Modal decompositions and point scatterer approximations near the Minnaert resonance frequencies

F. Feppon and H. Ammari

Research Report No. 2021-25
August 2021

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding: The work of F. Feppon was supported by ETH Zürich through the Hermann-Weyl fellowship of the Institute for Mathematical Research (FIM)

MODAL DECOMPOSITIONS AND POINT SCATTERER APPROXIMATIONS NEAR THE MINNAERT RESONANCE FREQUENCIES

F. FEPPON[1,*], H. AMMARI[1]

[1] *Department of Mathematics, ETH Zürich, Switzerland.*

ABSTRACT. As a continuation of the previous works [13, 4, 15], this paper provides several contributions to the mathematical analysis of subwavelength resonances in a high-contrast medium containing N acoustic obstacles. Our approach is based on an exact decomposition formula which reduces the solution of the sound scattering problem to that of a N dimensional linear system, and characterizes resonant frequencies as the solutions to a N-dimensional nonlinear eigenvalue problem. Under a simplicity assumptions on the eigenvalues of the capacitance matrix, we prove the analyticity of the scattering resonances with respect to the square root of the contrast parameter, and we provide a deterministic algorithm allowing to compute all terms of the corresponding Puiseux series. We then establish a nonlinear modal decomposition formula for the scattered field as well as point scatterer approximations for the far field pattern of the sound wave scattered by N bodies. As a prerequisite to our analysis, a first part of the work establishes various novel results about the capacitance matrix, since qualitative properties of the resonances, such as the leading order of the scattering frequencies or of the corresponding far field pattern are closely related to its spectral decomposition.

Keywords. Subwavelength resonance, high-contrast medium, modal decomposition, point scatterer approximation, capacitance matrix, holomorphic integral operators.

AMS Subject classifications. 35B34, 35P25, 35J05, 35C20, 46T25.

CONTENTS

1. INTRODUCTION

There has been a growing interest in the past few years for subwavelength resonant systems [63, 60, 64, 47, 73, 38, 42, 31, 55], which have the remarkable property of strongly amplifying incident electromagnetic, elastic or acoustic waves through the scattering with objects of size much smaller than the wavelength. In acoustics, an instance of subwavelength resonant system was first evidenced by Minnaert who studied the interaction of acoustic waves with air bubbles in water [69]. In electromagnetics, such systems encompass plasmonic particles and high contrast dielectric particles [33, 18, 27, 28, 29, 22, 20, 62]. In linear elasticity, an example of a subwavelength resonator consists in a solid core material with relatively high density and a coating of elastically

* Corresponding author. Email: florian.feppon@sam.math.ethz.ch.

The work of F. Feppon was supported by ETH Zürich through the Hermann-Weyl fellowship of the Institute for Mathematical Research (FIM).

soft material [63]. In general, the resonant property at subwavelength scales is the consequence of a large contrast between physical parameters of the medium, for instance between the air density and the density of water in the case of Minnaert bubbles, or between the width of the opening and the size of a Helmholtz resonators [74]. Subwavelength resonances offer promising potentialities for the design of wave systems in a large variety of interesting applications such as superresolution [32, 24, 25, 19], sensing [79, 7], focusing [58, 12], the design of negative refractive index metamaterials [49, 15], meta-surfaces [11, 1] and cloaking [61].

The contribution of this paper is to enhance the mathematical analysis of subwavelength resonances in acoustic high-contrast media which has already been initiated in the previous articles [13, 1, 2, 37, 5, 8, 6]. More precisely, we consider an acoustic medium $D \subset \mathbb{R}^3$ constituted of N smooth connected components B_i (the "bubbles" or acoustic resonators):

$$D = \bigcup_{i=1}^{N} B_i.$$

We refer to Figure 1 for an illustration of the setting. The background medium $\mathbb{R}^3 \backslash \overline{D}$ is a homogeneous

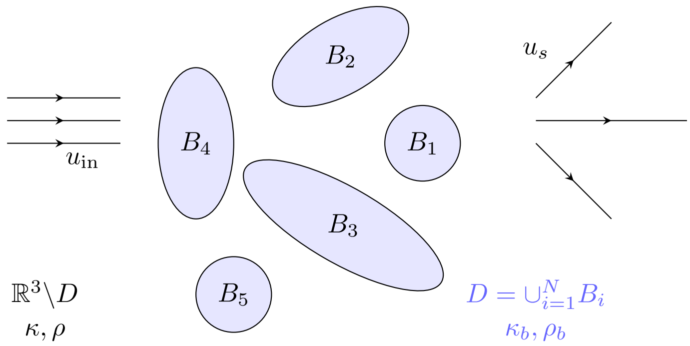

FIGURE 1. Distribution of acoustic obstacles in the three-dimensional space $\mathbb{R}^3$. An incident wave u_{in} is propagating with frequency ω and generates a total wave field u_{tot}.

acoustic material characterized by a homogeneous density ρ and bulk modulus κ. The "bubbles" are acoustic heterogeneities with homogeneous density ρ_b and bulk modulus κ_b. We are interested in the scattering of an incoming wave u_{in} propagating through the bulk material with frequency ω. We denote by

$$v = \sqrt{\frac{\kappa}{\rho}}, \quad v_b = \sqrt{\frac{\kappa_b}{\rho_b}}, \quad k = \frac{\omega}{v}, \quad k_b = \frac{\omega}{v_b}$$

the sound velocities v and v_b and the wave numbers k and k_b in respectively the background medium and the acoustic obstacles. We consider the high-contrast regime whereby the quantity

$$\delta := \frac{\rho_b}{\rho}$$

is asymptotically small: $\delta \to 0$. The incoming sound wave u_{in} is the solution to the Helmholtz equation in the free space $\mathbb{R}^3$; it satisfies

$$\nabla \cdot \left(\frac{1}{\rho} \nabla u_{\text{in}} \right) + \frac{\omega^2}{\kappa} u_{\text{in}} = 0 \text{ in } \mathbb{R}^3 \backslash D.$$

The wave u_{in} generates a scattered field u_s, which is such that the total field $u_{\text{tot}} := u_{\text{in}} + u_s$ is the solution to the following scattering problem:

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\rho_b} \nabla u_{\text{tot}} \right) + \dfrac{\omega^2}{\kappa_b} u_{\text{tot}} = 0 \text{ in } D, \\ \nabla \cdot \left(\dfrac{1}{\rho} \nabla u_{\text{tot}} \right) + \dfrac{\omega^2}{\kappa} u_{\text{tot}} = 0 \text{ in } \mathbb{R}^3 \backslash D, \\ u_{\text{tot},+} - u_{\text{tot},-} = 0 \text{ on } \partial D, \\ \dfrac{1}{\rho_b} \dfrac{\partial u_{\text{tot}}}{\partial n} \Big|_{-} = \dfrac{1}{\rho} \dfrac{\partial u_{\text{tot}}}{\partial n} \Big|_{+} \quad \text{on } \partial D, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k \right) (u_{\text{tot}} - u_{\text{in}}) = O\left(\dfrac{1}{|x|^2} \right) \text{ as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where $u_{\text{tot},+}$ and $u_{\text{tot},-}$ denote the trace of u_{tot} on respectively the outer and the inner boundaries of the obstacles ∂D, and $\partial u_{tot}/\partial n|_-$ and $\partial u_{tot}/\partial n|_+$ the inner and outer normal derivatives with the normal vector $\boldsymbol{n}$ pointing outward D. The last equation is the outgoing Sommerfeld radiation condition for the scattered field u_s.

As it is reviewed in [8], there exist N pairs of subwavelength resonances $(\omega_i^{\pm}(\delta))_{1\leq i\leq N}$ which are mathematically defined as complex values of ω for which the problem (1.1) admits non-trivial solutions. These have negative imaginary part and are positive with respect to the imaginary axis: $\omega_i^+(\delta) = -\overline{\omega_i^-(\delta)}$. Physically, the scattered field $u_s = u_{\text{tot}} - u_{\text{in}}$ is greatly enhanced as the (physical) real frequency $\omega > 0$ becomes close to one of the N complex resonances $(\omega_i^+(\delta))_{1\leq i\leq N}$. These resonances are called "subwavelength" because they occur in the low frequency regime: one can indeed prove that $\omega_i^{\pm}(\delta) \to 0$ as $\delta \to 0$. As we shall see below, the quantities $\omega_i^{\pm}(\delta)$ are the poles of the resolvent operator associated to (1.1); they are therefore a particular case of *scattering resonances*, which also occur in quantum physics or general relativity [80].

The main contributions of this article is to provide a thorough mathematical analysis of the subwavelength (Minnaert) resonant properties of the system (1.1). The main fundamental novelties, which are missing in the previous works [13, 15, 8, 6], can be summarized in the following contributions.

1. **Capacitance matrix:** one of the most enlightening concepts for understanding the qualitative properties of the subwavelength resonant problem (1.1) is the *capacitance matrix* $C \equiv (C_{ij})_{1\leq i,j\leq N}$. In the context where resonators are replaced with electric conductors with same shapes $(B_i)_{1\leq i\leq N}$, the entry C_{ij} is the charge that is accumulated by B_i when applying the unit potential $(\delta_{ij})_{1\leq j\leq N}$ on the boundaries of the resonators/conductors $(B_j)_{1\leq j\leq N}$. As it has been fruitfully exploited in [5, 3, 9, 8], the spectral decomposition of the capacitance matrix enables to predict the values of the scattering frequencies and the resulting scattered field.
   Section 2 is dedicated to the main properties of the capacitance matrix and establishes novel results such as a Perron-Frobenius type theorem, new spectral bounds and properties on the coefficients in the case of symmetries. We obtain that the vector of ones $\mathbf{1} = (1)_{1\leq i\leq N}$, which plays a particularly important roles in the analysis of the scattering resonances, is an eigenvector of the capacitance matrix when the system of resonators admits enough symmetries. However, we also prove that this situation is in general exceptional in the sense that this property is lost (at least when $N \geq 3$) after small perturbations of the shape of the resonators.
2. **Asymptotic analysis of the resonances:** Section 3 extends the analysis and the methods of [24, 13] to obtain a fully explicit representation formula for the total field u_{tot}. Using integral operator theory, we prove that u_{tot} can be decomposed as

$$u_{\text{tot}} = \sum_{i=1}^{N} x_i v_i(\omega, \delta) + w(\omega, \delta), \tag{1.2}$$

   where the variables $v_i(\omega, \delta)$ and $w(\omega, \delta)$ are holomorphic fields in ω and δ (hence uniformly bounded) and $v_i(\omega, \delta)$ is independent of u_{in}. (1.2) is a decomposition of u_{tot} into a resonant part featuring amplitude coefficients x_i which blow up at the resonances, and the non-resonant field $w(\omega, \delta)$. The complex coefficients $\boldsymbol{x} \equiv (x_i)_{1\leq i\leq N} \equiv (x_i(\omega, \delta))_{1\leq i\leq N}$ are indeed N meromorphic functions of ω whose poles are the $2N$ Minnaert resonances $(\omega_i^{\pm}(\delta))_{1\leq i\leq N}$. The vector $\boldsymbol{x}$ is characterized as the solution to an $N \times N$ linear system

$$A(\omega, \delta)\boldsymbol{x} = \boldsymbol{F}(\omega, \delta),$$

   for explicit $N \times N$ holomorphic matrix $A(\omega, \delta) \in \mathbb{C}^{N\times N}$ and right-hand side $\boldsymbol{F}(\omega, \delta) \in \mathbb{C}^N$, while the scattering resonances $(\omega_i^{\pm}(\delta))$ solve the following nonlinear eigenvalue problem:

$$\text{find } \boldsymbol{x} \neq 0 \text{ such that } A(\omega_i^{\pm}(\delta), \delta)\boldsymbol{x} = 0. \tag{1.3}$$

   With the help of the implicit function theorem applied to (1.3), we prove the holomorphic dependence of the scattering frequencies $(\omega_i^{\pm}(\delta))_{1\leq i\leq N}$ with respect to the quantity $\delta^{\frac{1}{2}}$ by assuming a simplicity assumption of the eigenvalues of the capacitance matrix (which is a stronger result than the existence theorem provided by Gohberg–Sigal theory [41, 17] as in [13, 8]). Furthermore, we provide a systematic algorithm which allows, in principle, to compute all the coefficients $(\omega_{i,p}^{\pm})_{p\geq 1}$ of the Puiseux expansion

$$\omega_i^{\pm}(\delta) = \sum_{p=1}^{+\infty} \omega_{i,p}^{\pm} \delta^{\frac{1}{2}}. \tag{1.4}$$

3. **Modal decompositions:** we then provide, in Section 4, a pole decomposition of the meromorphic coefficients $(x_i)_{1\leq i\leq N}$ of (1.2), which allow us to establish a modal decomposition for the solution u_{tot}. More precisely, we prove the existence of N generalized eigenmodes $(u_i(\omega, \delta))_{1\leq i\leq N}$ and of N linear forms $E_i(\omega, \delta) : L^2(\partial D) \times L^2(\partial D) \to \mathbb{C}$ satisfying $E_i(\omega, \delta) = O(1)$, independent on the incident field

u_{in}, such that the solution u_{tot} to the scattering problem (1.1) reads:

$$u_{\text{tot}}(\omega,\delta)(x)=\sum_{i=1}^{N}\frac{\delta E_i(\omega,\delta)\begin{pmatrix}u_{\text{in}}\\ \frac{\partial u_{\text{in}}}{\partial n}\end{pmatrix}}{\omega^2-|\omega_i^{\pm}(\delta)|^2-2\mathrm{i}\omega\Im(\omega_i^{\pm}(\delta))}u_i(\omega,\delta)(x)+w(\omega,\delta). \tag{1.5}$$

The main originality of (1.5) lies in the fact that it does not resort to any approximation; in particular, it is visible that $u_{\text{tot}}(\omega,\delta)$ blows up *exactly* when ω coincides with one of the resonant frequencies $(\omega_i^{\pm}(\delta))_{1\leq i\leq N}$. In contrast, the modal decomposition obtained in [4] (Lemma 2.11) involved only approximate denominators which might be very inaccurate for complex values of ω close to the actual resonances, since these vanish at approximate frequencies different to $\omega_i^{\pm}(\delta)$. Moreover, (1.5) highlights that the (negative) imaginary part $\Im(\omega_i^{\pm}(\delta))$ of the scattering frequency $\omega_i^{\pm}(\delta)$ explicitly controls the damping of the resonance at physical, positive frequencies $\omega>0$.

4. **Point scatterer approximations:** Finally, we derive in Section 5 a point scatterer approximation for the scattered field $u_s(x)$ propagating through the medium with N general resonators by computing the far field expansion of the functions $(u_i(\omega,\delta))_{1\leq i\leq N}$ and $w(\omega,\delta)$. We improve and generalize thus previous results established in [13, 15] for the particular cases where D is a single bubble ($N=1$) or a dimer (constituted of $N=2$ identical spherical resonators). We find that in the general case, the group of resonators D behaves in the far field as a monopole scatterer (a point source) at all resonant frequencies. Dipole or multipole modes may occur only under exceptional circumstances such as enough symmetries of the distribution D of acoustic obstacles, and more precisely when the vector of ones $\mathbf{1}=(1)_{1\leq i\leq N}$ is an eigenvector of the capacitance matrix C.

We stress that the analysis of this paper is not exhaustive: several interesting questions related to the Minnaert resonances remain to be explored. Notably, most of our conclusions strongly rely on the simplicity assumption of the eigenvalues of the capacitance matrix. In the general case, multiple eigenvalues could occur, for instance when the system of resonators D has many symmetries. The modal decomposition (1.5) could then potentially feature resonant denominators elevated to higher order exponents (see the discussion in Remark 3.4 below), and one would need to characterize differently the Puiseux expansion of the frequencies $\omega_i^{\pm}(\delta)$ which would not need to be analytic in $\delta^{\frac{1}{2}}$. This issue would require to analyse the splitting of multiple eigenvalues and exceptional points of the nonlinear system (1.3), which is known to be particularly delicate, see e.g. [23, 68, 7, 48] for some treatments in various contexts.

Finally, we mention that our work is a preamble to a subsequent paper concerned with the derivation of an effective medium theory for a system $D\equiv\cup_{1\leq i\leq N}(y_i+sB_i)$ constituted of a growing number of small resonators, namely the present setting with $N\to+\infty$ and where each resonator centered at $y_i\in\mathbb{R}^3$ is rescaled by a small parameter $s\to 0$ [40]. This question has been examined previously in the case where D is made of identical resonators in [26, 1, 2], or identical dimers (using formal arguments) in [15]. Relying on the exact formula (1.2), our subsequent work [40] establishes a quantitative homogenization theory for a system constituted of a large number of tiny clusters of identical packets of N resonators distributed randomly in a bounded domain.

Notation conventions. In all what follows, vectors are written in bold face notation. $(\boldsymbol{e}_i)_{1\leq i\leq N}$ is the canonical basis of $\mathbb{R}^N$. The purely imaginary number is denoted by (a straight) i.

$\mathcal{S}_D^k$ and $\mathcal{K}_D^{k*}$ denote respectively the single layer potential and the adjoint of the Neumann-Poincaré operator on D: for any $\phi\in L^2(\partial D)$,

$$\sigma\sigma(y), \qquad x\in\mathbb{R}^3, \tag{1.6}$$

$$\mathcal{K}_D^{k*}[\phi](x):=\int_{\partial D}\nabla_x\Gamma^k(x-y)\cdot n(x)\phi(y)\,\mathrm{d}\sigma(y), \qquad x\in\partial D, \tag{1.7}$$

where $\Gamma^k(x):=-e^{ik|x|}/4\pi|x|$ is the fundamental solution to the Helmholtz equation and $\mathrm{d}\sigma$ is the surface measure of ∂D. We use the notations $\mathcal{S}_D^{k_b}$ and $\mathcal{K}_D^{k_b*}$ for the same operator with k replaced with k_b, and we use the short-hand notation $\Gamma:=\Gamma^0$, $\mathcal{S}_D:=\mathcal{S}_D^0$ and $\mathcal{K}_D^*:=\mathcal{K}_D^{0*}$ for the fundamental solution of the Laplace operator, its associated single layer potential and Neumann-Poincaré operator. We recall that $\mathcal{S}_D^k$ is an invertible operator from $L^2(\partial D)$ to $H^1(\partial D)$, whose inverse is denoted by $(\mathcal{S}_D^k)^{-1}: H^1(\partial D)\to L^2(\partial D)$, and that $\mathcal{K}_D^{k*}$ is a compact operator on $L^2(\partial D)$ [17].

Finally, a few notation conventions related to tensors (which follow [39]) are used in Sections 4 and 5 and are now detailed. For a given integer $p\in\mathbb{N}$, we denote by $\mathcal{X}^p\equiv(\mathcal{X}^p_{i_1\dots i_p})_{1\leq i_1\dots i_p\leq 3}$ the p-th order scalar tensor with p derivative indices $1\leq i_1\dots i_p\leq 3$. We also consider *vector* tensors $\boldsymbol{M}^p\equiv(M^p_{i_1\dots i_p,l})_{1\leq i_1\dots i_p\leq 3}$ with p derivative indices $1\leq i_1\dots i_p\leq 3$ and one spatial index $1\leq l\leq 3$, and we denote by $\boldsymbol{M}^p_{i_1\dots i_p}$ the vector-valued components of $\boldsymbol{M}^p$. For any function f, we denote by y^p and $\nabla^p f(x)$ the p-th order tensors

$$y^p=(y^p_{i_1\dots i_p})_{1\leq i_1\dots i_p\leq 3}, \quad \nabla^p f(x):=(\partial^p_{x_{i_1}\dots x_{i_p}}f(x))_{1\leq i_1\dots i_p\leq 3}, \qquad x\in\mathbb{R}^3,$$

and by $\mathcal{X}^p \cdot \nabla^p f(x)$ and $\boldsymbol{M}^p \cdot \nabla^p f(x)$ the contractions

$$\mathcal{X}^p \cdot \nabla^p f(x) = \sum_{1\le i_1\dots i_p\le 3} \mathcal{X}^p_{i_1\dots i_p} \partial^p_{x_{i_1}\dots x_{i_p}} f(x), \qquad \boldsymbol{M}^p \cdot \nabla^p f(x) = \sum_{1\le i_1\dots i_p\le 3} \boldsymbol{M}^p_{i_1\dots i_p} \partial^p_{x_{i_1}\dots x_{i_p}} f(x).$$

Note that $\mathcal{X}^p \cdot \nabla^p f(x)$ is a scalar function while $\boldsymbol{M}^p \cdot \nabla^p f(x)$ is a rank-one vector field. For $1 \le j \le 3$, e_j denotes the first order scalar tensor whose components are given by $e_{j,i_1} = \delta_{ji_1}$ where δ is the Kronecker symbol. Finally, $\mathcal{X}^p \otimes e_j$ denotes the tensor product between the p-th order tensor $\mathcal{X}^p$ and e_j; it is a scalar tensor of order $p+1$ whose components are

$$(\mathcal{X}^p \otimes e_j)_{i_1\dots i_{p+1}} = \mathcal{X}^p_{i_1\dots i_p}\delta_{ji_{p+1}}, \qquad 1 \le i_1 \dots i_{p+1} \le 3.$$

2. Properties of the capacitance matrix

A recurrent property exploited in the analysis of high-contrast media [4, 9, 8] is the fact that subwavelength resonances of the system (1.1) can be predicted by the eigenvalues of the capacitance matrix associated to the N connected components $(B_i)_{1\le i\le N}$. The capacitance matrix $C \equiv (C_{ij})_{1\le i,j\le N}$ is defined by

$$C_{ij} := -\int_{\partial B_j} \frac{\partial u_i}{\partial n}\,\mathrm{d}\sigma, \quad \forall 1 \le i,j \le N, \tag{2.1}$$

where $\boldsymbol{n}$ is the outward normal to D and u_i is the solution to the exterior problem

$$\begin{cases} -\Delta u_i = 0 \text{ in } \mathbb{R}^3\backslash\partial D, \\ \quad u_i = \delta_{ij} \text{ on } \partial B_j \text{ for any } 1 \le j \ne i \le N, \\ u_i(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases} \tag{2.2}$$

The capacitance matrix is usually considered in the physical context where the resonators $(B_i)_{1\le i\le N}$ are replaced with perfect electric conductors $(B_i)_{1\le i\le N}$ with same shapes, e.g. [44]. Then the variable u_i is the electric potential in the free space $\mathbb{R}^3$ satisfying $u_i = 1$ uniformly inside the conductor B_i, and $u_i = 0$ inside the other conductors $(B_j)_{1\le j\ne i\le N}$. The coefficient C_{ij} is the electric charge stored at equilibrium on the boundary of the conductor B_i induced by the potential u_i. In the setting of (1.1) whereby B_i are acoustic obstacles, u_i can be interpreted as an acoustic pressure and C_{ij} is the jump of the normal velocity through the obstacle in the perfectly reflecting regime (corresponding to $\delta = 0$). Note that the solution to (2.2) can also be explicitly written as the single layer potential $u_i = \mathcal{S}_D[(\mathcal{S}_D)^{-1}[1_{\partial B_i}]]$.

Let V be the (positive definite) diagonal matrix whose entries are the volumes of the resonators $(B_i)_{1\le i\le N}$:

$$V := \mathrm{diag}((|B_i|)_{1\le i\le N}). \tag{2.3}$$

This section dedicated on the capacitance matrix is motivated by the importance of the following generalized eigenvalue problem in the analysis of (1.1):

$$C\boldsymbol{a}_i = \lambda_i V \boldsymbol{a}_i. \tag{2.4}$$

Notably, the eigenvalues $(\lambda_i)_{1\le i\le N}$ enable to predict the frequencies of the subwavelength resonances thanks to the asymptotic $\omega_i^{\pm}(\delta) \sim \pm v_b \lambda_i^{\frac{1}{2}}\delta^{\frac{1}{2}}$ as $\delta \to 0$ (Corollary 3.1 below). Note that the eigenvalues $(\lambda_i)_{1\le i\le N}$ are also the eigenvalues of the "weighted capacitance matrix" $V^{-1}C$, which was rather considered in [4, 8].

The section outlines as follows. Section 2.1 recalls well-known positivity and symmetry properties of the matrix C which imply the existence of N positive eigenvalues $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \ldots \ge \lambda_N > 0$. An important consequence is a Perron-Frobenius type theorem for (2.4), which to the best of our knowledge, had not been stated in previous works. The next Section 2.2 establishes spectral bounds for the eigenvalues. Then, Section 2.3 examines the structure of the matrix C in case of symmetries of the system of resonators $D = \cup_{1\le i\le N} B_i$. We obtain in particular that when D has enough symmetries, the first eigenvector $\boldsymbol{a}_1$ coincides with the vector of ones, $\mathbf{1} := (1)_{1\le i\le N}$. This fact implies exceptional properties for the resonant system (1.1), such as dipole or multipole far field patterns at the resonance. The structure of the capacitance matrix for several particular cases such as the dimer, trimer, and quadrimer, is explicitly derived. Finally, Section 2.4 establishes that the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$ is, at least for $N \ge 3$ acoustic obstacles, in general not an eigenvector of the capacitance matrix. Indeed, we prove thanks to differential shape calculus that almost any shape perturbations of the resonators (up to a finite dimensional subset) suffices to cancel this property.

In the next parts of this section, we make use of the function u solution to the problem

$$\begin{cases} -\Delta u = 0 \text{ in } \mathbb{R}^3\backslash\partial D, \\ \quad u = 1 \text{ on } \partial B_i \text{ for any } 1 \le i \le N, \\ u(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty, \end{cases} \tag{2.5}$$

which is related to the capacity $\mathrm{cap}(D)$ of the set D, defined by the formula

$$\mathrm{cap}(D) := -\int_{\partial D} \frac{\partial u}{\partial n}\,\mathrm{d}\sigma = \int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x. \tag{2.6}$$

Substituting $u = \sum_{i=1}^N u_i$ in (2.6), we note that the quantity $\mathrm{cap}(D)$ satisfies the identity

$$\mathrm{cap}(D) = \sum_{1\le i,j\le N} C_{ij} = \mathbf{1}^T C\mathbf{1}.$$

2.1. A Perron-Frobenius type theorem for the capacitance matrix

Let us start by recalling that the physical definition (2.1) makes sense because the system (2.2) is well-posed in the homogeneous space $\mathcal{D}^{1,2}(\mathbb{R}^3\backslash D) = \{v \,|\, \nabla v \in L^2(\mathbb{R}^3\backslash D)\}$; each function u_i satisfies the variational equality

$$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^3\backslash D),\ \int_{\mathbb{R}^3\backslash D} \nabla u_i \cdot \nabla v\,\mathrm{d}x = -\int_{\partial D} \frac{\partial u_i}{\partial n} v\,\mathrm{d}\sigma, \tag{2.7}$$

see e.g. [71]. Using (2.1), (2.2) and (2.7), it is immediate to see that C_{ij} could be equivalently defined by the following formulas:

$$C_{ij} := -\int_{\partial D} u_j \frac{\partial u_j}{\partial n}\,\mathrm{d}\sigma = \int_{\mathbb{R}^3} \nabla u_i \cdot \nabla u_j\,\mathrm{d}x, \tag{2.8}$$

$$C_{ij} = -\int_{\partial B_i} \psi_j^*\,\mathrm{d}\sigma \text{ where } \psi_i^* := \mathcal{S}_D^{-1}[1_{\partial B_i}] = \left.\frac{\partial u_i}{\partial n}\right|_+ \text{ on } \partial D, \tag{2.9}$$

where, throughout the paper, $1_{\partial B_i} \in L^2(\partial D)$ denotes the characteristic function of the bubble ∂B_i on the boundary ∂D:

$$1_{\partial B_i} := \begin{cases} 1 \text{ on } \partial B_i \\ 0 \text{ on } \partial B_j \text{ for } 1 \le j \ne i \le N. \end{cases} \tag{2.10}$$

The following lemma states symmetry and positivity properties of the capacitance matrix which are rather classical, however not often stated in the literature; we are only aware of similar statements in [44] which considers a slightly different setting in which the radiation condition of (2.2) is replaced with a zero Dirichlet condition on a boundary enclosing the conductors of D. We give a proof for the case of conductors in the free space $\mathbb{R}^3$.

