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HIGH ORDER HOMOGENIZED STOKES MODELS CAPTURE ALL THREE REGIMES*

FLORIAN FEPPON† AND WENJIA JING‡

Abstract. This article is a sequel to our previous work [13] concerned with the derivation of high-order homogenized models for the Stokes equation in a periodic porous medium. We provide an improved asymptotic analysis of the coefficients of the higher order models in the low-volume fraction regime whereby the periodic obstacles are rescaled by a factor η which converges to zero. By introducing a new family of order k corrector tensors with a controlled growth as $\eta \to 0$ uniform in $k \in \mathbb{N}$, we are able to show that both the infinite order and the finite order models converge in a coefficient-wise sense to the three classical asymptotic regimes. Namely, we retrieve the Darcy model, the Brinkman equation or the Stokes equation in the homogeneous cubic domain depending on whether η is respectively larger, proportional to, or smaller than the critical size $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$. For completeness, the paper first establishes the analogous results for the perforated Poisson equation, considered as a simplified scalar version of the Stokes system.

Key words. Homogenization, higher order models, perforated Poisson problem, Stokes system, low volume fraction asymptotics, strange term.

AMS subject classifications. 35B27, 76M50, 35330

1. Introduction.

The homogenization of the Stokes system has attracted a lot of attention recently, regarding random or complex domains [17, 10], extensions to inhomogeneous viscosity or different kinds of boundary conditions [7, 16, 15], and new unified and quantitative homogenization approaches [21, 19] in the periodic setting.

The goal of this paper is to show that higher order effective models provide a unified understanding for the homogenization for the Stokes system in a periodic porous medium:

$$\begin{cases} -\Delta \boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \boldsymbol{f} \text{ in } D_\varepsilon \\ \operatorname{div}(\boldsymbol{u}_\varepsilon) = 0 \text{ in } D_\varepsilon \\ \boldsymbol{u}_\varepsilon = 0 \text{ on } \partial\omega_\varepsilon \\ (\boldsymbol{u}_\varepsilon, p_\varepsilon) \text{ is } D\text{–periodic} \end{cases} \tag{1.1}$$

where $D_\varepsilon = D\backslash\overline{\omega_\varepsilon}$ is a d-dimensional cubic domain $D = (0, L)^d$ perforated with periodic obstacles $\omega_\varepsilon := \varepsilon(\mathbb{Z}^d + \eta T) \cap D$ (represented on Figure 1) and the right-hand side $\boldsymbol{f} \in \mathcal{C}^\infty_{\text{per}}(D, \mathbb{R}^d)$ is a smooth D–periodic vector field. D_ε is the union of periodic cells of size $\varepsilon := L/N$ where $N \in \mathbb{N}$ is a large integer. Each cell contains an obstacle $\varepsilon\eta T$ where $\eta > 0$ is a rescaling of the obstacles. This parameter η allows to consider the so-called low volume fraction regime corresponding to the situation where the obstacles disappear at a rate $\eta \to 0$ which possibly depends on ε. We assume the total fluid domain D_ε to be connected, as well as the fluid component $Y = P\backslash(\eta T)$

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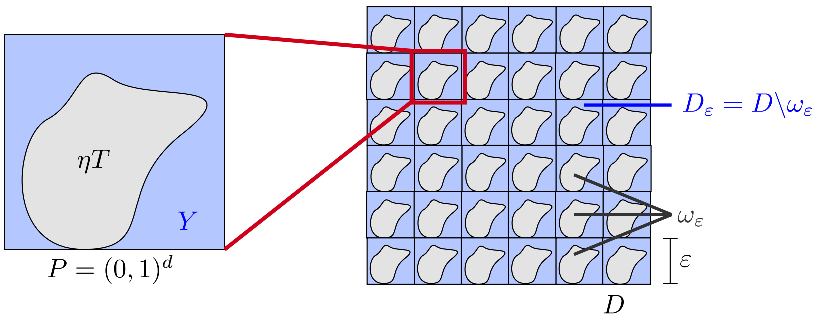

FIG. 1. *The perforated domain $D_\varepsilon = D\backslash\overline{\omega_\varepsilon}$ and the unit cell $Y = P\backslash(\eta\overline{T})$.*

of the rescaled unit cell $P := (-1/2, 1/2)^d$. The first assumption ensures that the pressure variable p_ε of (1.1) is uniquely determined up to a single additive constant while the second is used when considering cell problems in Y. For simplicity, the domain is assumed to be at least three-dimensional: $d \geq 3$.

In [13], we have derived a formal "infinite-order" homogenized system for (1.1) which reads in terms of averaged velocity and pressure $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ as

$$\begin{cases} \displaystyle\sum_{k=0}^{+\infty} \varepsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\varepsilon^* + \nabla p_\varepsilon^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}_\varepsilon^*) = 0 \text{ in } D \\ (\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*) \text{ is } D\text{–periodic.} \end{cases} \tag{1.2}$$

In (1.2), $(M^k)_{k\in\mathbb{N}}$ is a family of matrix valued tensors which can be explicitly constructed by a procedure involving cell problems that we review below, and k denotes the order of the tensor M^k. For a given $k \in \mathbb{N}$, $M^k \cdot \nabla^k$ is the differential operator defined for any $\boldsymbol{v} \in \mathcal{C}^\infty(R, \mathbb{R}^d)$ by

$$(M^k \cdot \nabla^k \boldsymbol{v})_l := M^k_{i_1\ldots i_k, lm} \partial^k_{i_1\ldots i_k} v_m$$

where we assume the Einstein summation convention over the repeated indices $1 \leq i_1 \ldots i_k \leq d$ and $1 \leq l, m \leq d$.

In order to obtain effective models suitable for numerical computations, we have proposed a truncation procedure for (1.2) inspired from [27]. For any integer $K \in \mathbb{N}$, it yields a well-posed higher order homogenized model of *finite* order $2K + 2$, which reads

$$\begin{cases} \displaystyle\sum_{k=0}^{2K+2} \varepsilon^{k-2} \mathbb{D}^k_K \cdot \nabla^k \boldsymbol{v}_{\varepsilon,K}^* + \nabla q_{\varepsilon,K}^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{v}_{\varepsilon,K}^*) = 0 \text{ in } D \\ (\boldsymbol{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*) \text{ is } D\text{–periodic,} \end{cases} \tag{1.3}$$

where the coefficients $(\mathbb{D}^k_K)_{0\leq k\leq 2K+2}$ is another family of matrix valued tensors. The system (1.3) is indeed a truncated version of (1.2) because the first half of the coefficients coincide, namely $\mathbb{D}^k_K = M^k$ for $0 \leq k \leq K$. The remaining higher order coefficients $(\mathbb{D}^k_K)_{K+1\leq k\leq 2K+2}$ are in general different from $(M^k)_{K+1\leq k\leq 2K+2}$; they ensure that (1.3) is well-posed. It is then possible to show that, for a fixed $\eta > 0$, $\boldsymbol{v}^*_{\varepsilon,K}$ and $q^*_{\varepsilon,K}$ yield approximations of $\boldsymbol{u}_\varepsilon$ and p_ε at orders $O(\varepsilon^{K+3})$ and $O(\varepsilon^{K+1})$ in the $L^2(D_\varepsilon)$ norm respectively. Similar results hold for the Laplace problem with a smooth periodic right-hand side $f \in \mathcal{C}^\infty_{\text{per}}(D)$,

$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } D_\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\omega_\varepsilon \\ u_\varepsilon \text{ is } D\text{–periodic}, \end{cases} \tag{1.4}$$

which we considered in [12]. In fact, it turns out that in scalar context of (1.4), free of the divergence constraint, the approximation error on the solution u_ε committed by the homogenized model of order $2K+2$ improves rather surprisingly up to the order $O(\varepsilon^{2K+4})$.

Still in [13], we have analyzed the asymptotic behaviors of the tensors M^k and $\mathbb{D}^k_K$ in the low volume fraction regime $\eta \to 0$. Assuming $d \geq 3$ for simplicity, we have found (see Corollary 5.5 of this reference)

$$M^0 \sim \eta^{d-2} F \tag{1.5}$$

$$M^1 = o(\eta^{d-2}) \tag{1.6}$$

$$M^2 \to -I \tag{1.7}$$

$$\forall k \geq 2, M^{2k} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right), \tag{1.8}$$

$$\forall k \geq 1, M^{2k+1} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right), \tag{1.9}$$

as well as equivalent results for the tensors $(\mathbb{D}^k_K)$. The first result (1.5) has been known since the work of Allaire on the continuity of the Darcy equation [3], it involves a $d \times d$ dimensional matrix $F \equiv (F_{ij})_{1 \leq i,j \leq d}$ which can be retrieved by solving an exterior problem in $\mathbb{R}^d \backslash T$ (the definition is recalled in (4.10) below). In the scalar case, the same results hold with F being replaced by the capacity $\text{Cap}(\partial T)$ of the obstacle.

The motivation for seeking these asymptotics in [13] was to investigate whether the high order models (1.2) and (1.3) have the potential to unify the three classical homogenized regimes acknowledged by the literature. Standard homogenization theory [26, 24, 9, 2, 4, 1, 5, 22, 23] states that $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ (or a suitable rescaling) converges in some sense to the solution $(\boldsymbol{u}^*, p^*)$ to three possible limit equations as $\varepsilon \to 0$, depending on how η compares with respect to the critical size $\eta_{\text{crit}} := \varepsilon^{2/(d-2)}$. The limiting equation is either the Darcy, the Brinkman or the Stokes model in the homogeneous domain D.

As far as we are concerned with the present periodic setting, we can read from (1.5)–(1.9), the following coefficient-wise convergences of (1.2) (or (1.3)) as $\eta \to 0$ and $\varepsilon \to 0$:

- if $1 \gg \eta \gg \varepsilon^{2/(d-2)}$, namely the holes are large, then the limiting equation for $(\eta^{d-2}\varepsilon^{-2}\boldsymbol{u}_\varepsilon, p_\varepsilon)$ is the Darcy problem

$$\begin{cases} F\boldsymbol{u}^* + \nabla p^* = \boldsymbol{f} \text{ in } D \\ \text{div}(\boldsymbol{u}^*) = 0 \text{ in } D \\ \boldsymbol{u}^* \text{ is } D\text{–periodic}; \end{cases} \tag{1.10}$$

- if $\eta \sim c\varepsilon^{2/(d-2)}$, namely the holes are exactly proportional to the critical diameter $a_{\rm crit} := \eta_{\rm crit}\varepsilon = \varepsilon^{d/(d-2)}$, then the limiting equation for $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ is the Brinkman problem

$$\begin{cases} -\Delta \boldsymbol{u}^* + cF\boldsymbol{u}^* + \nabla p^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}^*) = 0 \text{ in } D \\ (\boldsymbol{u}^*, p^*) \text{ is } D\text{–periodic,} \end{cases} \tag{1.11}$$

where in both (1.10) and (1.11), F is the matrix appearing in (1.5).

The coefficient-wise convergence of (1.2) towards either (1.10) and (1.11) is consistent with the literature which asserts that the solutions $(\boldsymbol{u}^*, p^*)$ to either (1.10) or (1.11) is the limit of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ in the corresponding regimes. This allowed us to conclude in [13] that the high order homogenization process captures both the Darcy and the Brinkman regimes (1.10) and (1.11).

Finally, the literature states that in the subcritical regime $\eta = o << \varepsilon^{2/(d-2)}$, $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges in some sense, as $\varepsilon \to 0$, to the solution $(\boldsymbol{u}^*, p^*)$ of the Stokes equation in the homogeneous domain D (without holes):

$$\begin{cases} -\Delta \boldsymbol{u}^* + \nabla p^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}^*) = 0 \text{ in } D \\ (\boldsymbol{u}^*, p^*) \text{ is } D\text{–periodic.} \end{cases} \tag{1.12}$$

Intuitively, this means that when $\eta \ll \varepsilon^{2/(d-2)}$, the holes are too small to be actually sensed by the effective model. However, the analysis that we performed in [11] is not sufficient to retrieve this result as a coefficient-wise convergence of the higher order models (1.2) or (1.3) to the homogeneous Stokes system (1.12). Indeed, although (1.5)–(1.7) allows to infer that the right convergence holds for the first three coefficients $M^0\varepsilon^{-2}$, $M^1\varepsilon^{-1}$ and M^2, the asymptotic bounds (1.8) and (1.9) only enable to obtain that the coefficient $\varepsilon^{2k-2}M^{2k}$ is bounded when $k \geq 2$ by the quantity $(\varepsilon^{2/(d-2)}/\eta)^{(k-1)(d-2)}$ which grows to infinity as $\eta \to 0$.

In this perspective, the purpose of this article is to propose a different asymptotic analysis of [13] which allows to substantially improve the asymptotic convergences of (1.5)–(1.9). Our main results are stated in Corollary 4.6 and Proposition 4.10 where we obtain that in fact, $M^k \to 0$ and $\mathbb{D}_K^k \to 0$ for any $k > 2$ with a convergence rate not bigger than $O(\eta^{d-2})$. This implies in particular the coefficient-wise convergence of the high-order models (1.2) and (1.3) towards the Stokes equation (1.12) not only in the subcritical regime $\eta = o(\varepsilon^{2/(d-2)})$ as $\varepsilon \to 0$, but also in the situation where the size of the periodic cell ε (and so their number) is *fixed* while the holes disappear as $\eta \to 0$.

All in all, this paper demonstrates that at least in the sense of coefficient-wise convergence, the effective models (1.2) and (1.3) have indeed the potential to yield high order homogenized approximations of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ that are valid in all possible regimes of size of holes. A more formal statement would require to improve the error bounds of [13] involving $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{u}_\varepsilon^*$, so as to obtain error results with bounding constants uniform with respect to η. We expect this could be done by using e.g. the unified approach proposed in [18] in the context of the homogenization of the Poisson system; a precise treatment is left for future works.

