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Research Report No. 2021-41

December 2021
Latest revision: July 2022

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
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Funding: Austrian Science Fund (FWF) by the special research program

WEIGHTED ANALYTIC REGULARITY FOR THE INTEGRAL FRACTIONAL LAPLACIAN IN POLYGONS

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Abstract. We prove weighted analytic regularity of solutions to the Dirichlet problem for the integral fractional Laplacian in polygons with analytic right-hand side. We localize the problem through the Caffarelli-Silvestre extension and study the tangential differentiability of the extended solutions, followed by bootstrapping based on Caccioppoli inequalities on dyadic decompositions of vertex, edge, and edge-vertex neighborhoods.

Key word. fractional Laplacian, analytic regularity, corner domains, weighted Sobolev spaces

AMS subject classifications. 26A33, 35A20, 35B45, 35J70, 35R11.

1. Introduction. In this work, we study the regularity of solutions to the Dirichlet problem for the integral fractional Laplacian

$$(-\Delta)^s u = f \text{ on } \Omega, \qquad u = 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega}, \tag{1.1}$$

with $0 < s < 1$, where we consider the case of a polygonal Ω and a source term f that is analytic. We derive weighted analytic-type estimates for the solution u, with vertex and edge weights that vanish on the domain boundary $\partial\Omega$.

Unlike their integer order counterparts, solutions to fractional Laplace equations are known to lose regularity near $\partial\Omega$, even when the source term and $\partial\Omega$ are smooth (see, e.g., [Gru15]). After the establishment of low-order Hölder regularity up to the boundary for $C^{1,1}$ domains in [RS14], solutions to the Dirichlet problem for the integral fractional Laplacian have been shown to be smooth (after removal of the boundary singularity) in C^∞ domains [Gru15]. Subsequent results have filled in the gap between low and high regularity in Sobolev [AG20] and Hölder spaces [ARO20], with appropriate assumptions on the regularity of the domain. Besov regularity of weak solutions u of (1.1) has recently been established in [BN21] in Lipschitz domains Ω. Finally, for polygonal Ω, the precise characterization of the singularities of the solution in vertex, edge, and edge-vertex neighborhoods is the focus of the Mellin-based analysis of [GSŠ21, Što20].

For smooth geometries, [Gru15] characterizes the mapping properties of the integral fractional Laplacian, exhibiting in particular the anisotropic nature of solutions near the boundary. Interior regularity results have been obtained in [Coz17, BWZ17, FKM22] and, under analyticity assumptions on the right-hand side, (interior) analyticity of the solution has been derived even for certain nonlinear problems [KRS19, DFØS12, DFØS13] and more general integro-differential operators [AFV15]. The loss of regularity near the boundary can be accounted for by weights in the context of isotropic Sobolev spaces [AB17]. While all the latter references focus on the Dirichlet integral fractional Laplacian, which is also the topic of the present work, corresponding regularity results for the Dirichlet spectral fractional Laplacian are also available, see, e.g., [CS16].

The purpose of the present work is a description of the regularity of the solution of (1.1) for piecewise analytic input data that reflects both the interior analyticity and the anisotropic nature of the solution near the boundary. This is achieved in Theorem 2.1 through the use of appropriately weighted Sobolev spaces. Unlike local elliptic operators in polygons, for which vertex-weighted spaces allow for analytic regularity shifts (e.g., [BG88, MR10]), corresponding results for fractional operators in polygons require additionally edge-weights [Gru15].

An observation that was influential in the analysis of elliptic fractional diffusion problems is their *localization through a local, divergence form, elliptic degenerate operator in higher dimension.* First pointed out in [CS07], it subsequently inspired many developments in the analysis of fractional problems. While not falling into the standard elliptic setting (see, e.g., the discussion in [Gru15]), the localization via a higher-dimensional local elliptic boundary value problem does allow one to leverage tools from

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Funding: The research of JMM is funded by the Austrian Science Fund (FWF) by the special research program *Taming complexity in PDE systems* (grant SFB F65). The research of CM was performed during a PostDoctoral fellowship at the Seminar for Applied Mathematics, ETH Zürich, in 2020-2021.

elliptic regularity theory. Indeed, the present work studies the regularity of the higher-dimensional local degenerate elliptic problem and infers from that the regularity of (1.1) by taking appropriate traces.

Our analysis is based on Caccioppoli estimates and bootstrapping methods for the higher-dimensional elliptic problem. Such arguments are well-known to require (under suitable assumptions on the data) a basic regularity shift for variational solutions from the energy space of the problem (in the present case, a fractional order, nonweighted Sobolev space) into a slightly smaller subspace (with a fixed order increase in regularity). This is subsequently used to iterate in a bootstrapping manner local regularity estimates of Caccioppoli type on appropriately scaled balls in a Besicovitch covering of the domain. In the classical setting of non-degenerate elliptic problems, the initial regularity shift (into a vertex-weighted Sobolev space) is achieved by localization and a Mellin type analysis at vertices, as presented, e.g., in [MR10] and the references there. The subsequent bootstrapping can then lead to analytic regularity as established in a number of references for local non-degenerate elliptic boundary value problems (see, e.g., [BG88, GB97a, GB97b, CDN12] and the references there). The bootstrapping argument of the present work structurally follows these approaches.

While delivering sharp ranges of indices for regularity shifts (as limited by poles in the Mellin resolvent), the Mellin-based approach will naturally meet with difficulties in settings with multiple, non-separated vertices (as arise, e.g., in general Lipschitz polygons). Here, an alternative approach to extract some finite amount of regularity in nonweighted Besov-Triebel-Lizorkin spaces was proposed in [Sav98]; it is based on difference-quotient techniques and compactness arguments. In the present work, our initial regularity shift is obtained with the techniques of [Sav98]. In contrast to the Mellin approach, the technique of [Sav98] leads to regularity shifts even in Lipschitz domains but does not, as a rule, give better shifts for piecewise smooth geometries such as polygons. While this could be viewed as mathematically non-satisfactory, we argue in the present note that it can be quite adequate as a base shift estimate in establishing analytic regularity in vertex- and boundary-weighted Sobolev spaces, where quantitative control of constants under scaling takes precedence over the optimal range of smoothness indices.

1.1. Impact on numerical methods. The mathematical analysis of efficient numerical methods for the numerical approximation of fractional diffusion has received considerable attention in recent years. We only mention the surveys [DDG+20, BBN+18, BLN20, LPG+20] and the references there for broad surveys on recent developments in the analysis and in the discretization of nonlocal, fractional models. At this point, most basic issues in the numerical analysis of discretizations of linear, elliptic fractional diffusion problems are rather well understood, and convergence rates of variational discretizations based on finite element methods on regular simplicial meshes have been established, subject to appropriate regularity hypotheses. Regularity in isotropic Sobolev/Besov spaces is available, [BN21], leading to certain algebraically convergent methods based on shape-regular simplicial meshes. As discussed above, the expected solution behavior is anisotropic so that edge-refined meshes can lead to improved convergence rates. Indeed, a sharp analysis of vertex and edge singularities via Mellin techniques is the purpose of [GSŠ21, Što20] and allows for unravelling the optimal mesh grading for algebraically convergent methods. The analytic regularity result obtained in Theorem 2.1 captures both the anisotropic behavior of the solution and its analyticity so that *exponentially convergent* numerical methods for integral fractional Laplace equations in polygons can be developed in our follow-up work [FMMS22b]; see also [FMMS22a] for the corresponding convergence theory in 1D.

1.2. Structure of this text. After having introduced some basic notation in the forthcoming subsection, in Section 2 we present the variational formulation of the nonlocal boundary value problem. We also introduce the scales of boundary-weighted Sobolev spaces on which our regularity analysis is based. In Section 2.2, we state our main regularity result, Theorem 2.1. The rest of this paper is devoted to its proof, which is structured as follows.

Section 3 develops regularity estimates for the localized extension. In Section 4, we establish along the lines of [Sav98], a local regularity shift for the tangential derivatives of the solution of the extension problem, in a vicinity of (smooth parts of) the boundary. These estimates are combined in Section 5 with covering arguments and scaling to establish the weighted analytic regularity.

Section 6 provides a brief summary of our main results, and outlines generalizations and applications of the present results.

1.3. Notation. For open $\omega \subseteq \mathbb{R}^d$ and $t \in \mathbb{N}_0$, the spaces $H^t(\omega)$ are the classical Sobolev spaces of order t. For $t \in (0,1)$, fractional order Sobolev spaces are given in terms of the Aronstein-Slobodeckij seminorm $|\cdot|_{H^t(\omega)}$ and the full norm $\|\cdot\|_{H^t(\omega)}$ by

$$(1.2)\qquad |v|^2_{H^t(\omega)} = \int_{x\in\omega}\int_{z\in\omega} \frac{|v(x)-v(z)|^2}{|x-z|^{d+2t}}\,dz\,dx, \qquad \|v\|^2_{H^t(\omega)} = \|v\|^2_{L^2(\omega)} + |v|^2_{H^t(\omega)},$$

where we denote the Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$. For bounded Lipschitz domains $\Omega\subset\mathbb{R}^d$ and $t\in(0,1)$, we introduce additionally

$$\widetilde{H}^t(\Omega) := \{u\in H^t(\mathbb{R}^d) \,:\, u\equiv 0 \text{ on } \mathbb{R}^d\backslash\overline{\Omega}\}, \quad \|v\|^2_{\widetilde{H}^t(\Omega)} := \|v\|^2_{H^t(\Omega)} + \left\|v/r^t_{\partial\Omega}\right\|^2_{L^2(\Omega)},$$

where $r_{\partial\Omega}(x) := \operatorname{dist}(x,\partial\Omega)$ denotes the Euclidean distance of a point $x\in\Omega$ from the boundary $\partial\Omega$. On $\widetilde{H}^t(\Omega)$ we have, by combining [Gri11, Lemma 1.3.2.6] and [AB17, Proposition 2.3], the estimate

$$(1.3)\qquad \forall u\in\widetilde{H}^t(\Omega):\quad \|u\|_{\widetilde{H}^t(\Omega)} \le C|u|_{H^t(\mathbb{R}^d)}$$

for some $C>0$ depending only on t and Ω. For $t\in(0,1)\backslash\{\frac12\}$, the norms $\|\cdot\|_{\widetilde{H}^t(\Omega)}$ and $\|\cdot\|_{H^t(\Omega)}$ are equivalent on $\widetilde{H}^t(\Omega)$, see, e.g., [Gri11, Sec. 1.4.4]. Furthermore, for $t>0$, the space $H^{-t}(\Omega)$ denotes the dual space of $\widetilde{H}^t(\Omega)$, and we write $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ for the duality pairing that extends the $L^2(\Omega)$-inner product.

We denote by $\mathbb{R}_+$ the positive real numbers. For subsets $\omega\subset\mathbb{R}^d$, we will use the notation $\omega^+ := \omega\times\mathbb{R}_+$. For any multi index $\beta=(\beta_1,\dots,\beta_d)\in\mathbb{N}_0^d$, we denote $\partial_x^\beta = \partial_{x_1}^{\beta_1}\cdots\partial_{x_d}^{\beta_d}$ and $|\beta|=\sum_{i=1}^d\beta_i$. We adhere to convention that empty sums are null, i.e., $\sum_{j=a}^b c_j = 0$ when $b<a$; this even applies to the case where the terms c_j may not be defined. We also follow the standard convention $0^0=1$.

Throughout this article, we use the notation $\lesssim$ to abbreviate $\le$ up to a generic constant $C>0$ that does not depend on critical parameters in our analysis.

2. Setting.

There are several different ways to define the fractional Laplacian $(-\Delta)^s$ for $s\in(0,1)$. A classical definition on the full space $\mathbb{R}^d$ is in terms of the Fourier transformation $\mathcal{F}$, i.e., $(\mathcal{F}(-\Delta)^s u)(\xi) = |\xi|^{2s}(\mathcal{F}u)(\xi)$. Alternative, equivalent definitions of $(-\Delta)^s$ are, e.g., via spectral, semi-group, or operator theory, [Kwa17] or via singular integrals.

In the following, we consider the integral fractional Laplacian defined pointwise for sufficiently smooth functions u as the principal value integral

$$(2.1)\qquad (-\Delta)^s u(x) := C(d,s)\ \text{P.V.}\int_{\mathbb{R}^d}\frac{u(x)-u(z)}{|x-z|^{d+2s}}\,dz \quad\text{with}\quad C(d,s) := -2^{2s}\frac{\Gamma(s+d/2)}{\pi^{d/2}\Gamma(-s)},$$

where $\Gamma(\cdot)$ denotes the Gamma function. We investigate the fractional differential equation

$$(2.2a)\qquad (-\Delta)^s u = f \qquad \text{in}\,\Omega,$$

$$(2.2b)\qquad u = 0 \qquad \text{in}\,\Omega^c := \mathbb{R}^d\backslash\overline{\Omega},$$

where $s\in(0,1)$ and $f\in H^{-s}(\Omega)$ is a given right-hand side. Equation (2.2) is understood in weak form: Find $u\in\widetilde{H}^s(\Omega)$ such that

$$(2.3)\qquad a(u,v) := \langle(-\Delta)^s u, v\rangle_{L^2(\mathbb{R}^d)} = \langle f,v\rangle_{L^2(\Omega)} \qquad \forall v\in\widetilde{H}^s(\Omega).$$

The bilinear form a has the alternative representation

$$(2.4)\qquad a(u,v) = \frac{C(d,s)}{2}\int\!\!\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{d+2s}}\,dz\,dx \qquad \forall u,v\in\widetilde{H}^s(\Omega).$$

Existence and uniqueness of $u\in\widetilde{H}^s(\Omega)$ follow from the Lax–Milgram Lemma for any $f\in H^{-s}(\Omega)$, upon the observation that the bilinear form $a(\cdot,\cdot):\widetilde{H}^s(\Omega)\times\widetilde{H}^s(\Omega)\to\mathbb{R}$ is continuous and coercive.

2.1. The Caffarelli-Silvestre extension.

A very influential interpretation of the fractional Laplacian is provided by the so-called *Caffarelli-Silvestre extension*, due to [CS07]. It showed that the nonlocal operator $(-\Delta)^s$ can be be understood as a Dirichlet-to-Neumann map of a degenerate, *local* elliptic PDE on a half space in $\mathbb{R}^{d+1}$. Throughout the following text, we let

$$(2.5)\qquad \alpha := 1-2s.$$

2.1.1. Weighted spaces for the Caffarelli-Silvestre extension. Throughout the text, we single out the last component of points in $\mathbb{R}^{d+1}$ by writing them as (x,y) with $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, $y\in\mathbb{R}$. We introduce, for open sets $D\subset\mathbb{R}^d\times\mathbb{R}_+$, the weighted L^2-norm $\|\cdot\|_{L^2_\alpha(D)}$ via

$$\|U\|^2_{L^2_\alpha(D)} := \int_{(x,y)\in D} y^\alpha\,|U(x,y)|^2\,dx\,dy. \tag{2.6}$$

We denote by $L^2_\alpha(D)$ the space of functions on D that are square-integrable with respect to the weight y^α. We introduce $H^1_\alpha(D) := \{U\in L^2_\alpha(D) : \nabla U\in L^2_\alpha(D)\}$ as well as the Beppo-Levi space $\mathrm{BL}^1_\alpha := \{U\in L^2_{loc}(\mathbb{R}^d\times\mathbb{R}_+) : \nabla U\in L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)\}$. For elements of the Beppo-Levi space BL^1_α, one can give meaning to their trace at $y=0$, which is denoted $\operatorname{tr}U$. Recalling $\alpha=1-2s$, one has in fact $\operatorname{tr}U\in H^s_{loc}(\mathbb{R}^d)$ (see, e.g., [KM19, Lem. 3.8]). If $\operatorname{supp}\operatorname{tr}U\subset\overline{\Omega}$ for some bounded Lipschitz domain Ω, then $\operatorname{tr}U\in\widetilde{H}^s(\Omega)$ and

$$\|\operatorname{tr}U\|_{\widetilde{H}^s(\Omega)} \overset{(1.3)}{\lesssim} |\operatorname{tr}U|_{H^s(\mathbb{R}^d)} \overset{\text{[KM19, Lem. 3.8]}}{\lesssim} \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \tag{2.7}$$

with an implied constant depending on s and Ω.

2.1.2. The Caffarelli-Silvestre extension. Given $u\in\widetilde{H}^s(\Omega)$, let $U=U(x,y)$ denote the minimum norm extension of u to $\mathbb{R}^d\times\mathbb{R}_+$, i.e., $U=\operatorname{argmin}\{\|\nabla U\|^2_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \mid U\in\mathrm{BL}^1_\alpha,\ \operatorname{tr}U=u \text{ in } H^s(\mathbb{R}^d)\}$.

The function U is indeed unique in BL^1_α (see, e.g., [KM19, p. 2900]). The Euler-Lagrange equations corresponding to this extension problem read

$$\operatorname{div}(y^\alpha\nabla U)=0 \qquad \text{in } \mathbb{R}^d\times(0,\infty), \tag{2.8a}$$
$$U(\cdot,0)=u \qquad \text{in } \mathbb{R}^d. \tag{2.8b}$$

Henceforth, when referring to solutions of (2.8), we will additionally understand that $U\in\mathrm{BL}^1_\alpha$.

The relevance of (2.8) is due to the fact that the fractional Laplacian applied to $u\in\widetilde{H}^s(\Omega)$ can be recovered as distributional normal trace of the extension problem [CS07, Section 3], [CS16]:

$$(-\Delta)^s u = -d_s\lim_{y\to 0^+} y^\alpha\partial_y U(x,y), \qquad d_s=2^{2s-1}\Gamma(s)/\Gamma(1-s). \tag{2.9}$$

2.2. Main result: weighted analytic regularity for polygonal domains in $\mathbb{R}^2$. The following theorem is the main result of this article. It states that, provided the data f is analytic in $\overline{\Omega}$, we obtain analytic regularity for the solution u of (2.2) in a scale of weighted Sobolev spaces. In order to specify these weighted spaces, we need additional notation.