Lemma 2.1. *The capacitance matrix C_{ij} defined by (2.8) satisfies the following properties:*

(i) C is symmetric, i.e. $C_{ij} = C_{ji}$ for any $1 \le i, j \le N$, and positive definite;
(ii) the diagonal entries of C are positive: $C_{ii} > 0$ for any $1 \le i \le N$;
(iii) the extra-diagonal entries of C are negative: $C_{ij} < 0$ for any $1 \le i \ne j \le N$;
(iv) C is diagonally dominant:

$$C_{ii} > \sum_{j\ne i} |C_{ij}|, \text{ for any } 1 \le i \le N.$$

Proof. (i) and (ii) are obvious from the definition (2.8). The positive definiteness comes from the fact that if there exists $\boldsymbol{x} \in \mathbb{R}^N$ such that $\boldsymbol{x}^T C\boldsymbol{x} = 0$, then the function $u := \sum_{i=1}^N x_i u_i$ satisfies $\nabla u = 0$. The Poincaré inequality in $\mathcal{D}^{1,2}(\mathbb{R}^3\backslash D)$ implies $u = 0$ in $\mathbb{R}^3\backslash D$, and then $x_i = 0$ for any $1 \le i \le N$ because it holds $x_i = u(y)$ for any $y \in \partial B_i$.
(iii) The maximum principle implies the inequality $0 \le u_i(x) \le 1$. Hence, it is necessary that $\nabla u_i \cdot \boldsymbol{n} > 0$ on ∂B_j with $j \ne i$ and $\nabla u_i \cdot \boldsymbol{n} < 0$ on ∂B_i. The result follows from (2.9).
(iv) The solution u to the problem (2.5) is given by $u = \sum_{1\le i\le N} u_i$. By the maximum principle, it holds $0 \le u(x) \le 1$ in $\mathbb{R}^3\backslash D$, which implies $\nabla u \cdot \boldsymbol{n} < 0$ on ∂D. Therefore the following inequality holds for any $1 \le i \le N$:

$$0 < -\int_{\partial B_i} \frac{\partial u}{\partial n}\,\mathrm{d}\sigma = -\sum_{j=1}^N \int_{\partial B_i} \frac{\partial u_j}{\partial n}\,\mathrm{d}\sigma = \sum_{j=1}^N C_{ij} = C_{ii} - \sum_{j\ne i} |C_{ij}|.$$

□

The positivity and symmetry properties of both matrices C and V imply the existence of eigenvalues $(\lambda_i)_{1\le i\le N}$ and eigenvectors $(\boldsymbol{a}_i)_{1\le i\le N}$ solving (2.4). More precisely, we recall the following classical result [77, 78] about generalized eigenvalue problems, whose proof is given for the convenience of the reader.

Proposition 2.1. *Let C and V be two $n \times n$ symmetric matrix with V definite positive. There exists a basis of generalized eigenvectors $(\boldsymbol{a}_i)_{1\le i\le N}$ and real eigenvalues $(\lambda_i)_{1\le i\le N}$ such that:*

(i) $(\boldsymbol{a}_i)_{1\le i\le N}$ is orthogonal for the inner product of V:

$$\boldsymbol{a}_i^T V\boldsymbol{a}_j = \delta_{ij}, \quad 1 \le i, j \le N,$$

where T denotes the transpose.

(ii) $C\boldsymbol{a}_i = \lambda_i V \boldsymbol{a}_i$ *for any* $1 \le i \le N$.

Equivalently, let P *be the transition matrix*

$$P := \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_N \end{bmatrix},$$

and $\Lambda := \mathrm{diag}((\lambda_i)_{1\le i\le N})$. *Then the following identities hold:*

$$P^T V P = I, \quad PP^T = V^{-1}, \quad CP = VP\Lambda.$$

Proof. Introduce $V^{1/2}$ the unique positive definite matrix satisfying $(V^{1/2})^2 = V$. Let $(\boldsymbol{v}_i)_{1\le i\le N}$ and $(\lambda_i)_{1\le i\le N}$ be respectively an orthonormal basis of eigenvectors and the associated eigenvalues of the symmetric matrix $V^{-1/2}CV^{-1/2}$. Then the family $(\boldsymbol{a}_i)_{1\le i\le N}$ defined by

$$\boldsymbol{a}_i = V^{-1/2}\boldsymbol{v}_i$$

satisfies the desired properties. Indeed,

$$\boldsymbol{a}_i^T V \boldsymbol{a}_j = \boldsymbol{v}_i^T V^{-1/2} V V^{1/2} \boldsymbol{v}_j = \boldsymbol{v}_i^T \boldsymbol{v}_j = \delta_{ij},$$

and

$$C\boldsymbol{a}_i = CV^{-1/2}\boldsymbol{v}_i = V^{1/2}(\lambda_i \boldsymbol{v}_i) = \lambda_i V \boldsymbol{a}_i.$$

□

The diagonally dominant property (iv) of Lemma 2.1 and the previous proposition imply the following Perron-Frobenius type theorem:

Proposition 2.2. *The lowest eigenvalue* λ_1 *of* C *of the problem* (2.4) *is of multiplicity one and is associated to an eigenvector* $\boldsymbol{a}_1 \equiv (a_{1,i})_{1\le i\le N}$ *which can be selected with positive coefficients* $a_{1,i} > 0$ *for any* $1 \le i \le N$. *Furthermore, the following lower bound holds:*

$$\lambda_1 \ge \frac{\min_{1\le i\le N} C_{ii}}{\max_{1\le i\le N} |B_i|}\left(1 - \sup_{1\le i\le N} \sum_{j\ne i} \frac{|C_{ij}|}{C_{ii}}\right). \tag{2.11}$$

Proof. It is classical that the inverse of a diagonally dominant matrix has positive coefficients and hence implies the Perron-Frobenius theorem [45]. We give a full proof for the convenience of the reader. Consider the triple norm $|||\cdot|||$ defined on matrices $A \equiv (a_{ij})_{1\le i,j\le N}$ by

$$|||A||| = \sup_{1\le i\le N} \sum_{j=1}^{N} |a_{ij}|.$$

$|||\cdot|||$ is a triple norm because $||Au||_\infty \le |||A|||\,||u||_\infty$ where $||u||_\infty := \max_{1\le i\le N} |u_i|$. Let us write $C = A - B$ with $A := \mathrm{diag}(C_{ii})_{1\le i\le N}$ being the diagonal entries of C and $B := A - C$. From Lemma 2.1, A is positive definite, and B is a nonnegative symmetric matrix. Furthermore, the diagonally dominant property implies $|||A^{-1}B||| < 1$. Therefore the Neumann series converges and it holds

$$(A-B)^{-1} = (I - A^{-1}B)^{-1}A^{-1} = \sum_{i=0}^{+\infty} (A^{-1}B)^i A^{-1}. \tag{2.12}$$

Then (2.12) implies that $C^{-1} = (A-B)^{-1}$ has positive coefficients. Since the matrix $V^{1/2}C^{-1}V^{1/2}$ satisfies the assumptions of the Perron-Frobenius theorem, one can find a positive eigenvector $\boldsymbol{v}_1$ associated with the maximum eigenvalue of $V^{1/2}C^{-1}V^{1/2}$, i.e. λ_1^{-1}, which is of multiplicity one (this result can be obtained by considering the method of power iterations to (2.12)). Then $\boldsymbol{a}_1 := V^{-1/2}\boldsymbol{v}_1$ has positive coefficients and satisfies $C\boldsymbol{a}_1 = \lambda_1 V \boldsymbol{a}_1$. The bound on λ_1 comes from

$$\lambda_1^{-1} \le |||V^{1/2}C^{-1}V^{1/2}||| \le |||V^{1/2}|||^2 |||(A-B)^{-1}||| = \max_{1\le i\le N} |B_i| \frac{|||A^{-1}|||}{1 - |||A^{-1}B|||}.$$

□

We are unaware of previous works mentioning the result of Proposition 2.2 for the capacitance matrix C. For our applications, we shall retain the following important consequences:

(1) the lowest eigenvalue λ_1 of C is always associated to a fully positive or fully negative distribution of charges $(a_{1,i})_{1\le i\le N}$ on the resonators $(B_i)_{1\le i\le N}$; we say $\boldsymbol{a}_1$ is a *monopole* mode.

(2) If the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$ is an eigenvector, then it is necessarily associated to the lowest eigenvalue λ_1 (or in other words, $\boldsymbol{a}_1$ is proportional to $\mathbf{1}$).

2.2. **Spectral bounds for the lowest eigenvalue of the capacitance matrix**

The purpose of this section is to relate the discrete eigenvalue problem (2.4) to a natural functional Riesz minimization problem. The main result of this part is the inequality

$$\frac{1}{\mu_1} \leq \lambda_1 \leq \frac{\mathrm{cap}(D)}{|D|}, \tag{2.13}$$

where μ_1 is the leading eigenvalue of the Newton potential (see [50] for further related results on the spectrum of the Newton potential). The upper and lower bounds are proved in Propositions 2.4 and 2.5 respectively.

Proposition 2.3. *Let us denote by $\mathcal{V}$ the subspace*

$$\mathcal{V} = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \,|\, \nabla u = 0 \text{ in } D\}.$$

The smallest eigenvalue λ_1 of C satisfies the Riesz minimization property

$$\lambda_1 = \min_{u\in\mathcal{V}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x}{\int_D |u|^2 \,\mathrm{d}x}. \tag{2.14}$$

Proof. Let us denote by λ the minimal value of (2.14) and u an associated minimizer. The existence of u minimizing (2.14) is a consequence of the strong convexity of the functional $J(u) = \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x$ on the convex set $\{u \in \mathcal{V} \,|\, \int_D |u|^2 \,\mathrm{d}x \leq 1\}$. Furthermore λ satisfies that for any $v \in \mathcal{V}$,

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\mathbb{R}^3\setminus D} \nabla u \cdot \nabla v \,\mathrm{d}x = \lambda \int_D uv \,\mathrm{d}x. \tag{2.15}$$

This implies $-\Delta u = 0$ in $\mathbb{R}^3\setminus D$. Then denoting $x_i := u_{|B_i}$, we obtain the identity

$$u(x) = \sum_{i=1}^N x_i u_i(x), \qquad x \in \mathbb{R}^3.$$

Therefore, considering $v = \sum_{i=1}^N y_i u_i(x)$ for an arbitrary $\boldsymbol{y} = (y_i)_{1\leq i\leq N}$, (2.15) rewrites as

$$\boldsymbol{y}^T C\boldsymbol{x} = \lambda \boldsymbol{y}^T V \boldsymbol{x} \text{ for any } \boldsymbol{y} \in \mathbb{R}^N,$$

which yields that λ is an eigenvalue of C and $\boldsymbol{x}$ an associated eigenvector, with $\lambda \geq \lambda_1$. Reciprocally, the first eigenvalue λ_1 of (2.4) is the solution to the Riesz minimization problem

$$\lambda_1 = \min_{\boldsymbol{x}\in\mathbb{R}^N} \frac{\boldsymbol{x}^T C\boldsymbol{x}}{\boldsymbol{x}^T V\boldsymbol{x}} = \min_{\boldsymbol{x}\in\mathbb{R}^N} \frac{\int_{\mathbb{R}^3} \left|\nabla \left(\sum_{i=1}^N x_i u_i\right)\right|^2 \,\mathrm{d}x}{\int_D \left|\sum_{i=1}^N x_i u_i\right|^2 \,\mathrm{d}x} \geq \min_{u\in\mathcal{V}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x}{\int_D |u|^2 \,\mathrm{d}x} = \lambda.$$

This shows that $\lambda = \lambda_1$ and the proof is complete. □

Proposition 2.4. *For any set of resonators D the following inequality holds*

$$\lambda_1 \leq \frac{\mathrm{cap}(D)}{|D|} = \frac{\sum_{1\leq i,j\leq N} C_{ij}}{\sum_{i=1}^N |B_i|},$$

where the equality is attained if and only if $\mathbf{1} = (1)_{1\leq i\leq N}$ is an eigenvector of (2.4).

Proof. It suffices to remark that

$$\lambda_1 \leq \frac{\mathbf{1}^T C\mathbf{1}}{\mathbf{1}^T V\mathbf{1}} = \frac{\int_{\mathbb{R}^3\setminus D} |\nabla u|^2 \,\mathrm{d}x}{\int_D |u|^2 \,\mathrm{d}x},$$

where u is the function defined by (2.5). The result follows from the definition (2.6) of the capacity. □

We now use (2.14) to obtain a lower bound on λ_1 different from (2.11).

Proposition 2.5. *Let $\mu_1 > 0$ be the greatest eigenvalue of the Newtonian potential $T : L^2(D) \to L^2(D)$ defined by*

$$T\phi := \int_D \Gamma(\cdot - y)\phi(y) \,\mathrm{d}\sigma(y), \qquad \phi \in L^2(D). \tag{2.16}$$

Then the minimum eigenvalue λ_1 of (2.4) satisfies

$$\lambda_1 \geq \frac{1}{\mu_1}. \tag{2.17}$$

Proof. It is obvious from (2.14) and $\mathcal{V} \subset \mathcal{D}^{1,2}(\mathbb{R}^3)$ that $\lambda_1 \geq \beta$ where β is the minimizer to the Riesz problem

$$\beta := \min_{v \in \mathcal{D}^{1,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |\nabla v|^2 \, \mathrm{d}x}{\int_D |v|^2 \, \mathrm{d}x}. \tag{2.18}$$

It remains to show that $\beta \geq 1/\mu_1$. The Euler-Lagrange equation for the minimizer w of (2.18) reads in the following variational form:

$$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^3), \int_{\mathbb{R}^3} \nabla w \cdot \nabla v \, \mathrm{d}x = \beta \int_D wv \, \mathrm{d}x. \tag{2.19}$$

Hence such a minimizer w is solution to the exterior problem

$$\begin{cases} \Delta w = \beta w \text{ in } D, \\ \Delta w = 0 \text{ in } \mathbb{R}^3 \backslash D, \\ w_+ = w_- \text{ on } \partial D, \\ \left.\dfrac{\partial w}{\partial n}\right|_+ = \left.\dfrac{\partial w}{\partial n}\right|_- \text{ on } \partial D, \\ w(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases} \tag{2.20}$$

The function $\widetilde{w} := \beta T w$ satisfies then $\Delta \widetilde{w} = \beta w$ in D and all the other properties of (2.20). Hence (2.19) and another integration by parts yield $\int_{\mathbb{R}^3} |\nabla (w - \widetilde{w})|^2 \, \mathrm{d}x = 0$. Remembering that w and $\widetilde{w}$ belong to the space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ (they vanish at infinity), we obtain $\widetilde{w} = w = \beta T w$. Consequently, the quantity $1/\beta$ is an eigenvalue of the Newtonian potential T, and it must hold $1/\beta \leq \mu_1$. □

2.3. Properties of the capacitance matrix in case of symmetries

We now examine how particular circumstances of symmetries on the distribution of resonators $D = \cup_{1 \leq i \leq N} B_i$ reflect on the structure of the capacitance matrix C and on its spectral decomposition. General results are stated for arbitrary distributions of resonators in Section 2.3.1, and for a rotationally invariant chain of resonators in Section 2.3.2. These are applied in the next Sections 2.3.3 to 2.3.6 to characterize conveniently the effective coefficients of the associated capacitance matrix for a dimer constituted of two symmetrical resonators, a trimer constituted of three symmetrical resonators, and for acoustic obstacles arranged at the vertices of a square or of a regular tetrahedron.

Throughout this part, $\mathfrak{S}_N$ denotes the group of permutations of the set $\{1, 2, \dots, N\}$.

2.3.1. *Invariance properties with respect to arbitrary group of symmetries*

Our first proposition establishes that any symmetry of the distribution of resonators D implies the commutation of the capacitance matrix with a related permutation matrix.

Proposition 2.6. *Assume that there exists an isometry S such that $SD = D$. Then:*

(i) there exists a permutation $\sigma \in \mathfrak{S}_N$ such that

$$SB_i = B_{\sigma(i)}.$$

(ii) For any $1 \leq i \leq N$, the solution u_i to (2.2) satisfies

$$u_{\sigma(i)} \circ S = u_i.$$

(iii) The coefficients of the capacitance matrix and of the volume matrix V satisfy

$$C_{\sigma(i)\sigma(j)} = C_{ij}, \quad V_{\sigma(i)\sigma(j)} = V_{ij}, \qquad \forall 1 \leq i,j \leq N.$$

In other words, C and V commutes with the permutation matrix $P_\sigma := (\delta_{\sigma(i)j})_{1 \leq i,j \leq N}$:

$$P_\sigma C = C P_\sigma, \qquad P_\sigma V = V P_\sigma.$$

Proof. (i) An isometry preserves connected components so such a permutation must exist.
(ii) $u_{\sigma(i)} \circ S$ satisfies $-\Delta(u_{\sigma(i)} \circ S) = 0$ and $u_{\sigma(i)} \circ S = \delta_{\sigma(i)\sigma(j)} = \delta_{ij}$ on ∂B_j, hence $u_{\sigma(i)} \circ S = u_i$.
(iii) By using a change of variables, we obtain

$$C_{\sigma(i)\sigma(j)} = \int_{\mathbb{R}^3 \backslash D} \nabla u_{\sigma(i)} \cdot \nabla u_{\sigma(j)} \, \mathrm{d}\sigma = \int_{\mathbb{R}^3 \backslash D} \nabla u_i \cdot \nabla u_j \, \mathrm{d}\sigma = C_{ij}.$$

For the matrix V, it is enough to remark that

$$V_{\sigma(i)\sigma(j)} = \delta_{\sigma(i)\sigma(j)} |B_{\sigma(i)}| = \delta_{ij} |B_i| = V_{ij}.$$

□

The commutation of C with the permutation matrix P_σ has several consequences on its coefficients and its spectrum, which can be characterized thanks to the following result from Stuart and Weaver [76].

Proposition 2.7 (Stuart and Weaver [76]). *Let*

$$\sigma = (i_{11}i_{12}\dots i_{1h_1})(i_{21}i_{22}\dots i_{2h_2})\dots(i_{m1}i_{m2}\dots i_{mh_m}) \tag{2.21}$$

be the decomposition of σ *into products of* $m \geq 1$ *disjoint cycles. For* $1 \leq p \leq m$, *let* U_p *be the* $N \times h_p$ *selection matrix*

$$U_p = \begin{bmatrix} \boldsymbol{e}_{i_{p1}} & \dots & \boldsymbol{e}_{i_{ph_p}} \end{bmatrix},$$

where $(\boldsymbol{e}_i)_{1\leq i\leq N}$ *is the canonical basis of* $\mathbb{R}^N$. *Since* $P_\sigma C = CP_\sigma$:

(i) For any $1 \leq p \leq m$, $U_p^T C U_p$ *is a* $h_p \times h_p$ *circulant matrix of the form*

$$U_p^T C U_p = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{h_p-1} \\ a_{h_p-1} & a_0 & a_1 & \dots & \dots & a_{h_p-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ a_1 & a_2 & a_3 & \dots & a_{h_p-1} & a_0 \end{bmatrix}.$$

(ii) For any $1 \leq p, q \leq m$ *with* $p \neq q$, *let* $g = \gcd(h_p, h_q)$. $U_p^T C U_q$ *is a* $h_p \times h_q$ *matrix of the form*

$$U_p^T C U_q = \left[\begin{array}{ccccccc|ccc} b_0 & b_1 & b_2 & \dots & \dots & \dots & b_{g-1} & b_0 & b_1 & \dots \\ b_{g-1} & b_0 & b_1 & \dots & \dots & \dots & b_{g-2} & b_{g-1} & b_0 & \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & & & \ddots \\ b_2 & & & & & & b_1 & b_2 & & \\ b_1 & b_2 & \dots & & \dots & b_{g-1} & b_0 & b_1 & & \\ \hline b_0 & b_1 & b_2 & \dots & \dots & \dots & b_{g-1} & \ddots & & \\ \vdots & \ddots & & & & & & & & \end{array}\right],$$

where the main block is a $g \times g$ *circulant matrix copied* $h_p/g \times h_q/g$ *times.*

(iii) Since $C = C^T$, *it additionally holds, for each block:*

$$a_i = a_{h_p-i} \text{ for any } 1 \leq i \leq h_p, \qquad \text{and } U_q^T C U_p = (U_p^T C U_q)^T \text{ for } p \neq q.$$

Let us recall that the eigenvalues of P_σ are the h_p roots of the unity

$$\mathrm{sp}(P_\sigma) = \bigcup_{1\leq j\leq m} \{1, \omega_{h_j}, \dots, \omega_{h_j}^{h_j-1}\},$$

where $\omega_{h_j} := e^{\frac{2i\pi}{h_j}}$. The associated eigenvectors are

$$U_j \begin{pmatrix} 1 \\ \omega_{h_j}^p \\ \vdots \\ \omega_{h_j}^{p(h_j-1)} \end{pmatrix} U_j^T \text{ for } 0 \leq p \leq h_j - 1, \quad 1 \leq j \leq m.$$

We emphasize that P_σ may admit multiple eigenvalues as recalled in the following remark.

Remark 2.1. (i) The integer 1 is always an eigenvalue of P_σ with multiplicity the number of cycles m occuring in the decomposition (2.21): a basis of eigenvectors is $(U_j \mathbf{1} U_j^T)_{1\leq j\leq m}$ where $\mathbf{1} = (1)_{1\leq i\leq h_m}$.

(ii) An eigenvalue $\omega_{h_j}^p$ with $0 < p \leq h_j - 1$ and $1 \leq j \leq m$ is multiple if and only if there exists another cycle such that $\gcd(h_m, h_l) \neq 1$ and $p = nh_m/\gcd(h_m, h_l)$ for an integer $1 \leq n < \gcd(h_m, h_l)$.

It is well-known that matrices which commute share the same invariant subspaces [56, 77]. The identity $P_\sigma C = CP_\sigma$ implies the following constraints for the spectral decomposition of C.

Proposition 2.8. *(i) Denote by* $\alpha_1, \dots, \alpha_m$ *the distinct eigenvalues of* P_σ *with multiplicities* $n_1, \dots, n_m$ *and* $E_1 \oplus \dots \oplus E_m$ *the associated eigenspace decomposition. There exists a basis adapted to this decomposition in which* C *is block diagonal on each space* E_i, $1 \leq i \leq m$.

(ii) If $\boldsymbol{a}_i$ *is an eigenvector of* (2.4) *for* $1 \leq i \leq N$, *then the vector* $P_\sigma \boldsymbol{a}_i$ *is also an eigenvector of* (2.4) *associated to the same eigenvalue* λ_i.

(iii) In particular, the first eigenvector $\boldsymbol{a}_1$ *associated to the simple eigenvalue* λ_1 *satisfies* $P_\sigma \boldsymbol{a}_1 = \boldsymbol{a}_1$.

Proof. (i) is a direct consequence of the commutation property $P_\sigma C = CP_\sigma$.
(ii) If $C\boldsymbol{a}_i = \lambda_i \boldsymbol{a}_i$, then $CP_\sigma \boldsymbol{a}_i = P_\sigma C\boldsymbol{a}_i = \lambda_i P_\sigma \boldsymbol{a}_i$.
(iii) is obtained by using (ii) and the fact that the first eigenvalue λ_1 is simple. □

Remark 2.2. The same result holds for the generalized eigenvalue problem (2.4).

The previous result yields a simple sufficient condition to obtain that the vector of ones is an eigenvector of C.

Corollary 2.1. *If for any resonator B_i with $1 \le i \le N$, there is an isometry S such that $SD = D$ and $SB_1 = B_i$, then the eigenvector $\boldsymbol{a}_1$ associated to the smallest eigenvalue λ_1 is proportional to the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$.*

Proof. The assumption implies that for any $1 \le i \le N$, one can find a permutation σ such that $\sigma(1) = i$ and $P_\sigma C = CP_\sigma$. The previous proposition states that $P_\sigma \boldsymbol{a}_1 = \boldsymbol{a}_1$, which implies

$$a_{1,1} = a_{1,\sigma(1)} = a_{1,i}.$$

Consequently, $\boldsymbol{a}_1 = a_{1,1}\mathbf{1}$ and the result is obtained. □

The next subsections apply the above results to characterize the spectral decomposition of the capacitance matrix for particular systems of resonators.

2.3.2. *A general chain of N resonators with rotational symmetry*

In what follows, we examine the case where $D = \cup_{i=1}^N R^i B$ is invariant by the rotation R of angle $2\pi/N$ around the one-dimensional axis $\boldsymbol{e}_3$, which yields the invariance by the permutation group generated by the N cycle

$$\sigma = \begin{pmatrix} 0 & \dots & N-1 \end{pmatrix}.$$

An occurence of such set of resonator D is illustrated on the schematic of Figure 2.

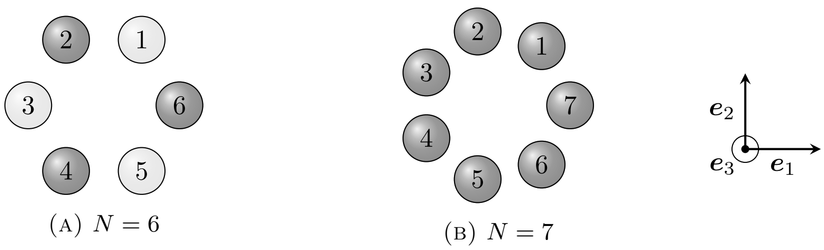

FIGURE 2. Chains of respectively (three-dimensional) 6 and 7 spherical resonators rotationally invariant around the axis $\boldsymbol{e}_3$. The rotational symmetry entails the invariance of the geometry by the permutation group generated by the N-cycle $\sigma = \begin{pmatrix} 0 & \dots & N-1 \end{pmatrix}$. The alternate colors visible on Figure 2a with $N = 6$ illustrates the fact that the sign changing vector $((-1)^i)_{1\le i\le N}$ is an eigenvector of the capacitance matrix C when N is even.

Propositions 2.7 and 2.8 together imply the following result.

Proposition 2.9. *Assume that the set of resonator is rotationally invariant, in the sense that Proposition 2.6 holds with the cyclic permutation σ given by*

$$\sigma = \begin{pmatrix} 0 & \dots & N-1 \end{pmatrix}.$$

Then:

(i) C is a $N \times N$ circulant matrix of the form

$$C = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \dots & \dots & a_{h_p-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ a_1 & a_2 & a_3 & \dots & a_{N-1} & a_0 \end{bmatrix},$$

with $a_i = a_{N-i}$ for any $0 \le i < N$.

(ii) The spectrum of C is given by

$$\mathrm{sp}(C) = \left\{ \bigcup_{0\le p\le N-1} \sum_{i=0}^{N-1} a_i \omega^{pi} \right\},$$

where $\omega = \exp(2\mathrm{i}\pi/N)$ with respective eigenvectors $(\omega^{pi})_{1\le i\le N}$ for $0 \le p \le N-1$.

Moreover, if N is even, this can be rewritten as the following set of real eigenvalues

$$\mathrm{sp}(C) = \left\{ \sum_{i=0}^{N-1} a_i \right\} \cup \left\{ \sum_{i=0}^{N-1} (-1)^i a_i \right\} \cup \left\{ a_0 + (-1)^p a_{N/2} + \sum_{i=1}^{N/2-1} 2\cos\left(\frac{2ip\pi}{N}\right) a_i, \quad 0 < p < \frac{N}{2} \right\},$$

with respective eigenvectors

$$\mathbf{1} = (1)_{1\le i\le N}, ((-1)^i)_{1\le i\le N}, \left\{ (\cos(2ip\pi/N))_{1\le i\le N}, (\sin(2ip\pi/N))_{1\le i\le N} \quad 0 < p < \frac{N}{2} \right\},$$

while if N is odd, the spectrum of C reads instead

$$\mathrm{sp}(C) = \left\{ \sum_{i=0}^{N-1} a_i \right\} \cup \left\{ a_0 + \sum_{i=1}^{(N-1)/2} 2\cos\left(\frac{2ip\pi}{N}\right) a_i, \quad 1 \le p \le \frac{N-1}{2} \right\},$$

with respective eigenvectors

$$\mathbf{1} = (1)_{1\le i\le N}, \left\{ \left(\cos\left(\frac{2ip\pi}{N}\right)\right)_{1\le i\le N}, \left(\sin\left(\frac{2ip\pi}{N}\right)\right)_{1\le i\le N} \quad 1 \le p \le \frac{N-1}{2} \right\}.$$

Proof. See [76]. The rewriting for N even is obtained by using $a_i = a_{N-i}$ for $0 \le i \le N$. ☐

Corollary 2.2. *In particular, for any set D of N resonators having a rotational invariance symmetry, the vector $\mathbf{1} = (1)_{1\le i\le N}$ is an eigenvector for the general eigenvalue problem (2.4) associated with the eigenvalue $\lambda_1 = \mathrm{cap}(D)/|D|$.*

The next Sections 2.3.3 to 2.3.5 apply this result explicitly for the particular cases $N = 2, 3, 4$.

2.3.3. *Example 1: the dimer*

FIGURE 3. A dimer made of two symmetrical resonators. The alternate color emphasize the occurrence of the dipole mode $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$.

The structure of the capacitance matrix for a dimer made of two identical spheres (Figure 3) has already been obtained in [59]. Proposition 2.9 enables to retrieve very quickly this result even in the case where the resonators are not necessary spherical but only symmetrical. It states that the matrix C has the form

$$C = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}, \tag{2.22}$$

for two positive constants $a > b > 0$. The distinct eigenvalues of C are

$$a - b, \quad a + b, \tag{2.23}$$

with associated eigenvectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.24}$$

corresponding respectively to monopole and dipole modes.

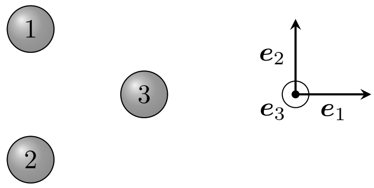

FIGURE 4. A trimer made of three resonators rotationally invariant around the axis $\boldsymbol{e}_3$.

