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FIRST-KIND BOUNDARY INTEGRAL EQUATIONS FOR THE DIRAC OPERATOR IN 3D LIPSCHITZ DOMAINS

ERICK SCHULZ AND RALF HIPTMAIR

Abstract. We develop novel first-kind boundary integral equations for Euclidean Dirac operators in 3D Lipschitz domains comprising square-integrable potentials and involving only weakly singular kernels. Generalized Gårding inequalities are derived and we establish that the obtained boundary integral operators are Fredholm of index zero. Their finite dimensional kernels are characterized and we show that their dimension is equal to the number of topological invariants of the domain's boundary, in other words to the sum of its Betti numbers. This is explained by the fundamental discovery that the associated bilinear forms agree with those induced by the 2D surface Dirac operators for $H^{-1/2}$ surface de Rham Hilbert complexes whose underlying inner-products are the non-local inner products defined through the classical single-layer boundary integral operators for the Laplacian. Decay conditions for well-posedness in natural energy spaces of the Dirac system in unbounded exterior domains are also presented.

Keywords. Dirac, Hodge–Dirac, potential representation, representation formula, jump relations, first-kind boundary integral equations, coercive boundary integral equations

1. Introduction

We develop first-kind boundary integral equations for the first-order linear system of partial differential equations

$$\begin{aligned} -\operatorname{div}\mathbf{U}_1 &= F_0,\\ \nabla U_0 + \mathbf{curl}\,\mathbf{U}_2 &= \mathbf{F}_1,\\ -\nabla U_3 + \mathbf{curl}\,\mathbf{U}_1 &= \mathbf{F}_2,\\ \operatorname{div}\mathbf{U}_2 &= F_3. \end{aligned} \tag{1.1}$$

We will consider both interior and exterior problems. Precisely, the equations (1.1) are either posed on a bounded domain $\Omega = \Omega^-$ having a Lipschitz boundary $\Gamma := \partial\Omega$, or on the unbounded complement $\Omega^+ := \mathbb{R}^3\backslash\overline{\Omega}$. In the later case, suitable decay conditions will be imposed.

The system of equations (1.1) will be written as

$$\mathsf{D}\vec{\mathbf{U}} = \vec{\mathbf{F}}, \tag{1.2}$$

where the Dirac operator

$$\mathsf{D} := \begin{pmatrix} 0 & -\operatorname{div} & \mathbf{0}^\top & 0\\ \nabla & \mathbf{0}_{3\times3} & \mathbf{curl} & \mathbf{0}\\ \mathbf{0} & \mathbf{curl} & \mathbf{0}_{3\times3} & -\nabla\\ 0 & \mathbf{0}^\top & \operatorname{div} & 0 \end{pmatrix} \tag{1.3}$$

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acts on vector fields $\vec{\mathbf{U}} = (U_0, \mathbf{U}_1, \mathbf{U}_2, U_3)$ with 8 components.

Remark 1.1. *The ordering of components of the vector fields is intentional. As hinted by the notation, we view 8-dimensional vectors as a grouping of two scalar fields U_0, U_3, and two 3-dimensional vector fields $\mathbf{U}_1$, $\mathbf{U}_2$. While the ordering of the fields U_0, $\mathbf{U}_1$, $\mathbf{U}_2$ and U_3 is at this level theoretically arbitrary, the choice made in (1.1) —which results in the particular structure shown in (1.3)— corresponds to the formulation in components of the Hodge-Dirac operator as it acts on the full graded exterior algebra of differential forms [6, 7, 23]. The same system also appears when calculating the action of the Dirac operator on the coefficients of Euclidean Clifford algebra-valued functions as presented in the literature of hypercomplex analysis [24, 27, 29].*

This particular ordering of the components reveals that the Dirac operator is related to the Fredholm Hilbert co-chain complex of unbounded densely-defined operators [1, 8]

$$L^2(\Omega) \xrightarrow{\nabla} \mathbf{L}^2(\Omega) \xrightarrow{\mathbf{curl}} \mathbf{L}^2(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)\,, \tag{1.4}$$

whose domain complex is the de Rham cochain complex

$$H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \tag{1.5}$$

that supplies natural function spaces to study the Dirac system of equations (1.1).

The Dirac operator

$$\mathsf{D} := \mathbf{d} + \boldsymbol{\delta} \tag{1.6}$$

is composed of two main ingredients: lower and upper diagonal operators

$$\mathbf{d} := \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ \nabla & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{curl} & \mathbf{0}_{3\times3} & \mathbf{0} \\ 0 & \mathbf{0}^\top & \mathrm{div} & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\delta} := \begin{pmatrix} 0 & -\mathrm{div} & \mathbf{0}^\top & 0 \\ \mathbf{0} & \mathbf{0}_{3\times3} & \mathbf{curl} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\nabla \\ 0 & \mathbf{0}^\top & \mathbf{0}^\top & 0 \end{pmatrix}, \tag{1.7}$$

called exterior derivative and coderivative, respectively.

The decomposition (1.6) emphasizes the geometric relevance of the Dirac operator for harmonic analysis and topology. Its square

$$\mathsf{D}^2 = \cancelto{0}{\mathbf{d}^2} + \mathbf{d}\,\boldsymbol{\delta} + \boldsymbol{\delta}\,\mathbf{d} + \cancelto{0}{\boldsymbol{\delta}^2} = \begin{pmatrix} -\Delta & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ \mathbf{0} & \mathbf{curl}\,\mathbf{curl} - \nabla\mathrm{div} & \mathbf{0}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{3\times3} & \mathbf{curl}\,\mathbf{curl} - \nabla\mathrm{div} & \mathbf{0} \\ 0 & \mathbf{0}^\top & \mathbf{0}^\top & -\Delta \end{pmatrix}, \tag{1.8}$$

is a vector Laplacian; therefore, the coefficients of vector fields in its kernel are harmonic functions.

The operator matrix D represents the Hodge–Dirac operator from [6, 7, 23] when the Fredholm-nilpotent operator is taken to be the exterior derivative acting on the full graded exterior algebra of square-integrable differential forms in 3D. The abstract inf-sup inequalities derived by P. Leopardi and A. Stern in [23, Sec. 2] guarantee that the domain-based mixed variational formulation associated with the Hodge decomposition problem of finding $\vec{\mathbf{U}}, \vec{\mathbf{P}} \in \mathbf{H}(\mathsf{D}, \Omega)$ satisfying

$$\mathsf{D}\vec{\mathbf{U}} + \vec{\mathbf{P}} = \vec{\mathbf{F}} \in \left(L^2(\Omega)\right)^8, \qquad \vec{\mathbf{U}} \in \ker(\mathsf{D})^\perp \qquad \text{and} \qquad \vec{\mathbf{P}} \in \ker(\mathsf{D})\,, \tag{1.9}$$

is well-posed in *bounded* Lipschitz domains for both collections of boundary conditions

$$U_0 = 0, \qquad \mathbf{n} \times (\mathbf{U}_1 \times \mathbf{n}) = \mathbf{0}, \qquad \mathbf{U}_2 \cdot \mathbf{n} = 0, \qquad \text{on } \Gamma, \tag{1.10a}$$

$$\mathbf{U}_1 \cdot \mathbf{n} = 0, \qquad \mathbf{U}_2 \times \mathbf{n} = \mathbf{0}, \qquad U_3 = 0, \qquad \text{on } \Gamma. \tag{1.10b}$$

In (1.9), the finite dimensional kernel of D can be characterized as the space of harmonic (or monogenic) 8D vector fields [2, 23].

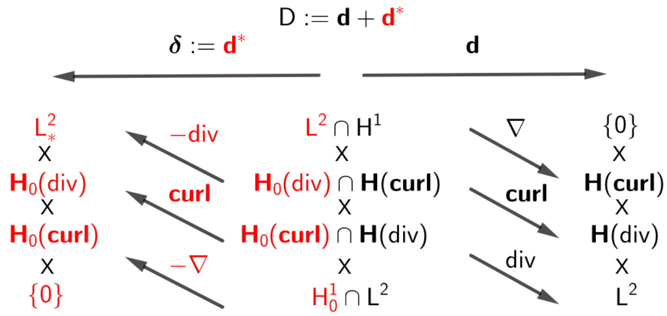

FIGURE 1. This diagram shows the mapping properties of the exterior derivative and its Hilbert space adjoint in the case of *natural* boundary conditions (1.10b). In the figure, the operators on the left-hand side are to be understood as the adjoint operators $-\mathrm{div} = \nabla^*$, $\mathbf{curl} = \mathbf{curl}^*$ and $-\nabla = \mathrm{div}^*$. Here, $L^2_* = \{V \in L^2 \mid \int v \,\mathrm{d}\mathbf{x} = 0\}$ and $\mathsf{D} = \mathbf{d} + \mathbf{d}^*$.

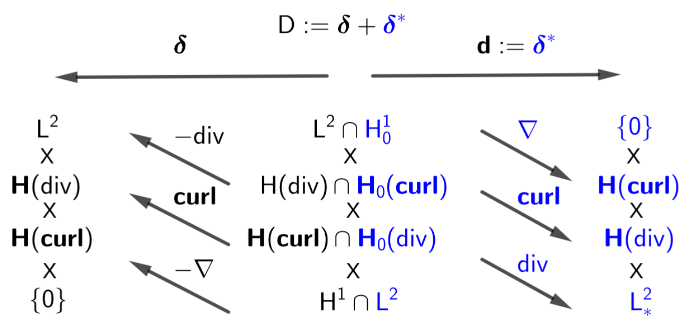

FIGURE 2. This diagram shows the mapping properties of the coderivative and its Hilbert space adjoint corresponding to the case of *essential* boundary conditions (1.10a). In the figure, the operators on the left-hand side are to be understood as the adjoint operators $\nabla = -\mathrm{div}^*$, $\mathbf{curl} = \mathbf{curl}^*$ and $\mathrm{div} = -\nabla$. Here, $L^2_* = \{V \in L^2 \mid \int v \,\mathrm{d}\mathbf{x} = 0\}$ and $\mathsf{D} := \boldsymbol{\delta} + \boldsymbol{\delta}^*$.

Lemma 1.1 (see [23, Sec. 2.1] and [1, Sec. 1.1.3])**.** *The Dirac operators of Figure 1 and Figure 2 both satisfy*

$$\ker(\mathsf{D}) = \ker(\mathbf{d}) \cap \ker(\boldsymbol{\delta}) = \ker(\boldsymbol{\Delta}) \subset \left(L^2(\Omega)\right)^8. \tag{1.11}$$

The precise function spaces for the domain of the Dirac operator $\mathbf{H}(\mathsf{D},\Omega) := \mathbf{H}(\mathbf{d},\Omega) \cap \mathbf{H}(\boldsymbol{\delta},\Omega)$ associated with the natural boundary conditions (1.10b) appear in Figure 1. In this context, $\boldsymbol{\delta}$ is the operator matrix representation of the Hilbert space adjoint $\mathbf{d}^*$ and $\mathsf{D} = \mathbf{d} + \mathbf{d}^*$.

As seen from Figure 2, the essential boundary conditions (1.10a) could be obtained upon restricting the domain of the exterior derivative explicitly—and this is the reason we call them *essential.* However, we prefer adopting the dual point of view. Starting from the Fredholm Hilbert chain complex

$$L^2(\Omega) \xleftarrow[-\mathrm{div}]{} \mathbf{L}^2(\Omega) \xleftarrow[\mathbf{curl}]{} \mathbf{L}^2(\Omega) \xleftarrow[-\nabla]{} L^2(\Omega), \tag{1.12}$$

we choose the domain of the coderivative $\boldsymbol{\delta}$ to be the product of the *full* spaces in the domain complex

$$L^2(\Omega) \xleftarrow[-\operatorname{div}]{} \mathbf{H}(\operatorname{div},\Omega) \xleftarrow[\mathbf{curl}]{} \mathbf{H}(\mathbf{curl},\Omega) \xleftarrow[-\nabla]{} \mathbf{H}^1(\Omega). \tag{1.13}$$

associated to (1.12). Then, we define the exterior derivative $\mathbf{d} := \boldsymbol{\delta}^*$ as the Hilbert space adjoint of the coderivative, which sets $\mathsf{D} = \boldsymbol{\delta} + \boldsymbol{\delta}^*$. In Section 7, this perspective will reveal an elegant correspondence between the structures of the 3D volume equations and the 2D boundary integral equations.

Remark 1.2. *Literature discussing the Dirac operators from the point of view of Hodge theory offers at the moment solutions to* (1.9) *and related eigenvalue problems based on domain variational formulations [13, 23].*

Remark 1.3. *The operator matrix in* (1.1) *appears under a change of variables in the works of M. Taskinen [37] as Picard's extended Maxwell operator. It was originally assembled by Picard by combining the first-order Maxwell operator with the principal part of the equations of linear acoustics [30, 31]. The ranges of these operators are L^2-orthogonal and the complement of their union is finite dimensional. Hence, since the intersection of their kernels is also finite, the idea of Picard was for them to 'mutually elliptize' [22]. In related literature, the Dirac operator enters "Helmholtz-like" boundary value problems that were tackled in [36–38] with the development of second-kind boundary integral equations.*

Remark 1.4. *Eigenvalue problems related to acoustic and electromagnetic scattering, that is transmission problems for the so-called perturbed Dirac operator, have also guided the study of second-kind boundary integral equations in the literature of harmonic and hyper-complex analysis. Important contributions were made in that direction by A. Axelsson [4], E. Marmolejo-Olea and M. Mitrea [24], and A. Rosén [19, 32]. There, the block operator D enters larger systems of equations that correspond to [19] or encompass [24] Maxwell's equations. An extensive body of work created by these authors and A. McIntosh [26, 27] is devoted to the harmonic analysis of the Dirac operator in L_p spaces [5–7, 25].*

In this work, we derive novel *first-kind* boundary integral equations for the Dirac equation (1.1). Two boundary integral operators are obtained and shown to satisfy generalized Gårding inequalities, making them Fredholm of index 0. Their finite dimensional kernels are characterized in Section 6, where we show that their dimension equals the number of topological invariants of the boundary—counted as the sum of its Betti numbers. Indeed, the integral representations of their associated bilinear forms turn out to be related to the variational formulations of the surface Dirac operators introduced in Section 7. Recognizing these surface operators will simultaneously reveal how the boundary integral operators introduced in Section 4, which are related to two different set of boundary conditions, arise as "rotated" versions of one another. The exterior representation formula of Proposition 3.3 and the condition at infinity identified in (3.72) eventually lead, together with the coercivity results of Section 5, to well-posedness of the Euclidean Dirac exterior problem in natural energy spaces.

The new integral formulas display desirable properties: the surface potentials are square-integrable and the kernels of the bilinear forms associated with the boundary integral operators are merely weakly singular. Nevertheless, we want to emphasize that the most surprising novelty is the discovery that they relate to the surface Hodge-Dirac operators in the de Rham $H^{-1/2}$ Hilbert complexes equipped with the common non-local inner product defined by integrating the classical single-layer potential for the Laplacian. As a consequence, we already know a lot about these first-kind boundary integral operators for the Dirac operator. Moreover, it suggests that the first-kind boundary integral operators for the Dirac operator are related to the first-kind boundary integral operators for the Hodge-Laplacian.

For the sake of readability, we adopt throughout this work the framework of classical vector analysis. It is in this framework that the structural relationship between the following development and the standard theory for second-order elliptic operators seemed most explicit.

In summary, our main contributions are:

- We derive representation formulas for the Dirac system of equations (1.1) posed on domains having a Lipschitz boundary by following the approach pioneered by M. Costabel [16]. The novelty here is to follow and extend the elegant strategy used in [14]—there to find a representation formula for Hodge-Laplace/Helmholtz operators—that leads to potentials having simple explicit expressions. By adapting the arguments in the now classical monographs by W. McLean [28, Chap. 7] and A. Sauter and C. Schwab [33, Chap. 3], we also establish an exterior representation formula. We will observe that the development of this theory is possible due to the strong structural similarity between integration by parts for the first-order Dirac operator and Green's second formula for second-order elliptic operators.

- A sneak peak at the potentials presented in (3.45) and (3.48) will already convince the reader that the approach we have adopted leads to simple formulas for the *square-integrable* potentials involved in the representation formula. Some terms are recognizable from [14, 15], while the others belong to well-known theory for elliptic second-order operators. The simplicity that comes with the calculation procedure provided by Lemma 3.4 allows for a straightforward analysis of their mapping and jump properties.
- Given the previous items, it is not suprising that conditions for the exterior problem posed on the unbounded domain Ω^+ can be easily established by adapting the approach for second-order elliptic operators presented in [28, Chap. 7].
- The crux of our calculations are the formulas (4.6) and (4.7) for the bilinear forms associated with the obtained weakly-singular first-kind boundary integral operators. We provide generalized Gårding inequalities for the two operators and characterize their null-spaces.
- Our main discovery is presented in Section 7, where we expose the relationship between these boundary integral operators and surface Dirac operators in an Hilbert complex framework.

2. Function spaces and traces

As usual, $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ denote the Hilbert spaces of complex square-integrable scalar and vector-valued functions defined over Ω. We denote their inner products using round brackets, e.g. $(\cdot,\cdot)$. The spaces $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ refer to the corresponding Sobolev spaces. The notation $C^\infty(\Omega)$ is used for smooth functions. The subscript in $C_0^\infty(\Omega)$ further specifies that these smooth functions have compact support in Ω. $C^\infty(\overline{\Omega})$ is defined as the space of uniformly continuous functions over Ω that have uniformly continuous derivatives of all order. A subscript is used to identify spaces of locally integrable functions/vector fields, e.g. $U \in L^2_{\mathrm{loc}}(\Omega)$ if and only if ϕU is square-integrable for all $\phi \in C_0^\infty(\mathbb{R}^3)$. We denote with an asterisk the spaces of functions with zero mean, e.g. $H^1_*(\Omega)$.

