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Spurious Quasi-Resonances in Boundary Integral Equations for the Helmholtz Transmission Problem

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Abstract

We consider the Helmholtz transmission problem with piecewise-constant material coefficients, and the standard associated direct boundary integral equations. For certain coefficients and geometries, the norms of the inverses of the boundary integral operators grow rapidly through an increasing sequence of frequencies, even though this is not the case for the solution operator of the transmission problem; we call this phenomenon that of *spurious quasi-resonances*. We give a rigorous explanation of why and when spurious quasi-resonances occur, and propose modified boundary integral equations that are not affected by them.

AMS subject classification: 35J05, 35J25, 45A05, 78A45

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1 Introduction and summary of the main results

1.1 The Helmholtz transmission scattering problem

We consider the scattering of an incident time-harmonic acoustic wave by a penetrable homogeneous object that occupies the region of space $\Omega^- \subset \mathbb{R}^d$, $d = 2, 3$, which is a bounded Lipschitz open set. We first introduce notation necessary for a precise mathematical statement of this *transmission problem*. Let $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$, $\Gamma := \partial\Omega^- = \partial\Omega^+$, and let $\mathbf{n}$ be the unit normal vector field on Γ pointing from Ω^- into Ω^+. For any $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, we let $\varphi^- := \varphi|_{\Omega^-}$ and $\varphi^+ := \varphi|_{\Omega^+}$. With $H^1_{\mathrm{loc}}(\Omega^\pm, \Delta) := \{v : \chi v \in H^1(\Omega^\pm), \Delta(\chi v) \in L^2(\Omega^\pm) \text{ for all } \chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)\}$, we define the Dirichlet and Neumann trace operators

$$\gamma_D^\pm : H^1_{\mathrm{loc}}(\Omega_\pm) \to H^{1/2}(\Gamma) \quad \text{and} \quad \gamma_N^\pm : H^1_{\mathrm{loc}}(\Omega_\pm, \Delta) \to H^{-1/2}(\Gamma),$$

with $\gamma_D^\pm v := v^\pm|_\Gamma$ and $\gamma_N^\pm$ such that if $v \in H^2_{\mathrm{loc}}(\Omega_\pm)$ then $\gamma_N^\pm v = \mathbf{n} \cdot \gamma_D^\pm(\nabla v)$. Let $\boldsymbol{\gamma}_C^\pm := (\gamma_D^\pm, \gamma_N^\pm)$ be the Cauchy trace, which satisfies

$$\boldsymbol{\gamma}_C^\pm : H^1_{\mathrm{loc}}(\Omega^\pm, \Delta) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma).$$

Given $\varphi \in C^1(\mathbb{R}^d \setminus B_R)$, for some ball $B_R := \{|\mathbf{x}| < R\}$, and $\kappa > 0$, φ satisfies the *Sommerfeld radiation condition* if

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial \varphi(\mathbf{x})}{\partial r} - \mathrm{i}\kappa\varphi(\mathbf{x}) \right) = 0 \tag{1.1}$$

uniformly in all directions, where $r = |\mathbf{x}|$; we then write $\varphi \in \mathrm{SRC}(\kappa)$.

Given $n_i, n_o > 0$ and frequency $k > 0$, the **Helmholtz transmission scattering problem** is that of finding the complex amplitude u of the sound pressure, with $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$ the solution of

$$\begin{aligned} (\Delta + k^2 n_i)u^- &= 0 && \text{in } \Omega^-, \\ (\Delta + k^2 n_o)u^+ &= 0 && \text{in } \Omega^+, \\ \boldsymbol{\gamma}_C^- u^- &= \boldsymbol{\gamma}_C^+ u^+ + \boldsymbol{\gamma}_C^\pm u^I && \text{on } \Gamma. \\ u^+ &\in \mathrm{SRC}(k\sqrt{n_o}), \end{aligned} \tag{1.2}$$

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where the incident wave u^I is an entire solution of the homogeneous Helmholtz equation in $\mathbb{R}^d$,

$$(\Delta + k^2 n_o)u^I = 0 \quad \text{in } \mathbb{R}^d. \tag{1.3}$$

This set up means that u^- is the total field in Ω^- and u^+ the scattered field in Ω^+.

In principle, the jump $\boldsymbol{\gamma}_C^+ u^+ - \boldsymbol{\gamma}_C^- u^-$ of the Cauchy trace of u across Γ can be more general than the Cauchy trace of an incident wave. This leads to the following generic Helmholtz transmission problem.

Definition 1.1. (The Helmholtz transmission problem.) *Given positive real numbers k, n_i, and n_o and $\mathbf{f} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^1_{\rm loc}(\mathbb{R}^d \setminus \Gamma) \cap \mathrm{SRC}(k\sqrt{n_o})$ such that,*

$$\begin{aligned}
(\Delta + k^2 n_i)u^- &= 0 && \text{in } \Omega^-,\\
(\Delta + k^2 n_o)u^+ &= 0 && \text{in } \Omega^+,\\
\boldsymbol{\gamma}_C^- u^- &= \boldsymbol{\gamma}_C^+ u^+ + \mathbf{f} && \text{on } \Gamma.
\end{aligned} \tag{1.4}$$

The following well-posedness result is proved in, e.g., [23, Lemma 2.2 and Appendix A].

Lemma 1.2. *The solution of the transmission problem of Definition 1.1 exists and is unique. Moreover, if $\mathbf{f} \in H^1(\Gamma) \times L^2(\Gamma)$ then $\boldsymbol{\gamma}_C^\pm u^\pm \in H^1(\Gamma) \times L^2(\Gamma)$.*

Remark 1.3. The transmission problem of Definition 1.1 is not the most general form of the transmission problem. If the transmission condition in (1.4) is replaced by

$$\boldsymbol{\gamma}_C^- u^- = D\boldsymbol{\gamma}_C^+ u^+ + \mathbf{f}, \qquad \text{where } D := \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \tag{1.5}$$

for α a constant, then this covers all possible constant-coefficient transmission problems; see, e.g., [23, Page 322]. In Appendix A we outline how our results extend this more general transmission problem. We see that, although the main ideas remain the same, more notation and technicalities are required, hence why we have chosen to focus on the simpler problem of Definition 1.1.

1.2 Solution operators and quasi-resonances

Definition 1.4. (Solution operators.) *Given positive real numbers k, c_i, and c_o, let*

$$S(c_i, c_o)\mathbf{f} := \boldsymbol{\gamma}_C^- u,$$

where $u \in H^1_{\rm loc}(\mathbb{R}^d \setminus \Gamma) \cap \mathrm{SRC}(k\sqrt{c_o})$ is the solution of the Helmholtz transmission problem

$$\begin{aligned}
(\Delta + k^2 c_i)u^- &= 0 && \text{in } \Omega^-,\\
(\Delta + k^2 c_o)u^+ &= 0 && \text{in } \Omega^+,\\
\boldsymbol{\gamma}_C^- u^- &= \boldsymbol{\gamma}_C^+ u^+ + \mathbf{f} && \text{on } \Gamma.
\end{aligned} \tag{1.6}$$

Lemma 1.2 implies that $S(c_i, c_o)$ is well defined and bounded on *either* $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *or* $H^1(\Gamma) \times L^2(\Gamma)$. We introduce the abbreviations

$$S_{io} := S(n_i, n_o) \quad \text{and} \quad S_{oi} := S(n_o, n_i).$$

We refer to S_{io} as the "physical" solution operator, since it corresponds to the transmission problem of Definition 1.1, and S_{oi} as the "unphysical" solution operator, since it corresponds to the transmission problem where the indices n_i and n_o are swapped compared to those in Definition 1.1.

Recall that the high-frequency behaviour of S_{io} depends on which of n_i and n_o is larger. Indeed, if $n_i < n_o$ and Ω^- is Lipschitz and star-shaped with respect to a ball, then Lemma 4.5 below shows that the norm of S_{io} has, at worst, mild algebraic growth in k; this result uses the bounds on the solution operator from [23], with analogous bounds obtained for smooth, convex Ω^- with strictly positive curvature in [9]. If $n_i > n_o$ and Ω^- is smooth and convex with strictly positive curvature, then Lemma 4.6 below, based on the results of [28], shows that there exists $0 < k_1 < k_2 < \ldots$ with $k_j \to \infty$ such that the norm of S_{io} blows up superalgebraically through k_j as $j \to \infty$. (Similar results in the particular case when Ω^- is a ball were obtained in [7,8], and summarised in [1, Chapter 5]).

We call these real frequencies k_j *quasi-resonances*, since they can be understood as real parts of complex resonances of the transmission problem lying close to the real axis (with this terminology also used in, e.g., [1, 7, 8]); the particular functions on which S_{io} at $k = k_j$ blows up are called *quasimodes*. The relationship between quasimodes and resonances is a classic topic in scattering theory; see [31–33], [17, §7.3]. The Weyl-type bound on the number of resonances of the transmission problem when Ω^- is smooth and convex with strictly positive curvature in [10, Theorem 1.3] implies that the number of quasi-resonances in $[0, K]$ in this case grows like K^d as $K \to \infty$.

Remark 1.5. The physical reason for the existence of quasi-resonances when $n_i > n_o$ is that, in this case, geometric-optic rays can undergo total internal reflection when hitting Γ from Ω^-. Rays "hugging" the boundary via a large number of bounces with total internal reflection correspond to solutions of the transmission problem localised near Γ; in the asymptotic-analysis literature these solutions are known as "whispering gallery" modes; see, e.g., [3, 4]. The existence of quasi-resonances of the transmission problem has only been rigorously established when Ω^- is smooth and convex with strictly positive curvature. The understanding above via rays suggests that such quasi-resonances and quasimodes do not exist for polyhedral Ω^- (since sharp corners prevent rays from moving parallel to the boundary), although solutions with localisation qualitatively similar to that of quasimodes can be seen when Ω^- is a pentagon [20, Figure 13] or a hexagon [6, Figure 23].

1.3 Calderón projectors and the standard first- and second-kind direct boundary integral equations (BIEs)

Since all the layer potentials and integral operators depend on k, we omit this k-dependence in the notation. Let the Helmholtz fundamental solutions be given by

$$\Phi_{i/o}(\mathbf{x},\mathbf{y}) := \frac{\mathrm{i}}{4} H_0^{(1)}\big(k\sqrt{n_{i/o}}|\mathbf{x}-\mathbf{y}|\big) \quad \text{for } d=2, \quad \text{and} \quad \Phi_{i/o}(\mathbf{x},\mathbf{y}) := \frac{\mathrm{e}^{\mathrm{i}k\sqrt{n_{i/o}}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \quad \text{for } d=3,$$

where $H_0^{(1)}$ is the Hankel function of the first kind and order zero; see [29, Section 3.1].

As in [29, Equation 3.6], the single-layer, adjoint-double-layer, double-layer, and hypersingular operators are defined for $\phi \in L^2(\Gamma)$ and $\psi \in H^1(\Gamma)$ by

$$V_{i/o}\phi(\mathbf{x}) := \int_\Gamma \Phi_{i/o}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad K'_{i/o}\phi(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi_{i/o}(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{1.7}$$

$$K_{i/o}\phi(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi_{i/o}(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad W_{i/o}\psi(\mathbf{x}) := -\frac{\partial}{\partial n(\mathbf{x})}\int_\Gamma \frac{\partial \Phi_{i/o}(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{1.8}$$

for $\mathbf{x} \in \Gamma$ (note that the sign of the hypersingular operator is swapped compared to, e.g., [11]). When Γ is Lipschitz, the integrals defining $K_{i/o}$ and $K'_{i/o}$ must be understood as Cauchy principal values (see, e.g., [11, Equation 2.33]), and the integral defining $W_{i/o}$ is understood as a non-tangential limit (see, e.g., [11, Equation 2.36]) or finite-part integral (see, e.g., [21, Theorem 7.4 (iii)]), but we do not need the details of these definitions in this paper.