2.3.4. *Example 2: the trimer*

If D is a trimer constituted of three rotationally invarariant, not necessary spherical resonators (illustrated on Figure 4), the result of Proposition 2.9 states that there exists $a, b > 0$ such that C has the form

$$C = \begin{bmatrix} a & -b & -b \\ -b & a & -b \\ -b & -b & a \end{bmatrix}.$$

The eigenvalues are

$$a - 2b, \quad a + b,$$

with respective multiplicities 1 and 2, and associated eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

2.3.5. *Example 3: the quadrimer*

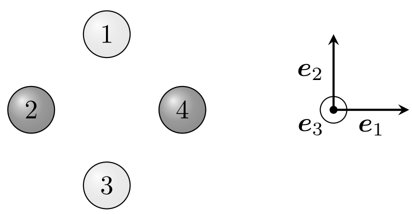

FIGURE 5. A quadrimer constituted of four resonators rotationally invariant around the axis $\boldsymbol{e}_3$. The alternate colors emphasize the existence of the mode $\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

If D is a quadrimer, i.e. four resonators rotationally invariant located at the vertices of a square (illustrated on Figure 5), not necessary with spherical shapes, the result of Proposition 2.9 states that there exist three effective coefficients $a, b, c > 0$ such that C has the form

$$C = \begin{bmatrix} a & -b & -c & -b \\ -b & a & -b & -c \\ -c & -b & a & -b \\ -b & -c & -b & a \end{bmatrix}. \tag{2.25}$$

The eigenvalues are

$$a - 2b - c, \quad a + 2b - c, \quad a + c,$$

where the last one is double, with respective eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{2.26}$$

2.3.6. *Example 4: identical spheres arranged at the vertices of a regular tetrahedron*

We finally consider the case where D is made of four identical spheres arranged at the vertices of a regular thetrahedron (illustrated on Figure 6). The symmetries on the middle axes entail the invariance by any trans-

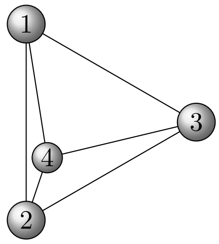

FIGURE 6. A set of four identical resonators arranged at the vertices of a regular tetrahedron.

position, hence C commutes with the matrix P_σ for any permutation $\sigma \in \mathfrak{S}_4$. This implies the existence of two effective coefficients $a, b > 0$ such that C has the form

$$C = \begin{bmatrix} a & -b & -b & -b \\ -b & a & -b & -b \\ -b & -b & a & -b \\ -b & -b & -b & a \end{bmatrix}. \tag{2.27}$$

The eigenvalues are

$$a - 3b, \quad a + b,$$

where $a + b$ is a triple eigenvalue, with the same respective eigenvectors given by (2.26).

2.4. **The vector of ones is generically not an eigenvector**

We finally close this section devoted to the capacitance matrix by establishing a result which shows that, for generic domains, the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$ is in general not an eigenvector for the problem (2.4), at least for a set of $N \ge 3$ resonators. As we shall see in the next sections, this has several consequences for the resonant system (1.1), one of them being that a system of N resonators behaves in general as a monopole scatterer in the far field near any of the resonant frequencies $(\omega_i^\pm(\delta))_{1\le i\le N}$ (see Proposition 5.3 below). We conjecture that the result remains true for $N = 2$, although our arguments are not sufficient to be conclusive in this case.

Our result is precisely formulated in Proposition 2.11 below; it states that if $N \ge 3$ and if the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$ is an eigenvector of (2.4), then most of the deformation $(I + \boldsymbol{\theta})D$ of the shape of the resonators D by a small vector field $\boldsymbol{\theta} \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ do not satisfy this property. Throughout this section, $W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ is the Sobolev space

$$W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3) := \{\boldsymbol{\theta} \,|\, \boldsymbol{\theta} \in L^\infty(\mathbb{R}^3, \mathbb{R}^3) \text{ and } \nabla\boldsymbol{\theta} \in L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3})\}.$$

We assume throughout this section that D is a smooth domain of class at least $\mathcal{C}^2$ which is such that $\mathbf{1}$ is an eigenvector of (2.4). For any small vector field $\boldsymbol{\theta} \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$, we consider $(B_{i,\boldsymbol{\theta}})_{1\le i\le N} := ((I + \boldsymbol{\theta})B_i)_{1\le i\le N}$ the deformed acoustic obstacles and $D_{\boldsymbol{\theta}} := (I + \boldsymbol{\theta})D$ the deformed domain obtained by moving the points of ∂D according to the vector field $\boldsymbol{\theta}$. We denote by $C(\boldsymbol{\theta})$ and $V(\boldsymbol{\theta})$ the associated capacitance and volume matrices (obtained by substituting D with $D_{\boldsymbol{\theta}}$ in the definitions (2.1) to (2.3)), and we consider the generalized eigenvalue problem of finding $(\boldsymbol{a}_i(\boldsymbol{\theta}))_{1\le i\le N}$ and $0 < \lambda_1(\boldsymbol{\theta}) < \lambda_2(\boldsymbol{\theta}) \le \ldots \le \lambda_N(\boldsymbol{\theta})$ such that

$$C(\boldsymbol{\theta})\boldsymbol{a}_i(\boldsymbol{\theta}) = \lambda_i(\boldsymbol{\theta})V(\boldsymbol{\theta})\boldsymbol{a}_i(\boldsymbol{\theta}), \qquad 1 \le i \le N. \tag{2.28}$$

Since the first eigenvalue λ_1 is simple and $\boldsymbol{\theta} \mapsto C(\boldsymbol{\theta})$ and $\boldsymbol{\theta} \mapsto V(\boldsymbol{\theta})$ are continuous, it is possible to choose $\boldsymbol{a}_1(\boldsymbol{\theta})$ such that the function $\boldsymbol{\theta} \mapsto \boldsymbol{a}_1(\boldsymbol{\theta})$ is continuous (in fact, smooth) and $\boldsymbol{a}_1(0) = \boldsymbol{a}_1$ [57]. Assuming $\boldsymbol{a}_1$ is proportional to the vector of ones, i.e.

$$\boldsymbol{a}_1 \equiv \boldsymbol{a}_1(0) = |D|^{-1/2}\mathbf{1} \text{ and } \lambda_1 \equiv \lambda_1(0) = \text{cap}(D)/|D|, \tag{2.29}$$

our strategy is to compute the Fréchet derivative $\boldsymbol{a}_1' : W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}^N$ at $\boldsymbol{\theta} = 0$ and to characterize the vector fields $\boldsymbol{\theta} \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ for which $\boldsymbol{a}_1'(\boldsymbol{\theta}) \ne 0$. Since the first variation of $\boldsymbol{a}_1(\boldsymbol{\theta})$ is not zero while $\boldsymbol{a}_1(0) = \mathbf{1}$, this is enough to obtain that the vector of ones $\mathbf{1}$ is not an eigenvector of (2.28) associated to the perturbed set of resonators $D_{\boldsymbol{\theta}}$.

Our analysis heavily relies on shape differential calculus; the reader is referred to the textbooks [43, 75, 36] for a detailed introduction to this topic.

Proposition 2.10. *The first order asymptotics of $\lambda_1(\boldsymbol{\theta})$ and $\boldsymbol{a}_1(\boldsymbol{\theta})$ as $\boldsymbol{\theta}\to 0$ are given by*

$$\lambda_1(\boldsymbol{\theta}) = \frac{\mathrm{cap}(D)}{|D|} + \frac{1}{|D|}\int_{\partial D}\left[\left|\frac{\partial u}{\partial n}\right|^2 - \frac{\mathrm{cap}(D)}{|D|}\right]\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + o(\boldsymbol{\theta})_{W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)}, \tag{2.30}$$

$$\boldsymbol{a}_1(\boldsymbol{\theta}) = \boldsymbol{a}_1 + \sum_{i>1}\sum_{j=1}^{N}\frac{1}{\lambda_1-\lambda_i}\frac{\boldsymbol{a}_i^T\boldsymbol{e}_j}{|D|^{\frac{1}{2}}}\left[\int_{\partial D}\left[\frac{\partial u_j}{\partial n}\frac{\partial u}{\partial n} - \frac{\mathrm{cap}(D)}{|D|}1_{\partial B_j}\right]\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma\right]\boldsymbol{a}_i + o(\boldsymbol{\theta})_{W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)}, \tag{2.31}$$

where we recall the definitions (2.2) and (2.5) of the functions $(u_i)_{1\le i\le N}$ and u.

Proof. We may assume without loss of generality that $\boldsymbol{a}_1(\boldsymbol{\theta})$ is a unit vector with respect to the inner product V:

$$\boldsymbol{a}_1(\boldsymbol{\theta})^T V(\boldsymbol{\theta})\boldsymbol{a}_1(\boldsymbol{\theta}) = 1. \tag{2.32}$$

Using standard shape differential calculus, see e.g. [43, 36], we find that the matrix $V(\boldsymbol{\theta})$ is differentiable and that the Fréchet derivative of its coefficients at $\boldsymbol{\theta}=0$ is given by:

$$V'_{ij}(\boldsymbol{\theta}) = \delta_{ij}\int_{\partial B_i}\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma.$$

We then sketch the computation of the Fréchet derivative of the capacitance coefficients $(C'_{ij}(\boldsymbol{\theta}))_{1\le i,j\le N}$. Differentiating the identity $C_{ij} = \int_{\mathbb{R}^3\setminus D}\nabla u_i\cdot\nabla u_j\,\mathrm{d}x$ (eqn. (2.8)), we find

$$C'_{ij}(\boldsymbol{\theta}) = \int_{\mathbb{R}^3\setminus D}(\mathrm{div}(\boldsymbol{\theta})I - \nabla\boldsymbol{\theta} - \nabla\boldsymbol{\theta}^T)\nabla u_i\cdot\nabla u_j\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus D}(\nabla u'_i(\boldsymbol{\theta})\cdot\nabla u_j + \nabla u_i\cdot\nabla u'_j(\boldsymbol{\theta}))\,\mathrm{d}x,$$

where $u'_i(\boldsymbol{\theta})$ and $u'_j(\boldsymbol{\theta})$ are the Lagrangian shape derivatives of u_i and u_j. Since these derivatives satisfy $u'_i(\boldsymbol{\theta})=0$ and $u'_j(\boldsymbol{\theta})=0$ on ∂D, (2.7) implies that the second integral vanishes, and we obtain after an integration by parts:

$$C'_{ij}(\boldsymbol{\theta}) = -\int_{\partial D}\left(\nabla u_i\cdot\nabla u_j - 2\frac{\partial u_i}{\partial n}\frac{\partial u_j}{\partial n}\right)\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = \int_{\partial D}\frac{\partial u_i}{\partial n}\frac{\partial u_j}{\partial n}\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma.$$

Differentiating now (2.28) and (2.32) with respect to $\boldsymbol{\theta}$, we obtain

$$\begin{aligned}(C-\lambda_1 V)\boldsymbol{a}'_1(\boldsymbol{\theta}) &= \lambda'_1(\boldsymbol{\theta})V\boldsymbol{a}_1 - (C'(\boldsymbol{\theta}) - \lambda_1 V'(\boldsymbol{\theta}))\boldsymbol{a}_1\\ 2\boldsymbol{a}'_1(\boldsymbol{\theta})^T V\boldsymbol{a}_1 &= -\boldsymbol{a}_1^T V'(\boldsymbol{\theta})\boldsymbol{a}_1.\end{aligned} \tag{2.33}$$

We compute $\lambda'_1(\boldsymbol{\theta})$: differentiating the identity $\lambda_1(\boldsymbol{\theta}) = \boldsymbol{a}_1(\boldsymbol{\theta})^T C(\boldsymbol{\theta})\boldsymbol{a}_1(\boldsymbol{\theta})$, we find

$$\begin{aligned}\lambda'_1(\boldsymbol{\theta}) &= 2\boldsymbol{a}'_1(\boldsymbol{\theta})^T C\boldsymbol{a}_1 + \boldsymbol{a}_1^T C'(\boldsymbol{\theta})\boldsymbol{a}_1 = 2\lambda_1\boldsymbol{a}'_1(\boldsymbol{\theta})^T V\boldsymbol{a}_1 + \boldsymbol{a}_1^T C'(\boldsymbol{\theta})\boldsymbol{a}_1\\ &= \boldsymbol{a}_1^T(C'(\boldsymbol{\theta}) - \lambda_1 V'(\boldsymbol{\theta}))\boldsymbol{a}_1.\end{aligned} \tag{2.34}$$

Then, inverting (2.33) yields

$$\begin{aligned}\boldsymbol{a}'_1(\boldsymbol{\theta}) &= \sum_{i>1}\frac{1}{\lambda_1-\lambda_i}[\boldsymbol{a}_i^T(I - V\boldsymbol{a}_1\boldsymbol{a}_1^T)(C'-\lambda_1 V')\boldsymbol{a}_1]\boldsymbol{a}_i\\ &= \sum_{i>1}\frac{1}{\lambda_1-\lambda_i}[\boldsymbol{a}_i^T(C'-\lambda_1 V')\boldsymbol{a}_1]\boldsymbol{a}_i.\end{aligned} \tag{2.35}$$

The identities (2.30) and (2.31) follow by substituting the values of $C'_{ij}(\boldsymbol{\theta})$ and $V'_{ij}(\boldsymbol{\theta})$ into (2.34) and (2.35). ☐

We deduce the existence of a dense set of shape perturbations which cancel the property of $\mathbf{1}$ being an eigenvector of (2.4).

Proposition 2.11. *(i) If $\boldsymbol{a}_1 = |D|^{-1/2}\mathbf{1}$ is an eigenvector of (2.4), then $\boldsymbol{a}'_1(\boldsymbol{\theta}) = 0$ if and only if*

$$\int_{\partial D}(f_1 - f_i)\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0 \text{ for any } 1 < i \le N, \tag{2.36}$$

where $(f_i)_{1\le i\le N}$ are the functions defined by

$$f_i := \frac{\partial u_i}{\partial n}\frac{\partial u}{\partial n} - \lambda_1 1_{\partial B_i}.$$

(ii) If $N>2$, then none of the functions $f_1 - f_i$ vanishes identically on ∂D and the set of deformations $\boldsymbol{\theta}$ which do not fulfill the condition (2.36) form a dense subset of $W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)$.

Proof. 1. Let $\boldsymbol{v}(\boldsymbol{\theta}) = (v_i(\boldsymbol{\theta}))_{1\le i\le N}$ be the vector defined by

$$v_i(\boldsymbol{\theta}) := \int_{\partial D} f_i\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma, \qquad 1\le i\le N.$$

From (2.35) and the fact that the vectors $(\boldsymbol{a}_i)_{1\le i\le N}$ form an orthonormal basis, $\boldsymbol{a}'_1(\boldsymbol{\theta}) = 0$ is equivalent to

$$\boldsymbol{a}_i^T\boldsymbol{v}(\boldsymbol{\theta}) = 0 \text{ for any } 1 < i \le N,$$

i.e. $\boldsymbol{v}(\boldsymbol{\theta})$ must be proportional to the vector $\boldsymbol{a}_1 = |D|^{-1/2}\mathbf{1}$. Therefore, $\boldsymbol{a}_1'(\boldsymbol{\theta}) = 0$ if and only if $v_1(\boldsymbol{\theta}) - v_i(\boldsymbol{\theta}) = 0$ for any $1 < i \leq N$ which is equivalent to (2.36).
(ii) It is sufficient to prove that at least one of the linear forms of (2.36) is not zero to obtain that all deformations $\boldsymbol{\theta}$ do not fulfill (2.36) up to a finite-dimensional subset. We prove that it is not possible that $f_1 - f_i = 0$ on ∂D for any $1 < i \leq N$. If it is the case and assuming $N > 2$, then it must hold

$$\frac{\partial(u_1 - u_i)}{\partial n}\frac{\partial u}{\partial n} = 0 \text{ on } \partial B_l.$$

for any $l \notin \{1, i\}$. Since $\frac{\partial u}{\partial n} < 0$ on ∂B_l, the function $w := u_1 - u_i$ satisfies the overdetermined relations $\frac{\partial w}{\partial n} = 0$ on ∂B_l and $w = 0$ in ∂B_l. However, recalling that w can be expressed as a single layer potential:

$$w = \mathcal{S}_D[\phi] \text{ with } \phi = (\mathcal{S}_D^{-1})[1_{\partial B_1} - 1_{\partial B_i}], \tag{2.37}$$

the jump relation for the normal derivative of $\mathcal{S}_D[\phi]$ yields

$$\phi = \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] := \left.\frac{\partial w}{\partial n}\right|_+ - \left.\frac{\partial w}{\partial n}\right|_- = 0 \text{ on } \partial B_l.$$

Therefore the representation (2.37) of w as a single layer potential implies that w is smooth in the vicinity of ∂B_l. Consequently, w is harmonic in $\mathbb{R}^3 \backslash (B_1 \cup B_i)$ and satisfies $w = 0$ on B_l. By the unique continuation principle (see e.g. [35, 54]), we obtain that w vanishes identically in $\mathbb{R}^3 \backslash (B_1 \cup B_i)$, which contradicts $w = 1$ on ∂B_1. □

Remark 2.3. The above proof does not allow to obtain a version of this result for the case $N = 2$. However, we may conjecture that the equality $f_1 = f_2$ occurs only for exceptional shapes even if $N = 2$, which would lead to the same conclusions.

3. Asymptotic analysis of the scattering resonances

We now turn to the analysis of the subwavelength resonances of the scattering problem (1.1). Following [13, 8], the solution u_{tot} can be represented as single layer potentials in D and $\mathbb{R}^3 \backslash D$:

$$u_{\text{tot}}(x) = \begin{cases} \mathcal{S}_D^{k_b}[\phi](x) \text{ if } x \in \overline{D}, \\ u_{\text{in}}(x) + \mathcal{S}_D^{k}[\psi](x) \text{ if } x \in \mathbb{R}^3 \backslash D, \end{cases} \tag{3.1}$$

where the functions $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ solve the integral equation

$$\mathcal{A}(\omega, \delta) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} u_{\text{in}} \\ \delta \frac{\partial u_{\text{in}}}{\partial n} \end{bmatrix}, \tag{3.2}$$

with the operator $\mathcal{A}(\omega, \delta)$ being given by

$$\mathcal{A}(\omega, \delta) = \begin{bmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^{k} \\ -\frac{1}{2}I + \mathcal{K}_D^{k_b *} & -\delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right) \end{bmatrix}.$$

Due to the Sommerfeld radiation condition, the problem (3.2) can be shown to admit a unique solution for any real frequency provided the wave number $k = \omega/v$ is not a Dirichlet eigenvalue of the domain D [14]. This assumption is naturally satisfied in the regime $\omega \to 0$.

In this section, we solve explicitly the integral formulation (3.2) by computing the inverse of the holomorphic operator $\mathcal{A}(\omega, \delta)$. This allows us to characterize the Minnaert resonances $(\omega_i^{\pm}(\delta))_{1 \leq i \leq N}$ as the poles of the meromorphic operator $\omega \mapsto \mathcal{A}(\omega, \delta)^{-1}$, and to compute full asymptotic expansions as $\delta \to 0$. Our analysis outlines as follows. We show in Section 3.1 that the invertibility of $\mathcal{A}(\omega, \delta)$ reduces to the one of a complex $N \times N$ Schur complement matrix $A(\omega, \delta)$, which is holomorphic in ω and δ. We provide an explicit formula for the inverse of $\mathcal{A}(\omega, \delta)$ and we obtain the decomposition (1.2). The scattering resonances $\omega_i^{\pm}(\delta)$ can consequently also be characterized as the solutions to the nonlinear eigenvalue problem

$$A(\omega, \delta)\boldsymbol{x} = 0. \tag{3.3}$$

After computing the asymptotic expansion of $A(\omega, \delta)$ at the order $O((\omega^2 + \delta)^2)$ in Section 3.2, we find that the nonlinear eigenvalue problem (3.3) is a perturbation of the generalized, linear eigenvalue problem (2.4). Finally, Section 3.3 applies the implicit function theorem to (3.3), which yields the Puiseux expansion (1.4) of the resonance $(\omega_i^{\pm}(\delta))$ under a simplicity assumption on the eigenvalues of the capacitance matrix C.

3.1. **Explicit inversion of the scattering operator**

In order to compute the inverse of $\mathcal{A}(\omega,\delta)$, we solve the following linear system (3.2) which reads explicitly

$$\begin{cases} \mathcal{S}_D^{k_b}[\phi]-\mathcal{S}_D^{k}[\psi]=u_{\text{in}}, \\ \left(-\frac{1}{2}I+\mathcal{K}_D^{k_b*}\right)[\phi]-\delta\left(\frac{1}{2}I+\mathcal{K}_D^{k*}\right)[\psi]=\delta\frac{\partial u_{\text{in}}}{\partial n}. \end{cases} \tag{3.4}$$

Reducing (3.4) to a single equation by using the invertibility of $\mathcal{S}_D^k$ (as in [4, 8]), we are left with

$$\begin{cases} \psi=(\mathcal{S}_D^{k})^{-1}\mathcal{S}_D^{k_b}[\phi]-(\mathcal{S}_D^{k})^{-1}[u_{\text{in}}], \\ \left(-\frac{1}{2}I+\mathcal{K}_D^{k_b*}-\delta\left(\frac{1}{2}I+\mathcal{K}_D^{k*}\right)(\mathcal{S}_D^{k})^{-1}\mathcal{S}_D^{k_b}\right)[\phi]=\delta\frac{\partial u_{\text{in}}}{\partial n}-\delta\left(\frac{1}{2}I+\mathcal{K}_D^{k*}\right)(\mathcal{S}_D^{k})^{-1}[u_{\text{in}}]. \end{cases} \tag{3.5}$$

So the invertibility of $\mathcal{A}(\omega,\delta)$ is equivalent to that of the operator

$$\mathcal{L}(\omega,\delta):=-\frac{1}{2}I+\mathcal{K}_D^{k_b*}-\delta\left(\frac{1}{2}I+\mathcal{K}_D^{k*}\right)(\mathcal{S}_D^{k})^{-1}\mathcal{S}_D^{k_b}. \tag{3.6}$$

The operator $\mathcal{L}(\omega,\delta)$ is holomorphic in the variables ω and δ. Indeed, we recall the following classical expansions of the potential (see e.g. [15, 14]).

Proposition 3.1. *The following expansions hold for the single layer potential and the Neumann-Poincaré operator as $k=\omega/v\to 0$:*

$$\mathcal{S}_D^k=\sum_{p=0}^{+\infty}k^p\mathcal{S}_{D,p}=\mathcal{S}_D+k\mathcal{S}_{D,1}+k^2\mathcal{S}_{D,2}+\dots, \tag{3.7}$$

$$\mathcal{K}_D^{k*}=\sum_{p=0}^{+\infty}k^p\mathcal{K}_{D,p}^*=\mathcal{K}_D^*+k^2\mathcal{K}_{D,2}^*+k^3\mathcal{K}_{D,3}^*+\dots, \tag{3.8}$$

where the series converges in operator norms, and where the operators $\mathcal{S}_{D,p}$ and $\mathcal{K}_{D,p}^*$ are defined by

$$\mathcal{S}_{D,p}[\phi]:=-\frac{\mathrm{i}^p}{4\pi p!}\int_{\partial D}|x-y|^{p-1}\phi(y)\,\mathrm{d}\sigma(y),\qquad \phi\in L^2(\partial D),\ p\in\mathbb{N}, \tag{3.9}$$

$$\mathcal{K}_{D,p}^*[\phi]:=-\frac{\mathrm{i}^p}{4\pi p!}\int_{\partial D}\boldsymbol{n}(x)\cdot\nabla_x|x-y|^{p-1}\phi(y)\,\mathrm{d}\sigma(y),\qquad \phi\in L^2(\partial D),\ p\in\mathbb{N}. \tag{3.10}$$

Furthermore, we have the identities

(i) $\Delta\mathcal{S}_{D,0}[\phi]=\Delta\mathcal{S}_{D,1}[\phi]=0$ *and* $\Delta\mathcal{S}_{D,p}[\phi]=-\mathcal{S}_{D,p-2}[\phi]$ *for any* $p\geq 2$,

(ii) $\mathcal{K}_{D,p}[\phi](x)=\boldsymbol{n}(x)\cdot\nabla_x\mathcal{S}_{D,p}[\phi]$ *for* $p\geq 1$, *and*

$$\int_{\partial B_i}\mathcal{K}_D^*[\phi]\,\mathrm{d}\sigma=\frac{1}{2}\int_{\partial B_i}\phi\,\mathrm{d}\sigma \text{ and } \int_{\partial B_i}\mathcal{K}_{D,p}^*[\phi]\,\mathrm{d}\sigma=-\int_{B_i}\mathcal{S}_{D,p-2}[\phi]\,\mathrm{d}\sigma \text{ for } p\geq 2.$$

In view of (3.8) we find that (3.6) can be rewritten as

$$\mathcal{L}(\omega,\delta)=-\frac{1}{2}I+\mathcal{K}_D^*+\omega^2\mathcal{B}_1(\omega)+\delta\mathcal{B}_2(\omega), \tag{3.11}$$

where $\mathcal{B}_1(\omega)$ and $\mathcal{B}_2(\omega)$ are the holomorphic and compact operators defined by

$$\mathcal{B}_1(\omega):=\sum_{p=0}^{+\infty}\frac{\omega^p}{v_b^{p+2}}\mathcal{K}_{D,p}^*,\qquad \mathcal{B}_2(\omega):=\left(\frac{1}{2}I+\mathcal{K}_D^{k*}\right)(\mathcal{S}_D^{k})^{-1}\mathcal{S}_D^{k_b}. \tag{3.12}$$

The operator $\mathcal{L}(\omega,\delta)$ is a compact perturbation of the Fredholm operator $-\frac{1}{2}I+\mathcal{K}_D^*$, which has a finite dimensional kernel, as recalled in the following proposition (see e.g. [71, 65, 14]):

Proposition 3.2. *The kernel of the operator $-\frac{1}{2}I+\mathcal{K}_D^*$ is the N-dimensional space*

$$\operatorname{Ker}\left(-\frac{1}{2}I+\mathcal{K}_D^*\right)=\operatorname{span}((\psi_i^*)_{1\leq i\leq N}),$$

where $(\psi_i^*)_{1\leq i\leq N}$ are the functions defined by

$$\psi_i^*=\mathcal{S}_D^{-1}[1_{\partial B_i}],\qquad 1\leq i\leq N.$$

The range of the operator $-\frac{1}{2}I+\mathcal{K}_D^*$ is the space of zero average square integrable functions $L_0^2(\partial D)$:

$$\operatorname{Ran}\left(-\frac{1}{2}I+\mathcal{K}_D^*\right)=L_0^2(\partial D),$$

where $L^2_0(\partial D) := \left\{ \phi \in L^2(\partial D) \,\middle|\, \int_{\partial B_i} \phi \, \mathrm{d}\sigma = 0 \text{ for any } 1 \le i \le N \right\}$*. Furthermore, we have the direct sum decomposition*

$$L^2(\partial D) = L^2_0(\partial D) \oplus \mathrm{Ker}\left(-\frac{1}{2}I + \mathcal{K}^*_D\right),$$

and $-\frac{1}{2}I + \mathcal{K}^*_D$ *is invertible as an operator* $L^2_0(\partial D) \to L^2_0(\partial D)$.

Classically, the computation of the inverse of the holomorphic Fredholm operator $\mathcal{L}(\omega, \delta)$ reduces to that of a finite dimensional holomorphic Schur complement matrix after introducing suitable projections on the kernel and coimage [66, 67]. In our context, we compute $\mathcal{L}(\omega, \delta)^{-1}$ by using a method inspired from [24] (also used in the proof of Theorem 2.1 of [13] in the case $N = 1$ of a single bubble), which consists in introducing a constant finite-range operator $\mathcal{H}$ making the operator $-\frac{1}{2}I + \mathcal{K}^*_D + \mathcal{H}$ invertible.