In general, given an operator L acting on square-integrable fields in the sense of distributions, we equip

$$\mathbf{H}(\mathsf{L},\Omega) := \{\mathbf{U} \in \left(\mathbf{L}^2(\Omega)\right)^{\bullet} \big| \, \mathsf{L}\mathbf{U} \in \left(\mathbf{L}^2(\Omega)\right)^{\dagger}\} \tag{2.1}$$

with the obvious graph norm, where $\bullet = 8$ or 3 and $\dagger = 8$, 3 or 1. The Hilbert spaces

$$\mathbf{H}(\mathrm{div},\Omega) := \left\{\mathbf{U} \in \left(L^2(\Omega)\right)^3 \mid \mathrm{div}(\mathbf{U}) \in L^2(\Omega)\right\}, \tag{2.2}$$

$$\mathbf{H}(\mathbf{curl},\Omega) := \left\{\mathbf{U} \in \left(L^2(\Omega)\right)^3 \mid \mathbf{curl}(\mathbf{U}) \in \left(L^2(\Omega)\right)^3\right\}, \tag{2.3}$$

are of prime importance in the following. Evidently, $\Omega =: \Omega^-$ can be replaced in all of the above definitions by $\mathbb{R}^3$, $\Omega^+ := \mathbb{R}^3 \backslash \Omega^-$ or any other domain that may come under consideration. We understand restrictions in the sense of distributions when working with domains having disconnected components. For example, in line with the above notation we mean in particular

$$\mathbf{H}\left(\mathsf{D},\mathbb{R}^3\backslash\Gamma\right) := \mathbf{H}(\mathsf{D},\Omega) \times \mathbf{H}\left(\mathsf{D},\mathbb{R}^3\backslash\Omega\right) \subset \left(L^2(\mathbb{R}^3)\right)^8. \tag{2.4}$$

We use a prime superscript to denote dual spaces, for instance $C_0^\infty(\Omega)'$ is the space of distributions in Ω. Throughout, angular brackets indicates duality pairing, e.g. $\langle\cdot,\cdot\rangle_\Omega$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\Gamma$. The former will be used for domain-based quantities in Ω, while the latter will pair spaces on Γ.

Trace-related theory for Lipschitz domains can be found in [9–11] and [18, 28], where it is established that the traces

$$\gamma W := \mathbf{W}\big|_{\Gamma}, \qquad \forall\, W \in C^{\infty}(\overline{\Omega}), \tag{2.5a}$$

$$\gamma_n \mathbf{W} := \gamma \mathbf{W} \cdot \mathbf{n}, \qquad \forall\, \mathbf{W} \in \mathbf{C}^{\infty}(\overline{\Omega}), \tag{2.5b}$$

$$\gamma_\tau \mathbf{W} := \gamma \mathbf{W} \times \mathbf{n}, \qquad \forall\, \mathbf{W} \in \mathbf{C}^{\infty}(\overline{\Omega}), \tag{2.5c}$$

$$\gamma_t \mathbf{W} := \mathbf{n} \times (\gamma_\tau \mathbf{W}), \qquad \forall\, \mathbf{W} \in \mathbf{C}^{\infty}(\overline{\Omega}), \tag{2.5d}$$

extend to continuous and surjective linear operators

$$\gamma : H^1(\Omega) \to H^{1/2}(\Gamma), \tag{2.6a}$$

$$\gamma_n : \mathbf{H}(\mathrm{div}, \Omega) \to H^{-1/2}(\Gamma), \tag{2.6b}$$

$$\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \tag{2.6c}$$

$$\gamma_t : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma). \tag{2.6d}$$

Here, $\mathbf{n} \in \mathbf{L}^{\infty}(\Gamma)$ is the essentially bounded unit normal vector field on Γ directed toward the exterior of Ω^-. Similarly as for the Hodge–Laplace operator [14, 15, 34, 35], a theory of differential equations for the Hodge–Dirac problem in three dimensions entails partitioning our collection of traces into two “dual” pairs. Accordingly, we assemble the traces into

$$\gamma_{\mathsf{T}}\left(\vec{\mathbf{U}}\right) := \begin{pmatrix} \gamma(U_0) \\ \gamma_t(\mathbf{U}_1) \\ \gamma_n(\mathbf{U}_2) \end{pmatrix} \qquad \text{and} \qquad \gamma_{\mathsf{R}}\left(\vec{\mathbf{U}}\right) := \begin{pmatrix} \gamma_n(\mathbf{U}_1) \\ \gamma_\tau(\mathbf{U}2) \\ \gamma(U_3) \end{pmatrix}. \tag{2.7}$$

Warning. *We want to highlight that in spite of the notation, γ_{T} and γ_{R} are **not** defined as in [14], [15] and related work.*

The trace spaces

$$\mathcal{H}_{\mathsf{T}} := H^{1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma), \tag{2.8a}$$

$$\mathcal{H}_{\mathsf{R}} := H^{-1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma), \tag{2.8b}$$

are dual to each other with respect to the $L^2(\Gamma)$ duality pairing.

Naturally, the traces can also be taken from the exterior domain. The extensions (2.6) will be distinguished (only when required to avoid confusion) with a minus subscript, e.g. γ^-, by opposition to the extensions obtained from (2.5) by replacing Ω with $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega}$, which we will denote with a plus superscript, e.g. γ^+.

Lemma 2.1. *The linear mappings*

$$\gamma^{\pm}_{\mathsf{T}} : \mathbf{H}_{\mathrm{loc}}\left(\mathsf{D}, \Omega^{\pm}\right) \to \mathcal{H}_{\mathsf{T}}, \qquad \gamma^{\pm}_{\mathsf{R}} : \mathbf{H}_{\mathrm{loc}}\left(\mathsf{D}, \Omega^{\pm}\right) \to \mathcal{H}_{\mathsf{R}}, \tag{2.9}$$

defined by (2.7) *are continuous and surjective.*

Detailed definitions for the individual spaces entering the above trace spaces can be found in [9–11] together with a study of the involved surface differential operators. Short practical summaries are also provided in [12, 14, 21, 34].

Lemma 2.2 (See [14, Lem. 6.4])**.** *The surface divergence extends to a continuous surjection* $\mathrm{div}_\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to H^{-1/2}_*(\Gamma)$*, while* $\mathbf{curl}_\Gamma : H^{1/2}_* \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *is a bounded injection with closed range such that* $\mathbf{curl}_\Gamma(\xi) = \nabla_\Gamma(\xi) \times \mathbf{n}$ *for all* $\xi \in H^{1/2}(\Gamma)$*. These operators satisfy* $\mathrm{div}_\Gamma \circ \mathbf{curl}_\Gamma = 0$.

Lemma 2.3 (See [14, Sec. 2.5])**.** *There exist continuous lifting maps* $\mathscr{E}_{\mathsf{T}} : \mathscr{H}_{\mathsf{T}} \to \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \overline{\Omega})$ *and* $\mathscr{E}_{\mathsf{R}} : \mathscr{H}_{\mathsf{R}} \to \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \overline{\Omega})$ *such that* $\gamma_{\mathsf{T}} \circ \mathscr{E}_{\mathsf{T}} = \mathrm{Id}$ *and* $\gamma_{\mathsf{R}} \circ \mathscr{E}_{\mathsf{R}} = \mathrm{Id}$.

Lemma 2.4 (Integration by parts)**.** *For all* $\mathbf{U} \in \mathbf{H}(\mathrm{d}, \Omega^{\mp})$ *and* $\mathbf{V} \in \mathbf{H}(\delta, \Omega^{\mp})$,

$$\int_{\Omega^{\mp}} \mathrm{d}\mathbf{U} \cdot \mathbf{V} \, \mathrm{d}\mathbf{x} = \int_{\Omega^{\mp}} \mathbf{U} \cdot \delta \mathbf{V} \, \mathrm{d}\mathbf{x} \pm \langle\!\langle \gamma_{\mathsf{T}} \mathbf{U}, \gamma_{\mathsf{R}} \mathbf{V} \rangle\!\rangle. \tag{2.10}$$

Proof. We integrate by parts using Green's identities to obtain

$$\begin{aligned}
\int_{\Omega^{\mp}} \mathrm{d}\mathbf{U} \cdot \mathbf{V} \, \mathrm{d}\mathbf{x} &= \int_{\Omega^{\mp}} \nabla U_0 \cdot \mathbf{V}_1 \, \mathrm{d}\mathbf{x} + \int_{\Omega^{\mp}} \mathbf{curl}\,(\mathbf{U}_1) \cdot \mathbf{V}_2 \, \mathrm{d}\mathbf{x} + \int_{\Omega^{\mp}} \mathrm{div}\,(\mathbf{U}_2)\, V_3 \, \mathrm{d}\mathbf{x} \\
&= -\int_{\Omega^{\mp}} \mathbf{U}_0 \,\mathrm{div}\,(\mathbf{V}_1) \, \mathrm{d}\mathbf{x} + \int_{\Omega^{\mp}} \mathbf{U}_1 \cdot \mathbf{curl}\,(\mathbf{V}_2) \, \mathrm{d}\mathbf{x} - \int_{\Omega^{\mp}} \mathbf{U}_2 \cdot \nabla V_3 \, \mathrm{d}\mathbf{x} \\
&\quad + \langle \gamma\,(U_0)\,, \gamma_n\,(\mathbf{V}_1) \rangle_{\Gamma} + \langle \gamma_t\,(\mathbf{U}_1)\,, \gamma_\tau\,(\mathbf{V}_2) \rangle_{\tau} + \langle \gamma_n\,(\mathbf{U}_2)\,, \gamma\,(V_3) \rangle \\
&= \int_{\Omega^{\mp}} \mathbf{U} \cdot \delta \mathbf{V} \, \mathrm{d}\mathbf{x} + \langle\!\langle \gamma_{\mathsf{T}} \mathbf{U}, \gamma_{\mathsf{R}} \mathbf{V} \rangle\!\rangle.
\end{aligned} \tag{2.11}$$

□

Corollary 2.1. *(Green's second formula for the Dirac operator) For all* $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in \mathbf{H}(\mathsf{D}, \Omega^{\mp})$, *we have*

$$\int_{\Omega^{\mp}} \mathsf{D}\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} \, \mathrm{d}\mathbf{x} = \int_{\Omega^{\mp}} \vec{\mathbf{U}} \cdot \mathsf{D}\vec{\mathbf{V}} \, \mathrm{d}\mathbf{x} \pm \langle\!\langle \gamma_{\mathsf{T}} \vec{\mathbf{U}}, \gamma_{\mathsf{R}} \vec{\mathbf{V}} \rangle\!\rangle \mp \langle\!\langle \gamma_{\mathsf{T}} \vec{\mathbf{V}}, \gamma_{\mathsf{R}} \vec{\mathbf{U}} \rangle\!\rangle. \tag{2.12}$$

Remark 2.1. *It is remarkable that* ***despite the fact that*** D ***is a first-order operator****, Equation* (2.12) *nevertheless resembles Green's classical second formula for the Laplacian. This structure induces profound structural similarity between the representation formula, potentials and boundary integral equations for the Dirac operator established in the next sections and the already well-known theory for second-order elliptic operators. As emphasized in [35], a formula such as eq.* (2.12) *paves the way for harnessing valuable established techniques.*

We will indicate with curly brackets the average $\{\gamma_\bullet\} := \frac{1}{2}(\gamma_\bullet^+ + \gamma_\bullet^-)$ of a trace and with square brackets its jump $[\gamma_\bullet] := \gamma_\bullet^- - \gamma_\bullet^+$ over the interface Γ.

Warning. *Notice the sign in the jump* $[\gamma] = \gamma^- - \gamma^+$, *which is often taken to be the opposite in the literature!*

3. Representation formula

In this section, we derive a representation formula for solutions of the Dirac equation and express it through known boundary potentials. We also compute the jump properties of these potentials across Γ.

3.1. **Fundamental solution.** In the following, convolution of a vector field $\vec{\mathbf{U}} : \mathbb{R}^3 \to \mathbb{R}^8$ by an integrable matrix-valued function $\mathsf{L} : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}^{8,8}$ depending on one space variable is given by

$$\left(\mathsf{L} * \vec{\mathbf{U}}\right)(\mathbf{x}) := \int_{\mathbb{R}^3} \mathsf{L}(\mathbf{x} - \mathbf{y}) \vec{\mathbf{U}}\,(\mathbf{y}) \, \mathrm{d}\mathbf{y} \in \mathbb{R}^8 \tag{3.1}$$

when the integral is well-defined. In some cases, if the integrand has a singularity, then the right hand side of eq. (3.1) must be understood as a Cauchy integral.

The main task of this section is to find a matrix-valued function $\mathfrak{G} : (\mathbb{R}^3) \backslash \{\mathbf{0}\} \to \mathbb{R}^{8\times 8}$ such that, whenever $\vec{\mathbf{U}} \in (C_0^\infty(\mathbb{R}^3)^8)'$, the formal evaluation

$$\langle \mathfrak{G} * \mathsf{D}\vec{\mathbf{U}}, \vec{\mathbf{V}} \rangle = \langle \vec{\mathbf{U}}, \vec{\mathbf{V}} \rangle \tag{3.2}$$

is valid for all $\vec{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$. It is in that sense that such a matrix-valued function $\mathfrak{G}$ may be called a fundamental solution for the Dirac operator. In other words, we are looking for a matrix-valued function satisfying

$$\mathfrak{G} * \mathsf{D} = \mathrm{Id} \tag{3.3}$$

in the sense of distributions. When found, convolution

$$\mathfrak{N}\vec{\mathbf{U}}(\mathbf{x}) := \left(\mathfrak{G} * \vec{\mathbf{U}}\right)(\mathbf{x}) \tag{3.4}$$

with that particular matrix-valued function in turn spawns a Newton potential operator $\mathfrak{N} : C_0^\infty(\mathbb{R}^3)^8 \to C^\infty(\mathbb{R}^3)^8$ for the Dirac equation whose role—once its domain is extended to distributions—is akin to the classical Newton operator that arises in the study of conventional scalar Laplace or Poisson problems.

Let $G : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}$ be the fundamental solution for the scalar Laplace operator $G(\mathbf{z}) := \left(4\pi|\mathbf{z}|\right)^{-1}$. We define $\mathsf{G} : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}^{8,8}$ as the matrix-valued function such that

$$\mathsf{G}(\mathbf{z})\,\vec{\mathbf{C}} = G(\mathbf{z})\,\vec{\mathbf{C}}, \qquad \mathbf{z} \in \mathbb{R}^3 \backslash \{\mathbf{0}\} \tag{3.5}$$

for all $\vec{\mathbf{C}} \in \mathbb{R}^8$. The right hand side of eq. (3.5) simply scales the vector argument by a scalar, and thus if we explicitly denote the fundamental solution of the vector Laplace operator by

$$\mathbf{G}(\mathbf{z}) := G(\mathbf{z}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3,3}, \qquad \mathbf{z} \neq \mathbf{0}, \tag{3.6}$$

then we can write the matrix representation

$$\mathsf{G}(\mathbf{z}) = \begin{pmatrix} G(\mathbf{z}) & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ \mathbf{0} & \mathbf{G}(\mathbf{z}) & \mathbf{0}_{3\times 3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{3\times 3} & \mathbf{G}(\mathbf{z}) & \mathbf{0} \\ 0 & \mathbf{0}^\top & \mathbf{0}^\top & G(\mathbf{z}) \end{pmatrix} \in \mathbb{R}^{8,8}, \qquad \mathbf{z} \neq \mathbf{0}. \tag{3.7}$$

Remark 3.1. *In the spirit of [37], one may want to apply the Dirac operator to the above matrix-valued function directly and write (at least formally)*

$$\mathsf{D}\mathsf{G} = \begin{pmatrix} 0 & -\operatorname{div}\mathbf{G} & \mathbf{0}^\top & 0 \\ \nabla G & \mathbf{0}_{3\times 3} & \mathbf{curl}\,\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{curl}\,\mathbf{G} & \mathbf{0}_{3\times 3} & -\nabla G \\ 0 & \mathbf{0}^\top & \operatorname{div}\mathbf{G} & 0 \end{pmatrix}. \tag{3.8}$$

Equation (3.8) needs to be made more precise, but the idea that should come across is that from (1.8), setting $\mathfrak{G} = \mathsf{D}\mathsf{G}$ *formally yields*

$$\mathsf{D}\mathfrak{G}(\mathbf{z})\,\vec{\mathbf{C}} = \mathsf{D}^2(\mathsf{G}\vec{\mathbf{C}})(\mathbf{z}) = \delta_{\mathbf{0}}(\mathbf{z})\,\vec{\mathbf{C}} \tag{3.9}$$

in the sense of distributions for all constant vector $\vec{\mathbf{C}} \in \mathbb{R}^8$*. In eq. (3.9),* $\delta_{\mathbf{0}}$ *is the Dirac distribution centered at* $\mathbf{0}$.