For completeness and in a pedagogical purpose, we prove the results first in the context of the Laplace problem (1.4), which can be considered as a simplified scalar version of the full Stokes system (1.1). In a second part, we shall state how the results actually extend to (1.1) with an emphasis on the differences that occur due to the vectorial context and to the zero divergence constraint.

The paper outlines as follows. Notation conventions and the definitions of various families of tensors (including M^k and $\mathbb{D}_K^k$) related to the high order homogenization process are reviewed in section 2 for both the Poisson equation (1.4) and the Stokes system (1.1). Section 3 provides our new asymptotic analysis for the tensors M^k and $\mathbb{D}_K^k$ in the context of the Poisson equation (1.4). Treating first the scalar case allows us to highlight the key arguments in a simplified setting, namely the introduction of a new family of cell tensors $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$ in $P\backslash(\eta T)$ whose averages $(\mathcal{Y}^{k*})_{k\in\mathbb{N}}$ remain of the same order $O(\eta^{2-d})$ uniformly in $k \in \mathbb{N}$ (Proposition 3.3). Finally, the Stokes case is treated in section 4. The main differences of the asymptotic analysis are related to the vectorial setting and the presence of the pressure, which require to consider vector and matrix valued tensors rather than scalar tensors. Furthermore, the asymptotic analysis of the coefficients $\mathbb{D}_K^k$ requires an additional treatment due to the fact that, in contrast with the scalar case, half of the coefficients (for $K+1 \leq k \leq 2K+1$) do not coincide with the corresponding tensors M^k.

2. Setting, notation and review of available results. In this section, we review the notation conventions used for tensors and the definitions of the tensors $(M^k)_{k\in\mathbb{N}}$ and $(\mathbb{D}_K^k)_{0\leq k\leq 2K+2}$ in both contexts of the Poisson equation (1.4) and the Stokes system (1.1). Both situations involve the solutions of partial differential equations posed in the perforated unit cell $Y = P\backslash(\eta T)$ where $P = (-1/2, 1/2)^d$, T is an obstacle centered in the cell (i.e. $0 \in T$) and $\eta > 0$ is the rescaling. When considering the low-volume fraction regime $\eta \to 0$, we also assume that the hole is strictly included in the cell for $0 < \eta \leq 1$: $T \subset P$. The setting is illustrated on Figure 2.

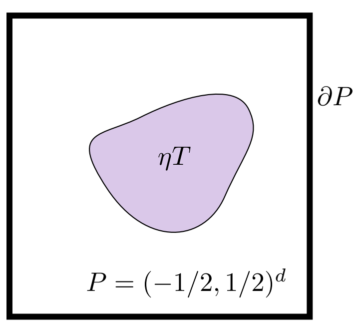

Fig. 2. *Schematic of the cell P and the obstacle ηT.*

2.1. Notation conventions. In the whole paper, we use the same notation conventions for tensor related operations as in our previous works [12, 13]. These are summarized in the nomenclature below. These notations allow us to systematically avoid writing indices for partial derivatives (e.g. $1 \leq i_1 \dots i_k \leq d$), and to distinguish them from spatial indices (e.g. $1 \leq l, m \leq d$) associated with vector or matrix components.

We recall that unless otherwise specified, the Einstein summation convention over repeated *subscript* indices is assumed (but never on *superscript* indices). Vectors $\boldsymbol{b} \in \mathbb{R}^d$ are written in bold face notation.

Scalar, vector, and matrix valued tensors and their coordinates

$\boldsymbol{b} \equiv (b_j)_{1\le j\le d}$ — Vector of $\mathbb{R}^d$

b^k — Scalar valued tensor of order k ($b^k_{i_1\dots i_k} \in \mathbb{R}$ for $1 \le i_1,\dots,i_k \le d$)

$\boldsymbol{b}^k$ — Vector valued tensor of order k ($\boldsymbol{b}^k_{i_1\dots i_k} \in \mathbb{R}^d$ for $1 \le i_1,\dots,i_k \le d$)

B^k — Matrix valued tensor of order k ($B^k_{i_1\dots i_k} \in \mathbb{R}^{d\times d}$ for $1 \le i_1,\dots,i_k \le d$)

$(b^k_j)_{1\le j\le d}$ — Coordinates of the vector valued tensor $\boldsymbol{b}^k$ (b^k_j is a *scalar* tensor of order k).

$(B^k_{lm})_{1\le l,m\le d}$ — Coefficients of the matrix valued tensor B^k (B^k_{lm} is a *scalar* tensors of order k).

$b^k_{i_1\dots i_k,j}$ — Coefficient of the vector valued tensor $\boldsymbol{b}^k$ ($1 \le i_1,\dots i_k, j \le d$).

$B^k_{i_1\dots i_k,lm}$ — Coefficients of the matrix valued tensor B^k ($1 \le i_1,\dots i_k, l, m \le d$).

Tensor products

$b^p \otimes c^{k-p}$ — Tensor product of scalar tensors b^p and c^{k-p}:

$$(b^p \otimes c^{k-p})_{i_1\dots i_k} := b^p_{i_1\dots i_p} c^{k-p}_{i_{p+1}\dots i_k}. \tag{2.1}$$

$B^p \otimes C^{k-p}$ — Tensor product of matrix valued tensors B^p and C^{k-p}:

$$(B^p \otimes C^{k-p})_{i_1\dots i_k,lm} := B^p_{i_1\dots i_p,lj} C^{k-p}_{i_{p+1}\dots i_k,jm}. \tag{2.2}$$

Hence a matrix product is implicitly assumed in the notation $B^p \otimes C^{k-p}$.

$\boldsymbol{b}^p \cdot \boldsymbol{c}^{k-p}$ — Tensor product and inner product of vector valued tensors $\boldsymbol{b}^p$ and $\boldsymbol{c}^{k-p}$:

$$(\boldsymbol{b}^p \cdot \boldsymbol{c}^{k-p})_{i_1\dots i_k} := b^p_{i_1\dots i_p,m} c^{k-p}_{i_{p+1}\dots i_k,m}. \tag{2.3}$$

$B^p \cdot \boldsymbol{c}^{k-p}$ — Tensor product of a matrix tensor B^p and a vector tensors $\boldsymbol{c}^{k-p}$:

$$(B^p \cdot \boldsymbol{c}^{k-p})_{i_1\dots i_k,l} := B^p_{i_1\dots i_p,lm} c^{k-p}_{i_{p+1}\dots i_k,m}. \tag{2.4}$$

Hence a matrix-vector product is implicitly assumed in $B^p \cdot \boldsymbol{c}^{k-p}$.

Contraction with partial derivatives

$b^k \cdot \nabla^k$ — Differential operator of order k associated with a scalar tensor b^k: for any smooth scalar field $v \in \mathcal{C}^\infty_{per}(D,\mathbb{R}^d)$,

$$b^k \cdot \nabla^k v := b^k_{i_1\dots i_k} \partial^k_{i_1\dots i_k} v. \tag{2.5}$$

$\boldsymbol{b}^k \cdot \nabla^k$ — Differential operator of order k associated with a vector tensor $\boldsymbol{b}^k$: for any smooth vector field $\boldsymbol{v} \in \mathcal{C}^\infty_{per}(D,\mathbb{R}^d)$,

$$\boldsymbol{b}^k \cdot \nabla^k \boldsymbol{v} = b^k_{i_1\dots i_k,l} \partial^k_{i_1\dots i_k} v_l. \tag{2.6}$$

$B^k \cdot \nabla^k$ — Differential operator of order k associated with a matrix valued tensor B^k: for any smooth vector field $\boldsymbol{v} \in \mathcal{C}^\infty_{per}(D,\mathbb{R}^d)$,

$$(B^k \cdot \nabla^k \boldsymbol{v})_l = B^k_{i_1\dots i_k,lm} \partial^k_{i_1\dots i_k} v_m. \tag{2.7}$$

In (2.5)–(2.7) above, the reader may equivalently think $\nabla^k v$ and $\nabla^k \boldsymbol{v}$ as scalar valued and vector valued tensors of order k and the dot $\cdot$ notation as the contraction operator of two order k tensors.

Special tensors

$(\boldsymbol{e}_j)_{1\leq j\leq d}$	Vectors of the canonical basis of $\mathbb{R}^d$.
e_j	Scalar valued tensor of order 1 given by $e_{j,i_1} := \delta_{i_1 j}$ (with $1 \leq j \leq d$).
δ_{ij}	Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.
I	Scalar-valued identity tensor of order 2: $$I_{i_1 i_2} = \delta_{i_1 i_2}.$$ The identity tensor is another notation for the Kronecker tensor and it holds $I = e_j \otimes e_j$ with summation on the index $1 \leq j \leq d$. With a small abuse of notation and when the context is clear, we also denote by I the *matrix-valued* second order tensor $I \equiv (I_{i_1 i_2, lm})_{1\leq i_1, i_2, l, m \leq d}$ defined by $$I_{i_1 i_2, lm} := \delta_{i_1 i_2}\delta_{lm}. \quad (2.8)$$ This notation is used in (1.7), (4.18), and (4.32).

In the whole paper, we consider zeroth order tensors which are scalar, vector or matrices devoid of partial derivative indices; e.g. $b^0 \in \mathbb{R}$ if b^0 is scalar, $\boldsymbol{b}^0 \in \mathbb{R}^d$ if $\boldsymbol{b}^0(y)$ is a vector field, and so on. Then the various possible tensor products involving of a zero-th order tensor make sense and follow the same conventions as in eqn. (2.1) to (2.4).

Since a k-th order tensor b^k (scalar, vector or matrix valued) truly makes sense when contracted with k partial derivatives, as in (2.5)–(2.7), all the tensors considered throughout this work are identified to their symmetrization:

$$b^k_{i_1\ldots i_k} \equiv \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} b_{i_{\sigma(1)}\ldots i_{\sigma(k)}},$$

where $\mathfrak{S}_k$ is the permutation group of order k. Consequently, the order in which the derivative indices $i_1, \ldots, i_k$ are written in $b^k_{i_1\ldots i_k}$ does not matter. This alleviates the need for specifying the order of the indices in tensor product notations such as in (2.15) below.

In the paper, the star–"$*$"– symbol is used to indicate that a quantity is "macroscopic" in the sense that it does not depend on the fast variable x/ε; such as $(\boldsymbol{u}^*_\varepsilon, p^*_\varepsilon)$ or $(\boldsymbol{v}^*_{\varepsilon,K}, q^*_{\varepsilon,K})$ in (1.2) and (1.3). In the particular case where a quantity $\mathcal{X}(y)$ is given as a P–periodic function of $Y = P\backslash(\eta T)$ extended by 0 on the obstacle $\partial(\eta T)$, then $\mathcal{X}^*$ denotes the average of $y \mapsto \mathcal{X}(y)$ with respect to the y variable:

$$\mathcal{X}^*(y) := \int_P \mathcal{X}(y)\mathrm{d}y = \int_{P\backslash(\eta T)} \mathcal{X}(y)\mathrm{d}y.$$

At some places we find occasionally more convenient to write the cell average with the more usual angle bracket symbols:

$$\langle\mathcal{X}\rangle := \int_P \mathcal{X}(y)\mathrm{d}y.$$

Finally, we write C or C_K to denote universal constants that do not depend on ε or η but whose values may be redefined from lines to lines.

Remark 2.1. In a limited number of places, the superscript or subscript indices $p, q \in \mathbb{N}$ are used. Naturally, these are not to be confused with the pressure variables p_ε or $q^*_{\varepsilon,K}$ introduced in (1.1) and (1.3).

Remark 2.2. In all what follows, the various tensors coming at play such as $\mathcal{X}^k$, $\mathcal{X}^{k*}$, M^k, $\mathbb{D}^k_K$ etc., depend on the scaling of the obstacle η, but this dependence is made implicit for notational simplicity.

2.2. High order effective models for the perforated Poisson equation. For the Poisson equation, the homogenized equations of respectively "infinite" order and of order $2K+2$ read respectively

$$\begin{cases} \displaystyle\sum_{k=0}^{+\infty} \varepsilon^{2k-2} M^{2k} \cdot \nabla^{2k} u^*_\varepsilon = f \text{ in } D \\ u^*_\varepsilon \text{ is } D\text{–periodic,} \end{cases} \tag{2.9}$$

$$\begin{cases} \displaystyle\sum_{k=0}^{K+1} \varepsilon^{2k-2} \mathbb{D}^{2k}_K \cdot \nabla^{2k} v^*_{\varepsilon,K} = f \text{ in } D \\ v^*_{\varepsilon,K} \text{ is } D\text{–periodic.} \end{cases} \tag{2.10}$$

Note that in this scalar context, (2.9) and (2.10) feature no *odd* order differential operators, i.e. $M^{2k+1} = 0$ and $\mathbb{D}^{2k+1}_K = 0$. The coefficients $(M^k)_{k\in\mathbb{N}}$ and $(\mathbb{D}^k_{2K+1})$ are defined by a procedure involving cell tensors $(\mathcal{X}^k(y))_{k\in\mathbb{N}}$ and $(N^k(y))_{k\in\mathbb{N}}$ with $y \in Y$.