In order to introduce the operator $\mathcal{H}$, we introduce a new basis of functions $(\phi^*_i)_{1\le i\le N}$ of $\mathrm{Ker}(-\frac{1}{2}I + \mathcal{K}^*)$ defined by

$$\phi^*_i := -\sum_{j=1}^{N} (C^{-1})_{ij} \psi^*_j, \qquad 1 \le i \le N, \tag{3.13}$$

where C is the *capacitance* matrix of (2.9). The definition (3.13) ensures the property

$$\int_{\partial B_i} \phi^*_j \, \mathrm{d}\sigma = \delta_{ij} \text{ for any } 1 \le i, j \le N. \tag{3.14}$$

Definition 3.1. We denote by $\mathcal{H} : L^2(\partial D) \to L^2(\partial D)$ the unique projection operator satisfying $\mathrm{Ker}(\mathcal{H}) = L^2_0(\partial D)$ and $\mathrm{Ran}(\mathcal{H}) = \mathrm{Ker}(-\frac{1}{2}I + \mathcal{K}^*_D)$. For any $\phi \in L^2(\partial D)$, the value of $\mathcal{H}[\phi]$ reads explicitly

$$\mathcal{H}[\phi] = \sum_{i=1}^{N} \left(\int_{\partial B_i} \phi \, \mathrm{d}\sigma \right) \phi^*_i. \tag{3.15}$$

Proposition 3.3. *The operator* $\mathcal{L}(\omega, \delta)$ *defined in (3.6) can be decomposed as*

$$\mathcal{L}(\omega, \delta) = \mathcal{L}_0 - \mathcal{H} + \omega^2 \mathcal{B}_1(\omega) + \delta \mathcal{B}_2(\omega), \tag{3.16}$$

where $\mathcal{L}_0 := -\frac{1}{2}I + \mathcal{K}^*_D + \mathcal{H}$ *is an invertible Fredholm operator. The inverse of* $\mathcal{L}_0$ *reads explicitly:*

$$\mathcal{L}_0^{-1}[\phi] = \left(-\frac{1}{2}I + \mathcal{K}^*_D\right)^{-1} \left(\phi - \sum_{i=1}^{N} \left(\int_{\partial B_i} \phi \, \mathrm{d}\sigma \right) \phi^*_i \right) + \sum_{i=1}^{N} \left(\int_{\partial B_i} \phi \, \mathrm{d}\sigma \right) \phi^*_i, \quad \phi \in L^2(\partial D), \tag{3.17}$$

where $(-\frac{1}{2}I + \mathcal{K}^*_D)^{-1}$ *is the inverse of the operator* $-\frac{1}{2}I + \mathcal{K}^*_D : L^2_0(\partial D) \to L^2_0(\partial D)$*. Furthermore, the following properties hold true:*

- $\mathcal{H}[\phi^*_i] = \mathcal{L}_0[\phi^*_i] = \phi^*_i$ *for any* $1 \le i \le N$.
- $\int_{\partial B_i} \mathcal{L}_0^{-1}[\phi] \, \mathrm{d}\sigma = \int_{\partial B_i} \phi \, \mathrm{d}\sigma$ *for any* $1 \le i \le N$ *and* $\phi \in L^2(\partial B_i)$.
- $\phi = (\phi - \mathcal{H}[\phi]) + \mathcal{H}[\phi]$ *is the direct sum decomposition of* $\phi \in L^2(\partial D)$ *on* $L^2_0(\partial D) \oplus \mathrm{Ker}(-\frac{1}{2}I + \mathcal{K}^*_D)$.

The decomposition (3.16) reads

$$\mathcal{L}(\omega, \delta) = \mathcal{G}(\omega, \delta) - \mathcal{H}, \tag{3.18}$$

where $\mathcal{G}(\omega, \delta)$ is the operator

$$\mathcal{G}(\omega, \delta) := \mathcal{L}_0 + \omega^2 \mathcal{B}_1(\omega) + \delta \mathcal{B}_2(\omega).$$

Since $\mathcal{L}_0$ is invertible, $\mathcal{G}(\omega, \delta)$ is a holomorphic invertible Fredholm operator, whose inverse can be easily computed thanks to a Neumann series.

Lemma 3.1. *The operator* $\mathcal{G}(\omega, \delta)$ *is invertible for sufficiently small* ω *and* δ*: more explicitly, the inverse reads*

$$(\mathcal{G}(\omega, \delta))^{-1} = \mathcal{L}_0^{-1} - \mathcal{C}(\omega, \delta), \tag{3.19}$$

where $\mathcal{C}(\omega, \delta)$ *is the compact operator of order* $O(\omega^2 + \delta)$ *defined by the following Neumann series:*

$$\mathcal{C}(\omega, \delta) := \sum_{p=1}^{+\infty} (-1)^{p+1} \mathcal{L}_0^{-1}((\omega^2 \mathcal{B}_1(\omega) + \delta \mathcal{B}_2(\omega)) \mathcal{L}_0^{-1})^p. \tag{3.20}$$

Equation (3.18) is analogous to the "pole-pencil decomposition" considered in [24, 19], in the sense that $\mathcal{L}(\omega, \delta)$ is the sum of a holomorphic operator easily invertible and a constant finite-rank operator. This feature together with the previous Lemma 3.1 allow to solve conveniently the problem (3.4).

Proposition 3.4. *The operator $\mathcal{A}(\omega,\delta)$ is invertible if and only if the $N\times N$ matrix $A(\omega,\delta)\equiv(A(\omega,\delta)_{ij})_{1\le i,j\le N}$ defined by*

$$A(\omega,\delta)_{ij} := \int_{\partial B_i} \mathcal{C}(\omega,\delta)[\phi_j^*]\,\mathrm{d}y, \qquad 1\le i,j\le N, \tag{3.21}$$

is invertible. When it is the case, the solution (ϕ,ψ) to the problem (3.4) reads

$$\begin{cases} \phi = \displaystyle\sum_{i=1}^N x_i \mathcal{G}^{-1}(\omega,\delta)[\phi_i^*] + \mathcal{G}^{-1}(\omega,\delta)[f], \\ \psi = \displaystyle\sum_{i=1}^N x_i (\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)\phi_i^* + (\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)[f] - (\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}], \end{cases} \tag{3.22}$$

where $f\in L^2(\partial D)$ is the function

$$f := \delta\frac{\partial u_{\mathrm{in}}}{\partial n} - \delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right)(\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}], \tag{3.23}$$

and where the coefficients $\boldsymbol{x} := (x_i)_{1\le i\le N}$ are the solutions to the finite dimensional problem

$$A(\omega,\delta)\boldsymbol{x} = \boldsymbol{F} \text{ with } \boldsymbol{F} := \left(\int_{\partial B_i} \mathcal{G}^{-1}(\omega,\delta)[f]\,\mathrm{d}\sigma\right)_{1\le i\le N}. \tag{3.24}$$

Proof. The second line of equation (3.5) reads

$$\mathcal{G}(\omega,\delta)[\phi] - \mathcal{H}[\phi] = f \tag{3.25}$$

with f given by (3.23). This equation is equivalent to

$$\phi - \mathcal{G}^{-1}(\omega,\delta)\mathcal{H}[\phi] = \mathcal{G}^{-1}(\omega,\delta)[f]. \tag{3.26}$$

Observing that $\mathcal{G}^{-1}(\omega,\delta)\mathcal{H}$ is a finite-rank operator, we decompose $\phi=\sum_{i=1}^n x_i\phi_i^* + \widetilde{\phi}$ on $\mathrm{span}(\mathcal{H})\oplus\mathrm{Ker}(\mathcal{H})$. Then integrating (3.26) over each resonator ∂B_i (namely, projecting on $\mathrm{span}(\mathcal{H})$), we find that the coefficients $\boldsymbol{x} := (x_i)_{1\le i\le N}$ must solve the finite dimensional system

$$\begin{aligned} \int_{\partial B_i} \mathcal{G}^{-1}(\omega,\delta)[f]\,\mathrm{d}\sigma &= \left(\delta_{ij} - \int_{\partial B_i}\mathcal{G}^{-1}(\omega,\delta)\mathcal{H}[\phi_j^*]\,\mathrm{d}\sigma\right)x_j = \left(\delta_{ij} - \int_{\partial B_i}\mathcal{G}^{-1}(\omega,\delta)[\phi_j^*]\,\mathrm{d}\sigma\right)x_j \\ &= \left(\delta_{ij} - \int_{\partial B_i}(\mathcal{L}_0^{-1}[\phi_j^*] - \mathcal{C}(\omega,\delta)[\phi_j^*])\,\mathrm{d}\sigma\right)x_j \\ &= \left(\int_{\partial B_i}\mathcal{C}(\omega,\delta)[\phi_j^*]\,\mathrm{d}\sigma\right)x_j. \end{aligned}$$

Therefore it is necessary that the linear system (3.24) be invertible for (3.25) being invertible. Reciprocally, this condition is sufficient because if $\boldsymbol{x}=(x_i)_{1\le i\le N}$ is the solution to (3.24), then one obtains from (3.26) that ϕ given by the first line of (3.22) is solution to (3.4). Then the formula for ψ follows from the first line of (3.5). ☐

Remark 3.1. Inserting (3.22) into the integral representation formula (3.1) yields the decomposition (1.2) for the total field u_{tot} with

$$v_i(\omega,\delta)(x) = \begin{cases} \mathcal{S}_D^{k_b}[\mathcal{G}^{-1}(\omega,\delta)\phi_i^*](x) \text{ if } x\in D, \\ \mathcal{S}_D^{k}[(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)\phi_i^*](x) \text{ if } x\in\mathbb{R}^3\backslash D, \end{cases}$$

$$w(\omega,\delta)(x) = \begin{cases} \mathcal{S}_D^{k_b}[\mathcal{G}^{-1}(\omega,\delta)f](x) \text{ if } x\in D, \\ \mathcal{S}_D^{k}[(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)f](x) - \mathcal{S}_D^k[(\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}]](x) \text{ if } x\in\mathbb{R}^3\backslash D. \end{cases}$$

Remark 3.2. We shall see below in Section 4 that (3.22) can be interpreted as a modal expansion of ϕ, where the modes are linear combinations of the functions $v_i(\omega,\delta)\phi_i^*$ and the functions $x_i\equiv x_i(\omega,\delta,f)$ are scattering amplitudes with poles being the resonant frequencies $(\omega_i^{\pm}(\delta))_{1\le i\le N}$.

Remark 3.3. There exist as many possibilities for the choice of operator $\mathcal{H}$ enabling to compute $\mathcal{L}(\omega,\delta)^{-1}$ as there are invertible operators from $\mathrm{Ker}((-\frac{1}{2}I+\mathcal{K}_D^*))$ to a complement subspace of $\mathrm{Ran}((-\frac{1}{2}I+\mathcal{K}_D^*))$. Since $\mathrm{Ker}((-\frac{1}{2}I+\mathcal{K}_D^*))$ is itself a complement subspace of $\mathrm{Ran}((-\frac{1}{2}I+\mathcal{K}_D^*))$, the definition (3.15) is natural and leads to several simplifications in the computations performed below.

The previous proposition yields a convenient definition and characterization of the scattering resonances $(\omega_i^{\pm}(\delta))_{1\le i\le N}$.

Proposition 3.5. *There exists $2N$ complex frequencies $(\omega_i^{\pm}(\delta))_{1\leq i\leq N}$ which are defined as the characteristic values of the operator $\mathcal{A}(\omega,\delta)$, i.e. the values for which $\mathcal{A}(\omega_i^{\pm}(\delta),\delta)$ has a non-trivial kernel. Equivalently, these are also the characteristic values of the $N\times N$ matrix $A(\omega,\delta)$ of (3.21), i.e. eigenvalues of the non-linear eigenvalue problem (1.3).*

Proof. Proposition 3.6 below reveals that $A(\omega,0)\sim\omega^2VC^{-1}$ with VC^{-1} invertible. Hence the generalized Rouché theorem [41, 14] implies that $A(\omega,\delta)$ has exactly $2N$ characteristic values $(\omega_i^{\pm})_{1\leq i\leq N}$ in a neighborhood of zero for δ sufficiently small (see also [8]. □

3.2. **Asymptotic expansion of the Schur complement** $A(\omega,\delta)$

Scattering resonances $(\omega_i^{\pm}(\delta))_{1\leq i\leq N}$ are the characteristic values of the operator $\mathcal{A}(\omega,\delta)$, i.e. values of ω for which $\mathcal{A}(\omega,\delta)$ has a non-trivial kernel. The previous Proposition 3.4 obtained that they are equivalently the characteristic values of the holomorphic matrix $A(\omega,\delta)$. The first terms of the asymptotic of $A(\omega,\delta)$ as $\omega\to0$ and $\delta\to0$ are explicited in the next proposition, which is consistent with a similar computation performed in [4, Theorem 2.7].

Proposition 3.6. *The following asymptotic holds true as $\omega\to0$ and $\delta\to0$:*

$$A(\omega,\delta)=\frac{\omega^2}{v_b^2}VC^{-1}-\delta I+\frac{\mathrm{i}\omega^3}{4\pi v_b^3}V\mathbf{1}\mathbf{1}^T-\frac{\mathrm{i}\delta\omega}{4\pi}\left(\frac{1}{v_b}-\frac{1}{v}\right)C\mathbf{1}\mathbf{1}^T+O(\omega^2+\delta)^2, \tag{3.27}$$

where C is the capacitance matrix given by (2.1), V is the volume matrix (2.3) and $\mathbf{1}=(1)_{1\leq i\leq N}$ is the vector of ones.

Proof. By the definition (3.12), the fourth-order asymptotic expansion of $\omega^2\mathcal{B}_(\omega)$ reads

$$\omega^2B_1(\omega)=\frac{\omega^2}{v_b^2}\mathcal{K}_{D,2}^*+\frac{\omega^3}{v_b^3}\mathcal{K}_{D,3}^*+O(\omega^4), \tag{3.28}$$

and a computation yields

$$\begin{aligned}\delta\mathcal{B}_2(\omega)&=-\delta\left(\frac{1}{2}I+\mathcal{K}_D^{k*}\right)(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\\&=-\delta\left(\frac{1}{2}I+\mathcal{K}_D^*+O(\omega^2)\right)\left(\mathcal{S}_D+\frac{\omega}{v}\mathcal{S}_{D,1}+O(\omega^2)\right)^{-1}\left(\mathcal{S}_D+\frac{\omega}{v_b}\mathcal{S}_{D,1}+O(\omega^2)\right)\\&=-\delta\left(\frac{1}{2}I+\mathcal{K}_D^*\right)\left(\mathcal{S}_D^{-1}-\frac{\omega}{v}\mathcal{S}_D^{-1}\mathcal{S}_{D,1}\mathcal{S}_D^{-1}\right)\left(\mathcal{S}_D+\frac{\omega}{v_b}\mathcal{S}_{D,1}\right)+O(\omega^2\delta)\\&=-\delta\left(\frac{1}{2}I+\mathcal{K}_D^*\right)+\delta\omega\left(\frac{1}{v}-\frac{1}{v_b}\right)\left(\frac{1}{2}I+\mathcal{K}_D^*\right)\mathcal{S}_D^{-1}\mathcal{S}_{D,1}+O(\omega^2\delta).\end{aligned} \tag{3.29}$$

Then equations (3.20) and (3.21) allow to obtain

$$\begin{aligned}A(\omega,\delta)_{ij}=\int_{\partial B_i}\mathcal{C}(\omega,\delta)[\phi_j^*]\,\mathrm{d}\sigma&=\int_{\partial B_i}\mathcal{L}_0^{-1}(\omega^2\mathcal{B}_1(\omega)+\delta\mathcal{B}_2(\omega))[\phi_j^*]\,\mathrm{d}\sigma+O(\omega^2+\delta)^2\\&=\int_{\partial B_i}(\omega^2\mathcal{B}_1(\omega)+\delta\mathcal{B}_2(\omega))[\phi_j^*]\,\mathrm{d}\sigma+O(\omega^2+\delta)^2,\end{aligned}$$

where the third equality is a consequence of (3.17). From the expansions (3.28) and (3.29), we obtain

$$\begin{aligned}&\int_{\partial B_i}\omega^2\mathcal{B}_1(\omega)[\phi_j^*]=\frac{\omega^2}{v_b^2}\int_{\partial B_i}\mathcal{K}_{D,2}^*[\phi_j^*]\,\mathrm{d}\sigma+\frac{\omega^3}{v_b^3}\int_{\partial B_i}\mathcal{K}_{D,3}^*[\phi_j^*]\,\mathrm{d}\sigma+O(\omega^4)\\&=-\frac{\omega^2}{v_b^2}\int_{B_i}\mathcal{S}_D[\phi_j^*]\,\mathrm{d}\sigma+\frac{\mathrm{i}\omega^3}{v_b^3}\frac{|B_i|}{4\pi}\int_{\partial D}\phi_j^*\,\mathrm{d}\sigma+O(\omega^4)\\&=\frac{\omega^2}{v_b^2}|B_i|C_{ji}^{-1}+\frac{\mathrm{i}\omega^3}{v_b^3}\frac{|B_i|}{4\pi}+O(\omega^4).\end{aligned}$$

For the computation of $\delta\mathcal{B}_2(\omega)$, we remark that $\mathcal{S}_{D,1}[\phi_j^*]=-\mathrm{i}/(4\pi)1_{\partial D}$ and $\mathcal{S}_D^{-1}[1_{\partial D}]=\sum_i\psi_i^*$, which implies

$$\int_{\partial B_i}\mathcal{S}_D^{-1}\mathcal{S}_{D,1}[\phi_j^*]\,\mathrm{d}\sigma=\frac{\mathrm{i}}{4\pi}\sum_{l=1}^N C_{il}.$$

Substituting into (3.29) yields then

$$\begin{aligned}\int_{\partial B_i} \delta \mathcal{B}_2(\omega)[\phi_j^*]\,\mathrm{d}\sigma &= -\delta \int_{\partial B_i} \left(\frac{1}{2}I + \mathcal{K}_D^*\right)[\phi_j^*]\,\mathrm{d}\sigma + \delta\omega\left(\frac{1}{v} - \frac{1}{v_b}\right)\int_{\partial B_i}\left(\frac{1}{2}I + \mathcal{K}_D^*\right)\mathcal{S}_D^{-1}\mathcal{S}_{D,1}[\phi_j^*]\,\mathrm{d}\sigma + O(\omega^2\delta)\\ &= -\delta\delta_{ij} + \delta\omega\left(\frac{1}{v} - \frac{1}{v_b}\right)\int_{\partial B_i}\mathcal{S}_D^{-1}\mathcal{S}_{D,1}[\phi_j^*]\,\mathrm{d}\sigma + O(\omega^2\delta)\\ &= -\delta\delta_{ij} - \frac{\mathrm{i}\delta\omega}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right)\sum_{l=1}^N C_{il} + O(\omega^2\delta).\end{aligned}.$$

Hence the asymptotic of $A(\omega,\delta)_{ij}$ reads

$$A(\omega,\delta)_{ij} = \frac{\omega^2}{v_b^2}|B_i|C_{ji}^{-1} - \delta\delta_{ij} + \frac{\mathrm{i}\omega^3}{v_b^3}\frac{|B_i|}{4\pi} - \frac{\mathrm{i}\delta\omega}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right)\sum_{l=1}^N C_{il} + O(\omega^2+\delta)^2 \tag{3.30}$$

which is the result to obtain. □

Remark 3.4. With symbolic computations, we find that the term of the asymptotic expansion of $A(\omega,\delta)_{ij}$ of order $O(\omega^2+\delta)^2$ is

$$\begin{aligned}&\frac{\omega^2\delta}{16\pi^2 v}\left(\frac{1}{v} - \frac{1}{v_b}\right)\mathbf{1}^T C\mathbf{1}\sum_{l=1}^N C_{il} + \frac{\omega^2\delta}{v_b^2}|B_i|C_{ij}^{-1}\delta_{ij} + \omega^2\delta|B_i|\left(\frac{1}{v_b^2} - \frac{1}{v^2}\right)C_{ij}^{-1}\\ &\quad + \omega^2\delta\left(\frac{1}{v_b^2} - \frac{1}{v^2}\right)\sum_{l=1}^N C_{il}\int_{\partial D}\phi_l^*\mathcal{S}_{D,2}[\phi_j^*]\,\mathrm{d}\sigma - \delta^2\delta_{ij} + \frac{\omega^4}{v_b^4}\int_{B_i}[\mathcal{S}_D\mathcal{L}_0^{-1}\mathcal{K}_{D,2}^* - \mathcal{S}_{D,2}][\phi_j^*]\,\mathrm{d}\sigma.\end{aligned} \tag{3.31}$$

It is worth mentioning that the terms involving $\mathcal{K}_{D,2}^*$ and $\mathcal{S}_{D,2}^*$ are non-Hermitian, making an explicit analysis at higher orders in ω and δ delicate.

Right multiplying (3.27) by C, we obtain that the nonlinear eigenvalue problem (3.3) reads at first order

$$-\left(C - \frac{\omega^2}{v_b^2\delta}V\right)\boldsymbol{x} + O(\omega) = 0.$$

The nonlinear eigenvalue problem (3.3) is therefore a perturbation of the generalized eigenvalue problem (2.4), and the i-th resonance reads at first order $\omega_i^2 = \lambda_i v_b^2\delta$, for λ_i a generalized eigenvalue. This result was already obtained in [13, 4, 8], is rigorously justified by mean of the implicit function theorem in the next subsection.

3.3. **Full asymptotic expansions of the resonances**

We now propose a procedure which enables in principle to compute full asymptotic expansions of the resonant frequencies $\omega_i^{\pm}(\delta)$ when the eigenvalues $(\lambda_i)_{1\le i\le N}$ of the capacitance matrix C are simple. In contrast with [4, 8], our procedure does not require to assume the existence of a formal ansatz; it relies on a change of variables and on the implicit function theorem to derive the Puiseux expansion (1.4), which is inspired from [17, 24] and which can be related to Newton diagrams [30, 48]. In particular, we prove the analyticity of $\omega_i^{\pm}(\delta)$ with respect to $\delta^{1/2}$, a result that was implicitly assumed in the aforementioned works.

From there and in the next sections, we assume that the eigenvalues $(\lambda_i)_{1\le i\le N}$ of the generalized eigenvalue problem (2.4) are simple:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N. \tag{3.32}$$

We discuss shortly in Remark 3.4 what could happen in the case of multiple eigenvalues.

We start by rewriting (3.27) as an explicit perturbation of the linear eigenvalue problem (2.4). Introducing the variable $\lambda = \omega^2/(v_b^2\delta)$ and recalling the definitions (3.20) and (3.21), the coefficients $A(\omega,\delta)_{ij}$ for $1\le i,j\le N$ can be rewritten in terms of λ and ω as

$$\begin{aligned}A(\omega,\delta)_{ij} &= \sum_{p=1}^{+\infty}\int_{\partial B_i}(\mathcal{L}_0^{-1}(\omega^2\mathcal{B}_1(\omega) + \delta\mathcal{B}_2(\omega)))^p[\phi_j^*]\,\mathrm{d}\sigma\\ &= \sum_{p=1}^{+\infty} v_b^{2p}\delta^p\int_{\partial B_i}\left(\mathcal{L}_0^{-1}\left(\frac{\omega^2}{v_b^2\delta}\mathcal{B}_1(\omega) + v_b^{-2}\mathcal{B}_2(\omega)\right)\right)^p[\phi_j^*]\,\mathrm{d}\sigma.\\ &= \sum_{p=1}^{+\infty}\frac{\omega^{2p}}{\lambda^p}\int_{\partial B_i}\left(\mathcal{L}_0^{-1}(\lambda\mathcal{B}_1(\omega) + v_b^{-2}\mathcal{B}_2(\omega))\right)^p[\phi_j^*]\,\mathrm{d}\sigma.\end{aligned}$$

We define $\widetilde{A}(\lambda, \omega)$ to be the $N \times N$ complex matrix holomorphic in ω given by the following power series with $\omega = o(1)$ and $\lambda = O(1)$:

$$\widetilde{A}(\lambda, \omega) := \frac{v_b^2 \lambda}{\omega^2} A\left(\omega, \frac{\omega^2}{v_b^2 \lambda}\right) C = \left(\sum_{l=1}^{N} \sum_{p=0}^{+\infty} \frac{v_b^2 \omega^{2p}}{\lambda^p} \int_{\partial B_i} \left(\mathcal{L}_0^{-1}(\lambda \mathcal{B}_1(\omega) + v_b^{-2} \mathcal{B}_2(\omega))\right)^{p+1} C_{lj}[\phi_l^*] \, \mathrm{d}\sigma \right)_{1 \le i,j \le N}. \tag{3.33}$$

Equation (3.27) implies that $\widetilde{A}(\lambda, \omega)$ reads more explicitly at second order

$$\widetilde{A}(\lambda, \omega) = \lambda V - C + \frac{\mathrm{i}\omega\lambda}{4\pi v_b} V \mathbf{1}\mathbf{1}^T C - \frac{\mathrm{i}\omega}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right) C \mathbf{1}\mathbf{1}^T C + O(\omega^2). \tag{3.34}$$

From the definition (3.33), it is straightforward to observe that ω is a characteristic value of $A(\omega, \delta)$ if and only if $\lambda = \omega^2/(v_b^2 \delta)$ is a characteristic value of $\lambda \mapsto \widetilde{A}(\lambda, \omega)$. The interest of introducing $\widetilde{A}(\lambda, \omega)$ lies in the fact that under the simplicity assumption (3.32), there exist N characteristic functions $(\lambda_i(\omega))_{1 \le i \le N}$ which are holomorphic in ω. This enables to solve the splitting of the scattering frequencies $\omega_i^{\pm}(\delta)$, by inverting the relation $\omega_i^{\pm}(\delta)^2 = \delta v_b^2 \lambda_i(\omega)$ for $1 \le i \le N$.

The existence of N holomorphic characteristic functions $(\lambda_i(\omega))_{1 \le i \le N}$ is guaranteed by the analytic implicit function theorem (see e.g. Chapter 0 of [52]).

Proposition 3.7. *Assume (3.32). There exist N generalized eigenfunctions $(\lambda_i(\omega))_{1 \le i \le N}$ and N associated eigenvectors $(\boldsymbol{a}_i(\omega))_{1 \le i \le N}$ which are holomorphic in an open neighborhood $\mathcal{V}$ of $\omega = 0$ and which satisfy, for any $1 \le i \le N$ and $\omega \in \mathcal{V}$:*

(i) $\widetilde{A}(\lambda_i(\omega), \omega)\boldsymbol{a}_i(\omega) = 0$,
(ii) $\boldsymbol{a}_i(\omega)^T V \boldsymbol{a}_i(\omega) = 1$,
(iii) $\lambda_i(0) = \lambda_i$ *and* $\boldsymbol{a}_i(0) = \boldsymbol{a}_i$.

Proof. (i) and (ii) are equivalent to find holomorphic functions $\lambda_i(\omega)$ and $\boldsymbol{a}_i(\omega)$ such that $F((\lambda_i(\omega), \boldsymbol{a}_i(\omega)), \omega) = 0$ for any $\omega \in \mathcal{V}$, where $F : (\mathbb{C} \times \mathbb{C}^N) \times \mathbb{C} \to \mathbb{C}^N \times \mathbb{C}$ is the functional

$$F((\lambda, \boldsymbol{x}), \omega) := (\widetilde{A}(\lambda, \omega)\boldsymbol{x}, \boldsymbol{x}^T V \boldsymbol{x} - 1), \qquad \lambda \in \mathbb{C},\ \boldsymbol{x} \in \mathbb{C}^N. \tag{3.35}$$

Since (3.34) is equivalent to $F((\lambda_i, \boldsymbol{a}_i), 0) = 0$, the result follows from the implicit function theorem whose hypotheses are satisfied as soon as we prove that the differential of $(\lambda, \boldsymbol{x}) \mapsto F((\lambda, \boldsymbol{x}), 0)$ is invertible at $(\lambda, \boldsymbol{x}) = (\lambda_i, \boldsymbol{a}_i)$. A straightforward computation yields

$$\mathrm{D}F((\lambda_i, \boldsymbol{a}_i), 0)(\lambda', \boldsymbol{x}') = (\lambda' V \boldsymbol{a}_i + (\lambda_i V - C)\boldsymbol{x}', 2\boldsymbol{a}_i^T V \boldsymbol{x}').$$

Assuming (3.32), we find that $\mathrm{D}F((\lambda_i, \boldsymbol{a}_i), 0)$ is indeed invertible, with the inverse given by

$$[\mathrm{D}F((\lambda_i, \boldsymbol{a}_i), 0)]^{-1}(\boldsymbol{\alpha}, \beta) = \left(\boldsymbol{a}_i^T \boldsymbol{\alpha}, \frac{\beta}{2}\boldsymbol{a}_i + \sum_{j \ne i} \frac{\boldsymbol{a}_j^T \boldsymbol{\alpha}}{\lambda_i - \lambda_j} \boldsymbol{a}_j \right), \qquad (\boldsymbol{\alpha}, \beta) \in \mathbb{C}^N \times \mathbb{C}. \tag{3.36}$$

□

Since the functions $(\lambda_i(\omega))_{1 \le i \le N}$ and $(\boldsymbol{a}_i(\omega))_{1 \le i \le N}$ are holomorphic, they can be written as

$$\lambda_i(\omega) = \lambda_{i,0} + \sum_{p=1}^{+\infty} \lambda_{i,p}\omega^p, \quad \boldsymbol{a}_i(\omega) = \boldsymbol{a}_{i,0} + \sum_{p=1}^{+\infty} \boldsymbol{a}_{i,p}\omega^p, \qquad 1 \le i \le N. \tag{3.37}$$

The coefficients $(\lambda_{i,p})_{p \ge 0}$, $(\boldsymbol{a}_{i,p})_{p \ge 0}$ can be computed explicitly by solving a triangular system obtained by differentiating p times the equation $F((\lambda_i(\omega), \boldsymbol{a}_i(\omega)), \omega) = 0$ with respect to ω at $\omega = 0$ [53]. Furthermore, the coefficients satisfy the following properties:

Proposition 3.8. *The coefficients $\lambda_{i,p}$ and $\boldsymbol{a}_{i,p}$ are real and imaginary for respectively even and odd values of p:*

$$\forall 1 \le i \le N,\ \forall p \in \mathbb{N},\ \ \lambda_{i,2p} \in \mathbb{R} \text{ and } \lambda_{i,2p+1} \in \mathrm{i}\mathbb{R}, \tag{3.38}$$

$$\forall 1 \le i \le N,\ \forall p \in \mathbb{N},\ \ \boldsymbol{a}_{i,2p} \in \mathbb{R}^N \text{ and } \boldsymbol{a}_{i,2p+1} \in \mathrm{i}\mathbb{R}^N. \tag{3.39}$$

Proof. It is straightforward to verify that $\overline{\mathcal{A}(\omega, \delta)} = \mathcal{A}(-\overline{\omega}, \overline{\delta})$. This implies in turn $\overline{\widetilde{A}(\lambda, \omega)} = \widetilde{A}(\overline{\lambda}, -\overline{\omega})$, which can be shown to entail $\overline{\lambda_i(\omega)} = \lambda_i(-\overline{\omega})$ and $\overline{\boldsymbol{a}_i(\omega)} = \boldsymbol{a}_i(-\overline{\omega})$. These properties reflect on the coefficients of the expansion (3.37) in the identities (3.38) and (3.39). □

The next and final step is to invert the equation $\omega^2 = \delta v_b^2 \lambda_i(\omega)$ for $1 \le i \le N$, which can be achieved as follows.