However, even though eq. (3.8) can be a valuable mnemonic device for computations, there is some ambiguity in the expression. On the right hand side, the notation does not make explicitly clear that differentiation is only to be carried out on the components of the fundamental solutions G *and* $\mathbf{G}$*, even when* $\mathfrak{G}$ *acts on non-constant vector-fields.* ***In other words, the Dirac operator is applied to the columns of*** G. *It is more appropriate to rigorously define* $\mathfrak{G}$ *as follows.*

We define $\mathfrak{G} : \mathbb{R}^3 \backslash (\mathbf{0}) \to \mathscr{L}(\mathbb{R}^8)$ as the matrix-valued function satisfying

$$\mathfrak{G}(\mathbf{z})\,\vec{\mathbf{C}} = \mathsf{D}(\mathsf{G}\vec{\mathbf{C}})\Big|_{\mathbf{z}}, \qquad \mathbf{z} \neq \mathbf{0}, \tag{3.10}$$

for all $\vec{\mathbf{C}} \in \mathbb{R}^8$. The subtlety is also resolved when the more precise matrix representation

$$\mathfrak{G}(\mathbf{z}) = \begin{pmatrix} 0 & -(\nabla G)^{\top}(\mathbf{z}) & \mathbf{0}^{\top} & 0 \\ (\nabla G)(\mathbf{z}) & \mathbf{0}_{3\times 3} & \mathsf{C}_{3\times 3}(\mathbf{z}) & \mathbf{0} \\ \mathbf{0} & \mathsf{C}_{3\times 3}(\mathbf{z}) & \mathbf{0}_{3\times 3} & -(\nabla G)(\mathbf{z}) \\ 0 & \mathbf{0}^{\top} & (\nabla G)^{\top}(\mathbf{z}) & 0 \end{pmatrix} \in \mathbb{R}^{8\times 8}, \qquad \mathbf{z} \neq \mathbf{0}, \tag{3.11}$$

is used. In eq. (3.11), the blocks

$$\mathsf{C}_{3\times 3}(\mathbf{z}) := \begin{pmatrix} 0 & -(\partial_3 G)(\mathbf{z}) & (\partial_2 G)(\mathbf{z}) \\ (\partial_3 G)(\mathbf{z}) & 0 & -(\partial_1 G)(\mathbf{z}) \\ -(\partial_2 G)(\mathbf{z}) & (\partial_1 G)(\mathbf{z}) & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \qquad \mathbf{z} \neq \mathbf{0}, \tag{3.12}$$

associated with the curl operator are anti-symmetric, and the next lemma can immediately be read from eq. (3.11).

Lemma 3.1. *For* $\mathbf{z} \neq \mathbf{0}$,

$$\mathfrak{G}(\mathbf{z})\,\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} = -\vec{\mathbf{U}} \cdot \mathfrak{G}(\mathbf{z})\,\vec{\mathbf{V}}, \qquad \forall\,\vec{\mathbf{U}}, \vec{\mathbf{V}} \in \mathbb{R}^8. \tag{3.13}$$

Combined with Lemma 3.1, the next lemma will allow us to extend the domain of $\mathfrak{N}$ to the space of distributions.

Lemma 3.2. *For* $\mathbf{z} \neq 0$,

$$\mathfrak{G}(-\mathbf{z}) = -\mathfrak{G}(\mathbf{z}). \tag{3.14}$$

Proof. Let $\mathsf{s} : \mathbb{R}^3 \to \mathbb{R}^3$ be the sign flip operation $\mathsf{s}(\mathbf{z}) = -\mathbf{z}$. We simply verify that for any $\vec{\mathbf{C}} \in \mathbb{R}^8$,

$$\begin{aligned} \mathfrak{G}(-\mathbf{z})\,\vec{\mathbf{C}} = \mathsf{D}\left(\mathsf{G}\vec{\mathbf{C}}\right)\Big|_{\mathsf{s}(\mathbf{z})} &= -\mathsf{D}_{\mathbf{x}}\left(\mathsf{G}\left(\mathsf{s}\left(\mathbf{x}\right)\right)\vec{\mathbf{C}}\right)\Big|_{\mathbf{x}=\mathbf{z}} \\ &= -\mathsf{D}_{\mathbf{x}}\left(G\left(\mathsf{s}\left(\mathbf{x}\right)\right)\vec{\mathbf{C}}\right)\Big|_{\mathbf{x}=\mathbf{z}} = -\mathsf{D}_{\mathbf{x}}\left(G\left(\mathbf{x}\right)\vec{\mathbf{C}}\right)\Big|_{\mathbf{x}=\mathbf{z}} = -\mathfrak{G}(\mathbf{z})\,\vec{\mathbf{C}}, \end{aligned} \tag{3.15}$$

relying on the fact that the fundamental solution $G(\mathbf{x}) = G\left(|\mathbf{x}|\right)$ of the Laplace operator is symmetric. □

Combined with Lemma 3.1, the last lemma allows us to extend the domain of $\mathfrak{N}$ to the space of distributions.

Lemma 3.3. *For all* $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$,

$$\left(\mathfrak{N}\,\vec{\mathbf{U}}, \vec{\mathbf{V}}\right) = \left(\vec{\mathbf{U}}, \mathfrak{N}\,\vec{\mathbf{V}}\right). \tag{3.16}$$

Proof. By Fubini's theorem, we can change the order of integration and evaluate

$$\begin{aligned} \left(\mathfrak{N}\vec{\mathbf{U}}, \vec{\mathbf{V}}\right) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{x}-\mathbf{y})\,\vec{\mathbf{U}}(\mathbf{y}) \cdot \vec{\mathbf{V}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} &\overset{(*)}{=} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \vec{\mathbf{U}}(\mathbf{y}) \cdot \mathfrak{G}(\mathbf{y}-\mathbf{x})\,\vec{\mathbf{V}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^3} \vec{\mathbf{U}}(\mathbf{y}) \cdot \int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{y}-\mathbf{x})\,\vec{\mathbf{V}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = \left(\vec{\mathbf{U}}, \mathfrak{N}\,\vec{\mathbf{V}}\right), \end{aligned} \tag{3.17}$$

where both Lemma 3.1 and Lemma 3.2 were used in $(*)$. □

Lemma 3.3 reflects the fact that D is symmetric, even self-adjoint in $(L^2(\mathbb{R}^3))^8$. The extension

$$\mathfrak{N} : (C^\infty(\mathbb{R}^3)^8)' \to (C_0^\infty(\mathbb{R}^3)^8)' \tag{3.18}$$

is obtained as in [33] via the dual mapping by defining the action of the distribution $\mathfrak{N}\vec{\mathbf{U}} \in (C_0^\infty(\mathbb{R}^3)^8)'$ by

$$\langle \mathfrak{N}\,\vec{\mathbf{U}}, \vec{\mathbf{V}} \rangle := \langle \vec{\mathbf{U}}, \mathfrak{N}\,\vec{\mathbf{V}} \rangle \tag{3.19}$$

for all $\vec{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$. The goal of the next proposition is to show that this extension satisfy eq. (3.3).

Proposition 3.1. *For a compactly supported distribution $\vec{\mathbf{U}} \in (C^\infty(\mathbb{R}^3)^8)'$ and $\vec{\mathbf{V}} \in C_0^\infty(\mathbb{R}^3)^8$,*

$$\langle \mathfrak{N}\,\mathsf{D}\vec{\mathbf{U}}, \vec{\mathbf{V}}\rangle = \langle \vec{\mathbf{U}}, \vec{\mathbf{V}}\rangle. \tag{3.20}$$

Proof. The following argument is inspired by [17, Thm.1]. Let $\mathbf{e}_i \in \mathbb{R}^3$ be the vector with 1 at the i-th entry and zeros elsewhere, $i = 1, 2, 3$. Since

$$\mathfrak{N}\vec{\mathbf{V}} = \int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{x}-\mathbf{y})\,\vec{\mathbf{V}}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{y})\,\vec{\mathbf{V}}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{3.21}$$

we have

$$\frac{\mathfrak{N}\vec{\mathbf{V}}(\mathbf{x}+h\mathbf{e}_i) - \mathfrak{N}\vec{\mathbf{V}}(\mathbf{x})}{h} = \int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{y})\,\frac{\vec{\mathbf{V}}(\mathbf{x}+h\mathbf{e}_i-\mathbf{y}) - \vec{\mathbf{V}}(\mathbf{x}-\mathbf{y})}{h}\,\mathrm{d}\mathbf{y}. \tag{3.22}$$

Hence,

$$\mathsf{D}_{\mathbf{x}}\mathfrak{N}\vec{\mathbf{V}}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{y})\,\mathsf{D}\vec{\mathbf{V}}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{3.23}$$

because the assumption that $\vec{\mathbf{V}}$ is smooth and compactly supported guarantees that

$$\frac{\vec{\mathbf{V}}(\mathbf{x}+h\mathbf{e}_i-\mathbf{y}) - \vec{\mathbf{V}}(\mathbf{x}-\mathbf{y})}{h} \to \frac{\partial}{\partial \mathbf{x}_i}\vec{\mathbf{V}}(\mathbf{x}-\mathbf{y}) \tag{3.24}$$

uniformly for $h \to 0$. The main idea behind our strategy is to isolate $\mathfrak{G}$'s singularity at $\mathbf{0}$ within a ball $B_\epsilon(\mathbf{0})$ of radius ϵ by splitting the right hand side of eq. (3.23) into two integrals as

$$\mathsf{D}_{\mathbf{x}}\mathfrak{N}\vec{\mathbf{V}}(\mathbf{x}) = \underbrace{\int_{B_\epsilon(\mathbf{0})} \mathfrak{G}(\mathbf{y})\,\mathsf{D}\vec{\mathbf{V}}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \mathfrak{G}(\mathbf{y})\,\mathsf{D}\vec{\mathbf{V}}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}}_{J_\epsilon} \tag{3.25}$$

whose limits as $\epsilon \to 0$ we can control. A difficulty that becomes apparent here is that we cannot immediately mimic the standard proof commonly given for the Poisson equation as the integration by parts for the product of two vectors as supplied by eq. (2.12) is not applicable to the matrix-vector multiplications forming the integrand of eq. (3.25).

The analysis has to be carried out component-wise. Using the explicit representation of eq. (3.11) we obtain

$$\mathfrak{G}(\mathbf{y})\,\mathsf{D}\vec{\mathbf{V}}(\mathbf{x}-\mathbf{y}) = \begin{pmatrix} -\nabla G(\mathbf{y})\cdot\nabla V_0(\mathbf{x}-\mathbf{y}) - \nabla G(\mathbf{y})\cdot\mathbf{curl}\,\mathbf{V}_1(\mathbf{x}-\mathbf{y}) \\ -\operatorname{div}\mathbf{V}_1(\mathbf{x}-\mathbf{y})\,\nabla G(\mathbf{y}) - \nabla G(\mathbf{y})\times\nabla V_3(\mathbf{x}-\mathbf{y}) + \nabla G(\mathbf{y})\times\mathbf{curl}\,\mathbf{V}_1(\mathbf{x}-\mathbf{y}) \\ -\nabla G(\mathbf{y})\times\nabla V_0(\mathbf{x}-\mathbf{y}) + \nabla G(\mathbf{y})\times\mathbf{curl}\,\mathbf{V}_2(\mathbf{x}-\mathbf{y}) - \operatorname{div}\mathbf{V}_2(\mathbf{x}-\mathbf{y})\,\nabla G(\mathbf{y}) \\ -\nabla G(\mathbf{y})\cdot\nabla V_3(\mathbf{x}-\mathbf{y}) + \nabla G(\mathbf{y})\cdot\mathbf{curl}\,\mathbf{V}_2(\mathbf{x}-\mathbf{y}) \end{pmatrix} \tag{3.26}$$

There are four different types of terms whose limit need to be investigated. Let $\mathbf{V} \in (C_0^\infty(\mathbb{R}^3))^3$ and $V \in C_0^\infty(\mathbb{R}^3)$ be arbitrary fields. We denote by $\mathbf{n}_\epsilon$ the unit normal vector field pointing towards the interior of $B_\epsilon(\mathbf{0})$.

Integrating by parts using that $\Delta G = 0$ in $\mathbb{R}^3\setminus(\mathbf{0})$ and $\mathbf{curl}\circ\nabla \equiv \mathbf{0}$, we find that

$$\begin{aligned} \int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y})\cdot\nabla V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} &= \int_{\partial B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y})\cdot\mathbf{n}_\epsilon(\mathbf{y})\,V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \\ &= \frac{1}{4\pi}\int_{\partial B_\epsilon(\mathbf{0})} \frac{V(\mathbf{x}-\mathbf{y})}{|\mathbf{y}|^3}\left(-\mathbf{y}\cdot\frac{\mathbf{y}}{|\mathbf{y}|}\right)\mathrm{d}\sigma(\mathbf{y}) = -\frac{1}{4\pi\epsilon^2}\int_{\partial B_\epsilon(\mathbf{0})} V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \\ &= -⨍_{\partial B_\epsilon(\mathbf{x})} V(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \xrightarrow[\epsilon\to 0]{} -V(\mathbf{x}) \end{aligned} \tag{3.27}$$

and

$$\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \cdot \mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{\partial B_\epsilon(\mathbf{0})} \left(\nabla G(\mathbf{y}) \times \mathbf{n}_\epsilon(\mathbf{y})\right)\cdot \mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) = -\frac{1}{4\pi\epsilon^4}\int_{\partial B_\epsilon(\mathbf{0})} (\mathbf{y}\times\mathbf{y})\cdot\mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) = 0. \tag{3.28}$$

Similarly, integrating by parts component-wise yields

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \nabla V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} &= \int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \begin{pmatrix} \partial_2 G(\mathbf{y})\,\partial_3 V(\mathbf{x}-\mathbf{y}) - \partial_3 G(\mathbf{y})\,\partial_2 V(\mathbf{x}-\mathbf{y}) \\ \partial_3 G(\mathbf{y})\,\partial_1 V(\mathbf{x}-\mathbf{y}) - \partial_1 G(\mathbf{y})\,\partial_3 V(\mathbf{x}-\mathbf{y}) \\ \partial_1 G(\mathbf{y})\,\partial_2 V(\mathbf{x}-\mathbf{y}) - \partial_2 G(\mathbf{y})\,\partial_1 V(\mathbf{x}-\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \begin{pmatrix} G(\mathbf{y})\,\partial_2\partial_3 V(\mathbf{x}-\mathbf{y}) - G(\mathbf{y})\,\partial_3\partial_2 V(\mathbf{x}-\mathbf{y}) \\ G(\mathbf{y})\,\partial_3\partial_1 V(\mathbf{x}-\mathbf{y}) - G(\mathbf{y})\,\partial_1\partial_3 V(\mathbf{x}-\mathbf{y}) \\ G(\mathbf{y})\,\partial_1\partial_2 V(\mathbf{x}-\mathbf{y}) - G(\mathbf{y})\,\partial_2\partial_1 V(\mathbf{x}-\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} \\ &\quad + \int_{\partial B_\epsilon(\mathbf{0})} \begin{pmatrix} -(\mathbf{n}_\epsilon)_2(\mathbf{y})\,G(\mathbf{y})\,\partial_3 V(\mathbf{x}-\mathbf{y}) + (\mathbf{n}_\epsilon)_3(\mathbf{y})\,G(\mathbf{y})\,\partial_2 V(\mathbf{x}-\mathbf{y}) \\ -(\mathbf{n}_\epsilon)_2(\mathbf{y})\,G(\mathbf{y})\,\partial_1 V(\mathbf{x}-\mathbf{y}) + (\mathbf{n}_\epsilon)_1(\mathbf{y})\,G(\mathbf{y})\,\partial_3 V(\mathbf{x}-\mathbf{y}) \\ -(\mathbf{n}_\epsilon)_1(\mathbf{y})\,G(\mathbf{y})\,\partial_2 V(\mathbf{x}-\mathbf{y}) + (\mathbf{n}_\epsilon)_2(\mathbf{y})\,G(\mathbf{y})\,\partial_1 V(\mathbf{x}-\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y}.\end{aligned} \tag{3.29}$$

Since V is smooth everywhere in $\mathbb{R}^3$, partial derivatives commute and the volume integral vanishes, leading to

$$\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \nabla V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})\,\mathbf{n}_\epsilon(\mathbf{y}) \times \nabla V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}). \tag{3.30}$$

This integral vanishes under the limit $\epsilon \to 0$, because

$$\sup_{\mathbf{x}\in\mathbb{R}^3} \left| \int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})\,\mathbf{n}_\epsilon(\mathbf{y}) \times \nabla V(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \right| \leq \|\nabla V\|_\infty \int_{\partial B_\epsilon(\mathbf{0})} \left|G(\mathbf{y})\right| \mathrm{d}\sigma(\mathbf{y}) = \mathcal{O}(\epsilon). \tag{3.31}$$

Moving on to the next term, one eventually obtains from similar calculations that

$$\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} G(\mathbf{y})\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y}) \left(\mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y}) \times \mathbf{n}_\epsilon(\mathbf{y})\right) \mathrm{d}\sigma(\mathbf{y}). \tag{3.32}$$

Since $\|\mathbf{curl}\,\mathbf{V}\|_\infty < \infty$, the boundary integral on the right hand side vanishes under the limit by repeating the argument of eq. (3.31). Finally, commuting partial derivatives after integrating by parts also yields

$$\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \mathrm{div}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\,\nabla G(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} G(\mathbf{y})\,\nabla \mathrm{div}\mathbf{V}(\mathbf{x}-\mathbf{y}) - \int_{\partial B_\epsilon(\mathbf{0})} G(\mathbf{y})\,\mathrm{div}\mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathbf{n}_\epsilon(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \tag{3.33}$$

Putting the two previous calculations together, we find that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} \nabla G(\mathbf{y}) \times \mathbf{curl}\,\mathbf{V}(\mathbf{x}-\mathbf{y}) - \mathrm{div}\,\mathbf{V}(\mathbf{x}-\mathbf{y})\,\nabla G(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\lim_{\epsilon\to 0}\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{0})} G(\mathbf{y})\,\boldsymbol{\Delta}\mathbf{V}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{V}(\mathbf{x}), \tag{3.34}$$

where we recognized the vector (Hodge-) Laplace operator $-\boldsymbol{\Delta} \equiv \mathbf{curl}\,\mathbf{curl} - \nabla\,\mathrm{div}$.