DEFINITION 2.3. *The cell tensors $(\mathcal{X}^k(y))_{k\in\mathbb{N}}$ are defined recursively as the solutions to the following cascade of equations:*

$$\begin{cases} -\Delta\mathcal{X}^0 = 1 \textit{ in } P\backslash(\eta T) \\ -\Delta\mathcal{X}^1 = 2\partial_j \mathcal{X}^0 \otimes e_j \textit{ in } P\backslash(\eta T) \\ -\Delta\mathcal{X}^{k+2} = 2\partial_j \mathcal{X}^{k+1} \otimes e_j + \mathcal{X}^k \otimes I \textit{ in } P\backslash(\eta T), \quad k \geq 0 \\ \mathcal{X}^k = 0 \textit{ on } \partial(\eta T), \quad k \geq 0 \\ \mathcal{X}^k \textit{ is } P\text{–}\textit{periodic.} \end{cases} \tag{2.11}$$

We then denote by $\mathcal{X}^{k*}$ the average of the tensor field $\mathcal{X}^k$:

$$\mathcal{X}^{k*} := \int_{P\backslash(\eta T)} \mathcal{X}^k(y)\mathrm{d}y. \tag{2.12}$$

Remark 2.4. Owing to our notation convention of subsection 2.1, the third equation of (2.11) can be equivalently written

$$\begin{aligned} -\Delta\mathcal{X}^{k+2}_{i_1\ldots i_{k+2}} &= 2\partial_j \mathcal{X}^{k+1}_{i_1\ldots i_{k+1}} \delta_{j i_{k+2}} + \mathcal{X}^k_{i_1\ldots i_k} \delta_{i_{k+1} i_{k+2}} \\ &= 2\partial_{i_{k+2}} \mathcal{X}^{k+1}_{i_1\ldots i_{k+1}} + \mathcal{X}^k_{i_1\ldots i_k} \delta_{i_{k+1} i_{k+2}}. \end{aligned}.$$

In particular, the repeated index k in the equation is not summed over, but the repeated index j *is*.

DEFINITION 2.5. *The family of constant scalar tensors $(M^k)_{k\in\mathbb{N}}$ is defined by the following recursive formula*

$$M^k := \begin{cases} (\mathcal{X}^{0*})^{-1} \text{ if } k=0 \\ -(\mathcal{X}^{0*})^{-1} \sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p \text{ if } k \geq 1. \end{cases} \tag{2.13}$$

The definition (2.13) is valid because the tensor $\mathcal{X}^{0*} = \int_Y |\nabla \mathcal{X}^0|^2 \mathrm{d}y > 0$ is a positive number; it rewrites equivalently as

$$\sum_{p=0}^{k} \mathcal{X}^{p*} \otimes M^{k-p} = \begin{cases} 1 \text{ if } k=0 \\ 0 \text{ if } k \geq 1. \end{cases} \tag{2.14}$$

For $k \geq 1$, M^k can be computed by the following explicit formula, see [12] (Proposition 6):

$$M^k = \sum_{p=1}^{k} (-1)^p \sum_{\substack{i_1+\dots i_p=k \\ 1\leq i_1 \dots i_p \leq k}} (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1}. \tag{2.15}$$

In [12] (Proposition 3), we have found that the definitions (2.11)–(2.13) imply that the odd order coefficient tensors vanish, namely $\mathcal{X}^{2k+1*} = 0$ and $M^{2k+1} = 0$ for all $k \geq 0$.

The Cauchy product of the tensors $(M^k)_{k\in\mathbb{N}}$ and $(\mathcal{X}^k(y))_{k\in\mathbb{N}}$ then yields an additional and important family of cell tensors $(N^k(y))_{k\in\mathbb{N}}$.

DEFINITION 2.6. *For any $k \in \mathbb{N}$, we define the k-th order cell tensor N^k by*

$$N^k(y) := \sum_{p=0}^{k} M^p \otimes \mathcal{X}^{k-p}(y), \qquad y \in Y. \tag{2.16}$$

Remark 2.7. Equation (2.14) states that the averages of the tensors $(N^k)_{k\in\mathbb{N}}$ are given respectively by

$$\int_Y N^k(y)\mathrm{d}y = \begin{cases} 1 \text{ if } k=0 \\ 0 \text{ if } k \geq 1. \end{cases} \tag{2.17}$$

The tensors $(N^k(y))_{k\in\mathbb{N}}$ allow to reconstruct the oscillating solution u_ε of (1.4) from its high order homogenized approximations u_ε^* or $v_{\varepsilon,K}^*$ given by (2.9) and (2.10). Indeed, the following identity holds at least in a formal sense,

$$u_\varepsilon(x) = \sum_{k=0}^{+\infty} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k u_\varepsilon^*(x), \qquad x \in D_\varepsilon \tag{2.18}$$

and likewise, we proved in [12] (Corollary 5) that the reconstructed function

$$W_{\varepsilon,2K+1}(v_{\varepsilon,K}^*) := \sum_{k=0}^{2K+1} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k v_{\varepsilon,K}^*(x), \qquad x \in D_\varepsilon$$

approximates u_ε up to a remainder of order $O(\varepsilon^{2K+4})$ in the $L^2(D_\varepsilon)$ norm. The identity (2.18) relating u_ε to u_ε^* is somewhat remarkable. We have called it a "criminal ansatz" based on similar observations which hold in the context of the conductivity or wave equation [8, 6].

Finally, the tensors $(N^k(y))_{k\in\mathbb{N}}$ determine the coefficients $(\mathbb{D}_K^k)_{0\le k\le 2K+2}$ of (2.10) (see [12], Proposition 13).

DEFINITION 2.8. *For any $K \ge 0$ and $0 \le k \le 2K+2$, the coefficient $\mathbb{D}_K^k$ is defined by:*

$$\mathbb{D}_K^k = M^k \text{ for any } 0 \le k \le 2K+1 \tag{2.19}$$

$$\mathbb{D}_K^{2K+2} = (-1)^{K+1}\int_Y N^K(y)\otimes N^K(y)\otimes I\mathrm{d}y \tag{2.20}$$

where $N^K(y)$ is the cell tensor given by (2.16).

2.3. High order effective models for the Stokes system in a porous medium. The construction of the tensors $(M^k)_{k\in\mathbb{N}}$ and $(\mathbb{D}_K^k)_{0\le k\le 2K+2}$ for the effective Stokes systems (1.2) and (1.3) follow the same construction as in the scalar case, up to the following differences:

1. due to the vectorial nature of $\boldsymbol{u}_\varepsilon$, the tensors M^k, $\mathbb{D}_K^k$, $\mathcal{X}^k(y)$, $\mathcal{X}^{k*}$ and $N^k(y)$ become *matrix valued*. They include therefore k partial derivatives indices $i_1\dots i_k$, and two spatial indices $1\le l,m\le d$ which follow the notation conventions of subsection 2.1;
2. the presence of the pressure p_ε and of the divergence constraint $\operatorname{div}(\boldsymbol{u}_\varepsilon)=0$ in (1.1) reflects in the introduction of *vector valued* tensorial pressure fields $\boldsymbol{\alpha}^k(y)$, $\boldsymbol{\beta}^k(y)$ coming along $\mathcal{X}^k(y)$ and $N^k(y)$. The vector valued tensors $\boldsymbol{\alpha}^k(y)$ and $\boldsymbol{\beta}^k(y)$ are therefore characterized by k partial derivative indices $1\le i_1\dots i_k\le d$ and one spatial index $1\le l\le d$.

The starting point is the definition of the solution tensors $(\boldsymbol{\mathcal{X}}^k(y),\alpha^k(y))$ to a hierarchy of Stokes systems analogous to (2.11):

DEFINITION 2.9. *For any $k\ge 0$, we define respectively the vector valued tensors $(\boldsymbol{\mathcal{X}}_j^k(y))_{1\le j\le d}$ and the scalar valued tensors $(\alpha_j^k(y))_{1\le j\le d}$ to be the unique solutions in $H^1_{\mathrm{per}}(Y,\mathbb{R}^d)\times L^2(Y)$ to the following cell problems:*

$$\begin{cases}-\Delta_{yy}\boldsymbol{\mathcal{X}}_j^0+\nabla_y\alpha_j^0=\boldsymbol{e}_j \text{ in } Y,\\ \operatorname{div}_y(\boldsymbol{\mathcal{X}}_j^0)=0 \text{ in } Y\end{cases} \tag{2.21}$$

$$\begin{cases}-\Delta_{yy}\boldsymbol{\mathcal{X}}_j^1+\nabla_y\alpha_j^1=(2\partial_l\boldsymbol{\mathcal{X}}_j^0-\alpha_j^0\boldsymbol{e}_l)\otimes e_l \text{ in } Y\\ \operatorname{div}_y(\boldsymbol{\mathcal{X}}_j^1)=-(\boldsymbol{\mathcal{X}}_j^0-\langle\boldsymbol{\mathcal{X}}_j^0\rangle)\cdot\boldsymbol{e}_l\otimes e_l \text{ in } Y,\end{cases} \tag{2.22}$$

$$\begin{cases}-\Delta_{yy}\boldsymbol{\mathcal{X}}_j^{k+2}+\nabla_y\alpha_j^{k+2}=(2\partial_l\boldsymbol{\mathcal{X}}_j^{k+1}-\alpha_j^{k+1}\boldsymbol{e}_l)\otimes e_l+\boldsymbol{\mathcal{X}}_j^k\otimes I \text{ in } Y\\ \operatorname{div}_y(\boldsymbol{\mathcal{X}}_j^{k+2})=-(\boldsymbol{\mathcal{X}}_j^{k+1}-\langle\boldsymbol{\mathcal{X}}_j^{k+1}\rangle)\cdot\boldsymbol{e}_l\otimes e_l \text{ in } Y\end{cases}\quad \forall k\ge 0, \tag{2.23}$$

supplemented with the following boundary conditions:

$$\begin{cases}\displaystyle\int_Y\alpha_j^k\mathrm{d}y=0\\ \boldsymbol{\mathcal{X}}_j^k=0 \text{ on } \partial(\eta T)\\ (\boldsymbol{\mathcal{X}}_j^k,\alpha_j^k) \text{ is } P\text{–periodic}\end{cases}\quad \forall k\ge 0. \tag{2.24}$$

The k-th order matrix valued tensor field $\mathcal{X}^k(y)$ is then assembled by gathering the d vector valued tensors $(\boldsymbol{\mathcal{X}}_j^k(y))_{1\leq j\leq d}$ into columns:

$$\mathcal{X}^k(y) := \begin{bmatrix} \boldsymbol{\mathcal{X}}_1^k(y) & \dots & \boldsymbol{\mathcal{X}}_d^k(y) \end{bmatrix}, \forall y \in Y, \quad \forall k \geq 0,$$

or in other words $\mathcal{X}_{ij}^k(y) = \boldsymbol{\mathcal{X}}_j^k(y) \cdot \boldsymbol{e}_i$. Similarly, we define $\boldsymbol{\alpha}^k(y)$ to be the k-th order vector valued tensor whose coordinates are the scalar tensors $\alpha_j^k(y)$:

$$\boldsymbol{\alpha}^k(y) := (\alpha_j^k(y))_{1\leq j\leq d}, \forall y \in Y, \quad \forall k \geq 0.$$

Following (2.12), the *matrix* valued tensor $\mathcal{X}^{k*}$ is then defined as the average of $\mathcal{X}^k(y)$ over the perforated cell:

$$\mathcal{X}^{k*} := \int_Y \mathcal{X}^k(y)\mathrm{d}y, \forall k \geq 0. \tag{2.25}$$

Note that by the definition (2.24), $\boldsymbol{\alpha}^k(y)$ is of zero average for any $k \geq 0$. Similarly, the porosity matrix $\mathcal{X}^{0*}$ is known to be symmetric definite positive [26]. Therefore, the following definition makes sense:

Definition 2.10. *The family of matrix valued tensors $(M^k)_{k\in\mathbb{N}}$ is defined by the following recursive formula:*

$$\begin{cases} M^0 = (\mathcal{X}^{0*})^{-1} \\ M^k = -(\mathcal{X}^{0*})^{-1} \displaystyle\sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p, \quad \forall k \geq 1. \end{cases} \tag{2.26}$$

Note that in contrast with the scalar case, matrix products take place between the tensors $\mathcal{X}^{k-p*}$ and M^p in (2.26). The explicit formula (2.15) still holds under the same convention.

Contrarily to the scalar case, odd order tensors $\mathcal{X}^{2k+1*}$ and M^{2k+1} do not vanish in general (they do in case the obstacle ηT is symmetric with respect to the cell axes). Instead, we find the following symmetry properties (Proposition 3.5 of [13]):

Proposition 2.11. *The k-th order tensors $\mathcal{X}^{k*}$ and M^k are symmetric and antisymmetric matrix valued for respectively even and odd values of k; i.e.*

$$\mathcal{X}^{2k*}_{i_1\dots i_k,lm} = \mathcal{X}^{2k*}_{i_1\dots i_k,ml}, \qquad M^{2k}_{i_1\dots i_k,lm} = M^{2k}_{i_1\dots,i_k,ml},$$
$$\mathcal{X}^{2k+1*}_{i_1\dots i_k,lm} = -\mathcal{X}^{2k+1}_{i_1\dots i_k,ml}, \qquad M^{2k+1}_{i_1\dots i_k,lm} = -M^{2k+1}_{i_1\dots i_k,ml},$$

where $1 \leq i_1, \dots, i_k \leq d$ and $1 \leq l, m \leq d$ denote respectively the partial derivative and the spatial indices.

From the Cauchy product of M^k and $\mathcal{X}^k(y)$, we define matrix and vector valued cell tensors $N^k(y)$ and $\boldsymbol{\beta}^k(y)$ (Proposition 3.9 in [13]).