Proposition 3.9. *For $\delta > 0$ sufficiently small, the matrix $A(\omega, \delta)$ has $2N$ characteristic values $(\omega_i^{\pm}(\delta))_{1\le i\le N}$ which are the implicit solutions to the equations*

$$\omega_i^+(\delta) = \delta^{\frac{1}{2}} v_b \sqrt{\lambda_i(\omega_i^+(\delta))} \text{ and } \omega_i^-(\delta) = -\delta^{\frac{1}{2}} v_b \sqrt{\lambda_i(\omega_i^+(\delta))},$$

where $z \mapsto \sqrt{z}$ denotes the analytic continuation of the square root to $\mathbb{C}\backslash\mathbb{R}_-$. The functions $\omega_i^{\pm}(\delta)$ are analytic in $\delta^{\frac{1}{2}}$, i.e. there exists coefficients $(\omega_{i,p}^{\pm})_{p\ge 2}$ such that

$$\omega_i^{\pm}(\delta) = \pm v_b \lambda_i^{\frac{1}{2}} \delta^{\frac{1}{2}} + \sum_{p=2}^{+\infty} \omega_{i,p}^{\pm} \delta^{\frac{p}{2}}. \tag{3.40}$$

Furthermore:

(i) the coefficients $\omega_{i,p}^{\pm}$ are real and imaginary for respectively even and odd values of p:

$$\omega_{i,2p}^{\pm} \in \mathbb{R} \text{ and } \omega_{i,2p+1}^{\pm} \in \mathrm{i}\mathbb{R}, \qquad \forall 1 \le i \le N, \forall p \ge 2;$$

(ii) $\omega_i^-(\delta) = -\overline{\omega}_i^+(\delta)$, i.e. $\omega_i^+(\delta)$ and $\omega_i^-(\delta)$ are symmetrical with respect to the imaginary axis, or in other words

$$\omega_{i,2p}^+ = -\omega_{i,2p}^- \text{ and } \omega_{i,2p+1}^+ = \omega_{i,2p+1}^-, \qquad \forall 1 \le i \le N, \forall p \ge 2.$$

Proof. We apply once again the analytic implicit function theorem to the functions $F_i^{\pm}$ defined by

$$F_i^{\pm}(\omega, z) := \omega \mp z v_b \sqrt{\lambda_i(\omega)}.$$

Obviously $F_i^{\pm}(0,0) = 0$ and $\partial_\omega F_i^{\pm}(0,0) = 1 \neq 0$ so the hypotheses of the implicit function theorem are satisfied and there exists a unique local solution $z \mapsto \omega_i^{\pm}(z)$ to $F_i^{\pm}(\omega_i^{\pm}(z), z) = 0$ which is analytic for z belonging to a neighborhood of 0. Then by definition of the scattering resonance $\omega_i^{\pm}(\delta)$, it holds

$$A(\omega_i^{\pm}(\delta), \delta)\boldsymbol{a}_i(\omega_i^{\pm}(\delta)) = 0 \text{ with } \boldsymbol{a}_i(\omega_i^{\pm}(\delta))^T V \boldsymbol{a}_i(\omega_i^{\pm}(\delta)) = 1,$$

which shows that $(\omega_i^{\pm}(\delta))_{1\le i\le N}$ are $2N$ characteristic values of $A(\omega, \delta)$. Formula (3.40) determines then all characteristic values of $A(\omega, \delta)$. Then, noticing that $\overline{F_i^+(\omega, z)} = -F_i^-(-\overline{\omega}, \overline{z})$ and $F_i^+(\omega, -z) = F_i^-(\omega, z)$, we obtain respectively $-\overline{\omega_i^+(z)} = \omega_i^-(\overline{z})$ and $\omega_i^+(-z) = \omega_i^-(z)$, which easily imply both properties (i) and (ii) on the coefficients $(\omega_{i,p})_{p\ge 2}$. ☐

Remark 3.5. If D is such that one of the eigenvalues $(\lambda_i(\omega))_{1\le i\le N}$ of (2.4) is of multiplicity $m > 1$, (for instance if $N > 1$ and if there are enough symmetries as in the examples of Section 2.3), then a much more subtle analysis is required, because this eigenvalue $\lambda_i(\omega)$ split *a priori* into m distinct eigenvalues; see e.g. [30, 51] about the characterization of the splitting of linear eigenvalues and [46, 57, 72, 66, 48] in the nonlinear case. Then the frequency $\omega_i^{\pm}(\delta) := \pm v_b \lambda_i(\omega)^{\frac{1}{2}} \delta^{\frac{1}{2}}$ would not need to be holomorphic in $\delta^{\frac{1}{2}}$ and a different Puiseux series than (1.4) would need to be computed. Notably, the present analysis could be affected by the occurence of poles $(\omega^2 - \delta\lambda_i(\omega))^p$ of order $p > 1$ in the modal expansion (4.1) below.

Let us finally retrieve the leading terms of the asymptotic expansion of $\omega_i^{\pm}(\delta)$ based on the previous propositions.

Proposition 3.10. *The asymptotic expansions of the eigenvalue $\lambda_i(\omega)$ and of its associated eigenvector $\boldsymbol{a}_i(\omega)$ read at first order:*

$$\lambda_i(\omega) = \lambda_i - \frac{\mathrm{i}\omega\lambda_i^2}{4\pi v}(\boldsymbol{a}_i^T V \mathbf{1})^2 + O(\omega^2), \qquad 1 \le i \le N,$$

$$\boldsymbol{a}_i(\omega) = \boldsymbol{a}_i - \sum_{j\neq i} \frac{\mathrm{i}\omega\lambda_i}{4\pi(\lambda_i - \lambda_j)} \left(\frac{\lambda_i}{v_b} - \lambda_j \left(\frac{1}{v_b} - \frac{1}{v}\right)\right) (\boldsymbol{a}_j^T V \mathbf{1})(\boldsymbol{a}_i^T V \mathbf{1})\boldsymbol{a}_j + O(\omega^2).$$

Proof. With the notation of Proposition 3.7, we have from (3.27)

$$\mathrm{D}F((\lambda_i, \boldsymbol{a}_i), 0)(\lambda_{i,1}, \boldsymbol{a}_{i,1}) = -\mathrm{D}_\omega F((\lambda_i, \boldsymbol{a}_i), 0) = -\left(\left(\frac{\mathrm{i}\lambda_i}{4\pi v_b} V\mathbf{1}\mathbf{1}^T C - \frac{\mathrm{i}}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right) C\mathbf{1}\mathbf{1}^T C\right)\boldsymbol{a}_i, 0\right).$$

From (3.36), the inversion of the above equation yields

$$\begin{aligned} \lambda_{i,1} &= -\frac{\mathrm{i}\lambda_i}{4\pi}\left(\frac{\lambda_i}{v_b}\boldsymbol{a}_i^T V\mathbf{1}\mathbf{1}^T V\boldsymbol{a}_i - \left(\frac{1}{v_b} - \frac{1}{v}\right)\boldsymbol{a}_i^T C\mathbf{1}\mathbf{1}^T V\boldsymbol{a}_i\right) \\ &= -\frac{\mathrm{i}\lambda_i^2}{4\pi v}\boldsymbol{a}_i^T V\mathbf{1}\mathbf{1}^T V\boldsymbol{a}_i, \end{aligned} \tag{3.41}$$

$$\boldsymbol{a}_{i,1} = -\sum_{j\neq i} \frac{\mathrm{i}\lambda_i}{4\pi(\lambda_i - \lambda_j)}\left(\frac{\lambda_i}{v_b} - \lambda_j\left(\frac{1}{v_b} - \frac{1}{v}\right)\right)(\boldsymbol{a}_j^T V\mathbf{1}\mathbf{1}^T V\boldsymbol{a}_i)\boldsymbol{a}_j.$$

☐

Inserting these values in (3.40) we eventually retrieve the asymptotic claimed in [8]:

Corollary 3.1. *Assume that the eigenvalues of the (weighted) capacitance matrix are simple (hypothesis (3.32)). The subwavelength resonances $\omega_k^{\pm}(\delta)$ admit the following asymptotic expansions:*

$$\omega_i^{\pm}(\delta) = \pm\delta^{\frac{1}{2}} v_b \lambda_i^{\frac{1}{2}} - \frac{\mathrm{i} v_b^2 \lambda_i^2}{8\pi v}(\boldsymbol{a}_i^T V \mathbf{1})^2 \delta + O(\delta^{\frac{3}{2}}). \tag{3.42}$$

Proof. (3.42) is obtained by writing

$$\begin{aligned} \omega_i^{\pm}(\delta) &= \pm\delta^{\frac{1}{2}} v_b \sqrt{\lambda_i + \lambda_{i,1}\omega_i^{\pm}(\delta) + O(\omega_i^{\pm}(\delta))} = \pm\delta^{\frac{1}{2}} v_b \sqrt{\lambda_i}\sqrt{1 \pm \frac{\lambda_{i,1}}{\lambda_i}\lambda_i^{\frac{1}{2}} v_b \delta^{\frac{1}{2}} + O(\delta)} \\ &= \pm\delta^{\frac{1}{2}} v_b \sqrt{\lambda_i} + \frac{1}{2}\delta v_b^2 \lambda_{i,1} + O(\delta^{\frac{3}{2}}), \end{aligned} \tag{3.43}$$

which yields the result. □

Remark 3.6. If D is a single resonator $D \equiv B$ ($N = 1$), we have $C = \mathrm{cap}(B)$ and $V = |B|$ hence $\lambda_1 = \mathrm{cap}(B)/|B|$ and $\boldsymbol{a}_1 = |B|^{-1/2}$. Then (3.42) reads more explicitly as

$$\omega_1^{\pm}(\delta) = \pm\delta^{\frac{1}{2}} v_b \sqrt{\frac{\mathrm{cap}(B)}{|B|}} - \frac{\mathrm{i} v_b^2\, \mathrm{cap}(B)^2}{8\pi v |B|}\delta + O(\delta^{\frac{3}{2}}), \tag{3.44}$$

which is the result obtained in [13].

Remark 3.7. From (3.42), we see that when the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$ is an eigenvector of the weighted capacitance matrix (e.g. when there are enough symmetries as in the cases considered in Section 2.3), then $\boldsymbol{a}_i^T V \mathbf{1} = 0$ for $i > 1$ and the first order varation of the resonant frequency vanishes: $\omega_i^{\pm}(\delta) = \pm\delta^{\frac{1}{2}} v_b \lambda_i^{\frac{1}{2}} + O(\delta^{\frac{3}{2}})$, i.e. the corresponding frequencies $\omega_i^{\pm}(\delta)$ are quite robust to the variations of δ.

4. Modal decomposition

This section takes advantage of the explicit formula (3.22) in order to establish a (nonlinear) modal decomposition of the form of (1.5) for the solution u_{tot} to the scattering problem (1.1). This is achieved in two steps: we start by computing in Section 4.1 a pole expansion of the meromorphic coefficients $\boldsymbol{x} = (x_i)_{1\le i\le N}$ solutions to the finite-dimensional problem (3.24). In Section 4.2, we substitute this expansion into (3.22) to obtain the modal decomposition (1.5) for the scattered field u_{tot}. Finally, Section 4.3 states a few remarks regarding the estimation of the magnitude of the resonances when considering physical, real frequencies $\omega > 0$.

4.1. Pole expansion of the resonant amplitudes $(x_i)_{1\le i\le N}$

We first establish a pole decomposition of the meromorphic solution $\boldsymbol{x}$ to the finite-dimensional problem (3.24), whose leading order expansion can be expressed in terms of the generalized eigenvalues and eigenvectors $(\lambda_j(\omega))_{1\le j\le N}$ and $(\boldsymbol{a}_j(\omega))_{1\le j\le N}$ of Proposition 3.7.

Proposition 4.1. *The following modal decomposition holds for the solution $\boldsymbol{x}(\omega,\delta)$ to (3.24):*

$$\boldsymbol{x}(\omega,\delta) = \sum_{i=1}^N \frac{v_b^2}{\omega^2 - \delta v_b^2 \lambda_i(\omega)}(\boldsymbol{a}_i^T \boldsymbol{F} - \mathrm{i}\omega \boldsymbol{b}_i^T \boldsymbol{F} + O(\omega^2+\delta)^T \boldsymbol{F}) C \boldsymbol{a}_i(\omega), \tag{4.1}$$

where the mode $\boldsymbol{a}_i(\omega)$ is given by Proposition 3.7, $\boldsymbol{b}_i$ is the vector defined by

$$\boldsymbol{b}_i := \frac{\lambda_i}{4\pi v_b}(\boldsymbol{a}_i^T V \mathbf{1})^2 \boldsymbol{a}_i + \frac{1}{4\pi v}\sum_{j\ne i}^N \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)}(\boldsymbol{a}_i^T V \mathbf{1})(\boldsymbol{a}_j^T V \mathbf{1})\boldsymbol{a}_j, \tag{4.2}$$

and where $O(\omega^2+\delta)$ is a holomorphic vector field in ω and δ which can be written fully explicitly.

Proof. Using the quantity $\lambda = \omega^2/(v_b\delta)$, (3.24) rewrites $\widetilde{A}(\lambda,\omega)\boldsymbol{y} = \frac{1}{\delta}\boldsymbol{F}$ with $\boldsymbol{y} = C^{-1}\boldsymbol{x}$. By continuity of the determinant, $(\boldsymbol{a}_i(\omega))_{1\le i\le N}$ is a basis of $\mathbb{C}^N$ for ω sufficiently small. Let us consider the decomposition of $\boldsymbol{y} \equiv \boldsymbol{y}(\omega,\delta)$ onto this basis with coefficients $(y_i(\omega,\delta))_{1\le i\le N}$:

$$\boldsymbol{y}(\omega,\delta) = \sum_{i=1}^N y_i(\omega,\delta)\boldsymbol{a}_i(\omega). \tag{4.3}$$

By Proposition 3.7, it holds $\widetilde{A}(\lambda_i(\omega),\omega)\boldsymbol{a}_i(\omega) = 0$ for any $1 \le j \le N$. Therefore

$$\widetilde{A}(\lambda,\omega)\boldsymbol{y} = \sum_{j=1}^N (\widetilde{A}(\lambda,\omega) - \widetilde{A}(\lambda_j(\omega),\omega))\boldsymbol{a}_j(\omega) y_j(\lambda,\omega). \tag{4.4}$$

We factorize $(\lambda-\lambda_j(\omega))$ in this expression. From (3.33), there exist coefficients $(A_{p,l,m})_{p,m\geq 0, 0\leq l\leq p+1}$ such that $\widetilde{A}(\omega,\lambda)$ can be expanded as

$$\widetilde{A}(\lambda,\omega)=\sum_{\substack{p,l,m\geq 0\\ 0\leq l\leq p+1}} A_{p,l,m}\frac{\omega^{2p}}{\lambda^p}\omega^m\lambda^l.$$

Let us then write a fully explicit expansion of the difference $\widetilde{A}(\lambda,\omega)-\widetilde{A}(\lambda_j(\omega),\omega)$:

$$\begin{aligned}\widetilde{A}(\lambda,\omega)-\widetilde{A}(\lambda_j(\omega),\omega) &= \sum_{\substack{p,l,m\geq 0\\ 0\leq l\leq p+1}} A_{p,l,m}\omega^{2p+m}(\lambda^{l-p}-\lambda_j(\omega)^{l-p})\\ &= \sum_{p,m\geq 0} A_{p,p+1,m}\omega^{2p+m}(\lambda-\lambda_j(\omega))+\sum_{\substack{p\geq 1\\ l,m\geq 0\\ 0\leq l\leq p-1}} A_{p,l,m}\omega^{2p+m}\left(\frac{1}{\lambda^{p-l}}-\frac{1}{\lambda_j(\omega)^{p-l}}\right)\\ &= (\lambda-\lambda_j(\omega))\left[\sum_{p,m\geq 0} A_{p,p+1,m}\omega^{2p+m}-\sum_{\substack{p\geq 1\\ l,m\geq 0\\ 0\leq l\leq p-1}} A_{p,l,m}\omega^{2p+m}\sum_{q=0}^{p-l-1}\frac{1}{\lambda^{q+1}\lambda_j(\omega)^{p-l-q}}\right],\end{aligned}\tag{4.5}$$

where the third equality is a consequence of the identity

$$\frac{1}{a^{p+1}}-\frac{1}{b^{p+1}}=-(a-b)\sum_{q=0}^{p}\frac{1}{a^{q+1}b^{p-q+1}},\qquad \forall a,b\in\mathbb{C},\forall p\in\mathbb{N}.$$

Substituting $\lambda=\omega^2/(v_b^2\delta)$ reveals that the second term of (4.5) is a holomorphic function of ω and δ,

$$\omega^{2p+m}\sum_{q=0}^{p-l-1}\frac{v_b^{2(q+1)}\delta^{q+1}}{\omega^{2(q+1)}\lambda_j(\omega)^{p-l-q}}=\sum_{q=0}^{p-l-1}\frac{v_b^{2(q+1)}}{\lambda_j(\omega)^{p-l-q}}\delta^{q+1}\omega^{2(p-q-1)+m},$$

which is also smaller than $O(\delta)$. Therefore the only term of (4.5) which is bigger than $O(\omega^2+\delta)$ is $A_{0,1,0}+A_{0,1,1}\omega$ with $A_{0,1,0}=V$ and $A_{0,1,1}\omega=\mathrm{i}\omega/(4\pi v_b)V\mathbf{1}\mathbf{1}^TC$ according to (3.34). Hence (4.5) reads with $\lambda=\omega^2/(v_b^2\delta)$:

$$\widetilde{A}(\lambda,\omega)-\widetilde{A}(\lambda_j(\omega),\omega)=(\lambda-\lambda_j(\omega))\left(V+\frac{\mathrm{i}\omega}{4\pi v_b}V\mathbf{1}\mathbf{1}^TC+O(\omega^2+\delta)\right).$$

Coming back to (4.4), we obtain

$$\begin{aligned}\widetilde{A}(\lambda,\omega)\boldsymbol{y} &= \sum_{j=1}^{N}(\lambda-\lambda_j(\omega))\left(V+\frac{\mathrm{i}\omega}{4\pi v_b}V\mathbf{1}\mathbf{1}^TC+O(\omega^2+\delta)\right)y_j(\omega,\delta)\boldsymbol{a}_j(\omega)\\ &= \sum_{j=1}^{N}(\lambda-\lambda_j(\omega))y_j(\omega,\delta)\left(V\boldsymbol{a}_j(\omega)+\mathrm{i}\omega\frac{\lambda_j}{4\pi v_b}V\mathbf{1}\mathbf{1}^TV\boldsymbol{a}_j+O(\omega^2+\delta)\right).\end{aligned}$$

Left multiplying (4.4) by $\boldsymbol{a}_i^T$ and using Proposition 3.10, we find that the vector $\boldsymbol{z}(\omega,\delta)$ given by

$$\boldsymbol{z}(\omega,\delta):=((\lambda-\lambda_j(\omega))y_j(\omega,\delta))_{1\leq j\leq N}$$

is the solution to the linear system

$$(I+\mathrm{i}\omega G+O(\omega^2+\delta))\boldsymbol{z}(\omega,\delta)=\frac{1}{v_b^2\delta}\widetilde{\boldsymbol{F}},\tag{4.6}$$

where $\widetilde{\boldsymbol{F}}:=(\boldsymbol{a}_i^T\boldsymbol{F})_{1\leq i\leq N}$ and G is the matrix

$$\begin{aligned}G:=(G_{ij})_{1\leq i,j\leq N} &:= \left(\left[\frac{\lambda_j}{4\pi v_b}-\frac{\lambda_j\delta_{i\neq j}}{4\pi(\lambda_j-\lambda_i)}\left(\frac{\lambda_j}{v_b}-\lambda_i\left(\frac{1}{v_b}-\frac{1}{v}\right)\right)\right](\boldsymbol{a}_i^TV\mathbf{1})(\boldsymbol{a}_j^TV\mathbf{1})\right)_{1\leq i,j\leq N}\\ &= \left(\left(\frac{\lambda_j\delta_{ij}}{4\pi v_b}+\frac{\delta_{i\neq j}}{4\pi v}\frac{\lambda_i\lambda_j}{(\lambda_i-\lambda_j)}\right)(\boldsymbol{a}_i^TV\mathbf{1})(\boldsymbol{a}_j^TV\mathbf{1})\right)_{1\leq i,j\leq N}.\end{aligned}$$

By using a Neumann series to invert (4.6), we arrive at

$$\boldsymbol{z}=\frac{1}{\delta}(I-\mathrm{i}\omega G+O(\omega^2+\delta))\widetilde{\boldsymbol{F}}$$

and then, since $y_i(\omega,\delta)=\boldsymbol{e}_i^Tz(\omega,\delta)/(\lambda-\lambda_i(\omega))$, we finally obtain

$$y_i(\omega,\delta)=\frac{1}{\delta}\frac{1}{\lambda-\lambda_i(\omega)}(\boldsymbol{a}_i^T\boldsymbol{F}-\mathrm{i}\omega\boldsymbol{e}_i^TG\widetilde{\boldsymbol{F}}+O(\omega^2+\delta)^T\boldsymbol{F}),\qquad 1\leq i\leq N.\tag{4.7}$$

The result follows from $\lambda = \omega^2/(v_b^2\delta)$ and by substituting the value of G and $\widetilde{\boldsymbol{F}}$ in $\boldsymbol{b}_i = \sum_{j=1}^N G_{ij}\boldsymbol{a}_j$. $\square$

Remark 4.1. If D is constituted of a single resonator, i.e. $N = 1$ and $D \equiv B$, the vector $\boldsymbol{b}_1$ of (4.2) reads:

$$\boldsymbol{b}_1 = \frac{\operatorname{cap}(B)}{4\pi v_b |B|^{\frac{1}{2}}}.$$

Remark 4.2. It is interesting to note that the modes $(\boldsymbol{a}_i(\omega))_{1\le i\le N}$ featured in the decomposition (4.1) are not the modes $(\boldsymbol{a}_i(\omega_i^{\pm}(\delta)))$ that can be built from the resonant frequencies $(\omega_i^{\pm}(\delta))_{1\le i\le N}$, as one could have expected.

Remark 4.3. The interest of the formula (4.1) lies in that there is no approximation made in the denominators, which vanish exactly at the "true" resonant frequencies $\omega_i^{\pm}(\delta))_{1\le i\le N}$. (4.1) is more explicit than a Laurent series expansion of $A[\delta,\omega]^{-1}$ in the neighborhood of only one of these poles.

Remark 4.4. It is clear from the proof that (4.1) is a nonlinear continuation of the more standard modal decomposition

$$\widehat{\boldsymbol{x}} = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} (\boldsymbol{a}_j^T \boldsymbol{F}) C \boldsymbol{a}_j,$$

which is the solution of the linear problem

$$\widetilde{A}(\lambda, 0) C^{-1} \widehat{\boldsymbol{x}} = \boldsymbol{F} \text{ with } \widetilde{A}(\lambda, 0) = \lambda V - C.$$

We now factorize the denominators of (4.1) in order to make the poles $(\omega_i^{\pm}(\delta))_{1\le i\le N}$ appear more explicitly in (4.1).

Proposition 4.2. *We can write the following factorization for the poles of (4.1) in the regime $\omega, \delta \to 0$:*

$$\omega^2 - \delta v_b^2 \lambda_i(\omega) = (1 + O(\delta))(\omega - \omega_i^+(\delta))(\omega - \omega_i^-(\delta)), \tag{4.8}$$

where the remainder $O(\delta)$ is a holomorphic function of ω and $\delta^{\frac{1}{2}}$, which can be written fully explicitly, and where $\omega_j^+(\delta)$ and $\omega_j^-(\delta)$ are the resonant frequencies characterized by Proposition 3.9.

Proof. It is clear that

$$\begin{aligned}\omega^2 - \delta v_b^2 \lambda_i(\omega) &= \left(\omega - v_b \delta^{\frac{1}{2}} \sqrt{\lambda_i(\omega)}\right)\left(\omega + v_b \delta^{\frac{1}{2}} \sqrt{\lambda_i(\omega)}\right) \\ &= \left[\omega - \omega_i^+(\delta) - v_b\delta^{\frac{1}{2}}\left(\sqrt{\lambda_i(\omega)} - \sqrt{\lambda_i(\omega_i^+(\delta))}\right)\right]\left[\omega - \omega_i^-(\delta) + v_b\delta^{\frac{1}{2}}\left(\sqrt{\lambda_i(\omega)} - \sqrt{\lambda_i(\omega_i^-(\delta))}\right)\right].\end{aligned}$$

Let us consider the holomorphic function $g_i(\omega) := \sqrt{\lambda_i(\omega)}$ and denote by $(g_{i,p})_{p\in\mathbb{N}}$ the coefficients of its Taylor expansion at $\omega = 0$:

$$g_i(\omega) = \sum_{p=0}^{+\infty} g_{i,p}\omega^p.$$

We consider the function $h_i(\omega, \delta)$ defined as follows:

$$1 + h_i(\omega,\delta) := \frac{\omega^2 - \delta\lambda_i(\omega)}{(\omega - \omega_i^+(\delta))(\omega - \omega_i^-(\delta))} = \left[1 - v_b\delta^{\frac{1}{2}} \frac{g_i(\omega) - g_i(\omega_i^+(\delta))}{\omega - \omega_i^+(\delta)}\right]\left[1 + v_b\delta^{\frac{1}{2}} \frac{g_i(\omega) - g_i(\omega_i^-(\delta))}{\omega - \omega_i^-(\delta)}\right]. \tag{4.9}$$

Then the result (4.8) is proved if we show that $h_i(\omega, \delta)$ is holomorphic in ω and $\delta^{\frac{1}{2}}$ with $h_i(\omega, \delta) = O(\delta)$. By using the identity $a^p - b^p = (a - b)(a^{p-1} + a^{p-2}b + \cdots + b^{p-2}a + b^{p-1})$ valid for any $a, b \in \mathbb{C}$ and $p \in \mathbb{N}$, we can write the full asymptotic expansion of (4.9):

$$\begin{aligned}\frac{g_i(\omega) - g_i(\omega_i^+(\delta))}{\omega - \omega_i^+(\delta)} &= \sum_{p=1}^{+\infty}\sum_{l=0}^{p-1} g_{i,p}\omega_i^+(\delta)^{p-1-l}\omega^l = \sum_{l=0}^{+\infty}\sum_{p=l+1}^{+\infty} g_{i,p}\omega_i^+(\delta)^{p-1-l}\omega^l \\ &= \sum_{p=0}^{+\infty}\left(\sum_{l=0}^{+\infty} g_{i,l+p+1}\omega_i^+(\delta)^l\right)\omega^p = g_{i,1} + O(\delta^{\frac{1}{2}}),\end{aligned} \tag{4.10}$$

and similarly

$$\frac{g_i(\omega) - g_i(\omega_i^-(\delta))}{\omega - \omega_i^-(\delta)} = \sum_{p=0}^{+\infty}\left(\sum_{l=0}^{+\infty} g_{i,l+p+1}\omega_i^-(\delta)^l\right)\omega^p = g_{i,1} + O(\delta^{\frac{1}{2}}). \tag{4.11}$$

Hence, coming back to (4.9), we see that the function h_i is holomorphic in ω and $\delta^{\frac{1}{2}}$, and we obtain

$$1 + h_i(\omega,\delta) = (1 - v_b\delta^{\frac{1}{2}} g_{i,1} + O(\delta))(1 + v_b\delta^{\frac{1}{2}} g_{i,1} + O(\delta)) = 1 + O(\delta).$$

$\square$

For any $p \in \mathbb{N}$, we define the p-th order vector-valued tensor $\boldsymbol{M}^p$ by the formula

$$\boldsymbol{M}^p := \left(\int_{\partial D} y^p \phi_i^* \, \mathrm{d}\sigma \right)_{1 \le i \le N}. \tag{4.12}$$

We recall that $\boldsymbol{M}^p$ is a vector tensor in the sense that for any p derivative indices $1 \le i_1, \dots, i_p \le 3$, $\boldsymbol{M}^p_{i_1 \dots i_p}$ is the vector

$$\boldsymbol{M}^p_{i_1 \dots i_p} = \left(\int_{\partial D} y^p_{i_1 \dots i_p} \phi_i^* \, \mathrm{d}\sigma \right)_{1 \le i \le N}.$$

$\boldsymbol{M}^p$ is analogous to a polarization or moment tensor [16]; it appears in the asymptotic expansion of the right hand-side $\boldsymbol{F}$ of (3.24). By the property (3.14), it holds $\boldsymbol{M}^0 = \mathbf{1}$.