We have found that $J_\epsilon \longrightarrow \vec{\mathbf{V}}(\mathbf{x})$ as $\epsilon \to 0$. Meanwhile,

$$\|I_\epsilon\|_\infty \leq \left\|\mathsf{D}\vec{\mathbf{V}}\right\|_\infty \int_{B_\epsilon(\mathbf{0})} \|\mathfrak{G}\|_\infty\,\mathrm{d}\mathbf{y} = \mathcal{O}\left(\int_{B_\epsilon(\mathbf{0})} \|\nabla G\|_\infty\,\mathrm{d}\mathbf{y}\right) = \mathcal{O}(\epsilon). \tag{3.35}$$

□

Corollary 3.1 (Fundamental solution)**.** *For all compactly supported distributions* $\vec{\mathbf{U}} \in (C^\infty(\mathbb{R}^3)^8)'$,

$$\mathfrak{N}\,\mathsf{D}\,\vec{\mathbf{U}} = \vec{\mathbf{U}} = \mathsf{D}\,\mathfrak{N}\,\vec{\mathbf{U}} \qquad \text{in } (C_0^\infty(\mathbb{R}^3)^8)'. \tag{3.36}$$

Proof. The calculations for $\vec{\mathbf{U}} = \mathsf{D}\,\mathfrak{N}\,\vec{\mathbf{U}}$ follow similarly starting from (3.23). □

3.2. **Surface potentials.** In line with the perspective on first-kind boundary integral operators from [16], [28], [33] and [14]—there in the study of second-order elliptic operators—we define here for the first-order Dirac operator the surface potentials

$$\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) := \mathfrak{N}\gamma_{\mathsf{T}}'\vec{\mathbf{a}}, \qquad \forall\, \vec{\mathbf{a}} = (a_0, \mathbf{a}_1, a_2) \in \mathcal{H}_{\mathsf{R}}, \tag{3.37}$$

$$\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) := -\mathfrak{N}\gamma_{\mathsf{R}}'\vec{\mathbf{b}}, \qquad \forall\, \vec{\mathbf{b}} = (b_0, \mathbf{b}_1, b_2) \in \mathcal{H}_{\mathsf{T}}. \tag{3.38}$$

The mappings $\gamma_{\mathsf{T}}' : \mathcal{H}_{\mathsf{R}} \to \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3\backslash\overline{\Omega})'$ and $\gamma_{\mathsf{R}}' : \mathcal{H}_{\mathsf{T}} \to \mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3\backslash\overline{\Omega})'$ are adjoint to the trace operators γ_{T} and γ_{R} defined in eq. (2.7).

It will be convenient to denote by $\mathfrak{G}_{\mathbf{x}}$ the map $\mathbf{y} \mapsto \mathfrak{G}(\mathbf{x} - \mathbf{y})$. Let $\vec{\mathbf{E}}_j \in \mathbb{R}^8$ denote the constant vector with 1 at the j-th entry and zeros elsewhere, $j = 1, ..., 8$. Similarly for $\mathbf{E}_k \in \mathbb{R}^3$, $k = 1, 2, 3$.

Adapting the calculations found in [14, Sec. 4.2], we will establish integral representation formulas for these potentials by splitting the pairings into their components.

Lemma 3.4 (Integral representations of surface potentials)**.** *Given* $\vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ *and* $\vec{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$*, it holds for* $\mathbf{x} \in \Omega\backslash\Gamma$ *that*

$$\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}})(\mathbf{x}) \cdot \vec{\mathbf{E}}_j = -\langle\!\langle \vec{\mathbf{a}}\,,\, \gamma_{\mathsf{T}}^- \left(\mathfrak{G}_{\mathbf{x}}\,\vec{\mathbf{E}}_j\right) \rangle\!\rangle, \tag{3.39a}$$

$$\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}})(\mathbf{x}) \cdot \vec{\mathbf{E}}_j = \langle\!\langle \vec{\mathbf{b}}\,,\, \gamma_{\mathsf{R}}^- \left(\mathfrak{G}_{\mathbf{x}}\,\vec{\mathbf{E}}_j\right) \rangle\!\rangle. \tag{3.39b}$$

Proof. Let $V \in C_0^\infty(\mathbb{R}^3)$ and suppose that $\vec{\mathbf{a}}$ is the trace of a smooth 8-dimensional vector-field. Using Fubini's theorem and the fact that $\mathfrak{G}$ is smooth away from the origin,

$$\langle \mathfrak{N}\left(\gamma_{\mathsf{T}}^-\right)'\vec{\mathbf{a}},\, V\vec{\mathbf{E}}_j\rangle = \langle\!\langle \vec{\mathbf{a}}, \gamma_{\mathsf{T}}\mathfrak{N}\left(V\vec{\mathbf{E}}_j\right)\rangle\!\rangle = \int_\Gamma \vec{\mathbf{a}}(\mathbf{y}) \cdot \gamma_{\mathsf{T}} \overline{\int_{\mathbb{R}^3} \mathfrak{G}(\mathbf{y} - \mathbf{x})\, V(\mathbf{x})\, \vec{\mathbf{E}}_j(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\sigma(\mathbf{y})} \tag{3.40}$$

$$\overset{(*)}{=} -\int_{\mathbb{R}^3} \overline{V(\mathbf{x})} \left(\int_\Gamma \vec{\mathbf{a}}(\mathbf{y}) \cdot \gamma_{\mathsf{T}}\mathfrak{G}(\mathbf{x} - \mathbf{y})\vec{\mathbf{E}}_j\, \mathrm{d}\sigma(\mathbf{y})\right) \mathrm{d}\mathbf{x}, \tag{3.41}$$

where the sign was obtained in $(*)$ because of Lemma 3.2. The integrals on the right-hand side of (3.41) can be extended to duality pairings by a standard density argument exploiting Lemma 2.1.

Similar calculations can be carried out for $\mathcal{L}_{\mathsf{R}}$. □

In particular,

$$\mathfrak{G}_{\mathbf{x}}(\mathbf{y})\;\vec{\mathbf{E}}_1 = \begin{pmatrix} 0 \\ \nabla G(\mathbf{x} - \mathbf{y}) \\ \mathbf{0} \\ 0 \end{pmatrix}, \qquad \mathfrak{G}_{\mathbf{x}}(\mathbf{y})\;\vec{\mathbf{E}}_8 = \begin{pmatrix} 0 \\ \mathbf{0} \\ -\nabla G(\mathbf{x} - \mathbf{y}) \\ 0 \end{pmatrix}, \tag{3.42}$$

$$\mathfrak{G}_{\mathbf{x}}(\mathbf{y})\;\vec{\mathbf{E}}_i = \begin{pmatrix} -\frac{\partial}{\partial \mathbf{z}_{\mu(i)}} G(\mathbf{z}) \\ \mathbf{0} \\ \nabla G(\mathbf{z}) \times \mathbf{E}_{\mu(i)} \\ 0 \end{pmatrix}\Bigg|_{\mathbf{z}=\mathbf{x}-\mathbf{y}}, \qquad \mathfrak{G}_{\mathbf{x}}(\mathbf{y})\;\vec{\mathbf{E}}_k = \begin{pmatrix} 0 \\ \nabla G(\mathbf{z}) \times \mathbf{E}_{\nu(k)} \\ \mathbf{0} \\ \frac{\partial}{\partial \mathbf{z}_{\nu(k)}} G(\mathbf{z}) \end{pmatrix}\Bigg|_{\mathbf{z}=\mathbf{x}-\mathbf{y}}, \tag{3.43}$$

for $i = 2, 3, 4$, $k = 5, 6, 7$, $\mu(i) = i - 1$ and $\nu(k) = k - 4$.

Therefore, we can evaluate

$$\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}})(\mathbf{x})\cdot\vec{\mathbf{E}}_1 = -\int_\Gamma \mathbf{a}_1(\mathbf{y})\cdot\nabla G(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \tag{3.44a}$$

$$\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}})(\mathbf{x})\cdot\vec{\mathbf{E}}_i = \int_\Gamma a_0(\mathbf{y})\,\partial_{\mu(i)}G(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) - \int_\Gamma a_2(\mathbf{y})\,\left(\nabla G(\mathbf{x}-\mathbf{y})\times\mathbf{E}_{\mu(i)}\right)\cdot\mathbf{n}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \tag{3.44b}$$

$$= \partial_{\mu(i)}\int_\Gamma a_0(\mathbf{y})\,G(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) + \mathbf{E}_{\mu(i)}\cdot\int_\Gamma a_2(\mathbf{y})\,\nabla G(\mathbf{x}-\mathbf{y})\times\mathbf{n}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \tag{3.44c}$$

$$\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}})(\mathbf{x})\cdot\vec{\mathbf{E}}_k = -\int_\Gamma \mathbf{a}_1(\mathbf{y})\cdot\left(\nabla G(\mathbf{x}-\mathbf{y})\times\mathbf{E}_{\nu(k)}\right)\mathrm{d}\sigma(\mathbf{y}) \tag{3.44d}$$

$$= \mathbf{E}_{\nu(k)}\cdot\int_\Gamma \nabla_{\mathbf{y}}G(\mathbf{x}-\mathbf{y})\times\mathbf{a}_1(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) \tag{3.44e}$$

$$\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}})(\mathbf{x})\cdot\vec{\mathbf{E}}_8 = \int_\Gamma a_2(\mathbf{y})\,\nabla_{\mathbf{y}}G_{\mathbf{x}}(\mathbf{y})\cdot\mathbf{n}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,, \tag{3.44f}$$

where we have used the fact that $\mathbf{a}_1 \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ was "tangential" to safely drop the trace γ_t everywhere. Evidently, similarly as in the proof of Lemma 3.4, all these integrals should be understood as duality pairings and the following explicit representations not only in the sense of distributions by pointwise on $\mathbb{R}^3\backslash\Gamma$.

We collect the above entries to obtain

$$\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}}) = \begin{pmatrix} -\operatorname{div}\mathbf{\Psi}(\mathbf{a}_1) \\ \nabla\psi(a_0) + \mathbf{curl}\,\mathbf{\Upsilon}(a_2) \\ \mathbf{curl}\,\mathbf{\Psi}(\mathbf{a}_1) \\ \operatorname{div}\mathbf{\Upsilon}(a_2) \end{pmatrix}, \quad \text{pointwise on } \mathbb{R}^3\backslash\Gamma, \tag{3.45}$$

where we respectively recognize in

$$\psi(q)(\mathbf{x}) := \int_\Gamma q(\mathbf{y})G_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x}\in\mathbb{R}^3\backslash\Gamma, \tag{3.46a}$$

$$\mathbf{\Psi}(\mathbf{p})(\mathbf{x}) := \int_\gamma \mathbf{p}(\mathbf{y})G_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x}\in\mathbb{R}^3\backslash\Gamma, \tag{3.46b}$$

$$\mathbf{\Upsilon}(q)(\mathbf{x}) := \int_\Gamma q(\mathbf{y})\,\mathbf{G}_{\mathbf{x}}(\mathbf{y})\,\mathbf{n}\,\mathrm{d}\sigma(\mathbf{y}) \qquad \mathbf{x}\in\mathbb{R}^3\backslash\Gamma, \tag{3.46c}$$

the well-known single layer, vector single layer and normal vector single layer potentials. They notably enter eq. (3.45) in the expression for the classical double layer potential div $\mathbf{\Upsilon}(q)$ and for the Maxwell double layer potential $\mathbf{curl}\mathbf{\Psi}(\mathbf{p})$ as they arise in acoustic and electromagnetic scattering respectively.

Similarly, for $i = 2, 3, 4$ and $k = 5, 6, 7$,

$$\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}})(\mathbf{x}) \cdot \vec{\mathbf{E}}_1 = \int_\Gamma b_0(\mathbf{y}) \nabla G(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47a}$$

$$\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}})(\mathbf{x}) \cdot \vec{\mathbf{E}}_i = \int_\Gamma \mathbf{b}_1(\mathbf{y}) \cdot \left(\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\mu(i)}\right) \times \mathbf{n}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47b}$$

$$= \int_\Gamma \left(\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\mu(i)}\right) \cdot \mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47c}$$

$$= \mathbf{E}_{\mu(i)} \cdot \int_\Gamma \left(\mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y})\right) \times \nabla G(\mathbf{x}-\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47d}$$

$$\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}})(\mathbf{x}) \cdot \vec{\mathbf{E}}_k = \int_\Gamma b_0(\mathbf{y}) \left(\nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{E}_{\nu(k)}\right) \cdot \mathbf{n}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) + \int_\Gamma \mathscr{b}_2(\mathbf{y})\, \partial_j G(\mathbf{x}-\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47e}$$

$$= \mathbf{E}_{\nu(k)} \cdot \int_\Gamma b_0(\mathbf{y})\, \mathbf{n}(\mathbf{y}) \times \nabla G(\mathbf{x}-\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) + \int_\Gamma \mathscr{b}_2(\mathbf{y})\, \partial_j G(\mathbf{x}-\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47f}$$

$$\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}})(\mathbf{x}) \cdot \vec{\mathbf{E}}_8 = -\int_\Gamma \mathbf{b}_1(\mathbf{y}) \cdot \nabla G(\mathbf{x}-\mathbf{y}) \times \mathbf{n}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47g}$$

$$= -\int_\Gamma \nabla G(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \times \mathbf{b}_1(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}) \tag{3.47h}$$

so that we have

$$\mathscr{L}_{\mathsf{R}}\left(\vec{\mathbf{b}}\right) = \begin{pmatrix} \mathrm{div}\boldsymbol{\Upsilon}(b_0) \\ \mathbf{curl}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n}) \\ -\mathbf{curl}\boldsymbol{\Upsilon}(b_0) + \nabla\psi(\mathscr{b}_2) \\ \mathrm{div}\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n}) \end{pmatrix}, \quad \text{pointwise on } \mathbb{R}^3\backslash\Gamma. \tag{3.48}$$

3.3. **Mapping properties of the surface potentials.** Fortunately, we already know a lot about each potential entering eq. (3.45) and eq. (3.48).

Lemma 3.5. *The potentials $\mathscr{L}_{\mathsf{T}} : \mathscr{H}_{\mathsf{R}} \to \mathbf{H}(\mathsf{D}, \mathbb{R}^3\backslash\Gamma)$ and $\mathscr{L}_{\mathsf{R}} : \mathscr{H}_{\mathsf{T}} \to \mathbf{H}(\mathsf{D}, \mathbb{R}^3\backslash\Gamma)$ explicitly given by eq. (3.45) and eq. (3.48) are continuous.*

Proof. Recall that if $\mathbf{b}_1 \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, then $\mathbf{n} \times \mathbf{b}_1 \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. The proof simply boils down to extracting from the discussion of Section 5 in [14] the mapping properties

$$\nabla\psi : H^{-1/2}(\Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \mathbb{R}^3\backslash\Gamma) \cap \mathbf{H}_{\mathrm{loc}}(\nabla\mathrm{div}, \mathbb{R}^3\backslash\Gamma), \tag{3.49a}$$

$$\mathrm{div}\boldsymbol{\Upsilon} : H^{1/2} \to H^1_{\mathrm{loc}}(\Delta, \mathbb{R}^3\backslash\Gamma), \tag{3.49b}$$

$$\mathbf{curl}\,\boldsymbol{\Upsilon} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3\backslash\Gamma), \tag{3.49c}$$

$$\mathrm{div}\,\boldsymbol{\Psi} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\Gamma), \tag{3.49d}$$

$$\mathbf{curl}\,\boldsymbol{\Psi} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3\backslash\Gamma). \tag{3.49e}$$

Since $\mathrm{div} \circ \mathbf{curl} \equiv 0$, we have in particular

$$\mathbf{curl}\,\boldsymbol{\Upsilon} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3\backslash\Gamma) \cap \mathbf{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3\backslash\Gamma), \tag{3.50a}$$

$$\mathbf{curl}\,\boldsymbol{\Psi} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3\backslash\Gamma) \cap \mathbf{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3\backslash\Gamma). \tag{3.50b}$$

For $\mathbf{z} \neq \mathbf{0}$, the kernels of the two surface potentials decay as $\left\|\nabla G(\mathbf{z})\right\|_\infty \lesssim \|\mathbf{z}\|_\infty^{-2}$. Therefore, they are not only locally square-integrable, but they in fact truly lie in $(L^2(\mathbb{R}^3\backslash\Gamma))^8$. The next lemma guarantees that they provide solutions of the homogeneous Dirac equation.