Let the Calderón projectors $P^\pm_{i/o}$ be defined by

$$P^\pm_{i/o} := \frac{1}{2}I \pm M_{i/o}, \quad \text{where} \quad M_{i/o} := \begin{bmatrix} K_{i/o} & -V_{i/o} \\ -W_{i/o} & -K'_{i/o} \end{bmatrix}; \tag{1.9}$$

see, e.g., [29, Section 3.6], [11, Page 117]. Basic results about $P^\pm_{i/o}$ (including that they are indeed projectors) are in §2, but we record here that

$$P^+_{i/o} + P^-_{i/o} = I. \tag{1.10}$$

Let the boundary integral operators (BIOs) A_{I} and A_{II} be defined by

$$A_{\mathrm{I}} := P^-_o - P^+_i = P^-_i - P^+_o = \begin{bmatrix} -(K_i + K_o) & V_i + V_o \\ W_i + W_o & K'_i + K'_o \end{bmatrix} \tag{1.11}$$

and

$$A_{\mathrm{II}} := P^-_o + P^+_i = 2I - P^+_o - P^-_i = I + \begin{bmatrix} K_i - K_o & -(V_i - V_o) \\ -(W_i - W_o) & -(K'_i - K'_o) \end{bmatrix}. \tag{1.12}$$

Lemma 1.6. *If u is the solution of the Helmholtz transmission problem of Definition 1.1, then*

$$A_{\rm I}(\boldsymbol{\gamma}_C^- u^-) = P_o^- \mathbf{f} \quad \text{and} \quad A_{\rm II}(\boldsymbol{\gamma}_C^- u^-) = P_o^- \mathbf{f}. \tag{1.13}$$

In particular, if u solves the Helmholtz transmission scattering problem (1.2), then

$$A_{\rm I}(\boldsymbol{\gamma}_C^- u^-) = \boldsymbol{\gamma}_C^- u^I, \quad \text{and} \quad A_{\rm II}(\boldsymbol{\gamma}_C^- u^-) = \boldsymbol{\gamma}_C^- u^I. \tag{1.14}$$

These boundary integral equations (BIEs) are called single-trace formulations (STFs). The *first-kind* BIEs in (1.13) and (1.14) appeared in [16], [37], and are also derived in, e.g., [13, Section 3.3]. Their counterparts for electromagnetic scattering are known as the PMCHWT (Poggio–Miller–Chang–Harrington–Wu–Tsai) formulation [27]. The *second-kind* BIEs in (1.13) and (1.14) can be found in, e.g., [14] and are known as the Müller formulation in computational electromagnetics [25].

Lemma 1.7. *(i) Both $A_{\rm I}$ and $A_{\rm II}$ are bounded and invertible on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.*
(ii) $A_{\rm II}$ is bounded and invertible on $H^1(\Gamma) \times L^2(\Gamma)$.

The proofs of Lemmas 1.6 and 1.7 are contained in §2.

1.4 Spurious quasi-resonances for the standard BIOs

Lemma 1.7 shows that the BIEs of (1.13) and (1.14) are well-posed. It is then reasonable to believe that the solution operators of these BIEs inherit the behaviour (with respect to frequency) of the solution operator of the transmission problem. The following numerical results, however, show that this is not the case. [1]

Example 1.8. If Γ is a circle for $d = 2$ or a sphere for $d = 3$ all boundary integral operators $V_{i/o}$, $K'_{i/o}$, $K_{i,o}$, and $W_{i/o}$ can be "diagonalized" by switching to a "modal" $L^2(\Gamma)$-orthogonal basis of Fourier harmonics in 2D or spherical harmonics in 3D, respectively. The corresponding eigenvalues can be found in, e.g., [2] for $d = 2$ and in, e.g., [36] for $d = 3$. All relevant norms have a simple sum representation with respect to these bases. Therefore we can compute the norms of the solution operators as the maximum of the Euclidean norms of 2×2-matrices, one for every mode. We did this in MATLAB for the modes of order at most 100, which seems to be sufficient, because the maximal norm was invariably found among the modes of order ≤ 25.

We report the computed norms of the solution operator S_{io} along with the norms of $A_{\rm I}^{-1}$ and $A_{\rm II}^{-1}$ (i.e., the solution operators for the BIEs (1.14)) on the space $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, where we use the weighted norm $\| \cdot \|_{H_k^{1/2}(\Gamma) \times H_k^{-1/2}(\Gamma)}$ defined in §4.1. We plot these norms for different frequencies k and give the results for $d = 2$ in Figure 1 and for $d = 3$ in Figure 2.

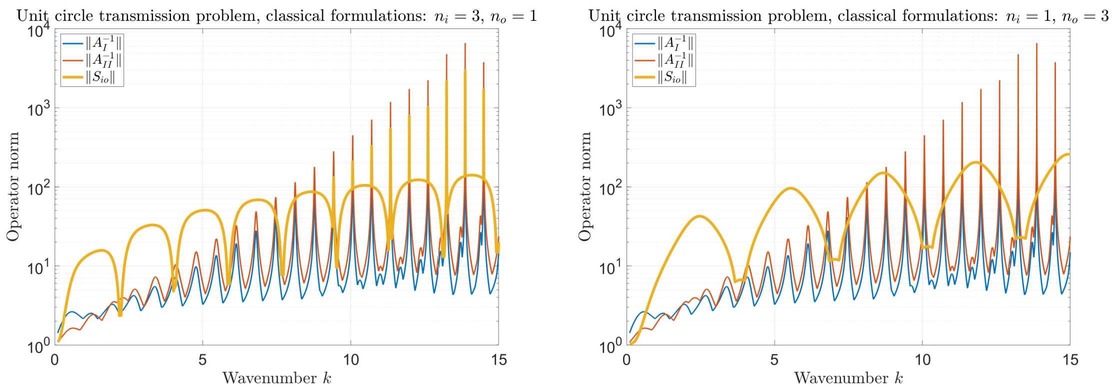

Figure 1: Γ unit circle: Norms of operators S_{io}, $A_{\rm I}^{-1}$, and $A_{\rm II}^{-1}$ on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ for $n_i = 3$, $n_o = 1$ (left) and $n_i = 1$, $n_o = 3$ (right)

[1]The code used to produce the numerical results is available at https://github.com/moiola/TransmissionBIE-OpNorms

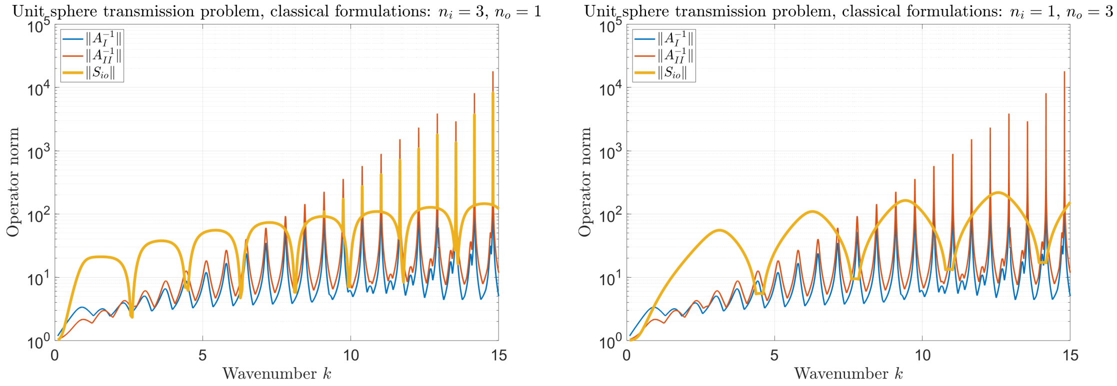

Figure 2: Γ unit sphere: Norms of operators S_{io}, $A_{\rm I}^{-1}$, and $A_{\rm II}^{-1}$ on $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ for $n_i = 3$, $n_o = 1$ (left) and $n_i = 1$, $n_o = 3$ (right)

When $n_i > n_o$ (plots on the left) we see the typical spikes in the norms as a function of k, expected because of the results recalled in §1.2; these spikes are caused by quasi-resonances. Conversely, for $n_i < n_o$ (right plots) the norm of S_{io} (in yellow) does not have any spikes, whereas the spikes persist in the norms of $A_{\rm I}^{-1}$ and $A_{\rm II}^{-1}$.

The observations made in Example 1.8 provide evidence of *spurious quasi-resonances* of $A_{\rm I}$ and $A_{\rm II}$ when $n_i < n_o$: for certain frequencies these boundary integral operators are ill-conditioned though for the same frequencies the solution operator is stable.

On rare occasions such spurious quasi-resonances have been noticed before. Indeed, the paper [22] computed the complex eigenvalues of $A_{\rm I}$ and $A_{\rm II}$ and pointed out in [22, Section 2.3] the existence of "fictitious eigenvalues", i.e., non-physical poles of the resolvent operators. Although [22] did not give a rigorous explanation for this phenomenon, [22] attempted to remedy it by modifying the BIEs; these new BIEs, however, still have issues with poles with small imaginary part – see the discussion in [22, §4]. Non-physical spikes in the condition numbers of discretized BIEs for Helmholtz transmission problems were also reported in [34, Section 4.4], but no deeper investigation was attempted.

The observation of the spurious quasi-resonances of Example 1.8 was the starting point for this paper – we wanted to understand precisely why they affect $A_{\rm I}$ and $A_{\rm II}$. We also wanted to find alternative BIEs immune to spurious quasi-resonances. The remainder of this paper reports our progress towards these goals.

Remark 1.9. For the standard first and second-kind BIEs for the exterior Dirichlet and Neumann problems for the Helmholtz operator (modelling acoustic scattering by impenetrable objects), the occurrence of spurious (true) resonances is well-known; see, e.g., [29, Section 3.9.2]: the solutions of the BIEs are not unique for an infinite sequence of distinct ks, although the boundary-value problems have unique solutions for all k. The standard remedies for this are recalled (and linked to the results of the present paper) in Remark 1.16 below.

1.5 The main results

1.5.1 The relationship between the BIOs and the solution operators

Theorem 1.10. *As an operator on* $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$, $A_{\rm I}^{-1}$ *has the decomposition*

$$A_{\rm I}^{-1} = S_{io} + S_{oi} - I \tag{1.15}$$

and, as an operator on either $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ or $H^{1}(\Gamma)\times L^{2}(\Gamma)$, $A_{\rm II}^{-1}$ *has the decomposition*

$$A_{\rm II}^{-1} = I - S_{io} - S_{oi} + 2S_{io}S_{oi}. \tag{1.16}$$

The following result uses (1.15) and results about the behaviour of S_{io} and S_{oi} in Lemmas 4.5 and 4.6 below to prove that if $n_i \neq n_o$, then the norm of $A_{\rm I}$ blows up through the quasi-resonances of the transmission problem (1.6) with $c_i = \max\{n_i, n_o\}$ and $c_o = \min\{n_i, n_o\}$. This result explains rigorously the experiments in Figures 1 and 2. The result is stated using the weighted norm $\|\cdot\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)}$ defined in §4.1, with the operator norm

$$\|\cdot\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)\to H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)} \quad \text{abbreviated to} \quad \|\cdot\|_{H_k^{1/2}\times H_k^{-1/2}}. \tag{1.17}$$

Theorem 1.11. (Superalgebraic blow up of $\|A_{\rm I}^{-1}\|$ for Ω^- smooth and convex.) *If Ω^- is C^∞ with strictly-positive curvature and $n_i \neq n_o$, then there exist frequencies $0 < k_1 < k_2 < \ldots$ with $k_j \to \infty$ such that given any $N > 0$ there exists C_N such that*

$$\|A_{\rm I}^{-1}\|_{H^{1/2}_{k_j}\times H^{-1/2}_{k_j}} \geq C_N k_j^N \quad \textit{for all } j.$$

The reason we only prove blow up of $A_{\rm I}$, and not of $A_{\rm II}$, is that Theorem 1.10 shows that $A_{\rm II}^{-1}$ involves not only S_{io} and S_{oi} but also the composition of S_{io} and S_{oi} (whereas $A_{\rm I}$ does not), and we do not currently know how to show that this extra term does not cancel out the blow up of one of S_{io} or S_{oi}.

