\mathrm{q}}}^{\widetilde{\mathfrak{a}}})(\boldsymbol{y}) = \begin{pmatrix} \widetilde{\mathfrak{a}}(\boldsymbol{y})(\boldsymbol{x}_1^{(q)}) \\ \vdots \\ \widetilde{\mathfrak{a}}(\boldsymbol{y})(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}) \end{pmatrix}.$$

Let $\widetilde{u}_{\boldsymbol{y}} \in X$ denote, for each $\boldsymbol{y} \in \mathcal{P}$, the solution to

$$\nabla \cdot (\widetilde{\mathfrak{a}}(\boldsymbol{y}) \nabla \widetilde{u}_{\boldsymbol{y}}) = f, \qquad \text{in } \Omega.$$

Thanks to (5.19), (5.20), and to Theorem 5.7, there exist a constant $C_2$ independent of $\varepsilon$, $n_{\mathrm{q}} \in \mathbb{N}$ such that $n_{\mathrm{q}} \le C_2(1 + |\log \varepsilon|)$, and networks $\widetilde{\Phi}_{\varepsilon/3}^{\mathrm{br}}$ and $\widetilde{\Phi}_{\varepsilon/3}^{\mathrm{tr}}$ such that, for all $\boldsymbol{y} \in \mathcal{P}$,

$$\|\widetilde{u}_{\boldsymbol{y}} - \left(\mathrm{R}(\widetilde{\Phi}_{\varepsilon/3}^{\mathrm{br}}) \circ \mathrm{R}(\Phi_{\varepsilon_p, n_{\mathrm{p}}, n_{\mathrm{q}}}^{\widetilde{\mathfrak{a}}})\right)(\boldsymbol{y}) \cdot \mathrm{R}(\widetilde{\Phi}_{\varepsilon/3}^{\mathrm{tr}})\|_{H^1(\Omega)} \le \frac{\varepsilon}{3}.$$

Furthermore, for all $\boldsymbol{y} \in \mathcal{P}$, $\mathfrak{a}(\boldsymbol{y})$ and $\widetilde{\mathfrak{a}}(\boldsymbol{y})$ satisfy the conditions in (5.16), hence

$$
\begin{aligned}
\|u_{\boldsymbol{y}} - \widetilde{u}_{\boldsymbol{y}}\|_{H^1(\Omega)} &\le C_L \|\mathfrak{a}(\boldsymbol{y}) - \widetilde{\mathfrak{a}}(\boldsymbol{y})\|_{L^\infty(\Omega)} \\
&\le C_L \left( \|\mathfrak{a}(\boldsymbol{y}) - \sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y}) \psi_i\|_{L^\infty(\Omega)} + \| \sum_{i=1}^{n_{\mathrm{p}}} \left( a_i(\boldsymbol{y}) - \left[\mathrm{R}(\Phi_{\varepsilon, n_{\mathrm{p}}}^{a, \mathrm{coef}})(\boldsymbol{y})\right]_i \right) \psi_i\|_{L^\infty(\Omega)} \right) \\
&\le \frac{\varepsilon}{3} + C_L n_{\mathrm{p}} \varepsilon_{\mathrm{p}} A_\psi \\
&\le \frac{2}{3} \varepsilon.
\end{aligned}
$$

We deduce that

$$\sup_{\boldsymbol{y} \in \mathcal{P}} \|u_{\boldsymbol{y}} - \left(\mathrm{R}(\widetilde{\Phi}_{\varepsilon/3}^{\mathrm{br}}) \circ \mathrm{R}(\Phi_{\varepsilon_p, n_{\mathrm{p}}, n_{\mathrm{q}}}^{\widetilde{\mathfrak{a}}})\right)(\boldsymbol{y}) \cdot \mathrm{R}(\widetilde{\Phi}_{\varepsilon/3}^{\mathrm{tr}})\|_{H^1(\Omega)} \le \varepsilon,$$

which is Item (ii), with

$$\Phi_\varepsilon^{\mathrm{br}} := \widetilde{\Phi}_{\varepsilon/3}^{\mathrm{br}} \odot \Phi_{\varepsilon_p, n_{\mathrm{p}}, n_{\mathrm{q}}}^{\widetilde{\mathfrak{a}}}, \qquad \Phi_\varepsilon^{\mathrm{tr}} := \widetilde{\Phi}_{\varepsilon/3}^{\mathrm{tr}}.$$

***Depth and size bounds.*** The bounds on the depth and size of $\Phi_\varepsilon^{\mathrm{tr}}$ can be inferred directly from Theorem 5.7. To compute bounds on the size and depth of $\Phi_\varepsilon^{\mathrm{br}}$, note that, by Lemma 5.9, there exist $C_3, C_4, C_5, C_6$ independent of $\varepsilon$ such that

$$(5.23) \qquad \mathrm{L}(\Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}}) \le C_3(1 + |\log \varepsilon_{\mathrm{p}}|)(1 + \log |\log \varepsilon_{\mathrm{p}}|) \le C_4(1 + |\log \varepsilon|)(1 + \log |\log \varepsilon|)$$

and

$$(5.24) \qquad \mathrm{M}(\Phi_{\varepsilon_p, n_{\mathrm{p}}}^{a, \mathrm{coef}}) \le C_5(1 + |\log \varepsilon_{\mathrm{p}}|^{d_{\mathrm{p}}+1}) n_p \le C_6(1 + |\log \varepsilon|^{d_{\mathrm{p}}+1+1/\alpha_{\mathrm{p}}}).$$