Definition 2.12. *For any $k \in \mathbb{N}$, let N^k and $\boldsymbol{\beta}^k$ be respectively the $k-th$ order matrix valued and vector valued tensors defined by*

$$N^k(y) := \sum_{p=0}^{k} \mathcal{X}^{k-p}(y) \otimes M^p, \qquad \boldsymbol{\beta}^k(y) := \sum_{p=0}^{k} (-1)^p M^p \cdot \boldsymbol{\alpha}^{k-p}(y), \qquad \forall y \in Y.$$

Remark that a matrix product and a matrix-vector product take place in the respective definitions of $N^k(y)$ and $\boldsymbol{\beta}^k(y)$. We have the following property analogous to (2.17) in this vectorial context:

$$\int_Y N^k(y)\mathrm{d}y = \begin{cases} I \text{ if } k=0 \\ 0 \text{ if } k \geq 1. \end{cases} \tag{2.27}$$

It is useful to consider $(\boldsymbol{N}_j^k)_{1\leq j\leq d}$ and $(\beta_j^k)_{1\leq j\leq d}$ which are respectively the column vectors and the coefficients of $N^k(y)$ and $\boldsymbol{\beta}^k(y)$:

$$\forall 1\leq i,j\leq d,\ \boldsymbol{N}_j^k(y) := N^k(y)\boldsymbol{e}_j \text{ and } \beta_j^k(y) := \boldsymbol{\beta}^k(y)\cdot \boldsymbol{e}_j, \qquad y\in Y. \tag{2.28}$$

Similar to the scalar case, these new tensors allow to reconstruct the oscillating velocity and pressure $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ solutions to (1.1) from their homogenized approximations $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ or $(\boldsymbol{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ given by (1.2) and (1.3). We have indeed the following formal identities

$$\begin{cases} \boldsymbol{u}_\varepsilon(x) = \displaystyle\sum_{i=0}^{+\infty} \varepsilon^i N^i(x/\varepsilon)\cdot\nabla^i \boldsymbol{u}_\varepsilon^*(x) \\ p_\varepsilon(x) = p_\varepsilon^*(x) + \displaystyle\sum_{i=0}^{+\infty}\varepsilon^{i-1}\boldsymbol{\beta}^i(x/\varepsilon)\cdot\nabla^i\boldsymbol{u}_\varepsilon^*(x), \end{cases} \qquad \forall x\in D_\varepsilon. \tag{2.29}$$

Likewise, we proved in [13] that the reconstructed velocity and pressures

$$\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v}_{\varepsilon,K}^*)(x) := \sum_{k=0}^{K}\varepsilon^k N^k(x/\varepsilon)\cdot\nabla^k\boldsymbol{v}_{\varepsilon,K}^*(x), \qquad x\in D_\varepsilon \tag{2.30}$$

$$Q_{\varepsilon,K-1}(\boldsymbol{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)(x/\varepsilon) := q_{\varepsilon,K}^*(x) + \sum_{k=0}^{K-1}\varepsilon^{k-1}\boldsymbol{\beta}^k(x/\varepsilon)\cdot\nabla^k\boldsymbol{v}_{\varepsilon,K}^*(x), \quad x\in D_\varepsilon \tag{2.31}$$

yield approximations of $\boldsymbol{u}_\varepsilon$ and p_ε of respective order $O(\varepsilon^{K+3})$ and $O(\varepsilon^{K+1})$ in the $L^2(D_\varepsilon)$ norm. Unfortunately and in contrast with the scalar case, we do not obtain an error estimate of order $O(\varepsilon^{2K+4})$ for the velocity as it could have been expected, because only half of the coefficients $\mathbb{D}_K^k$ obtained from the well-posed truncation process of [13] turn to be equal to the M^k.

The latter coefficients $(\mathbb{D}_K^k)_{0\leq k\leq 2K+2}$ are indeed given by the following formulas (Proposition 4.10 of [13]):

DEFINITION 2.13. *For any $K\geq 0$ and $0\leq k\leq 2K+2$, the coefficient $\mathbb{D}_K^k$ is defined by*

$$\mathbb{D}_{K,ij}^k = \begin{cases} M^k \text{ if } 0\leq k\leq K \\ M^k + \mathbb{A}_K^k \text{ if } K+1\leq k\leq 2K+1 \\ (-1)^{K+1}\displaystyle\int_Y \boldsymbol{N}_i^K\cdot\boldsymbol{N}_j^K\otimes I\mathrm{d}y \text{ if } k=2K+2. \end{cases} \tag{2.32}$$

where the matrix valued tensor $\mathbb{A}_K^k$ is given for any $K+1\leq k\leq 2K+1$ by

$$\mathbb{A}_{K,ij}^k := (-1)^{K+1}\int_Y (\nabla\beta_j^{k-K-1}\cdot\boldsymbol{N}_i^{K+1} + (-1)^k\nabla\beta_i^{k-K-1}\cdot\boldsymbol{N}_j^{K+1})\mathrm{d}y, \tag{2.33}$$

remembering the definition (2.28) of the vector valued and scalar valued tensors $\boldsymbol{N}^k(y)$ and $\beta_i^k(y)$.

3. Low volume fraction asymptotic of the high order homogenized Laplace model.

In this section, we are concerned with the scalar context of the perforated Laplace problem (1.4); the setting is therefore the one considered in subsection 2.2. We aim at establishing the coefficient-wise convergence of both higher order models (2.9) and (2.10) of respectively infinite and finite orders, in the low-volume fraction regime $\eta \to 0$, or in other words, the convergence of the tensors M^k and $\mathbb{D}_K^k$.

The main results of this section are Corollary 3.8 and Proposition 3.12 where we effectively obtain the asymptotics of these coefficient tensors.

3.1. Cell tensors $\mathcal{Y}^k(y)$ of controlled growth.

The key ingredient which was missing in our previous analysis [12, 13] is the introduction of a new family of cell tensors $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$ of controlled growth with respect to k.

Definition 3.1. *We define the family of cell tensors $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$ by induction as the solutions to the following cascade of equations:*

$$\left\{\begin{aligned} -\Delta\mathcal{Y}^0 &= 1 \text{ in } P\backslash(\eta T)\\ -\Delta\mathcal{Y}^1 &= 2\partial_j\mathcal{Y}^0\otimes e_j\\ -\Delta\mathcal{Y}^{k+2} &= 2\partial_j\mathcal{Y}^{k+1}\otimes e_j + (\mathcal{Y}^k-\mathcal{Y}^{k*})\otimes I \text{ in } P\backslash(\eta T)\\ \mathcal{Y}^k &= 0 \text{ on } \partial(\eta T)\\ \mathcal{Y}^k &\text{ is } P\text{–periodic.}\end{aligned}\right. \tag{3.1}$$

where for any $k\in\mathbb{N}$, we denote by $\mathcal{Y}^{k*}$ the average of these tensors in the unit cell:

$$\mathcal{Y}^{k*} := \int_{P\backslash(\eta T)} \mathcal{Y}^k(y)\mathrm{d}y. \tag{3.2}$$

The benefit of introducing $\mathcal{Y}^k(y)$ lies in the fact that the mean $\mathcal{Y}^{k*}$ remains not bigger than $O(\eta^{2-d})$ as $\eta\to 0$ uniformly in $k\in\mathbb{N}$. The proof relies on the following classical Poincaré estimates in the perforated cell [3, 18] which is recalled in the next lemma.

Lemma 3.2. *For any $v\in H^1(P\backslash(\eta T))$ which is P–periodic and vanishes on the hole $\partial(\eta T)$, the following Poincaré inequality holds:*

$$||v||_{L^2(P\backslash(\eta T))} \leq C\eta^{1-d/2}||\nabla v||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} \tag{3.3}$$

for a constant $C>0$ independent of η and v. Furthermore, for any $v\in H^1(P\backslash(\eta T))$, the following Poincaré-Wirtinger inequality holds:

$$||v-\langle v\rangle||_{L^2(P\backslash(\eta T))} \leq C||\nabla v||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} \tag{3.4}$$

These inequalities entail the following result for the tensors $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$:

Proposition 3.3. *For any integer $k\geq 0$, there exists a constant $C_k>0$ independent of η such that*

$$||\nabla\mathcal{Y}^k||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} \leq C_k\eta^{1-d/2} \tag{3.5}$$

$$||\mathcal{Y}^k-\mathcal{Y}^{k*}||_{L^2(P\backslash(\eta T))} \leq C_k\eta^{1-d/2} \tag{3.6}$$

where, with a little abuse of notation, it is understood that every component $\mathcal{Y}^k_{i_1\dots i_k}$ *with* $1\leq i_1\dots i_k\leq d$ *satisfies* (3.5) *and* (3.6). *In addition, there exists a constant* $\alpha>0$ *independent of* k *and* η *such that*

$$0<C_k<\alpha(1+\sqrt{2})^kC^k$$

where C *is the Poincaré constant of* (3.3) *and* (3.4).

Proof. We proceed by induction.
Case $k=0$: multiply the first equation of (3.1) by $\mathcal{Y}^0$, then integrate by parts to obtain

$$\begin{aligned}||\nabla\mathcal{Y}^0||^2_{L^2(P\backslash(\eta T),\mathbb{R}^d)}=\int_{P\backslash(\eta T)}\mathcal{Y}^0\mathrm{d}y\leq||\mathcal{Y}^0||_{L^2(P\backslash(\eta T))}\\ \leq C\eta^{1-d/2}||\nabla\mathcal{Y}^0||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}.\end{aligned}$$

Case $k=1$: multiply the second equation of (3.1) by $\mathcal{Y}^1$, then integrate by parts to obtain

$$\begin{aligned}||\nabla\mathcal{Y}^1||^2_{L^2(P\backslash(\eta T),\mathbb{R}^d)}\leq 2||\nabla\mathcal{Y}^0||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}||\mathcal{Y}^1-\mathcal{Y}^{1*}||_{L^2(P\backslash(\eta T))}\\ \leq 2C^2\eta^{1-d/2}||\nabla\mathcal{Y}^1||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}.\end{aligned}$$

Case $k+2$ *with* $k\geq 0$: assuming the result is true till rank $k+1$, multiply the third equation of (3.1) by $\mathcal{Y}^{k+2}$, then integrate by parts to obtain

$$\begin{aligned}&||\nabla\mathcal{Y}^{k+2}||^2_{L^2(P\backslash(\eta T),\mathbb{R}^d)}\\ &\quad\leq(2||\nabla\mathcal{Y}^{k+1}||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}+||\mathcal{Y}^k-\mathcal{Y}^{k*}||_{L^2(P\backslash(\eta T))})||\mathcal{Y}^{k+2}-\mathcal{Y}^{k+2*}||_{L^2(P\backslash(\eta T))}\\ &\quad\leq(2C_{k+1}+C_k)C\eta^{1-d/2}||\nabla\mathcal{Y}^{k+2}||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}.\end{aligned}$$

This implies (3.5). Then (3.6) follows from (3.3) and (3.4). ∎

Using the Cauchy-Schwarz inequality and (3.5), we can infer from the above result that $|\mathcal{Y}^{k*}|\leq C_k\eta^{2-d}$. The next proposition provides more precise asymptotics for the mean $\mathcal{Y}^{k*}$ (eqn. (3.2)). In particular, we find that in fact, $\mathcal{Y}^{k*}=o(\eta^{2-d})$ for $k\geq 1$.

PROPOSITION 3.4. *The following asymptotic convergences hold for the mean tensors* $(\mathcal{Y}^{k*})_{k\in\mathbb{N}}$ *as* $\eta\to 0$*:*

$$\mathcal{Y}^{0*}\sim\frac{\eta^{2-d}}{\mathrm{Cap}(\partial T)}\text{ and }\mathcal{Y}^{k*}=o(\eta^{2-d})\text{ for }k\geq 1. \tag{3.7}$$

Proof. Since $\mathcal{Y}^{0*}=\mathcal{X}^{0*}$, the result for $k=0$ is standard and can be found in [3, 18]. The case $k=1$ (and in fact for any odd value of k) is trivial since $\mathcal{Y}^{1*}=\mathcal{X}^{1*}=0$. In order to prove that $\mathcal{Y}^{k+2*}=o(\eta^{2-d})$ for any $k\geq 0$, we follow the lines of the proof of [12], Proposition 14.

Let us denote $\widetilde{\mathcal{Y}}^k:=\eta^{d-2}\mathcal{Y}^k$ for any $k\geq 0$. Then $\widetilde{Y}^k$ is the solution to

$$\begin{cases}-\Delta\widetilde{\mathcal{Y}}^{k+2}=2\eta\partial_j\widetilde{\mathcal{Y}}^{k+1}\otimes e_j+\eta^2(\widetilde{\mathcal{Y}}^k-\langle\widetilde{\mathcal{Y}}^k\rangle)\otimes I\text{ in }\eta^{-1}P\backslash T\\ \quad\mathcal{Y}^{k+2}=0\text{ on }\partial T\\ \quad\mathcal{Y}^{k+2}\text{ is }\eta^{-1}P\text{–periodic,}\end{cases} \tag{3.8}$$

with $\langle\widetilde{\mathcal{Y}}^k\rangle := \eta^d \int_{\eta^{-1}P\backslash T} \widetilde{\mathcal{Y}}^k \mathrm{d}x$. From the previous proposition, there exists a constant $C > 0$ independent of η such that

$$||\nabla\widetilde{\mathcal{Y}}^{k+2}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)} \leq C \text{ and } |\langle\widetilde{\mathcal{Y}}^{k+2}\rangle| \leq C.$$

Hence, up to extracting a subsequence, we may assume the existence of order $k+2$ field and scalar valued tensors $\Psi^{k+2}(x) \in H^1_{loc}(\eta^{-1}P\backslash T)$ and $\gamma^{k+2} \in \mathbb{R}$ such that

$$\widetilde{\mathcal{Y}}^{k+2} \rightharpoonup \Psi^{k+2} \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \text{ and } \langle\widetilde{\mathcal{Y}}^{k+2}\rangle \to \gamma^{k+2} \text{ as } \eta \to 0.$$

Furthermore, the lower-semi continuity of the $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$ norm (see the proof of Theorem 3.1 in [3] for a detailed justification) implies that $\Psi^{k+2} - \gamma^{k+2}$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$. Multiplying (3.8) by a compactly supported test function ϕ, integrating by part and passing to the limit implies that Ψ^{k+2} is the solution to the exterior problem

$$\begin{cases} -\Delta\Psi^{k+2} = 0 \in \mathbb{R}^d\backslash T \\ \Psi^{k+2} = 0 \text{ on } \partial T \\ \Psi^{k+2} \to \gamma^{k+2} \text{ at infinity.} \end{cases} \tag{3.9}$$

Therefore $\Psi^{k+2} = \gamma^{k+2}\phi^*$ where ϕ^* is the solution to

$$\begin{cases} -\Delta\phi^* = 0 \in \mathbb{R}^d\backslash T \\ \phi^* = 0 \text{ on } \partial T \\ \phi^* \to 1 \text{ at infinity.} \end{cases} \tag{3.10}$$

To identify γ^{k+2}, we multiply (3.8) by the constant function 1 and we integrate by part to obtain that

$$0 = -\int_{\partial T} \frac{\partial\widetilde{\mathcal{Y}}^{k+2}}{\partial\boldsymbol{n}}\mathrm{d}y$$

because the right-hand side of (3.8) is of average zero. Using now the continuity of the normal flux with respect to the $H^1_{loc}(\mathbb{R}^d\backslash T)$ weak convergence, we obtain by passing to the limit as $\eta \to 0$:

$$0 = -\lim_{\eta\to 0}\int_{\partial T} \gamma^{k+2}\frac{\partial\phi^*}{\partial\boldsymbol{n}}\mathrm{d}y = \mathrm{Cap}(\partial T)\gamma^{k+2},$$

whence $\gamma^{k+2} = 0$. This implies that the whole sequence $(\langle\widetilde{\mathcal{Y}}^{k+2}\rangle)_{\eta>0}$ converges to zero, and then (3.7) by rescaling. □

We now find that the tensors $(\mathcal{X}^k(y))_{k\in\mathbb{N}}$ to $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$ are related by a Cauchy product identity.