The next result shows that, on appropriate subspaces, $A_{\rm I}^{-1}$ and $A_{\rm II}^{-1}$ involve only the physical solution operator S_{io}. In particular, this result demonstrates that, because of the specific form of the right-hand sides in (1.13), only the physical solution operator S_{io} is involved in the solution of the boundary value problem of Definition 1.1, as expected. The results for $A_{\rm II}^{-1}$ hold on either $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or $H^1(\Gamma) \times L^2(\Gamma)$, but the results for $A_{\rm I}^{-1}$ hold only on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ (since we have not proved that $A_{\rm I}^{-1}$ exists on $H^1(\Gamma) \times L^2(\Gamma)$). We use the notation that $R(P)$ is the range of the operator P.

Theorem 1.12. ($A_{\rm I}$ and $A_{\rm II}$ as operators $R(P_i^-) \to R(P_o^-)$.)
(i) $A_{\rm I}^{-1} P_o^- = A_{\rm II}^{-1} P_o^- = S_{io} P_o^-$.
(ii) Both $A_{\rm I}$ and $A_{\rm II}$ are bounded and invertible from $R(P_i^-) \to R(P_o^-)$ with $A_{\rm I}^{-1} = A_{\rm II}^{-1} = S_{io}$ as operators from $R(P_o^-) \to R(P_i^-)$.

1.5.2 Augmented BIEs

We now propose a simple way to suppress spurious quasi-resonances in the BIEs without resorting to products of integral operators. We work in the Hilbert space $\mathcal{H}$ where $\mathcal{H} := H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ for the results involving $A_{\rm I}$, and $\mathcal{H}$ equals *either* $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *or* $H^1(\Gamma) \times L^2(\Gamma)$ for the results involving $A_{\rm II}$; the norm $\|\cdot\|_{\mathcal{H}}$ is then either $\|\cdot\|_{H^{1/2}_k \times H^{-1/2}_k}$ or $\|\cdot\|_{H^1_k \times L^2}$. We equip the space $\mathcal{H}\times\mathcal{H}$ with the norm

$$\|\boldsymbol{\psi}\|^2_{\mathcal{H}\times\mathcal{H}} := \|\boldsymbol{\psi}_1\|^2_{\mathcal{H}} + \|\boldsymbol{\psi}_2\|^2_{\mathcal{H}},$$

where $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ with $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \mathcal{H}$.

Define the augmented BIOs $\widetilde{A}_{\rm I}$ and $\widetilde{A}_{\rm II} : \mathcal{H} \to \mathcal{H}\times\mathcal{H}$ by

$$\widetilde{A}_{\rm I} := \begin{pmatrix} A_{\rm I} \\ P_i^+ \end{pmatrix} \quad \text{and} \quad \widetilde{A}_{\rm II} := \begin{pmatrix} A_{\rm II} \\ P_i^+ \end{pmatrix}. \tag{1.18}$$

The idea behind introducing these augmented systems is that the solution $\boldsymbol{\gamma}_C^- u^-$ to the BIEs (1.13) satisfies $P_i^+ \boldsymbol{\gamma}_C^- u^- = 0$ (we see this below in (2.10) in the proof of Lemma 1.6).

Lemma 1.13. (Solutions of augmented BIEs.) *Let $\widetilde{A}_*$ be one of $\widetilde{A}_{\rm I}$ and $\widetilde{A}_{\rm II}$. Given $\mathbf{g} \in \mathcal{H}$, if the solution $\boldsymbol{\phi}$ to the augmented system*

$$\widetilde{A}_* \boldsymbol{\phi} = \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix} \tag{1.19}$$

exists, then $\mathbf{g}$ satisfies

$$\mathbf{g} = S_{oi}\mathbf{g} \tag{1.20}$$

and $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi} = S_{io}\mathbf{g}. \tag{1.21}$$

Lemma 1.13 shows that the solution of the augmented system (1.19), if it exists, only involves the physical solution operator S_{io}. Note that if $\mathbf{g} = P_o^- \mathbf{f}$, i.e., the right-hand side of the first- and second-kind BIEs (1.13), then (1.20) is satisfied; indeed, it follows from Lemma 3.2 below that $(S_{oi} - I)P_o^- = 0$.

Example 1.14. As in Example 1.8 we perform a "diagonalization" of the augmented BIOs of (1.18) to compute the operator norms of their pseudo-inverses in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ numerically (i.e., we compute the inverse of the smallest singular value of the block-diagonal matrix arising from truncating the Fourier/spherical-harmonic expansion). These norms as functions of the frequency

k are plotted in Figure 3 for the case $n_1 = 1$, $n_o = 3$, in which the physical solution operator S_{io} has small norm for all values of k considered (as shown by the right-hand plots of Figures 1 and 2).

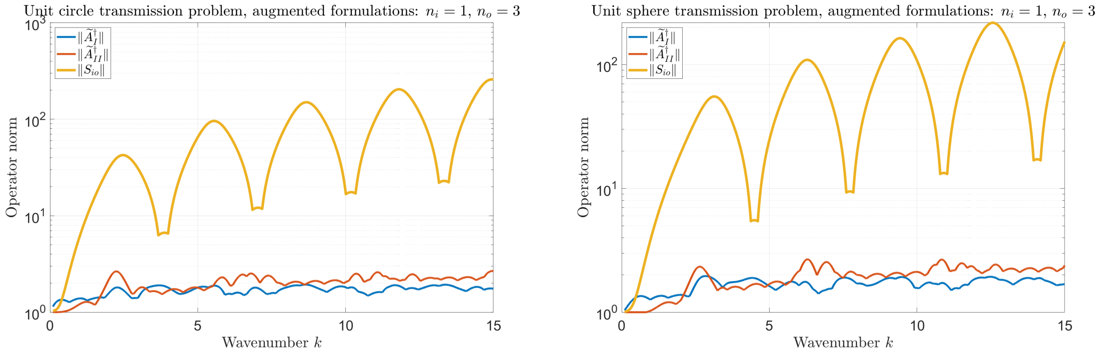

Figure 3: Plots of the operator norms of the pseudo-inverses $\widetilde{A}_{\mathrm{I}}^\dagger$, $\widetilde{A}_{\mathrm{II}}^\dagger$ of the augmented BIOs

As an agreeable surprise, we see that the norms of the pseudo-inverses of the augmented BIOs are smaller than those of S_{io} for the range of frequencies considered – augmentation has successfully removed any spurious quasi-resonances!

The following theorem rigorously explains the results in Figure 3.

Theorem 1.15. (Stability of augmented BIEs.)

$$\inf_{\phi\in\mathcal{H}\setminus\{\mathbf{0}\}} \sup_{\psi\in\mathcal{H}\setminus\{\mathbf{0}\}} \frac{\left|(\widetilde{A}_{\mathrm{I}}\phi,\psi)_{\mathcal{H}\times\mathcal{H}}\right|}{\|\phi\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}\times\mathcal{H}}} \geq \frac{1}{\sqrt{2}\max\left\{\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}, 1\right\}} \tag{1.22}$$

and

$$\inf_{\phi\in\mathcal{H}\setminus\{\mathbf{0}\}} \sup_{\psi\in\mathcal{H}\setminus\{\mathbf{0}\}} \frac{\left|(\widetilde{A}_{\mathrm{II}}\phi,\psi)_{\mathcal{H}\times\mathcal{H}}\right|}{\|\phi\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}\times\mathcal{H}}} \geq \frac{1}{\sqrt{6+4\sqrt{2}}\,\max\left\{\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}, 1\right\}}. \tag{1.23}$$

This theorem reveals that the operator norms of the pseudo-inverses $\widetilde{A}_{\mathrm{I}}^\dagger$ and $\widetilde{A}_{\mathrm{II}}^\dagger$ are bounded by $C\max\left\{\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}, 1\right\}$ for some k-independent constant $C > 0$. Hence, if the physical solution operator S_{io} is well-conditioned, then this well-conditioning carries over to the BIOs of the augmented formulations.

Remark 1.16. (The analogue of Theorem 1.10 for BIOs for scattering by impenetrable obstacles.) The analogous formulae to those in Theorem 1.10 for second-kind combined-field BIOs for solving the exterior Dirichlet, Neumann, and impedance problems were given in [11, Theorem 2.33], with formulae for certain BIOs involving operator preconditioning given in [5, Lemma 6.1]. (We note that [5, Lemma 6.1] introduced the idea of obtaining these formulae via Calderón projectors, and we prove Theorem 1.10 using this idea in §3.)

For example, the standard direct second-kind combined-field BIO for solving the exterior Dirichlet problem involves the operator $A'_\eta := \frac{1}{2}I + K' - \mathrm{i}\eta S$, for $\eta \in \mathbb{R}\setminus\{0\}$, and [11, Theorem 2.33] and [5, Lemma 6.1] prove that

$$\left(A'_\eta\right)^{-1} = I - \left(\mathrm{DtN}^+ - \mathrm{i}\eta\right)\mathrm{ItD}^{-,\eta}, \tag{1.24}$$

where DtN^+ is the exterior Dirichlet-to-Neumann map for solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition (1.1), and $\mathrm{ItD}^{-,\eta}$ is the interior impedance-to-Dirichlet map (where the impedance boundary condition is $\gamma_N^- u - \mathrm{i}\eta\gamma_D^- u = g$). Recalling that A'_η is also the standard indirect second-kind BIO for solving the interior impedance problem, we see that (1.24) expresses $(A'_\eta)^{-1}$ in terms of the solution operators for the appropriate exterior and interior problems solved using A'_η.

The standard *indirect* second-kind combined-field BIO for solving the exterior Dirichlet problem involves the operator $A_\eta := \frac{1}{2}I + K - \mathrm{i}\eta S$; this operator is also the standard direct second-kind BIO for solving the interior impedance problem, and, correspondingly,

$$\left(A_\eta\right)^{-1} = I - \mathrm{ItD}^{-,\eta}\left(\mathrm{DtN}^+ - \mathrm{i}\eta\right).$$

Remark 1.17. (Indirect BIEs) In this paper, we have considered only direct BIEs for the Helmholtz transmission problem, i.e., BIEs where the unknown is the Cauchy data of the solution. It is reasonable to expect that similar results hold for indirect BIEs for the transmission problem, just as similar decompositions into solution operators hold for the inverses of the direct BIOs for scattering by impenetrable obstacles (see the previous remark and [11, Theorem 2.33]), but we have not investigated this.

Remark 1.18. (Spurious quasi-resonances for electromagnetic BIEs) We expect that the phenomenon of spurious quasi-resonances also occurs for the BIEs for time-harmonic electromagnetic scattering; we have not pursued this in this paper however.

2 Recap of results about layer potentials, BIOs, and Calderón projectors

The single-layer and double-layer potentials, $\mathcal{V}_{i/o}$ and $\mathcal{K}_{i/o}$ respectively, are defined for $\varphi \in L^1(\Gamma)$ by

$$\mathcal{V}_{i/o}\varphi(\mathbf{x}) = \int_\Gamma \Phi_{i/o}(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\mathrm{d}s(\mathbf{y}) \quad \text{for all } \mathbf{x}\in\mathbb{R}^d\setminus\Gamma, \quad \text{and} \tag{2.1}$$

$$\mathcal{K}_{i/o}\varphi(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi_{i/o}(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\varphi(\mathbf{y})\mathrm{d}s(\mathbf{y}) \quad \text{for all } \mathbf{x}\in\mathbb{R}^d\setminus\Gamma; \tag{2.2}$$

these definitions for $\varphi \in L^1(\Gamma)$ naturally extend to $\varphi \in H^{-s}(\Gamma)$ for $s\in[0,1]$ by continuity (see, e.g., [11, Page 109]).