□

Lemma 3.6. *For all* $\vec{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$ *and* $\vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$*, it holds on* $\mathbb{R}^3 \backslash \Gamma$ *that*

$$\mathsf{D}\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) \equiv \vec{\mathbf{0}}, \tag{3.51a}$$

$$\mathsf{D}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) \equiv \vec{\mathbf{0}}. \tag{3.51b}$$

Proof. The well-known vector and scalar potentials of (3.46) are harmonic. Hence, since $\operatorname{div} \circ \operatorname{\mathbf{curl}} \equiv 0$ and $\operatorname{\mathbf{curl}} \circ \nabla \equiv 0$, we directly evaluate

$$\begin{aligned}
\mathsf{D}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) &= \begin{pmatrix} -\operatorname{div} \nabla\psi(\mathsf{a}_0) - \operatorname{div} \operatorname{\mathbf{curl}} \boldsymbol{\Upsilon}(a_2) \\ -\nabla \operatorname{div} \boldsymbol{\Psi}(\mathbf{a}_1) + \operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{\Psi}(\mathbf{a}_1) \\ \operatorname{\mathbf{curl}} \nabla\psi(\mathsf{a}_0) + \operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{\Upsilon}(a_2) - \nabla \operatorname{div} \boldsymbol{\Upsilon}(a_2) \\ \operatorname{div} \operatorname{\mathbf{curl}} \boldsymbol{\Psi}(\mathbf{a}_1) \end{pmatrix} \\
&= \begin{pmatrix} -\Delta\psi(\mathsf{a}_0) \\ -\nabla \operatorname{div} \boldsymbol{\Psi}(\mathbf{a}_1) + \operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{\Psi}(\mathbf{a}_1) \\ -\nabla \operatorname{div} \boldsymbol{\Upsilon}(a_2) + \operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{\Upsilon}(a_2) \\ 0 \end{pmatrix} = \vec{\mathbf{0}}.
\end{aligned} \tag{3.52}$$

A similar calculation holds for $\mathsf{D}\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}})$. □

Remark 3.2. *The above lemma was proven using the explicit representations* (3.45) *and* (3.48)*. As such, the technique we have used has revealed some of the structure behind the two boundary potentials. However, also notice that adapting the argument found in the poof of [33, Thm. 3.1.6], the desired result could also be obtained by observing that*

$$\gamma_{\mathsf{T}}' : (\mathcal{H}_{\mathsf{R}}) \to (\mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \mathbb{R}^3 \backslash \Gamma))' \subset (C^\infty(\mathbb{R}^3 \backslash \Gamma)^8)', \tag{3.53}$$

together with Proposition 3.1, guarantees that $\mathsf{D}\mathcal{L}_{\mathsf{T}}\vec{\mathbf{a}} = \gamma_{\mathsf{T}}'\vec{\mathbf{a}}$ *in the sense of a functional on* $C_0^\infty(\mathbb{R}^3 \backslash \Gamma)$*.*

Remark 3.3. *We want to highlight that it is a nice and unusual property for the potentials* $\mathcal{L}_{\mathsf{T}}$ *and* $\mathcal{L}_{\mathsf{R}}$ *to be everywhere square-integrable in the exterior domain, as opposed to being only locally* $(L^2(\Omega^+))^8$*. We see from Lemma 3.4 that it is the result of two ingredients: the stronger singularity of the Dirac fundamental solution combined with the absence of differential operators in the relevant traces.*

Lemma 3.7 (Jump relations)**.** *For all* $\vec{\mathbf{a}} \in \mathcal{H}_{\mathsf{R}}$ *and* $\vec{\mathbf{b}} \in \mathcal{H}_{\mathsf{T}}$*,*

$$[\gamma_{\mathsf{T}}]\, \mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) = \vec{\mathbf{0}}, \qquad [\gamma_{\mathsf{R}}]\, \mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) = \mathrm{Id}, \tag{3.54}$$

$$[\gamma_{\mathsf{T}}]\, \mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) = \mathrm{Id}, \qquad [\gamma_{\mathsf{R}}]\, \mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) = \vec{\mathbf{0}}. \tag{3.55}$$

Remark 3.4. *It is striking that the abstract structure of these jump relations for the potentials of the Dirac operator is the same as for the jump identities of the potentials associated with*

- *scalar second-order elliptic operators*
- *Hodge–Laplace and Hodge–Helmholtz operators*
- *Maxwell equations in frequency domain (e.g. electric wave operator).*

Proof. For the most part, the following jump relations can be inferred from known theory. We carefully evaluate

$$[\gamma_{\mathsf{T}}]\,\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}}) = \begin{pmatrix} -[\gamma]\operatorname{div}\boldsymbol{\Psi}\,(\mathbf{a}_1) \\ [\gamma_t]\,\nabla\psi\,(a_0) + [\gamma_t]\,\mathbf{curl}\boldsymbol{\Upsilon}\,(a_2) \\ [\gamma_n]\,\mathbf{curl}\boldsymbol{\Psi}\,(\mathbf{a}_1) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}, \tag{3.56a}$$

$$[\gamma_{\mathsf{R}}]\,\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}}) = \begin{pmatrix} [\gamma_n]\,\nabla\psi\,(\mathsf{a}_0) + [\gamma_n]\,\mathbf{curl}\boldsymbol{\Upsilon}\,(a_2) \\ [\gamma_\tau]\,\mathbf{curl}\boldsymbol{\Psi}\,(\mathbf{a}_1) \\ [\gamma]\operatorname{div}\boldsymbol{\Upsilon}\,(a_2) \end{pmatrix} = \begin{pmatrix} a_0 \\ \mathbf{a}_1 \\ a_2 \end{pmatrix}, \tag{3.56b}$$

$$[\gamma_{\mathsf{T}}]\,\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}}) = \begin{pmatrix} [\gamma]\operatorname{div}\boldsymbol{\Upsilon}\,(b_0) \\ [\gamma_t]\,\mathbf{curl}\boldsymbol{\Psi}\,(\mathbf{b}_1\times\mathbf{n}) \\ -[\gamma_n]\,\mathbf{curl}\boldsymbol{\Upsilon}\,(b_0) + [\gamma_n]\,\nabla\psi\,(b_2) \end{pmatrix} = \begin{pmatrix} b_0 \\ \mathbf{b}_1 \\ b_2 \end{pmatrix}, \tag{3.56c}$$

$$[\gamma_{\mathsf{R}}]\,\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}}) = \begin{pmatrix} [\gamma_n]\,\mathbf{curl}\boldsymbol{\Psi}\,(\mathbf{b}_1\times\mathbf{n}) \\ -[\gamma_\tau]\,\mathbf{curl}\boldsymbol{\Upsilon}\,(b_0) + [\gamma_\tau]\,\nabla\psi\,(b_2) \\ [\gamma]\operatorname{div}\Psi\,(\mathbf{b}_1\times\mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}. \tag{3.56d}$$

The individual terms appearing in the above calculations can be found in [14, Sec. 5] and [20, Sec. 4], possibly up to tangential rotation by 90°. Some terms slightly differ. In both eq. (3.56b) and eq. (3.56b), we are particularly concerned with the normal jump of $\mathbf{curl}\,\boldsymbol{\Upsilon}$ across Γ. Fortunately, we know that the restriction of $\boldsymbol{\Upsilon}$ to $H^{1/2}(\Gamma)$ is a continuous map with codomain $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2,\Omega)$. It's image is therefore regular enough for the identity

$$[\gamma_n]\,\mathbf{curl}\,\boldsymbol{\Upsilon}\,(q) = \operatorname{div}_\Gamma\big([\gamma_\tau]\,\boldsymbol{\Upsilon}\,(q)\big) = 0$$

to hold for all $q\in H^{1/2}(\Gamma)$ [12, Eq. 8]. □

3.4. **Representation by surface potentials.** We mimic the approach of Costabel [16]. Corollary 2.1 here plays the role of Green's second identity.

Proposition 3.2 (Representation formula for compactly supported distributions)**.** *For* $\vec{\mathbf{U}}\in\mathbf{H}_{\mathrm{loc}}(\mathsf{D},\mathbb{R}^3\backslash\Gamma)$ *with compact support and* $\vec{\mathbf{F}}\in(L^2(\mathbb{R}^3))^8$ *satisfying* $\vec{\mathbf{F}}\big|_\Omega = (\mathsf{D}\mathbf{U})\big|_\Omega$ *and* $\vec{\mathbf{F}}\big|_{\mathbb{R}^3\backslash\overline{\Omega}} = (\mathsf{D}\mathbf{U})\big|_{\mathbb{R}^3\backslash\overline{\Omega}}$ *in the sense of distributions, then*

$$\vec{\mathbf{U}} = \mathfrak{G}*\vec{\mathbf{F}} + \mathscr{L}_{\mathsf{T}}\left[\gamma_{\mathsf{R}}\vec{\mathbf{U}}\right] + \mathscr{L}_{\mathsf{R}}\left[\gamma_{\mathsf{T}}\vec{\mathbf{U}}\right] \tag{3.57}$$

as a functional on $(C_0^\infty(\mathbb{R}^3))^8$.

Proof. According to eq. (2.12),

$$\begin{aligned}\langle\mathsf{D}\vec{\mathbf{U}},\vec{\mathbf{V}}\rangle \overset{(*)}{=} \int_\Omega \vec{\mathbf{U}}\cdot\mathsf{D}\vec{\mathbf{V}}\,\mathrm{d}\mathbf{x} &= \int_\Omega \vec{\mathbf{F}}\cdot\vec{\mathbf{V}}\,\mathrm{d}\mathbf{x} + \langle\!\langle\gamma_{\mathsf{R}}^-\vec{\mathbf{U}},\gamma_{\mathsf{T}}^-\vec{\mathbf{V}}\rangle\!\rangle - \langle\!\langle\gamma_{\mathsf{T}}^-\vec{\mathbf{U}},\gamma_{\mathsf{R}}^-\vec{\mathbf{V}}\rangle\!\rangle \\ &\quad + \int_{\mathbb{R}^3\backslash\overline{\Omega}} \vec{\mathbf{F}}\cdot\vec{\mathbf{V}}\,\mathrm{d}\mathbf{x} - \langle\!\langle\gamma_{\mathsf{R}}^+\vec{\mathbf{U}},\gamma_{\mathsf{T}}^+\vec{\mathbf{V}}\rangle\!\rangle + \langle\!\langle\gamma_{\mathsf{T}}^+\vec{\mathbf{U}},\gamma_{\mathsf{R}}^+\vec{\mathbf{V}}\rangle\!\rangle \\ &= \int_{\mathbb{R}^3} \vec{\mathbf{F}}\cdot\vec{\mathbf{V}}\,\mathrm{d}\mathbf{x} + \langle\!\langle\left[\gamma_{\mathsf{R}}\vec{\mathbf{U}}\right],\gamma_{\mathsf{T}}\vec{\mathbf{V}}\rangle\!\rangle - \langle\!\langle\left[\gamma_{\mathsf{T}}\vec{\mathbf{U}}\right],\gamma_{\mathsf{R}}\vec{\mathbf{V}}\rangle\!\rangle\end{aligned} \tag{3.58}$$

for all $\vec{\mathbf{V}}\in(C^\infty(\mathbb{R}^3))^8$. The regularity assumptions on $\vec{\mathbf{U}}$ guarantee that the traces are well-defined. We have used the fact that $\vec{\mathbf{V}}$ is smooth across the boundary to obtain the last equality, because smoothness guarantees that $\gamma_{\mathsf{T}}^-\vec{\mathbf{V}} = \gamma_{\mathsf{T}}^+\vec{\mathbf{V}}$ and $\gamma_{\mathsf{R}}^-\vec{\mathbf{V}} = \gamma_{\mathsf{R}}^+\vec{\mathbf{V}}$. Therefore, in the sense of distributions, we have

$$\mathsf{D}\vec{\mathbf{U}} = \mathbf{F} + \left(\gamma_{\mathsf{T}}^-\right)'\left[\gamma_{\mathsf{R}}^-\vec{\mathbf{U}}\right] - \left(\gamma_{\mathsf{R}}^-\right)'\left[\gamma_{\mathsf{T}}^-\vec{\mathbf{U}}\right]. \tag{3.59}$$

Since $\vec{\mathbf{U}}$ is assumed to have compact support, convolution with the fundamental solution $\mathfrak{G}$ for the Dirac operator using Proposition 3.1 leads to the desired representation formula. □

Remark 3.5. *Proposition 3.2 is stated in terms of an identity between functionals, but Lemma 3.5 says that in fact the equality holds in* $(L^2(\mathbb{R}^3))^8$.

In the following, we will work over the domains defined as the interior B_ρ and exterior B_ρ^+ of an open ball of radius ρ. Therefore, we must introduce the traces γ_{T}^ρ and γ_{R}^ρ that extend the restrictions defined in (2.5) where Γ is replaced by the boundary ∂B_ρ of the open ball. The surface potentials $\mathcal{L}_{\mathsf{R}}^\rho$ and $\mathcal{L}_{\mathsf{T}}^\rho$ are defined accordingly with respect to these trace mappings. Similarly, a dagger † will refer to any given Lipschitz domain $\Omega_\dagger \subset \mathbb{R}^3$. The following development is to be compared with [28, Sec. 7].

Lemma 3.8. *For* $\vec{\mathbf{U}} \in (C_0^\infty(\Omega^+)^8)'$ *such that* $\mathsf{D}\vec{\mathbf{U}}$ *has compact support in* Ω^+*, there exists a* ***unique*** *vector field* $\mathcal{M}\vec{\mathbf{U}} \in (C^\infty(\mathbb{R}^3))^8$ *such that*

$$\mathcal{M}\vec{\mathbf{U}}\,(\mathbf{x}) = \mathcal{L}_{\mathsf{T}}^\dagger\left(\gamma_{\mathsf{R}}^\dagger\vec{\mathbf{U}}\right)(\mathbf{x}) + \mathcal{L}_{\mathsf{R}}^\dagger\left(\gamma_{\mathsf{T}}^\dagger\vec{\mathbf{U}}\right)(\mathbf{x}) \tag{3.60}$$

for all $\mathbf{x}$ *inside* ***any*** *bounded Lipschitz domain* $\Omega_\dagger$ *such that*

$$\overline{\Omega} \cup \operatorname{supp}(\mathsf{D}\vec{\mathbf{U}}) \Subset \Omega_\dagger. \tag{3.61}$$

Remark 3.6. *It is key in the statement of Lemma 3.8 that the vector field* $\mathcal{M}\vec{\mathbf{U}}$ *is* ***independent*** *of* $\Omega_\dagger$.

Proof. Under the above hypotheses, $\vec{\mathbf{U}}$ is an harmonic distribution in $\Omega^+\backslash\operatorname{supp}(\mathsf{D}\vec{\mathbf{U}})$, because $\mathsf{D}\vec{\mathbf{U}} = \vec{\mathbf{0}}$ implies that $\vec{\boldsymbol{\Delta}}\vec{\mathbf{U}} = \mathsf{D}^2\vec{\mathbf{U}} = \vec{\mathbf{0}}$ as in (1.8). Standard elliptic regularity theory [28, Thm. 6.4] further tells us that $\vec{\mathbf{U}}$ is a regular distribution whose coefficients are smooth in that domain. Therefore, we can *define* $\mathcal{M}\vec{\mathbf{U}}$ in B_{ρ_1} as in the right hand side of (3.60) by

$$\mathcal{M}\vec{\mathbf{U}}\,(\mathbf{x}) := \mathcal{L}_{\mathsf{T}}^{\rho_1}\left(\gamma_{\mathsf{R}}^{\rho_1}\vec{\mathbf{U}}\right)(\mathbf{x}) + \mathcal{L}_{\mathsf{R}}^{\rho_1}\left(\gamma_{\mathsf{T}}^{\rho_1}\vec{\mathbf{U}}\right)(\mathbf{x}), \tag{3.62}$$

where the radius ρ_1 is large enough that $\overline{\Omega} \cup \operatorname{supp}(\mathsf{D}\vec{\mathbf{U}}) \Subset B_{\rho_1}$.

Applying eq. (2.12) inside $B_{\rho_2}\backslash\overline{B}_{\rho_1}$ with $\rho_1 < \rho_2$ eventually shows that this definition is independent of the radius. Indeed, for any $\mathbf{x} \in B_{\rho_1}$, $\mathfrak{G}_{\mathbf{x}}$ is a smooth matrix in $\mathbb{R}^3\backslash\overline{B_{\rho_1}}$, and thus $\operatorname{supp}(\mathsf{D}\vec{\mathbf{U}}) \Subset B_{\rho_1}$ guarantees for $i = 1, ..., 8$ that

$$\begin{aligned} 0 &= \int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \mathfrak{G}\,(\mathbf{x}-\mathbf{y})\,\mathsf{D}\vec{\mathbf{U}}\,(\mathbf{y})\,\mathrm{d}\mathbf{y}\cdot\vec{\mathbf{E}}_i = \int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \mathfrak{G}_{i,:}\,(\mathbf{x}-\mathbf{y})\,\mathsf{D}\vec{\mathbf{U}}\,(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &\overset{(*)}{=} -\int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \mathfrak{G}_{:,i}\,(\mathbf{x}-\mathbf{y})\cdot\mathsf{D}\vec{\mathbf{U}}\,(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{B_{\rho_2}\backslash\overline{B}_{\rho_1}} \mathsf{D}_{\mathbf{y}}\mathfrak{G}_{:,i}\,(\mathbf{x}-\mathbf{y})\cdot\vec{\mathbf{U}}\,(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &\quad - \langle\!\langle\gamma_{\mathsf{T}}^{\rho_2}\vec{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_2}\mathfrak{G}_{:,i}\,(\mathbf{x}-\cdot)\rangle\!\rangle - \langle\!\langle\gamma_{\mathsf{T}}^{\rho_2}\mathfrak{G}_{:,i}\,(\mathbf{x}-\cdot)\,, \gamma_{\mathsf{R}}^{\rho_2}\vec{\mathbf{U}}\rangle\!\rangle \\ &\quad + \langle\!\langle\gamma_{\mathsf{T}}^{\rho_1}\vec{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_1}\mathfrak{G}_{:,i}\,(\mathbf{x}-\cdot)\rangle\!\rangle + \langle\!\langle\gamma_{\mathsf{T}}^{\rho_1}\mathfrak{G}_{:,i}\,(\mathbf{x}-\cdot)\,, \gamma_{\mathsf{R}}^{\rho_1}\vec{\mathbf{U}}\rangle\!\rangle, \end{aligned} \tag{3.63}$$

where $\mathfrak{G}_{i,:}$ corresponds to the i-th row of $\mathfrak{G}$, $\mathfrak{G}_{:,j}$ to its j-th column, and Lemma 3.1 was used to obtain $(*)$.