Let $\Omega\subset\mathbb{R}^2$ be a bounded, polygonal Lipschitz domain with finitely many vertices and (straight) edges (also, connectedness of the boundary is not necessary in the following). We denote by $\mathcal{V}$ the set of vertices and by $\mathcal{E}$ the set of the (open) edges. For $\mathbf{v}\in\mathcal{V}$ and $\mathbf{e}\in\mathcal{E}$, we define the distance functions

$$r_{\mathbf{v}}(x) := |x-\mathbf{v}|, \qquad r_{\mathbf{e}}(x) := \inf_{y\in\mathbf{e}}|x-y|, \qquad \rho_{\mathbf{ve}}(x) := r_{\mathbf{e}}(x)/r_{\mathbf{v}}(x).$$

For each vertex $\mathbf{v}\in\mathcal{V}$, we denote by $\mathcal{E}_{\mathbf{v}} := \{\mathbf{e}\in\mathcal{E} : \mathbf{v}\in\overline{\mathbf{e}}\}$ the set of all edges that meet at $\mathbf{v}$. For any $\mathbf{e}\in\mathcal{E}$, we define $\mathcal{V}_{\mathbf{e}} := \{\mathbf{v}\in\mathcal{V} : \mathbf{v}\in\overline{\mathbf{e}}\}$ as set of endpoints of $\mathbf{e}$. For fixed, sufficiently small $\xi>0$ and for $\mathbf{v}\in\mathcal{V}$, $\mathbf{e}\in\mathcal{E}$, we define vertex, edge-vertex and edge neighborhoods by

$$\begin{aligned}
\omega^\xi_{\mathbf{v}} &:= \{x\in\Omega : r_{\mathbf{v}}(x)<\xi \quad\wedge\quad \rho_{\mathbf{ve}}(x)\ge\xi \quad \forall\mathbf{e}\in\mathcal{E}_{\mathbf{v}}\},\\
\omega^\xi_{\mathbf{ve}} &:= \{x\in\Omega : r_{\mathbf{v}}(x)<\xi \quad\wedge\quad \rho_{\mathbf{ve}}(x)<\xi\},\\
\omega^\xi_{\mathbf{e}} &:= \{x\in\Omega : r_{\mathbf{v}}(x)\ge\xi \quad\wedge\quad r_{\mathbf{e}}(x)<\xi^2 \quad \forall\mathbf{v}\in\mathcal{V}_{\mathbf{e}}\}.
\end{aligned}$$

Figure 1 illustrates this notation near a vertex $\mathbf{v}\in\mathcal{V}$ of the polygon. Throughout the paper, we will assume that ξ is small enough so that $\omega^\xi_{\mathbf{v}}\cap\omega^\xi_{\mathbf{v}'}=\emptyset$ for all $\mathbf{v}\neq\mathbf{v}'$, that $\omega^\xi_{\mathbf{e}}\cap\omega^\xi_{\mathbf{e}'}=\emptyset$ for all $\mathbf{e}\neq\mathbf{e}'$ and $\omega^\xi_{\mathbf{ve}}\cap\omega^\xi_{\mathbf{v}'\mathbf{e}'}=\emptyset$ for all $\mathbf{v}\neq\mathbf{v}'$ and all $\mathbf{e}\neq\mathbf{e}'$. We will drop the superscript ξ unless strictly necessary.

We can decompose the Lipschitz polygon Ω into sectoral neighborhoods of vertices $\mathbf{v}$ which are unions of vertex and edge-vertex neighborhoods (as depicted in Figure 1), edge neighborhoods (that are away from a vertex), and an interior part Ω_{int}, i.e.,

$$\Omega = \bigcup_{\mathbf{v}\in\mathcal{V}}\left(\omega_{\mathbf{v}}\cup\bigcup_{\mathbf{e}\in\mathcal{E}_{\mathbf{v}}}\omega_{\mathbf{ve}}\right)\cup\bigcup_{\mathbf{e}\in\mathcal{E}}\omega_{\mathbf{e}}\cup\Omega_{\mathrm{int}}.$$

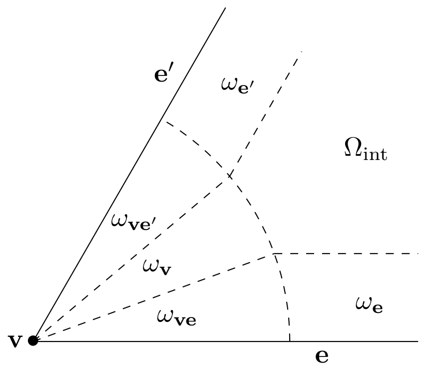

Fig. 1: Notation near a vertex $\mathbf{v}$.

Each sectoral and edge neighborhood may have a different value ξ. However, since only finitely many different neighborhoods are needed to decompose the polygon, the interior part $\Omega_{\text{int}} \subset \Omega$ has a positive distance from the boundary.

In a given edge neighborhood $\omega_{\mathbf{e}}$ or an edge-vertex neighborhood $\omega_{\mathbf{ve}}$, we let $\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ be two unit vectors such that $\mathbf{e}_\parallel$ is tangential to $\mathbf{e}$ and $\mathbf{e}_\perp$ is normal to $\mathbf{e}$. We introduce the differential operators

$$D_{x_\parallel} v \coloneqq \mathbf{e}_\parallel \cdot \nabla_x v, \qquad\qquad D_{x_\perp} v \coloneqq \mathbf{e}_\perp \cdot \nabla_x v$$

corresponding to differentiation in the tangential and normal direction. Inductively, we can define higher order tangential and normal derivatives by $D^j_{x_\parallel} v \coloneqq D_{x_\parallel}(D^{j-1}_{x_\parallel} v)$ and $D^j_{x_\perp} v \coloneqq D_{x_\perp}(D^{j-1}_{x_\perp} v)$ for $j > 1$.

Our main result provides local analytic regularity in edge- and vertex-weighted Sobolev spaces.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal Lipschitz domain. Let the data $f \in C^\infty(\overline{\Omega})$ satisfy*

$$\sum_{|\beta|=j} \|\partial_x^\beta f\|_{L^2(\Omega)} \le \gamma_f^{j+1} j^j \qquad \forall j \in \mathbb{N}_0 \tag{2.10}$$

with a constant $\gamma_f > 0$. Let $\mathbf{v} \in \mathcal{V}$, $\mathbf{e} \in \mathcal{E}$ and $\omega_{\mathbf{v}}$, $\omega_{\mathbf{ve}}$, $\omega_{\mathbf{e}}$ be fixed vertex, edge-vertex and edge-neighborhoods. Let u be the weak solution of (2.2).

Then, there is $\gamma > 0$ depending only on γ_f, s, and Ω such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ (depending only on ε and Ω) such that for all $p \in \mathbb{N}_0$ and for all $\beta \in \mathbb{N}_0^2$ with $|\beta| = p$

$$\left\| r_{\mathbf{v}}^{p-1/2-s+\varepsilon} \partial_x^\beta u \right\|_{L^2(\omega_{\mathbf{v}})} \le C_\varepsilon \gamma^{p+1} p^p, \tag{2.11}$$

and for all $(p_\perp, p_\parallel) \in \mathbb{N}_0^2$, with $p = p_\perp + p_\parallel$

$$\left\| r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{e}})} \le C_\varepsilon \gamma^{p+1} p^p, \tag{2.12}$$

$$\left\| r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{ve}})} \le C_\varepsilon \gamma^{p+1} p^p. \tag{2.13}$$

Finally, for the interior part Ω_{int} and all $p \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^2$ with $|\beta| = p$, we have

$$\left\| \partial_x^\beta u \right\|_{L^2(\Omega_{\text{int}})} \le \gamma^{p+1} p^p. \tag{2.14}$$

Remark 2.2. Inequalities (2.12) and (2.13) can be written in compact form: For all $\nu > -1/2 - s$ there exists $C_\nu > 0$ such that for $\bullet \in \{\mathbf{e}, \mathbf{ve}\}$

$$\|r_{\mathbf{v}}^{p+\nu} \rho_{\mathbf{ve}}^{p_\perp+\nu} D^{p_\parallel}_{x_\parallel} D^{p_\perp}_{x_\perp} u\|_{L^2(\omega_\bullet)} \le C_\nu \gamma^{p+1} p^p \qquad \forall (p_\perp, p_\parallel) \in \mathbb{N}_0^2 \text{ with } p = p_\parallel + p_\perp. \tag{2.15}$$

■

Remark 2.3. (i) Stirling's formula implies $p^p \le C p! e^p$. Therefore, there exists a constant $\widetilde{C}_\nu$ such that (2.15) can also be written as

$$\|r_{\mathbf{v}}^{p+\nu}\rho_{\mathbf{ve}}^{p_\perp+\nu}D_{x_\parallel}^{p_\parallel}D_{x_\perp}^{p_\perp}u\|_{L^2(\omega_\bullet)} \le \widetilde{C}_\nu(\gamma e)^{p+1}p!. \tag{2.16}$$

and the same can also be done for (2.11) and (2.14) in Theorem 2.1.

(ii) We note that $(p_\parallel+p_\perp)^{p_\parallel+p_\perp} \le p_\parallel^{p_\parallel}p_\perp^{p_\perp}e^{p_\parallel+p_\perp}$. Together with $p^p \le Cp!e^p$ (using Stirling's formula), one can also formulate the estimates (2.15) as follows: There are constants $\widetilde{C}_\nu$ and $\widetilde{\gamma} > 0$ such that for all $(p_\parallel, p_\perp) \in \mathbb{N}_0^2$,

$$\|r_{\mathbf{v}}^{p+\nu}\rho_{\mathbf{ve}}^{p_\perp+\nu}D_{x_\parallel}^{p_\parallel}D_{x_\perp}^{p_\perp}u\|_{L^2(\omega_\bullet)} \le \widetilde{C}_\nu\widetilde{\gamma}^{p_\perp+p_\parallel}p_\perp!\,p_\parallel!. \tag{2.17}$$

(iii) The assumption (2.10) on the data f expresses analyticity in $\overline{\Omega}$ (combine Morrey's embedding [Gri11, eq. (1,4,4,6)] to see $f \in C^\infty$ with [Mor66, Lemma 5.7.2]). Inspection of the proof (in particular Lemmas 5.5 and 5.7) shows that f could be admitted to be in vertex or edge-weighted classes of analytic functions. For simplicity of exposition, we do not explore this further.

(iv) Inspection of the proofs also shows that, in order to obtain weighted regularity of fixed, finite order p, only finite regularity of the data f is required.

(v) By Morrey's embedding, e.g., [Gri11, eq. (1,4,4,6)], estimate (2.14) implies that the solution $u \in C^\infty(\overline{\Omega_{\rm int}})$ as well as analyticity of u in $\overline{\Omega}_{\rm int}$, [Mor66, Lemma 5.7.2]. Other results on interior analytic regularity of more general, linear integro-differential operators are, e.g., in [AFV15], for $1/2 < s < 1$. ∎

3. Regularity results for the extension problem.

In this section, we derive local (higher order) regularity results for solutions to the Caffarelli-Silvestre extension problem. As the techniques employed are valid in any space dimension, we formulate our results for general $d \in \mathbb{N}$.

Fix $H > 0$. Given $F \in L^2_{-\alpha}(\mathbb{R}^d\times(0,H))$ and $f \in H^{-s}(\Omega)$, consider the problem to find the minimizer $U = U(x,y)$ with $x \in \mathbb{R}^d$ and $y \in \mathbb{R}_+$ of

$$\text{minimize } \mathcal{F} \text{ on } \mathrm{BL}^1_{\alpha,0,\Omega} := \{U \in \mathrm{BL}^1_\alpha \,:\, \operatorname{tr} U = 0 \ \text{ on } \Omega^c\} \quad , \tag{3.1}$$

where

$$\mathcal{F}(U) := \frac{1}{2}b(U,U) - \int_{\mathbb{R}^d\times(0,H)} FU\,dx\,dy - \int_\Omega f \operatorname{tr} U\,dx, \quad b(U,V) := \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha \nabla U\cdot\nabla V\,dx\,dy. \tag{3.2}$$

We have the following Poincaré type estimate:

Lemma 3.1. *(i) The map $\mathrm{BL}^1_{\alpha,0,\Omega} \ni U \mapsto \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}$ is a norm, and $\mathrm{BL}^1_{\alpha,0,\Omega}$ endowed with this norm is a Hilbert space with corresponding inner-product given by the bilinear form $b(\cdot,\cdot)$ in* (3.2).

(ii) For every $H \in (0,\infty)$, there is $C_{H,\alpha} > 0$ such that

$$\forall U \in \mathrm{BL}^1_{\alpha,0,\Omega}: \quad \|U\|_{L^2_\alpha(\mathbb{R}^d\times(0,H))} \le C_{H,\alpha}\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}. \tag{3.3}$$

Proof. Details of the proof are given in Appendix B. □

With Lemma 3.1 in hand, existence and uniqueness of solutions of (3.1) follows from the Lax-Milgram Lemma since, for $F \in L^2_{-\alpha}(\mathbb{R}^d\times(0,H))$ and $f \in H^{-s}(\Omega)$, the map $U \mapsto \int_{\mathbb{R}^d\times(0,H)} FU + \int_\Omega f \operatorname{tr} U$ in (3.2) extends to a bounded linear functional on $\mathrm{BL}^1_{\alpha,0,\Omega}$. In view of (3.3) and the trace estimate (2.7), the minimization problem (3.1) admits by Lax-Milgram a unique solution $U \in \mathrm{BL}^1_{\alpha,0,\Omega}$ with the *a priori* estimate

$$\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \le C\left[\|F\|_{L^2_{-\alpha}(\mathbb{R}^d\times(0,H))} + \|f\|_{H^{-s}(\Omega)}\right] \tag{3.4}$$

with constant C dependent on $s \in (0,1)$ and $H > 0$.

The Euler-Lagrange equations formally satisfied by the solution U of (3.1) are:

$$-\operatorname{div}(y^\alpha\nabla U) = F \qquad \text{in } \mathbb{R}^d\times(0,\infty), \tag{3.5a}$$

$$\partial_{n_\alpha}U(\cdot,0) = f \qquad \text{in } \Omega, \tag{3.5b}$$

$$\operatorname{tr} U = 0 \qquad \text{on } \Omega^c, \tag{3.5c}$$

where $\partial_{n_\alpha} U(x,0) = -d_s \lim_{y\to 0} y^\alpha \partial_y U(x,y)$ and we implicitly extended F to $\mathbb{R}^d \times \mathbb{R}_+$. In view of (2.9) together with the fractional PDE $(-\Delta)^s u = f$, this is a Neumann-type Caffarelli-Silvestre extension problem with an additional source F.

Remark 3.2. (i) The system (3.5) is understood in a weak sense, i.e., to find $U \in \mathrm{BL}^1_{\alpha,0,\Omega}$ such that

$$\forall V \in \mathrm{BL}^1_{\alpha,0,\Omega}: \quad b(U,V) = \int_{\mathbb{R}^d\times\mathbb{R}_+} FV\,dx\,dy + \int_\Omega f\,\mathrm{tr}\,V\,dx. \tag{3.6}$$

Due to (3.3), the integral $\int_{\mathbb{R}^d\times\mathbb{R}_+} FV\,dx\,dy$ is well-defined.

(ii) For the notion of solution of (3.5), the support requirement $\operatorname{supp} F \subset \mathbb{R}^d \times [0,H]$ can be relaxed e.g., to $F \in L^2_{-\alpha}(\mathbb{R}^d\times\mathbb{R}_+)$ by testing with $V \in H^1_{\alpha,0,\Omega}(\mathbb{R}^d\times\mathbb{R}_+) := H^1_\alpha(\mathbb{R}^d\times\mathbb{R}_+) \cap \mathrm{BL}^1_{\alpha,0,\Omega}$. In this case, the integral $\int_{\mathbb{R}^d\times\mathbb{R}_+} FV\,dx\,dy$ is well-defined by Cauchy-Schwarz.

(iii) For open $\omega \subset \mathbb{R}^d$ and $F \in L^2_{-\alpha}(\omega^+)$, we call U a solution to (3.5) on ω^+ if (3.6) holds for all test functions $V \in \{V \in H^1_{\alpha,0,\Omega} \,|\, \operatorname{supp} V \subset \overline{\omega}^+\}$.

(iv) We note that working with functions supported in $\mathbb{R}^d \times [0,H]$ induces an implicit dependence on H of all constants, which is due to the Poincaré type estimate (3.3). Alternatively to restricting the test space, one could also circumvent this by introducing suitable weights that control the behavior of F at infinity; we do not develop this here. ∎

3.1. Global regularity: a shift theorem. The following lemma provides additional regularity of the extension problem in the x–direction. The argument uses the technique developed in [Sav98] that has recently been used in [BN21] to show a closely related shift theorem for the Dirichlet fractional Laplacian; the technique merely assumes Ω to be a Lipschitz domain in $\mathbb{R}^d$. On a technical level, the difference between [BN21] and Lemma 3.3 below is that Lemma 3.3 studies (tangential) differentiability properties of the extension U, whereas [BN21] focuses on the trace $u = \mathrm{tr}\,U$.

For functions U, F, f, it is convenient to introduce the abbreviation

$$N^2(U,F,f) := \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \left(\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + \|F\|_{L^2_{-\alpha}(\mathbb{R}^d\times(0,H))} + \|f\|_{H^{1-s}(\Omega)} \right). \tag{3.7}$$

In view of the a *priori* estimate (3.4), we have the simplified bound (with updated constant C)

$$N^2(U,F,f) \le C \left(\|f\|^2_{H^{1-s}(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^d\times(0,H))} \right). \tag{3.8}$$

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $B_{\widetilde{R}} \subset \mathbb{R}^d$ be a ball with $\Omega \subset B_{\widetilde{R}}$. For $t \in [0,1/2)$, there is $C_t > 0$ (depending only on s, t, Ω, $\widetilde{R}$, and H) such that for $f \in C^\infty(\overline{\Omega})$, $F \in L^2_{-\alpha}(\mathbb{R}^d \times (0,H))$ the solution U of (3.1) satisfies*

$$\int_{\mathbb{R}_+} y^\alpha \|\nabla U(\cdot,y)\|^2_{H^t(B_{\widetilde{R}})}\,dy \le C_t N^2(U,F,f)$$

with $N^2(U,F,f)$ given by (3.7).

Proof. The idea is to apply the difference quotient argument from [Sav98] only in the x-direction.

Let $x_0 \in \overline{\Omega}$ be arbitrary. For $h \in \mathbb{R}^d$ denote $T_h U := \eta U_h + (1-\eta)U$, where $U_h(x,y) := U(x+h,y)$ and η is a cut-off function that localizes to a suitable ball $B_{2\rho}(x_0)$, i.e, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_\rho(x_0)$ and $\operatorname{supp}\eta \subset B_{2\rho}(x_0)$. In Steps 1–5 of this proof, we will abbreviate $B_{\rho'}$ for $B_{\rho'}(x_0)$ for $\rho' > 0$.

The main result of [Sav98] is that estimates for the modulus $\omega(U)$ defined with the quadratic functional $\mathcal{F}$ as in (3.2) by

$$\omega(U) := \sup_{h\in D\setminus\{0\}} \frac{\mathcal{F}(T_hU) - \mathcal{F}(U)}{|h|} = \omega_b(U) + \omega_F(U) + \omega_f(U),$$

$$\omega_b(U) := \frac{1}{2} \sup_{h\in D\setminus\{0\}} \frac{b(T_hU, T_hU) - b(U,U)}{|h|},$$

$$\omega_F(U) := \sup_{h\in D\setminus\{0\}} \frac{\int_{\mathbb{R}^d\times(0,H)} F(T_hU - U)}{|h|}, \qquad \omega_f(U) := \sup_{h\in D\setminus\{0\}} \frac{\int_\Omega f\,\mathrm{tr}(T_hU - U)}{|h|},$$

can be used to derive regularity results in Besov spaces.