Furthermore, there exists $C_7$ independent of $\varepsilon$ such that

$$(5.25) \qquad \|\mathbf{V}_{n_{\mathrm{q}},n_{\mathrm{p}}}\|_0 \leq n_{\mathrm{p}}n_{\mathrm{q}} \leq C_7(1 + |\log \varepsilon|^{1+1/\alpha_{\mathrm{P}}}).$$

From (5.24) and (5.25) it follows that

$$(5.26) \quad \mathrm{L}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_p,n_{\mathrm{p}},n_{\mathrm{q}}}) \leq C_8(1 + |\log \varepsilon|)(1 + \log |\log \varepsilon|), \qquad \mathrm{M}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_p,n_{\mathrm{p}},n_{\mathrm{q}}}) \leq C_9(1 + |\log \varepsilon|^{d_{\mathrm{p}}+1+1/\alpha_{\mathrm{P}}}),$$

for constants $C_8, C_9$ independent of $\varepsilon$. Combining the bounds in (5.26) with the bounds on the depth and size of $\widetilde{\Phi}^{\mathrm{br}}_{\varepsilon/3}$ coming from Theorem 5.7 concludes the proof. $\qquad\square$

*Remark* 5.11. If each function $a_i$ does not depend on all the parameters but only on a subset of them, the size bound of Theorem 5.10 results in an overestimation. Specifically, for all $i \in \mathbb{N}$, let $\mathcal{P}_i$ be the domain of $a_i$ and denote $d_{\mathrm{p},i} := \dim(\mathcal{P}_i)$. Then, with a slight modification of Lemma 5.9 we obtain in Theorem 5.10 that there exists a constant $c > 0$ independent of $\varepsilon$ such that for $\varepsilon \downarrow 0$,

$$\mathrm{M}(\Phi^{\mathrm{br}}_{\varepsilon}) = \mathcal{O}\left(|\log \varepsilon|^{3d+2}(\log|\log \varepsilon|)^2 + \sum_{i=1}^{c|\log \varepsilon|^{1/\alpha_{\mathrm{P}}}} |\log \varepsilon|^{1+d_{\mathrm{p},i}}\right).$$

Clearly, setting $d_{\mathrm{p},i} = d_{\mathrm{p}}$ for all $i$ in the equation above gives the estimate in Theorem 5.10.

*Remark* 5.12. Similar results to Theorem 5.10 can be obtained through the technique in [11], by using the exponential convergence of polynomial approximations to the functions in the solution manifold $\mathcal{M} = \{u(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{P}\}$ to derive an upper bound on the $n$-width of $\mathcal{M}$.

**6. Generalizations.** All steps of the analysis of ONet emulation rates for the coefficient-to-solution map of (2.3) directly generalize to other, structurally similar, linear divergence-form elliptic PDEs. We illustrate the extension of the preceding result by two of these: anisotropic diffusion-reaction equations and linear elastostatics.

**6.1. Linear anisotropic diffusion-reaction equations.**

**6.1.1. Definition of the problem.** We consider again the torus $\Omega = (\mathbb{R}/\mathbb{Z})^d$. For a constant $A_{\mathcal{D}^{\mathrm{rd}}} > 0$, introduce the set of admissible data

$$\mathcal{D}^{\mathrm{rd}} \subset \mathrm{Hol}(\Omega; A_{\mathcal{D}^{\mathrm{rd}}})^{d\times d} \times \mathrm{Hol}(\Omega; A_{\mathcal{D}^{\mathrm{rd}}})$$

of pairs $(\mathbf{A}, c)$ and suppose there exist $Q_0 \subset Q$, $a_{\min}, c_{\min} > 0$ such that for all $(\mathbf{A}, c) \in \mathcal{D}^{\mathrm{rd}}$,
- $\mathbf{A}$ is symmetric and is uniformly positive definite, i.e., $\mathbf{A}_{ij} = \mathbf{A}_{ji}$ and

$$\forall \boldsymbol{x} \in \Omega, \ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad \boldsymbol{\xi}^\top \mathbf{A}(\boldsymbol{x})\boldsymbol{\xi} \geq a_{\min}|\boldsymbol{\xi}|^2,$$

- $c(\boldsymbol{x}) \geq c_{\min}$ for all $\boldsymbol{x} \in Q_0$.