Lemma 2.1. *(i) If $\phi \in H^{s-1/2}(\Gamma)$ with $|s|\leq 1/2$, then $\mathcal{V}_{i/o}\phi \in H^{s+1}_{\mathrm{loc}}(\mathbb{R}^d)\cap C^2(\mathbb{R}^d\setminus\Gamma)\cap \mathrm{SRC}(k\sqrt{n_{i/o}})$.*

(i) If $\psi \in H^{s+1/2}(\Gamma)$ with $|s|\leq 1/2$, then $\mathcal{K}_{i/o}\psi \in H^{s+1}_{\mathrm{loc}}(\mathbb{R}^d\setminus\Gamma)\cap C^2(\mathbb{R}^d\setminus\Gamma)\cap \mathrm{SRC}(k\sqrt{n_{i/o}})$.

References for the proof. See, e.g., [11, Theorem 2.15]; we note that the mapping properties for $|s| = 1/2$ crucially use the harmonic analysis results of [15], [35]. □

The potentials (2.1) and (2.2) are related to the integral operators in (1.7) and (1.8) via the jump relations

$$\gamma_D^\pm\mathcal{V}_{i/o} = V_{i/o}, \quad \gamma_N^\pm\mathcal{V}_{i/o} = \mp\frac{1}{2}I + K'_{i/o}, \quad \gamma_D^\pm\mathcal{K}_{i/o} = \pm\frac{1}{2}I + K_{i/o}, \quad \gamma_N^\pm\mathcal{K}_{i/o} = -W_{i/o}; \tag{2.3}$$

see, e.g., [21, §7, Page 219]. Recall the mapping properties, valid when Γ is Lipschitz, $k\in\mathbb{C}$, and $|s|\leq 1/2$,

$$\begin{aligned} V_{i/o} &: H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma), \qquad & W_{i/o} &: H^{s+1/2}(\Gamma)\to H^{s-1/2}(\Gamma),\\ K_{i/o} &: H^{s+1/2}(\Gamma)\to H^{s+1/2}(\Gamma), \qquad & K'_{i/o} &: H^{s-1/2}(\Gamma)\to H^{s-1/2}(\Gamma); \end{aligned} \tag{2.4}$$

see, e.g., [11, Theorems 2.17 and 2.18] (similar to the results of Lemma 2.1, the mapping properties for $|s| = 1/2$ crucially use the harmonic analysis results of [15], [35]). The mapping properties (2.4) imply that $P^\pm_{i/o}$ is a bounded operator from $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ to itself and from $H^1(\Gamma)\times L^2(\Gamma)$ to itself.

We use the following notation for spaces of Helmholtz solutions:

$$\begin{aligned} \mathsf{H}^-(\kappa) &:= \big\{v\in H^1(\Omega^-)\cap C^2(\Omega^-),\ (\Delta+\kappa^2)v = 0\big\},\\ \mathsf{H}^+(\kappa) &:= \big\{v\in H^1_{\mathrm{loc}}(\Omega^+)\cap C^2(\Omega^+)\cap\mathrm{SRC}(\kappa),\ (\Delta+\kappa^2)v = 0\big\}. \end{aligned}$$

Lemma 2.2. $R(P^\pm_{i/o}) = \boldsymbol{\gamma}_C^\pm\mathsf{H}^\pm(k\sqrt{n_{i/o}})$.

Proof. By the jump relations (2.3) and the definitions of $P^\pm_{i/o}$ (1.9), with $\boldsymbol{\phi} = (\phi_1,\phi_2)$,

$$P^\pm_{i/o}\boldsymbol{\phi} = \pm\boldsymbol{\gamma}_C^\pm\big(\mathcal{K}_{i/o}\phi_1 - \mathcal{V}_{i/o}\phi_2\big); \tag{2.5}$$

see, e.g., [11, Equation 2.49]. Both when $\boldsymbol{\phi}\in H^1(\Gamma)\times L^2(\Gamma)$ and when $\boldsymbol{\phi}\in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$, the right-hand side is then the trace of an element of $\mathsf{H}^\pm(k\sqrt{n_{i/o}})$ by Lemma 2.1, so that $R(P^\pm_{i/o})\subset \boldsymbol{\gamma}_C^\pm\mathsf{H}^\pm(k\sqrt{n_{i/o}})$. To prove the reverse inclusion, given $u^\pm\in\mathsf{H}^\pm(k\sqrt{n_{i/o}})$, $u^\pm = \pm(\mathcal{K}_{i/o}\gamma_D^\pm u - \mathcal{V}_{i/o}\gamma_N^\pm u)$ by Green's integral representation (see, e.g., [11, Theorems 2.20 and 2.21]); (2.5) with $\phi_1 = \gamma_D^\pm u$ and $\phi_2 = \gamma_N^\pm u$ then implies that $\mathsf{H}^\pm(k\sqrt{n_{i/o}})\subset R(P^\pm_{i/o})$. □

The following two lemmas are proved in, e.g., [11, Page 118 and Lemma 2.22], respectively.[2]

Lemma 2.3. *$(P^+_{i/o})^2 = P^+_{i/o}$ and $(P^-_{i/o})^2 = P^-_{i/o}$ as operators* either *on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$* or *on $H^1(\Gamma) \times L^2(\Gamma)$.*

Lemma 2.4. *(i) If $v \in \mathsf{H}^-(k\sqrt{n_{i/o}})$ then*

$$P^-_{i/o}\gamma^-_C v = \gamma^-_C v. \tag{2.6}$$

(ii) If $v \in \mathsf{H}^+(k\sqrt{n_{i/o}})$ then

$$P^+_{i/o}\gamma^+_C v = \gamma^+_C v. \tag{2.7}$$

The next lemma is a converse to Lemma 2.4.

Lemma 2.5. *Let $\boldsymbol{\phi} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$* or *$H^1(\Gamma) \times L^2(\Gamma)$.*
(i) If $P^-_{i/o}\boldsymbol{\phi} = \boldsymbol{\phi}$, then $\boldsymbol{\phi} = \gamma^-_C v$ for some $v \in \mathsf{H}^-(k\sqrt{n_{i/o}})$.
(ii) If $P^+_{i/o}\boldsymbol{\phi} = \boldsymbol{\phi}$, then $\boldsymbol{\phi} = \gamma^+_C v$ for some $v \in \mathsf{H}^+(k\sqrt{n_{i/o}})$.

Proof. (i) Given $\boldsymbol{\phi}$ such that $P^-_{i/o}\boldsymbol{\phi} = \boldsymbol{\phi}$, let

$$v(\mathbf{x}) = -\big(\mathcal{K}_{i/o}\phi_1 - \mathcal{V}_{i/o}\phi_2\big)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega^-. \tag{2.8}$$

By Lemma 2.1, $v \in \mathsf{H}^-(k\sqrt{n_{i/o}})$. We therefore only need to check that $\boldsymbol{\phi} = \gamma^-_C v$. However, by (2.5) and the definition of v (2.8), $\boldsymbol{\phi} = P^-_{i/o}\boldsymbol{\phi} = \gamma^-_C v$.

(ii) Given $\boldsymbol{\phi}$ such that $P^+_{i/o}\boldsymbol{\phi} = \boldsymbol{\phi}$, let

$$v(\mathbf{x}) = \big(\mathcal{K}_{i/o}\phi_1 - \mathcal{V}_{i/o}\phi_2\big)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega^+. \tag{2.9}$$

Similar to in (i), $v \in \mathsf{H}^+(k\sqrt{n_{i/o}})$, and, by (2.5) and the definition of v (2.9), $\boldsymbol{\phi} = P^+_{i/o}\boldsymbol{\phi} = \gamma^+_C v$. ☐

Proof of Lemma 1.6. By (1.4) and (2.6), $P^-_i\gamma^-_C u^- = \gamma^-_C u^-$, so that, by (1.10),

$$P^+_i\gamma^-_C u^- = 0. \tag{2.10}$$

Similarly, by (1.4), (2.7), and (1.10),

$$P^-_o\gamma^+_C u^+ = 0. \tag{2.11}$$

Applying P^-_o to the transmission condition $\gamma^-_C u^- = \gamma^+_C u^+ + \mathbf{f}$ in (1.4) and using (2.11), we find that

$$P^-_o(\gamma^-_C u^-) = P^-_o\mathbf{f}. \tag{2.12}$$

Subtracting (2.10) from (2.12), we obtain the first-kind BIE in (1.13). Adding (2.10) to (2.12), we obtain the second-kind BIE in (1.13). ☐

Remark 2.6. If $\mathbf{f} = \gamma^-_C u^I$ with u^I satisfying (1.3), then $P^-_o\gamma^-_C u^I = \gamma^-_C u^I$ by (2.6), and thus the right-hand sides of (1.13) are just $\gamma^-_C u^I$.

Lemma 2.7. *Let A_* equal either $A_{\rm I}$ or $A_{\rm II}$. Then A_* is an injective, bounded operator on* either *$H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$* or *$H^1(\Gamma) \times L^2(\Gamma)$.*

Proof. The boundedness of A_* follows from the expressions (1.11)/(1.12) and the boundedness of $P^\pm_{i/o}$. Injectivity follows by repeating the arguments in the proof of Theorem 1.10 below with $\mathbf{g} = \mathbf{0}$ (these arguments use uniqueness of the Helmholtz transmission problem of Definition 1.1). ☐

Proof of Lemma 1.7. The result for $A_{\rm I}$ follows from Lemma 2.7 combined with the coercivity result of, e.g., [13, Theorem 7.27] (see [13, Corollary 7.28]).

The results for $A_{\rm II}$ follows from Lemma 2.7 combined with the fact that $A_{\rm II} - I$ is compact on both $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ dand $H^1(\Gamma) \times L^2(\Gamma)$. This latter result follows if we can show that

- $K_i - K_o$ is compact $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $H^1(\Gamma) \to H^1(\Gamma)$,

[2] Strictly speaking, [11, §2.5] only considers $P^\pm_{i/o}$ as operators on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, but the proofs of the results on $H^1(\Gamma) \times L^2(\Gamma)$ are the same.

- $V_i - V_o$ is compact $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $L^2(\Gamma) \to H^1(\Gamma)$,
- $W_i - W_o$ is compact $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $H^1(\Gamma) \to L^2(\Gamma)$, and
- $K_i' - K_o'$ is compact $H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $L^2(\Gamma) \to L^2(\Gamma)$.

Since $\Phi_i - \Phi_o = (\Phi_i - \Phi_0) - (\Phi_o - \Phi_0)$, where Φ_0 is the Laplace fundamental solution, these mapping properties follow from the bounds on the difference of the Helmholtz and Laplace fundamental solutions in [11, Equation 2.25] and the fact that the inclusion $H^s(\Gamma) \to H^t(\Gamma)$ is compact for $-1 \le t \le s \le 1$. □

3 Proof of Theorems 1.10 and 1.12

Lemma 3.1. *Given* $\boldsymbol{\phi}, \mathbf{f}$ *in* either $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or $H^1(\Gamma) \times L^2(\Gamma)$,

$$\boldsymbol{\phi} = S_{io}\mathbf{f} \quad \text{if and only if} \quad \begin{cases} P_i^- \boldsymbol{\phi} = \boldsymbol{\phi}, \text{ and} \\ P_o^-(\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0}. \end{cases} \tag{3.1}$$

Similarly,

$$\boldsymbol{\phi} = S_{oi}\mathbf{f} \quad \text{if and only if} \quad \begin{cases} P_o^- \boldsymbol{\phi} = \boldsymbol{\phi}, \text{ and} \\ P_i^-(\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0}. \end{cases} \tag{3.2}$$

Proof. We prove (3.1); the proof of (3.2) is the same with i and o swapped.