Here, $D \subset \mathbb{R}^d$ denotes a set of admissible directions h. These directions are chosen such that the function $T_h U$ is an admissible test function, i.e., $T_h U \in \mathrm{BL}^1_{\alpha,0,\Omega}$. For this, we have to require $\operatorname{supp}\operatorname{tr}(T_h U) \subset \overline{\Omega}$. In [Sav98, (30)] a description of this set is given in terms of a set of admissible outward pointing vectors $\mathcal{O}_\rho(x_0)$, which are directions h with $|h| \le \rho$ such that for all $t \in [0,1]$ the translate $B_{3\rho}(x_0)\backslash\Omega + th$ is completely contained in Ω^c.

Step 1. (Estimate of $\omega_b(U)$). The definition of the bilinear form $b(\cdot,\cdot)$, $h \in D$, and the definition of T_h give

$$\begin{aligned}
b(T_hU, T_hU) - b(U,U) &= \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha(|\nabla T_h U|^2 - |\nabla U|^2)\,dx\,dy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha(|\nabla \eta(U_h - U) + T_h\nabla U|^2 - |\nabla U|^2)\,dx\,dy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha(|\nabla \eta(U_h - U)|^2 + 2T_h\nabla U\cdot\nabla\eta(U_h-U))\,dx\,dy \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha(|T_h\nabla U|^2 - |\nabla U|^2)\,dx\,dy \\
&=: T_1 + T_2.
\end{aligned}$$

For the first integral T_1, we use the support properties of η and that $\|U(\cdot,y) - U_h(\cdot,y)\|_{L^2(B_{2\rho})} \lesssim |h|\,\|\nabla U(\cdot,y)\|_{L^2(B_{3\rho})}$, which gives

$$\begin{aligned}
T_1 &\lesssim \int_{\mathbb{R}_+} y^\alpha(|h|^2\,\|\nabla U(\cdot,y)\|^2_{L^2(B_{3\rho})} + |h|\,\|\nabla U(\cdot,y)\|_{L^2(B_{3\rho})}\,\|T_h\nabla U(\cdot,y)\|_{L^2(B_{2\rho})})\,dy \\
&\lesssim |h| \int_{B^+_{3\rho}} y^\alpha\,|\nabla U|^2\,dx\,dy.
\end{aligned}$$

For the term T_2, we first note $|T_h\nabla U|^2 \le \eta\,|\nabla U_h|^2 + (1-\eta)\,|\nabla U|^2$ since $0 \le \eta \le 1$. Using the variable transformation $z = x + h$ together with $B_{2\rho}(x_0) + h \subset B_{3\rho}(x_0)$ we obtain

$$\begin{aligned}
T_2 &= \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha(|T_h\nabla U|^2 - |\nabla U|^2)\,dx\,dy \le \int_{\mathbb{R}_+}\int_{B_{2\rho}} y^\alpha\eta(|\nabla U_h|^2 - |\nabla U|^2)\,dx\,dy \\
&\le \int_{\mathbb{R}_+}\int_{B_{3\rho}} y^\alpha(\eta(x-h) - \eta(x))\,|\nabla U|^2\,dx\,dy \lesssim |h| \int_{B^+_{3\rho}} y^\alpha\,|\nabla U|^2\,dx\,dy.
\end{aligned}$$

Altogether we get from the previous estimates that

$$\omega_b(U) \lesssim \int_{B^+_{3\rho}} y^\alpha\,|\nabla U|^2\,dx\,dy.$$

Step 2. (Estimate of $\omega_F(U)$). Using the definition of T_h, we can write $U - T_hU = \eta(U - U_h)$, and $\operatorname{supp}\eta \subset B_{2\rho}(x_0)$ implies

$$\begin{aligned}
\left|\int_{\mathbb{R}^d\times(0,H)} F(U - T_hU)\,dx\,dy\right| &= \left|\int_{\mathbb{R}^d\times(0,H)} F\eta(U-U_h)\,dx\,dy\right| \le \|F\|_{L^2_{-\alpha}(B_{2\rho}\times(0,H))}\,\|U - U_h\|_{L^2_\alpha(B^+_{2\rho})} \\
&\lesssim |h|\,\|F\|_{L^2_{-\alpha}(B_{2\rho}\times(0,H))}\,\|\nabla U\|_{L^2_\alpha(B^+_{3\rho})}, \qquad (3.9)
\end{aligned}$$

which produces

$$\omega_F(U) \lesssim \|F\|_{L^2_{-\alpha}(B_{3\rho}\times(0,H))}\|\nabla U\|_{L^2_\alpha(B^+_{3\rho})}.$$

Step 3. (Estimate of $\omega_f(U)$). For the trace term, we use a second cut-off function $\widetilde{\eta} \in C_0^\infty(\mathbb{R}^{d+1})$ with $\widetilde{\eta} \equiv 1$ on $B_{2\rho}(x_0)$ and $\operatorname{supp}(\widetilde{\eta}) \subset B_{3\rho}(x_0)\times(-H,H)$ and get with the trace inequality (2.7) and the estimate (3.3)

$$\left|\int_\Omega f\,\mathrm{tr}(U-T_hU)\,dx\right| = \left|\int_{B_{2\rho}} f\eta\,\mathrm{tr}(U-U_h)\,dx\right| = \left|\int_{B_{3\rho}} (f\eta-(f\eta)_{-h})\,\mathrm{tr}(\widetilde{\eta}U)\,dx\right|$$

$$\le \|f\eta-(f\eta)_{-h}\|_{H^{-s}(B_{3\rho})}\,\|\mathrm{tr}(\widetilde{\eta}U)\|_{\widetilde{H}^s(B_{3\rho})}$$

$$\overset{(2.7),(3.3)}{\lesssim} |h|\,\|f\|_{H^{1-s}(B_{4\rho})}\,\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}, \tag{3.10}$$

where the estimate $\|f\eta-(f\eta)_{-h}\|_{H^{-s}(B_{3\rho})} \lesssim |h|\|f\|_{H^{1-s}(B_{4\rho})}$ can be seen, for example, by interpolating the estimates $\|f\eta-(f\eta)_{-h}\|_{H^{-1}(\mathbb{R}^d)} \lesssim |h|\|\eta f\|_{L^2(\mathbb{R}^d)}$ and $\|f\eta-(f\eta)_{-h}\|_{L^2(\mathbb{R}^d)} \lesssim |h|\|\eta f\|_{H^1(\mathbb{R}^d)}$, see, e.g., [Tar07]. We have thus obtained

$$\omega_f(U) \lesssim \|f\|_{H^{1-s}(B_{4\rho})}\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}.$$

Step 4. (Application of the abstract framework of [Sav98]). We introduce the seminorms $[U]^2 := \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha|\nabla U|^2\,dxdy$. By the coercivity of $b(\cdot,\cdot)$ on $\mathrm{BL}^1_{\alpha,0,\Omega}$ with respect to $[\cdot]^2$ and the abstract estimates in [Sav98, Sec. 2], we have

$$[U-T_hU]^2 \overset{[\mathrm{Sav98}]}{\lesssim} \omega(U)|h| \lesssim |h|\,(\omega_b(U)+\omega_F(U)+\omega_f(U))$$

$$\overset{\text{steps 1-3}}{\le} |h|\left(\|\nabla U\|^2_{L^2_\alpha(B^+_{3\rho})} + \|F\|_{L^2_{-\alpha}(B^+_{2\rho})}\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + \|f\|_{H^{1-s}(B_{4\rho})}\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}\right)$$

$$=: |h|\,\widetilde{C}^2_{U,F,f}.$$

Using that $\eta\equiv 1$ on $B^+_\rho(x_0)$, we get

$$\int_{B^+_\rho} y^\alpha|\nabla U-\nabla U_h|^2\,dx\,dy \le \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha|\nabla(\eta U-\eta U_h)|^2\,dx\,dy = [U-T_hU]^2 \le |h|\,\widetilde{C}^2_{U,F,f}. \tag{3.11}$$

Step 5: (Removing the restriction $h\in D$). The set D contains a truncated cone $C=\{x\in\mathbb{R}^d : |x\cdot e_D|>\delta|x|\}\cap B_{R'}(0)$ for some unit vector e_D and $\delta\in(0,1)$, $R'>0$. Geometric considerations then show that there is $c_D>0$ sufficiently large such that for arbitrary $h\in\mathbb{R}^d$ sufficiently small, $h+c_D|h|e_D\in D$. For a function v defined on $\mathbb{R}^d$, we observe

$$v(x)-v_h(x) = v(x)-v(x+h) = v(x)-v(x+(h+c_D|h|e_D)) + v((x+h)+c_D|h|e_D) - v(x+h).$$

We may integrate over $B_{\rho'}(x_0)$ and change variables to get

$$\|v-v_h\|^2_{L^2(B_{\rho'})} \le 2\left\|v-v_{h+c_D|h|e_D}\right\|^2_{L^2(B_{\rho'})} + 2\left\|v-v_{c_D|h|e_D}\right\|^2_{L^2(B_{\rho'}+h)}.$$

Selecting $\rho'=\rho/2$ and for $|h|\le\rho/2$, we obtain

$$\|v-v_h\|^2_{L^2(B_{\rho/2})} \le 2\left\|v-v_{h+c_D|h|e_D}\right\|^2_{L^2(B_\rho)} + 2\left\|v-v_{c_D|h|e_D}\right\|^2_{L^2(B_\rho)}.$$

Applying this estimate with $v=\nabla U$ and using that $h+c_D|h|e_D\in D$ and $c_D|h|e_D\in D$, we get from (3.11) that

$$\|\nabla U-\nabla U_h\|^2_{L^2_\alpha(B^+_{\rho/2})} \lesssim |h|\,\widetilde{C}^2_{U,F,f}.$$

The fact that Ω is a Lipschitz domain implies that the value of ρ and the constants appearing in the definition of the truncated cone C can be controlled uniformly in $x_0\in\Omega$. Hence, covering the ball $B_{2\widetilde{R}}$ (with twice the radius as the ball $B_{\widetilde{R}}$) by finitely many balls $B_{\rho/2}$, we obtain with the constant $N(U,F,f)$ of (3.7)

$$\|\nabla U-\nabla U_h\|^2_{L^2_\alpha(B_{2\widetilde{R}})} \lesssim |h|\,N^2(U,F,f) \tag{3.12}$$

for all $h\in\mathbb{R}^d$ with $|h|\le\delta'$ for some fixed $\delta'>0$.

Step 6: ($H^t(B_{\widetilde{R}})$–estimate). For $t < 1/2$, we estimate with the Aronstein-Slobodecki seminorm

$$\int_{\mathbb{R}_+} |\nabla U(\cdot,y)|^2_{H^t(B_{\widetilde{R}})}\,dy \le \int_{\mathbb{R}_+}\int_{x\in B_{\widetilde{R}}}\int_{|h|\le 2\widetilde{R}} \frac{|\nabla U(x+h,y)-\nabla U(x,y)|^2}{|h|^{d+2t}}\,dh\,dx\,dy.$$

The integral in h is split into the range $|h| \le \varepsilon$ for some fixed $\varepsilon > 0$, for which (3.12) can be brought to bear, and $\varepsilon < |h| < 2\widetilde{R}$, for which a triangle inequality can be used. We obtain

$$\begin{aligned}\int_{\mathbb{R}_+} |\nabla U(\cdot,y)|^2_{H^t(B_{\widetilde{R}})}\,dy &\lesssim N^2(U,F,f)\int_{|h|\le\varepsilon}|h|^{1-d-2t}\,dh + \|\nabla U\|^2_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}\int_{\varepsilon<|h|<2\widetilde{R}}|h|^{-d-2t}\,dh\\ &\lesssim N^2(U,F,f),\end{aligned}$$

which is the sought estimate. ∎

Remark 3.4. The regularity assumptions on F and f can be weakened by interpolation techniques as described in [Sav98, Sec. 4]. For example, by linearity, we may write $U = U_F + U_f$, where U_F and U_f solve (3.5) for data $(F, 0)$ and $(0, f)$. The *a priori* estimate (3.4) gives $\|\nabla U_f\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \le C\|f\|_{H^{-s}(\Omega)}$ so that we have

$$\begin{aligned}\int_{\mathbb{R}_+} |\nabla U_f(\cdot,y)|^2_{H^t(B_{\widetilde{R}})}\,dy &\le C_t\left(\|\nabla U_f\|^2_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + \|f\|_{H^{1-s}(\Omega)}\|\nabla U_f\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}\right)\\ &\lesssim \|f\|^2_{H^{-s}(\Omega)} + \|f\|_{H^{1-s}(\Omega)}\|f\|_{H^{-s}(\Omega)} \lesssim \|f\|_{H^{1-s}(\Omega)}\|f\|_{H^{-s}(\Omega)}.\end{aligned}$$

By, e.g., [Tar07, Lemma 25.3], the mapping $f \mapsto U_f$ then satisfies

$$\int_{\mathbb{R}_+} |\nabla U_f(\cdot,y)|^2_{H^t(B_{\widetilde{R}})}\,dy \le C_t\|f\|^2_{B^{1/2-s}_{2,1}(\Omega)},$$

where $B^{1/2-s}_{2,1}(\Omega) = (H^{-s}(\Omega), H^{1-s}(\Omega))_{1/2,1}$ is an interpolation space (K-method). We mention that $B^{1/2-s}_{2,1}(\Omega) \subset H^{1/2-s-\varepsilon}(\Omega)$ for every $\varepsilon > 0$.

A similar estimate could, in principle, be obtained for U_F; however, the pertinent interpolation space is less tractable. ■

3.2. Interior regularity for the extension problem. In the following, we derive localized interior regularity estimates, also called Caccioppoli inequalities, for solutions to the extension problem (3.5), where second order derivatives on some ball $B_R(x_0) \subset \Omega$ can be controlled by first order derivatives on some ball with a (slightly) larger radius.

The following Caccioppoli type inequality provides local control of higher order x-derivatives and is structurally similar to [FMP21, Lem. 4.4].

LEMMA 3.5 (Interior Caccioppoli inequality). *Let $B_R := B_R(x_0) \subset \Omega \subset \mathbb{R}^d$ be an open ball of radius $R > 0$ centered at $x_0 \in \Omega$, and let B_{cR} be the concentric scaled ball of radius cR with $c \in (0,1)$. Let $\zeta \in C_0^\infty(B_R)$ with $0 \le \zeta \le 1$ and $\zeta \equiv 1$ on B_{cR} as well as $\|\nabla\zeta\|_{L^\infty(B_R)} \le C_\zeta((1-c)R)^{-1}$ for some $C_\zeta > 0$ independent of c, R. Let U satisfy (3.5a), (3.5b) on B_R^+ with given data f and F (see Remark 3.2(iii)).*

Then, there is $C_{\rm int} > 0$ independent of R and c such that for $i \in \{1,\dots,d\}$

$$(3.13)\qquad \|\partial_{x_i}(\nabla U)\|^2_{L^2_\alpha(B^+_{cR})} \le C^2_{\rm int}\left(((1-c)R)^{-2}\|\nabla U\|^2_{L^2_\alpha(B^+_R)} + \|\zeta\partial_{x_i}f\|^2_{H^{-s}(\Omega)} + \|F\|^2_{L^2_{-\alpha}(B^+_R)}\right).$$

In particular, $\|\zeta\partial_{x_i}f\|_{H^{-s}(\Omega)} \le C_{\rm loc}\|\partial_{x_i}f\|_{L^2(B_R)}$ for some $C_{\rm loc} > 0$ independent of R, c, and f (cf. Lemma A.1).

Proof. The function ζ is defined on $\mathbb{R}^d$; through the constant extension we will also view it as a function on $\mathbb{R}^d \times \mathbb{R}_+$. With the unit vector e_{x_i} in the x_i-coordinate and $\tau \in \mathbb{R}\backslash\{0\}$, we define the difference quotient

$$D^\tau_{x_i}w(x) := \frac{w(x+\tau e_{x_i}) - w(x)}{\tau}.$$

For $|\tau|$ sufficiently small, we may use the test function $V = D^{-\tau}_{x_i}(\zeta^2 D^{\tau}_{x_i} U)$ in the weak formulation of (3.5) (observe that this test function is in $H^1_{\alpha,0,\Omega}$ and has support in $\overline{B^+_R}$) and compute

$$\operatorname{tr} V = -\frac{1}{\tau^2}\Big(\zeta^2(x-\tau e_{x_i})(u(x)-u(x-\tau e_{x_i})) + \zeta^2(x)(u(x)-u(x+\tau e_{x_i}))\Big) = D^{-\tau}_{x_i}(\zeta^2 D^{\tau}_{x_i} u).$$

Integration by parts in (3.5) over $\mathbb{R}^d \times \mathbb{R}_+$ and using that the Neumann trace (up to the constant d_s from (2.9)) produces the fractional Laplacian gives

$$\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{R}_+} FV\,dx\,dy - \frac{1}{d_s}\int_{\mathbb{R}^d}(-\Delta)^s u \operatorname{tr} V\,dx &= \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha \nabla U\cdot\nabla V\,dx\,dy\\
&= \int_{\mathbb{R}^d\times\mathbb{R}_+} D^{\tau}_{x_i}(y^\alpha\nabla U)\cdot\nabla(\zeta^2 D^{\tau}_{x_i}U)\,dx\,dy\\
&= \int_{B^+_R} y^\alpha D^{\tau}_{x_i}(\nabla U)\cdot\big(\zeta^2\nabla D^{\tau}_{x_i}U + 2\zeta\nabla\zeta D^{\tau}_{x_i}U\big)\,dx\,dy\\
&= \int_{B^+_R} y^\alpha\zeta^2 D^{\tau}_{x_i}(\nabla U)\cdot D^{\tau}_{x_i}(\nabla U)\,dx\,dy + \int_{B^+_R} 2y^\alpha\zeta\nabla\zeta\cdot D^{\tau}_{x_i}(\nabla U) D^{\tau}_{x_i}U\,dx\,dy.
\end{aligned}$$

We recall that by, e.g., [Eva98, Sec. 6.3], we have uniformly in τ

$$\|D^{\tau}_{x_i} v\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)} \lesssim \|\partial_{x_i} v\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)}. \tag{3.14}$$