Proposition 3.5. *The tensors $\mathcal{Y}^k(y)$ can be rewritten in terms of the tensors $\mathcal{X}^k(y)$ and $\mathcal{X}^{k*}$ according to the following recursive formula:*

$$\begin{cases} \mathcal{Y}^0(y) = \mathcal{X}^0(y) \\ \mathcal{Y}^1(y) = \mathcal{X}^1(y) \\ \mathcal{Y}^k(y) = \mathcal{X}^k(y) - \sum_{l=0}^{k-2} \mathcal{Y}^l(y)\otimes\mathcal{X}^{k-l-2*}\otimes I \textit{ for } k\geq 2, \end{cases} \quad y \in P\backslash(\eta T). \tag{3.11}$$

Proof. Let us denote by $\mathcal{Y}^k(y)$ the tensors defined according to (3.11). We prove that the tensors $\mathcal{Y}^k$ referring to this definition solve the cascade of partial differential equations (3.1), which implies the result by uniqueness. Obviously (3.1) is true for $\mathcal{Y}^k$ with $k=0$ or $k=1$. Assuming the third equation is true till $k-1$ with $k\geq 0$ (with the convention $\mathcal{Y}^{-1}=0$), we then prove that it still holds at rank k. We compute

$$\begin{aligned}
-\Delta\mathcal{Y}^{k+2} &= -\Delta\mathcal{X}^{k+2} - \sum_{l=0}^{k}(-\Delta\mathcal{Y}^l)\otimes\mathcal{X}^{k-l*}\otimes I\\
&= 2\partial_j\mathcal{X}^{k+1}\otimes e_j + \mathcal{X}^k\otimes I - \mathcal{X}^{k*}\otimes I - 2\partial_j\mathcal{Y}^0\otimes e_j\otimes\mathcal{X}^{k-1*}\otimes I\\
&\quad -\sum_{l=2}^{k}(2\partial_j\mathcal{Y}^{l-1}\otimes e_j + (\mathcal{Y}^{l-2}-\mathcal{Y}^{l-2*})\otimes I)\otimes\mathcal{X}^{k-l*}\otimes I\\
&= 2\partial_j\left(\mathcal{X}^{k+1} - \sum_{l=0}^{k-1}\mathcal{Y}^l\otimes\mathcal{X}_\eta^{k-l-1*}\otimes I\right)\otimes e_j\\
&\quad + \left(\mathcal{X}^k\otimes I - \mathcal{X}^{k*}\otimes I - \sum_{l=0}^{k-2}(\mathcal{Y}^l-\mathcal{Y}^{l*})\otimes\mathcal{X}^{k-l-2*}\otimes I\otimes I\right)\\
&= 2\partial_j\mathcal{Y}^{k+1}\otimes e_j + (\mathcal{Y}^k - \mathcal{Y}^{k*})\otimes I,
\end{aligned}$$

which implies the result. ∎

Remark 3.6. Let us comment the consequences of Propositions 3.3 and 3.5. From (3.11), we have obtained, for any $p\in\mathbb{N}$,

$$\mathcal{X}^{p*} = \mathcal{Y}^{p*} + \sum_{l=0}^{p-2}\mathcal{Y}^{p-2-l*}\otimes\mathcal{X}^{l*}\otimes I. \tag{3.12}$$

Since $\mathcal{Y}^{p-2-l*}=O(\eta^{1-d/2})$, a simple recursive argument, (3.12) yields the following asymptotic for the tensors $\mathcal{X}^{2k*}$ for $k\geq 1$:

$$\mathcal{X}^{2k*} = \frac{\eta^{(2-d)(k+1)}}{\mathrm{Cap}(\partial T)^{k+1}}\underbrace{I\otimes I\otimes\cdots\otimes I}_{k\text{ times}} + o(\eta^{(2-d)k}), \qquad k\geq 1, \tag{3.13}$$

which is a slight quantitative improvement of the result of [12], Proposition 14. In our previous works [12, 13], our asymptotics (1.5) to (1.9) were obtained by inserting the estimate (3.13) into the explicit formula (2.15) for the tensor M^k. However this leads to suboptimal bounds due to the fact that the mean of $\mathcal{X}^{2k}$ is growing with k like $\mathcal{X}^{2k*}=O(\eta^{-(d-2)(k+1)})$.

Since from (3.7), $\mathcal{Y}^{k*}$ has a controlled growth with respect to η (namely $\mathcal{Y}^{k*} = O(\eta^{2-d})$ independently of k), we obtain in the next section improved asymptotic estimates for the coefficient tensors M^k by relying on the *exact* identity (3.12). Note that (3.12) can be interpreted as an asymptotic expansion for $\mathcal{X}^{p*}$, because the terms $\mathcal{Y}^{p-2-2l*}\otimes\mathcal{X}^{2l*}$ of the expansion have an increasing magnitude $O(\eta^{-(d-2)(l+2)})$.

3.2. Low-volume fraction asymptotics of the infinite order homogenized equation.

PROPOSITION 3.7. *The following identity holds for any $k\geq 1$:*

$$\sum_{p=0}^{k}\mathcal{Y}^{p*}\otimes M^{k-p} = -\mathcal{Y}^{k-2*}\otimes I \tag{3.14}$$

with the convention $\mathcal{Y}^{-1*} = 0$.

Proof. Let us multiply (3.12) by M^{k-p} and compute the summation for $0 \le p \le k$:

$$\begin{aligned}\sum_{p=0}^{k} \mathcal{X}^{p*} \otimes M^{k-p} &= \sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} + \sum_{p=0}^{k}\sum_{l=0}^{p-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{p-2-l*} \otimes M^{k-p} \otimes I \\ &= \sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} + \sum_{l=0}^{k-2}\sum_{p=l+2}^{k} \mathcal{Y}^{l*} \otimes \mathcal{X}^{p-2-l*} \otimes M^{k-p} \otimes I \\ &= \sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} + \sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \left(\sum_{p=0}^{k-l-2} \mathcal{X}^{p*} \otimes M^{k-l-2-p}\right) \otimes I\end{aligned} \quad . \tag{3.15}$$

Using now (2.14), the second terms of the above equation vanishes except for $k-l-2 = 0$ where it is equal to one. Since the above quantity is also zero for $k \ge 1$, we obtain therefore, for $k \ge 2$:

$$0 = \sum_{p=0}^{k} \mathcal{X}^{p*} \otimes M^{k-p} = \sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} + \mathcal{Y}^{k-2*} \otimes I$$

which is the result (3.14). ◻

Identity (3.14) is a recursive formula for the tensors $(M^k)_{k\in\mathbb{N}}$. This allows to obtain the following asymptotic estimates.

COROLLARY 3.8. *The tensors* M^k *satisfy the following asymptotics as* $\eta \to 0$*:*

$$M^0 \sim \text{Cap}(\partial T)\eta^{d-2} \tag{3.16}$$

$$M^2 = -I + o(\eta^{d-2}) \tag{3.17}$$

$$M^{2k} = o(\eta^{d-2}) \text{ for any } k \ge 2. \tag{3.18}$$

Proof. The first asymptotic is already known. For $k = 1$, (3.14) reads

$$M^2 = (\mathcal{Y}^{0*})^{-1}(-\mathcal{Y}^{0*} \otimes I - \mathcal{Y}^{2*} \otimes M^0) = -I + (M^0)^2\mathcal{Y}^{2*}.$$

Since $M^0 = O(\eta^{d-2})$ and $\mathcal{Y}^{2*} = o(\eta^{2-d})$, we obtain (3.17).

Then for $k \ge 2$, we rewrite (3.14) as

$$\begin{aligned}M^{2k} &= -(\mathcal{Y}^{0*})^{-1}\left(\mathcal{Y}^{2k-2*} \otimes I + \sum_{p=1}^{k} \mathcal{Y}^{2p*} \otimes M^{2(k-p)}\right) \\ &= -M^0\left(\mathcal{Y}^{2k*} \otimes M^0 + \mathcal{Y}^{2k-2*} \otimes (M^2 + I) + \mathcal{Y}^{2k-4*} \otimes M^4 + \cdots + \mathcal{Y}^{2*} \otimes M^{2k-2}\right).\end{aligned}$$

Assuming the results holds till the rank $k-1$, we see that all the terms in the parenthesis are of order $o(1)$. Therefore, (3.18) follows by induction, since $M^0 = O(\eta^{d-2})$. ◻

Remark 3.9. We now have the full picture of how (2.9) behaves in the low volume fraction limit. Indeed, we have obtained, as $\eta \to 0$

$$\varepsilon^{-2}M^0 \sim \eta^{d-2}\varepsilon^{-2}\text{Cap}(\partial T) \tag{3.19}$$

$$\varepsilon^0 M^2 \to -I \tag{3.20}$$

$$\varepsilon^{2k-2}M^{2k} = o(\varepsilon^{2k-2}\eta^{d-2}) \text{ for } k \geq 2. \tag{3.21}$$

Therefore we obtain the coefficient-wise convergence of the infinite order homogenized equation (2.9) to the three classical limiting equations depending on how η compares with the critical scaling $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$:

- if $\eta \gg \varepsilon^{2/(d-2)}$, then the zero-th order term remains dominant and the limiting equation for $\varepsilon^{-2}\eta^{d-2}u_\varepsilon$ is the zero-th order model

$$\begin{cases} \text{Cap}(\partial T)u^* = f \text{ in } D \\ \qquad u^* \text{ is } D\text{–periodic,} \end{cases} \tag{3.22}$$

  which is the scalar analogue of the Darcy equation (1.10);
- if $\eta = c\varepsilon^{2/(d-2)}$ for some constant $c > 0$, then $\varepsilon^{-2}M^0$ converges to $c\text{Cap}(\partial T)$ and (2.9) converges coefficient-wisely to the Poisson equation with "strange term"

$$\begin{cases} -\Delta u^* + c\text{Cap}(\partial T)u^* = f \text{ in } D \\ \qquad u^* \text{ is } D\text{–periodic.} \end{cases} \tag{3.23}$$

  This is the analogue of the Brinkman regime (1.11).
- Finally, if $\eta = o(\varepsilon^{2/(d-2)})$, then $\varepsilon^{-2}M^0 \to 0$, $\varepsilon^0 M^2 \to -I$ and $\varepsilon^{2k-2}M^{2k} \to 0$ for $k \geq 2$. We obtain therefore the Poisson equation in the homogeneous domain D as the limit model:

$$\begin{cases} -\Delta u^* = f \text{ in } D, \\ \quad u^* \text{ is } D\text{–periodic,} \end{cases} \tag{3.24}$$

  which is the analogue of the unperturbed Stokes regime (1.12).

3.3. Low volume fraction asymptotics of the truncated higher order homogenized equation. We finally terminate this section by showing that the homogenized model (2.10) of finite order $2K+2$ has the same asymptotic behavior as (2.9) in the low-volume fraction regime $\eta \to 0$.

According to Definition 2.8, it is sufficient to examine the asymptotic of the coefficient $\mathbb{D}_K^{2K+2}$ only, since $\mathbb{D}_K^k = M^k$ for $0 \leq k \leq 2K+1$. From (2.20), this requires to estimate the tensor $N^K(y)$ defined in (2.16). This can be achieved by conveniently rewriting $N^K(y)$ in terms of the tensors $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$.

PROPOSITION 3.10. *For any $k \geq 0$, the tensor $N^k(y)$ reads in terms of $\mathcal{Y}^k(y)$ as follows:*

$$\begin{aligned} N^k(y) &= \sum_{p=0}^{k} \mathcal{Y}^p(y) \otimes M^{k-p} + \mathcal{Y}^{k-2}(y) \otimes I \\ &= \mathcal{Y}^k \otimes M^0 + \mathcal{Y}^{k-1} \otimes M^1 + \mathcal{Y}^{k-2} \otimes (M^2 + I) + \mathcal{Y}^{k-3} \otimes M^3 + \cdots + \mathcal{Y}^0 \otimes M^k \end{aligned} \tag{3.25}$$

where $\mathcal{Y}^{k-2} := 0$ for $0 \leq k \leq 1$ by convention.