We first prove the forward implication in (3.1). Given $\mathbf{f}$, let u be as in the definition of S_{io} (Definition 1.4), i.e., u satisfies (1.6) with $c_i = n_i$ and $c_o = n_o$. By definition $\boldsymbol{\phi} = \boldsymbol{\gamma}_C^- u$, so $P_i^- \boldsymbol{\phi} = \boldsymbol{\phi}$ by (2.6). The jump condition in (1.6) implies that $\boldsymbol{\phi} - \mathbf{f} = \boldsymbol{\gamma}_C^+ u$, and (2.7) then implies that $P_o^+(\boldsymbol{\phi} - \mathbf{f}) = \boldsymbol{\phi} - \mathbf{f}$.

For the reverse implication in (3.1), given $\boldsymbol{\phi}$ satisfying the right-hand side of (3.1), Part (i) of Lemma 2.5 implies that $\boldsymbol{\phi} = \boldsymbol{\gamma}_C^- w^-$ for some $w^- \in \mathsf{H}^-(k\sqrt{n_i})$. Similarly, Part (ii) of Lemma 2.5 implies that $\boldsymbol{\phi} - \mathbf{f} = \boldsymbol{\gamma}_C^+ w^+$ for some $w^+ \in \mathsf{H}^+(k\sqrt{n_o})$. Let $w := w^+$ in Ω^+ and $w := w^-$ in Ω^-. Then $\boldsymbol{\gamma}_C^- w^- - \boldsymbol{\gamma}_C^+ w^+ = \boldsymbol{\phi} - (\boldsymbol{\phi} - \mathbf{f}) = \mathbf{f}$. Since the solution of the transmission problem is unique, w equals the function u in the definition of S_{io} (i.e., Definition 1.4 with $c_i = n_i$ and $c_o = n_o$), and $\boldsymbol{\phi} = \boldsymbol{\gamma}_C^- w^- = \boldsymbol{\gamma}_C^- u^- = S_{io}\mathbf{f}$. □

We now prove Theorem 1.10.

Proof of the result (1.15) *in Theorem 1.10.* Assume that $\boldsymbol{\psi}, \mathbf{g} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or $H^1(\Gamma) \times L^2(\Gamma)$ with $A_{\mathrm{I}}\boldsymbol{\psi} = \mathbf{g}$, i.e.,

$$(P_i^- - P_o^+)\boldsymbol{\psi} = \mathbf{g}. \tag{3.3}$$

Step 1: apply P_i^- *to* (3.3).
Applying P_i^- to (3.3) and using the fact that P_i^- is a projection (by Lemma 2.3), we have

$$P_i^-\big(\boldsymbol{\psi} - P_o^+\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0},$$

that is, by (1.10),

$$P_i^-\big(P_o^-\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0}, \tag{3.4}$$

Let $\boldsymbol{\phi} = P_o^-\boldsymbol{\psi}$ and let $\mathbf{f} = \mathbf{g}$. Then by Lemma 3.1 and the fact that P_o^- is a projection, $\boldsymbol{\phi} = S_{oi}\mathbf{g}$, i.e.

$$P_o^-\boldsymbol{\psi} = S_{oi}\mathbf{g}. \tag{3.5}$$

Step 2: apply P_o^- *to* (3.3).
Applying P_o^- to (3.3) and using the fact that P_o^- is a projection (so that, in particular, $P_o^- P_o^+ = 0$), we have

$$P_o^-\big(P_i^-\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0}. \tag{3.6}$$

Let $\boldsymbol{\phi} = P_i^-\boldsymbol{\psi}$ and let $\mathbf{f} = \mathbf{g}$. Then by Lemma 3.1 and the fact that P_i^- is a projection, $\boldsymbol{\phi} = S_{io}\mathbf{g}$, i.e.

$$P_i^-\boldsymbol{\psi} = S_{io}\mathbf{g}. \tag{3.7}$$

Step 3: use (1.10) *and* (3.3) *and the results of Steps 1 and 2.*
By (1.10), (3.5), (3.3), and (3.7) (in that order),

$$\boldsymbol{\psi} = (P_o^- + P_o^+)\boldsymbol{\psi} = S_{oi}\mathbf{g} + P_i^-\boldsymbol{\psi} - \mathbf{g} = (S_{oi} + S_{io} - I)\mathbf{g}, \tag{3.8}$$

which is the result (1.15). □

Proof of the result (1.16) *in Theorem 1.10.* Assume that $\boldsymbol{\psi}, \mathbf{g} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or $H^1(\Gamma) \times L^2(\Gamma)$ with $A_{\rm II}\boldsymbol{\psi} = \mathbf{g}$, i.e.,

$$(P_o^- + P_i^+)\boldsymbol{\psi} = \mathbf{g}. \tag{3.9}$$

Step 1: apply P_i^- to (3.9).
Applying P_i^- to (3.9) and using the fact that P_i^- is a projection (by Lemma 2.3), we see that (3.4) holds. Let $\boldsymbol{\phi} = P_o^-\boldsymbol{\psi}$ and let $\mathbf{f} = \mathbf{g}$. Then by Lemma 3.1 and the fact that P_o^- is a projection, $\boldsymbol{\phi} = S_{oi}\mathbf{g}$, i.e., (3.5) holds.

Step 2: apply P_o^- to (3.9).
Applying P_o^- to (3.9) and using the fact that P_o^- is a projection, we see that

$$P_o^-\big(\boldsymbol{\psi} + P_i^+\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0}.$$

Let $\widetilde{\boldsymbol{\phi}} = P_i^-\boldsymbol{\psi}$, so that

$$P_o^-\big(\widetilde{\boldsymbol{\phi}} + 2P_i^+\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0}.$$

Let $\widetilde{\mathbf{f}} = -2P_i^+\boldsymbol{\psi} + \mathbf{g}$. Then by Lemma 3.1 and the fact that P_i^- is a projection, $\widetilde{\boldsymbol{\phi}} = S_{io}\widetilde{\mathbf{f}}$, i.e.,

$$P_i^-\boldsymbol{\psi} = S_{io}\big(-2P_i^+\boldsymbol{\psi} + \mathbf{g}\big).$$

Using (3.9) and then (3.5), which holds by *Step 1*, we have

$$P_i^-\boldsymbol{\psi} = S_{io}\big(2P_o^-\boldsymbol{\psi} - \mathbf{g}\big) = S_{io}\big(2S_{oi}\mathbf{g} - \mathbf{g}\big). \tag{3.10}$$

Step 3: use (1.10), (3.9)*, and the results of Steps 1 and 2.*
By (1.10) and (3.5),

$$\boldsymbol{\psi} = (P_o^- + P_o^+)\boldsymbol{\psi} = S_{oi}\mathbf{g} + P_o^+\boldsymbol{\psi}. \tag{3.11}$$

Using (1.10) in (3.9) and rearranging, we have

$$P_o^+\boldsymbol{\psi} = -P_i^-\boldsymbol{\psi} + 2\boldsymbol{\psi} - \mathbf{g},$$

and using this in (3.11) we find that

$$\boldsymbol{\psi} = \big(I - S_{oi}\big)\mathbf{g} + P_i^-\boldsymbol{\psi};$$

the result (1.16) then follows from using (3.10). □

To prove Theorem 1.12, we need the following consequences of the definitions of S_{io} and S_{oi}.

Lemma 3.2. *$S_{io}P_o^+ = 0$ and $S_{io}P_i^- = P_i^-$ as operators on* either $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or $H^1(\Gamma) \times L^2(\Gamma)$*. Similarly, $S_{oi}P_i^+ = 0$ and $S_{oi}P_o^- = P_o^-$.*

Proof. We prove the relationships involving S_{io}; the proofs of those involving S_{oi} are completely analogous. Given $\mathbf{f} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ or $H^1(\Gamma) \times L^2(\Gamma)$, by Definition 1.4, $S_{io}P_o^+\mathbf{f} = \boldsymbol{\gamma}_C^- v^-$ where

$$v^+ \in \mathsf{H}^+(k\sqrt{n_o}), \quad v^- \in \mathsf{H}^-(k\sqrt{n_i}), \quad \text{and} \quad \boldsymbol{\gamma}_C^- v^- = \boldsymbol{\gamma}_C^+ v^+ + P_o^+\mathbf{f}. \tag{3.12}$$

By Lemma 2.2 and (2.5), there exists $w^+ \in \mathsf{H}^+(k\sqrt{n_o})$ such that $\boldsymbol{\gamma}_C^+ w = P_o^+\mathbf{f}$. Thus $v^- = 0$ and $v^+ = -w$ is a solution of (3.12), and by uniqueness of the Helmholtz transmission problem (Lemma 1.2) it is the only solution. Therefore $S_{io}P_o^+\mathbf{f} = \boldsymbol{\gamma}_C^- v^- = \mathbf{0}$.

The proof that $S_{io}P_i^- = P_i^-$ is similar. Indeed, again using uniqueness of the Helmholtz transmission problem, we have $S_{io}P_i^-\mathbf{f} = \boldsymbol{\gamma}_C^- w^-$ with $w^- \in \mathsf{H}^-(k\sqrt{n_i})$ and $\boldsymbol{\gamma}_C^- w^- = P_i^-\mathbf{f}$. □

Proof of Theorem 1.12. Part (i): By (1.15), $A_{\rm I}^{-1}P_o^- = (S_{io} + S_{oi} - I)P_o^-$. Lemma 3.2 shows that $(S_{oi} - I)P_o^- = 0$, and thus $A_{\rm I}^{-1}P_o^- = S_{io}P_o^-$. The equation $A_{\rm II}^{-1}P_o^- = S_{io}P_o^-$ follows similarly.

Part (ii): By the second equality in (1.11), Lemma 2.3, and (1.10),

$$A_{\rm I}P_i^- = (P_i^- - P_o^+)P_i^- = (I - P_o^+)P_i^- = P_o^-P_i^-,$$

so that $A_{\rm I} : R(P_i^-) \to R(P_o^-)$. Similarly, by (1.12),

$$A_{\rm II}P_i^- = (P_o^- + P_i^+)P_i^- = P_o^-P_i^-,$$

so that $A_{\rm II} : R(P_i^-) \to R(P_o^-)$. By Definition 1.4, S_{io} maps into the space of Cauchy data of $\mathsf{H}^-(k\sqrt{n_i})$; by (2.6) and Lemma 2.2, this space is $R(P_i^-)$. Therefore, by Part (i), both $A_{\rm I}^{-1}$ and $A_{\rm II}^{-1}$ map $R(P_o^-) \to R(P_i^-)$ and both equal S_{io} as operators between these spaces. □

4 Proof of Theorem 1.11

Throughout this section we use the notation that $a \lesssim b$ if there exists $C > 0$, independent of k, such that $a \leq Cb$. We write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$.

4.1 Definitions of k-weighted norms and associated results

For $\boldsymbol{\phi} \in H^1(\Gamma) \times L^2(\Gamma)$ with $\boldsymbol{\phi} = (\phi_1, \phi_2)$, let $\nabla_T \phi_1$ be the tangential gradient of ϕ_1 on Γ and

$$\|\phi_1\|^2_{H^1_k(\Gamma)} := \|\nabla_T \phi_1\|^2_{L^2(\Gamma)} + k^2 \|\phi_1\|^2_{L^2(\Gamma)}, \qquad \|\boldsymbol{\phi}\|^2_{H^1_k(\Gamma)\times L^2(\Gamma)} := \|\phi_1\|^2_{H^1_k(\Gamma)} + \|\phi_2\|^2_{L^2(\Gamma)}.$$

Define $H^{1/2}_k(\Gamma)$ by interpolation between $H^1_k(\Gamma)$ and $L^2(\Gamma)$ and then $H^{-1/2}_k(\Gamma)$ by duality. As in §1, we use the abbreviation (1.17).