Using the equation $(-\Delta)^s u = f$ on Ω, Young's inequality, and the Poincaré inequality together with the trace estimate (2.7), we get the existence of constants $C_j > 0$, $j \in \{1,\dots,5\}$, such that

$$\begin{aligned}
\big\|\zeta D^{\tau}_{x_i}(\nabla U)\big\|^2_{L^2_\alpha(B^+_R)} &\le C_1\Bigg(\bigg|\int_{B^+_R} y^\alpha\zeta\nabla\zeta\cdot D^{\tau}_{x_i}(\nabla U)D^{\tau}_{x_i}U\,dx\,dy\bigg| + \bigg|\int_{\mathbb{R}^d\times\mathbb{R}_+} F\, D^{-\tau}_{x_i}\zeta^2 D^{\tau}_{x_i}U\,dx\,dy\bigg|\\
&\qquad + \bigg|\int_{\mathbb{R}^d} D^{\tau}_{x_i} f(\zeta^2 D^{\tau}_{x_i}u)\,dx\bigg|\Bigg)\\
&\le \frac14\big\|\zeta D^{\tau}_{x_i}(\nabla U)\big\|^2_{L^2_\alpha(B^+_R)} + C_2\Big(\|\nabla\zeta\|^2_{L^\infty(B_R)}\big\|D^{\tau}_{x_i}U\big\|^2_{L^2_\alpha(B^+_R)}\\
&\qquad + \|F\|_{L^2_{-\alpha}(B^+_R)}\|\partial_{x_i}(\zeta^2 D^{\tau}_{x_i}U)\|_{L^2_\alpha(B^+_R)} + \big\|\zeta D^{\tau}_{x_i} f\big\|_{H^{-s}(\Omega)}\big\|\zeta D^{\tau}_{x_i}u\big\|_{H^s(\mathbb{R}^d)}\Big)\\
&\le \frac12\big\|\zeta D^{\tau}_{x_i}(\nabla U)\big\|^2_{L^2_\alpha(B^+_R)} + C_3\Big(\|\nabla\zeta\|^2_{L^\infty(B_R)}\|\nabla U\|^2_{L^2_\alpha(B^+_R)} + \|F\|^2_{L^2_{-\alpha}(B^+_R)}\\
&\qquad + \big\|\zeta D^{\tau}_{x_i} f\big\|_{H^{-s}(\Omega)}\big|\zeta D^{\tau}_{x_i}u\big|_{H^s(\mathbb{R}^d)}\Big)\\
&\overset{(2.7)}{\le} \frac12\big\|\zeta D^{\tau}_{x_i}(\nabla U)\big\|^2_{L^2_\alpha(B^+_R)} + C_4\Big(\|\nabla\zeta\|^2_{L^\infty(B_R)}\|\nabla U\|^2_{L^2_\alpha(B^+_R)} + \|F\|^2_{L^2_{-\alpha}(B^+_R)}\\
&\qquad + \big\|\zeta D^{\tau}_{x_i} f\big\|_{H^{-s}(\Omega)}\big\|\nabla(\zeta D^{\tau}_{x_i}U)\big\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}\Big)\\
&\le \frac34\big\|\zeta D^{\tau}_{x_i}(\nabla U)\big\|^2_{L^2_\alpha(B^+_R)}\\
&\qquad + C_5\Big(\|\nabla\zeta\|^2_{L^\infty(B_R)}\|\nabla U\|^2_{L^2_\alpha(B^+_R)} + \|F\|^2_{L^2_{-\alpha}(B^+_R)} + \big\|\zeta D^{\tau}_{x_i} f\big\|^2_{H^{-s}(\Omega)}\Big).
\end{aligned}$$

Absorbing the first term of the right-hand side in the left-hand side and taking the limit $\tau \to 0$, we obtain the sought inequality for the second derivatives since $\|\nabla\zeta\|_{L^\infty(B_R)} \lesssim ((1-c)R)^{-1}$. ∎

Remark that the constant C_{int} of (3.13) depends on s, due to the usage of (2.7) in the proof above.

The Caccioppoli inequality in Lemma 3.5 can be iterated on concentric balls to provide control of higher order derivatives by lower order derivatives locally, in the interior of the domain.

Corollary 3.6 (High order interior Caccioppoli inequality). *Let $B_R := B_R(x_0) \subset \Omega \subset \mathbb{R}^d$ be an open ball of radius $R > 0$ centered at $x_0 \in \Omega$, and let B_{cR} be the concentric scaled ball of radius cR with $c \in (0,1)$. Let U satisfy (3.5a), (3.5b) on B^+_R with given data f and F (cf. Remark 3.2(iii)).*

Then, there is $\gamma > 0$ (depending only on s, Ω, and c) such that for all $\beta \in \mathbb{N}_0^d$ with $|\beta| = p$, we have

$$\left\|\partial_x^\beta \nabla U\right\|^2_{L^2_\alpha(B^+_{cR})} \leq (\gamma p)^{2p} R^{-2p} \left\|\nabla U\right\|^2_{L^2_\alpha(B^+_R)} + \sum_{j=1}^{p} (\gamma p)^{2(p-j)} R^{2(j-p)} \left(\max_{|\eta|=j} \|\partial_x^\eta f\|^2_{L^2(B_R)} + \max_{|\eta|=j-1} \|\partial_x^\eta F\|^2_{L^2_{-\alpha}(B^+_R)} \right). \tag{3.15}$$

Proof. We start by noting that the case $p = 0$ is trivially true since empty sums are zero and $0^0 = 1$. For $p \geq 1$, we fix a multi index β such that $|\beta| = p$. As the x-derivatives commute with the differential operator in (3.5), we have that $\partial_x^\beta U$ solves equation (3.5) with data $\partial_x^\beta F$ and $\partial_x^\beta f$. For given $c > 0$, let

$$c_i = c + (i-1)\frac{1-c}{p}, \qquad i = 1, \ldots, p+1.$$

Then, we have $c_{i+1}R - c_iR = \frac{(1-c)R}{p}$ and $c_1R = cR$ as well as $c_{p+1}R = R$. For ease of notation and without loss of generality, we assume that $\beta_1 > 0$. Applying Lemma 3.5 iteratively on the sets $B^+_{c_iR}$ for $i > 1$ provides

$$\begin{aligned}
&\left\|\partial_x^\beta \nabla U\right\|^2_{L^2_\alpha(B^+_{cR})} \\
&\leq C^2_{\rm int} \left(\frac{p^2}{(1-c)^2} R^{-2} \left\|\partial_x^{(\beta_1-1,\beta_2)} \nabla U\right\|^2_{L^2_\alpha(B^+_{c_2R})} + C^2_{\rm loc} \left\|\partial_x^\beta f\right\|^2_{L^2(B_{c_2R})} + \left\|\partial_x^{(\beta_1-1,\beta_2)} F\right\|^2_{L^2_{-\alpha}(B^+_{c_2R})} \right) \\
&\leq \left(\frac{C_{\rm int} p}{(1-c)}\right)^{2p} R^{-2p} \left\|\nabla U\right\|^2_{L^2_\alpha(B^+_R)} + C^2_{\rm loc} \sum_{j=1}^{p} \left(\frac{C_{\rm int} p}{(1-c)}\right)^{2p-2j} R^{-2p+2j} \max_{|\eta|=j} \|\partial_x^\eta f\|^2_{L^2(B_{c_{p-j+2}R})} \\
&\quad + \sum_{j=0}^{p-1} \left(\frac{C_{\rm int} p}{(1-c)}\right)^{2p-2j-2} R^{-2p+2j+2} \max_{|\eta|=j} \|\partial_x^\eta F\|^2_{L^2_{-\alpha}(B^+_{c_{p-j+1}R})}.
\end{aligned}$$

Choosing $\gamma = \max(C^2_{\rm loc}, 1) C_{\rm int}/(1-c)$ concludes the proof. ∎

4. Local tangential regularity for the extension problem in 2d. Lemma 3.3 provides global regularity for the solution U of (3.5). In this section, we derive a localized version of Lemma 3.3 for tangential derivatives of U, where we solely consider the case $d = 2$.

Lemma 3.5 is formulated as an interior regularity estimate as the balls are assumed to satisfy $B_R(x_0) \subset \Omega$. Since $u = 0$ on Ω^c (i.e., u satisfies "homogeneous boundary conditions"), one obtains estimates near $\partial\Omega$ for derivatives in the direction of an edge.

LEMMA 4.1 (Boundary Caccioppoli inequality). *Let $\mathbf{e} \subset \partial\Omega$ be an edge of the polygon Ω. Let $B_R := B_R(x_0)$ be an open ball with radius $R > 0$ and center $x_0 \in \mathbf{e}$ such that $B_R(x_0) \cap \Omega$ is a half-ball, and let B_{cR} be the concentric scaled ball of radius cR with $c \in (0,1)$. Let $\zeta \in C_0^\infty(B_R)$ be a cut-off function with $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on B_{cR} as well as $\|\nabla\zeta\|_{L^\infty(B_R)} \leq C_\zeta((1-c)R)^{-1}$ for some $C_\zeta > 0$ independent of c, R. Let U satisfy (3.5) on B^+_R with given data f and F (cf. Remark 3.2(iii)).*

Then, there exists a constant $C > 0$ (independent of R, c, and the data F, f) such that

$$\left\|D_{x_\parallel} \nabla U\right\|^2_{L^2_\alpha(B^+_{cR})} \leq C \left(((1-c)R)^{-2} \left\|\nabla U\right\|^2_{L^2_\alpha(B^+_R)} + \left\|\zeta D_{x_\parallel} f\right\|^2_{H^{-s}(\Omega)} + \|F\|^2_{L^2_{-\alpha}(B^+_R)} \right). \tag{4.1}$$

In particular, $\|\zeta D_{x_\parallel} f\|_{H^{-s}(\Omega)} \leq C_{\rm loc} \|D_{x_\parallel} f\|_{L^2(B_R\cap\Omega)}$ for some $C_{\rm loc} > 0$ independent of R (cf. Lemma A.1).

Proof. The proof is almost verbatim the same as that of Lemma 3.5. The key observation is that $V = D^{-\tau}_{x_\parallel}(\zeta^2 D^\tau_{x_\parallel} U)$ with the difference quotient

$$D^\tau_{x_\parallel} w(x) := \frac{w(x + \tau\mathbf{e}_\parallel) - w(x)}{\tau}$$

is an admissible test function. ∎

Iterating the boundary Caccioppoli equation provides an estimate for higher order tangential derivatives.

COROLLARY 4.2 (High order boundary Caccioppoli inequality). *Let $\mathbf{e} \subset \partial\Omega$ be an edge of Ω. Let $B_R := B_R(x_0)$ be an open ball with radius $R > 0$ and center $x_0 \in \mathbf{e}$ such that $B_R(x_0) \cap \Omega$ is a half-ball, and let B_{cR} be the concentric scaled ball of radius cR with $c \in (0, 1)$. Let U satisfy (3.5) on B_R^+ with given data f and F (cf. Remark 3.2(iii)).*

Let $p \in \mathbb{N}_0$. Then, there is $\gamma > 0$ independent of p, R, and the data f, F such that

$$\|D^p_{x_\parallel} \nabla U\|^2_{L^2_\alpha(B^+_{cR})} \leq (\gamma p)^{2p} R^{-2p} \|\nabla U\|^2_{L^2_\alpha(B^+_R)} + \sum_{j=1}^{p} (\gamma p)^{2(p-j)} R^{2(j-p)} \left(\|D^j_{x_\parallel} f\|^2_{L^2(B_R)} + \|D^{j-1}_{x_\parallel} F\|^2_{L^2_{-\alpha}(B^+_R)} \right). \tag{4.2}$$

Proof. The statement follows from Lemma 4.1 in the same way as Corollary 3.6 follows from Lemma 3.5. ∎

The term $\|\nabla U\|_{L^2_\alpha(B^+_R)}$ in (4.2) is actually small for $R \to 0$ in the presence of regularity of U, which was asserted in Lemma 3.3; this is quantified in the following lemma.

LEMMA 4.3. *Let $S_R := \{x \in \Omega \,:\, r_{\partial\Omega}(x) < R\}$ be the tubular neighborhood of $\partial\Omega$ of width $R > 0$. Then, for $t \in [0, 1/2)$, there exists $C_{\rm reg} > 0$ depending only on t and Ω such that the solution U of (3.1) satisfies*

$$R^{-2t} \|\nabla U\|^2_{L^2_\alpha(S^+_R)} \leq \|r^{-t}_{\partial\Omega} \nabla U\|^2_{L^2_\alpha(\Omega^+)} \leq C_{\rm reg} C_t N^2(U, F, f) \tag{4.3}$$

with the constant $C_t > 0$ from Lemma 3.3 and $N^2(U, F, f)$ given by (3.7).

Proof. The first estimate in (4.3) is trivial. For the second bound, we start by noting that the shift result Lemma 3.3 gives the global regularity

$$\int_{\mathbb{R}_+} y^\alpha \, \|\nabla U(\cdot, y)\|^2_{H^t(\Omega)} \, dy \leq C_t N^2(U, F, f). \tag{4.4}$$

For $t \in [0, 1/2)$ and any $v \in H^t(\Omega)$, we have by, e.g., [Gri11, Thm. 1.4.4.3] the embedding result $\|r^{-t}_{\partial\Omega} v\|_{L^2(\Omega)} \leq C_{\rm reg} \|v\|_{H^t(\Omega)}$. Applying this embedding to $\nabla U(\cdot, y)$, multiplying by y^α, and integrating in y yields (4.3). ∎

The following lemma provides a shift theorem for localizations of tangential derivatives of U.

LEMMA 4.4 (High order localized shift theorem). *Let U be the solution of (3.1). Let $x_0 \in \mathbf{e}$ for an edge $\mathbf{e} \in \mathcal{E}$ of the polygon Ω. Let $R \in (0, 1/2]$, and assume that $B_R(x_0) \cap \Omega$ is a half-ball. Let $\eta_x \in C^\infty_0(B_R(x_0))$ with $\|\nabla^j \eta_x\|_{L^\infty(B_R(x_0))} \leq C_\eta R^{-j}$, $j \in \{0, 1, 2\}$, with a constant $C_\eta > 0$ independent of R. Let $\eta_y \in C^\infty_0((-H, H))$ with $\eta_y \equiv 1$ in $(-H/2, H/2)$ and $\|\partial^j_y \eta_y\|_{L^\infty(-H,H)} \leq C_\eta H^{-j}$, with a constant $C_\eta > 0$ independent of H. Let $\eta(x, y) := \eta_x(x)\eta_y(y)$. Then, for $t \in [0, 1/2)$, there is $C > 0$ independent of R and x_0 such that, for each $p \in \mathbb{N}$, the function $\widetilde{U}^{(p)} := \eta D^p_{x_\parallel} U$ satisfies*

$$\int_{\mathbb{R}_+} y^\alpha \left\| \nabla \widetilde{U}^{(p)}(\cdot, y) \right\|^2_{H^t(\Omega)} dy \leq C R^{-2p-1+2t} (\gamma p)^{2p} (1 + \gamma p) \widetilde{N}^{(p)}(F, f), \tag{4.5}$$

where γ is the constant in Corollary 4.2 and

$$\widetilde{N}^{(p)}(F, f) := \|f\|^2_{H^1(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^2 \times (0,H))} + \sum_{j=2}^{p+1} (\gamma p)^{-2j} \left(2^j \max_{|\beta|=j} \|\partial^\beta_x f\|^2_{L^2(\Omega)} + 2^{j-1} \max_{|\beta|=j-1} \|\partial^\beta_x F\|^2_{L^2_{-\alpha}(\mathbb{R}^2 \times (0,H))} \right). \tag{4.6}$$

In addition,

$$\int_{\mathbb{R}_+} y^\alpha \|r^{-t}_{\partial\Omega} \nabla \widetilde{U}^{(p)}(\cdot, y)\|^2_{L^2(\Omega)} \, dy \leq C R^{-2p-1+2t} (\gamma p)^{2p} (1 + \gamma p) \widetilde{N}^{(p)}(F, f). \tag{4.7}$$

Proof. We abbreviate $U^{(p)}_{x_\parallel} := D^p_{x_\parallel} U$, $\widetilde{U}^{(p)}(x, y) := \eta(x) D^p_{x_\parallel} U(x, y)$, $F^{(p)}_{x_\parallel} = D^p_{x_\parallel} F$, and $f^{(p)}_{x_\parallel} = D^p_{x_\parallel} f$. Throughout the proof we will use the fact that, for all $j \in \mathbb{N}$ and all sufficiently smooth functions v, we have

$$|D^j_{x_\parallel} v| \leq 2^{j/2} \max_{|\beta|=j} |\partial^\beta_x v|.$$

We also note that the assumptions on $\eta(x,y) = \eta_x(x)\eta_y(y)$ imply the existence of $\tilde{C}_\eta > 0$ (which absorbes the dependence on H that we do not further track) such that

$$\|\nabla_x^j \partial_y^{j'} \eta\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \leq \tilde{C}_\eta R^{-j}, \qquad j \in \{0,1,2\}, j' \in \{0,1,2\}. \tag{4.8}$$

Step 1. (Localization of the equation). Using that U solves the extension problem (3.5), we obtain that the function $\widetilde{U}^{(p)} = \eta U_{x_\parallel}^{(p)}$ satisfies the equation

$$\begin{aligned}
\operatorname{div}(y^\alpha \nabla \widetilde{U}^{(p)}) &= y^\alpha \operatorname{div}_x(\nabla_x \widetilde{U}^{(p)}) + \partial_y(y^\alpha \partial_y \widetilde{U}^{(p)}) \\
&= y^\alpha \left((\Delta_x \eta) U_{x_\parallel}^{(p)} + 2\nabla_x \eta \cdot \nabla_x U_{x_\parallel}^{(p)} + \eta \Delta_x U_{x_\parallel}^{(p)} \right) + \eta \partial_y(y^\alpha \partial_y U_{x_\parallel}^{(p)}) + \partial_y(y^\alpha U_{x_\parallel}^{(p)} \partial_y \eta) + y^\alpha \partial_y U_{x_\parallel}^{(p)} \partial_y \eta \\
&= y^\alpha \left((\Delta_x \eta) U_{x_\parallel}^{(p)} + 2\nabla_x \eta \cdot \nabla_x U_{x_\parallel}^{(p)} \right) + \partial_y(y^\alpha U_{x_\parallel}^{(p)} \partial_y \eta) + y^\alpha \partial_y U_{x_\parallel}^{(p)} \partial_y \eta + \eta \operatorname{div}(y^\alpha \nabla U_{x_\parallel}^{(p)}) \\
&= y^\alpha \left((\Delta_x \eta) U_{x_\parallel}^{(p)} + 2\nabla_x \eta \cdot \nabla_x U_{x_\parallel}^{(p)} \right) + \partial_y(y^\alpha U_{x_\parallel}^{(p)} \partial_y \eta) + y^\alpha \partial_y U_{x_\parallel}^{(p)} \partial_y \eta + \eta F_{x_\parallel}^{(p)} =: \widetilde{F}^{(p)}
\end{aligned}$$

as well as the boundary conditions

$$\begin{aligned}
\partial_{n_\alpha} \widetilde{U}^{(p)}(\cdot,0) &= \eta(\cdot,0) D_{x_\parallel}^p f =: \widetilde{f}^{(p)} && \text{on } \Omega, \\
\operatorname{tr} \widetilde{U}^{(p)} &= 0 && \text{on } \Omega^c.
\end{aligned}$$