Lemma 4.1. *The following expansion holds for the vector* $\boldsymbol{F}$ *of* (3.24)*:*

$$\boldsymbol{F} = \delta u_{\mathrm{in}}(0) \left(1 - \frac{\mathrm{i}\omega}{4\pi v} \operatorname{cap}(D) \right) C\mathbf{1} + \delta C \boldsymbol{M}^1 \cdot \nabla u_{\mathrm{in}}(0) + O((\omega^2 + \delta)\delta), \tag{4.13}$$

where we recall $\boldsymbol{M}^1 \cdot \nabla u_{in}(0) := \sum_{1 \le i_1 \le N} \boldsymbol{M}^1_{i_1} \partial_{i_1} u_{in}(0)$.

Proof. We have the following expansion for the right-hand side f of (3.23):

$$\begin{aligned} f &= \delta \frac{\partial u_{\mathrm{in}}}{\partial \boldsymbol{n}} - \delta \left(\frac{1}{2} I + \mathcal{K}_D^{k*} \right) (\mathcal{S}_D^k)^{-1} [u_{\mathrm{in}}] \\ &= \delta \nabla u_{\mathrm{in}}(0) \cdot \boldsymbol{n} - \delta \left(\frac{1}{2} I + \mathcal{K}_D^* \right) \left((\mathcal{S}_D)^{-1} - \frac{\omega}{v} (\mathcal{S}_D)^{-1} \mathcal{S}_{D,1} (\mathcal{S}_D)^{-1} \right) [u_{\mathrm{in}}] + O(\omega^2 \delta), \end{aligned}$$

where we use the fact that u_{in} satisfies $(\Delta + k^2) u_{\mathrm{in}}$ in $\mathbb{R}^3$, which implies (e.g. by using the Fourier transform) that $\nabla^p u_{\mathrm{in}}$ is of order ω^p on any bounded set and any $p \in \mathbb{N}$. From (3.19) and (3.20), we obtain $\mathcal{G}^{-1}(\omega, \delta) f = O(\delta)$ and hence the point (i) of the proposition.
Then for any $1 \le i \le N$, we have

$$\begin{aligned} &\int_{\partial B_i} \mathcal{G}^{-1}(\delta, \omega)[f] \, \mathrm{d}\sigma \\ &= - \int_{\partial B_i} \delta \mathcal{L}_0^{-1} \left(\frac{1}{2} I + \mathcal{K}_D^* \right) \left((\mathcal{S}_D)^{-1} - \frac{\omega}{v} (\mathcal{S}_D)^{-1} \mathcal{S}_{D,1} (\mathcal{S}_D)^{-1} \right) [u_{\mathrm{in}}] \, \mathrm{d}\sigma + \int_{\partial B_i} \delta \nabla u_{\mathrm{in}}(0) \cdot \boldsymbol{n} \, \mathrm{d}\sigma + O((\omega^2 + \delta)\delta) \\ &= -\delta \int_{\partial B_i} \left((\mathcal{S}_D)^{-1} - \frac{\omega}{v} (\mathcal{S}_D)^{-1} \mathcal{S}_{D,1} (\mathcal{S}_D)^{-1} \right) [u_{\mathrm{in}}] \, \mathrm{d}\sigma + O((\omega^2 + \delta)\delta) \\ &= -\delta \int_{\partial D} u_{\mathrm{in}} \psi_i^* \, \mathrm{d}\sigma + \delta \frac{\omega}{v} u_{\mathrm{in}}(0) \int_{\partial D} \psi_i^* \, \mathrm{d}\sigma \left(-\frac{\mathrm{i}}{4\pi} \right) \int_{\partial D} \sum_{j=1}^N \psi_j^* \, \mathrm{d}\sigma + O((\omega^2 + \delta)\delta) \\ &= \delta u_{\mathrm{in}}(0) \sum_{j=1}^N C_{ij} + \delta \nabla u_{in}(0) \cdot \int_{\partial D} y^1 \sum_{j=1}^N C_{ij} \phi_j^* \, \mathrm{d}\sigma - \delta \frac{\mathrm{i}\omega}{4\pi v} \sum_{j=1}^N C_{ij} \sum_{1 \le j, l \le N} C_{jl} u_{\mathrm{in}}(0) + O((\omega^2 + \delta)\delta). \end{aligned}$$

Hence the right-hand side $\boldsymbol{F} = \left(\int_{\partial B_i} \mathcal{G}^{-1}(\delta, \omega)[f] \, \mathrm{d}\sigma \right)_{1 \le i \le N}$ of (3.24) is given by (4.13). □

Remark 4.5. The point 0 at which the Taylor expansion of u_{in} is computed in (4.13) does not matter, it can be replaced by any other fixed given point $x_0 \in \mathbb{R}^3$. However, as implicitly used in [26, 15], it needs to be replaced by the center of the resonator in the dilute setting where its size shrinks to zero [40].

Remark 4.6. More generally, higher order terms in (4.13) depend on the tensors $\boldsymbol{M}^p$ as well as the operators $\mathcal{S}_{D,p}$ and $\mathcal{K}^*_{D,p}$ for larger values of $p \in \mathbb{N}$.

Inserting (4.8) and (4.13) into (4.1), we obtain the following result.

Corollary 4.1. *The following pole expansion holds as* $\delta \to 0$ *and* $\omega \to 0$ *for the solution* $\boldsymbol{x}(\omega, \delta)$ *of* (3.24)*:*

$$\boldsymbol{x}(\omega, \delta) = \sum_{i=1}^N \frac{\delta g_i^0(\omega) u_{\mathrm{in}}(0) + \delta g_i^1(\omega) \cdot \nabla u_{\mathrm{in}}(0) + O((\omega^2 + \delta)\delta)}{(\omega - \omega_i^+(\delta))(\omega - \omega_i^-(\delta))} C \boldsymbol{a}_i(\omega), \tag{4.14}$$

where g_i^0 *and* g_i^1 *are the zero-th and first order scalar tensors*

$$g_i^0(\omega) := v_b^2 \left(1 - \frac{\mathrm{i}\omega}{4\pi v} \operatorname{cap}(D) \right) (\boldsymbol{a}_i^T - \mathrm{i}\omega \boldsymbol{b}_i^T) C\mathbf{1}, \quad g_i^1(\omega) := v_b^2 \boldsymbol{a}_i^T C \boldsymbol{M}^1, \qquad 1 \le i \le N, \tag{4.15}$$

for $\boldsymbol{a}_i(\omega)$ *and* $\boldsymbol{b}_i$ *defined in* Proposition 3.7 *and* (4.2)*. Furthermore, the remainder* $O(\omega^2 + \delta)$ *is a holomorphic function in* ω *and* $\delta^{\frac{1}{2}}$ *which can be fully explicited.*

Remark 4.7. The result of Corollary 4.1 is true without assuming $\omega = O(\delta^{\frac{1}{2}})$.

4.2. **Modal decomposition of the scattered field**

We now use the result of Corollary 4.1 in order to obtain a modal decomposition for the field u_{tot}, solution to (1.1). We start by defining "resonant states" as root functions to the integral problem (3.4), or to the scattering problem (1.1). Since $A(\omega,\delta)C\boldsymbol{a}_i(\omega) = 0$ when $\omega = \omega_i^\pm(\delta)$ for any $1 \leq i \leq N$ (Proposition 3.7), these are obtained by substituting $(x_i)_{1\leq i\leq N}$ with the coordinates of the vector $\boldsymbol{a}_i(\omega) = (\boldsymbol{a}_i(\omega)^T C\boldsymbol{e}_j)_{1\leq j\leq N}$ in (3.22) with $u_{\text{in}} = 0$:

Definition 4.1. For $1 \leq i \leq N$, We define "resonant states" $\phi_i^*(\omega,\delta), \psi_i^*(\omega,\delta) \in L^2(\partial D)$ associated to the eigenvector $\boldsymbol{a}_i(\omega)$ by the formulas

$$\phi_i^*(\omega,\delta) := \sum_{j=1}^N (\boldsymbol{e}_j^T C\boldsymbol{a}_i(\omega))\mathcal{G}^{-1}(\omega,\delta)[\phi_j^*], \qquad \psi_i^*(\omega,\delta) := (\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}[\phi_i^*(\omega,\delta)], \qquad 1 \leq i \leq N. \tag{4.16}$$

We then define the "physical" resonant mode $u_i(\omega,\delta)$ as a function of $\mathbb{R}^3$ by

$$u_i(\omega,\delta)(x) = \begin{cases} \mathcal{S}_D^{k_b}[\phi_i^*(\omega,\delta)](x) \text{ if } x \in D, \\ \mathcal{S}_D^{k}[\psi_i^*(\omega,\delta)](x) \text{ if } x \in \mathbb{R}^3\backslash D. \end{cases} \tag{4.17}$$

Remark 4.8. By definition, the mode $(\phi_i^*(\omega,\delta), \psi_i^*(\omega,\delta))$ is a root function of the scattering problem (1.1), and $u_i(\omega_i^\pm(\delta),\delta)$ is a non-trivial solution to (1.1) with $u_{\text{in}} = 0$.

Remark 4.9. The resonant states $\phi_i^*(\omega,\delta)$ and $\psi_i^*(\omega,\delta)$ with $1 \leq i \leq N$ are to the first order linear combinations of the potentials $(\phi_i^*)_{1\leq i\leq N}$ (definition (3.13)). More precisely, it holds, due to (3.19):

$$\phi_i^*(\omega,\delta) = \sum_{j=1}^N (\boldsymbol{e}_j^T C\boldsymbol{a}_i(\omega))\phi_j^* + O(\omega^2+\delta), \tag{4.18}$$

$$\psi_i^*(\omega,\delta) = \sum_{j=1}^N \left(\boldsymbol{e}_j^T C\boldsymbol{a}_i(\omega) + \frac{\mathrm{i}\omega}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right)\boldsymbol{1}^T C\boldsymbol{a}_i \sum_{l=1}^N C_{lj}\right)\phi_j^* + O(\omega^2+\delta), \tag{4.19}$$

where the second identity is a consequence of

$$(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}[\phi_j^*] = \phi_j^* - \frac{\mathrm{i}\omega}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right)\sum_{l=1}^N \psi_l^* \text{ with } \psi_l^* = -\sum_{j=1}^N C_{lj}\phi_j^*. \tag{4.20}$$

The resonant states enable to write a modal decomposition of the solution to (3.4). It is obtained by reading first a modal decomposition for the potential ϕ and ψ of (3.22):

Corollary 4.2. *The solution (ϕ,ψ) to the scattering problem (3.2) admits the following modal decomposition as $\omega \to 0$ and $\delta \to 0$:*

$$\phi = \mathcal{G}^{-1}(\omega,\delta)[f] + \sum_{i=1}^N \frac{\delta g_i^0(\omega)u_{\text{in}}(0) + \delta g_i^1(\omega)\cdot\nabla u_{\text{in}}(0) + O((\omega^2+\delta)\delta)}{(\omega-\omega_i^+(\delta))(\omega-\omega_i^-(\delta))}\phi_i^*(\omega,\delta), \tag{4.21}$$

$$\psi = (\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)[f] - (\mathcal{S}_D^k)^{-1}[u_{\text{in}}] + \sum_{i=1}^N \frac{\delta g_i^0(\omega)u_{\text{in}}(0) + \delta g_i^1(\omega)\cdot\nabla u_{\text{in}}(0) + O((\omega^2+\delta)\delta)}{(\omega-\omega_i^+(\delta))(\omega-\omega_i^-(\delta))}\psi_i^*(\omega,\delta), \tag{4.22}$$

where $\mathcal{G}^{-1}(\omega,\delta)[f] = O(\delta)$; g_i^0 and g_i^1 are given by (4.15) and $O((\omega^2+\delta)\delta)$ is a holomorphic function of ω and $\delta^{1/2}$.

Proof. Using (3.22) and (4.3) with $\boldsymbol{x}(\omega,\delta) = C\boldsymbol{y}(\omega,\delta)$, the potential ϕ reads

$$\begin{aligned}\phi &= \mathcal{G}^{-1}(\omega,\delta)[f] + \sum_{j=1}^N \boldsymbol{e}_j^T\boldsymbol{x}(\omega,\delta)\mathcal{G}^{-1}(\omega,\delta)[\phi_j^*] = \mathcal{G}^{-1}(\omega,\delta)[f] + \sum_{j=1}^N\sum_{i=1}^N y_i(\omega,\delta)\boldsymbol{e}_j^T C\boldsymbol{a}_i(\omega)\mathcal{G}^{-1}(\omega,\delta)[\phi_j^*] \\ &= \mathcal{G}^{-1}(\omega,\delta)[f] + \sum_{i=1}^N y_i(\omega,\delta)\phi_i^*(\omega,\delta).\end{aligned} \tag{4.23}$$

The result (4.21) follows from the expression (4.15) determining the value of $y_i(\omega,\delta)$. Then (4.22) is obtained by inserting (4.21) into the first line of (3.5). □

Remark 4.10. The identity (4.21) improves the result of [13, 8] in the fact that it clearly highlights the structure of the inverse of the operator $\mathcal{A}(\omega,\delta)$ in terms of the resonant poles $\omega_i^\pm(\delta)$ and modes $\phi_i^*(\omega,\delta)$. Furthermore the scattering amplitude is known up to the order $O(\omega^2+\delta)$ instead of $O(\omega+\delta^{\frac{1}{2}})$.

In order to propagate asymptotic expansions from the boundary potentials ψ and ϕ to the fields $\mathcal{S}_D^k[\psi]$ and $\mathcal{S}_D^{k_b}[\phi]$, we need to bound $\mathcal{S}_D^k[\phi]$ as a function on $\mathbb{R}^3$ in terms of ϕ. Following [71], we consider the space

$$H = \left\{ u \,|\, \frac{u}{(1+r^2)^{1/2}} \in L^2(\mathbb{R}^3),\ \frac{\nabla u}{(1+r^2)^{1/2}} \in L^2(\mathbb{R}^3),\ \frac{\partial u}{\partial r} - \mathrm{i}ku \in L^2(\mathbb{R}^3) \right\}.$$

Let $R > 0$ be a sufficiently large positive number such that D is contained in the ball of radius R: $D \subset B(0,R)$. We recall the following facts (see [71, Theorem 2.6.6]).

Proposition 4.3. *(i) Let $u \in H^1_{loc}(\mathbb{R}^3)$ be a function solving the homogeneous Helmholtz equation on the exterior domain $\mathbb{R}^3 \backslash B(0,R)$:*

$$\begin{cases} -\Delta u - k^2 u = 0 \ \text{in } \mathbb{R}^3\backslash B(0,R), \\ \left(\dfrac{\partial u}{\partial |x|} - \mathrm{i}ku\right)(x) = o(|x|^{-1}) \ \text{as } |x| \to +\infty. \end{cases} \tag{4.24}$$

Then $u \in H$ and there exists constants $\alpha, \alpha' > 0$ depending only on R such that

$$||u||_H \leq \alpha ||u||_{H^{-\frac{1}{2}}(\partial B(0,R))} \leq \alpha' ||u||_{H^1(B(0,R))}.$$

(ii) In particular, $\mathcal{S}_D^k[\phi] \in H$ for any $\phi \in L^2(\partial D)$ and there exists a constant $\alpha > 0$ depending only on $R > 0$ such that

$$||\mathcal{S}_D^k[\phi]||_H \leq \alpha ||\phi||_{L^2(\partial D)}.$$

Proof. The point (i) is obtained in Theorem 2.6.3 of [71]. Point (ii) is obtained by writing that $u = \mathcal{S}_D^k[\phi]$ satisfies

$$\begin{cases} -\Delta u - k^2 u = 0 \text{ in } B(0,R), \\ \left[\!\left[\dfrac{\partial u}{\partial n} \right]\!\right] = \phi \text{ on } \partial D, \\ \dfrac{\partial u}{\partial n} - T_R u = 0 \text{ on } \partial B(0,R), \end{cases}$$

where $T_R : H^{\frac{1}{2}}(\partial B(0,R)) \to H^{-\frac{1}{2}}(\partial B(0,R))$ is the capacity operator. The variational formulation of this Fredholm problem reads :

find $u \in H^1(B(0,R))$ such that for any $v \in H^1(B(0,R))$,

$$\int_{B(0,R)} \nabla u \cdot \nabla \overline{v} \, \mathrm{d}x - \int_{\partial B(0,R)} T_R u \overline{v} \, \mathrm{d}\sigma = \int_{\partial D} \phi \overline{v} \, \mathrm{d}\sigma, \tag{4.25}$$

where $\overline{v}$ is the complex conjugate of the test function v. The bilinear form associated to (4.25) is injective and independent of ∂D. Hence, setting $v = u$ and using the Banach-Nečas-Babuška theorem [65], we can obtain the existence of a constant C independent of ∂D such that

$$||u||_{H^1(B(0,R))} \leq \alpha ||\phi||_{L^2(\partial D)},$$

from where the result is derived. □

Since $\omega_i^-(\delta) = -\overline{\omega}_i^+(\delta)$, the denominators of (4.21) and (4.22) read:

$$(\omega - \omega_i^+(\delta))(\omega - \omega_i^-(\delta)) = (\omega^2 - |\omega_i^\pm(\delta)|^2) - 2\mathrm{i}\omega\Im(\omega_i^\pm(\delta)). \tag{4.26}$$

Gathering (3.1), (4.21) and (4.22) yields the following result:

Corollary 4.3. *The solution u_{tot} to the scattering problem (1.1) admits the following modal decomposition as $\omega \to 0$ and $\delta \to 0$:*

$$u_{\text{tot}} = \sum_{i=1}^N \frac{\delta g_i^0(\omega) u_{\text{in}}(0) + \delta g_i^1(\omega) \cdot \nabla u_{\text{in}}(0) + O(\delta(\delta + \omega^2))}{\omega^2 - |\omega_i^\pm(\delta)|^2 - 2\mathrm{i}\omega\Im(\omega_i^\pm(\delta))} u_i(\omega,\delta) + (u_{\text{in}} - \mathcal{S}_D^k[(\mathcal{S}_D^k)^{-1}[u_{\text{in}}]]) 1_{\mathbb{R}^3\backslash D} + O(\delta), \tag{4.27}$$

where $O(\delta)$ is a function holomorphic in ω and δ such that $||O(\delta)||_H/\delta \to 0$ as $\omega, \delta \to 0$, and g_i^0 and g_i^1 are given by (4.15).

Remark 4.11. Considering only real and positive values of ω, we see that $-2\Im(\omega_i^\pm(\delta))/|\omega_i^\pm(\delta)|$ plays the role of a damping constant.

Remark 4.12. Corollary 4.3 is a clarification of Lemma 2.11 in [4], whereby the resonant denominator of (4.27) vanishes exactly at the scattering frequency $\omega_i^\pm(\delta)$, and the norm (as well as the space H) measuring the smallness of $O(\delta)$ is specified.

Remark 4.13. Using (3.22) and (4.1) without approximating the vector $\boldsymbol{F}$ by using the result of Lemma 4.1, one obtains the existence of N linear forms $E_i(\omega,\delta) : L^2(\partial D)\times L^2(\partial D)\to\mathbb{R}$, for $1\le i\le N$, such that

$$\delta g_i^0(\omega)u_{\text{in}}(0)+\delta g_i^1(\omega)\cdot\nabla u_{\text{in}}(0)+O(\delta(\delta+\omega^2))=E_i(\omega,\delta)\begin{pmatrix}u_{\text{in}}\\ \frac{\partial u_{\text{in}}}{\partial n}\end{pmatrix}.$$

Inserting this expression into (4.27) yields the modal decomposition (1.5) claimed in the introduction. Furthermore, the previous analysis shows that $E_i(\omega,\delta)$ is holomorphic as a function of ω and $\delta^{\frac{1}{2}}$ and verifies $E_i(\omega,\delta)=O(1)$ in the operator norm.

Remark 4.14. Applying the inverse Fourier transform to the resonant part of (4.27), it is possible to rewrite (4.27) in the form of a modal decomposition in the time domain, see e.g. [4, 3, 21, 8].

4.3. **Estimation of the magnitude of the resonances for real frequencies**

Physically, the parameter ω is real, hence the resonant part of u_{tot} in (4.27) featuring resonant poles

$$\frac{\delta}{(\omega-\omega_i^+(\delta))(\omega-\omega_i^-(\delta))}=\frac{\delta}{\omega^2-|\omega_i^\pm(\delta)|^2-2\mathrm{i}\omega\Im(\omega_i^\pm(\delta))}\tag{4.28}$$

has a bounded magnitude for these frequencies (despite it blows up for $\omega=\omega_i^\pm(\delta)$). Furthermore, the resonant frequencies $\omega_i^\pm(\delta)$ are replaced by approximations in the scattering amplitude (4.28) [26, 13, 15, 8], and it is important to estimate the induced error as ω is real (this error being unbounded for complex frequencies). In this part, we gather a few results which allow to estimate the magnitude of the resonant poles of u_{tot} at real frequencies, and to estimate the error induced by replacing resonant denominators with approximate ones.

Lemma 4.2. *Let $a,b\in\mathbb{R}$ satisfying $|a|>|b|$. The following inequality holds:*

$$\forall\omega\in\mathbb{R},\ |\omega^2-a^2+2\mathrm{i}b\omega|\ge 2|b|\sqrt{a^2-b^2}.\tag{4.29}$$

Proof. We consider the real function f defined by

$$f(\omega):=|\omega^2-a^2+2\mathrm{i}b\omega|^2=(\omega^2-a^2)^2+4\omega^2b^2,$$

whose derivative reads

$$f'(\omega)=4\omega(\omega^2-a^2)+8b^2\omega=4\omega(\omega^2-a^2+2b^2).$$

Consequently, f reaches therefore its minimum when $\omega^2=a^2-2b^2$ and we obtain

$$\forall\omega\in\mathbb{R},\ f(\omega)\ge 4b^4+4b^2(a^2-2b^2)=4b^2(a^2-b^2).$$

□

The bound (4.29) enables to estimate the amplitude of the resonances in (4.27).

Corollary 4.4. *For real frequencies $\omega\in\mathbb{R}$, the resonant amplitudes of (4.27) are of order:*

$$\frac{\delta}{\omega^2-|\omega_i^\pm(\delta)|^2-2\mathrm{i}\omega\Im(\omega_i^\pm(\delta))}=O\left(\frac{\delta^{\frac{1}{2}}}{\Im(\omega_i^\pm(\delta))}\right).\tag{4.30}$$

Remark 4.15. The enhancement coefficient of the resonance is therefore determined by the imaginary part of $\omega_i^\pm(\delta)$. In most situations, $\Im(\omega_i^\pm(\delta))=O(\delta)$ inducing an enhancement is of order $O(\delta^{-\frac{1}{2}})$. However, $\Im(\omega_i^\pm(\delta))$ may be of order smaller or equal to $O(\delta^2)$ when $\boldsymbol{a}_i^TV\mathbf{1}=0$ (eqn. (3.42)), e.g. in case of symmetries. Furthermore, the modal decomposition (4.27) reveals that this amplification can be reduced by a factor ω in the situation where $g_i^0(\omega)=0$.

Remark 4.16. The reader may retain that the magnitude of the resonance of (4.30) is obtained by setting $\omega=|\omega_i^\pm(\delta)|$ in (4.28).

We conclude this part by a remark on the error committed by the approximation of a resonant ratio (considered at real frequencies $\omega>0$) by a different one.

Lemma 4.3. *Let $a,b\in\mathbb{R}$ satisfying $|b|\ll|a|$ and $\Delta a,\Delta b\in\mathbb{R}$ such that $|\Delta a|\ll|a|$ and $|\Delta b|\ll|b|$. If further*

$$|\Delta a|\ll|b|,\tag{4.31}$$

then the following approximations hold as $\omega=O(|a|)$:

$$\frac{1}{\omega^2-(a+\Delta a)^2+2\mathrm{i}(b+\Delta b)\omega}=\frac{1}{\omega^2-a^2+2\mathrm{i}b\omega}\left(1+O\left(\frac{\max(|\Delta a|,|\Delta b|)}{|b|}\right)\right).\tag{4.32}$$

Proof. We write, by using (4.29) and the assumptions of the lemma:

$$\begin{aligned}\frac{1}{\omega^2-(a+\Delta a)^2+2\mathrm{i}(b+\Delta b)\omega} &= \frac{1}{\omega^2-a^2+2\mathrm{i}b\omega}\frac{1}{1+O\left(\left|\frac{\Delta a}{b}\right|+\left|\frac{\Delta b}{b}\right|\frac{\omega}{|a|}\right)} \\ &= \frac{1}{\omega^2-a^2+2\mathrm{i}b\omega}\left(1+O\left(\left|\frac{\Delta a}{b}\right|+\left|\frac{\Delta b}{b}\right|\frac{\omega}{|a|}\right)\right),\end{aligned} \tag{4.33}$$

from where the result follows. □

Remark 4.17. The condition (4.31) is very important for the approximation (4.32) to be valid. Recalling the expansion (3.42) of $\omega_i^{\pm}(\delta)$, we find that in the generic case where $\boldsymbol{a}_i^T V\mathbf{1}\neq 0$ (i.e. corresponding to $i=1$ or when the vector of ones $\mathbf{1}=(1)_{1\leq i\leq N}$ is not an eigenvector of (2.4)), then $\Im(\omega_i^{\pm}(\delta))=O(\delta)$ and

$$\frac{\delta}{\omega^2-|\omega_i^{\pm}(\delta)|^2-2\mathrm{i}\omega\Im(\omega_i^{\pm}(\delta))}=O\left(\delta^{-\frac{1}{2}}\right).$$

Furthermore, we have the approximation

$$\frac{\delta}{\omega^2-|\omega_i^{\pm}(\delta)|^2-2\mathrm{i}\omega\Im(\omega_i^{\pm}(\delta))}=\frac{\delta}{\omega^2-v_b^2\lambda_i\delta+\frac{\mathrm{i}\omega\delta}{4\pi v}v_b^2\lambda_i^2(\boldsymbol{a}_i^TV\mathbf{1})^2}\left(1+O(\delta^{\frac{1}{2}})\right).$$

Remark 4.18. For the dimer considered in Section 2.3.3, the vector of ones $\mathbf{1}$ is an eigenvector of (2.4), hence $\boldsymbol{a}_2^TC\mathbf{1}=0$ and the second resonant frequency (see also [15]) $\omega_2^{\pm}(\delta)$ has the form

$$\omega_2^{\pm}(\delta)=\pm v_b\lambda_2^{\frac{1}{2}}\delta^{\frac{1}{2}}\pm\eta_1\delta^{3/2}-\eta_2 i\delta^2+O(\delta^{\frac{5}{2}}), \tag{4.34}$$

for coefficients η_1,η_2 a priori positive. The resonant amplitude (4.30) is therefore of order $O(\delta^{-\frac{3}{2}})$. Therefore, the coefficient η_1 needs to be kept in the denominator if one desires to approximate the amplitude with vanishing errors: for ω real,

$$\frac{1}{\omega^2-|\omega_2^{\pm}(\delta)|^2-2\mathrm{i}\omega\Im(\omega_2^{\pm}(\delta))}=\frac{1}{\omega^2-|\sqrt{\lambda_2\delta}+\pm\eta_1\delta^{3/2})|^2-2\mathrm{i}\omega\eta_2\delta^2}(1+O(\delta^{\frac{1}{2}})),$$

however if we do not keep η_1, we arrive instead at

$$\frac{1}{\omega^2-|\omega_2^{\pm}(\delta)|^2-2\mathrm{i}\omega\Im(\omega_2^{\pm}(\delta))}=\frac{1}{\omega^2-|\sqrt{\lambda_2\delta}|^2-2\mathrm{i}\omega\eta_2\delta^2}(1+O(\delta^{-\frac{1}{2}})).$$

5. Point scatterer approximations

In this final section, we compute the leading asymptotics of the far field pattern for the field scattered by the resonant medium D in the subwavelength regime. The case of N resonators is treated in Section 5.1, where the far field of the scattered field is computed in Proposition 5.3. We find that in this generic situation, the group of N resonators behaves in the far field as a monopole scatterer (a point source) as the frequency ω gets close to any of the resonant frequencies $\omega_i^{\pm}(\delta)$: $u_{\mathrm{tot}}(x)$ is approximately proportional to the fundamental solution $\Gamma^k(x)$ as $|x|\to+\infty$. Multipole behaviors, i.e. a far field pattern proportional to $\nabla^p\Gamma^k(x)$ with $p\geq 1$, can be obtained under sufficient symmetry of the system of resonators D, requiring among other $\mathbf{1}$ to be an eigenvector of the eigenvalue problem (2.4). Then, the result obtained for N bodies is applied to the case $N=1$ with a single resonator in Section 5.2, and finally to a dimer of $N=2$ spherical identical resonators in Section 5.3. In both cases $N=1$ and $N=2$ we retrieve and propose simplifications to the corresponding formulas obtained in [13, 15]. We still assume the simplicity (3.32) of the eigenvalues of the (weighted) capacitance matrix. We emphasize the connexion with generalized moment tensors [16] and we retrieve some results from [13, 15, 10].