For all $(\mathbf{A}, c) \in \mathcal{D}^{\mathrm{rd}}$, the bilinear form $\mathfrak{b}^{(\mathbf{A},c)}(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$\mathfrak{b}^{(\mathbf{A},c)}(w, v) := \int_\Omega ((\mathbf{A}\nabla w) \cdot \nabla v + cwv)$$

is coercive, i.e., there exists a constant $\alpha_0 > 0$ independent of $(\mathbf{A}, c)$ such that

$$\forall v \in H^1(\Omega), \quad \mathfrak{b}^{(\mathbf{A},c)}(v, v) \geq \alpha_0\|v\|^2_{H^1(\Omega)}.$$

The continuity of the form $\mathfrak{b}^{(\mathbf{A},c)}(\cdot, \cdot)$ on $H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ being evident, the Lax-Milgram Lemma implies that for every $f \in \mathrm{Hol}(\Omega)$ there exists a unique solution

$$u \in H^1(\Omega) : \quad \mathfrak{b}^{(\mathbf{A},c)}(u, v) = (f, v) \quad \forall v \in H^1(\Omega).$$

For given, fixed $f \in \mathrm{Hol}(\Omega)$, the coefficient-to-solution map

$$S^{\mathrm{rd}} : (\mathbf{A}, c) \mapsto u$$

is analytic. Furthermore, there exists $A_{\mathcal{U}^{\mathrm{rd}}} > 0$ such that

$$S^{\mathrm{rd}}(\mathcal{D}^{\mathrm{rd}}) \subset \mathrm{Hol}(\Omega; A_{\mathcal{U}^{\mathrm{rd}}}),$$

which can be proven as in Lemma 4.1.

**6.1.2. Operator network approximation.** We introduce, for all $n_{\mathrm{q}} \in \mathbb{N}$ such that $q := n_{\mathrm{q}}^{1/d} \in \mathbb{N}$, the encoding operator $\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}^{\mathrm{rd}} : C(\Omega)^{d \times d} \times C(\Omega) \to \mathbb{R}^{d^2 n_{\mathrm{q}} + n_{\mathrm{q}}}$ such that

$$\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}^{\mathrm{rd}}(\mathbf{A}, c) = \begin{pmatrix} \mathbf{vec}(\mathbf{A}(\boldsymbol{x}_1^{(q)})) \\ \vdots \\ \mathbf{vec}(\mathbf{A}(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)})) \\ c(\boldsymbol{x}_1^{(q)}) \\ \vdots \\ c(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}) \end{pmatrix},$$

where $\boldsymbol{x}_{\mathrm{enc}} = \boldsymbol{x}_1^{(q)}, \dots, \boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}$ are the points from Section 4.2. Theorem 5.7 can then be extended to this class of reaction-diffusion equations.

THEOREM 6.1. *Theorem 5.7 holds with $a \in \mathcal{D}$ replaced by $(\mathbf{A}, c) \in \mathcal{D}^{\mathrm{rd}}$, $S(a)$ replaced by $S^{\mathrm{rd}}(\mathbf{A}, c)$, and $\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}(a)$ replaced by $\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}^{\mathrm{rd}}(\mathbf{A}, c)$.*

For all $p \in \mathbb{N}$, writing $\widetilde{n}_b = (p+1)^d$, we consider $L^2(\Omega)$-orthonormal basis functions $\{\widetilde{\varphi}_i\}_{i=1}^{\widetilde{n}_b}$ of $\mathbb{Q}_p(\Omega)$, i.e.,

$$\int_\Omega \widetilde{\varphi}_i \widetilde{\varphi}_j = \delta_{ij}, \quad \forall (i, j) \in \{1, \dots, \widetilde{n}_b\}^2.$$

Let then

$$\widetilde{X}_{n_{\mathrm{b}}} := \mathrm{span}(\{\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_{\widetilde{n}_b}\}) \subset \widetilde{X} := H^1(\Omega).$$

In order to prove Theorem 6.1, we have to replace the input layer network introduced in Lemma 5.1 with an input layer adapted for anisotropic diffusion-reaction problems, as introduced in Lemma 6.2. For $k \in \{1, \dots, n_{\mathrm{q}}\}$, we introduce $\widetilde{\mathbf{D}}(\boldsymbol{x}_k^{(q)})$ such that

$$\widetilde{\mathbf{D}}_{mn}^{ij}(\boldsymbol{x}_k^{(q)}) = w_k^{(q)}(\partial_{x_n}\widetilde{\varphi}_j)(\boldsymbol{x}_k^{(q)})(\partial_{x_m}\widetilde{\varphi}_i)(\boldsymbol{x}_k^{(q)}), \qquad (i, j) \in \{1, \dots, n_{\mathrm{b}}\}^2, (m, n) \in \{1, \dots, d\}^2.$$

Furthermore, let $\mathfrak{v} : \mathbb{N}^2 \to \mathbb{N}$ be the reordering such that for any matrix $\mathbf{A}$,

(6.1) $$\mathbf{vec}(\mathbf{A})_{\mathfrak{v}(i,j)} = \mathbf{A}_{ij}.$$

We introduce the operation $\widetilde{\mathbf{vec}} : \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{b}} \times d \times d} \to \mathbb{R}^{n_{\mathrm{b}}^2 \times d^2}$

$$\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}_k^{(q)}))_{\mathfrak{v}(i,j)\mathfrak{v}(m,n)} = \widetilde{\mathbf{D}}_{mn}^{ij}(\boldsymbol{x}_k^{(q)}).$$