By the support properties of the cut-off function η, we have $\operatorname{supp} \widetilde{F}^{(p)} \subset \overline{B_R}(x_0) \times [0,H] \subset \mathbb{R}^2 \times [0,H]$. By Lemma 3.3, for all $t \in [0,1/2)$, there is a $C_t > 0$ such that

$$\int_{\mathbb{R}_+} y^\alpha \|\nabla \widetilde{U}^{(p)}(\cdot,y)\|_{H^t(B_{\widetilde{R}})}^2 dy \leq C_t N^2(\widetilde{U}^{(p)}, \widetilde{F}^{(p)}, \widetilde{f}^{(p)}), \tag{4.9}$$

where $B_{\widetilde{R}}$ is a ball containing Ω. By (3.7), we have to estimate $N^2(\widetilde{U}^{(p)}, \widetilde{F}^{(p)}, \widetilde{f}^{(p)})$, i.e., $\|\nabla \widetilde{U}^{(p)}\|_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)}$, $\|\widetilde{F}^{(p)}\|_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))}$, and $\|\widetilde{f}^{(p)}\|_{H^{1-s}(\Omega)}$. Let γ be the constant introduced in Corollary 4.2. We note that by (3.8) there exists $C_N > 0$ such that, for all $p \in \mathbb{N}_0$,

$$N^2(U,F,f) \leq C_N \widetilde{N}^{(p)}(F,f). \tag{4.10}$$

Step 2. (Estimate of $\|\nabla \widetilde{U}^{(p)}\|_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)}$). We write

$$\begin{aligned}
\|\nabla \widetilde{U}^{(p)}\|_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)}^2 &\leq 2\|\nabla\eta\|_{L^\infty}^2 \|\nabla_x U_{x_\parallel}^{(p-1)}\|_{L^2_\alpha(R_R^+)}^2 + 2\|\eta\|_{L^\infty}^2 \|\nabla U_{x_\parallel}^{(p)}\|_{L^2_\alpha(B_R^+)}^2 \\
&\leq 2\tilde{C}_\eta^2 \left(R^{-2} \|\nabla U_{x_\parallel}^{(p-1)}\|_{L^2_\alpha(B_R^+)}^2 + \|\nabla U_{x_\parallel}^{(p)}\|_{L^2_\alpha(B_R^+)}^2 \right).
\end{aligned} \tag{4.11}$$

We employ Corollary 4.2 with a ball B_{2R} and $c = 1/2$ as well as Lemma 4.3 to obtain for $p \in \mathbb{N}_0$

$$\begin{aligned}
\|\nabla U_{x_\parallel}^{(p)}\|_{L^2_\alpha(B_R^+)}^2 &\leq (2R)^{-2p}(\gamma p)^{2p} \left(\|\nabla U\|_{L^2_\alpha(B_{2R}^+)}^2 + \sum_{j=1}^p (2R)^{2j}(\gamma p)^{-2j} \left(\|D_{x_\parallel}^j f\|_{L^2(B_{2R})}^2 + \|D_{x_\parallel}^{j-1} F\|_{L^2_{-\alpha}(B_{2R}^+)}^2 \right) \right) \\
&\leq (2R)^{-2p}(\gamma p)^{2p} \Bigg(\|\nabla U\|_{L^2_\alpha(B_{2R}^+)}^2 \\
&\qquad + (2R)^2 \sum_{j=1}^p (2R)^{2(j-1)}(\gamma p)^{-2j} \left(2^j \max_{|\beta|=j} \|\partial_x^\beta f\|_{L^2(B_{2R})}^2 + 2^{j-1} \max_{|\beta|=j-1} \|\partial_x^\beta F\|_{L^2_{-\alpha}(B_{2R}^+)}^2 \right) \Bigg) \\
&\overset{R\leq 1/2, \text{L.}4.3}{\leq} (2R)^{-2p}(\gamma p)^{2p} \Big(\left(C_{\text{reg}} C_t R^{2t} + (2R)^2 2\gamma^{-2} \right) N^2(U,F,f) + (2R)^2 \widetilde{N}^{(p)}(F,f) \Big) \\
&\overset{t<1/2, (4.10)}{\leq} (2R)^{-2p}(\gamma p)^{2p} \underbrace{\left(C_{\text{reg}} C_t (1 + 8\gamma^{-2}) C_N + 4 \right)}_{=:C_{\text{reg,N}}} R^{2t} \widetilde{N}^{(p)}(F,f).
\end{aligned} \tag{4.12}$$

For $p \in \mathbb{N}$, we apply (4.12) to the $(p-1)^{th}$ derivative and exploit the structure of the expression $(\gamma(p-1))^{2p-2} \widetilde{N}^{(p-1)}(F,f)$ to get

$$\begin{aligned}
\|\nabla U_{x_\parallel}^{(p-1)}\|_{L^2_\alpha(B_R^+)}^2 &\leq (2R)^{-2(p-1)} C_{\text{reg,N}} (\gamma(p-1))^{2(p-1)} \widetilde{N}^{(p-1)}(F,f) \\
&\leq (2R)^{-2(p-1)} C_{\text{reg,N}} R^{2t} (\gamma p)^{2p} \widetilde{N}^{(p)}(F,f).
\end{aligned} \tag{4.13}$$

Inserting (4.12) and (4.13) into (4.11) provides the estimate

$$\|\nabla\widetilde{U}^{(p)}\|^2_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)} \le CR^{-2p+2t}(\gamma p)^{2p}\widetilde{N}^{(p)}(F,f)$$

with a constant $C>0$ depending only on the constants $C_{\rm reg}$, C_t, $\tilde{C}_\eta$, and C_N.

Step 3. (Estimate of $\|\widetilde{F}^{(p)}\|_{L^2_{-\alpha}(\mathbb{R}^2\times\mathbb{R}_+)}$). We treat the five terms appearing in $\|\widetilde{F}^{(p)}\|_{L^2_{-\alpha}(\mathbb{R}^2\times\mathbb{R}_+)}$ separately. With (4.12), we obtain

$$\begin{aligned}\left\|y^\alpha\nabla_x\eta\cdot\nabla_x U^{(p)}_{x_\parallel}\right\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} &= \left\|\nabla_x\eta\cdot\nabla_x U^{(p)}_{x_\parallel}\right\|^2_{L^2_{\alpha}(\mathbb{R}^2\times\mathbb{R}_+)} \le C_\eta^2\frac{1}{R^2}\left\|\nabla_x U^{(p)}_{x_\parallel}\right\|^2_{L^2_\alpha(B_R^+)}\\ &\overset{(4.12)}{\le} (2R)^{-2p}(\gamma p)^{2p}C_\eta^2 C_{\rm reg,N}R^{-2+2t}\widetilde{N}^{(p)}(F,f).\end{aligned}$$

Similarly, we get

$$\begin{aligned}\left\|y^\alpha(\Delta_x\eta)U^{(p)}_{x_\parallel}\right\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} &= \left\|(\Delta_x\eta)U^{(p)}_{x_\parallel}\right\|^2_{L^2_\alpha(B_R^+)} \le C_\eta^2\frac{1}{R^4}\left\|\nabla U^{(p-1)}_{x_\parallel}\right\|^2_{L^2_\alpha(B_R^+)}\\ &\overset{(4.13)}{\le} 4(2R)^{-2p}(\gamma p)^{2p}C_\eta^2 C_{\rm reg,N}R^{-2+2t}\widetilde{N}^{(p)}(F,f).\end{aligned}$$

Next, we estimate

$$\|\eta F^{(p)}_{x_\parallel}\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} \le \|F^{(p)}_{x_\parallel}\|^2_{L^2_{-\alpha}(B_R^+)} \le 2^p\max_{|\beta|=p}\|\partial_x^\beta F\|^2_{L^2_{-\alpha}(B_R^+)} \le (\gamma p)^{2p+2}\widetilde{N}^{(p)}(F,f).$$

Finally, for the term $\partial_y(y^\alpha U^{(p)}_{x_\parallel}\partial_y\eta)+y^\alpha\partial_y U^{(p)}_{x_\parallel}\partial_y\eta$, we observe that $\partial_y\eta$ vanishes near $y=0$ so that the weight y^α does not come into play as it can be bounded from above and below by positive constants depending only on H. We arrive at

$$\begin{aligned}\left\|\partial_y(y^\alpha U^{(p)}_{x_\parallel}\partial_y\eta)+y^\alpha\partial_y U^{(p)}_{x_\parallel}\partial_y\eta\right\|_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} &\le C\left(H^{-2}\|U^{(p)}_{x_\parallel}\|_{L^2_\alpha(B_R\times(0,H))}+H^{-1}\|\nabla U^{(p)}_{x_\parallel}\|_{L^2_\alpha(B_R^+)}\right)\\ &\overset{(4.12),(4.13)}{\le} C_H(\gamma p)^{2p}R^{-2p+2t}\widetilde{N}^{(p)}(F,f),\end{aligned}$$

for suitable $C_H>0$ depending on H.

Step 4. (Estimate of $\|\widetilde{f}^{(p)}\|_{H^{1-s}(\Omega)}$.) Here, we use Lemma A.1 and $R<1/2$ together with $s<1$ to obtain

$$\begin{aligned}\|\widetilde{f}^{(p)}\|^2_{H^{1-s}(\Omega)} &\le 2C^2_{\rm loc,2}C^2_\eta\left(9R^{2s-2}\|D^p_{x_\parallel}f\|^2_{L^2(\Omega)}+|D^p_{x_\parallel}f|^2_{H^{1-s}(\Omega)}\right)\\ &\le CC^2_{\rm loc,2}C^2_\eta R^{2s-2}\left(2^p\max_{|\beta|=p}\|\partial_x^\beta f\|^2_{L^2(\Omega)}+2^{p+1}\max_{|\beta|=p+1}\|\partial_x^\beta f\|^2_{L^2(\Omega)}\right)\\ &\le CC^2_{\rm loc,2}C^2_\eta R^{2s-2}(\gamma p)^{2p}(1+(\gamma p)^2)\widetilde{N}^{(p)}(F,f)\end{aligned}$$

with a constant $C>0$ depending only on Ω and s.

Step 5. (Putting everything together.) Combining the above estimates, we obtain that there exists a constant $C>0$ depending only on $C_{\rm reg}$, C_t, $\tilde{C}_\eta$, C_N, $C_{\rm loc,2}$, and H such that

$$\begin{aligned}&N^2(\widetilde{U}^{(p)},\widetilde{F}^{(p)},\widetilde{f}^{(p)})\\ &\quad=\left(\|\nabla\widetilde{U}^{(p)}\|^2_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)}+\|\nabla\widetilde{U}^{(p)}\|_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)}\|\widetilde{F}^{(p)}\|_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))}+\|\nabla\widetilde{U}^{(p)}\|_{L^2_\alpha(\mathbb{R}^2\times\mathbb{R}_+)}\|\widetilde{f}^{(p)}\|_{H^{1-s}(\Omega)}\right)\\ &\quad\le C\left(R^{-2p+2t}(\gamma p)^{2p}+R^{-p+t}(\gamma p)^pR^{-p-1+t}(\gamma p)^p(1+\gamma p)+R^{-p+t}(\gamma p)^pR^{s-1}(\gamma p)^p(1+\gamma p)\right)\widetilde{N}^{(p)}(F,f)\\ &\quad\overset{R\le 1,t<1/2}{\le} CR^{-2p-1+2t}(\gamma p)^{2p}(1+\gamma p)\widetilde{N}^{(p)}(F,f).\end{aligned}$$

Inserting this estimate in (4.9) concludes the proof of (4.5).

Step 6: The estimate (4.7) follows from [Gri11, Thm. 1.4.4.3], which gives

$$\int_{\mathbb{R}_+}y^\alpha\|r^{-t}_{\partial\Omega}\nabla\widetilde{U}^{(p)}(\cdot,y)\|^2_{L^2(\Omega)}\,dy \le C\int_{\mathbb{R}_+}y^\alpha\|\nabla\widetilde{U}^{(p)}(\cdot,y)\|^2_{H^t(\Omega)}\,dy,$$

and from (4.5). ∎

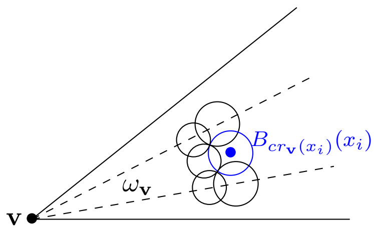

Fig. 2: Covering of "vertex cones" such as $\omega_{\mathbf{v}}$ by union of balls $B_{cr_{\mathbf{v}}(x_i)}(x_i)$ with fixed $c \in (0,1)$.

5. Weighted H^p-estimates in polygons. In this section, we derive higher order weighted regularity results, at first for the extension problem and finally for the fractional PDE. This is our main result, Theorem 2.1.

5.1. Coverings. A main ingredient in our analysis are suitable localizations of *vertex neighborhoods* $\omega_{\mathbf{v}}$ and *edge-vertex neighborhoods* $\omega_{\mathbf{ve}}$ near a vertex $\mathbf{v}$ and of *edge neighborhoods* $\omega_{\mathbf{e}}$ near an edge $\mathbf{e}$. This is achieved by covering such neighborhoods by balls or half-balls with the following two properties: a) their diameter is proportional to the distance to vertices or edges and b) scaled versions of these balls/half-balls satisfy a locally finite overlap property.

We start by recalling a lemma that follows from Besicovitch's Covering Theorem:

Lemma 5.1 ([MW12, Lemma A.1], [HMW13, Lemma A.1]). *Let $\omega \subset \mathbb{R}^d$ be bounded, open and $M \subset \partial\omega$ be closed. Fix c, $\zeta \in (0,1)$ such that $1 - c(1+\zeta) =: c_0 > 0$. For each $x \in \omega$, let $B_x := \overline{B}_{c\,\mathrm{dist}(x,M)}(x)$ be the closed ball of radius $c\,\mathrm{dist}(x,M)$ centered at x, and let $\widehat{B}_x := \overline{B}_{(1+\zeta)c\,\mathrm{dist}(x,M)}(x)$ be the stretched closed ball of radius $(1+\zeta)c\,\mathrm{dist}(x,M)$ centered at x. Then, there is a countable set $(x_i)_{i\in\mathcal{I}} \subset \omega$ (for some suitable index set $\mathcal{I} \subset \mathbb{N}$) and a number $N \in \mathbb{N}$ depending solely on d, c, ζ with the following properties:*

1. *(covering property) $\bigcup_i B_{x_i} \supset \omega$.*
2. *(finite overlap) for $x \in \mathbb{R}^d$ it holds that $\mathrm{card}\{i \,|\, x \in \widehat{B}_{x_i}\} \le N$.*

Proof. The lemma is taken from [MW12, Lemma A.1] except that there $x \in \omega$ in the condition of finite overlap is assumed. Inspection of the proof shows that this condition can be relaxed as given here. Note that the proof of [MW12, Lemma A.1] required the balls B_{x_i} to be non-degenerate, which is ensured in the present setting of $M \subset \partial\omega$. ∎

In the next lemma, we introduce a covering of $\omega_{\mathbf{v}}$, see Figure 2.

Lemma 5.2 (covering of $\omega_{\mathbf{v}}$). *Given $\mathbf{v} \in \mathcal{V}$ and $\xi > 0$, there are $0 < c < \widehat{c} < 1$ and points $(x_i)_{i\in\mathbb{N}} \subset \omega_{\mathbf{v}} = \omega_{\mathbf{v}}^{\xi}$ such that the collections $\mathcal{B} := \{B_i := B_{c\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \,|\, i \in \mathbb{N}\}$ and $\widehat{\mathcal{B}} := \{\widehat{B}_i := B_{\widehat{c}\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \,|\, i \in \mathbb{N}\}$ of (open) balls satisfy the following conditions: the balls from $\mathcal{B}$ cover $\omega_{\mathbf{v}}$; the balls from $\widehat{\mathcal{B}}$ satisfy a finite overlap property with overlap constant N depending only on the spatial dimension $d = 2$ and c, $\widehat{c}$; the balls from $\widehat{\mathcal{B}}$ are contained in Ω. Furthermore, for every $\delta > 0$ there is $C_\delta > 0$ (depending additionally on δ) such that with the radii $R_i := \widehat{c}\,\mathrm{dist}(x_i,\mathbf{v})$ it holds that*

$$\sum_i R_i^\delta \le C_\delta. \tag{5.1}$$

Proof. Apply Lemma 5.1 with $M = \{\mathbf{v}\}$ and sufficiently small parameters c, $\zeta > 0$. Note that by possibly slightly increasing the parameter c, one can ensure that the open balls rather than the closed balls given by Lemma 5.1 cover $\omega_{\mathbf{v}}$. Also, since $c < 1$, the index set $\mathcal{I}$ of Lemma 5.1 cannot be finite so that $\mathcal{I} = \mathbb{N}$.

To see (5.1), we compute with the spatial dimension $d = 2$

$$\sum_i R_i^\delta = \sum_i R_i^{\delta-d} R_i^d \lesssim \sum_i \int_{\widehat{B}_i} r_{\mathbf{v}}^{\delta-d}\,dx \overset{\text{finite overlap}}{\lesssim} \int_\Omega r_{\mathbf{v}}^{\delta-d}\,dx < \infty.$$ ∎

We now introduce a covering of edge-vertex neighborhoods $\omega_{\mathbf{ve}}$. We start by a covering of half-balls resting on the edge $\mathbf{e}$ and with size proportional to the distance from the vertex, see Figure 3 (left).

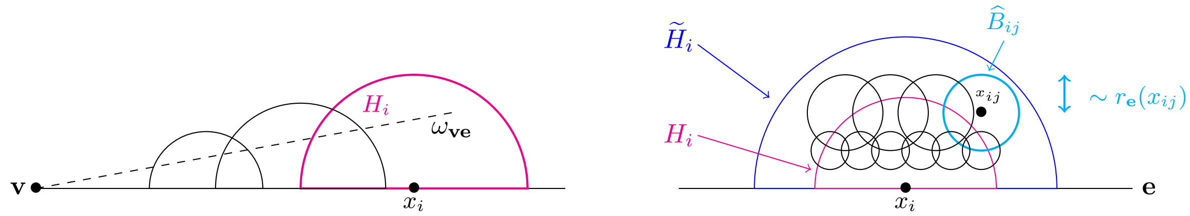

Fig. 3: Covering of $\omega_{\mathbf{ve}}$. Left: the half-balls H_i constructed in Lemma 5.3. Right: covering of H_i by balls B_{ij} such that the larger balls $\widehat{B}_{ij}$ are contained in a ball $\widetilde{H}_i$. For better illustration, only the larger balls $\widehat{B}_{ij}$ are shown, the balls B_{ij} are included therein and still provide a covering of H_i.

Lemma 5.3 (covering of $\omega_{\mathbf{ve}}$). *Given $\mathbf{v} \in \mathcal{V}$, $\mathbf{e} \in \mathcal{E}(\mathbf{v})$, there is $\xi > 0$ and parameters $0 < c < \widehat{c} < 1$ as well as points $(x_i)_{i\in\mathbb{N}} \subset \mathbf{e}$ such that the following holds:*

(i) the sets $H_i := B_{c\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \cap \Omega$ are half-balls and the collection $\mathcal{B} := \{H_i \,|\, i \in \mathbb{N}\}$ covers $\omega_{\mathbf{ve}} = \omega^{\xi}_{\mathbf{ve}}$.