For a bounded Lipschitz open set $D \subset \mathbb{R}^d$, let

$$\|v\|^2_{H^1_k(D)} := \|\nabla v\|^2_{L^2(D)} + k^2 \|v\|^2_{L^2(D)}.$$

Fix $k_0 > 0$. Then, with $H^{1/2}_k(\partial D)$ defined above, by, e.g., [26, Theorem 5.6.4],

$$\|\gamma_D v\|_{H^{1/2}_k(\partial D)} \lesssim \|v\|_{H^1_k(D)} \qquad \text{for all } v \in H^1(D) \text{ and } k \geq k_0, \tag{4.1}$$

and there exists $E : H^{1/2}(D) \to H^1(D)$ such that

$$\gamma_D E\phi = \phi \quad \text{and} \quad \|E\phi\|_{H^1_k(D)} \lesssim \|\phi\|_{H^{1/2}_k(\partial D)}. \tag{4.2}$$

Lemma 4.1. *If $v \in H^1(D, \Delta)$ with $(\Delta + k^2 c)v = 0$ and $k \geq k_0$, then*

$$\|\gamma_N v\|_{H^{-1/2}_k(\partial D)} \lesssim \|v\|_{H^1_k(D)} \tag{4.3}$$

(where the omitted constant depends on c and k_0).

Sketch proof of Lemma 4.1. This follows by repeating the argument in, e.g., [21, Lemma 4.3] (which starts from the definition of the Neumann trace via Green's identity) and then using weighted norms and, in particular, the bound (4.2). □

4.2 From resolvent estimates to bounds on S_{io} and S_{oi}

Lemma 4.2. *Given c_o, c_i, R positive real numbers and $\mathbf{g} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, let $v \in H^1_{\rm loc}(\mathbb{R}^d \setminus \Gamma) \cap {\rm SRC}(k\sqrt{c_o})$ satisfy*

$$\begin{aligned}(\Delta + k^2 c_i)v^- &= 0 && \text{in } \Omega^-,\\ (\Delta + k^2 c_o)v^+ &= 0 && \text{in } \Omega^+,\\ \boldsymbol{\gamma}^-_C v^- &= \boldsymbol{\gamma}^+_C v^+ + \mathbf{g} && \text{on } \Gamma.\end{aligned}$$

Assume that, for all $f \in L^2(\mathbb{R}^d)$ with ${\rm supp}\, f \subset B_R$ and $w \in H^1_{\rm loc}(\mathbb{R}^d \setminus \Gamma) \cap {\rm SRC}(k\sqrt{c_o})$ that satisfy

$$\begin{aligned}(\Delta + k^2 c_i)w^- &= f^- && \text{in } \Omega^-,\\ (\Delta + k^2 c_o)w^+ &= f^+ && \text{in } \Omega^+,\\ \boldsymbol{\gamma}^-_C w^- &= \boldsymbol{\gamma}^+_C w^+ && \text{on } \Gamma,\end{aligned}$$

the following bound holds:

$$\|w\|_{H^1_k(B_R)} \leq C_{\rm sol}(k, R, c_i, c_o)\, \|f\|_{L^2(B_R)}. \tag{4.4}$$

Then

$$\|v\|_{H^1_k(B_R)} \lesssim k\big(1 + C_{\rm sol}(k, R, c_i, c_o)\big)\, \|g\|_{H^{1/2}_k(\Gamma)\times H^{-1/2}_k(\Gamma)}. \tag{4.5}$$

Corollary 4.3. *Under the assumptions of Lemma 4.2,*

$$\|S(c_i, c_o)\|_{H^{1/2}_k \times H^{-1/2}_k} \lesssim k\big(1 + C_{\rm sol}(k, R, c_i, c_o)\big).$$

Proof of Corollary 4.3 from Lemma 4.2. This follows from combining the result of Lemma 4.2 and the trace results (4.1) and (4.3). □

Proof of Lemma 4.2. Given R' satisfying $\mathrm{diam}(\Omega^-) < R' < R$, let $u \in H^1(B_{R'} \setminus \Gamma)$ be the solution to

$$\begin{aligned} (\Delta + (k^2 + \mathrm{i}k)c_i)u^- &= 0 && \text{in } \Omega^-, \\ (\Delta + (k^2 + \mathrm{i}k)c_o)u^+ &= 0 && \text{in } \Omega^+ \cap B_{R'}, \\ \boldsymbol{\gamma}_C^- u^- &= \boldsymbol{\gamma}_C^+ u^+ + \mathbf{g} && \text{on } \Gamma, \\ \gamma_D u^+ &= 0 && \text{on } \partial B_{R'}. \end{aligned} \tag{4.6}$$

(this choice of auxiliary problem is motivated by the proof of [5, Theorem 3.5] using [5, Lemma 3.3]). We prove below that

$$\|u\|_{H^1_k(B_{R'})} \lesssim k \|\mathbf{g}\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)}. \tag{4.7}$$

Extend u by zero to a function in $H^1(\mathbb{R}^d \setminus \Gamma)$ and let $w = v - u$. Then $w \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma) \cap \mathrm{SRC}(c_o k)$ satisfies

$$\begin{aligned} (\Delta + k^2 c_i)w^- &= \mathrm{i}k c_i u^- && \text{in } \Omega^-, \\ (\Delta + k^2 c_o)w^+ &= \mathrm{i}k c_o u^+ && \text{in } \Omega^+, \\ \boldsymbol{\gamma}_C^- w^- &= \boldsymbol{\gamma}_C^+ w^+ && \text{on } \Gamma. \end{aligned}$$

Using the fact that $w = v - u$, the fact that $\mathrm{supp}\, u \subset B_R$ (by construction), and the bound (4.4), we have

$$\|v\|_{H^1_k(B_R)} \le \|w\|_{H^1_k(B_R)} + \|u\|_{H^1_k(B_R)} \le C_{\mathrm{sol}}(k, R, c_i, c_o) k \|u\|_{L^2(B_R)} + \|u\|_{H^1_k(B_R)};$$

the result (4.5) then follows from the bound (4.7).

It therefore remains to prove (4.7). Applying Green's identity to u^- in Ω^- and u^+ in $\Omega^+ \cap B_{R'}$, we obtain

$$\left\|\nabla u^-\right\|^2_{L^2(\Omega^-)} - (k^2 + \mathrm{i}k)c_i \left\|u^-\right\|^2_{L^2(\Omega^-)} = \langle \gamma_N^- u^-, \gamma_D^- u^- \rangle_\Gamma, \tag{4.8}$$

$$\left\|\nabla u^+\right\|^2_{L^2(\Omega^+\cap B_{R'})} - (k^2 + \mathrm{i}k)c_o \left\|u^+\right\|^2_{L^2(\Omega^+\cap B_{R'})} = -\langle \gamma_N^+ u^+, \gamma_D^+ u^+ \rangle_\Gamma. \tag{4.9}$$

The jump condition in (4.6) implies that, with $\mathbf{g} = (g_D, g_N)$,

$$\langle \gamma_N^- u^-, \gamma_D^- u^- \rangle_\Gamma - \langle \gamma_N^+ u^+, \gamma_D^+ u^+ \rangle_\Gamma = \langle g_N, \gamma_D^- u^- \rangle_\Gamma + \langle \gamma_N^+ u, g_D \rangle_\Gamma. \tag{4.10}$$

Therefore, adding (4.8) and (4.9), taking the imaginary part, and then using the Cauchy-Schwarz inequality on terms arising from (4.10), we obtain

$$\min\{c_i, c_o\} k \|u\|^2_{L^2(B_{R'})} \le \|g_N\|_{H_k^{-1/2}(\Gamma)} \left\|\gamma_D^- u^-\right\|_{H_k^{1/2}(\Gamma)} + \left\|\gamma_N^+ u^+\right\|_{H_k^{-1/2}(\Gamma)} \|g_D\|_{H_k^{1/2}(\Gamma)}. \tag{4.11}$$

Adding (4.8) and (4.9), taking the real part, adding an sufficiently-large multiple of k times (4.11), and then using the Cauchy-Schwarz inequality on terms arising from (4.10), we have

$$\|u\|^2_{H^1_k(B_{R'})} \lesssim k\Big(\|g_N\|_{H_k^{-1/2}(\Gamma)} \left\|\gamma_D^- u^-\right\|_{H_k^{1/2}(\Gamma)} + \left\|\gamma_N^+ u^+\right\|_{H_k^{-1/2}(\Gamma)} \|g_D\|_{H_k^{1/2}(\Gamma)} \Big). \tag{4.12}$$

The bound (4.7) then follows from using the inequality

$$2ab \le \epsilon a^2 + \epsilon^{-1} b^2, \quad a, b, \epsilon > 0, \tag{4.13}$$

and the trace bounds (4.1) and (4.3) in the right-hand side of (4.12). □

4.3 k-explicit bounds on S_{io} and S_{oi}

We recall the notions of *star-shaped* and *star-shaped with respect to a ball.*

Definition 4.4. *(i)* Ω^- *is* star-shaped with respect to the point $\mathbf{x}_0$ *if, whenever* $\mathbf{x} \in \Omega^-$*, the segment* $[\mathbf{x}_0, \mathbf{x}] \subset \Omega^-$.
(ii) Ω^- *is* star-shaped with respect to the ball $B_a(\mathbf{x}_0)$ *if it is star-shaped with respect to every point in* $B_a(\mathbf{x}_0)$.

Lemma 4.5. ("Good" behaviour of S_{io} when $n_i < n_o$.) *If* Ω^- *is star-shaped with respect to a ball and* $n_i < n_o$*, then, given* $k_0 > 0$,

$$\|S_{io}\|_{H_k^{1/2}\times H_k^{-1/2}\to H_k^{1/2}\times H_k^{-1/2}} \lesssim k \quad \textit{for all } k \geq k_0.$$

Proof. [23, Theorem 3.2] proves that (4.4) holds with $C_{\rm sol}(k) \sim 1$, and the result then follows from Corollary 4.3. □

Lemma 4.6. ("Bad" behaviour of S_{io} when $n_i > n_o$.) *If* Ω^- *is* C^∞ *with strictly-positive curvature and* $n_i > n_o$*, then there exist* $0 < k_1 < k_2 < \ldots$ *with* $k_j \to \infty$ *such that given any* $N > 0$ *there exists* $C_N > 0$ *such that*

$$\|S_{io}\|_{H_k^{1/2}\times H_k^{-1/2}} \geq C_N k_j^N \quad \textit{for all } j.$$

In the proof of Lemma 4.6, we use the notation that $a = O(k^{-\infty})$ as $k \to \infty$ if, given $N > 0$, there exists C_N, k_0 such that $|a| \leq C_N k^{-N}$ for all $k \geq k_0$, i.e. a decreases superalgebraically in k.

The ideas behind Lemma 4.6 are that (i) if there exist quasimodes with $O(k^{-\infty})$ remainder (in the sense of (4.15) below), then the norm of S_{io} has $O(k^\infty)$ blow up (immediately from the definitions of quasimodes and S_{io}), and (ii) if Ω^- is C^∞ with strictly positive curvature and $n_i > n_o$ then quasimodes with $O(k^{-\infty})$ remainder exist by [28]. To prove Lemma 4.6, we need the following bounds on the Newtonian potential, i.e., integration against the fundamental solution. Let

$$\mathcal{N}_{i/o} f(x) := \int_{\mathbb{R}^d} \Phi_{i/o}(x, y) f(y) \, {\rm d}y.$$

Lemma 4.7. *Given* $f \in L^2(\mathbb{R}^d)$ *with* $\operatorname{supp} f \subset B_R$ *and* $k_0 > 0$,

$$\frac{1}{k}|\mathcal{N}_{i/o} f|_{H^2(B_R)} + \|\mathcal{N}_{i/o} f\|_{H_k^1(B_R)} \lesssim \|f\|_{L^2(B_R)}$$

for all $k \geq k_0$*, where the omitted constant depends on* $n_{i/o}$ *and* R.