Finally, define

$$\widehat{\mathbf{M}}_{ij}(\boldsymbol{x}_k^{(q)}) = w_k^{(q)}\widetilde{\varphi}_i(\boldsymbol{x}_k^{(q)})\widetilde{\varphi}_j(\boldsymbol{x}_k^{(q)}).$$

LEMMA 6.2. *For all $\alpha \in \mathbb{R}$ the one layer NN*

$$\Phi_{n_b,n_q}^{(\mathbf{A},c),\alpha} := \left( \left( -\alpha \left[ \widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}_1^{(q)}))| \dots | \widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}_{n_q}^{(q)}))| \mathbf{vec}(\widehat{\mathbf{M}}(\boldsymbol{x}_1^{(q)}))| \dots | \mathbf{vec}(\widehat{\mathbf{M}}(\boldsymbol{x}_{n_q}^{(q)})) \right], \mathbf{0}_{n_b} \right) \right)$$

*is such that*

$$\mathbf{matr} \left( \mathrm{R}(\Phi_{n_b,n_q}^{(\mathbf{A},c),\alpha})(\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}^{\mathrm{rd}}(\mathbf{A},c)) \right)_{ij} = -\alpha \mathfrak{b}^{(\mathbf{A},c)}(\widetilde{\varphi}_j, \widetilde{\varphi}_i)$$

*and* $\mathrm{M}(\Phi_{n_b,n_q}^{(\mathbf{A},c),\alpha}) \leq (d^2+1)n_b^2 n_q$.

*Proof.* We have

$$\left[ \mathrm{R}\left( \Phi_{n_b,n_q}^{(\mathbf{A},c),\alpha} \right) \left( \mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}^{\mathrm{rd}}(\mathbf{A},c) \right) \right]_{\mathfrak{v}(i,j)}$$

$$= -\alpha \sum_{k=1}^{n_q} w_k^{(q)} \left( \sum_{m,n=1}^{d} \left[ \mathbf{A}_{mn}(\partial_{x_n}\widetilde{\varphi}_j)(\partial_{x_m}\widetilde{\varphi}_i) \right] (\boldsymbol{x}_k^{(q)}) + \left[ c\widetilde{\varphi}_j\widetilde{\varphi}_i \right] (\boldsymbol{x}_k^{(q)}) \right)$$

$$= -\alpha \sum_{k=1}^{n_q} w_k^{(q)} \left( \left[ (\mathbf{A}\nabla\widetilde{\varphi}_i) \cdot (\nabla\widetilde{\varphi}_j) \right] (\boldsymbol{x}_k^{(q)}) + \left[ c\widetilde{\varphi}_j\widetilde{\varphi}_i \right] (\boldsymbol{x}_k^{(q)}) \right),$$

hence the equality after matricization. The size bound follows from the fact that

$$\|\widetilde{\mathbf{D}}(\boldsymbol{x}_k^{(q)})\|_0 \leq d^2 n_b^2, \qquad \|\widehat{\mathbf{M}}(\boldsymbol{x}_k^{(q)})\|_0 \leq n_b^2,$$

for all $k \in \{1, \dots, n_q\}$. □

We can now prove Theorem 6.1.

*Proof of Theorem 6.1.* The proof follows along the same lines as the proof of Theorem 5.7. In particular, in the construction of $\Phi_\varepsilon^{\mathrm{br}}$, the input network $\Phi_{n_b,n_q}^{A,\alpha}$ and Lemma 5.1 are replaced by the network $\Phi_{n_b,n_q}^{(\mathbf{A},c),\alpha}$ and Lemma 6.2. Then, the spaces $X$ and $X_{n_b}$ are replaced by $\widetilde{X}$ and $\widetilde{X}_{\widetilde{n}_b}$. The basis $\{\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_{\widetilde{n}_b}\}$ is equal to $\{\varphi_1, \dots, \varphi_{n_b}\}$ with the addition of a constant function, which can be emulated exactly by deep ReLU neural networks. Hence, Proposition 5.6 can be extended to this case. Finally, the matrices $\mathbf{A}_{n_b,n_q}^a$ and $\mathbf{A}_{n_b,n_q}^1$ used in the proof of Theorem 5.7 are replaced, respectively, by the matrices with entries

$$\mathfrak{b}_{n_q}^{(\mathbf{A},c)}(\widetilde{\varphi}_j, \widetilde{\varphi}_i) \quad \text{and} \quad \mathfrak{b}_{n_q}^{(\mathbf{Id}_d,1)}(\widetilde{\varphi}_j, \widetilde{\varphi}_i), \qquad (i,j) \in \{1, \dots, \widetilde{n}_b\}^2,$$

where

$$\mathfrak{b}_{n_q}^{(\mathbf{A},c)}(u,v) := \sum_{k=1}^{n_q} w_k^{(q)} \left( \mathbf{A}(\boldsymbol{x}_k^{(q)})\nabla u(\boldsymbol{x}_k^{(q)}) \right) \cdot \nabla v(\boldsymbol{x}_k^{(q)}) + \sum_{k=1}^{n_q} w_k^{(q)} c(\boldsymbol{x}_k^{(q)}) u(\boldsymbol{x}_k^{(q)}) v(\boldsymbol{x}_k^{(q)}),$$

for all $u, v \in C^1(\Omega)$. Since the bilinear form $\mathfrak{b}^{(\mathbf{A},c)}$ is coercive and continuous on $H^1(\Omega)$, results equivalent to Lemmas 4.2 and 4.3 with the new matrices can be proven directly. The rest of the proof is the same as the proof of Theorem 5.7. □