On the one hand, for $\mathbf{x} \neq \mathbf{y}$, we obtain from (3.10) and (3.5) that

$$\begin{aligned} \mathsf{D}_{\mathbf{y}}\mathfrak{G}_{:,i}\,(\mathbf{x}-\mathbf{y})\cdot\vec{\mathbf{U}}\,(\mathbf{y}) &= \mathsf{D}_{\mathbf{x}}\left(\mathfrak{G}_{\mathbf{x}}\,(\mathbf{y})\,\vec{\mathbf{U}}\,(\mathbf{y})\right)\cdot\vec{\mathbf{E}}_i \\ &= \mathsf{D}_{\mathbf{x}}\mathsf{D}_{\mathbf{x}}\left(\mathsf{G}\,(\mathbf{x}-\mathbf{y})\,\vec{\mathbf{U}}\,(\mathbf{y})\right)\cdot\vec{\mathbf{E}}_i = \left(-\Delta_{\mathbf{x}}G\,(\mathbf{x}-\mathbf{y})\right)\vec{\mathbf{U}}\,(\mathbf{y})\cdot\vec{\mathbf{E}}_i = 0 \end{aligned} \tag{3.64}$$

as in (3.9). On the other hand,

$$\langle\!\langle\gamma_{\mathsf{T}}^{\rho_2}\vec{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_2}\mathfrak{G}_{:,i}\,(\mathbf{x}-\cdot)\rangle\!\rangle = -\langle\!\langle\gamma_{\mathsf{T}}^{\rho_2}\vec{\mathbf{U}}, \gamma_{\mathsf{R}}^{\rho_2}\left(\mathfrak{G}_{\mathbf{x}}\vec{\mathbf{E}}_i\right)\rangle\!\rangle = -\mathcal{L}_{\mathsf{R}}^{\rho_2}\left(\gamma_{\mathsf{T}}^{\rho_2}\vec{\mathbf{U}}\right)(\mathbf{x})\cdot\vec{\mathbf{E}}_j \tag{3.65}$$

by Lemma 3.4 and similarly for the remaining boundary terms. These two pieces of information together proves validity of the independence claim.

In fact, the same argument can be repeated in $B_{\rho_1}\backslash\overline{\Omega_\dagger}$ to confirm that (3.60) holds independently of the chosen Lipschitz domain satisfying the hypotheses.

Smoothness of $\mathcal{M}\vec{\mathbf{U}}$ is inherited from the smoothness of the integrands. □

Proposition 3.3 (Exterior representation formula)**.** *Suppose that* $\vec{\mathbf{F}} \in (L^2(\Omega^+))^8$ *has compact support and that* $\vec{\mathbf{U}} \in (C_0^\infty(\Omega^+)^8)'$ *satisfies* $\mathsf{D}\vec{\mathbf{U}} = \vec{\mathbf{F}}$ *on* Ω^+*. If the restriction (in the sense of distributions) of* $\vec{\mathbf{U}}$ *to* $\Omega^+ \cap B_\rho$ *belongs to* $\mathbf{H}(\mathsf{D}, \Omega^+ \cap B_\rho)$ *for some* ρ *large enough that* $\Omega \cup \Gamma \Subset B_\rho$ *and* $\operatorname{supp} \vec{\mathbf{F}} \Subset \Omega^+ \cap B_\rho$*, then*

$$\vec{\mathbf{U}} = \mathfrak{G} * \vec{\mathbf{F}} - \mathcal{L}_{\mathsf{T}} \gamma_{\mathsf{R}}^+ \vec{\mathbf{U}} - \mathcal{L}_{\mathsf{R}} \gamma_{\mathsf{T}}^+ \vec{\mathbf{U}} + \mathcal{M}\vec{\mathbf{U}} \tag{3.66}$$

in $\mathbf{H}(\mathsf{D}, \Omega^+)$.

Proof. Upon applying Proposition 3.2 to the distribution

$$\vec{\mathbf{U}}_0 := \begin{cases} \vec{\mathbf{0}} & \Omega \\ \vec{\mathbf{U}} & \Omega^+ \cap B_\rho \\ \vec{\mathbf{0}} & \mathbb{R}^3 \backslash \overline{B_\rho} \end{cases} \tag{3.67}$$

that is compactly supported and belongs to $\mathbf{H}_{\mathrm{loc}}\left(\mathsf{D}, \mathbb{R}^3 \backslash \left(\Gamma \cup \partial B_\rho\right)\right)$, we obtain

$$\vec{\mathbf{U}}_0 = \mathfrak{G} * \vec{\mathbf{F}} - \mathcal{L}_{\mathsf{T}} \left(\gamma_{\mathsf{R}}^+ \vec{\mathbf{U}}\right) - \mathcal{L}_{\mathsf{R}} \left(\gamma_{\mathsf{T}}^+ \vec{\mathbf{U}}\right) + \mathcal{L}_{\mathsf{T}}^\rho \left(\gamma_{\mathsf{R}}^\rho \vec{\mathbf{U}}\right) + \mathcal{L}_{\mathsf{R}}^\rho \left(\gamma_{\mathsf{T}}^\rho \vec{\mathbf{U}}\right) \tag{3.68}$$

as a functional on $(C_0^\infty(\mathbb{R}^3))^8$. Since B_ρ satisfies the hypotheses imposed on $\Omega_\dagger$ in the statement of Lemma 3.8, we recognize that

$$\mathcal{L}_{\mathsf{T}}^\rho \left(\gamma_{\mathsf{R}}^\rho \vec{\mathbf{U}}\right)(\mathbf{x}) + \mathcal{L}_{\mathsf{R}}^\rho \left(\gamma_{\mathsf{T}}^\rho \vec{\mathbf{U}}\right)(\mathbf{x}) = \mathcal{M}\vec{\mathbf{U}}(\mathbf{x}) \tag{3.69}$$

for all $\mathbf{x} \in B_\rho$. Hence,

$$\vec{\mathbf{U}} = \mathfrak{G} * \vec{\mathbf{F}} - \mathcal{L}_{\mathsf{T}} \gamma_{\mathsf{R}}^+ \vec{\mathbf{U}} - \mathcal{L}_{\mathsf{R}} \gamma_{\mathsf{T}}^+ \vec{\mathbf{U}} + \mathcal{M}\vec{\mathbf{U}} \tag{3.70}$$

in $\Omega^+ \cap B_\rho$.

As in Lemma 3.8, it follows from $\operatorname{supp} \vec{\mathbf{F}} \subset B_\rho$ that $\vec{\mathbf{U}}$ is harmonic in $\mathbb{R}^3 \backslash B_\rho$, and thus smooth everywhere outside the ball B_ρ by well-known elliptic regularity theory [28, Thm. 6.4]. Hence, the hypothesis that $\vec{\mathbf{U}} \in \mathbf{H}(\mathsf{D}, \Omega^+ \cap B_\rho)$ for at least one ball B_ρ satisfying the hypotheses in fact guarantees that it belongs to that space independently of the radius satisfying those same requirements. Therefore, (3.70) holds in the whole of Ω^+. Based on Lemma 3.8, the mapping properties of the potentials established in Lemma 3.5 and Corollary 3.1, we conclude that the equality (3.70) holds in fact not only in $\mathbf{H}_{\mathrm{loc}}(\mathsf{D}, \Omega^+)$, but in $\mathbf{H}(\mathsf{D}, \Omega^+)$—which is the desired result. □

Lemma 3.9 (Decay conditions)**.** *Under the hypotheses of Proposition 3.3,*

$$\mathcal{M}\vec{\mathbf{U}} = \vec{\mathbf{0}} \tag{3.71}$$

if and only if

$$\left\|\vec{\mathbf{U}}(\mathbf{z})\right\|_\infty \to 0 \text{ as } \mathbf{z} \to \infty. \tag{3.72}$$

Proof. The condition (3.72) is well-defined, because as in Proposition 3.3, there exists a radius ρ_1 large enough that the vector-field $\vec{\mathbf{U}}$ is smooth outside B_{ρ_1}. For the same reason, the traces of $\vec{\mathbf{U}}$ appearing in the following inequalities are smooth boundary fields.

For $\mathbf{z} \neq \mathbf{0}$,

$$\left\|\nabla G(\mathbf{z})\right\|_\infty \lesssim \|\mathbf{z}\|_\infty^{-2}. \tag{3.73}$$

Therefore, it is easily seen from (3.45) and (3.48) that if $\rho_2 > \rho_1$,

$$\left\|\mathcal{L}_\bullet^{\rho_2}\left(\gamma_\bullet^{\rho_2}\vec{\mathbf{U}}\right)(\mathbf{x})\right\|_\infty \lesssim \rho_2^{-2} \left\|\int_{\partial B_{\rho_2}} \gamma_\bullet \vec{\mathbf{U}}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\right\|_\infty \lesssim \max_{\mathbf{y} \in \partial B_{\rho_2}} \left\|\vec{\mathbf{U}}(\mathbf{y})\right\|_\infty \tag{3.74}$$

for all $\mathbf{x} \in B_{\rho_1}$, $\bullet = \mathsf{T}$ or R. Notice that on the left hand side of (3.74), the maximum norm is appropriate, because as in Lemma 3.8, Lemma 3.6 and (1.8) guarantee that away from the boundary ∂B_{ρ_2}, the potentials are smooth harmonic vector fields. No differential operator appears in the definition of the trace mappings γ_{R} and γ_{T}. The independence of $\mathcal{M}\vec{\mathbf{U}}$ from its domain of definition thus directly yield one implication of the lemma upon taking $\rho_2 \to \infty$.

The converse follows from the exterior representation formula (3.66) from Proposition 3.3 with $\mathcal{M}\vec{\mathbf{U}} = \vec{\mathbf{0}}$ and an analysis exploiting (3.73) that leads to an inequality similar to (3.74). However, this time the potentials are computed as integrals (duality pairings) on the fixed boundary Γ and an inverse square decay is inherited from the decay of the fundamental solution. $\square$

We refer to [28, Chap. 7] for the arguments, which based on the above results, lead to well-posedness of the exterior problem. At this point, all the necessary ingredients were established so that it is easy to see that the common proofs found in the theory for second-order elliptic operators can be mimicked without difficulties.

4. Boundary integral operators

The operator form of the interior and exterior Calderón projectors defined on $\mathcal{H}_{\mathsf{R}} \times \mathcal{H}_{\mathsf{T}}$, which we denote P^- and P^+ respectively, enter the Calderón identities

$$\underbrace{\begin{pmatrix} \{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{T}} + \frac{1}{2}\mathrm{Id} & \{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{R}} \\ \{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{T}} & \{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{R}} + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathsf{P}^-} \begin{pmatrix} \gamma_{\mathsf{R}}^-(\mathbf{U}) \\ \gamma_{\mathsf{T}}^-(\mathbf{U}) \end{pmatrix} = \begin{pmatrix} \gamma_{\mathsf{R}}^-(\mathbf{U}) \\ \gamma_{\mathsf{T}}^-(\mathbf{U}) \end{pmatrix}, \tag{4.1}$$

$$\underbrace{\begin{pmatrix} -\{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{T}} + \frac{1}{2}\mathrm{Id} & -\{\gamma_{\mathsf{R}}\}\,\mathcal{L}_{\mathsf{R}} \\ -\{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{T}} & -\{\gamma_{\mathsf{T}}\}\,\mathcal{L}_{\mathsf{R}} + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathsf{P}^+} \begin{pmatrix} \gamma_{\mathsf{R}}^+(\mathbf{U}) \\ \gamma_{\mathsf{T}}^+(\mathbf{U}) \end{pmatrix} = \begin{pmatrix} \gamma_{\mathsf{R}}^+(\mathbf{U}) \\ \gamma_{\mathsf{T}}^+(\mathbf{U}) \end{pmatrix}. \tag{4.2}$$

These were obtained by taking the traces γ_{R} and γ_{T} on both sides of either the representation formula given by (3.57) for interior domains or (3.66) with $\mathcal{M}\vec{\mathbf{U}} = \vec{\mathbf{0}}$ for exterior domains—that is imposing (3.72) as a decay condition at infinity. Lemma 2.3, the representation formula (3.57) and the jump relations of Lemma 3.7 will guarantee the important property that a pair of traces $(\vec{\boldsymbol{a}}, \vec{\mathbf{b}}) \in \mathcal{H}_{\mathsf{R}} \times \mathcal{H}_{\mathsf{T}}$ is valid interior or exterior Cauchy data if and only if it lies in the kernel of P^+ or P^-, respectively.

4.1. **Integral representations of the duality pairings.** Let us take a closer look at the bilinear forms naturally associated with the continuous first-kind boundary integral operators

$$\gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}} : \mathcal{H}_{\mathsf{R}} \to \mathcal{H}_{\mathsf{T}}, \tag{4.3}$$

$$\gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}} : \mathcal{H}_{\mathsf{T}} \to \mathcal{H}_{\mathsf{R}}, \tag{4.4}$$

that map trace spaces to their dual spaces.

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$ be the trial and test boundary vector fields lying in $\mathcal{H}_{\mathsf{R}}$ respectively, and similarly for $\vec{\mathbf{b}}$ and $\vec{\mathbf{d}}$ in $\mathcal{H}_{\mathsf{T}}$. Catching up with the calculations of section 3.2, we want to derive convenient integral formulas for

$$\langle\!\langle \vec{\mathbf{c}}, \gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}}) \rangle\!\rangle = -\langle c_0, \gamma \operatorname{div} \boldsymbol{\Psi}(\mathbf{a}_1)\rangle_\Gamma + \langle \mathbf{c}_1, \gamma_t \nabla\psi(a_0)\rangle_\tau + \langle \mathbf{c}_1, \gamma_t\, \mathbf{curl}\,\boldsymbol{\Upsilon}(a_2)\rangle_\tau + \langle c_2, \gamma_n\, \mathbf{curl}\,\boldsymbol{\Psi}(\boldsymbol{a}_1)\rangle_\Gamma$$

and

$$\langle\!\langle \vec{\mathbf{d}}, \gamma_{\mathsf{R}}\mathcal{L}_{\mathsf{R}}(\vec{\mathbf{b}}) \rangle\!\rangle = \langle d_0, \gamma_n\, \mathbf{curl}\,\boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma - \langle \mathbf{d}_1, \gamma_\tau\, \mathbf{curl}\,\boldsymbol{\Upsilon}(b_0)\rangle_\tau + \langle \mathbf{d}_1, \gamma_\tau \nabla\psi(b_2)\rangle_\tau + \langle d_2, \gamma \operatorname{div} \boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma.$$

In the course of our derivation, we will often rely implicitly on the fact that $\mathbf{a}_1$ and $\mathbf{b}_1$ are tangential vector fields

Using the fact that $\operatorname{div} \boldsymbol{\Psi}(\mathbf{a}_1) = \psi(\operatorname{div}_\Gamma \mathbf{a}_1)$ and $\operatorname{div} \boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n}) = \psi(\operatorname{curl}_\Gamma \mathbf{b}_1)$, we immediately find that

$$\langle c_0, \gamma \operatorname{div} \boldsymbol{\Psi}(\mathbf{a}_1)\rangle_\Gamma = \int_\Gamma \int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, c_0(\mathbf{x}) \operatorname{div}_\Gamma \mathbf{a}_1(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x})\, \mathrm{d}\sigma(\mathbf{y})$$

and

$$\langle d_2, \gamma \operatorname{div} \boldsymbol{\Psi}(\mathbf{b}_1 \times \mathbf{n})\rangle_\Gamma = \int_\Gamma \int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, d_2(\mathbf{x}) \operatorname{curl}_\Gamma \mathbf{b}_1(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}). \tag{4.5}$$

We know from [14, Sec. 6.4] that

$$\begin{aligned}\langle \mathbf{d}_1, \gamma_\tau \,\mathbf{curl}\boldsymbol{\Upsilon}(b_0)\rangle_\tau &= -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\big(\mathbf{n}(\mathbf{x})\times\mathbf{d}_1(\mathbf{x})\big)\cdot\big(\mathbf{n}(\mathbf{y})\times\nabla_\Gamma\, b_0(\mathbf{y})\big)\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\big(\mathbf{n}(\mathbf{x})\times\mathbf{d}_1(\mathbf{x})\big)\cdot\mathbf{curl}_\Gamma\, b_0(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x}).\end{aligned}$$