Proof. The proof is identical to that of Proposition 3.7: it suffices to replace $\mathcal{X}^{k-p}(y)$ with the formula given by (3.11) and to simplify the Cauchy product by using (2.14). ∎

Remark 3.11. It is visible that the identity (3.14) can also be obtained by computing the average of (3.25) and by using (2.17).

The estimates of Corollary 3.8 finally allow to prove that the truncated homogenized equation (2.10) of order $2K+2$ has the same limiting behavior as the infinite order homogenized equation (2.9) as $\eta \to 0$.

PROPOSITION 3.12. *We have the following asymptotics for the tensor* $\mathbb{D}_K^{2K+2}$ *as* $\eta \to 0$*:*

$$\mathbb{D}_0^2 = -I + O(\eta^{d-2}) \tag{3.26}$$

$$\mathbb{D}_K^{2K+2} = O(\eta^{d-2}) \text{ for } K \geq 1. \tag{3.27}$$

In particular, for any $K \in \mathbb{N}$*, the coefficients* $(\mathbb{D}_K^k)_{0\leq k\leq 2K+2}$ *of the higher homogenized equation* (2.10) *of order* $2K+2$ *satisfy the same asymptotics as the tensors* (M^k) *as* $\eta \to 0$*:*

$$\mathbb{D}_K^0 \sim \mathrm{Cap}(\partial T)\eta^{d-2}$$
$$\mathbb{D}_K^2 = -I + O(\eta^{d-2})$$
$$\mathbb{D}_K^{2k} = O(\eta^{d-2}) \text{ for any } k \geq 2.$$

Proof. Case $K = 0$: we have

$$\begin{aligned}\mathbb{D}_0^2 &= \left(-|M^0|^2 \int_Y |\mathcal{Y}^0|^2 \mathrm{d}y\right) I \\ &= \left(-\left(|\mathcal{Y}^{0*}|^2(1-\eta^d|T|) + ||\mathcal{Y}^0 - \mathcal{Y}^{0*}||^2_{L^2(P\setminus(\eta T))}\right)|M^0|^2\right) I \\ &= (-1 + O(\eta^{d-2}))I\end{aligned}$$

where the last estimate is a consequence of (3.6).
Case $K \geq 1$: we have

$$|\mathbb{D}^{2K+2}| = \left|\int_Y N^K \otimes N^K \otimes I \mathrm{d}y\right| \leq C_K ||N^K||^2_{L^2(P\setminus(\eta T))}$$

for a constant $C_K > 0$ which depends only on K. Since N^K is of average zero for $K \geq 1$, we can rewrite (3.25) as

$$\begin{aligned}N^K &= \sum_{p=0}^{K} (\mathcal{Y}^p - \mathcal{Y}^{p*}) \otimes M^{K-p} + (\mathcal{Y}^{K-2} - \mathcal{Y}^{K-2*}) \otimes I \\ &= (\mathcal{Y}^K - \mathcal{Y}^{K*}) \otimes M^0 + (\mathcal{Y}^{K-1} - \mathcal{Y}^{K-1*}) \otimes M^1 + (\mathcal{Y}^{K-2} - \mathcal{Y}^{K-2*}) \otimes (M^2 + I) \\ &\quad + (\mathcal{Y}^{K-3} - \mathcal{Y}^{K-3*}) \otimes M^3 + \cdots + (\mathcal{Y}^0 - \mathcal{Y}^{0*}) \otimes M^K.\end{aligned}.$$

Therefore by using again (3.6) and Corollary 3.8, we arrive at

$$||N^K||^2_{L^2(P\setminus(\eta T))} = O(\eta^{d-2})$$

which yields the result by using (2.20). ∎

Remark 3.13. We lost a bit in terms of speed of convergence: the high order coefficients $(\mathbb{D}_K^k)_{3\leq k\leq 2K+2}$ are only $O(\eta^{d-2})$ while $(M^k)_{k\geq 3}$ is of order $o(\eta^{d-2})$. However, since both quantities converge to zero due to our assumption $d \geq 3$, the conclusions of Remark 3.9 remain valid. Therefore the truncated model (2.10) converge as well to either of the three regimes (3.22)–(3.24) depending on whether η is greater, proportional to or lower than the critical size $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$.

4. The Stokes case. In this final section, we extend the asymptotic analysis of the previous section 3 to the Stokes system (1.1). We recall the homogenization setting reviewed in subsection 2.3, and our goal is to prove the coefficient-wise convergence of both the infinite order and the finite order effective models (1.2) and (1.3). We recall the Definitions 2.10 and 2.13 of their respective coefficients $(M^k)_{k\in\mathbb{N}}$ and $(\mathbb{D}^k)_{0\leq k\leq 2K+2}$.

The asymptotics of these coefficient tensors are obtained in Corollary 4.6 and Proposition 4.10. The proof follow the lines of section 3; the key ingredient is the introduction of matrix and vector valued cell tensors $(\mathcal{Y}^k(y),\boldsymbol{\omega}^k(y))_{k\in\mathbb{N}}$ with controlled growth, which generalize the family of scalar valued tensors $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$ introduced in subsection 3.1.

4.1. Cell tensors $(\mathcal{Y}^k(y),\boldsymbol{\omega}^k(y))_{k\in\mathbb{N}}$ of controlled growth. Recall the hierarchy of corrector systems (2.21)–(2.23) defining the cell tensors $(\boldsymbol{\mathcal{X}}_j^k(y),\alpha_j^k(y))_{k\in\mathbb{N}}$. We define the cell tensors $(\boldsymbol{\mathcal{Y}}_j^k(y),\omega_j^k(y))_{k\in\mathbb{N}}$ by an analogous recurrence.

DEFINITION 4.1. *For any $1\leq j\leq d$, we define a family of vector valued tensors $(\boldsymbol{\mathcal{Y}}_j^k(y))$ and scalar valued tensors $(\omega_j^k(y))_{k\in\mathbb{N}}$ as the unique solutions in $H^1_{\text{per}}(Y,\mathbb{R}^d)\times L^2(Y)$ to the following recursive systems:*

$$(4.1)\quad \begin{cases}-\Delta\boldsymbol{\mathcal{Y}}_j^0+\nabla\omega_j^0=\boldsymbol{e}_j \text{ in } Y,\\ \operatorname{div}(\boldsymbol{\mathcal{Y}}_j^0)=0 \text{ in } Y,\end{cases}$$

$$(4.2)\quad \begin{cases}-\Delta\boldsymbol{\mathcal{Y}}_j^1+\nabla\omega_j^1=(2\partial_l\boldsymbol{\mathcal{Y}}_j^0-\omega_j^0\boldsymbol{e}_l)\otimes e_l \text{ in } Y,\\ \operatorname{div}(\boldsymbol{\mathcal{Y}}_j^1)=-(\boldsymbol{\mathcal{Y}}_j^0-\langle\boldsymbol{\mathcal{Y}}_j^0\rangle)\cdot\boldsymbol{e}_l\otimes e_l \text{ in } Y,\end{cases}$$

$$(4.3)\quad \begin{cases}-\Delta\boldsymbol{\mathcal{Y}}_j^{k+2}+\nabla\omega_j^{k+2}=(2\partial_l\boldsymbol{\mathcal{Y}}_j^{k+1}-\omega_j^{k+1}\boldsymbol{e}_l)\otimes e_l+(\boldsymbol{\mathcal{Y}}_j^k-\langle\boldsymbol{\mathcal{Y}}_j^k\rangle)\otimes I, \text{ in } Y\\ \operatorname{div}(\boldsymbol{\mathcal{Y}}_j^{k+2})=-(\boldsymbol{\mathcal{Y}}_j^{k+1}-\langle\boldsymbol{\mathcal{Y}}_j^{k+1}\rangle)\cdot\boldsymbol{e}_l\otimes e_l \text{ in } Y,\end{cases}$$

supplemented with the following boundary conditions:

$$(4.4)\quad \begin{cases}\displaystyle\int_Y\omega_j^k\mathrm{d}y=0\\ \boldsymbol{\mathcal{Y}}_j^k=0 \text{ on } \partial(\eta T)\\ (\boldsymbol{\mathcal{Y}}_j^k,\omega_j^k) \text{ is } P\text{–periodic}\end{cases}\qquad\forall k\geq 0.$$

It is immediate to see that $(\boldsymbol{\mathcal{Y}}_j^k(y),\omega_j^k(y))$ and $(\boldsymbol{\mathcal{X}}_j^k(y),\alpha_j^k(y))$ coincide for $k=0,1$. In what follows, we also set $(\boldsymbol{\mathcal{Y}}^{-1}(y),\omega^{-1}(y))=(\boldsymbol{\mathcal{X}}_j^{-1}(y),\alpha_j^{-1}(y))=0$ by convention, so that (4.3) becomes valid for $k=-1$.

Our goal next is to obtain controlled estimates for $(\boldsymbol{\mathcal{Y}}_j^k(y),\omega_j^k(y))$ that are similar to those obtained in Proposition 3.3 in the Laplace case. We rely on the following result which allows to estimate the pressure term.

LEMMA 4.2. *Consider $\boldsymbol{h}\in L^2(P\backslash(\eta T),\mathbb{R}^d)$ and $g\in L^2(P\backslash(\eta T))$ a function satisfying $\int_{P\backslash(\eta T)}g\mathrm{d}x=0$. Let $(\boldsymbol{v},\phi)\in H^1(P\backslash(\eta T),\mathbb{R}^d)\times L^2(P\backslash(\eta T))$ be the unique* *solution to the following Stokes system:*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \phi = \boldsymbol{h} \text{ in } P\backslash(\eta T) \\ \operatorname{div}(\boldsymbol{v}) = g \text{ in } P\backslash(\eta T) \\ \displaystyle\int_{P\backslash(\eta T)} \phi \mathrm{d}x = 0 \\ \boldsymbol{v} = 0 \text{ on } \partial(\eta T) \\ \boldsymbol{v} \text{ is } P\text{–periodic.} \end{cases} \tag{4.5}$$

There exists a constant $C > 0$ independent of $(\boldsymbol{v}, \phi)$, η, $\boldsymbol{h}$ and g such that

$$||\nabla \boldsymbol{v}||_{L^2(P\backslash(\eta T),\mathbb{R}^{d\times d})} + ||\phi||_{L^2(P\backslash(\eta T))} \leq C(||\boldsymbol{h} - \langle \boldsymbol{h}\rangle||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} + \eta^{1-d/2}|\langle \boldsymbol{h}\rangle| + ||g||_{L^2(P\backslash(\eta T))}). \tag{4.6}$$

Proof. (4.6) is obtained by rescaling the estimates of Lemma 5.3 of [13] from the growing domain $\eta^{-1}P\backslash T$ to the perforated cell $P\backslash(\eta T)$. ∎

Using this lemma yields the fact that $(\boldsymbol{\mathcal{Y}}_j^k(y), \omega^k(y))$ has indeed a magnitude controlled with respect to k.

PROPOSITION 4.3. *For any $k \geq 0$ and $1 \leq j \leq d$, there exists a constant $C_k > 0$ independent of η such that*

$$\|\nabla \boldsymbol{\mathcal{Y}}_j^k\|_{L^2(P\backslash(\eta T),\mathbb{R}^d)} + \|\omega_j^k\|_{L^2(P\backslash(\eta T))} \leq C_k \eta^{1-d/2}, \tag{4.7}$$

$$||\boldsymbol{\mathcal{Y}}_j^k - \langle \boldsymbol{\mathcal{Y}}_j^k\rangle||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} \leq C_k \eta^{1-d/2}. \tag{4.8}$$

Proof. Again, we proceed by induction. Note that it is enough to prove (4.7) since (4.8) follows from the Poincaré-Wirtinger inequality (3.4).

Case $k = 0$: applying Lemma 4.2 to (4.1) yields

$$||\nabla \mathcal{Y}_j^0||_{L^2(P\backslash(\eta T),\mathbb{R}^{d\times d})} + ||\omega_j^0||_{L^2(P\backslash(\eta T))} \leq C\eta^{1-d/2}$$

since $\boldsymbol{e}_j = |\langle \boldsymbol{e}_j\rangle| = 1 - \eta^d|T|$.