## 6.2. Linear Elastostatics.

**6.2.1. Definition of the problem.** We assume $d = 2, 3$. Small, linear elastic deformation of a body occupying $Q = (0,1)^d$ with periodic boundary conditions and subject to a prescribed,

periodic body force $\boldsymbol{f} : \Omega = \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d$ can be described by the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ which satisfies the equilibrium of stress

$$(6.2) \qquad\qquad \operatorname{div} \boldsymbol{\sigma}[\boldsymbol{u}] + \boldsymbol{f} = 0 \quad \text{in } \Omega.$$

Here $\boldsymbol{\sigma}[\boldsymbol{u}] : \Omega \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ is symmetric matrix function, the so-called *stress tensor*. It depends on the displacement field $\boldsymbol{u}$ via the (linearized) *strain tensor* $\varepsilon[\boldsymbol{u}] : \Omega \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ which is given by

$$(6.3) \qquad \varepsilon[\boldsymbol{u}] := \frac{1}{2} \left( \operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^\top \right), \quad (\boldsymbol{\varepsilon}[\boldsymbol{u}])_{ij} := \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad i, j = 1, ..., d \,.$$

In the linearized theory, the tensors $\boldsymbol{\sigma}$ and $\varepsilon$ in (6.2), (6.3) are related by the linear constitutive stress-strain relation ("Hooke's law")

$$(6.4) \qquad\qquad\qquad\qquad \boldsymbol{\sigma} = \mathtt{A}\varepsilon.$$

In (6.4), $\mathtt{A}$ is a fourth order tensor field, i.e. $\mathtt{A} = \{\mathtt{A}_{ijkl} : i, j, k, l = 1, ..., d\}$, with certain symmetries: the $d^4$ component functions $\mathtt{A}_{ijkl}(x)$ are assumed analytic in $[0, 1]^d$ and 1-periodic with respect to each coordinate, and satisfy for every $x \in \Omega$,

$$(6.5) \qquad \forall \boldsymbol{\tau} \in \mathbb{R}^{d \times d}_{\mathrm{sym}}, \ \mathtt{A}(x)\boldsymbol{\tau} \in \mathbb{R}^{d \times d}_{\mathrm{sym}} \quad \text{and} \quad \forall \boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathbb{R}^{d \times d}_{\mathrm{sym}}, \ (\mathtt{A}(x)\boldsymbol{\tau}) : \boldsymbol{\sigma} = (\mathtt{A}(x)\boldsymbol{\sigma}) : \boldsymbol{\tau} \,.$$

Key assumption on $\mathtt{A}$ is *coercivity*: there exists a constant $a_{\min} > 0$ such that

$$(6.6) \qquad\qquad \forall x \in \Omega, \ \forall \boldsymbol{\tau} \in \mathbb{R}^{d \times d}_{\mathrm{sym}}, \quad (\mathtt{A}(x)\boldsymbol{\tau}) : \boldsymbol{\tau} \ge a_{\min}\|\boldsymbol{\tau}\|_2^2 \,.$$

see, e.g., [24] for details. Inserting (6.4) into (6.2), integrating by parts and noting the periodic boundary conditions, the so-called "primal variational formulation" of (6.2) reads: find $\boldsymbol{u}^{\mathtt{A}} \in [H^1(\Omega)/\mathbb{R}]^d$ such that

$$(6.7) \qquad \mathfrak{b}^{\mathtt{A}}(\boldsymbol{u}^{\mathtt{A}}, \boldsymbol{v}) := \int_\Omega \varepsilon[\boldsymbol{v}] : (\mathtt{A}\varepsilon[\boldsymbol{u}^{\mathtt{A}}]) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in [H^1(\Omega)/\mathbb{R}]^d \,.$$

Unique solvability of (6.7) is implied by the Lax-Milgram Lemma with (6.6) and Korn's inequality upon noticing that the space $X^d = [H^1(\Omega)/\mathbb{R}]^d$ does not contain rigid body motions: rigid body rotations are eliminated due to the periodicity of the present setting, and rigid body translations with the factoring of constants in each component. The Korn inequality and the Poincaré inequality (2.5) imply existence of a positive constant $c$ such that

$$\forall \boldsymbol{v} \in X^d : \quad \mathfrak{b}^{\mathtt{A}}(\boldsymbol{v}, \boldsymbol{v}) \ge ca_{\min}\|\boldsymbol{v}\|_{H^1(\Omega)}^2 \,.$$

For given, fixed $\boldsymbol{f} \in [\operatorname{Hol}(\Omega)/\mathbb{R}]^d$, which is 1-periodic in each argument, there exists a unique solution of (6.7). Furthermore, the coefficient-to-solution map $S^{\mathrm{el}} : \mathtt{A} \mapsto \boldsymbol{u}^{\mathtt{A}}$ is analytic from the set $\mathcal{D}^{\mathrm{el}} = \{\mathtt{A} \in \operatorname{Hol}(\Omega, A_{\mathcal{D}^{\mathrm{el}}})^{d^4} : (6.6) \text{ and } (6.5) \text{ hold}\}$ to $\mathcal{U}^{\mathrm{el}} = S^{\mathrm{el}}(\mathcal{D}^{\mathrm{el}}) \subset X^d \cap \operatorname{Hol}(\Omega, A_{\mathcal{U}^{\mathrm{el}}})^d$, for positive constants $A_{\mathcal{D}^{\mathrm{el}}}, A_{\mathcal{U}^{\mathrm{el}}}$.