References for the proof of Lemma 4.7. See, e.g., [17, Theorem 3.1] for $d = 3$ and [18, Theorem 14.3.7] for arbitrary dimension (note that [18, Theorem 14.3.7] is for fixed k, but a rescaling of the independent variable yields the result for arbitrary k). □

Proof of Lemma 4.6. We need to show that there exist $\mathbf{g}_j \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $j = 1, 2, \ldots$, such that the solutions v_j to (1.6) (with $c_i = n_i$ and $c_o = n_o$ and $\mathbf{f} = \mathbf{g}_j$) are such that given $N > 0$ there exists $C_N > 0$ such that

$$\big\|\boldsymbol{\gamma}_C^- v_j^-\big\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)} \geq C_N k_j^N \, \|\mathbf{g}_j\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)} \qquad \text{for all } j;$$

i.e., that

$$\frac{\|\mathbf{g}_j\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)}}{\big\|\boldsymbol{\gamma}_C^- v_j^-\big\|_{H_k^{1/2}(\Gamma)\times H_k^{-1/2}(\Gamma)}} = O(k_j^{-\infty}) \quad \text{as } j \to \infty. \tag{4.14}$$

By [28], there exist $k_j \in \mathbb{C}$, with $|k_j| \to \infty$, $0 > \Im k_j = O(k_j^{-\infty})$, $w_j^\pm \in C^\infty(\overline{\Omega^\pm})$ with the support of $w_j^\pm$ contained in a fixed compact neighbourhood of Γ, and such that $\|\gamma_D^- u_j^-\|_{L^2(\Gamma)} = 1$,

$$\big\|(\Delta + k_j^2 n_{i/o}) w_j^\pm\big\|_{L^2(\Omega^\pm)} = O(k_j^{-\infty}), \qquad \big\|\boldsymbol{\gamma}_C^- w_j^- - \boldsymbol{\gamma}_C^+ w_j^+\big\|_{H^2(\Gamma)\times H^2(\Gamma)} = O(k_j^{-\infty}), \tag{4.15}$$

as $j \to \infty$. We now claim that we can

1. change the normalisation from $\|\gamma_D^- u_j^-\|_{L^2(\Gamma)} = 1$ to $\|\gamma_D^- u_j^-\|_{H^{1/2}_{|k_j|}(\Gamma)} = |k_j|^{1/2}$ (or indeed any finite power of $|k_j|$), and
2. assume, without loss of generality, that $k_j \in \mathbb{R}$ for all j.

Indeed, [23, Corollary 6.1] shows that the results of [28] imply existence of a quasimode normalised by $\|\gamma_D^- u_j^-\|_{H^1_{|k_j|}(\Gamma)} = |k_j|$, and then [23, Corollary 6.2] shows that this implies existence of a quasimode with $k_j \in \mathbb{R}$ for all j, normalised by $\|\gamma_D^- u_j^-\|_{H^1_{k_j}(\Gamma)} = k_j$. The claim then follows by repeating these arguments with

- the bound on the Dirichlet-to-Neumann map from $H^1_k(\Gamma) \to L^2(\Gamma)$ from [24, Lemma 5] replaced by the analogous bound from $H_k^{1/2}(\Gamma) \to H_k^{-1/2}(\Gamma)$ obtained by interpolation (see, e.g., [12, Lemma 4.2]), and
- the bounds on the $L^2(\Gamma) \to L^2_{\rm comp}(\mathbb{R}^d)$ norms of $\mathcal{V}_{i/o}$ and $\mathcal{K}_{i/o}$ from [30, Lemma 4.3] replaced by analogous bounds on the $H_k^{-1/2}(\Gamma) \to L^2_{\rm comp}(\mathbb{R}^d)$ and $H_k^{1/2}(\Gamma) \to L^2_{\rm comp}(\mathbb{R}^d)$ norms, respectively; these bounds are proved using the bounds in Lemma 4.7, the trace result (4.1), and similar arguments to those in the proof of Lemma 4.1.

(Note that in both these points, the precise algebraic powers of k_j don't matter, since they are dominated by the $O(k_j^{-\infty})$ coming from the quasimode.)

With the changes in Points 1 and 2 above, we now let

$$v_j^- := w_j^- + \mathcal{N}_i\big((\Delta + k_j^2 n_i) w_j^-\big) \quad \text{and} \quad v_j^+ := w_j^+ + \mathcal{N}_o\big((\Delta + k_j^2 n_o) w_j^+\big) \tag{4.16}$$

(where the arguments of $\mathcal{N}_i$ and $\mathcal{N}_o$ are extended by zero outside their supports), and observe that, since w_j^+ has compact support, $v_j^+ \in \mathrm{SRC}(k_j\sqrt{n_o})$. Let

$$\mathbf{g}_j := \boldsymbol{\gamma}_C^- v_j^- - \boldsymbol{\gamma}_C^+ v_j^+;$$

then v satisfies (1.6) with $c_i = n_i$ and $c_o = n_o$ and $\mathbf{f} = \mathbf{g}_j$.

We now show that (4.14) holds. On the one hand, by the definition of $\mathbf{g}_j$, the second equation in (4.15), Lemma 4.7, and the first equation in (4.15).

$$\begin{aligned}
\|\mathbf{g}_j\|_{H^{1/2}_{k_j}(\Gamma)\times H^{-1/2}_{k_j}(\Gamma)} &\le \big\|\boldsymbol{\gamma}_C^- w_j^- - \boldsymbol{\gamma}_C^+ w_j^+\big\|_{H^{1/2}_{k_j}(\Gamma)\times H^{-1/2}_{k_j}(\Gamma)} + \big\|\boldsymbol{\gamma}_C^- \mathcal{N}_i\big((\Delta + k_j^2 n_i) w_j^-\big)\big\|_{H^{1/2}_{k_j}(\Gamma)\times H^{-1/2}_{k_j}(\Gamma)} \\
&\quad + \big\|\boldsymbol{\gamma}_C^+ \mathcal{N}_o\big((\Delta + k_j^2 n_o) w_j^+\big)\big\|_{H^{1/2}_{k_j}(\Gamma)\times H^{-1/2}_{k_j}(\Gamma)}, \\
&= O(k_j^{-\infty}) \quad \text{as } j \to \infty.
\end{aligned} \tag{4.17}$$

On the other hand, using (4.16), the normalisation $\|\gamma_D^- w_j^-\|_{H^{1/2}_{k_j}(\Gamma)} = k_j^{1/2}$, Lemma 4.7, and (4.15) (in that order), we have

$$\begin{aligned}
\big\|\boldsymbol{\gamma}_C^- v_j^-\big\|_{H^{1/2}_{k_j}(\Gamma)\times H^{-1/2}_{k_j}(\Gamma)} \ge \big\|\gamma_D^- v_j^-\big\|_{H^{1/2}_{k_j}(\Gamma)} &\ge \big\|\gamma_D^- w_j^-\big\|_{H^{1/2}_{k_j}(\Gamma)} - \big\|\gamma_D^- \mathcal{N}_i\big((\Delta + k_j^2 n_i) w_j^-\big)\big\|_{H^{1/2}_{k_j}(\Gamma)}, \\
&= k_j^{1/2} + O(k_j^{-\infty}) \quad \text{as } j \to \infty.
\end{aligned} \tag{4.18}$$

The bound (4.14) (and hence the result) follows by combining (4.17) and (4.18). □

Proof of Theorem 1.11. This follows by combining Theorem 1.10, Lemma 4.5, and Lemma 4.6. Indeed, if $n_i < n_o$, then S_{io} has "good" behaviour via Lemma 4.5, but S_{oi} has "bad" behaviour via Lemma 4.6. If $n_i > n_o$, then S_{io} has "bad" behaviour via Lemma 4.6, and S_{oi} has "good" behaviour via Lemma 4.5. □

We also record the following upper bound on $\big\|A_{\rm I}^{-1}\big\|_{H_k^{1/2}\times H_k^{-1/2}}$, valid for all Lipschitz Ω^-.

Theorem 4.8. (Inverse is algebraically bounded in frequency for almost all frequencies.) *Given positive real numbers n_i, n_o, k_0, δ, and ε, there exists a set $J \subset [k_0, \infty)$ with $|J| \le \delta$ and $C = C(\delta, \varepsilon, k_0)$ such that*

$$\big\|A_{\rm I}^{-1}\big\|_{H_k^{1/2}\times H_k^{-1/2}} \le C k^{2+5d/2+\varepsilon} \qquad \textit{for all } k \in [k_0, \infty) \setminus J.$$

This result shows that, at high frequency, the blow-up associated with spurious quasi-resonances is extremely localised in frequency, thus giving a possible reason why spurious quasi-resonances seem to have rarely been noticed in literature.

Proof of Theorem 4.8. This follows from Theorem 1.10, Corollary 4.3, and the results of [19]. Indeed, [19, Theorem 1.1] implies that, for arbitrary positive real numbers n_i, n_o, the assumptions of Lemma 4.2 (and hence also Corollary 4.3) are satisfied with $C_{\rm sol}(k) \sim k^{5d/2+1+\varepsilon}$. To see this, we note that [19, Theorem 1.1] holds for problems fitting in the "black-box scattering" framework, and the transmission problem fits in this framework by [19, Lemma 2.3 and Remark 2.4]. Furthermore, [19, Theorem 1.1] is an $L^2 \to L^2$ bound, but this implies a bound of the form (4.4) with $C_{\rm sol}(k) \sim k^{5d/2+1+\varepsilon}$ thanks to Green's identity – see the comments around [19, Equation 1.3]. □

5 Proofs of Lemma 1.13 and Theorem 1.15 (the results about the augmented BIEs)

Proof of Lemma 1.13. We first prove the result when $\widetilde{A}_* = \widetilde{A}_{\rm I}$. By the first equation in (1.19) $\boldsymbol{\phi} = A_{\rm I}^{-1}\mathbf{g}$. Then, by the second equation in (1.19) and the expression for $A_{\rm I}^{-1}$ (1.15),

$$\mathbf{0} = P_i^+\boldsymbol{\phi} = P_i^+ A_{\rm I}^{-1}\mathbf{g} = P_i^+(S_{io} + S_{oi} - I)\mathbf{g}.$$

By (3.1) $P_i^+ S_{io} = 0$, and then

$$P_i^+(S_{oi} - I)\mathbf{g} = \mathbf{0}. \tag{5.1}$$

Now, by (3.2), $P_i^-(S_{oi} - I)\mathbf{g} = \mathbf{0}$, and thus the constraint (1.20) follows by (1.10). Then, by the first equation in (1.19) and (1.15), $\boldsymbol{\phi} = A_{\rm I}^{-1}\mathbf{g} = (S_{io} + S_{oi} - I)\mathbf{g}$, and the result (1.21) follows by (1.20).