5.1. Point scatterer approximation for a system with N resonators

The far field expansions of the solution u_{tot} to the scattering problem (1.1) is obtained by expanding the kernel $\Gamma^k(x-y)$ involved in the single layer potential $\mathcal{S}_D^k[\psi](x)$ in the representation (3.1). The general formula for the expansion of $\Gamma^k(x-y)$ as $|x|\to+\infty$ and y remains in a bounded set is provided by the addition formula [34], which involves spherical Bessel functions. Here, we rather rely on a different multipole expansion of the fundamental solution, which yields a simpler formula in the regime $\omega\to 0$. In order to establish this expansion, we use the following result.

Lemma 5.1. *For any $p\in\mathbb{N}$, there exists a constant $\alpha>0$ such that*

$$\forall x\in\mathbb{R}^3,\quad \nabla^{p+1}\Gamma^k(x)\leq\alpha|x|^{-1}(|x|^{-p}+k^p). \tag{5.1}$$

Proof. The p-th derivative of Γ^k can be obtained by differentiating this function $p+1$ times using Leibniz and Faa Di Bruno formula [70], from where the estimate (5.1) can then be explicitly derived. We provide below a more constructive proof based on tensorial calculus.

Let us introduced the family of p-th order tensor $(f_j^p(x))_{0\le j\le p}$ defined by induction on p by

$$\begin{cases} f_j^{p+1}(x) = \displaystyle\sum_{l=1}^{3}\left(\partial_l f_{j-1}^p(x) + \mathrm{i}\frac{x_l}{|x|} f_j^p(x)\right)\otimes e_l \text{ with } 0\le j\le p+1, \\ f_0^0(x) = \dfrac{1}{|x|}, \end{cases}$$

with the convention $f_{p+1}^p(x) = f_{-1}^p(x) = 0$. By induction, we can prove that f_j^p is homogeneous of degree $-1-j$ and that the p-th derivative of Γ^k reads

$$\nabla^p\Gamma^k(x) = -\frac{1}{4\pi}\sum_{j=0}^{p} k^{p-j} f_j^p(x) e^{ik|x|}, \qquad x\in\mathbb{R}^3.$$

Since $f_j^p(x)$ is homogeneous of degree $-1-j$ and does not depend on the wave number k for $0\le j\le p$, there exists an independent constant $\alpha>0$ such that

$$\forall x\in\mathbb{R}^3,\quad |f_j^p(x)| < \alpha|x|^{-1-j} \text{ for any } 0\le j\le p.$$

Hence we obtain the bound

$$\begin{aligned} |\nabla^p\Gamma^k(x)| &\le \frac{\alpha}{4\pi}\sum_{j=0}^{p} k^{p-j}|x|^{-1-j} \le \frac{\alpha}{4\pi}|x|^{-1}k^p(p+1)\max(1,k^{-p}|x|^{-p}) \\ &\le \frac{\alpha(p+1)}{4\pi}|x|^{-1}(k^p + |x|^{-p}). \end{aligned}$$

□

Proposition 5.1. *The following multipole expansion holds for the kernel $\Gamma^k(x-y)$ as $\omega\to 0$, $|x|\to+\infty$ and $|y| = O(1)$: for any $p\in\mathbb{N}$,*

$$\Gamma^k(x-y) = \sum_{l=0}^{p}\frac{(-1)^l}{l!}\nabla^l\Gamma^k(x)\cdot y^l + O\left(\frac{1}{|x|^{p+2}}\right) + O\left(\frac{\omega^{p+1}}{|x|}\right). \tag{5.2}$$

Proof. The Taylor-Lagrange formula reads

$$\Gamma^k(x-y) = \sum_{l=0}^{p}\frac{(-1)^l}{l!}\nabla^l\Gamma^k(x)\cdot y^l + \frac{(-1)^{p+1}}{p!}\int_0^1 (1-t)^p\nabla^{p+1}\Gamma^k(x-ty)\cdot y^{p+1}\,\mathrm{d}t.$$

Therefore, as $|x|\to+\infty$ and $|y|$ remains bounded, the remainder of the Taylor sum truncated at order p is of order $O\left(\nabla^{p+1}\Gamma^k(x)\right)$ which is of order $O(|x|^{-1}(\omega^{p+1} + |x|^{-p+1})$ according to Lemma 5.1. □

The far field of the resonant modes $u_i(\omega,\delta)$ of (4.17) can be expressed in terms of polarization tensors $\boldsymbol{M}^p(\omega,\delta)$ which generalize the tensors $\boldsymbol{M}^p$ defined by (4.12), in the sense that $\boldsymbol{M}^p(0,0) = \boldsymbol{M}^p$.

Definition 5.1. We denote by $\boldsymbol{M}^p(\omega,\delta)$ the vector valued tensor

$$\boldsymbol{M}^p(\omega,\delta) := \left(\int_{\partial D} y^p(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)[\phi_i^*](y)\,\mathrm{d}\sigma(y)\right)_{1\le i\le N}. \tag{5.3}$$

Proposition 5.2. *The resonant mode $u_i(\omega,\delta)(x)$ of (4.17) admits the following multipole expansion as $|x|\to+\infty$ and $\omega\to 0$:*

$$u_i(\omega,\delta)(x) = \sum_{l=1}^{p}\frac{(-1)^l}{l!}\boldsymbol{a}_i(\omega)^T C\boldsymbol{M}^l(\omega,\delta)\cdot\nabla^l\Gamma^k(x) + O\left(\frac{1}{|x|^{p+2}}\right) + O\left(\frac{\omega^{p+1}}{|x|}\right). \tag{5.4}$$

Proof. Denote $\widehat{\phi}_i^*(\omega,\delta) := (\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)[\phi_i^*]$. By using (5.2), we can develop the single layer potential $\mathcal{S}_D^k[\widehat{\phi}_i^*]$ as $|x|\to+\infty$:

$$\mathcal{S}_D^k[\widehat{\phi}_i^*] = \int_{\partial D}\Gamma^k(x-y)\widehat{\phi}_i^*(y)\,\mathrm{d}\sigma(y) = \sum_{l=0}^{p}\frac{(-1)^l}{l!}\nabla^l\Gamma^k(x)\cdot M_i^l(\omega,\delta) + O(|x|^{-1}(|x|^{-p-1}+\omega^{p+1})).$$

Then (4.16) and (4.17) yield

$$
\begin{aligned}
u_i(\omega,\delta)(x) = \mathcal{S}_D^k[\psi_i^*(\omega,\delta)](x) &= \sum_{j=1}^N (\boldsymbol{e}_j^T C \boldsymbol{a}_i(\omega)) \mathcal{S}_D^k[\widehat{\phi}_j^*](x) \\
&= \sum_{l=0}^p \frac{(-1)^l}{l!} \nabla^l \Gamma^k(x) \cdot \sum_{j=1}^N (\boldsymbol{e}_j^T C \boldsymbol{a}_i(\omega)) M_j^l(\omega,\delta) + O(|x|^{-1}(|x|^{-p-1} + \omega^{p+1})) \\
&= \sum_{l=0}^p \frac{(-1)^l}{l!} \nabla^l \Gamma^k(x) \cdot \boldsymbol{a}_i(\omega)^T C \boldsymbol{M}^l(\omega,\delta) + O(|x|^{-1}(|x|^{-p-1} + \omega^{p+1})).
\end{aligned}
$$

□

We need to introduce a last tensor appearing in the point scatterer approximation of the scattered field $u_{\text{tot}} - u_{\text{in}}$.

Definition 5.2. We denote by K^2 the second order scalar tensor defined by

$$
K^2 := \int_{\partial D} y^1 \otimes (\mathcal{S}_D^k)^{-1}[y^1](y) \, \mathrm{d}\sigma(y). \tag{5.5}
$$

Proposition 5.3. *The following multipole expansion holds for the scattered field as $|x| \to +\infty$, $\omega \to 0$ and $\delta \to 0$:*

$$
\begin{aligned}
&u_{\text{tot}}(x) - u_{\text{in}}(x) \\
&= \sum_{i=1}^N \frac{\delta g_i^0(\omega) u_{\text{in}}(0) + \delta g_i^1(\omega) \cdot \nabla u_{\text{in}}(0)}{\omega^2 - |\omega_i^{\pm}(\delta)|^2 - 2\mathrm{i}\omega \Im(\omega_i^{\pm}(\delta))} \boldsymbol{a}_i(\omega)^T C \left[\boldsymbol{M}^0(\omega,\delta)\Gamma^k(x) - \boldsymbol{M}^1 \cdot \nabla \Gamma^k(x) \; + |x|^{-1} O(|x|^{-2} + \omega^2 + \delta) \right] \\
&+ \left[\left(1 - \frac{\mathrm{i}\omega}{4\pi v} \operatorname{cap}(D) \right) \operatorname{cap}(D) u_{\text{in}}(0) + \mathbf{1}^T C \boldsymbol{M}^1 \cdot \nabla u_{\text{in}}(0) \right] \Gamma^k(x) \\
&- u_{\text{in}}(0) \mathbf{1}^T C \boldsymbol{M}^1 \cdot \nabla \Gamma^k(x) + \nabla u_{\text{in}}(0) \cdot K^2 \cdot \nabla \Gamma^k(x) \\
&+ |x|^{-1} O(\omega^2 + |x|^{-2}) + O(\delta),
\end{aligned} \tag{5.6}
$$

where

$$
\nabla u_{\text{in}}(0) \cdot K^2 \cdot \nabla \Gamma^k(x) := \sum_{1 \le i_1, i_2 \le 3} K^2_{i_1 i_2} \partial_{y_{i_1}} u_{\text{in}}(0) \partial_{y_{i_2}} \Gamma^k(x)
$$

and where $g_i^0(\omega)$ and $g_i^1(\omega)$ are given by (4.15).

Proof. The point scatterer approximation of the resonant part is obtained by inserting (5.4) with $p = 1$ into the modal decomposition (3.26), where we further note that

$$
\boldsymbol{M}^1(\omega,\delta) \cdot \nabla \Gamma(x) = \boldsymbol{M}^1 \cdot \nabla \Gamma(x) + O((\omega + \delta)(|x|^{-1}(\omega + |x|^{-1}))) = \boldsymbol{M}^1 \cdot \nabla \Gamma(x) + O(|x|^{-1}(\omega^2 + \delta + |x|^{-2})).
$$

It remains to expand the non-resonant term $-\mathcal{S}_D^k[(\mathcal{S}_D^k)^{-1}[u_{\text{in}}]](x)$ of (4.27). Observe first that

$$
\begin{aligned}
(\mathcal{S}_D^k)^{-1}[u_{\text{in}}] &= (\mathcal{S}_D)^{-1}[u_{\text{in}}] - \frac{\omega}{v} (\mathcal{S}_D)^{-1} \mathcal{S}_{D,1} (\mathcal{S}_D)^{-1}[u_{\text{in}}] + O(\omega^2) \\
&= u_{\text{in}}(0) (\mathcal{S}_D)^{-1}[1_{\partial D}] + \nabla u_{\text{in}}(0) \cdot (\mathcal{S}_D)^{-1}[y^1] - u_{\text{in}}(0) \frac{\omega}{v} (\mathcal{S}_D)^{-1} \mathcal{S}_{D,1} (\mathcal{S}_D)^{-1}[1_{\partial D}] + O(\omega^2).
\end{aligned}
$$

Then, $(\mathcal{S}_D)^{-1}[1_{\partial D}] = \sum_{i=1}^N \psi_i^*$ and (3.9) entail

$$
(\mathcal{S}_D)^{-1} \mathcal{S}_{D,1} (\mathcal{S}_D)^{-1}[1_{\partial D}] = -\frac{\mathrm{i}}{4\pi} \sum_{i=1}^N \left(\int_{\partial D} \psi_i^* \right) \sum_{l=1}^N \psi_l^* = \frac{\mathrm{i}}{4\pi} \operatorname{cap}(D) \sum_{l=1}^N \psi_l^*.
$$

Therefore, we arrive at

$$
(\mathcal{S}_D^k)^{-1}[u_{\text{in}}] = u_{\text{in}}(0) \left(1 - \frac{\mathrm{i}\omega}{4\pi v} \operatorname{cap}(D) \right) \sum_{l=1}^N \psi_l^* + \nabla u_{\text{in}}(0) \cdot (\mathcal{S}_D)^{-1}[y^1] + O(\omega^2).
$$

Computing the far field of the single layer potential $\mathcal{S}_D^k[(\mathcal{S}_D^k)^{-1}[u_{\text{in}}]]$, we find that

$$\begin{aligned}-\mathcal{S}_D^k[(\mathcal{S}_D^k)^{-1}[u_{\text{in}}]](x) &= -\Gamma(x)\int_{\partial D}(\mathcal{S}_D^k)^{-1}[u_{\text{in}}]\,\mathrm{d}\sigma + \nabla\Gamma(x)\cdot\int_{\partial D} y^1(\mathcal{S}_D^k)^{-1}[u_{\text{in}}](y)\,\mathrm{d}\sigma(y) + O(|x|^{-1}(\omega^2+|x|^{-2}))\\ &= \left(u_{\text{in}}(0)\left(1-\frac{\mathrm{i}\omega}{4\pi v}\operatorname{cap}(D)\right)\operatorname{cap}(D) - \sum_{i=1}^N\int_{\partial D}\psi_i^*(y)y^1\,\mathrm{d}\sigma(y)\cdot\nabla u_{\text{in}}(0)\right)\Gamma^k(x)\\ &\quad+\left(u_{\text{in}}(0)\sum_{i=1}^N\int_{\partial D}\psi_i^*(y)y^1\,\mathrm{d}\sigma(y) + \nabla u_{\text{in}}(0)\cdot\int_{\partial D} y^1\otimes(\mathcal{S}_D^k)^{-1}[y^1](y)\,\mathrm{d}\sigma(y)\right)\cdot\nabla\Gamma^k(x)\\ &\quad+O(|x|^{-1}(\omega^2+|x|^{-2})).\end{aligned}$$

The result follows from the identity

$$\sum_{i=1}^N\int_{\partial D}\psi_i^*(y)y^1\,\mathrm{d}\sigma(y) = -\sum_{i=1}^N\sum_{j=1}^N\int_{\partial D}C_{ij}\phi_j^*(y)y^1\,\mathrm{d}\sigma(y) = -\mathbf{1}C\boldsymbol{M}^1,$$

where we recall the definition (4.12) of the tensor $\boldsymbol{M}^1$. □

Remark 5.1. From (3.19) and (4.20), we have the following expansion for $(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)[\phi_i^*]$:

$$(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\mathcal{G}^{-1}(\omega,\delta)[\phi_i^*] = \phi_i^* + \frac{\mathrm{i}\omega}{4\pi}\left(\frac{1}{v_b}-\frac{1}{v}\right)\sum_{1\le l,j\le N} C_{lj}\phi_j^* + O(\omega^2+\delta).$$

Recalling $\boldsymbol{M}^0 = \mathbf{1}$ and Section 5.2, the tensor $\boldsymbol{M}^0(\omega,\delta)$ occurring in the result of Proposition 5.3: has the following asymptotic expansion:

$$\boldsymbol{M}^0(\omega,\delta) = \left(1+\mathrm{i}\omega\frac{1}{4\pi}\left(\frac{1}{v_b}-\frac{1}{v}\right)\operatorname{cap}(D)\right)\mathbf{1} + O(\omega^2+\delta). \tag{5.7}$$

Remark 5.2. These expansions further highlight the fact that the vector of ones $\mathbf{1} = (1)_{1\le i\le N}$ plays a particular role if it turns to be an eigenvector of the capacitance matrix, as it occurs in case of many symmetries.

Remark 5.3. The multipole expansion (5.6) is in general of monopole type ($u_{\text{tot}}(x) - u_{\text{in}}(x)$ is proportional to $\Gamma^k(x)$ at leading order). A far field pattern of dipole type (proportional to $\nabla\Gamma^k(x)$) can be observed if $\boldsymbol{a}_i^T C\mathbf{1} = 0$ as ω becomes close to the resonance $\omega_i^\pm(\delta)$. Generating systems with higher order far field patterns (proportional to $\nabla^p\Gamma^k(x)$ with $p\ge 2$) does not seem trivial and would require to compute asymptotic expansions of u_{tot} at higher orders.

5.2. **Point scatterer approximation for a single resonator**

We now specialize the result of Proposition 5.3 to the case $N = 1$, where $D \equiv B$ is constituted of a single resonator. Using our accurate multipole expansion (5.6), we are able to retrieve and to clarify the results obtained for this context in [13, 8]. We also find a simplification of the damping constant of the complex scattering coefficient, which we find directly related to the imaginary part of the resonance.

Proposition 5.4. *Assume $N = 1$ and $D \equiv B$. The following monopole source behavior holds for the solution u_{tot} to the scattering problem as $|x| \to +\infty$, $\omega \to 0$ and $\delta \to 0$.*

$$u_{\text{tot}}(x) - u_{\text{in}}(x) = \left(\frac{\delta v_b^2|B|^{-1}\operatorname{cap}(B)}{\omega^2 - |\omega_1^\pm(\delta)|^2 - 2\mathrm{i}\omega\Im(\omega_1^\pm(\delta))}+1\right)\operatorname{cap}(B)u_{\text{in}}(0)(1+O(\omega+\delta+|x|^{-1}))\Gamma^k(x) + O(\delta). \tag{5.8}$$

Proof. In this context where $N = 1$, let us recall that $\boldsymbol{a}_1 = |B|^{-1/2}$ and $C = \operatorname{cap}(B)$. Neglecting the contributions in $O(\omega)$ in (4.15), we find

$$g_1^0(\omega) = v_b^2\boldsymbol{a}_1^T C\mathbf{1} + O(\omega) = v_b^2|B|^{-\frac12}\operatorname{cap}(B) + O(\omega).$$

The result follows from (5.6). □

Remark 5.4. We see from (5.8) that the scattering coefficient contains the contribution of the constant 1 and of a resonant coefficient which blows up exactly at the resonant frequency $\omega_1^\pm(\delta)$.

We now simplify the expression of the scattering coefficient in (5.8) in the case where $\omega \in \mathbb{R}$ is a physical, real frequency. Following [13], let us denote by

$$\omega_M := \omega_{1,1}^\pm\delta^{\frac12} = v_b\sqrt{\frac{\operatorname{cap}(B)}{|B|}}\delta^{\frac12} \tag{5.9}$$

the leading order of the resonant frequency $\omega_1^\pm(\delta)$.

Proposition 5.5. *Assuming that ω is real, (5.8) reduces to the following point-wise behavior for the solution u_{tot} to the scattering problem (1.1) as $|x| \to +\infty$, $\omega \to 0$ and $\delta \to 0$:*

$$u_{\text{tot}}(x) - u_{\text{in}}(x) = \left(\frac{1}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}} \right) (1 + O(\omega) + O(\delta^{\frac{1}{2}}) + O(|x|^{-1})) \operatorname{cap}(B) u_{\text{in}}(0) \Gamma^k(x) + O(\delta). \quad (5.10)$$

Proof. We approximate the resonant denominator by using (4.32) and (3.44):

$$\frac{\delta v_b^2 |B|^{-1} \operatorname{cap}(B)}{\omega^2 - |\omega_1^{\pm}(\delta)|^2 - 2\mathrm{i}\omega \Im(\omega_1^{\pm}(\delta))} = \frac{\omega_M^2}{\omega^2 - \omega_M^2 + O(\delta^{\frac{3}{2}}) + \mathrm{i}\omega \left(\delta \frac{v_b^2 \operatorname{cap}(B)^2}{4\pi v |B|} + O(\delta^2) \right)} = \frac{1 + O(\delta^{\frac{1}{2}})}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}}.$$

Then, (4.30) highlights that the above quantity is of order $O(\delta^{-\frac{1}{2}})$, hence the constant 1 of (5.8) is of the same order as the error of the approximation:

$$\frac{\delta v_b^2 |B|^{-1} \operatorname{cap}(B)}{\omega^2 - |\omega_1^{\pm}(\delta)|^2 - 2\mathrm{i}\omega \Im(\omega_1^{\pm}(\delta))} + 1 = \frac{1}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}} + O(1) =, \frac{1}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}} (1 + O(\delta^{\frac{1}{2}})),$$

from where the result follows. □

Remark 5.5. The expression (5.10) is somewhat simpler as the result of the original paper [13] (Theorem 3.1). The difference lies in the error term. First, the constant 1 of (5.8) was kept in [13], which does not change the approximation error. Keeping the constant 1 yields the scattering coefficient

$$\begin{aligned} \frac{1}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}} (1 + O(\delta^{\frac{1}{2}})) &= \left(\frac{1}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}} + 1 \right) (1 + O(\delta^{\frac{1}{2}})) \\ &= \frac{1}{1 - \frac{\omega_M^2}{\omega^2} + \mathrm{i} \frac{v_b^2 \operatorname{cap}(B)^2}{4\pi v |B|} \frac{\delta}{\omega}} (1 + O(\delta^{\frac{1}{2}})). \end{aligned} \quad (5.11)$$

Then, referring to the formula (2.28) for the damping constant in [8], one can verify that

$$\begin{aligned} &1 - \frac{v_b^2 \operatorname{cap}(B)}{|B|} \frac{\delta}{\omega^2} + \mathrm{i} \left(\frac{(v + v_b) \operatorname{cap}(B)}{8\pi v v_b} \omega - \frac{v - v_b}{v} \frac{v_b \operatorname{cap}(B)^2}{8\pi |B|} \frac{\delta}{\omega} \right) \\ &= \left(1 - \frac{v_b^2 \operatorname{cap}(B)}{|B|} \frac{\delta}{\omega^2} + \mathrm{i} \frac{v_b^2 \operatorname{cap}(B)^2}{4\pi v |B|} \frac{\delta}{\omega} \right) \left(1 + \mathrm{i} \frac{(v + v_b) \operatorname{cap}(B)}{8\pi v v_b} \omega + O(\omega^2) \right) \\ &= \left(1 - \frac{v_b^2 \operatorname{cap}(B)}{|B|} \frac{\delta}{\omega^2} + \mathrm{i} \frac{v_b^2 \operatorname{cap}(B)^2}{4\pi v |B|} \frac{\delta}{\omega} \right) (1 + O(\omega)). \end{aligned} \quad (5.12)$$

Hence, using (5.11) and (5.12), we obtain

$$\frac{1}{\frac{\omega^2}{\omega_M^2} - 1 + \mathrm{i}\omega \frac{\operatorname{cap}(B)}{4\pi v}} = \frac{1 + O(\delta^{\frac{1}{2}})}{1 - \frac{\omega_M^2}{\omega^2} + \mathrm{i} \left(\frac{(v + v_b) \operatorname{cap}(B)}{8\pi v v_b} \omega - \frac{v - v_b}{v} \frac{v_b \operatorname{cap}(B)^2}{8\pi |B|} \frac{\delta}{\omega} \right)}, \quad (5.13)$$

which shows that the damping constant of (5.10) and the one of [8, equation (2.28)] are equivalent for an approximation of the scattered field of the order $O(\delta^{\frac{1}{2}})$. However, let us remark than our formula (5.10) is more enlightening because it clearly shows that the damping coefficient can be taken to be positive and equal (at first order) to twice the opposite of the imaginary part of the complex resonant frequency $\omega_1^{\pm}(\delta)$.

5.3. **Point scatterer approximation for a dimer**

We now consider a dimer made of two identical spheres B_1 and B_2 of volume $|B_1| = |B_2| = |D|/2$, following the setting of Section 2.3.3. Without loss of generality, we assume that the axis of the dimer is aligned with the direction $\boldsymbol{e}_1$, and that the origin $O = (0, 0, 0)$ is the middle of the segment joining the two centers of B_1 and B_2. The setting is illustrated on Figure 7. In this final part, we retrieve and improve the results of

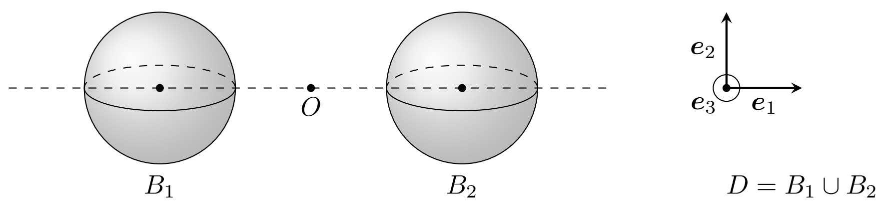

FIGURE 7. Setting of the dimer D constituted of two identical spheres.

[15] concerned with the derivation of point scatterer approximations for the field scattered by the dimer. Our

analysis emphasizes the fact that the "dipole" behavior of the resonance associated to the second scattering frequency $\omega_2^{\pm}(\delta)$ is closely related to the symmetry of the system and cannot be expected for arbitrary system of N resonators, where all resonances are in general of "monopole" type.

We recall that in the present context, the capacitance matrix C and the volume matrix V read

$$C = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}, \quad V = \frac{|D|}{2} I, \tag{5.14}$$

for two positive constants $a > b > 0$ (Section 2.3.3). The solutions to the generalized eigenvalue problem (2.4) are given by

$$\lambda_1 = \frac{2}{|D|}(a-b) = \frac{\mathrm{cap}(D)}{|D|}, \quad \lambda_2 = \frac{2}{|D|}(a+b), \tag{5.15}$$

with respective eigenvectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ normalized such that $\boldsymbol{a}_i^T V \boldsymbol{a}_j = \delta_{ij}$ for $1 \leq i, j \leq 2$:

$$\boldsymbol{a}_1 = |D|^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |D|^{-\frac{1}{2}} \mathbf{1}, \quad \boldsymbol{a}_2 = |D|^{-\frac{1}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.16}$$

Note that (5.14) to (5.16) hold as soon as the dimer is symmetric with respect to the median plane orthogonal to $\boldsymbol{e}_1$. However, we consider spherical resonators because the rotational invariance along the axis of the dimer is essential to obtain a far field of dipole type for ω close to the resonance $\omega_2^+(\delta)$.

In what follows, we denote by ω_1 and ω_2 the first order approximations of the real part of the resonant frequencies $(\omega_i^+(\delta))_{1\leq i\leq 2}$ (equation (3.42)):

$$\omega_1 := \delta^{\frac{1}{2}} v_b \sqrt{\frac{2}{|D|}(a-b)}, \quad \omega_2 := \delta^{\frac{1}{2}} v_b \sqrt{\frac{2}{|D|}(a+b)}.$$

The multipole expansion (5.6) involves $\boldsymbol{M}^0 = \mathbf{1}$ and $\boldsymbol{M}^1$. As we obtain below, the simplifications which entail the dipole behavior of the second resonance are induced by the algebraic symmetry properties of the tensor $\boldsymbol{M}^0$ and $\boldsymbol{M}^1$. The latter are the object of the next proposition, which examines more generally the symmetry properties of the polarization tensors $\boldsymbol{M}^p$ defined by (4.12) for an arbitrary $p \in \mathbb{N}$.