**6.2.2. Operator network approximation.** For the operator network approximation of the map $S^{\mathrm{el}}$, we introduce modified encoding and reconstruction operators. To construct the encoding operator, we extend the definition of the vectorization operation to fourth order tensors so that, for all $\mathtt{B} \in R^{n_1 \times \cdots \times n_4}$, $\mathbf{vec}(\mathtt{B}) \in \mathbb{R}^{n_1 \cdots n_4}$. We consequently extend the definition of the reordering function introduced in Section 6.1 to $\mathfrak{v} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$(6.8) \qquad\qquad\qquad\qquad \mathbf{vec}(\mathtt{B})_{\mathfrak{v}(m,n,p,q)} = \mathtt{B}_{mnpq}.$$

The modified encoding operator $\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}} : [C([0,1]^d)]^{d \times d \times d \times d} \to \mathbb{R}^{d^4 n_{\mathrm{q}}}$ is then given by

$$(6.9) \qquad \mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathtt{A}) := \begin{pmatrix} \mathbf{vec}(\mathtt{A}(\boldsymbol{x}^{(q)}_1)) \\ \vdots \\ \mathbf{vec}(\mathtt{A}(\boldsymbol{x}^{(q)}_{n_{\mathrm{q}}})) \end{pmatrix},$$

where $\boldsymbol{x}_{\mathrm{enc}} = \boldsymbol{x}^{(q)}_1, \ldots, \boldsymbol{x}^{(q)}_{n_{\mathrm{q}}}$ are the usual quadrature points. For all $m \in \mathbb{N}$, the modified reconstruction operator $\mathcal{R}^{\mathrm{el}} : \mathbb{R}^{dm} \to H^1(\Omega)^d$ is instead defined, given a neural network $\Phi^{\mathrm{branch}}$ such that $\mathrm{R}(\Phi^{\mathrm{branch}}) : \Omega \to \mathbb{R}^m$, as

$$(6.10) \qquad \mathcal{R}^{\mathrm{el}}_{\Phi^{\mathrm{branch}}}(\boldsymbol{c})(x) = \left( \mathbf{Id}_d \otimes \mathrm{R}(\Phi^{\mathrm{branch}})(x) \right)^\top \boldsymbol{c}, \qquad \forall x \in \Omega, \ \forall \boldsymbol{c} \in \mathbb{R}^{dm}.$$

We can now state the operator network approximation result for problem (6.2).

THEOREM 6.3. *Theorem 5.7 holds with $a \in \mathcal{D}$ replaced by $\mathtt{A} \in \mathcal{D}^{\mathrm{el}}$, $S(a)$ replaced by $S^{\mathrm{el}}(\mathtt{A})$, $\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}(a)$ replaced by $\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathtt{A})$, and $\mathcal{R}_{\Phi^{\mathrm{br}}_\varepsilon}$ replaced by $\mathcal{R}^{\mathrm{el}}_{\Phi^{\mathrm{br}}_\varepsilon}$.*

*Proof.* We construct a basis of the $dn_{\mathrm{b}}$-dimensional discrete space $X^d_{n_{\mathrm{b}}}$ approximating $X^d$ as

$$\boldsymbol{\psi}_1 = \begin{pmatrix} \varphi_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \boldsymbol{\psi}_{n_{\mathrm{b}}} = \begin{pmatrix} \varphi_{n_{\mathrm{b}}} \\ \vdots \\ 0 \end{pmatrix}, \ldots, \boldsymbol{\psi}_{(d-1)n_{\mathrm{b}}+1} = \begin{pmatrix} 0 \\ \vdots \\ \varphi_1 \end{pmatrix}, \boldsymbol{\psi}_{dn_{\mathrm{b}}} = \begin{pmatrix} 0 \\ \vdots \\ \varphi_{n_{\mathrm{b}}} \end{pmatrix},$$

where $\varphi_1, \ldots, \varphi_{n_{\mathrm{b}}}$ are the Legendre polynomials defined in Section 4.2. The trunk network $\Phi^{\mathrm{tr}}_\varepsilon$ is then constructed as in the proof of Theorem 5.7: it follows that the $j$th column of

$$\left( \mathbf{Id}_d \otimes \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon) \right)^\top$$

contains an approximation of $\boldsymbol{\psi}_j$, for each $j \in \{1, \ldots, dn_{\mathrm{b}}\}$.