We now prove the result when $\widetilde{A}_* = \widetilde{A}_{\rm II}$. Similar to before, by (1.19) and the expression for $A_{\rm II}^{-1}$ (1.16),

$$\mathbf{0} = P_i^+\boldsymbol{\phi} = P_i^+ A_{\rm II}^{-1}\mathbf{g} = P_i^+(I - S_{io} - S_{oi} + 2S_{io}S_{oi})\mathbf{g} = P_i^+(I - S_{oi})\mathbf{g},$$

since $P_i^+ S_{io} = 0$ by (3.1); we therefore obtain (5.1), and the constraint (1.20) follows exactly as before. The result (1.21) then follows from using the constraint (1.20) in the expression for $A_{\rm II}^{-1}$ (1.16). □

Proof of Theorem 1.15. We first prove (1.22). Let $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ for $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \mathcal{H}$. Then

$$\big(\widetilde{A}_{\rm I}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}} = \big(A_{\rm I}\boldsymbol{\phi}, \boldsymbol{\psi}_1\big)_{\mathcal{H}} + \big(P_i^+\boldsymbol{\phi}, \boldsymbol{\psi}_2\big)_{\mathcal{H}}.$$

Given $\boldsymbol{\phi}$, let $\boldsymbol{\psi}_2 := P_i^+\boldsymbol{\phi}$. Since $\boldsymbol{\phi} = P_i^+\boldsymbol{\phi} + P_i^-\boldsymbol{\phi}$,

$$\big(\widetilde{A}_{\rm I}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}} = \big(A_{\rm I}P_i^+\boldsymbol{\phi}, \boldsymbol{\psi}_1\big)_{\mathcal{H}} + \big(A_{\rm I}P_i^-\boldsymbol{\phi}, \boldsymbol{\psi}_1\big)_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}}.$$

Motivated by Theorem 1.12, let $\boldsymbol{\psi}_1 := S_{io}^* P_i^-\boldsymbol{\phi}$. By Theorem 1.12, $S_{io}A_{\rm I} = I$ as an operator $R(P_i^-) \to R(P_i^-)$, and thus

$$\begin{aligned}\big(\widetilde{A}_{\rm I}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}} &= \big(A_{\rm I}P_i^+\boldsymbol{\phi},\, S_{io}^*P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} + \big(S_{io}A_{\rm I}P_i^-\boldsymbol{\phi}, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}}\\ &= \big(A_{\rm I}P_i^+\boldsymbol{\phi},\, S_{io}^*P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} + \big\|P_i^-\boldsymbol{\phi}\big\|^2_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}}.\end{aligned}$$

Now, by the second equality in (1.11), Lemma 2.3, and (1.10),

$$\big(A_{\rm I}P_i^+\boldsymbol{\phi},\, S_{io}^*P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} = \big(S_{io}(P_i^- - P_o^+)P_i^+\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}},$$

which equals zero since $P_i^-P_i^+ = 0$ by (1.10) and $S_{io}P_o^+ = 0$ by Lemma 3.2. Therefore, with this choice of $\boldsymbol{\psi}$,

$$\frac{\big|\big(\widetilde{A}_{\rm I}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}}\big|}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\,\|\boldsymbol{\psi}\|_{\mathcal{H}\times\mathcal{H}}} = \frac{\big\|P_i^-\boldsymbol{\phi}\big\|^2_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}}}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\sqrt{\big\|S_{io}^*P_i^-\boldsymbol{\phi}\big\|^2_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}}}} \geq \frac{\sqrt{\big\|P_i^-\boldsymbol{\phi}\big\|^2_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}}}}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\max\big\{\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}, 1\big\}}. \tag{5.2}$$

By (1.10), the triangle inequality, and the inequality $2ab \leq a^2 + b^2$ for $a, b > 0$,

$$\big\|P_i^-\boldsymbol{\phi}\big\|^2_{\mathcal{H}} + \big\|P_i^+\boldsymbol{\phi}\big\|^2_{\mathcal{H}} \geq \frac{1}{2}\,\|\boldsymbol{\phi}\|^2_{\mathcal{H}}. \tag{5.3}$$

The result (1.22) then follows from combining (5.2) and (5.3).

We now prove (1.23). As above, let $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ for $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \mathcal{H}$. Motivated by the proof of (1.22), given $\boldsymbol{\phi} \in \mathcal{H}$, let $\boldsymbol{\psi}_1 := S_{io}^* P_i^- \boldsymbol{\phi}$. Then, by the definition of $\widetilde{A}_{\rm II}$ and Theorem 1.12,

$$\begin{aligned}\big(\widetilde{A}_{\rm II}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}} &= \big(S_{io}A_{\rm II}\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} + \big(P_i^+\boldsymbol{\phi}, \boldsymbol{\psi}_2\big)_{\mathcal{H}},\\ &= \big(S_{io}A_{\rm II}P_i^+\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} + \big\|P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 + \big(P_i^+\boldsymbol{\phi}, \boldsymbol{\psi}_2\big)_{\mathcal{H}}.\end{aligned} \tag{5.4}$$

Now, by the definition of $A_{\rm II}$ (1.12), Lemma 2.3, (1.10), and the fact that $S_{io}P_o^+ = 0$ by Lemma 3.2,

$$\big(S_{io}A_{\rm II}P_i^+\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} = \big(S_{io}\big(P_o^- + I\big)P_i^+\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} = 2\big(S_{io}P_i^+\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}}. \tag{5.5}$$

We now let $\boldsymbol{\psi}_2 := P_i^+\boldsymbol{\phi} - 2S_{io}^*P_i^-\boldsymbol{\phi}$. This definition along with (5.4) and (5.5) implies that

$$\begin{aligned}\big(\widetilde{A}_{\rm II}\boldsymbol{\phi}, \psi\big)_{\mathcal{H}\times\mathcal{H}} &= 2\big(S_{io}P_i^+\boldsymbol{\phi},\, P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}} + \big\|P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 + \big\|P_i^+\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 - 2\big(P_i^+\boldsymbol{\phi}, S_{io}^*P_i^-\boldsymbol{\phi}\big)_{\mathcal{H}},\\ &= \big\|P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 + \big\|P_i^+\boldsymbol{\phi}\big\|_{\mathcal{H}}^2.\end{aligned}$$

Therefore, with this choice of $\boldsymbol{\psi}$,

$$\frac{\big|\big(\widetilde{A}_{\rm II}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}}\big|}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\,\|\boldsymbol{\psi}\|_{\mathcal{H}\times\mathcal{H}}} = \frac{\big\|P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 + \big\|P_i^+\boldsymbol{\phi}\big\|_{\mathcal{H}}^2}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\sqrt{\big\|S_{io}^*P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 + \big\|(P_i^+ - 2S_{io}^*P_i^-)\boldsymbol{\phi}\big\|_{\mathcal{H}}^2}}.$$

We now use the triangle inequality and (4.13) to find that

$$\begin{aligned}\big\|S_{io}^*P_i^-\boldsymbol{\phi}\big\|^2 + \big\|(P_i^+ - 2S_{io}^*P_i^-)\boldsymbol{\phi}\big\|^2 &\le \|S_{io}^*\|^2\,\big\|P_i^-\boldsymbol{\phi}\big\|^2 + \Big(\big\|P_i^+\boldsymbol{\phi}\big\| + 2\,\|S_{io}^*\|\,\big\|P_i^-\boldsymbol{\phi}\big\|\Big)^2\\ &\le \|S_{io}^*\|^2\,\big\|P_i^-\boldsymbol{\phi}\big\|^2 + \big\|P_i^+\boldsymbol{\phi}\big\|^2 + 4\,\|S_{io}^*\|^2\,\big\|P_i^-\boldsymbol{\phi}\big\|^2 + 4\,\big\|P_i^+\boldsymbol{\phi}\big\|\,\|S_{io}^*\|\,\big\|P_i^-\boldsymbol{\phi}\big\|\\ &\le (1+2\epsilon)\,\big\|P_i^+\boldsymbol{\phi}\big\|^2 + (5 + 2/\epsilon)\,\|S_{io}^*\|^2\,\big\|P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2\,;\end{aligned}$$

if $\epsilon = 1+\sqrt{2}$, then $5 + 2\epsilon = 1 + 2\epsilon = 3 + 2\sqrt{2}$ and thus

$$\frac{\big|\big(\widetilde{A}_{\rm II}\boldsymbol{\phi}, \boldsymbol{\psi}\big)_{\mathcal{H}\times\mathcal{H}}\big|}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\,\|\boldsymbol{\psi}\|_{\mathcal{H}\times\mathcal{H}}} \ge \frac{\sqrt{\big\|P_i^-\boldsymbol{\phi}\big\|_{\mathcal{H}}^2 + \big\|P_i^+\boldsymbol{\phi}\big\|_{\mathcal{H}}^2}}{\|\boldsymbol{\phi}\|_{\mathcal{H}}\sqrt{3+2\sqrt{2}}\max\big\{\,\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}\,, 1\big\}}.$$

The result (1.23) follows from this inequality and (5.3). □

A Extension of the results to the more general form of the transmission problem

We now sketch how the decomposition (1.15) of the first-kind BIE extends to the more general transmission problem of Definition 1.1 with the transmission condition in (1.4) replaced by (1.5). The analogous extension of the decomposiiton (1.16) for the second-kind BIE is very similar.

Derivation of the first kind BIE. The equation (2.10) holds as before, but now the analogue of (2.12) is

$$P_o^- D^{-1}\boldsymbol{\gamma}_C^- u^- = P_o^- D^{-1}\mathbf{f}.$$

Therefore, the analogue of the first-kind BIE $A_{\rm I}$ in (1.13) is

$$A_{\rm I}^{\rm gen}\boldsymbol{\gamma}_C^- u^- = P_o^- D^{-1}\mathbf{f}$$

where

$$A_{\rm I}^{\rm gen} := P_o^- D^{-1} - D^{-1}P_i^+ = D^{-1}P_i^- - P_o^+ D^{-1} = \begin{bmatrix} -(K_i + K_o) & V_i + \alpha^{-1}V_o \\ \alpha^{-1}W_i + W_o & \alpha^{-1}(K_i' + K_o') \end{bmatrix}. \tag{A.1}$$

Solution operators. Let $S(c_i, c_o)$ be the solution operator of the BVP (1.6) with the transmission condition replaced by (1.5). Let $\widetilde{S}(c_i, c_o)$ be the solution operator to (1.6) with the transmission condition

$$\gamma_C^- u^- = D^{-1}\gamma_C^+ u^+ + \mathbf{f};$$

i.e., α is replaced by α^{-1} compared to $S(c_i, c_o)$. Let $S_{io} := S(n_i, n_o)$ and let $\widetilde{S}_{oi} := \widetilde{S}(n_o, n_i)$.

Lemma A.1.

$$\big(A_{\rm I}^{\rm gen}\big)^{-1} = S_{io}D + D\widetilde{S}_{oi} - D. \tag{A.2}$$

Sketch of the proof. The analogue of Lemma 3.1 is now that

$$\boldsymbol{\phi} = S_{io}\mathbf{f} \quad \text{if and only if} \quad \begin{cases} P_i^-\boldsymbol{\phi} = \boldsymbol{\phi}, \text{ and} \\ P_o^- D^{-1}(\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0}. \end{cases} \tag{A.3}$$

and

$$\boldsymbol{\phi} = \widetilde{S}_{oi}\mathbf{f} \quad \text{if and only if} \quad \begin{cases} P_o^-\boldsymbol{\phi} = \boldsymbol{\phi}, \text{ and} \\ P_i^- D(\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0}. \end{cases} \tag{A.4}$$

We then repeat the steps in the proof of Theorem 1.10. Assuming that $A_{\rm I}^{\rm gen}\boldsymbol{\psi} = \mathbf{g}$ and applying $P_i^- D$, we find that the analogue of (3.4) is that

$$P_i^- D\big(P_o^- D^{-1}\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0},$$

so that, by (A.4),

$$P_o^- D^{-1}\boldsymbol{\psi} = \widetilde{S}_{oi}\mathbf{g}. \tag{A.5}$$

Similarly applying P_o^- to $A_{\rm I}^{\rm gen}\boldsymbol{\psi} = \mathbf{g}$, we find that the analogue of (3.6) is

$$P_o^-\big(D^{-1}P_i^-\boldsymbol{\psi} - \mathbf{g}\big) = \mathbf{0},$$

so that, by (A.3),

$$P_i^-\boldsymbol{\psi} = S_{io}D\mathbf{g}. \tag{A.6}$$

Then, using that $A_{\rm I}^{\rm gen}\boldsymbol{\psi} = \mathbf{g}$, (A.1), (A.5), and (A.6), we find that the analogue of (3.8) is

$$\boldsymbol{\psi} = (P_i^+ + P_i^-)\boldsymbol{\psi} = DP_o^- D^{-1}\psi - D\mathbf{g} + P_i^-\boldsymbol{\psi} = D\widetilde{S}_{oi}\mathbf{g} - D\mathbf{g} + S_{io}D\mathbf{g},$$

which is (A.2). □

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