Adapting the arguments, we also obtain

$$\begin{aligned}\langle \mathbf{c}_1, \gamma_t \,\mathbf{curl}\boldsymbol{\Upsilon}(a_2)\rangle_\tau &= \langle \mathbf{c}_1\times\mathbf{n}, \gamma_\tau \,\mathbf{curl}\boldsymbol{\Upsilon}(a_0)\rangle_\tau\\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\Big(\mathbf{n}(\mathbf{x})\times\big(\mathbf{c}_1(\mathbf{x})\times\mathbf{n}(\mathbf{x})\big)\Big)\cdot\mathbf{curl}_\Gamma\, a_2(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathbf{c}_1(\mathbf{x})\cdot\mathbf{curl}_\Gamma\, a_2(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x}).\end{aligned}$$

Again, profiting from the same piece of literature, we can similarly extract

$$\begin{aligned}\langle c_2, \gamma_n \,\mathbf{curl}\boldsymbol{\Psi}(\mathbf{a}_1)\rangle_\Gamma &= -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathbf{a}_1(\mathbf{y})\cdot\big(\mathbf{n}(\mathbf{x})\times\nabla_\Gamma\, c_2(\mathbf{x})\big)\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\ &= \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathbf{a}_1(\mathbf{y})\cdot\mathbf{curl}_\Gamma\, c_2(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x}).\end{aligned}$$

and

$$\langle d_0, \gamma_n \,\mathbf{curl}\boldsymbol{\Psi}(\mathbf{b}_1\times\mathbf{n})\rangle_\Gamma = -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\big(\mathbf{n}(\mathbf{y})\times\mathbf{b}_1(\mathbf{y})\big)\cdot\mathbf{curl}_\Gamma\, d_0(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})$$

Finally, it follows almost directly by definition that

$$\langle \mathbf{c}_1, \gamma_t \nabla\psi(a_0)\rangle_\tau = -\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,a_0(\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{c}_1(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x}),$$

and

$$\langle \mathbf{d}_1, \gamma_\tau \nabla\psi(b_2)\rangle_\tau = \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,b_2(\mathbf{y})\,\mathrm{curl}_\Gamma\,\mathbf{d}_1(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x}).$$

Putting everything together yields the symmetric bilinear forms

$$\begin{aligned}\langle\!\langle \vec{\mathbf{c}}, \gamma_{\mathsf{T}}\mathcal{L}_{\mathsf{T}}(\vec{\mathbf{a}})\rangle\!\rangle = &-\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,c_0(\mathbf{x})\,\mathrm{div}_\Gamma\mathbf{a}_1(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\\ &-\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,a_0(\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{c}_1(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\ &+\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathbf{c}_1(\mathbf{x})\cdot\mathbf{curl}_\Gamma\, a_2(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\ &+\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\,\mathbf{a}_1(\mathbf{y})\cdot\mathbf{curl}_\Gamma\, c_2(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
\langle\!\langle \vec{\mathbf{d}}, \gamma_{\mathsf{R}}\mathscr{L}_{\mathsf{R}}(\vec{\mathbf{b}})\rangle\!\rangle = &-\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \big(\mathbf{n}(\mathbf{y})\times\mathbf{b}_1(\mathbf{y})\big)\cdot\mathbf{curl}_\Gamma\, d_0(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\
&-\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \big(\mathbf{n}(\mathbf{x})\times\mathbf{d}_1(\mathbf{x})\big)\cdot\mathbf{curl}_\Gamma\, b_0(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\
&+\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathscr{b}_2(\mathbf{y})\,\mathrm{curl}_\Gamma\,\mathbf{d}_1(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\\
&+\int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathscr{d}_2(\mathbf{x})\,\mathrm{curl}_\Gamma\mathbf{b}_1(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x}).
\end{aligned} \tag{4.7}
$$

The above integrals must be understood as duality pairings.

Remark 4.1. *Let us highlight here, as we have announced in the introduction, that these double integrals only feature weakly singular kernels!*

The non-local complex inner products

$$(u,v)_{-1/2} := \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, u(\mathbf{x})\, v(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}), \tag{4.8}$$

$$(\mathbf{u},\mathbf{v})_{-1/2,\mathsf{T}} := \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}), \tag{4.9}$$

$$(\mathbf{u},\mathbf{v})_{-1/2,\mathsf{R}} := \int_\Gamma\int_\Gamma G_{\mathbf{x}}(\mathbf{y})\, \big(\mathbf{n}(\mathbf{x})\times\mathbf{u}(\mathbf{x})\big)\cdot\big(\mathbf{n}(\mathbf{y})\times\mathbf{v}(\mathbf{y})\big)\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}), \tag{4.10}$$

respectively defined over $H^{-1/2}(\Gamma)$, $\mathbf{H}_{\mathsf{T}}^{-1/2}(\Gamma) := (\mathbf{H}_{\mathsf{T}}^{1/2}(\Gamma))'$ and $\mathbf{H}_{\mathsf{R}}^{-1/2}(\Gamma) := (\mathbf{H}_{\mathsf{R}}^{1/2}(\Gamma))'$, where

$$\mathbf{H}_{\mathsf{T}}^{1/2} := \gamma_t(\mathbf{H}^1(\Omega)) \qquad \text{and} \qquad \mathbf{H}_{\mathsf{R}}^{1/2} := \gamma_\tau(\mathbf{H}^1(\Omega)), \tag{4.11}$$

are positive definite Hermitian forms, and therefore induce equivalent norms on the trace spaces. In the following, we will therefore concern ourselves with the coercivity and geometric structure of the bilinear forms

$$
\begin{aligned}
\mathscr{B}_{\mathsf{R}}(\vec{\mathbf{a}},\vec{\mathbf{c}}) &:= \langle\!\langle \gamma_{\mathsf{T}}\mathscr{L}_{\mathsf{T}}(\vec{\mathbf{a}}), \vec{\mathbf{c}}\rangle\!\rangle\\
&= (-\mathrm{div}_\Gamma\,\mathbf{a}_1, c_0)_{-1/2} + (a_0, -\mathrm{div}_\Gamma\,\mathbf{c}_1)_{-1/2} + (\mathbf{curl}_\Gamma\, a_2, \mathbf{c}_1)_{-1/2,\mathsf{T}} + (\mathbf{a}_1, \mathbf{curl}_\Gamma c_2)_{-1/2,\mathsf{T}}
\end{aligned} \tag{4.12}
$$

and

$$
\begin{aligned}
\mathscr{B}_{\mathsf{T}}\left(\vec{\mathbf{b}},\vec{\mathbf{d}}\right) &:= \langle\!\langle \gamma_{\mathsf{R}}\mathscr{L}_{\mathsf{R}}\left(\vec{\mathbf{b}}\right), \vec{\mathbf{d}}\rangle\!\rangle\\
&= (\mathbf{b}_1, \nabla_\Gamma\, d_0)_{-1/2,\mathsf{R}} + (\nabla_\Gamma\, b_0, \mathbf{d}_1)_{-1/2,\mathsf{R}} + (\mathsf{b}_2, \mathrm{curl}_\Gamma\mathbf{d}_1)_{-1/2} + (\mathrm{curl}_\Gamma\mathbf{b}_1, \mathsf{d}_2)_{-1/2}\,.
\end{aligned} \tag{4.13}
$$

5. T-coercivity

Based on the space decomposition introduced by the next lemma, we design isomorphisms $\mathscr{H}_{\mathsf{R}}\to\mathscr{H}_{\mathsf{R}}$ and $\mathscr{H}_{\mathsf{T}}\to\mathscr{H}_{\mathsf{T}}$ that are instrumental for obtaining the desired generalized Gårding inequalities.

Lemma 5.1 (See [20, Sec. 7] and [14, Lem. 6.5])**.** *There exists a continuous projection* $\mathsf{Z}^\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\to\mathbf{H}_{\mathsf{R}}^{1/2}(\Gamma)$ *with*

$$\ker(\mathsf{Z}^\Gamma) = \ker(\mathrm{div}_\Gamma)\cap\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \tag{5.1}$$

and satisfying

$$\operatorname{div}_\Gamma\left(\mathsf{Z}^\Gamma(\mathbf{v})\right)=\operatorname{div}_\Gamma(\mathbf{v})\,. \tag{5.2}$$

The closed subspaces $\mathbf{X}(\operatorname{div}_\Gamma,\Gamma):=\mathsf{Z}^\Gamma\left(\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)\right)$ and $\mathbf{N}(\operatorname{div}_\Gamma,\Gamma):=\ker(\operatorname{div}_\Gamma)\cap\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ provide a stable direct regular decomposition $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)=\mathbf{X}(\operatorname{div}_\Gamma,\Gamma)\oplus\mathbf{N}(\operatorname{div}_\Gamma,\Gamma)$. Hence, it follows from (5.2) that

$$\mathbf{v}\mapsto\left\|\operatorname{div}_\Gamma(\mathbf{v})\right\|_{-1/2}+\left\|\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\mathbf{v}\right\|_{-1/2} \tag{5.3}$$

also defines an equivalent norm in $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$.

Note that since $\mathbf{H}^{1/2}_{\mathsf{R}}(\Gamma)$ compactly embeds by Rellich's embedding theorem in the space $\mathbf{L}^2_t(\Gamma):=\{\mathbf{u}\in\mathbf{L}^2(\Gamma)\mid\mathbf{u}\cdot\mathbf{n}\equiv 0\}$ of square-integrable tangential vector-fields, this is also the case for $\mathbf{X}(\operatorname{div}_\Gamma,\Gamma)$.

From Lemma 2.2, $\operatorname{div}_\Gamma:\mathbf{X}(\operatorname{div}_\Gamma,\Gamma)\to H^{-1/2}_*(\Gamma)$ is a continuous bijection, thus the bounded inverse theorem guarantees the existence of a continuous inverse $(\operatorname{div}_\Gamma)^\dagger: H^{-1/2}_*(\Gamma)\to\mathbf{X}(\operatorname{div}_\Gamma,\Gamma)$ such that

$$(\operatorname{div}_\Gamma)^\dagger\circ\operatorname{div}_\Gamma=\mathrm{Id}\Big|_{\mathbf{X}(\operatorname{div}_\Gamma,\Gamma)},\qquad\qquad \operatorname{div}_\Gamma\circ(\operatorname{div}_\Gamma)^\dagger=\mathrm{Id}\Big|_{H^{-1/2}_*(\Gamma)}.$$

The existence of an operator $\mathbf{curl}^\dagger_\Gamma:\mathbf{N}(\operatorname{div}_\Gamma,\Gamma)\to H^{1/2}_*(\Gamma)$ satisfying $\mathbf{curl}^\dagger_\Gamma\circ\mathbf{curl}_\Gamma=\mathrm{Id}$ and $\mathbf{curl}_\Gamma\circ\mathbf{curl}^\dagger_\Gamma=$ '$\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$-orthogonal projection onto (surface) divergence-free vector-fields' also follows by Lemma 2.2.

In the following, we will denote by Q_* both the projection $H^{1/2}(\Gamma)\to H^{1/2}_*(\Gamma)$ onto mean zero functions and the projection $H^{-1/2}(\Gamma)\to H^{-1/2}_*(\Gamma)$ onto the space of annihilators of the characteristic function.

Lemma 5.2. *The bounded linear operator*

$$\Xi: H^{-1/2}_*(\Gamma)\times\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)\times H^{1/2}_*(\Gamma)\to H^{-1/2}_*(\Gamma)\times\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)\times H^{1/2}_*(\Gamma) \tag{5.4}$$

defined by

$$\Xi\begin{pmatrix}a_0\\\mathbf{a}_1\\a_2\end{pmatrix}=\begin{pmatrix}-\operatorname{div}_\Gamma\mathbf{a}_1\\-(\operatorname{div}_\Gamma)^\dagger(Q_*a_0)+\mathbf{curl}_\Gamma(Q_*a_2)\\(\mathbf{curl}_\Gamma)^\dagger\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\mathbf{a}_1\right)\end{pmatrix}$$

is a continuous involution. In particular, Ξ is an isomorphism of Banach spaces.

Proof. We directly evaluate

$$\begin{aligned}\Xi^2\begin{pmatrix}a_0\\\mathbf{a}_1\\a_2\end{pmatrix}&=\Xi\begin{pmatrix}-\operatorname{div}_\Gamma\mathbf{a}_1\\-(\operatorname{div}_\Gamma)^\dagger(Q_*a_0)+\mathbf{curl}_\Gamma(Q_*a_2)\\(\mathbf{curl}_\Gamma)^\dagger\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\mathbf{a}_1\right)\end{pmatrix}\\&=\begin{pmatrix}\operatorname{div}_\Gamma\left((\operatorname{div}_\Gamma)^\dagger(Q_*a_0)\right)-\operatorname{div}_\Gamma\left(\mathbf{curl}_\Gamma(Q_*a_2)\right)\\(\operatorname{div}_\Gamma)^\dagger\left(Q_*(\operatorname{div}_\Gamma\mathbf{a}_1)\right)+\mathbf{curl}_\Gamma\left(Q_*(\mathbf{curl}_\Gamma)^\dagger\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\mathbf{a}_1\right)\right)\\-(\mathbf{curl}_\Gamma)^\dagger\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\left((\operatorname{div}_\Gamma)^\dagger(Q_*a_0)\right)\right)+(\mathbf{curl}_\Gamma)^\dagger\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\left(\mathbf{curl}_\Gamma(Q_*a_2)\right)\right)\end{pmatrix}\\&=\begin{pmatrix}Q_*a_0\\\mathsf{Z}^\Gamma\mathbf{a}_1+\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\mathbf{a}_1\\Q_*a_2\end{pmatrix}=\begin{pmatrix}a_0\\\mathbf{a}_1\\a_2\end{pmatrix}.\end{aligned}$$

□

Proposition 5.1. *There exists a constant $C > 0$ and a compact bilinear form $\mathcal{C} : \mathcal{H}_\mathsf{R} \times \mathcal{H}_\mathsf{R} \to \mathbb{R}$ such that*

$$\left| \langle\!\langle \Xi\, \vec{\mathbf{a}}, \gamma_\mathsf{T}\mathcal{L}_\mathsf{T}\left(\vec{\mathbf{a}}\right) \rangle\!\rangle_\times + \mathcal{C}\left(\vec{\mathbf{a}}, \vec{\mathbf{a}}\right) \right| \geq C \|\vec{\mathbf{a}}\|^2_{\mathcal{H}_\mathsf{R}}, \qquad \forall \vec{\mathbf{a}} \in \mathcal{H}_\mathsf{R}. \tag{5.5}$$

Proof. The operator $\mathbf{curl}_\Gamma : H^1_*(\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma)$ is a continuous injection with closed range, it is thus bounded below. Since the mean operator has finite rank, it is compact. Moreover, $(\mathrm{div}_\Gamma)^\dagger \left(H^{-1/2}_*(\Gamma)\right) \subset \mathbf{H}^{1/2}_\mathsf{R}(\Gamma)$ is compactly embedded in $\mathbf{L}^2_t(\Gamma)$. Hence, the proof ultimately follows from

$$\begin{aligned}\langle\!\langle \Xi\, \vec{\mathbf{a}}, \gamma_\mathsf{T}\mathcal{L}_\mathsf{T}\left(\vec{\mathbf{a}}\right) \rangle\!\rangle_\times \hat{=}\ & \left(\mathrm{div}_\Gamma\, \mathbf{a}_1, \mathrm{div}_\Gamma\, \mathbf{a}_1\right)_{-1/2} + \left(a_2, Q_* a_2\right)_{-1/2} \\ & + \left(\left(\mathrm{div}_\Gamma\right)^\dagger Q_* a_2, \mathbf{curl}_\Gamma\, a_0\right)_{-1/2} + \left(\mathbf{curl}_\Gamma\, Q_* a_0, \mathbf{curl}_\Gamma\, a_0\right)_{-1/2} + \left(\mathbf{a}_1, \left(\mathrm{Id} - \mathsf{Z}^\Gamma\right) \mathbf{a}_1\right)_{-1/2}\end{aligned}$$

and the opening observations of this section. ☐

Since $\mathrm{curl}_\Gamma\left(\mathbf{d}\right) = \mathrm{div}_\Gamma\left(\mathbf{n} \times \mathbf{d}\right)$ for all $\mathbf{d} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, tampering with the signs and introducing rotations in the definition of Ξ easily leads to an analogous generalized Gårding inequality for $\gamma_\mathsf{R}\mathcal{L}_\mathsf{R}$.

Corollary 5.1. *The boundary integral operators $\gamma_\mathsf{T}\mathcal{L}_\mathsf{T} : \mathcal{H}_\mathsf{R} \to \mathcal{H}_\mathsf{T}$ and $\gamma_\mathsf{R}\mathcal{L}_\mathsf{R} : \mathcal{H}_\mathsf{T} \to \mathcal{H}_\mathsf{R}$ are Fredholm of index* 0.

6. Kernels

We have learned in Corollary 5.1 that the kernels of $\gamma_\mathsf{T}\mathcal{L}_\mathsf{T}$ and $\gamma_\mathsf{R}\mathcal{L}_\mathsf{R}$ were finite dimensional. In this section, we proceed similarly as in [14, Sec. 7.1] and [15, Sec. 3] to characterize them explicitly.

Suppose that $\vec{\mathbf{a}} \in \mathcal{H}_\mathsf{R}$ is such that $\gamma_\mathsf{T}\mathcal{L}_\mathsf{T}\left(\vec{\mathbf{a}}\right) = 0$.