To construct the branch network $\Phi^{\mathrm{br}}_\varepsilon$, we replace the input layer used in the proof of Theorem 5.7, in a similar way as we did in Lemma 6.2. Define, for all $i, j \in \{1, \ldots, dn_{\mathrm{b}}\}$ and $m, n, p, q \in \{1, \ldots, d\}$,

$$\widetilde{\mathbf{D}}^{ij}_{mnpq}(\boldsymbol{x}^{(q)}_k) = w^{(q)}_k \left( \boldsymbol{\varepsilon}[\boldsymbol{\psi}_i](\boldsymbol{x}^{(q)}_k) \right)_{mn} \left( \boldsymbol{\varepsilon}[\boldsymbol{\psi}_j](\boldsymbol{x}^{(q)}_k) \right)_{pq}$$

and let $\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_k)) \in \mathbb{R}^{d^2 n^2_{\mathrm{b}} \times d^4}$ such that

$$\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_k))_{\mathfrak{v}(i,j)\mathfrak{v}(m,n,p,q)} = \widetilde{\mathbf{D}}^{ij}_{mnpq}(\boldsymbol{x}^{(q)}_k),$$

with $\mathfrak{v}$ defined in (6.1) and (6.8) for two and four arguments, respectively. Then,

$$\Phi^{\mathtt{A},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}} := \left( \left( -\alpha \left[ \widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_1)) | \ldots | \widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_{n_{\mathrm{q}}})) \right], \mathbf{0}_{n_{\mathrm{b}}} \right) \right)$$

is such that

$$\mathbf{matr} \left( \mathrm{R}(\Phi^{\mathtt{A},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}})(\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathtt{A})) \right)_{ij} = -\alpha \mathfrak{b}^{\mathtt{A}}(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i), \qquad \forall (i,j) \in \{1, \ldots, dn_{\mathrm{b}}\}^2.$$

We can then construct $\Phi^{\mathrm{tr}}_\varepsilon$ as in the proof of Theorem 5.7, with $\Phi^{\mathtt{A},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}$ replacing $\Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}$. The rest of the proof follows the same argument as the proof of Theorem 5.7. $\square$

**7. Conclusions.** We proved, in the periodic setting on $\Omega = \mathbb{R}^d/\mathbb{Z}^d$, the exponential convergence of deep operator network emulation of the coefficient-to-solution map of some linear elliptic equations, under the assumption of analytic coefficients $a$ and right-hand sides $f$. The proof used the analytic regularity of solutions $u^a$ of (2.3) implied by classical elliptic regularity results and the consequential exponential convergence of polynomial approximations of $a$ and $u^a$ and of fully discrete spectral-Galerkin numerical schemes. We have developed the analysis for isotropic diffusion equations and extended it to problems with parametric diffusion, with anisotropic diffusion and reaction, and to linear elastostatics.

**Appendix A. Convergence of fully discrete Spectral-Galerkin Solution.** We present here the exponential convergence of fully discrete Spectral-Galerkin solutions of the problems considered in this paper. The following classical approximation result will be useful.

LEMMA A.1. *Let $A > 0$. Then, there exist $C, b > 0$ such that for all $p \in \mathbb{N}_0$ and for all $v \in \mathrm{Hol}(\Omega; A)$,*

$$\inf_{w \in \mathbb{Q}_p(\Omega)} \|v - w\|_{L^\infty(\Omega)} \le C \exp(-bp).$$

*Proof.* The statement is a consequence of, e.g., [18, Remark 3.1 and Theorem 3.5]. □

The following lemma, then, concerns the convergence of fully discrete Spectral-Galerkin solutions for problems in $\Omega$, with analytic right-hand sides and coefficients.

LEMMA A.2. *Let $\Theta \in \{\mathcal{D}, \mathcal{D}^{\mathrm{rd}}, \mathcal{D}^{\mathrm{el}}\}$ and $\mathfrak{d} = d$ for linear elasticity, $\mathfrak{d} = 1$ otherwise. Let $f \in \mathrm{Hol}(\Omega)^{\mathfrak{d}}$ and, for coefficients $\theta \in \Theta$, let $\mathfrak{b}^\theta(\cdot, \cdot)$ be one of the bilinear forms defined in Sections 2.1, 6.1.1, or 6.2.1. There exists $C_1, C_2 > 0$ such that, for all $p \in \mathbb{N}$ and for all integer $q \ge p + 1$,*

$$\sup_{\theta \in \Theta} \|u^\theta - u^\theta_{n_{\mathrm{b}}, n_{\mathrm{q}}}\|_{H^1(\Omega)} \le C_1 \exp(-C_2 p),$$

*where $u^\theta_{n_{\mathrm{b}}, n_{\mathrm{q}}} \in \mathbb{Q}_p(\Omega)^{\mathfrak{d}}$ is such that $\mathfrak{b}^\theta_{q^d}(u^\theta_{n_{\mathrm{b}}, n_{\mathrm{q}}}, v) = (f, v)$ for all $v \in \mathbb{Q}_p(\Omega)^{\mathfrak{d}}$.*