(ii) The collection $\widehat{\mathcal{B}} := \{\widehat{H}_i := B_{\widehat{c}\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \cap \Omega\}$ is a collection of half-balls and satisfies a finite overlap property, i.e., there is $N > 0$ depending only on the spatial dimension $d = 2$ and the parameters c, $\widehat{c}$ such that for all $x \in \mathbb{R}^2$ it holds that $\mathrm{card}\{i \,|\, x \in \widehat{H}_i\} \le N$.

Furthermore, for every $\delta > 0$ there is $C_\delta > 0$ such that for the radii $R_i := \widehat{c}\,\mathrm{dist}(x_i, \mathbf{v})(x_i)$ it holds that $\sum_i R_i^\delta \le C_\delta$.

Proof. Let $\widetilde{\mathbf{e}}$ be the (infinite) line containing $\mathbf{e}$. We apply Lemma 5.1 to the 1D line segment $\mathbf{e} \cap B_\xi(\mathbf{v})$ (for some sufficiently small ξ) and $M := \{\mathbf{v}\}$ and the parameter c sufficiently small so that $B_{2c\,\mathrm{dist}(x,\mathbf{v})}(x) \cap \Omega$ is a half-ball for all $x \in \mathbf{e} \cap B_\xi(\mathbf{v})$. Lemma 5.1 provides a collection $(x_i)_{i\in\mathbb{N}} \subset \mathbf{e}$ such the balls $B_i := B_{c\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \subset \mathbb{R}^2$ and the stretched balls $\widehat{B}_i := B_{c(1+\zeta)\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \subset \mathbb{R}^2$ (for suitable, sufficiently small ζ) satisfy the following: the intervals $\{B_i \cap \widetilde{\mathbf{e}} \,|\, i \in \mathbb{N}\}$ cover $B_\xi(\mathbf{v}) \cap \mathbf{e}$, and the intervals $\{\widehat{B}_i \cap \widetilde{\mathbf{e}} \,|\, i \in \mathbb{N}\}$ satisfy a finite overlap condition on $\widetilde{\mathbf{e}}$. By possibly slightly increasing the parameter c (e.g., by replacing c with $c(1 + \zeta/2)$), the newly defined balls B_i then cover a set $\omega^{\xi}_{\mathbf{ve}}$ for a possibly reduced ξ. It remains to see that the balls $\widehat{B}_i$ satisfy a finite overlap condition on $\mathbb{R}^2$: given $x \in \widehat{B}_i$, its projection $x_{\mathbf{e}}$ onto $\widetilde{\mathbf{e}}$ satisfies $x_{\mathbf{e}} \in \widehat{B}_i \cap \widetilde{\mathbf{e}}$ since $x_i \in \mathbf{e} \subset \widetilde{\mathbf{e}}$. This implies that the overlap constants of the balls $\widehat{B}_i$ in $\mathbb{R}^2$ is the same as the overlap constant of the intervals $\widehat{B}_i \cap \widetilde{\mathbf{e}}$ in $\widetilde{\mathbf{e}}$. The half-balls $H_i := B_i \cap \Omega$ and $\widehat{H}_i := \widehat{B}_i \cap \Omega$ have the stated properties.

Finally, the convergence of the sum $\sum_i R_i^\delta$ is shown by the same arguments as in Lemma 5.2. ∎

We will also need a covering of the half-balls H_i constructed in Lemma 5.3, which we introduce in the next lemma. See also Figure 3 (right).

Lemma 5.4. *Let $\mathcal{B} = \{H_i \,|\, i \in \mathbb{N}\}$ and $\widehat{\mathcal{B}} = \{\widehat{H}_i \,|\, i \in \mathbb{N}\}$ be constructed in Lemma 5.3. Fix a $\widetilde{c} \in (c, \widehat{c})$ with c, $\widehat{c}$ from Lemma 5.3 and define the collection $\widetilde{\mathcal{B}} := \{\widetilde{H}_i := B_{\widetilde{c} r_{\mathbf{v}}(x_i)}(x_i) \cap \Omega \,|\, i \in \mathbb{N}\}$ of half-balls intermediate to the half-balls H_i and $\widehat{H}_i$.*

There are constants $0 < c_1 < \widehat{c}_1 < 1$ such that the following holds: for each i, there are points $(x_{ij})_{j\in\mathbb{N}} \subset H_i$ such that the collection $\mathcal{B}_i := \{B_{ij} := B_{c_1 r_{\mathbf{e}}(x_{ij})}(x_{ij})\}$ covers H_i and the collection $\widehat{\mathcal{B}}_i := \{\widehat{B}_{ij} := B_{\widehat{c}_1 r_{\mathbf{e}}(x_{ij})}(x_{ij})\}$ satisfies $\widehat{B}_{ij} \subset \widetilde{H}_i$ for all j as well as a finite overlap property, i.e., there is $N > 0$ independent of i such that for all $x \in \mathbb{R}^2$ it holds that $\mathrm{card}\{j \,|\, x \in \widehat{B}_{ij}\} \le N$.

Proof. We apply Lemma 5.1 with $M = \{\mathbf{e}\}$ and $\omega = H_i$. The parameters c and ζ are chosen small enough so that the balls B_x in Lemma 5.1 satisfy $\widehat{B}_x \subset \widetilde{H}_i$. Then, the lemma follows from Lemma 5.1. ∎

5.2. Weighted H^p-regularity for the extension problem. To illustrate the techniques, we start with the simplest case of estimates in vertex neighborhoods $\omega_{\mathbf{v}}$. It is worth stressing that we have

$$r_{\mathbf{e}} \sim r_{\mathbf{v}} \qquad \text{on } \omega_{\mathbf{v}}.$$

The following lemma provides higher order regularity estimates in a vertex weighted norm for solutions to the Caffarelli-Silvestre extension problem with smooth data.

Lemma 5.5 (Weighted H^p-regularity in $\omega_{\mathbf{v}}$). *Let $\omega_{\mathbf{v}} = \omega^{\xi}_{\mathbf{v}}$ be given for some $\xi > 0$ and $\mathbf{v} \in \mathcal{V}$. Let U be the solution of (3.1). There is $\gamma > 0$ depending only on s, Ω, and $\omega_{\mathbf{v}}$ and for every $\varepsilon \in (0, 1)$, there* *exists $C_\varepsilon > 0$ depending on ε, Ω, H such that, for all $\beta \in \mathbb{N}_0^2$ with $|\beta| = p \in \mathbb{N}_0$,*

$$\|r_{\mathbf{v}}^{p-1/2+\varepsilon}\partial_x^\beta \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{v}}^+)} \le C_\varepsilon \gamma^{2p+1}p^{2p}\Bigg(\|f\|^2_{H^1(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} + \sum_{j=2}^{p+1} p^{-2j}\left(\max_{|\eta|=j}\|\partial_x^\eta f\|^2_{L^2(\Omega)} + \max_{|\eta|=j-1}\|\partial_x^\eta F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))}\right)\Bigg).$$

Proof. The case $p=0$ follows from Lemma 4.3 and the estimates (3.7), (3.8). We therefore assume $p \in \mathbb{N}$.

Let the covering $\omega_{\mathbf{v}} \subset \bigcup_i B_i$ with $B_i = B_{c\,\mathrm{dist}(x_i,\mathbf{v})}(x_i)$ and stretched balls $\widehat{B}_i = B_{\widehat{c}\,\mathrm{dist}(x_i,\mathbf{v})}(x_i)$ be given by Lemma 5.2. It will be convenient to denote $R_i := \widehat{c}\,\mathrm{dist}(x_i,\mathbf{v})$ the radius of the ball $\widehat{B}_i$ and to note that, for some $C_B > 0$,

$$\forall i \in \mathbb{N} \quad \forall x \in \widehat{B}_i \qquad C_B^{-1}R_i \le r_{\mathbf{v}}(x) \le C_B R_i. \tag{5.2}$$

We assume (for convenience) that $R_i \le 1/2$ for all i.

Let β be a multi index such that $|\beta| = p$. By (4.10) there is $C_N > 0$ such that $N^2(U,F,f) \le C_N \widetilde{N}^{(p)}(F,f)$ for all $p \in \mathbb{N}_0$, where $\widetilde{N}^{(p)}$ is defined in (4.6). We employ Corollary 3.6 to the pair $(B_i, \widehat{B}_i)$ of concentric balls together with Lemma 4.3 for $t = 1/2 - \varepsilon/2$ and $N^2(U,F,f) \le C_N\widetilde{N}^{(p)}(F,f)$ to obtain, for suitable $\gamma > 0$,

$$\left\|\partial_x^\beta \nabla U\right\|^2_{L^2_\alpha(B_i^+)} \le \gamma^{2p+1}R_i^{-2p+1-\varepsilon}p^{2p}\widetilde{N}^{(p)}(F,f).$$

Summation over i (with very generous bounds for the data f, F) and (5.2) provides

$$\begin{aligned}
\|r_{\mathbf{v}}^{p-1/2+\varepsilon}\partial_x^\beta \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{v}}^+)} &\le C_B^{2p-1+2\varepsilon}\sum_i R_i^{2p-1+2\varepsilon}\|\partial_x^\beta\nabla U\|^2_{L^2_\alpha(B_i^+)} \\
&\le \gamma^{2p+1}C_B^{2p+1}p^{2p}\left(\sum_i R_i^\varepsilon\right)\widetilde{N}^{(p)}(F,f) \\
&\le C_\varepsilon(\gamma C_B)^{2p+1}p^{2p}\Bigg\{\|f\|^2_{H^1(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} \\
&\quad + \sum_{j=2}^{p+1}p^{-2j}\left(\max_{|\eta|=j}\|\partial_x^\eta f\|^2_{L^2(\Omega)} + \max_{|\eta|=j-1}\|\partial_x^\eta F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))}\right)\Bigg\},
\end{aligned}$$

since $\sum_i R_i^\varepsilon =: C_\varepsilon < \infty$ by Lemma 5.2. Relabelling γC_B as γ gives the result. ∎

We continue with the more involved case of edge-vertex neighborhoods.

LEMMA 5.6 (Weighted H^p-regularity in $\omega_{\mathbf{ve}}$). *Let $\xi > 0$ be sufficiently small. There exists $\gamma > 0$ depending only on s, ξ, and Ω and for any $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 0$ depending additionally on ε, and H such that the solution U of (3.1) satisfies, for all $p_\parallel$, $p_\perp \in \mathbb{N}_0$ with $p = p_\parallel + p_\perp$*

$$\begin{aligned}
&\left\|r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon/2}r_{\mathbf{v}}^{p_\parallel+\varepsilon}D_{x_\perp}^{p_\perp}D_{x_\parallel}^{p_\parallel}\nabla U\right\|^2_{L^2_\alpha((\omega^\xi_{\mathbf{ve}})^+)} \\
&\le C_\varepsilon\gamma^{2p+1}p^{2p}\left[\|f\|^2_{H^1(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} + \sum_{j=2}^{p+1}p^{-2j}\left(\max_{|\eta|=j}\|\partial_x^\eta f\|^2_{L^2(\Omega)} + \max_{|\eta|=j-1}\|\partial_x^\eta F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))}\right)\right].
\end{aligned}$$

Proof. As in the proof of Lemma 5.5, the case $p=0$ follows from Lemma 4.3 and the estimates (3.7), (3.8) so that we may assume $p \in \mathbb{N}$. By Lemma 5.4, for sufficiently small ξ, there is a covering of $\omega^\xi_{\mathbf{ve}}$ by half-balls $(H_i)_{i\in\mathbb{N}}$ with corresponding stretched half-balls $(\widehat{H}_i)_{i\in\mathbb{N}}$ and intermediate half-balls $(\widetilde{H}_i)_{i\in\mathbb{N}}$ such that each H_i is covered by balls $\mathcal{B}_i := \{B_{ij} \,|\, j \in \mathbb{N}\}$ with the stretched balls $\widehat{B}_{ij}$ satisfying a finite overlap condition and being contained in $\widetilde{H}_i$. We abbreviate the radii of the half-balls $\widehat{H}_i$ and the balls $\widehat{B}_{ij}$ by R_i and R_{ij} respectively. We note that the half-balls $\widehat{H}_i$ and the balls $\widehat{B}_{ij}$ satisfy for all i, j:

$$\forall x \in \widehat{H}_i: \qquad C_B^{-1}R_i \le r_{\mathbf{v}}(x) \le C_B R_i, \tag{5.3}$$

$$\forall x \in \widehat{B}_{ij}: \qquad C_B^{-1}R_{ij} \le r_{\mathbf{e}}(x) \le C_B R_{ij} \tag{5.4}$$

for some $C_B > 0$ depending only on $\omega^\xi_{\mathbf{ve}}$. For convenience, we assume that $R_i \leq 1/2$ for all i and that hence $R_{ij} \leq 1/2$ for all i, j.

Let $p_\|$, $p_\perp \in \mathbb{N}_0$. Since the balls $(B_{ij})_{i,j\in\mathbb{N}}$ cover $\omega^\xi_{\mathbf{ve}}$, we estimate using (5.3), (5.4)

$$\begin{aligned}
&\left\| r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon/2} r_{\mathbf{v}}^{p_\|+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\|}_{x_\|} \nabla U \right\|^2_{L^2_\alpha((\omega^\xi_{\mathbf{ve}})^+)} \\
&\qquad \leq C_B^{2p_\perp-1+\varepsilon+2p_\|+2\varepsilon} \sum_{i,j} R_i^{2p_\|+2\varepsilon} R_{ij}^{2p_\perp-1+\varepsilon} \left\| D^{p_\perp}_{x_\perp} D^{p_\|}_{x_\|} \nabla U \right\|^2_{L^2_\alpha(B^+_{ij})}. \qquad (5.5)
\end{aligned}$$

With the constant $\gamma > 0$ from Corollary 3.6, we abbreviate

$$\widehat{N}^{(p_\perp)}_{i,j}(F,f) := \sum_{n=1}^{p_\perp} (\gamma p_\perp)^{-2n} \left(\max_{|\eta|=n} \left\| \partial^\eta_x D^{p_\|}_{x_\|} f \right\|^2_{L^2(\widehat{B}_{ij})} + \max_{|\eta|=n-1} \left\| \partial^\eta_x D^{p_\|}_{x_\|} F \right\|^2_{L^2_{-\alpha}(\widehat{B}_{ij}\times(0,H))} \right),$$

$$\widehat{N}^{(p_\perp)}_{i}(F,f) := \sum_{n=1}^{p_\perp} (\gamma p_\perp)^{-2n} \left(\max_{|\eta|=n} \left\| \partial^\eta_x D^{p_\|}_{x_\|} f \right\|^2_{L^2(\widehat{H}_{i})} + \max_{|\eta|=n-1} \left\| \partial^\eta_x D^{p_\|}_{x_\|} F \right\|^2_{L^2_{-\alpha}(\widehat{H}_{i}\times(0,H))} \right).$$

Applying the interior Caccioppoli-type estimate (Corollary 3.6) for the pairs of concentric balls $(B_{ij}, \widehat{B}_{ij})$ (which are fully contained in Ω) and the function $D^{p_\|}_{x_\|} U$ (noting that this function satisfies (3.5) with data $D^{p_\|}_{x_\|} f$, $D^{p_\|}_{x_\|} F$) provides (we also use $R_i \leq 1/2 \leq 1$)

$$\begin{aligned}
(5.6) \qquad \left\| D^{p_\perp}_{x_\perp} \nabla D^{p_\|}_{x_\|} U \right\|^2_{L^2_\alpha(B^+_{ij})} &\leq 2^{p_\perp} \max_{|\beta|=p_\perp} \left\| \partial^\beta_x \nabla D^{p_\|}_{x_\|} U \right\|^2_{L^2_\alpha(B^+_{ij})} \\
&\leq (\sqrt{2}\gamma p_\perp)^{2p_\perp} R_{ij}^{-2p_\perp} \left(\left\| \nabla D^{p_\|}_{x_\|} U \right\|^2_{L^2_\alpha(\widehat{B}^+_{ij})} + R^2_{ij} \widehat{N}^{(p_\perp)}_{i,j}(F,f) \right) \\
&\overset{(5.4)}{\leq} C_B^{1-\varepsilon} (\sqrt{2}\gamma p_\perp)^{2p_\perp} R_{ij}^{-2p_\perp+1-\varepsilon} \left(\left\| r_{\mathbf{e}}^{-1/2+\varepsilon/2} \nabla D^{p_\|}_{x_\|} U \right\|^2_{L^2_\alpha(\widehat{B}^+_{ij})} + R^{1+\varepsilon}_{ij} \widehat{N}^{(p_\perp)}_{i,j}(F,f) \right).
\end{aligned}$$

Inserting this in (5.5), summing over all j, and using the finite overlap property as well as $R_{ij} \leq R_i$ yields

$$\begin{aligned}
&\left\| r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon/2} r_{\mathbf{v}}^{p_\|+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\|}_{x_\|} \nabla U \right\|^2_{L^2_\alpha((\omega^\xi_{\mathbf{ve}})^+)} \\
(5.7) \qquad &\lesssim C_B^{2p_\perp+2p_\|+2\varepsilon} (\sqrt{2}\gamma p_\perp)^{2p_\perp} \sum_i R_i^{2p_\|+2\varepsilon} \left(\| r_{\mathbf{e}}^{-1/2+\varepsilon/2} \nabla D^{p_\|}_{x_\|} U \|^2_{L^2_\alpha(\widetilde{H}^+_i)} + R_i^{1+\varepsilon} \widehat{N}^{(p_\perp)}_i(F,f) \right),
\end{aligned}$$

with the implied constant reflecting the overlap constant. Using again $R_i \leq 1$, we estimate the sum over the $\widehat{N}^{(p_\perp)}_i(F,f)$ (generously) by

$$\sum_i R_i^{2p_\|+2\varepsilon} R_i^{1+\varepsilon} \widehat{N}^{(p_\perp)}_i(F,f) \leq C \sum_{n=1}^{p_\perp} (\gamma p_\perp)^{-2n} \left(\max_{|\eta|=n} \| \partial^\eta_x D^{p_\|}_{x_\|} f \|^2_{L^2(\Omega)} + \max_{|\eta|=n-1} \| \partial^\eta_x D^{p_\|}_{x_\|} F \|^2_{L^2_{-\alpha}(\Omega\times(0,H))} \right).$$

The term involving $\| r_{\mathbf{e}}^{-1/2+\varepsilon} \nabla D^{p_\|}_{x_\|} U \|^2_{L^2_\alpha(\widetilde{H}^+_i)}$ in (5.7) is treated with Lemma 4.3 for the case $p_\| = 0$ and Lemma 4.4 for $p_\| > 0$. Considering first the case $p_\| = 0$, we estimate using the finite overlap property of the half-balls $\widehat{H}_i$ and $r_{\partial\Omega} \leq r_{\mathbf{e}}$

$$\sum_i R_i^{2p_\|+2\varepsilon} \| r_{\mathbf{e}}^{-1/2+\varepsilon/2} \nabla D^{p_\|}_{x_\|} U \|^2_{L^2_\alpha(\widetilde{H}^+_i)} \overset{\text{finite overlap}, p_\|=0}{\lesssim} \| r_{\partial\Omega}^{-1/2+\varepsilon/2} \nabla U \|^2_{L^2_\alpha(\Omega^+)} \overset{\text{L. 4.3}}{\lesssim} N^2(U,F,f).$$

For $p_\| > 0$, we use Lemma 4.4. To that end, we select, for each $i \in \mathbb{N}$, a cut-off function $\eta_i \in C^\infty_0(\mathbb{R}^2)$ with $\operatorname{supp} \eta_i \cap \Omega \subset \widehat{H}_i$ and $\eta_i \equiv 1$ on $\widetilde{H}_i$. Applying Lemma 4.4 with $t = 1/2 - \varepsilon/2$ there and using the finite overlap property we get for $\widetilde{U}^{(p_\|)}_i := \eta_i D^{p_\|}_{x_\|} U$ and $\widetilde{N}^{(p_\|)}(F,f)$ from (4.6)

$$\begin{aligned}
&\sum_i R_i^{2p_\|+2\varepsilon} \| r_{\mathbf{e}}^{-1/2+\varepsilon/2} \nabla D^{p_\|}_{x_\|} U \|^2_{L^2_\alpha(\widetilde{H}^+_i)} \leq \sum_i R_i^{2p_\|+2\varepsilon} \| r_{\partial\Omega}^{-1/2+\varepsilon/2} \nabla \widetilde{U}^{(p_\|)}_i \|^2_{L^2_\alpha(\widetilde{H}^+_i)} \\
&\lesssim \sum_i R_i^{2p_\|+2\varepsilon-2p_\|-1+2(1/2-\varepsilon/2)} (\gamma p_\|)^{2p_\|} (1+\gamma p_\|) \widetilde{N}^{(p_\|)}(F,f) \lesssim (\gamma p_\|)^{2p_\|} (1+\gamma p_\|) \widetilde{N}^{(p_\|)}(F,f);
\end{aligned}$$

here, we used that $\sum_i R_i^\varepsilon < \infty$ by Lemma 5.3.