- Since $\mathrm{div}_\Gamma\left(\mathbf{a}_1\right) \in H^{-1/2}\left(\Gamma\right)$, we can test the bilinear form of Equation (4.6) with $c_0 = \mathrm{div}_\Gamma\left(\mathbf{a}_1\right)$, $\mathbf{c}_1 = 0$ and $c_2 = 0$ to find that $\mathrm{div}_\Gamma\left(\mathbf{a}_1\right) = 0$.
- Testing with $c_0 = 0$ and $\mathbf{c}_1 = 0$ shows that $\left(\mathbf{a}_1, \mathbf{curl}_\Gamma\, v\right)_{-1/2} = 0$ for all $v \in H^{1/2}(\Gamma)$.
- Because $\mathrm{div}_\Gamma \circ \mathbf{curl}_\Gamma = 0$, we can choose $c_2 = 0$, $c_0 = 0$ and $\mathbf{c}_1 = \mathbf{curl}_\Gamma\left(a_2\right)$ to conclude that $\mathbf{curl}_\Gamma\left(a_2\right) = 0$.
- We are left with the observation that $\left(a_0, \mathrm{div}_\Gamma\left(\mathbf{v}\right)\right)_{-1/2} = 0$ for all $\mathbf{v} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

In $H^{1/2}\left(\Gamma\right)$, $\ker\left(\mathbf{curl}_\Gamma\right) = \ker\left(\nabla_\Gamma\right)$ is the space of functions $\mathcal{C}\left(\Gamma\right)$ that are constant over connected components of Γ. Defining $\mathbf{\Psi}_t := \gamma_t \mathbf{\Psi}$, we have found that

$$\ker\left(\gamma_\mathsf{T}\mathcal{L}_\mathsf{T}\right) = \left\{\vec{\mathbf{a}} \in \mathcal{H}_\mathsf{R} \;\middle|\; a_0 \in \mathcal{C}\left(\Gamma\right), \mathrm{curl}_\Gamma \mathbf{\Psi}_t\left(\mathbf{a}_1\right) = 0, \mathrm{div}_\Gamma\left(\mathbf{a}_1\right) = 0, \nabla_\Gamma \psi\left(a_0'\right) = 0\right\}. \tag{6.1}$$

Now, suppose that $\vec{\mathbf{b}} \in \mathcal{H}_\mathsf{T}$ is such that $\gamma_\mathsf{R}\mathcal{L}_\mathsf{R}(\vec{\mathbf{b}}) = 0$.

- As $\mathrm{curl}_\Gamma\left(\mathbf{b}_1\right) \in H^{-1/2}\left(\Gamma\right)$, we may test Equation (4.7) with $\mathscr{d}_2 = \mathrm{curl}_\Gamma\left(\mathbf{b}_1\right)$, $\mathbf{d}_1 = 0$ and $d_0 = 0$ to find that $\mathrm{curl}_\Gamma\left(\mathbf{b}_1\right) = 0$.
- Testing with $\mathscr{d}_2 = 0$ and $\mathbf{d}_1 = 0$, we find that $\left(\mathbf{n} \times \mathbf{b}_1, \mathbf{curl}_\Gamma\, v\right)_{-1/2} = 0$ for all $v \in H^{1/2}(\Gamma)$.
- Since $\mathrm{curl}_\Gamma \circ \nabla_\Gamma = 0$, we can choose $d_0 = 0$, $\mathscr{d}_2 = 0$ and $\mathbf{d}_1 = \nabla_\Gamma\left(b_0\right)$ to conclude that $\mathbf{curl}_\Gamma\left(b_0\right) = 0$.
- Finally, it follows that $\left(\mathscr{b}_2, \mathrm{curl}_\Gamma\left(\mathbf{v}\right)\right)_{-1/2} = 0$ for all $\mathbf{v} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$.

Notice that since $\nabla_\Gamma(v)$ is tangential for all $v \in H^{1/2}(\Gamma)$,

$$(\mathbf{n}\times\mathbf{b}_1, \mathbf{curl}_\Gamma\, v)_{-1/2} = \left(\mathbf{n}\times\mathbf{b}_1, \nabla_\Gamma(v)\times\mathbf{n}\right)_{-1/2} = \langle \mathbf{n}\times\mathbf{\Psi}(\mathbf{n}\times\mathbf{b}_1), \nabla_\Gamma v\rangle, \quad \forall v\in H^{1/2}(\Gamma).$$

Therefore, we let $\mathbf{\Psi}_\tau := -\gamma_\tau\mathbf{\Psi}$ and conclude that

$$\ker(\gamma_\mathsf{R}\mathcal{L}_\mathsf{R}) = \left\{\vec{\mathbf{b}}\in\mathcal{H}_\mathsf{T}\;\middle|\; b_0\in\mathcal{C}(\Gamma),\, \mathrm{curl}_\Gamma(\mathbf{b}_1)=0,\, \mathrm{div}_\Gamma\mathbf{\Psi}_\tau(\mathbf{n}\times\mathbf{b}_1)=0,\, \mathbf{curl}_\Gamma\psi\left(b_0'\right)=0\right\}. \tag{6.2}$$

We have characterized the kernels explicitly in terms of operator equations. However, we want to note that the variational formulations of the stated vanishing necessary and sufficiant conditions as they occur in the derivation will in fact become the most intuitive expressions when surface Dirac operators are introduced in the next section. Nevertheless, Equation (6.1) and Equation (6.2) together with the mapping properties of the scalar and vector single layer potentials allow us to determine as in [14] and [15] that the dimension of these kernels relate to the Betti numbers of Γ.

Proposition 6.1. *The dimensions of* $\ker(\gamma_\mathsf{T}\mathcal{L}_\mathsf{T})$ *and* $\ker(\gamma_\mathsf{R}\mathcal{L}_\mathsf{R})$ *are finite and equal to the sum of the Betti numbers* $\beta_0(\Gamma)+\beta_1(\Gamma)+\beta_2(\Gamma)$.

Remark 6.1. *The zeroth Betti number* $\beta_0(\Gamma)$ *indicates the number of connected components of* Γ*. The first Betti number* $\beta_1(\Gamma)$ *ammounts to the number of equivalence classes of non-bounding cycles in* Γ*. For the second Betti number, it holds that* $\beta_2(\Gamma)=\beta_2\left(\Omega^+\right)+\beta_2\left(\Omega^-\right)$.

7. Surface Dirac operators

In this section, we reveal the geometric structure behind the formulas of the bilinear forms $\mathcal{B}_\mathsf{R}$ and $\mathcal{B}_\mathsf{T}$ established in Section 4. They turn out to be associated with the 2D surface Dirac operators based on the Fredholm Hilbert co-chain and chain complexes of densely-defined unbounded operators

$$H^{-1/2}(\Gamma) \xrightarrow{\nabla_\Gamma} \left(\mathbf{H}_\mathsf{T}^{-1/2}(\Gamma), (\cdot,\cdot)_{-1/2,\mathsf{T}}\right) \xrightarrow{\mathrm{curl}_\Gamma} H^{-1/2}(\Gamma) \tag{7.1}$$

and

$$H^{-1/2}(\Gamma) \xleftarrow[-\mathrm{div}_\Gamma]{} \left(\mathbf{H}_\mathsf{R}^{-1/2}(\Gamma), (\cdot,\cdot)_{-1/2,\mathsf{R}}\right) \xleftarrow[\mathbf{curl}_\Gamma]{} H^{-1/2}(\Gamma), \tag{7.2}$$

whose associated domain complexes are the de Rham chain and co-chain complexes

$$H^{1/2}(\Gamma) \xrightarrow{\nabla_\Gamma} \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \xrightarrow{\mathrm{curl}_\Gamma} H^{-1/2}(\Gamma) \tag{7.3}$$

and

$$H^{-1/2}(\Gamma) \xleftarrow[-\mathrm{div}_\Gamma]{} \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \xleftarrow[\mathbf{curl}_\Gamma]{} H^{1/2}(\Gamma), \tag{7.4}$$

respectively.

Remark 7.1. *Notice that* (7.3) *and* (7.4) *are dual to each other with respect to the pivot space* L^2 *(in the sense of Banach space duality).*

The abstract theory for the complexes (7.1), (7.2), (7.3) and (7.3) is studied in [1–3, 8]. As mentioned in Section 1, the complexes can also be seen as built from a Fredholm-nilpotent operator in the sense of [7, 23]. The inner products of the Hilbert spaces involved in theses complexes are the non-local inner products (4.8), (4.9) and (4.10). We have shorthanded $H^{-1/2}(\Gamma)$ in place of the Hilbert space notation $(H^{-1/2}(\Gamma), (\cdot,\cdot)_{-1/2})$ to lighten notation.

We now show that ***the first-kind boundary integral operators spawned by the (volume) Dirac operators in 3D Euclidean space coincide with the (surface) Dirac operators associated with the de Rham Hilbert complexes* (7.1) *and* (7.2)*.***

The surface Dirac operators, akin to the volume Dirac operators of Figure 1 and Figure 2, act on the full product spaces $\mathscr{H}_{\mathsf{T}}$ and $\mathscr{H}_{\mathsf{R}}$. The spaces $\mathscr{H}_{\mathsf{T}}$ and $\mathscr{H}_{\mathsf{R}}$ can be viewed as the explicit vector representations of graded exterior algebra of differential forms on surfaces. The $H^{-1/2}$ Hilbert space adjoint $\mathbf{d}_{\mathsf{T}}^*$ and $\boldsymbol{\delta}_{\mathsf{R}}^*$ of the nilpotent operators $\mathbf{d}_{\mathsf{T}} : \mathscr{H}_{\mathsf{T}} \to \mathscr{H}_{\mathsf{T}}$ and $\boldsymbol{\delta}_{\mathsf{R}} : \mathscr{H}_{\mathsf{R}} \to \mathscr{H}_{\mathsf{R}}$ defined by

$$\mathbf{d}_{\mathsf{T}}\left(\vec{\mathbf{b}}\right) := \begin{pmatrix} 0 \\ \nabla_\Gamma b_0 \\ \operatorname{curl}_\Gamma(\mathbf{b}_1) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\delta}_{\mathsf{R}}\left(\vec{\mathbf{a}}\right) := \begin{pmatrix} -\operatorname{div}_\Gamma(\mathbf{a}_1) \\ \mathbf{curl}_\Gamma(a_0) \\ 0 \end{pmatrix}$$

are *non-local* operators. In terms of variational formulation, the bilinear forms associated with the operators $\mathsf{D}_{\mathsf{T}} := \mathbf{d}_{\mathsf{T}} + \mathbf{d}_{\mathsf{T}}^*$ and $\mathsf{D}_{\mathsf{R}} := \boldsymbol{\delta}_{\mathsf{R}} + \boldsymbol{\delta}_{\mathsf{R}}^*$ are precisely $\mathscr{B}_{\mathsf{T}}$ and $\mathscr{B}_R$ defined in (4.12) and (4.13), previously associated to the boundary integral operators $\gamma_{\mathsf{R}}\mathscr{L}_{\mathsf{R}}$ and $\gamma_{\mathsf{T}}\mathscr{L}_{\mathsf{T}}$:

$$\begin{aligned}\left(\mathsf{D}_{\mathsf{T}}\,\vec{\mathbf{b}},\vec{\mathbf{d}}\right)_{\mathscr{H}_{\mathsf{T}}} &= \left(\mathbf{d}_{\mathsf{T}}\vec{\mathbf{b}},\vec{\mathbf{d}}\right)_{\mathscr{H}_{\mathsf{T}}} + \left(\vec{\mathbf{b}},\mathbf{d}_{\mathsf{T}}\vec{\mathbf{d}}\right)_{\mathscr{H}_{\mathsf{T}}} \\ &= \left(\nabla_\Gamma\, b_0,\mathbf{d}_1\right)_{-1/2,\mathsf{R}} + \left(\operatorname{curl}_\Gamma\mathbf{b}_1,\mathsf{d}_2\right)_{-1/2} + \left(\mathbf{b}_1,\nabla_\Gamma\, d_0\right)_{-1/2,\mathsf{R}} + \left(\mathsf{b}_2,\operatorname{curl}_\Gamma\mathbf{d}_1\right)_{-1/2} \\ &= \mathscr{B}_{\mathsf{T}}\left(\vec{\mathbf{b}},\vec{\mathbf{d}}\right),\end{aligned} \tag{7.5}$$

and similarly

$$\begin{aligned}\left(\mathsf{D}_{\mathsf{R}}\,\vec{\mathbf{a}},\vec{\mathbf{c}}\right)_{\mathscr{H}_{\mathsf{R}}} &= \left(\widetilde{\mathbf{d}}_{\mathsf{R}}\vec{\mathbf{a}},\vec{\mathbf{c}}\right)_{\mathscr{H}_{\mathsf{R}}} + \left(\vec{\mathbf{a}},\widetilde{\mathbf{d}}_{\mathsf{R}}\vec{\mathbf{c}}\right)_{\mathscr{H}_{\mathsf{R}}} \\ &= \left(-\operatorname{div}_\Gamma\,\mathbf{a}_1,c_0\right)_{-1/2} + \left(a_0,-\operatorname{div}_\Gamma\,\mathbf{c}_1\right)_{-1/2} + \left(\mathbf{curl}_\Gamma\,a_2,\mathbf{c}_1\right)_{-1/2,\mathsf{T}} + \left(\mathbf{a}_1,\mathbf{curl}_\Gamma c_2\right)_{-1/2,\mathsf{T}} \\ &= \mathscr{B}_{\mathsf{R}}\left(\vec{\mathbf{a}},\vec{\mathbf{c}}\right).\end{aligned} \tag{7.6}$$

In other words, if $\mathsf{J}_{\mathsf{T}} : \mathscr{H}_R \to \mathscr{H}_{\mathsf{T}}$ and $\mathsf{J}_{\mathsf{R}} : \mathscr{H}_T \to \mathscr{H}_{\mathsf{R}}$ are Riesz maps, then

$$\mathsf{D}_{\mathsf{T}} = \mathsf{J}_{\mathsf{T}}\gamma_{\mathsf{R}}\mathscr{L}_{\mathsf{R}} \qquad \text{and} \qquad \mathsf{D}_{\mathsf{R}} = \mathsf{J}_{\mathsf{R}}\gamma_{\mathsf{T}}\mathscr{L}_{\mathsf{T}} \tag{7.7}$$

We conclude that the first-kind boundary integral equations for the 3D (volume) Dirac operator corresponds to surface Dirac operators: the set of boundary condition γ_{R} is associated with de Rham cohomology, while the set of boundary conditions γ_{T} relates to its homological dual.

This explains why the kernels of the first-kind boundary integral operators derived in the previous sections have the same dimension as the standard spaces of surface harmonic scalar and vector fields. Indeed, once the discovery of (7.5) and (7.6) is made, one finds in [23, Thm. 2.4] an inf-sup inequality for the mixed formulation associated with the bilinear forms of Equation (4.6) and Equation (4.7) in the space perpendicular to the (abstract) harmonic forms. In fact, we also read in [23, Sec. 4] that the related mixed problem for the Hodge-Laplace is well-posed; and we recognize in [23, Eq. 5] that the surface bilinear form in [15, Eq. 25] associated with the first-kind boundary integral equations for the Hodge-Laplace operator in 3D is a piece of that well-posed system.

8. Conclusion

First-kind boundary integral equations are appealing to the numerical analysis community because they lead to variational problems posed in natural "energy" trace spaces that are generally well-suited for Galerkin discretization. Therefore, the new equations on the one hand pave the way for development of new Galerkin boundary element methods. On the other hand, our results simultaneously open an original perspective towards the recent developments in boundary integral equations for Hodge-Laplace problems. As it stands, the rich theories of Hilbert complexes and nilpotent operators not only support our observations with the help of already established abstract inf-sup conditions, but in fact also supply the framework and harmonic analysis needed to relate the exposed non-standard surface Dirac operators to the mixed variational formulations associated with the first-kind boundary integral operators for the Hodge-Laplacian. In fact, this insight already led us to observe that the variational formulation [15, Eq. 25] is associated with the Laplace-Beltrami of the Hilbert complex (7.1). We note that [15, Eq. 34] also appears to be related to higher-order differential forms on surfaces.

For our presentation, we relied on classical vector analysis to emphasize the structural relationship between the above development and the standard theory for second-order elliptic operators. Integration by parts and a component-wise perspective allowed us to naturally identify relevant traces without extrapolation from the abstract geometric theory. Such perspective also permitted explicit use of results and techniques from known second-order theory. This is best recognized in Proposition 3.1, Lemma 3.8 and in the calculations leading to the integral representations of the surface potentials. Nevertheless, this detailed exposition also revealed a geometric structure that we expect can be fruitfully expressed in the language of differential forms, possibly generalizing the above results to arbitrary dimensions. Reformulation of the theory using exterior or Clifford calculus technology is another interesting venue of future investigations.

It would also be interesting to extend the above theory of first-order integral equations for the Dirac operator to the so-called *perturbed* Dirac operator, say of the form $\mathsf{D} + i\kappa\,\mathrm{Id}$, and relate this study with the works of [36–38] and [19, 24, 32].

Finally, the significant observation that our integral operators arise as non-standard surface Dirac operators associated to Hilbert complexes suggests a new analysis of Hodge-Dirac and Hodge-Laplace related first-kind boundary integral equations which has yet to be explored.

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