*Proof.* Strang's lemma [20, Lemma 10.1] implies that there exists $C > 0$ independent of $\theta \in \Theta$, $p$, and $q$, such that

$$\|u^\theta - u^\theta_{n_{\mathrm{b}}, n_{\mathrm{q}}}\|_{H^1(\Omega)} \le C \inf_{v \in \mathbb{Q}_p(\Omega)^{\mathfrak{d}}} \left( \|u^\theta - v\|_{H^1(\Omega)} + \sup_{w \in \mathbb{Q}_p(\Omega)^{\mathfrak{d}} \setminus \{\mathbf{0}\}} \frac{|\mathfrak{b}^\theta(v, w) - \mathfrak{b}^\theta_{n_{\mathrm{q}}}(v, w)|}{\|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}} \right).$$

By [3, Section 6.4.3], then, denoting $\widetilde{p} = \lfloor p/2 \rfloor$, there exists $\widetilde{C} > 0$ independent of $\theta \in \Theta$, $p$, and $q$ such that

$$\|u^\theta - u^\theta_{n_{\mathrm{b}}, n_{\mathrm{q}}}\|_{H^1(\Omega)} \le \widetilde{C} \left( \inf_{v \in \mathbb{Q}_{\widetilde{p}}(\Omega)^{\mathfrak{d}}} \|u^\theta - v\|_{H^1(\Omega)} + \inf_{v \in Y_{\widetilde{p}}} \|\theta - v\|_{L^\infty(\Omega)} \right),$$

where the space $Y_{\widetilde{p}}$ depends on the problem under consideration:

$$Y_{\widetilde{p}} = \begin{cases} \mathbb{Q}_{\widetilde{p}}(\Omega) & \text{if } \Theta = \mathcal{D}, \\ \left\{ \mathbf{A} \in \mathbb{Q}_{\widetilde{p}}(\Omega)^{d \times d} : \mathbf{A}_{ij} = \mathbf{A}_{ji} \right\} \times \mathbb{Q}_{\widetilde{p}}(\Omega) & \text{if } \Theta = \mathcal{D}^{\mathrm{rd}}, \\ \left\{ \mathbb{A} \in \mathbb{Q}_{\widetilde{p}}(\Omega)^{d \times d \times d \times d} : (6.5) \text{ holds} \right\} & \text{if } \Theta = \mathcal{D}^{\mathrm{el}}. \end{cases}$$

Since functions in $\Theta$ and in $S(\mathcal{D})$, $S^{\mathrm{rd}}(\mathcal{D}^{\mathrm{rd}})$, or $S^{\mathrm{el}}(\mathcal{D}^{\mathrm{el}})$ are analytic with uniform bounds on the norms at all orders, using Lemma A.1 concludes the proof. □

**Appendix B. Lipschitz continuity of the data-to-solution map.** For the readers' convenience, we provide a proof of the (known) Lipschitz dependence of the solution of the PDEs considered in this paper on the coefficients.

LEMMA B.1. *Let $X$ be a Hilbert space, let $Y$ be a Banach space, and let $\Theta \subset Y$. Let furthermore $\mathfrak{b}^\theta : X \times X \to \mathbb{R}$ be a bilinear form that is also linear with respect to the coefficient $\theta$. Suppose that*

$$\text{(B.1)} \qquad \mathfrak{b}^\theta(u, v) \leq \|\theta\|_Y \|u\|_X \|v\|_X, \qquad \forall u, v \in X, \ \forall \theta \in Y.$$

*Furthermore, suppose there exists $\theta_{\min} > 0$ such that*

$$\text{(B.2)} \qquad \mathfrak{b}^\theta(u, u) \geq \theta_{\min} \|u\|_X^2, \qquad \forall u \in X, \ \forall \theta \in \Theta.$$

*For fixed $f \in X'$ and for each $\theta \in \Theta$, define $u^\theta \in X$ as the function such that*

$$\text{(B.3)} \qquad \mathfrak{b}^\theta(u^\theta, v) = \langle f, v \rangle, \qquad \forall v \in X.$$

*Then, there exists $C_L > 0$ (depending only on $\theta_{\min}$ and $f$) such that*

$$\|u^{\theta_1} - u^{\theta_2}\|_X \leq C_L \|\theta_1 - \theta_2\|_Y, \qquad \forall \theta_1, \theta_2 \in \Theta.$$

*Proof.* Denote $u_i = u^{\theta_i}$, $i = 1, 2$. Using (B.2), (B.3), the continuity of the bilinear form with respect to the coefficient, and (B.1)

$$\begin{aligned}
\|u_1 - u_2\|_X^2 &\leq \frac{1}{\theta_{\min}} \mathfrak{b}^{\theta_1}(u_1 - u_2, u_1 - u_2) \\
&\leq \frac{1}{\theta_{\min}} \left( \mathfrak{b}^{\theta_2}(u_2, u_1 - u_2) - \mathfrak{b}^{\theta_1}(u_2, u_1 - u_2) \right) \\
&\leq \frac{1}{\theta_{\min}} \|\theta_2 - \theta_1\|_Y \|u_2\|_X \|u_1 - u_2\|_X.
\end{aligned}$$

The Lax-Milgram bound $\|u_2\|_X \leq \frac{1}{\theta_{\min}} \|f\|_{X'}$ concludes the proof. □

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