Combining the above estimates we have shown the existence of $C \geq 1$ independent of $p = p_\parallel + p_\perp$ such that

$$\left\| r_{\mathbf{e}}^{p_\perp - 1/2+\varepsilon/2} r_{\mathbf{v}}^{p_\parallel+\varepsilon} D_{x_\perp}^{p_\perp} D_{x_\parallel}^{p_\parallel} \nabla U \right\|^2_{L^2_\alpha((\omega^\xi_{\mathbf{ve}})^+)}$$
$$\leq C^{2p+1} \left[p_\perp^{2p_\perp} p_\parallel^{2p_\parallel+1} \widetilde{N}^{(p_\parallel)}(F,f) + \sum_{n=1}^{p_\perp} p_\perp^{2p_\perp - 2n} \left(\max_{|\eta|=n} \left\| \partial_x^\eta D_{x_\parallel}^{p_\parallel} f \right\|^2_{L^2(\Omega)} + \max_{|\eta|=n-1} \left\| \partial_x^\eta D_{x_\parallel}^{p_\parallel} F \right\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} \right) \right].$$

For $p_\perp \geq 1$ we estimate with $p_\perp \leq p$

$$\sum_{n=1}^{p_\perp} p_\perp^{2(p_\perp - n)} \max_{|\eta|=n} \|\partial_x^\eta D_{x_\parallel}^{p_\parallel} f\|^2_{L^2(\Omega)} \leq \sum_{n=1}^{p_\perp} p^{2(p_\perp-n)} \max_{|\eta|=n} \|\partial_x^\eta D_{x_\parallel}^{p_\parallel} f\|^2_{L^2(\Omega)} \leq \sum_{j=1+p_\parallel}^{p} p^{2(p-j)} \max_{|\eta|=j} \|\partial_x^\eta f\|^2_{L^2(\Omega)}$$

and analogously for the sum over the terms $\max_{|\eta|=n-1} \|\partial_x^\eta D_{x_\parallel}^{p_\parallel} F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))}$. Also by similar arguments, we estimate $p_\parallel^{2p_\parallel} \widetilde{N}^{(p_\parallel)}(F,f) \leq p^{2p_\parallel} \widetilde{N}^{(p)}(F,f)$. Using $p_\parallel + p_\perp = p$ as well as $|D_{x_\parallel}^{p_\parallel} v| \leq 2^{p_\parallel/2} \max_{|\beta|=p_\parallel} |\partial_x^\beta v|$ completes the proof of the edge-vertex case in view of the definition of $\widetilde{N}^{(p)}(F,f)$ from (4.6) and by suitably selecting γ. ∎

LEMMA 5.7 (Weighted H^p-regularity in $\omega_{\mathbf{e}}$). *Given $\xi > 0$ and $\mathbf{e} \in \mathcal{E}$, there is γ depending only on s, Ω, and $\omega_{\mathbf{e}} = \omega^\xi_{\mathbf{e}}$ such that for every $\varepsilon \in (0,1)$ there is $C_\varepsilon > 0$ depending additionally on ε and H such that the solution U of (3.1) satisfies, for all $p_\parallel$, $p_\perp \in \mathbb{N}_0$ with $p_\parallel + p_\perp = p$*

$$\left\| r_{\mathbf{e}}^{p_\perp - 1/2+\varepsilon} D_{x_\perp}^{p_\perp} D_{x_{p_\parallel}}^{p_\parallel} \nabla U \right\|^2_{L^2_\alpha(\omega^+_{\mathbf{e}})}$$
$$\leq C_\varepsilon \gamma^{2p} p^{2p} \left(\|f\|^2_{H^1(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} + \sum_{j=1}^{p} p^{-2j} \left(\max_{|\eta|=j} \|\partial_x^\eta f\|^2_{L^2(\Omega)} + \max_{|\eta|=j-1} \|\partial_x^\eta F\|^2_{L^2_{-\alpha}(\mathbb{R}^2\times(0,H))} \right) \right).$$

Proof. The proof is essentially identical to the case $p_\parallel = 0$ in the proof of Lemma 5.5 using a covering of $\omega_{\mathbf{e}}$ analogous to the covering of $\omega_{\mathbf{v}}$ given in Lemma 5.2 that is refined towards $\mathbf{e}$ rather than $\mathbf{v}$, see Figure 4. ∎

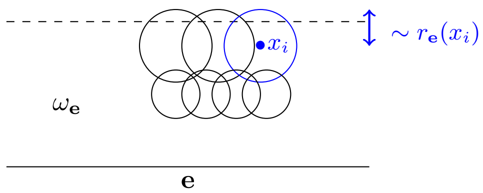

Fig. 4: Covering of edge-neighborhoods $\omega_{\mathbf{e}}$.

Remark 5.8. The assumption that ξ is sufficiently small in Lemma 5.6 can be dropped (as long as $\omega_{\mathbf{ve}}$ is well defined, as per Section 2.2). Indeed, for all ξ_1, ξ_2 such that $\xi_1 \geq \xi_2 > 0$ there exists $\xi_3 \geq \xi_2$ such that

$$\omega^{\xi_1}_{\mathbf{ve}} \subset \left(\omega^{\xi_2}_{\mathbf{ve}} \cup \omega^{\xi_3}_{\mathbf{v}} \cup \omega^{\xi_3}_{\mathbf{e}} \right). \tag{5.8}$$

In addition, there exists a constant $C_{\xi_3} > 0$ that depends only on ξ_3 and ε such that

$$\begin{aligned} \|r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\epsilon} D_{x_\perp}^{p_\perp} D_{x_{p_\parallel}}^{p_\parallel} \nabla U\|^2_{L^2_\alpha((\omega^{\xi_3}_{\mathbf{v}})^+)} &\leq 2^p \max_{|\beta|=p} \|r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\epsilon} \partial_x^\beta \nabla U\|^2_{L^2_\alpha((\omega^{\xi_3}_{\mathbf{v}})^+)} \\ &\leq C_{\xi_3}^{p+1} \max_{|\beta|=p} \|r_{\mathbf{v}}^{p-1/2+\varepsilon} \partial_x^\beta \nabla U\|^2_{L^2_\alpha((\omega^{\xi_3}_{\mathbf{v}})^+)} \end{aligned} \tag{5.9}$$

and that

$$\|r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\epsilon} D_{x_\perp}^{p_\perp} D_{x_{p_\parallel}}^{p_\parallel} \nabla U\|^2_{L^2_\alpha((\omega^{\xi_3}_{\mathbf{e}})^+)} \leq C_{\xi_3}^{p+1} \left\| r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} D_{x_\perp}^{p_\perp} D_{x_{p_\parallel}}^{p_\parallel} \nabla U \right\|^2_{L^2_\alpha((\omega^{\xi_3}_{\mathbf{e}})^+)}. \tag{5.10}$$

Given $\xi_1 > 0$, bounds in $\omega^{\xi_1}_{\mathbf{ve}}$ can therefore be derived by choosing ξ_2 such that Lemma 5.6 holds in $\omega^{\xi_2}_{\mathbf{ve}}$, exploiting the decomposition (5.8), using Lemmas 5.5 and 5.6 in $\omega^{\xi_3}_{\mathbf{v}}$ and $\omega^{\xi_3}_{\mathbf{e}}$, respectively, and concluding with (5.9) and (5.10). ∎

5.3. Proof of Theorem 2.1 – weighted H^p regularity for fractional PDE. In order to obtain regularity estimates for the solution u of $(-\Delta)^s u = f$, we have to take the trace $y \to 0$ in the weighted H^p-estimates for the Caffarelli-Silvestre extension problem provided by the previous subsection.

PROPOSITION 5.9. *Under the hypotheses of Theorem 2.1, there exists a constant $\widetilde{\gamma} > 0$ depending only on γ_f, s, and Ω such that for every $\varepsilon > 0$ there exists $\widetilde{C}_\varepsilon > 0$ (depending only on ε and Ω) such that for all $p \in \mathbb{N}$*

$$\left\| r_{\mathbf{e}}^{-1/2+\varepsilon} r_{\mathbf{v}}^{p-s+\varepsilon} D^p_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{ve}})} \leq C_\varepsilon \gamma^{p+1} p^p, \tag{5.11a}$$

and, for all $p_\parallel \in \mathbb{N}_0$, $p_\perp \in \mathbb{N}$ with $p_\parallel + p_\perp = p$,

$$\left\| r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{ve}})} \leq C_\varepsilon \gamma^{p+1} p^p. \tag{5.11b}$$

Moreover, for all $\beta \in \mathbb{N}_0^2$ with $|\beta| = p \geq 1$ and all $p_\parallel \in \mathbb{N}_0$, $p_\perp \in \mathbb{N}$ with $p_\parallel + p_\perp = p$,

$$\left\| r_{\mathbf{v}}^{p-1/2-s+\varepsilon} \partial^\beta_x u \right\|_{L^2(\omega_{\mathbf{v}})} \leq C_\varepsilon \gamma^{p+1} p^p, \tag{5.12}$$

$$\left\| r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{e}})} \leq C_\varepsilon \gamma^{p+1} p^p. \tag{5.13}$$

For $p_\parallel \in \mathbb{N}$, we have

$$\left\| r_{\mathbf{e}}^{-1/2+\varepsilon} D^{p_\parallel}_{x_\parallel} u \right\|_{L^2(\omega_{\mathbf{e}})} \leq C_\varepsilon \gamma^{p+1} p^p. \tag{5.14}$$

Finally, for the interior part Ω_{int} and all $p \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^2$ with $|\beta| = p$, we have

$$\left\| \partial^\beta_x u \right\|_{L^2(\Omega_{\text{int}})} \leq \gamma^{p+1} p^p. \tag{5.15}$$

Proof. We only show the estimates (5.11a) and (5.11b) using Lemma 5.6. The bounds (5.12) (using Lemma 5.5) and (5.13), (5.14) (using Lemma 5.7) follow with identical arguments. The bound in Ω_{int} follows directly from the interior Caccioppoli inequality, Corollary 3.6, and a trace estimate as below. (Note that the case $|\beta| = 0$ follows directly from the energy estimate $\|u\|_{L^2(\Omega_{\text{int}})} \leq \|u\|_{\widetilde{H}^s(\Omega)} \leq C \|f\|_{H^{-s}(\Omega)}$.)

Due to Lemma 5.6, applied with $F = 0$, and the assumption (2.10) on the data f, there exists a constant $C > 0$ such that for all $q_\perp$, $q_\parallel \in \mathbb{N}_0$ and $q_\perp + q_\parallel = q \in \mathbb{N}_0$ we have

$$\left\| r_{\mathbf{e}}^{q_\perp-1/2+\varepsilon} r_{\mathbf{v}}^{q_\parallel+\varepsilon} D^{q_\perp}_{x_\perp} D^{q_\parallel}_{x_\parallel} \nabla U \right\|^2_{L^2_\alpha(\omega^+_{\mathbf{ve}})} \leq C^{2q+1} q^{2q}. \tag{5.16}$$

The last step of the proof of [KM19, Lem. 3.7] gives the multiplicative trace estimate

$$|V(x,0)|^2 \leq C_{\text{tr}} \left(\|V(x,\cdot)\|^{1-\alpha}_{L^2_\alpha(\mathbb{R}_+)} \|\partial_y V(x,\cdot)\|^{1+\alpha}_{L^2_\alpha(\mathbb{R}_+)} + \|V(x,\cdot)\|^2_{L^2_\alpha(\mathbb{R}_+)} \right) \tag{5.17}$$

where, for univariate $v : \mathbb{R}_+ \to \mathbb{R}$, we write $\|v\|^2_{L^2_\alpha(\mathbb{R}_+)} := \int_{y=0}^\infty y^\alpha |v(y)|^2 \, dy$.

We have $p = p_\perp + p_\parallel \geq 1$. Suppose first $p_\perp \geq 1$ and $p_\parallel \geq 0$. Using the trace estimate (5.17) with $V = D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} U$ and additionally multiplying with the corresponding weight (using that $\alpha = 1 - 2s$) provides

$$\begin{aligned}
& r_{\mathbf{e}}^{2p_\perp-1-2s+2\varepsilon} r_{\mathbf{v}}^{2p_\parallel+2\varepsilon} \left| D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} U(x,0) \right|^2 \\
&\quad \leq C_{\text{tr}} \left\| r_{\mathbf{e}}^{p_\perp-3/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\varepsilon} \nabla D^{p_\perp-1}_{x_\perp} D^{p_\parallel}_{x_\parallel} U(x,\cdot) \right\|^{1-\alpha}_{L^2_\alpha(\mathbb{R}_+)} \left\| r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\varepsilon} D^{p_\perp}_{x_\perp} D^{p_\parallel}_{x_\parallel} \nabla U(x,\cdot) \right\|^{1+\alpha}_{L^2_\alpha(\mathbb{R}_+)} \\
&\qquad + C_{\text{tr}} \left\| r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} r_{\mathbf{v}}^{p_\parallel+\varepsilon} \nabla D^{p_\perp-1}_{x_\perp} D^{p_\parallel}_{x_\parallel} U(x,\cdot) \right\|^2_{L^2_\alpha(\mathbb{R}_+)},
\end{aligned}$$

where we have also used the fact that $(D_{x_\perp} v)^2 = (\mathbf{e}_\perp \cdot \nabla_x v)^2 \le |\nabla_x v|^2$ for all sufficiently smooth functions v. Integration over $\omega_{\mathbf{ve}}$ together with $r_{\mathbf{e}}^{-s} \lesssim r_{\mathbf{e}}^{-1}$ gives

$$
\begin{aligned}
&\left\| r_{\mathbf{e}}^{p_\perp - 1/2 - s + \varepsilon} r_{\mathbf{v}}^{p_\parallel + \varepsilon} D_{x_\perp}^{p_\perp} D_{x_\parallel}^{p_\parallel} u \right\|_{L^2(\omega_{\mathbf{ve}})}^2 \\
&\quad \le C_{\mathrm{tr}} \left\| r_{\mathbf{e}}^{p_\perp - 3/2 + \varepsilon} r_{\mathbf{v}}^{p_\parallel + \varepsilon} D_{x_\perp}^{p_\perp - 1} D_{x_\parallel}^{p_\parallel} \nabla U \right\|_{L^2_\alpha(\omega_{\mathbf{ve}}^+)}^{1-\alpha} \left\| r_{\mathbf{e}}^{p_\perp - 1/2 + \varepsilon} r_{\mathbf{v}}^{p_\parallel + \varepsilon} D_{x_\perp}^{p_\perp} D_{x_\parallel}^{p_\parallel} \nabla U \right\|_{L^2_\alpha(\omega_{\mathbf{ve}}^+)}^{1+\alpha} \\
&\qquad + C_{\mathrm{tr}} \left\| r_{\mathbf{e}}^{p_\perp - 1/2 - s + \varepsilon} r_{\mathbf{v}}^{p_\parallel + \varepsilon} D_{x_\perp}^{p_\perp - 1} D_{x_\parallel}^{p_\parallel} \nabla U \right\|_{L^2_\alpha(\omega_{\mathbf{ve}}^+)}^2 \\
&\quad \overset{(5.16)}{\le} C_{\mathrm{tr}} (C^{2p-1}(p-1)^{2(p-1)})^{(1-\alpha)/2} (C^{2p+1} p^{2p})^{(1+\alpha)/2} + C C_{\mathrm{tr}} C^{2p-1} (p-1)^{2(p-1)} \\
&\quad = C_{\mathrm{tr}} C^{2p+1+\alpha} p^{2p+\alpha} + C_{\mathrm{tr}} C^{2p-1} p^{2p} \le \gamma^{2p+1} p^{2p}
\end{aligned}
$$

for suitable $\gamma > 0$, which is estimate (5.11b). If $p_\perp = 0$, then $p_\parallel \ge 1$ and we have instead

$$
\begin{aligned}
&\left\| r_{\mathbf{e}}^{-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel - s + \varepsilon} D_{x_\parallel}^{p_\parallel} u \right\|_{L^2(\omega_{\mathbf{ve}})}^2 \\
&\quad \le C_{\mathrm{tr}} \left\| r_{\mathbf{e}}^{-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel - 1 + \varepsilon} \nabla D_{x_\parallel}^{p_\parallel - 1} U \right\|_{L^2_\alpha(\omega_{\mathbf{ve}}^+)}^{1-\alpha} \left\| r_{\mathbf{e}}^{-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel + \varepsilon} D_{x_\parallel}^{p_\parallel} \nabla U \right\|_{L^2_\alpha(\omega_{\mathbf{ve}}^+)}^{1+\alpha} \\
&\qquad + C_{\mathrm{tr}} \left\| r_{\mathbf{e}}^{-1/2+\varepsilon} r_{\mathbf{v}}^{p_\parallel - s + \varepsilon} \nabla D_{x_\parallel}^{p_\parallel - 1} U \right\|_{L^2_\alpha(\omega_{\mathbf{ve}}^+)}^2 .
\end{aligned}
$$

Again, inserting (5.16) into the right-hand side and proceeding similarly as above proves (5.11a). ∎

We now apply Proposition 5.9 to show our main result.