Lemma 5.2. *The polarization tensor $\boldsymbol{M}^p$ of (4.12) associated to the system of two identical spherical resonators has the following symmetries for any $p \in \mathbb{N}$:*

(i) there exists a scalar tensor $\mathcal{X}^p$ of order p such that for any indices $1 \leq i_1 \dots i_p \leq 3$,

$$\boldsymbol{M}^p_{i_1\dots i_p} = \mathcal{X}^p_{i_1\dots i_p} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ if } p \text{ is even,}$$

$$\boldsymbol{M}^p_{i_1\dots i_p} = \mathcal{X}^p_{i_1\dots i_p} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ if } p \text{ is odd.}$$

(ii) $\mathcal{X}^p_{i_1\dots i_p} = 0$ if 2 or 3 occurs an odd number of times in the indices $(i_1, \dots, i_p)$. More precisely,

$$\mathcal{X}^p_{i_1\dots i_p} = (-1)^{\delta_{i_1 j} + \dots + \delta_{i_p j}} \mathcal{X}^p_{i_1\dots i_p} \text{ for } j = 2, 3.$$

(iii) The numbers 2 and 3 can be permuted in the indices $(i_1, \dots, i_p)$. More precisely, if $\sigma \in \mathfrak{S}_3$ is the permutation defined by $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$, then

$$\mathcal{X}^p_{\sigma(i_1)\dots\sigma(i_p)} = \mathcal{X}^p_{i_1\dots i_p}.$$

Proof. We consider the planar symmetries S_1, S_2, S_3 defined by

$$S_i \boldsymbol{e}_i = -\boldsymbol{e}_i \text{ and } S_i \boldsymbol{e}_j = \boldsymbol{e}_j \text{ for } j \neq i \text{ with } 1 \leq i, j \leq 3,$$

and the symmetry S_{23} satisfying

$$S_{23}\boldsymbol{e}_1 = \boldsymbol{e}_1,\ S_{23}\boldsymbol{e}_2 = \boldsymbol{e}_3 \text{ and } S_{23}\boldsymbol{e}_3 = \boldsymbol{e}_2.$$

Due to the symmetries of D, the potential $\psi_l^* = \mathcal{S}_D^{-1}[1_{\partial B_l}]$ with $l = 1, 2$ satisfies

$$\psi_1^* \circ S_1 = \psi_2^*, \quad \psi_2^* \circ S_1 = \psi_1^*,$$

$$\psi_l^* \circ S_i = \psi_l^* \text{ for } l = 1, 2 \text{ and } i = 2, 3.$$

Let us consider the p-th order vector tensors $\widetilde{\boldsymbol{M}}^p \equiv (\widetilde{M}_i^p)_{1\leq i\leq 3}$ defined by

$$\widetilde{M}_i^p := \int_{\partial D} \psi_i^* y^p \, \mathrm{d}\sigma, \qquad 1 \leq i \leq 3. \tag{5.17}$$

By considering the change of variables $y = S_i(y')$ with $1 \le i \le 3$ in (5.17), we obtain respectively

$$\widetilde{M}^p_{1,i_1\dots i_p} = (-1)^{\delta_{i_1 1}+\dots+\delta_{i_p 1}} M^p_{2,i_1\dots i_p}, \tag{5.18}$$

$$\widetilde{M}^p_{l,i_1\dots i_p} = (-1)^{\delta_{i_1 j}+\dots+\delta_{i_p j}} \widetilde{M}^p_{l,i_1\dots i_p} \text{ for } l = 1,2 \text{ and } j = 2,3. \tag{5.19}$$

Furthermore, the change of variables $y = S_{23}(y')$ in (5.17) yields

$$\widetilde{M}^p_{l,i_1\dots i_p} = \widetilde{M}^p_{l,\sigma(i_1)\dots\sigma(i_p)}. \tag{5.20}$$

Property (5.18) implies that $\widetilde{\boldsymbol{M}}^p_{i_1\dots i_p}$ is proportional to $\begin{pmatrix}1 & 1\end{pmatrix}^T$ if there is an even number of 1 in $(i_1,\dots,i_p)$, and to $\begin{pmatrix}1 & -1\end{pmatrix}^T$ otherwise. The identity (5.19) states that $\widetilde{\boldsymbol{M}}^p_{i_1\dots i_p} = 0$ if there is an odd number of 2 or 3 in $(i_1,\dots,i_p)$. Finally, (5.20) implies that we can permute the numbers 2 and 3 in $(i_1,\dots,i_p)$, so properties (i), (ii) and (iii) hold for $\widetilde{\boldsymbol{M}}^p$ instead of $\boldsymbol{M}^p$. Since $\boldsymbol{M}^p = -C^{-1}\widetilde{\boldsymbol{M}}^p$ and $\begin{pmatrix}1 & 1\end{pmatrix}^T$ and $\begin{pmatrix}1 & -1\end{pmatrix}^T$ are eigenvectors of C, these results also hold for $\boldsymbol{M}^p$. □

Corollary 5.1. *The moment tensor $\boldsymbol{M}^1$ defined by (4.12) can be written*

$$\boldsymbol{M}^1 = M^1_{1,1} e_1 \begin{pmatrix}1\\-1\end{pmatrix},$$

for an effective constant $M^1_{1,1} \in \mathbb{C}$.

Remark 5.6. Let us further illustrate the result of Lemma 5.2 on the higher order tensors $\boldsymbol{M}^2$ and $\boldsymbol{M}^3$: using the properties (i), (ii) and (iii), we find that $\boldsymbol{M}^2$ and $\boldsymbol{M}^3$ depend respectively on only three effective coefficients:

$$\boldsymbol{M}^2 = (M^2_{1,11} e_1 \otimes e_1 + M^2_{1,22} e_2 \otimes e_2 + M^2_{1,33} e_3 \otimes e_3) \begin{pmatrix}1\\1\end{pmatrix},$$

$$\boldsymbol{M}^3 = (M^3_{1,111} e_1 \otimes e_1 \otimes e_1 + M^3_{1,122} e_1 \otimes e_2 \otimes e_2 + M^3_{1,122} e_1 \otimes e_3 \otimes e_3) \begin{pmatrix}1\\-1\end{pmatrix}.$$

We also need to determine the symmetry properties of the second order tensor K^2 of (5.5). We find that K^2 is a diagonal tensor.

Lemma 5.3. *The second order scalar tensor K^2 defined by (5.5) can be written*

$$K^2 = K^2_{11} e_1 \otimes e_1 + K^2_{22}(e_2 \otimes e_2 + e_3 \otimes e_3), \tag{5.21}$$

for two effective coefficients $K^2_{11}, K^2_{22} \equiv K^2_{33} \in \mathbb{C}$.

Proof. Due to the symmetry of the dimer, the potential $\zeta_i := (\mathcal{S}^k_D)^{-1}[y_i]$ with $1 \le i \le 3$ satisfies

$$\zeta_i \circ S_j = (\mathcal{S}^k_D)^{-1}[y_i \circ S_j] \text{ for } 1 \le j \le 3.$$

Consequently, we find by using the change of variables $y' = S_i y$ with $i = 1,2,3$:

$$K^2_{i_1 i_2} = (-1)^{\delta_{i_1 i}}(-1)^{\delta_{i_2 i}} K^2_{i_1 i_2},$$

which implies $K^2_{i_1 i_2} = 0$ as soon as $i_1 \neq i_2$. Then the change of variables $y' = S_{23}$ yields

$$K^2_{i_2 i_2} = K^2_{\sigma(i_2)\sigma(i_3)},$$

from where we obtain $K^2_{22} = K^2_{33}$. □

Proposition 5.6. *The following multipole expansion holds for the field $u_{\mathrm{tot}}(x)$ scattered through a dimer D constituted of two identical spherical resonators, as $|x| \to +\infty$ and $\omega, \delta \to 0$:*

$$\begin{aligned} u_{\mathrm{tot}}(x) - u_{\mathrm{in}}(x) & \\ = & \left[\frac{\omega_1^2(1-\mathrm{i}\omega\tau_v)}{\omega^2 - |\omega_1^{\pm}(\delta)|^2 - 2\mathrm{i}\omega\Im(\omega_1^{\pm}(\delta))} + 1\right] \mathrm{cap}(D)(1-\mathrm{i}\omega\tau_v) u_{\mathrm{in}}(0)(1 + O(\omega^2) + O(|x|^{-2}))\Gamma^k(x) \\ & + \frac{2(a+b)(M^1_{1,1})^2\omega_2^2}{\omega^2 - |\omega_2^{\pm}(\delta)|^2 - 2\mathrm{i}\omega\Im(\omega_2^{\pm}(\delta))} \partial_{y_1} u_{\mathrm{in}}(0) \partial_{y_1}\Gamma^k(x)(1 + O(\omega) + O(|x|^{-1})) \\ & + K^2_{11}\partial_{y_1} u_{\mathrm{in}}(0)\partial_{y_1}\Gamma^k(x) + \sum_{i=1}^{3} K^2_{ii}\partial_{y_i} u_{\mathrm{in}}(0)\partial_{y_i}\Gamma^k(x) + O(|x|^{-1}(\omega^2 + |x|^{-2})) + O(\delta), \end{aligned} \tag{5.22}$$

where $\tau_v := \mathrm{cap}(D)/(2\pi v)$ and $(K^2_{ii})_{1\le i\le 3}$ are the coefficients of the diagonal tensor K^2 of (5.21).

Proof. According to (4.2) and using the orthogonality $\boldsymbol{a}_2^T V \mathbf{1} = 0$, we find

$$\boldsymbol{b}_1 = \frac{\operatorname{cap}(D)}{4\pi v_b}\boldsymbol{a}_1, \quad \boldsymbol{b}_2 = 0.$$

This allows to compute the coefficients $g_i^0(\omega)$ and $g_i^1(\omega)$ of (4.15):

$$\begin{cases} g_1^0(\omega) = v_b^2\left(1 - \dfrac{\mathrm{i}\omega}{4\pi v}\operatorname{cap}(D)\right)\left(1 - \dfrac{\mathrm{i}\omega}{4\pi v_b}\operatorname{cap}(D)\right)\boldsymbol{a}_1 C\mathbf{1} \\ \qquad\quad = v_b^2\left(1 - \dfrac{\mathrm{i}\omega}{4\pi}\operatorname{cap}(D)\left(\dfrac{1}{v} + \dfrac{1}{v_b}\right)\right)|D|^{-\frac{1}{2}}\operatorname{cap}(D) + O(\omega^2), \\ g_1^1(\omega) = v_b^2 \boldsymbol{a}_1^T C\boldsymbol{M}^1 = 0, \\ g_2^0(\omega) = 0, \\ g_2^1(\omega) = v_b^2\boldsymbol{a}_2^T C\boldsymbol{M}^1 = v_b^2|D|^{\frac{1}{2}}\lambda_2 M_{1,1}^1 e_1 = 2v_b^2|D|^{-\frac{1}{2}}(a+b)M_{1,1}^1 e_1. \end{cases} \tag{5.23}$$

Furthermore, Proposition 3.10 imply that the first order perturbation of the eigenvectors are zero:

$$\boldsymbol{a}_i(\omega) = \boldsymbol{a}_i + O(\omega^2). \tag{5.24}$$

Therefore, with the help of (5.7), we find

$$\boldsymbol{a}_1(\omega)^T C\boldsymbol{M}^0(\omega,\delta)(\boldsymbol{M}^0(\omega,\delta)\Gamma^k(x) - \boldsymbol{M}^1\cdot\nabla\Gamma^k(x)) = \left(1 + \frac{\mathrm{i}\omega}{4\pi}\left(\frac{1}{v_b} - \frac{1}{v}\right)\operatorname{cap}(D)\right)|D|^{-\frac{1}{2}}\operatorname{cap}(D)\Gamma^k(x),$$

$$\boldsymbol{a}_2(\omega)^T C\boldsymbol{M}^0(\omega,\delta)(\boldsymbol{M}^0(\omega,\delta)\Gamma^k(x) - \boldsymbol{M}^1\cdot\nabla\Gamma^k(x)) = |D|^{\frac{1}{2}}\lambda_2 M_{1,1}^1\partial_{y_1}\Gamma^k(x) = 2|D|^{-\frac{1}{2}}(a+b)M_{1,1}^1\partial_{y_1}\Gamma^k(x).$$

This together with (5.23) yields the expressions of the resonant scattering amplitudes in (5.22). The result follows from (5.6) by using $\mathbf{1}^T C\boldsymbol{M}^1 = 0$. ☐

Proposition 5.6 yields a higher order asymptotic for the far field expansion of u_{tot} than [15, Theorem 4.2]. To conclude this section, we specialize (5.22) to the case where ω is real, using the approximation result of Lemma 4.3. We recall the definition (4.34) of the constants $\eta_1, \eta_2 > 0$ involved in the higher order expansion of the scattering frequency $\omega_2^{\pm}(\delta)$.

Proposition 5.7. *Assuming that ω is real, (5.22) reduces to the following multipole expansion for the solution u_{tot} to the scattering problem (1.1) as $|x| \to +\infty$, $\omega \to 0$ and $\delta \to 0$:*

$$\begin{aligned} u_{\text{tot}}(x) - u_{\text{in}}(x) &\\ = &\left[\frac{1}{\frac{\omega^2}{\omega_1^2} - 1 + \frac{\mathrm{i}\omega}{4\pi v}\operatorname{cap}(D)}\right]\operatorname{cap}(D)u_{\text{in}}(0)(1 + O(\omega) + O(\delta^{\frac{1}{2}}) + O(|x|^{-2}))\Gamma^k(x) \\ &+ \frac{2(a+b)(M_{1,1}^1)^2}{\frac{\omega^2}{\omega_2^2} - 1 - 2\frac{\eta_1}{\omega_2}\delta^{\frac{3}{2}} + 2i\eta_2\delta^2\frac{\omega}{\omega_2^2}}\partial_{y_1}u_{\text{in}}(0)\partial_{y_1}\Gamma^k(x)(1 + O(\omega) + O(|x|^{-1})). \end{aligned}$$

Remark 5.7. For real frequencies, the scattering coefficient for the dipole resonance is of order

$$\frac{\omega_2^2}{\omega^2 - |\omega_2^{\pm}(\delta)|^2 - 2i\Im(\omega_2^{\pm}(\delta))\omega} = O(\delta^{-\frac{3}{2}}),$$

which is greater than the amplitude coefficient of order $O(\delta^{-\frac{1}{2}})$ for the monopole mode. However, this magnitude is tempered by the derivative $\partial_{y_1}u_{\text{in}}(0)$ which is of order $O(\omega)$, and by the faster decay of $\partial_{y_1}\Gamma^k(x)$ as $O(|x|^{-1}(\omega + |x|^{-1}))$. Therefore, for $\omega > 0$ close to $\omega_2^+(\delta)$ (and so $\omega = O(\delta^{\frac{1}{2}})$), we find that the scattered field $u_{\text{tot}}(x)$ is of order

$$O(\delta^{-\frac{3}{2}}\delta^{\frac{1}{2}}|x|^{-1}(\delta^{\frac{1}{2}} + |x|^{-1})) = O(\delta^{-\frac{1}{2}}|x|^{-1} + \delta^{-1}|x|^{-2}).$$

Hence, the amplitude of the monopole and dipole modes are similar (of order $O(1)$) at the distance $|x| = O(\delta^{-\frac{1}{2}})$. However, the amplitude of the dipole mode is larger than the one of the monopole resonance for $|x| \to +\infty$ with $|x| \ll \delta^{-\frac{1}{2}}$.

Remark 5.8. The result of Proposition 5.7 coincides with the one of [15, Theorem 4.2] up to some rewriting of the denominators and by observing that the constant P of equation (4.3) of this reference is given by

$$P = \int_{\partial D} y_1(\psi_1^* - \psi_2^*)\,\mathrm{d}y = -\begin{pmatrix}1\\-1\end{pmatrix}^T C\boldsymbol{M}_1^1 = 2(a+b)M_{1,1}^1.$$

Remark 5.9. The non-resonant term $\nabla u_{\text{in}}(0)\cdot K^2\cdot\nabla\Gamma^k(x)$ appearing in (5.6) brings a contribution of order $O(1)$, which is smaller compared to the error of order $O(\omega\delta^{-\frac{3}{2}}) = O(\delta^{-1})$ committed on the resonant amplitude when considering real frequencies $\omega \in \mathbb{R}$ and truncating at order $O(\omega) + O(|x|^{-1})$.

References

[1] Ammari, H., Challa, D. P., Choudhury, A. P., and Sini, M. The point-interaction approximation for the fields generated by contrasted bubbles at arbitrary fixed frequencies. *Journal of Differential Equations 267*, 4 (aug 2019), 2104–2191.

[2] Ammari, H., Challa, D. P., Choudhury, A. P., and Sini, M. The Equivalent Media Generated by Bubbles of High Contrasts: Volumetric Metamaterials and Metasurfaces. *Multiscale Modeling & Simulation 18*, 1 (jan 2020), 240–293.

[3] Ammari, H., and Davies, B. A fully coupled subwavelength resonance approach to filtering auditory signals. *Proceedings of the Royal Society A 475*, 2228 (2019), 20190049.

[4] Ammari, H., and Davies, B. A biomimetic basis for auditory processing and the perception of natural sounds. *arXiv preprint arXiv:2005.12794* (2020).

[5] Ammari, H., and Davies, B. Mimicking the active cochlea with a fluid-coupled array of subwavelength Hopf resonators. *Proceedings of the Royal Society A 476*, 2234 (2020), 20190870.

[6] Ammari, H., Davies, B., and Hiltunen, E. O. Functional analytic methods for discrete approximations of subwavelength resonator systems. *arXiv preprint arXiv:2106.12301* (2021).

[7] Ammari, H., Davies, B., Hiltunen, E. O., Lee, H., and Yu, S. High-order exceptional points and enhanced sensing in subwavelength resonator arrays. *Studies in Applied Mathematics 146*, 2 (nov 2020), 440–462.

[8] Ammari, H., Davies, B., Hiltunen, E. O., Lee, H., and Yu, S. Wave interaction with subwavelength resonators. *arXiv preprint arXiv:2011.03575* (2020).

[9] Ammari, H., Davies, B., Hiltunen, E. O., and Yu, S. Topologically protected edge modes in one-dimensional chains of subwavelength resonators. *Journal de Mathématiques Pures et Appliquées 144* (2020), 17–49.

[10] Ammari, H., Davies, B., and Yu, S. Close-to-touching acoustic subwavelength resonators: eigenfrequency separation and gradient blow-up. *Multiscale Modeling & Simulation 18*, 3 (2020), 1299–1317.

[11] Ammari, H., Fitzpatrick, B., Gontier, D., Lee, H., and Zhang, H. A mathematical and numerical framework for bubble meta-screens. *SIAM Journal on Applied Mathematics 77*, 5 (2017), 1827–1850.

[12] Ammari, H., Fitzpatrick, B., Gontier, D., Lee, H., and Zhang, H. Sub-wavelength focusing of acoustic waves in bubbly media. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 473*, 2208 (2017), 20170469.

[13] Ammari, H., Fitzpatrick, B., Gontier, D., Lee, H., and Zhang, H. Minnaert resonances for acoustic waves in bubbly media. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire 35*, 7 (2018), 1975–1998.

[14] Ammari, H., Fitzpatrick, B., Kang, H., Ruiz, M., Yu, S., and Zhang, H. *Mathematical and Computational Methods in Photonics and Phononics*. American Mathematical Society, oct 2018.

[15] Ammari, H., Fitzpatrick, B., Lee, H., Yu, S., and Zhang, H. Double-negative acoustic metamaterials. *Quart. Appl. Math. 77*, 4 (2019), 767–791.

[16] Ammari, H., and Kang, H. *Polarization and moment tensors: with applications to inverse problems and effective medium theory*, vol. 162. Springer Science & Business Media, 2007.

[17] Ammari, H., Kang, H., and Lee, H. *Layer potential techniques in spectral analysis*. No. 153. American Mathematical Soc., 2009.

[18] Ammari, H., Li, B., and Zou, J. Mathematical Analysis of Electromagnetic Plasmonic Metasurfaces. *Multiscale Modeling & Simulation 18*, 2 (jan 2020), 758–797.

[19] Ammari, H., Li, B., and Zou, J. Superresolution in Recovering Embedded Electromagnetic Sources in High Contrast Media. *SIAM Journal on Imaging Sciences 13*, 3 (jan 2020), 1467–1510.

[20] Ammari, H., Millien, P., Ruiz, M., and Zhang, H. Mathematical analysis of plasmonic nanoparticles: the scalar case. *Arch. Ration. Mech. Anal 224*, 2 (2016), 597–658.

[21] Ammari, H., Millien, P., and Vanel, A. L. Modal expansion for plasmonic resonators in the time domain. *arXiv preprint arXiv:2003.09200* (2020).

[22] Ammari, H., Ruiz, M., Yu, S., and Zhang, H. Mathematical analysis of plasmonic resonances for nanoparticles: The full Maxwell equations. *Journal of Differential Equations 261*, 6 (sep 2016), 3615–3669.

[23] Ammari, H., and Triki, F. Splitting of resonant and scattering frequencies under shape deformation. *Journal of Differential Equations 202*, 2 (aug 2004), 231–255.

[24] Ammari, H., and Zhang, H. A mathematical theory of super-resolution by using a system of sub-wavelength Helmholtz resonators. *Communications in Mathematical Physics 337*, 1 (2015), 379–428.

[25] Ammari, H., and Zhang, H. Super-resolution in high-contrast media. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 471*, 2178 (2015), 20140946.

[26] Ammari, H., and Zhang, H. Effective medium theory for acoustic waves in bubbly fluids near Minnaert resonant frequency. *SIAM Journal on Mathematical Analysis 49*, 4 (2017), 3252–3276.

[27] Ando, K., and Kang, H. Analysis of plasmon resonance on smooth domains using spectral properties of the Neumann–Poincaré operator. *Journal of Mathematical Analysis and Applications 435*, 1 (mar 2016), 162–178.

[28] Ando, K., Kang, H., and Liu, H. Plasmon Resonance with Finite Frequencies: a Validation of the Quasi-static Approximation for Diametrically Small Inclusions. *SIAM Journal on Applied Mathematics 76*, 2 (jan 2016), 731–749.

[29] Ando, K., Kang, H., Miyanishi, Y., and Nakazawa, T. Surface Localization of Plasmons in Three Dimensions and Convexity. *SIAM Journal on Applied Mathematics 81*, 3 (jan 2021), 1020–1033.

[30] Baumgärtel, H. *Analytic perturbation theory for matrices and operators*, vol. 15. Birkhauser Verlag AG, 1985.

[31] Brunet, T., Poncelet, O., Aristégui, C., Leng, J., and Mondain-Monval, O. Soft 3D acoustic metamaterials. *The Journal of the Acoustical Society of America 138*, 3 (sep 2015), 1733–1733.

[32] Casse, B., Lu, W., Huang, Y., Gultepe, E., Menon, L., and Sridhar, S. Super-resolution imaging using a three-dimensional metamaterials nanolens. *Applied Physics Letters 96*, 2 (2010), 023114.

[33] Cherednichenko, K. D., Ershova, Y. Y., and Kiselev, A. V. Effective Behaviour of Critical-Contrast PDEs: Micro-resonances, Frequency Conversion, and Time Dispersive Properties. I. *Communications in Mathematical Physics 375*, 3 (feb 2020), 1833–1884.

[34] Coifman, R., Rokhlin, V., and Wandzura, S. The fast multipole method for the wave equation: a pedestrian prescription. *IEEE Antennas and Propagation Magazine 35*, 3 (1993), 7–12.

[35] Colton, D., and Kress, R. *Inverse acoustic and electromagnetic scattering theory*, vol. 93 of *Applied Mathematical Sciences*. Springer, Cham, 2019.

[36] Delfour, M. C., and Zolésio, J.-P. *Shapes and Geometries*. SIAM, 2011.

[37] Deng, Y., Liu, H., and Zheng, G.-H. Mathematical analysis of plasmon resonances for curved nanorods. *Journal de Mathématiques Pures et Appliquées* (jul 2021).

[38] Devaud, M., Hocquet, T., Bacri, J.-C., and Leroy, V. The Minnaert bubble: an acoustic approach. *European Journal of Physics 29*, 6 (sep 2008), 1263–1285.

[39] Feppon, F. High order homogenization of the Stokes system in a periodic porous medium. In preparation, 2020.

[40] Feppon, F., and Ammari, H. Scattering on wildly perturbed domains and homogenization of a system of subwavelength resonators. *Submitted* (2021).

[41] Gohberg, I. Operator extension of the logarithmic residue theorem and Rouché's theorem. *Math. USSR Sbornik 84* (1971), 607–642.

[42] Harazi, M., Rupin, M., Stephan, O., Bossy, E., and Marmottant, P. Acoustics of Cubic Bubbles: Six Coupled Oscillators. *Physical Review Letters 123*, 25 (dec 2019), 254501.

[43] Henrot, A., and Pierre, M. *Shape variation and optimization*, vol. 28 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2018. A geometrical analysis, English version of the French publication [MR2512810] with additions and updates.

[44] Herrera, W. J., and Diaz, R. A. The geometrical nature and some properties of the capacitance coefficients based on Laplace's equation. *American Journal of Physics 76*, 1 (2008), 55–59.

[45] Horn, R. A., and Johnson, C. R. *Matrix analysis*. Cambridge university press, 2012.

[46] Hryniv, R., and Lancaster, P. On the perturbation of analytic matrix functions. *Integral Equations and Operator Theory 34*, 3 (1999), 325–338.

[47] Jain, P. K., Lee, K. S., El-Sayed, I. H., and El-Sayed, M. A. Calculated Absorption and Scattering Properties of Gold Nanoparticles of Different Size, Shape, and Composition: Applications in Biological Imaging and Biomedicine. *The Journal of Physical Chemistry B 110*, 14 (apr 2006), 7238–7248.

[48] Jaiswal, R., Banerjee, A., and Narayan, A. Characterizing and Tuning Exceptional Points Using Newton Polygons. *arXiv preprint arXiv:2107.11649* (2021).

[49] Kaina, N., Lemoult, F., Fink, M., and Lerosey, G. Negative refractive index and acoustic superlens from multiple scattering in single negative metamaterials. *Nature 525*, 7567 (sep 2015), 77–81.

[50] Kalḿenov, T., Suragan, D., and Sabitbek, B. On eigenvalue inequalities of the Newton potential. In *AIP Conference Proceedings* (2015), vol. 1676, AIP Publishing LLC, p. 020052.

[51] Kato, T. *Perturbation theory for linear operators*, vol. 132. Springer Science & Business Media, 2013.

[52] Kaup, L., and Kaup, B. *Holomorphic functions of several variables: an introduction to the fundamental theory*, vol. 3. Walter de Gruyter, 2011.

[53] Koepf, W. Taylor polynomials of implicit functions, of inverse functions, and of solutions of ordinary differential equations. *Complex Variables and Elliptic Equations 25*, 1 (1994), 23–33.

[54] Kress, R. *Linear Integral Equations*. Springer New York, 2014.

[55] Kuznetsov, A. I., Miroshnichenko, A. E., Brongersma, M. L., Kivshar, Y. S., and Luk'yanchuk, B. Optically resonant dielectric nanostructures. *Science 354*, 6314 (nov 2016), aag2472.

[56] Lancaster, P. *The theory of matrices: with applications*. Academic Press, Orlando, 1985.

[57] Lancaster, P., Markus, A., and Zhou, F. Perturbation theory for analytic matrix functions: the semisimple case. *SIAM Journal on Matrix Analysis and Applications 25*, 3 (2003), 606–626.

[58] Lanoy, M., Pierrat, R., Lemoult, F., Fink, M., Leroy, V., and Tourin, A. Subwavelength focusing in bubbly media using broadband time reversal. *Physical Review B 91*, 22 (2015), 224202.

[59] Lekner, J. Capacitance coefficients of two spheres. *Journal of Electrostatics 69*, 1 (feb 2011), 11–14.

[60] Lemoult, F., Kaina, N., Fink, M., and Lerosey, G. Wave propagation control at the deep subwavelength scale in metamaterials. *Nature Physics 9*, 1 (nov 2012), 55–60.

[61] Li, J. A literature survey of mathematical study of metamaterials. *Int. J. Numer. Anal. Model 13*, 2 (2016), 230–243.

[62] Lin, J., and Zhang, H. Mathematical Analysis of Surface Plasmon Resonance by a Nano-Gap in the Plasmonic Metal. *SIAM Journal on Mathematical Analysis 51*, 6 (jan 2019), 4448–4489.

[63] Liu, Z., Zhang, X., Mao, Y., Zhu, Y., Yang, Z., Chan, C. T., and Sheng, P. Locally resonant sonic materials. *science 289*, 5485 (2000), 1734–1736.

[64] Ma, G., and Sheng, P. Acoustic metamaterials: From local resonances to broad horizons. *Science Advances 2*, 2 (feb 2016), e1501595.

[65] McLean, W. C. H. *Strongly elliptic systems and boundary integral equations*. Cambridge university press, 2000.

[66] Mennicken, R., and Möller, M. Root functions, eigenvectors, associated vectors and the inverse of a holomorphic operator function. *Archiv der Mathematik 42*, 5 (1984), 455–463.

[67] Mennicken, R., and Möller, M. *Non-self-adjoint boundary eigenvalue problems*, vol. 192. Gulf Professional Publishing, 2003.

[68] Michiels, W., Boussaada, I., and Niculescu, S.-I. An explicit formula for the splitting of multiple eigenvalues for nonlinear eigenvalue problems and connections with the linearization for the delay eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications 38*, 2 (2017), 599–620.

[69] Minnaert, M. XVI. On musical air-bubbles and the sounds of running water. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 16*, 104 (1933), 235–248.

[70] Mortini, R. The Faa di Bruno formula revisited. *arXiv preprint arXiv:1303.2023* (2013).

[71] Nédélec, J.-C. *Acoustic and electromagnetic equations: integral representations for harmonic problems*, vol. 144. Springer Science & Business Media, 2013.

[72] Qian, J., Chu, D., and Tan, R. C. A Systematic Analysis on Analyticity of Semisimple Eigenvalues of Matrix-Valued Functions. *SIAM Journal on Matrix Analysis and Applications 37*, 4 (2016), 1581–1599.

[73] Sarid, D., and Challener, W. A. *Modern introduction to surface plasmons: theory, Mathematica modeling, and applications*. Cambridge University Press, 2010.

[74] Schweizer, B. The low-frequency spectrum of small Helmholtz resonators. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 471*, 2174 (2015), 20140339.

[75] Sokolowski, J., and Zolesio, J.-P. Introduction to shape optimization. In *Introduction to Shape Optimization*. Springer Berlin Heidelberg, 1992, pp. 5–12.

[76] Stuart, J. L., and Weaver, J. R. Matrices that commute with a permutation matrix. *Linear Algebra and Its Applications 150* (1991), 255–265.
[77] Trefethen, L. N., and Bau III, D. *Numerical linear algebra*, vol. 50. Siam, 1997.
[78] Van Loan, C. F., and Golub, G. H. *Matrix computations*. Johns Hopkins University Press Baltimore, 1983.
[79] Wiersig, J. Sensors operating at exceptional points: general theory. *Physical Review A 93*, 3 (2016), 033809.
[80] Zworski, M. Mathematical study of scattering resonances. *arXiv preprint arXiv:1609.03550* (2016).