Case $k = 1$: since the right-hand side of (4.2) is of zero average, applying Lemma 4.2 yields

$$\begin{aligned} &||\nabla \boldsymbol{\mathcal{Y}}_j^1||_{L^2(P\backslash(\eta T),\mathbb{R}^{d\times d})} + ||\omega_j^1||_{L^2(P\backslash(\eta T))} \\ &\quad \leq C(2||\nabla \boldsymbol{\mathcal{Y}}_j^0||_{L^2(P\backslash(\eta T),\mathbb{R}^{d\times d})} + ||\omega_j^0||_{L^2(P\backslash(\eta T))} + ||\boldsymbol{\mathcal{Y}}_j^0 - \langle \boldsymbol{\mathcal{Y}}_j^0\rangle||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}). \\ &\quad \leq C_1 \eta^{1-d/2}. \end{aligned}$$

Case $k+2$ *with* $k \geq 0$: similarly, the right-hand side of (4.3) is of average zero. Therefore, assuming (4.7) and (4.8) holds till rank $k+1$ with $k \geq 0$, applying Lemma 4.2 yields

$$\begin{aligned} &||\nabla \boldsymbol{\mathcal{Y}}_j^{k+2}||_{L^2(P\backslash(\eta T),\mathbb{R}^{d\times d})} + ||\omega_j^{k+2}||_{L^2(P\backslash(\eta T))} \\ &\quad \leq C'(2||\nabla \boldsymbol{\mathcal{Y}}_j^{k+1}||_{L^2(P\backslash(\eta T),\mathbb{R}^{d\times d})} + ||\omega_j^{k+1}||_{L^2(P\backslash(\eta T))} + ||\boldsymbol{\mathcal{Y}}_j^k - \langle \boldsymbol{\mathcal{Y}}_j^k\rangle||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} \\ &\qquad + ||\boldsymbol{\mathcal{Y}}_j^{k+1} - \langle \boldsymbol{\mathcal{Y}}_j^{k+1}\rangle||_{L^2(P\backslash(\eta T),\mathbb{R}^d)}) \\ &\quad \leq C_{k+2}\eta^{1-d/2}. \end{aligned}$$

In the sequel, we consider the matrix-valued tensors $\mathcal{Y}^k$ and the vector-valued tensors $\boldsymbol{\omega}^k$ obtained by gathering the vector valued tensors $(\boldsymbol{\mathcal{Y}}_j^k(y))_{1\leq j\leq d}$ as columns and the scalar valued components $(\omega_j^k(y))_{1\leq j\leq d}$ as coordinates:

$$(\mathcal{Y}_{ij}^k(y))_{1\leq i,j\leq d} := \begin{bmatrix}\boldsymbol{\mathcal{Y}}_1^k(y) & \dots & \boldsymbol{\mathcal{Y}}_d^k(y)\end{bmatrix}_{ij}, \ \forall y\in Y, \quad \forall k\geq 0.$$

$$\boldsymbol{\omega}^k(y) := (\omega_j^k(y))_{1\leq j\leq d}, \ \forall y\in Y, \quad \forall k\geq 0.$$

As before, we introduce the mean matrix tensor $\mathcal{Y}^{k*}$ defined by

$$\mathcal{Y}^{k*} := \int_{P\backslash(\eta T)} \mathcal{Y}^k(y)\mathrm{d}y.$$

By using arguments similar to those of the proof of Proposition 3.4, we can precise the convergence of the mean $\mathcal{Y}^{k*}$. For any $1\leq j\leq d$, let us consider the unique solution $(\boldsymbol{\Psi}_j,\sigma_j)$ to the exterior Stokes problem

$$\begin{cases} -\Delta\boldsymbol{\Psi}_j + \nabla\sigma_j = 0 \text{ in } \mathbb{R}^d\backslash T\\ \operatorname{div}(\boldsymbol{\Psi}_j) = 0 \text{ in } \mathbb{R}^d\backslash T\\ \Psi_j = 0 \text{ on } \partial T\\ \boldsymbol{\Psi}_j \to \boldsymbol{e}_j \text{ at } \infty\\ \sigma_j\in L^2(\mathbb{R}^d\backslash T). \end{cases} \tag{4.9}$$

The existence and uniqueness of a solution to (4.9) is standard by using layer potential theory [20, 19] or variational arguments in homogeneous Sobolev spaces [14, 25] (also called Deny-Lions or Beppo-Levi spaces). We denote by $F := (F_{ij})_{1\leq i,j\leq d}$ the matrix collecting the drag force components:

$$F_{ij} := \int_{\mathbb{R}^d\backslash T} \nabla\boldsymbol{\Psi}_i : \nabla\boldsymbol{\Psi}_j\mathrm{d}x = -\int_{\partial T} \boldsymbol{e}_j\cdot(\nabla\boldsymbol{\Psi}_i - \sigma_i I)\cdot\boldsymbol{n}\mathrm{d}s, \tag{4.10}$$

where the normal $\boldsymbol{n}$ is pointing *inward* T. The matrix F is the analogue of the capacity $\mathrm{Cap}(\partial T)$ in the context of the Stokes equation. The following result holds.

PROPOSITION 4.4. *The mean matrix valued tensor $\mathcal{Y}^{k*}$ satisfy the following asymptotic convergences as $\eta\to 0$:*

$$\mathcal{Y}^{0*} \sim \eta^{2-d}F^{-1} \text{ and } \mathcal{Y}^{k*} = o(\eta^{2-d}) \text{ for } k\geq 1. \tag{4.11}$$

Proof. The convergence for $\mathcal{Y}^{0*}$ is a classical result and a proof can be found in [3]. The second estimate result from the fact that the right-hand sides of (4.2) and (4.3) are of zero average. The proof is obtained by repeating arguments similar to those of Proposition 3.4, see also the proof of Proposition 5.4 in [13]. ∎

The pairs $(\mathcal{Y}^k(y),\boldsymbol{\omega}^k(y))$ and $(\boldsymbol{\mathcal{X}}^k(y),\boldsymbol{\alpha}^k(y))$ are related by Cauchy-product identities analogous to (3.11).

PROPOSITION 4.5. *The matrix valued tensors* $(\mathcal{Y}^k(y))_{k\in\mathbb{N}}$ *and* $(\mathcal{X}^k(y))_{k\in\mathbb{N}}$ *are related through the following identities:*

$$\begin{cases} \mathcal{Y}^0(y) = \mathcal{X}^0(y), \quad \boldsymbol{\omega}^0(y) = \boldsymbol{\alpha}^0(y), \\ \mathcal{Y}^1(y) = \mathcal{X}^1(y), \quad \boldsymbol{\omega}^1(y) = \boldsymbol{\alpha}^1(y), \\ \mathcal{Y}^k(y) = \mathcal{X}^k(y) - \sum_{l=0}^{k-2} \mathcal{Y}^l(y) \otimes \mathcal{X}^{k-l-2*} \otimes I, \\ \boldsymbol{\omega}^k(y) = \boldsymbol{\alpha}^k(y) - \sum_{l=0}^{k-2} (-1)^{k-l} \mathcal{X}^{k-l-2*} \cdot \boldsymbol{\omega}^l(y) \otimes I, \end{cases} \quad y \in P\backslash(\eta T),\ k \geq 2. \tag{4.12}$$

In particular, we have the formula

$$\mathcal{X}^{k*}(y) = \mathcal{Y}^{k*} + \sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{k-l-2*} \otimes I \tag{4.13}$$

remembering Proposition 2.11 whereby $\mathcal{X}^{k*}$ *is symmetric when* k *is even and is antisymmetric when* k *is odd.*

Proof. The identities for $(\mathcal{Y}^k(y), \boldsymbol{\omega}^k(y)) = (\mathcal{X}^k(y), \boldsymbol{\alpha}^k(y))$, with $k = 1, 2$, are obvious from the definitions (4.1) and (4.2). By induction, (4.12) is obtained as soon as we prove

$$\begin{aligned} \boldsymbol{\mathcal{Y}}_j^{k+2} &= \boldsymbol{\mathcal{X}}_j^{k+2} - \sum_{l=0}^{k} \mathcal{Y}^l(y) \cdot \langle \boldsymbol{\mathcal{X}}_j^{k-l} \rangle \otimes I = \boldsymbol{\mathcal{X}}_j^{k+2} - \sum_{l=0}^{k} (\mathcal{X}_{ij}^{k-l*} \otimes I) \boldsymbol{\mathcal{Y}}_i^l(y), \\ \omega_j^{k+2} &= \alpha_j^{k+2} - \sum_{l=0}^{k} \boldsymbol{\omega}^l(y) \cdot \langle \boldsymbol{\mathcal{X}}_j^{k-l} \rangle \otimes I = \alpha_j^{k+2} - \sum_{l=0}^{k} (\mathcal{X}_{ij}^{k-l*} \otimes I) \omega_i^l(y), \end{aligned} \tag{4.14}$$

for $k \geq 0$, assuming these identities hold for lower values of k (remind the symmetry and antisymmetry properties of Proposition 2.11). Note that we use the implicit summation convention over the repeated index $1 \leq i \leq d$. Let $(\boldsymbol{v}, \phi)$ be the right-hand sides of the above equations. We compute

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla \phi &= (-\Delta \boldsymbol{\mathcal{X}}_j^{k+2} + \nabla \alpha_j^{k+2}) - \sum_{l=0}^{k} (\mathcal{X}_{ij}^{k-l*} \otimes I)(-\Delta \boldsymbol{\mathcal{Y}}_i^l + \nabla \omega_i^l) \\ &= (2\partial_l \boldsymbol{\mathcal{X}}_j^{k+1} - \alpha_j^{k+1} \boldsymbol{e}_l) \otimes e_l + \boldsymbol{\mathcal{X}}_j^k \otimes I - (\mathcal{X}_{ij}^{k*} \otimes I) \boldsymbol{e}_i \\ &\quad - (\mathcal{X}_{ij}^{k-1*} \otimes I)(2\partial_m \boldsymbol{\mathcal{X}}_i^0 - \alpha_i^0 \boldsymbol{e}_m) \otimes e_m \\ &\quad - \sum_{l=2}^{k} (\mathcal{X}_{ij}^{k-l*} \otimes I)[(2\partial_m \boldsymbol{\mathcal{Y}}_i^{l-1} - \omega_i^{l-1}) \otimes e_m + (\boldsymbol{\mathcal{Y}}_i^{l-2} - \langle \boldsymbol{\mathcal{Y}}_i^{l-2} \rangle) \otimes I] \\ &= (2\partial_m \boldsymbol{\mathcal{Y}}_j^{k+1} - \omega_j^{k+1} \boldsymbol{e}_m) \otimes e_m + (\boldsymbol{\mathcal{Y}}_j^k - \langle \boldsymbol{\mathcal{Y}}_j^k \rangle). \end{aligned}$$

In the last equality, we used the assumption that (4.14) holds when k is replaced by $k-1$ or $k-2$. By uniqueness of the defining problem for $(\boldsymbol{\mathcal{Y}}_j^{k+1}(y), \omega_j^{k+1}(y))$, we obtain that (4.14) holds. ∎

4.2. Low-volume fraction asymptotic of the infinite order homogenized Stokes system. We now obtain the asymptotic of the coefficients M^k by relating them to the mean tensors $\mathcal{Y}^{k*}$. Recall that the recursive definition (2.26) of the tensors M^k states that

$$\sum_{p=0}^{k} \mathcal{X}^{k-p*} \otimes M^p = \begin{cases} I, & \text{if } k=0, \\ 0, & \text{if } k \geq 1. \end{cases}$$

Using this result and repeating the proof of Proposition 3.7, we obtain that the identity (3.14) remains valid in the present vectorial context:

$$\sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} = -\mathcal{Y}^{k-2*} \otimes I \text{ for any } k \geq 2. \tag{4.15}$$

This identity implies the following results.

COROLLARY 4.6. *Let M^k be the tensors defined by (2.26) and $F \equiv (F_{ij})_{1\leq i,j^d}$ the drag force matrix defined by (4.10). Then as $\eta \to 0$,*

$$M^0 \sim \eta^{d-2} F \tag{4.16}$$

$$M^1 = o(\eta^{d-2}) \tag{4.17}$$

$$M^2 = -I + o(\eta^{d-2}) \tag{4.18}$$

$$M^k = o(\eta^{d-2}) \textit{ for any } k > 2. \tag{4.19}$$

Proof. The proof is identical to that of Corollary 3.8, except that some extra care must be taken because of non-commuting matrix products and non-zero odd order tensors. The result for $M^0 = (\mathcal{X}^{0*})^{-1}$ is a restatement of the first asymptotic convergence of (4.11). For $k = 1$, we have by definition

$$M^1 = -(\mathcal{Y}^{0*})^{-1} \otimes \mathcal{Y}^{1*} \otimes M^0 = -M^0 \otimes \mathcal{Y}^{1*} \otimes M^0.$$

Since $\mathcal{Y}^{1*} = o(\eta^{2-d})$ and $M^0 = O(\eta^{d-2})$, we obtain $M^1 = o(\eta^{d-2})$. For $k = 2$, the identity (4.15) yields

$$M^2 + I = -M^0 \otimes \left[\mathcal{Y}^{1*} \otimes M^1 + \mathcal{Y}^{2*} \otimes M^0\right]$$

which is also of order $o(\eta^{d-2})$. Finally, for $k > 2$, we rewrite (4.15) as

$$M^k = -M^0 \left(\mathcal{Y}^{k*} \otimes M^0 + \mathcal{Y}^{k-1*} \otimes M^1 + \mathcal{Y}^{k-2*} \otimes (M^2 + I) + \cdots + \mathcal{Y}^{1*} \otimes M^{k-1}\right).$$

By induction, we deduce from the above relation that $M^k = o(\eta^{d-2})$ for all $k \geq 2$, which completes the proof. ∎

Remark 4.7. We recall that there is a slight abuse of notation in the notation I featured in (4.18) because I is here the second-order matrix-valued defined by (2.8) and not the scalar valued tensor I of the other equations.

Remark 4.8. We have therefore obtained the first main result of the paper, i.e. the coefficient-wise convergence of the infinite order homogenized Stokes system (1.2) towards either the Darcy, Brinkman or Stokes regimes (1.10)–(1.12) for the various scalings of η when compared to the critical size $\varepsilon^{2/(d-2)}$. Indeed, the coefficients of (1.2) satisfy as $\eta \to 0$:

$$\varepsilon^{-2} M^0 \sim \eta^{d-2} \varepsilon^{-2} F \tag{4.20}$$

$$\varepsilon^{-1}M^1 = o(\varepsilon^{-1}\eta^{d-2}) \tag{4.21}$$

$$\varepsilon^{0}M^2 \to -I \tag{4.22}$$

$$\varepsilon^{k}M^k = o(\varepsilon^{k}\eta^{d-2}) \text{ for } k > 2. \tag{4.23}$$

Reasonning as in Remark 3.9 we obtain the coefficient-wise convergence of (1.2) towards the three regimes as $\varepsilon \to 0$ and $\eta \to 0$ for the three possible scalings of η. Note that we also obtain the coefficinet-wise convergence of the infinite order model (1.2) towards the homogeneous Stokes system (1.12) if ε is fixed while $\eta \to 0$.

4.3. Low-volume fraction asymptotic of the truncated homogenized Stokes system of order $2K+2$. We now come to the final result concerned with the coefficient-wise limit of the truncated homogenized model (1.3), or in other words with the limit of the tensors $\mathbb{D}^k_K$ as $\eta \to 0$. Similarly as in subsection 3.3 and by reading the definition (2.32), we need to find the asymptotic limits of the tensors $N^k(y)$ and $\boldsymbol{\beta}^k(y)$ of Definition 2.12. Using (4.13), we can represent them using the controlled tensors $\mathcal{Y}^k$ and $\boldsymbol{\omega}^k$, as shown in the next result.