Proof of Theorem 2.1. Proposition 5.9 already covers most of the statements in Theorem 2.1. Only some lowest-order cases $p = 0$ or $p_\perp = 0$ are missing. We consider the three inequalities (2.11), (2.12), and (2.13) separately by using a Hardy inequality and then appealing to Proposition 5.9.

Proof of (2.11). Equation (2.11) with $p = 0$ follows from the weighted Hardy inequality [KMR97, Lem. 7.1.3], which provides

$$
\| r_{\mathbf{v}}^{-1/2-s+\varepsilon} u \|_{L^2(\omega_{\mathbf{v}})} \le C_{\mathrm{H},1} \| r_{\mathbf{v}}^{1/2-s+\varepsilon} \nabla u \|_{L^2(\omega_{\mathbf{v}})} \overset{\text{Prop. } 5.9}{<} \infty .
$$

Proof of (2.12). Let $(x_\perp, x_\parallel)$ be the coordinate system associated with edge $\mathbf{e}$. For μ, $\xi > 0$ sufficiently small and an interval I_μ of length μ consider

$$
\omega_{\mathbf{e}}^{\xi} \subseteq \{(x_\perp, x_\parallel) : x_\parallel \in I_\mu, x_\perp \in (0, \xi^2)\} =: \widetilde{\omega}_{\mathbf{e}}^{\xi,\mu}
$$

The interval I_μ is chosen such that $\omega_{\mathbf{e}}^{\xi} \subset \widetilde{\omega}_{\mathbf{e}}^{\xi,\mu}$ and $\widetilde{\omega}_{\mathbf{e}}^{\xi,\mu}$ stays away from the vertices $\mathcal{V}$ and the edges $\mathcal{E} \setminus \{\mathbf{e}\}$ so that the assertions of Proposition 5.9 still hold for $\widetilde{\omega}_{\mathbf{e}}^{\xi,\mu}$–cf. Remark 5.8. We will show (2.12) for $\widetilde{\omega}_{\mathbf{e}}$ (dropping the superscripts ξ, μ).

Let $\widetilde{u}$ be the function such that $\widetilde{u}(x_\perp, x_\parallel) = u(x_1, x_2)$ in $\widetilde{\omega}_{\mathbf{e}}$. By Fubini-Tonelli's theorem, for almost all $x_\parallel \in I_\mu$,

$$
\left(x_\perp \mapsto r_{\mathbf{e}}^{1/2-s+\epsilon} D_{x_\perp} (D_{x_\parallel}^{p_\parallel} \widetilde{u})(x_\perp, x_\parallel) \right) \in L^2((0, \xi^2)). \tag{5.18}
$$

The fundamental theorem of calculus, the Cauchy-Schwarz inequality, and (5.18), imply that, for almost all $x_\parallel \in I_\mu$, one has for $\epsilon < s$ that $(D_{x_\parallel}^{p_\parallel} \widetilde{u})(\cdot, x_\parallel) \in C^{0,s-\epsilon}([0, \xi^2])$. As $u \in \widetilde{H}^s(\Omega)$, we infer the pointwise equality $(D_{x_\parallel}^{p_\parallel} \widetilde{u})(0, x_\parallel) = 0$ for almost all $x_\parallel$. We can apply [KMR97, Lem. 7.1.3] again, in one dimension: for almost all $x_\parallel \in I_\mu$,

$$
\| r_{\mathbf{e}}^{-1/2-s+\epsilon} (D_{x_\parallel}^{p_\parallel} \widetilde{u})(\cdot, x_\parallel) \|_{L^2((0,\xi^2))} \le C_{\mathrm{H},2} \| r_{\mathbf{e}}^{1/2-s+\epsilon} (D_{x_\perp} D_{x_\parallel}^{p_\parallel} \widetilde{u})(\cdot, x_\parallel) \|_{L^2((0,\xi^2))}.
$$

Squaring and integrating over $x_\parallel \in I_\mu$ concludes the proof of (2.12).

Proof of (2.13). We use the same notation as in the previous part of the proof, but assume that the coordinate system (x_1, x_2) and the coordinate system $(x_\perp, x_\parallel)$ associated with edge $\mathbf{e}$ satisfy

$x_1 = x_\parallel$ and $x_2 = x_\perp$. Correspondingly, we assume $I_\mu = (0,\mu)$. We introduce the equivalent edge-vertex neighborhood

$$\widetilde{\omega}^{\xi,\mu}_{\mathbf{ve}} = \{(x_\perp, x_\parallel) : x_\parallel \in (0,\mu), x_\perp \in (0, \xi x_\parallel)\}.$$

We remark that in $\widetilde{\omega}_{\mathbf{ve}}$ there exists $c \geq 1$ such that for all $(x_\perp, x_\parallel) \in \widetilde{\omega}_{\mathbf{ve}}$

$$x_\parallel \leq r_{\mathbf{v}}(x_\parallel, x_\perp) \leq c x_\parallel. \tag{5.19}$$

We note $r_{\mathbf{e}}(x_\perp, x_\parallel) = x_\perp$. Hence, for almost all $x_\parallel \in (0,\mu)$,

$$\left(x_\perp \mapsto r_{\mathbf{e}}^{1/2-s+\epsilon} (D_{x_\perp}(D^{p_\parallel}_{x_\parallel} \widetilde{u}))(x_\perp, x_\parallel) \right) \in L^2((0, \xi x_\parallel)). \tag{5.20}$$

By the same argument as above, it follows that, for almost all $x_\parallel \in (0,\mu)$, we have $(D^{p_\parallel}_{x_\parallel}\widetilde{u})(\cdot, x_\parallel) \in C^{0,s-\epsilon}([0,\xi x_\parallel])$ and hence $(D^{p_\parallel}_{x_\parallel}\widetilde{u})(0, x_\parallel) = 0$. Therefore, [KMR97, Lemma 7.1.3] gives for almost all $x_\parallel \in (0,\mu)$,

$$\|r_{\mathbf{e}}^{-1/2-s+\epsilon}(D^{p_\parallel}_{x_\parallel}\widetilde{u})(\cdot, x_\parallel)\|_{L^2((0,\xi x_\parallel))} \leq C_{\mathrm{H},3} \|r_{\mathbf{e}}^{1/2-s+\epsilon}(D_{x_\perp} D^{p_\parallel}_{x_\parallel}\widetilde{u})(\cdot, x_\parallel)\|_{L^2((0,\xi x_\parallel))},$$

with constant $C_{\mathrm{H},3}$ independent of $x_\parallel$. Multiplying by $r_{\mathbf{v}}^{p_\parallel+\epsilon}$, squaring, integrating over $x_\parallel \in (0,\mu)$, and using (5.19),

$$\|r_{\mathbf{e}}^{-1/2-s+\epsilon} r_{\mathbf{v}}^{p_\parallel+\epsilon} D^{p_\parallel}_{x_\parallel}\widetilde{u}\|_{L^2(\widetilde{\omega}_{\mathbf{ve}})} \leq c^{p_\parallel+\epsilon} C_{\mathrm{H},3} \|r_{\mathbf{e}}^{1/2-s+\epsilon} r_{\mathbf{v}}^{p_\parallel+\epsilon} D_{x_\perp} D^{p_\parallel}_{x_\parallel}\widetilde{u}\|_{L^2(\widetilde{\omega}_{\mathbf{ve}})}.$$

This completes the proof except for the fact that the region $\omega_{\mathbf{ve}} \setminus \widetilde{\omega}_{\mathbf{ve}}$ is not covered yet. This region is treated with the observations of Remark 5.8. ∎

6. Conclusions.

We briefly recapitulate the principal findings of the present paper, outline generalizations of the present results, and also indicate applications to the numerical analysis of finite element approximations of (2.2). We established analytic regularity of the solution u in a scale of edge- and vertex-weighted Sobolev spaces for the Dirichlet problem for the fractional Laplacian in a bounded polygon $\Omega \subset \mathbb{R}^2$ with straight sides, and for forcing f analytic in $\overline{\Omega}$.

While the analysis in Sections 4 and 5 was developed at present in two spatial dimensions, we emphasize that all parts of the proof can be extended to higher spatial dimension $d \geq 3$, and polytopal domains $\Omega \subset \mathbb{R}^d$. Details shall be presented elsewhere.

Likewise, the present approach is also capable of handling nonconstant, analytic coefficients similar to the setting considered (for the spectral fractional Laplacian) in [BMN+19]. Details on this extension of the present results, with the presently employed techniques, will also be developed in forthcoming work.

The weighted analytic regularity results obtained in the present paper can be used to establish *exponential convergence rates* with the bound $C\exp(-b\sqrt[4]{N})$ on the error for suitable hp-Finite Element discretizations of (2.2), with N denoting the number of degrees of freedom of the discrete solution in Ω. This will be proved in the follow-up work [FMMS22b]. Importantly, as already observed in [BMN+19], achieving this exponential rate of convergence mandates *anisotropic mesh refinements* near the boundary $\partial\Omega$.

Appendix A. Localization of Fractional Norms.

The following elementary observation on localization of fractional norms was used in several places.

LEMMA A.1. *Let $\eta \in C_0^\infty(B_R)$ for some ball $B_R \subset \Omega$ of radius R and $s \in (0,1)$. Then,*

$$\|\eta f\|_{H^{-s}(\Omega)} \leq C_{\mathrm{loc}} \|\eta\|_{L^\infty(B_R)} \|f\|_{L^2(B_R)}, \tag{A.1}$$

$$\|\eta f\|_{H^{1-s}(\Omega)} \leq C_{\mathrm{loc},2} \Big[\big(R^s \|\nabla\eta\|_{L^\infty(B_R)} + (R^{s-1}+1)\|\eta\|_{L^\infty(B_R)}\big) \|f\|_{L^2(\Omega)} + \|\eta\|_{L^\infty(B_R)} |f|_{H^{1-s}(\Omega)} \Big], \tag{A.2}$$

where the constants C_{loc}, $C_{\mathrm{loc},2}$ depend only on Ω and s.

Proof. (A.1) follows directly from the embedding $L^2 \subset H^{-s}$. For (A.2), we use the definition of the Slobodecki norm and the triangle inequality to write

$$|\eta f|^2_{H^{1-s}(\Omega)} = \int_\Omega \int_\Omega \frac{|\eta(x)f(x) - \eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx$$
$$\lesssim \int_\Omega \int_\Omega \frac{|\eta(x)f(x) - \eta(x)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx + \int_\Omega \int_\Omega \frac{|\eta(x)f(z) - \eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx.$$

The first term on the right-hand side can directly be estimated by $\|\eta\|_{L^\infty(B_R)}|f|_{H^{1-s}(\Omega)}$. For the second term, we split the integration over $\Omega \times \Omega$ into four subsets, $B_{2R} \times B_{3R}$, $B_{2R} \times B^c_{3R} \cap \Omega$, $B^c_{2R} \cap \Omega \times B_R$, $B^c_{2R} \cap \Omega \times B^c_R \cap \Omega$; here, we assume for simplicity for the concentric balls $B_R \subset B_{2R} \subset B_{3R} \subset \Omega$, otherwise one has to intersect all balls with Ω. For the last case, $B^c_{2R} \cap \Omega \times B^c_R \cap \Omega$, we have that $\eta(x) - \eta(z)$ vanishes and the integral is zero. For the case $B_{2R} \times B^c_{3R}$, we have $|x - z| \geq R$ there. This gives

$$\int_{B_{2R}} \int_{B^c_{3R}\cap\Omega} \frac{|\eta(x)f(z) - \eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx = \int_{B_{2R}} \int_{B^c_{3R}\cap\Omega} \frac{|\eta(x)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx$$
$$\leq R^{-d-2+2s} \|\eta\|^2_{L^\infty(B_R)} \int_{B_{2R}} \int_{B^c_{3R}\cap\Omega} |f(z)|^2 dz dx \lesssim R^{-2+2s} \|\eta\|^2_{L^\infty(B_R)} \|f\|^2_{L^2(\Omega)}.$$

For the integration over $B^c_{2R} \cap \Omega \times B_R$, we write using polar coordinates (centered at z)

$$\int_{B^c_{2R}\cap\Omega} \int_{B_R} \frac{|\eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx = \int_{B_R} |\eta(z)f(z)|^2 \int_{B^c_{2R}\cap\Omega} \frac{1}{|x-z|^{d+2-2s}}\, dx\, dz$$
$$\lesssim \int_{B_R} |\eta(z)f(z)|^2 \int_R^\infty \frac{1}{r^{3-2s}}\, dx\, dz \lesssim R^{2s-2} \|\eta\|^2_{L^\infty(B_R)} \|f\|^2_{L^2(\Omega)}.$$

Finally, for the integration over $B_{2R} \times B_{3R}$, we use that $|\eta(x) - \eta(z)| \leq \|\nabla\eta\|_{L^\infty(B_R)} |x - z|$ and polar coordinates (centered at z) to estimate

$$\int_{B_{2R}} \int_{B_{3R}} \frac{|\eta(x)f(z) - \eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx \leq \|\nabla\eta\|^2_{L^\infty(B_R)} \int_{B_{3R}} |f(z)|^2 \int_{B_{2R}} \frac{1}{|x-z|^{d-2s}}\, dx\, dz$$
$$\lesssim \|\nabla\eta\|^2_{L^\infty(B_R)} \int_{B_{3R}} |f(z)|^2 \int_0^{5R} r^{-1+2s}\, dr\, dz \lesssim \|\nabla\eta\|^2_{L^\infty(B_R)} \|f\|^2_{L^2(B_{3R})} R^{2s}.$$

The straightforward bound $\|\eta f\|_{L^2(\Omega)} \leq \|\eta\|_{L^\infty(B_R)}\|f\|_{L^2(\Omega)}$ concludes the proof. ∎

Appendix B. Proof of Lemma 3.1. *Proof of Lemma 3.1*: The proof follows from the arguments given in [KM19, Sec. 3]; a more general development of Beppo-Levi spaces is given in [DL54].

Proof of (i): Fix a (nondegenerate) hypercube $K = \prod_{i=1}^{d+1}(a_i, b_i)$ with $a_{d+1} = 0$. Elements of the Beppo-Levi space BL^1_α are locally in L^2, and one can equip the space BL^1_α with the norm $\|U\|^2_{\mathrm{BL}^1_\alpha} := \|U\|^2_{L^2_\alpha(K)} + \|\nabla U\|^2_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}$. Endowed with this norm, BL^1_α is a Hilbert space and $C^\infty(\mathbb{R}^d \times [0,\infty)) \cap \mathrm{BL}^1_\alpha$ is dense, [KM19, Lemma 3.2]. On the subspace $\mathrm{BL}^1_{\alpha,0,\Omega}$ we show the norm equivalence $\|U\|_{\mathrm{BL}^1_\alpha} \sim \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}$ using the bounded linear lifting operator $\mathcal{E} : H^s(\mathbb{R}^d) \to H^1_\alpha(\mathbb{R}^d \times \mathbb{R}_+)$ of [KM19, Lemma 3.9] and the norm equivalence of [KM19, Cor. 3.4]

$$\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \leq \|U\|_{\mathrm{BL}^1_\alpha} \leq \|U - \mathcal{E}\operatorname{tr} U\|_{\mathrm{BL}^1_\alpha} + \|\mathcal{E}\operatorname{tr} U\|_{\mathrm{BL}^1_\alpha}$$
$$\overset{\text{[KM19, Cor. 3.4]}}{\lesssim} \|\nabla(U - \mathcal{E}\operatorname{tr} U)\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + \|\mathcal{E}\operatorname{tr} U\|_{\mathrm{BL}^1_\alpha}$$
$$\overset{\text{[KM19, Lem. 3.9]}}{\lesssim} \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + \|\operatorname{tr} U\|_{H^s(\mathbb{R}^d)}$$
$$\overset{\operatorname{tr} U\in\widetilde{H}^s(\Omega),(1.3)}{\lesssim} \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + |\operatorname{tr} U|_{H^s(\mathbb{R}^d)} \overset{\text{[KM19, Lem. 3.8]}}{\lesssim} \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}.$$

Proof of (ii): From the fundamental theorem of calculus, we have for smooth univariate functions v and $x \in (0, H)$ the estimate $|v(x)| = |v(0) + \int_{t=0}^x v'(t)\, dt| \lesssim |v(0)| + \sqrt{\int_{t=0}^x t^\alpha |v'(t)|^2\, dt}$.

Fix a closed hypercube $K' \subset \mathbb{R}^d$ of side length $d_{K'} > 0$ with $K' \supset \Omega$. Define the translates $K_j := d_{K'} j + K'$ for $j \in \mathbb{Z}^d$. For smooth U, we infer from the 1D estimate that

$$\|U\|_{L^2_\alpha(K' \times (0,H))} \leq C_{K'} \left(\|\nabla U\|_{L^2_\alpha(K' \times (0,H))} + \|\operatorname{tr} U\|_{L^2(K')} \right). \tag{B.1}$$

By the density of $C^\infty(\mathbb{R}^d \times [0,\infty)) \cap \mathrm{BL}^1_\alpha$ in BL^1_α from the proof of part (i), the estimate (B.1) holds for all $U \in \mathrm{BL}^1_\alpha$. By translation invariance of the norms and spaces, (B.1) also holds for all $U \in \mathrm{BL}^1_\alpha$ and for all translates K_j, $j \in \mathbb{Z}^d$, with the same constant $C_{K'}$. For $U \in \mathrm{BL}^1_{\alpha,0,\Omega}$, we observe $\|\operatorname{tr} U\|_{L^2(K_0)} \leq \|\operatorname{tr} U\|_{\widetilde{H}^s(K_0)} \leq C_\Omega |\operatorname{tr} U|_{H^s(\mathbb{R}^d)}$ (cf. (1.3)) and $\operatorname{tr} U|_{K_j} = 0$ for $j \neq 0$. Hence, using the Kronecker $\delta_{j,0}$ we arrive at

$$\|U\|_{L^2_\alpha(K_j \times (0,H))} \leq C_{K'} \left(\|\nabla U\|_{L^2_\alpha(K_j \times (0,H))} + C_\Omega \delta_{j,0} |\operatorname{tr} U|_{H^s(\mathbb{R}^d)} \right).$$

Since $\mathbb{R}^d = \cup_{j \in \mathbb{Z}^d} K_j$ and the intersection $K_j \cap K_{j'}$ is a set of measure zero for $j \neq j'$, summation over all j implies

$$\|U\|_{L^2_\alpha(\mathbb{R}^d \times (0,H))} \lesssim \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d \times (0,H))} + |\operatorname{tr} U|_{H^s(\mathbb{R}^d)}.$$

The proof is completed by noting $|\operatorname{tr} U|_{H^s(\mathbb{R}^d)} \lesssim \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d \times \mathbb{R}_+)}$ by [KM19, Lemma 3.8]. ∎

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