PROPOSITION 4.9. *For $k \geq 1$ and with the convention $\mathcal{Y}^{-2*} = \mathcal{Y}^{-1*} = 0$ and $\boldsymbol{\omega}^{-2} = \boldsymbol{\omega}^{-1} = 0$, the following identities hold:*

$$N^k(y) = \sum_{p=0}^{k} \mathcal{Y}^{k-p}(y) \otimes M^p + \mathcal{Y}^{k-2}(y) \otimes I, \qquad y \in Y, \tag{4.24}$$

$$\boldsymbol{\beta}^k(y) = \sum_{p=0}^{k} (-1)^p M^p \cdot \boldsymbol{\omega}^{k-p}(y) + \boldsymbol{\omega}^{k-2}(y) \otimes I, \qquad y \in Y. \tag{4.25}$$

Proof. Both identities are proved following the arguments and computations of Proposition 3.7. We only provide the proof for the second identity. We left multiply (4.12) by $(-1)^p M^p$ and sum over $0 \leq p \leq k$:

$$\begin{aligned}
&\sum_{p=0}^{k}(-1)^p M^p \cdot \boldsymbol{\alpha}^{k-p} \\
&\quad = \sum_{p=0}^{k}(-1)^p M^p \cdot \boldsymbol{\omega}^{k-p} + \sum_{p=0}^{k}\sum_{l=0}^{k-p-2}(-1)^{k-l-2} M^p \otimes \mathcal{X}^{k-p-l-2} \cdot \boldsymbol{\omega}^l \otimes I \\
&\quad = \sum_{p=0}^{k}(-1)^p M^p \cdot \boldsymbol{\omega}^{k-p} + \sum_{l=0}^{k-2}(-1)^{k-l-2}\left[\sum_{p=0}^{k-l-2} M^p \otimes \mathcal{X}^{k-p-l-2}\right] \cdot \boldsymbol{\omega}^l \otimes I,
\end{aligned}$$

In view of (4.13), the summation in the brackets vanishes unless $l = k-2$ when it sums to I. This leads to

$$\boldsymbol{\beta}^k = \sum_{p=0}^{k}(-1)^p M^p \cdot \boldsymbol{\alpha}^{k-p} = \sum_{p=0}^{k}(-1)^p M^p \cdot \boldsymbol{\omega}^{k-p} + \boldsymbol{\omega}^{k-2} \otimes I \tag{4.26}$$

which is the desired result. □

With those formulas, we finally obtain the low volume fraction asymptotics of the tensors $(\mathbb{D}^k_K)_{0\leq k\leq 2K+2}$ of the high order truncated homogenized Stokes system (1.3). The analysis requires slightly more work than in the scalar case due to the presence of the tensors $(\mathbb{A}^k_K)_{K+1\leq k\leq 2K+1}$ induced by the divergence constraint.

Proposition 4.10. *We have the following asymptotics for the tensors* $\mathbb{D}_K^{2K+2}$ *and* $(\mathbb{A}_K^k)_{K+1\leq k\leq 2K+1}$ *defined by* (2.32) *as* $\eta\to 0$*:*

$$\mathbb{D}_0^2 = -I + O(\eta^{d-2}) \tag{4.27}$$

$$\mathbb{D}_K^{2K+2} = O(\eta^{d-2}) \text{ for } K\geq 1 \tag{4.28}$$

$$\mathbb{A}_K^k = O(\eta^{d-2}) \text{ for } K\geq 0 \text{ and } K+1\leq k\leq 2K+1. \tag{4.29}$$

Therefore, for any $K\in\mathbb{N}$*, the matrix-valued coefficient tensors* $(\mathbb{D}_K^k)_{0\leq k\leq 2K+2}$ *of the truncated homogenized Stokes system* (1.3) *satisfy the following convergences as* $\eta\to 0$*:*

$$\mathbb{D}_K^0 \sim \eta^{d-2}F, \tag{4.30}$$

$$\mathbb{D}_K^1 = O(\eta^{d-2}), \tag{4.31}$$

$$\mathbb{D}_K^2 = -I + O(\eta^{d-2}), \tag{4.32}$$

$$\mathbb{D}_K^k = O(\eta^{d-2}) \text{ for any } k>2. \tag{4.33}$$

Proof. 1. Asymptotic (4.27). By the definition (2.32) and by using (4.24), we have

$$\begin{aligned}\mathbb{D}_{ij}^2 &= -\int_Y \boldsymbol{N}_i^0\cdot\boldsymbol{N}_j^0\otimes I\mathrm{d}y = -M_{mi}^0M_{lj}^0\int_Y \boldsymbol{\mathcal{Y}}_m^0\cdot\boldsymbol{\mathcal{Y}}_l^0\otimes I\mathrm{d}y\\ &= -M_{mi}^0M_{lj}^0\left(\langle\boldsymbol{\mathcal{Y}}_m^0\rangle\cdot\langle\boldsymbol{\mathcal{Y}}_l^0\rangle(1-\eta^d|T|)+\int_Y(\boldsymbol{\mathcal{Y}}_m^0-\langle\boldsymbol{\mathcal{Y}}_m^0\rangle)\cdot(\boldsymbol{\mathcal{Y}}_l^0-\langle\boldsymbol{\mathcal{Y}}_l^0\rangle)\mathrm{d}y\right)\otimes I,\end{aligned}$$

with implicit summation over the repeated indices $1\leq l,m\leq d$. Then, we observe that $M_{mi}^0\langle\boldsymbol{\mathcal{Y}}_m^0\rangle = \mathcal{X}^{0*}M\boldsymbol{e}_i = \boldsymbol{e}_i$, and similarly $M_{lj}^0\langle\boldsymbol{\mathcal{Y}}_l^0\rangle = \boldsymbol{e}_j$; this implies

$$-M_{mi}^0M_{lj}^0\left(\langle\boldsymbol{\mathcal{Y}}_m^0\rangle\cdot\langle\boldsymbol{\mathcal{Y}}_l^0\rangle(1-\eta^d|T|)\right) = -\delta_{ij}I+O(\eta^d).$$

Finally, using (4.8), (4.16) and the Cauchy-Schwarz inequality allows to obtain

$$\begin{aligned}&-M_{mi}^0M_{lj}^0\left(\int_Y(\boldsymbol{\mathcal{Y}}_m^0-\langle\boldsymbol{\mathcal{Y}}_m^0\rangle)\cdot(\boldsymbol{\mathcal{Y}}_l^0-\langle\boldsymbol{\mathcal{Y}}_l^0\rangle)\mathrm{d}y\right)\otimes I\\ &\qquad\qquad = O(\eta^{d-2})O(\eta^{d-2})O(\eta^{2-d}) = O(\eta^{d-2})\end{aligned}$$

which implies (4.27).

2. Asympotic (4.28). We use (4.24) to rewrite, for any $k\geq 1$, $\boldsymbol{N}_i^k$ as

$$\begin{aligned}\boldsymbol{N}_i^k &= \sum_{p=0}^k \boldsymbol{\mathcal{Y}}_m^{k-p}\otimes M_{mi}^p + \boldsymbol{\mathcal{Y}}_i^{k-2}\otimes I\\ &= \sum_{p=0}^k(\boldsymbol{\mathcal{Y}}_m^{k-p}-\langle\boldsymbol{\mathcal{Y}}_m^{k-p}\rangle)\otimes M_{mi}^p + (\boldsymbol{\mathcal{Y}}_i^{k-2}-\langle\boldsymbol{\mathcal{Y}}_i^{k-2}\rangle)\otimes I\\ &= (\boldsymbol{\mathcal{Y}}_m^k-\langle\boldsymbol{\mathcal{Y}}_m^{k-1}\rangle)\otimes M_{mi}^0 + (\boldsymbol{\mathcal{Y}}_m^{k-1}-\langle\boldsymbol{\mathcal{Y}}_m^{k-1}\rangle)\otimes M_{mi}^1\\ &\quad+(\boldsymbol{\mathcal{Y}}_m^{k-2}-\langle\boldsymbol{\mathcal{Y}}_m^{k-2}\rangle)\otimes(M_{mi}^2+\delta_{mi}I)\\ &\quad+(\boldsymbol{\mathcal{Y}}_m^{k-3}-\langle\boldsymbol{\mathcal{Y}}_m^{k-3}\rangle)\otimes M_{mi}^3+\cdots+(\boldsymbol{\mathcal{Y}}_m^0-\langle\boldsymbol{\mathcal{Y}}_m^0\rangle)\otimes M_{mi}^k,\end{aligned}$$

where we used that $\langle \boldsymbol{N}_i^k \rangle = 0$ at the second equality. Therefore the result of Corollary 4.6 and the bound of (4.8) controlling $||\boldsymbol{\mathcal{Y}}_m^k - \langle \boldsymbol{\mathcal{Y}}_m^k \rangle||_{L^2(P\setminus(\eta T))}$ imply that

$$||\boldsymbol{N}_i^k||_{L^2(P\setminus(\eta T))} = O(\eta^{d/2-1}) \text{ for } k \geq 1.$$

Then (4.28) follows from the definition (2.32) and the Cauchy-Schwarz inequality.
3. Asymptotics (4.29). By integration by parts, the formula (2.33) for $\mathbb{A}_{K,ij}^k$ with $K+1 \leq k \leq 2K+1$ can be rewritten as

$$\mathbb{A}_{K,ij}^k = (-1)^K \int_Y \left(\beta_j^{k-K-1} \otimes \operatorname{div}(\boldsymbol{N}_i^{K+1}) + (-1)^k \beta_i^{k-K-1} \otimes \operatorname{div}(\boldsymbol{N}_j^{K+1})\right) \mathrm{d}y.$$

Therefore we need to control the L^2 norm of $\beta_j^k(y)$ for $0 \leq k \leq K$ and of $\operatorname{div}\boldsymbol{N}_i^k$ for any $k \geq 1$ and $1 \leq i, j \leq d$. Using (4.24) to compute the divergence, we obtain for any $k \geq 1$

$$\begin{aligned}
\operatorname{div}\boldsymbol{N}_i^k &= \sum_{p=0}^{k} \operatorname{div}\boldsymbol{\mathcal{Y}}_m^{k-p} \otimes M_{mi}^p + \operatorname{div}\boldsymbol{\mathcal{Y}}_i^{k-2} \otimes I \\
&= -\sum_{p=0}^{k} (\boldsymbol{\mathcal{Y}}_m^{k-p-1} - \langle \boldsymbol{\mathcal{Y}}_m^{k-p-1} \rangle) \cdot \mathbf{e}_l \otimes e_l \otimes M_{mi}^p - (\boldsymbol{\mathcal{Y}}_i^{k-3} - \boldsymbol{\mathcal{Y}}_i^{k-3*}) \cdot \boldsymbol{e}_l \otimes e_l \otimes I \\
&= [(\boldsymbol{\mathcal{Y}}_m^{k-1} - \langle \boldsymbol{\mathcal{Y}}_m^{k-1} \rangle) \otimes M_{mi}^0 + (\boldsymbol{\mathcal{Y}}_m^{k-2} - \langle \boldsymbol{\mathcal{Y}}_m^{k-2} \rangle) \otimes M_{mi}^1 \\
&\quad + (\boldsymbol{\mathcal{Y}}_m^{k-3} - \langle \boldsymbol{\mathcal{Y}}_m^{k-3} \rangle) \otimes (M_{mi}^2 + \delta_{mi} I) \\
&\quad + (\boldsymbol{\mathcal{Y}}_m^{k-4} - \langle \boldsymbol{\mathcal{Y}}_m^{k-4} \rangle) \otimes M_{mi}^3 + \cdots + (\boldsymbol{\mathcal{Y}}_m^{0} - \langle \boldsymbol{\mathcal{Y}}_m^{0} \rangle) \otimes M_{mi}^{k-1}] \cdot \boldsymbol{e}_l \otimes e_l,
\end{aligned}$$

still assuming the summation convention over the repeated index $1 \leq m \leq d$. By using the result of Corollary 4.6 and the bound (4.8), we obtain therefore that

$$||\operatorname{div}\boldsymbol{N}_i^K||_{L^2(P\setminus(\eta T))} = O(\eta^{d/2-1}) \text{ for any } K \in \mathbb{N}.$$

Similarly, (4.24) allows to rewrite β_j^k as

$$\begin{aligned}
\beta_j^k &= \sum_{p=0}^{k} \omega_m^{k-p} \otimes M_{mj}^p + \omega_j^{k-2} \otimes I \\
&= \omega_m^k \otimes M_{mj}^0 + \omega_m^{k-1} \otimes M_{mj}^1 + \omega_m^{k-2} \otimes (M_{mj}^2 + \delta_{mj} I) \\
&\quad + \omega_m^{k-3} \otimes M_{mj}^3 + \cdots + \omega_m^0 \otimes M_{mj}^k.
\end{aligned}$$

Therefore, the bound (4.7) controlling $||\omega_j^k||_{L^2(P\setminus(\eta T))}$ and Corollary 4.6 yield

$$||\beta_j^k||_{L^2(P\setminus(\eta T))} = O(\eta^{d/2-1}) \text{ for any } k \in \mathbb{N}.$$

Hence (4.29) follows by using the Cauchy-Schwarz inequality. ∎

The result of Proposition 4.10 implies that the conclusions of Remark 4.8 still hold for the truncated model (1.3), which converges therefore in the coefficent-wise sense towards either of the three models (1.10)–(1.12) depending on how the scaling η compares to the critical value $\eta_{\text{crit}} \sim \eta^{2/(d-2)}$ as claimed.

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