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Phase retrieval of bandlimited functions for the wavelet transform

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Abstract

We study the problem of phase retrieval in which one aims to recover a function f from the magnitude of its wavelet transform $|\mathcal{W}_\psi f|$. We consider bandlimited functions and derive new uniqueness results for phase retrieval, where the wavelet itself can be complex-valued. In particular, we prove the first uniqueness result for the case that the wavelet ψ has a finite number of vanishing moments. In addition, we establish the first result on unique reconstruction from samples of the wavelet transform magnitude when the wavelet coefficients are complex-valued.

1 Introduction

The term phase retrieval indicates a large class of problems in which one seeks to recover a signal from phaseless measurements. In particular, *wavelet phase retrieval* consists of recovering a signal from the magnitudes of its wavelet coefficients. The wavelet transform of $f \in L^p(\mathbb{R})$ associated to the wavelet $\psi \in L^1(\mathbb{R})$ is defined by

$$\mathcal{W}_\psi f(b,a) = \frac{1}{a}\int_{\mathbb{R}} f(x)\overline{\psi\left(\frac{x-b}{a}\right)}\mathrm{d}x,$$

for every $b \in \mathbb{R}$ and $a \in \mathbb{R}_+$. The problem of wavelet phase retrieval then consists of reconstructing f from the measurements

$$|\mathcal{W}_\psi f(b,a)|, \quad b \in \mathbb{R},\, a \in \mathbb{R}_+. \tag{1}$$

Since the wavelet transform is a linear operator, it is impossible to distinguish between f and λf, $\lambda \in \mathbb{T}$, from only the measurements (1). Our aim is therefore to reconstruct f up to a global phase factor. In the present paper, we study the injectivity of the operator $\mathcal{A}_\psi \colon L^p(\mathbb{R})/\sim \to [0,\infty)^{\mathbb{R}\times\mathbb{R}_+}$ given by

$$\mathcal{A}_\psi f = (|\mathcal{W}_\psi f(b,a)|)_{\mathbb{R}\times\mathbb{R}_+},$$

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where $\psi \in L^1(\mathbb{R})$ is fixed and $f \sim g$ if for some $\lambda \in \mathbb{T}$, $f = \lambda g$. This problem has already been investigated in [6]. There, the authors prove injectivity of the operator $\mathcal{A}_\psi$ when restricted to the space of *analytic signals* $\mathcal{H}_+ = \{f \in L^2(\mathbb{R}) : \mathrm{supp}\widehat{f} \subseteq \mathbb{R}_+\}$ and when ψ is a Cauchy wavelet, which is a special kind of *progressive wavelet*, that is a wavelet with only positive frequencies. In other words, they show that the magnitude of the Cauchy wavelet transform uniquely determines the *analytic representation* $f_+ \in L^2(\mathbb{R})$ up to global phase of any signal $f \in L^2(\mathbb{R})$, given by

$$\widehat{f_+}(\xi) := 2\widehat{f}(\xi)\mathbf{1}_{\xi>0}, \qquad \xi \in \mathbb{R}.$$

In the second part of the paper, precisely in Section 5, we work with the results in [6] and we provide some new insights into the reconstruction of the analytic representation of bandlimited signals from magnitude samples of the Cauchy wavelet transform.

It is worth observing that the analytic representation f_+ of a real-valued signal f completely characterises the signal itself since

$$\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)} = \frac{\overline{\widehat{f_+}(\xi)}}{2},$$

for all $\xi > 0$. Thus, in classical time-scale analysis, if one is in a setting in which all the signals are real-valued (which is the case for many audio processing applications), working with analytic or real-valued functions is equivalent. However, this is not true for phase retrieval. Indeed, suppose that we have two real-valued signals $f, g \in L^2(\mathbb{R})$ whose analytic representations agree up to global phase, i.e. $f_+ = \mathrm{e}^{\mathrm{i}\alpha} g_+$ for some $\alpha \in \mathbb{R}$. Then, it follows that

$$\widehat{f}(-\xi) = \frac{\overline{\widehat{f_+}(\xi)}}{2} = \mathrm{e}^{-\mathrm{i}\alpha}\frac{\overline{\widehat{g_+}(\xi)}}{2} = \mathrm{e}^{-\mathrm{i}\alpha}\widehat{g}(-\xi),$$

for all $\xi > 0$. So in fact, $\widehat{f}$ and $\widehat{g}$ do not necessarily agree up to global phase and therefore f and g do not necessarily agree up to global sign (see also Remark 10).

Expressed slightly differently, knowing the analytic representation of a real-valued signal up to global phase does not correspond to knowing a real-valued signal up to global sign. For this reason, recovering merely the analytic representation of a real-valued signal up to global phase may not be sufficient and one has to ask the uniqueness question (up to global sign) for the signals themselves. In [1], this problem has been studied in a *sign retrieval* setting: There, both the signal and the wavelet are assumed to be real-valued so that taking the magnitudes of the wavelet coefficients amounts to losing sign information instead of general phase information.

In this manuscript, we derive the first uniqueness results for the recovery of real-valued signals from wavelet magnitude measurements in the case that the wavelet itself is complex-valued, i.e. *phase* and not merely *sign* information is lost in the acquisition process. To be more precise, we prove the injectivity of the operator $\mathcal{A}_\psi$ when restricted to the space of (real-valued) bandlimited functions and when the wavelet ψ has a finite number of vanishing moments. Furthermore, we are also able to show uniqueness of the reconstruction from magnitude samples. To the best of our knowledge, this is the first uniqueness result for wavelet phase retrieval from samples when the wavelet coefficients are complex-valued.

Outline In Section 2, we recall the definition of the wavelet transform and of the Paley–Wiener space and we prove some auxiliary results that are needed later in the paper. In Section 3, we consider real-valued bandlimited signals and we prove that if the wavelet has a finite number of vanishing moments, then the magnitude of the wavelet transform uniquely determines real-valued bandlimited signals up to global sign. In addition, we establish uniqueness for wavelet phase retrieval from magnitude samples. In Section 4, we apply our results to the Morlet wavelet and the chirp wavelet. Finally, in Section 5, we consider analytic signals and we prove a sampling result for recovering bandlimited analytic signals from samples of the Cauchy wavelet transform magnitude.

2 Preliminaries

The translation and dilation operators act on a function $f : \mathbb{R} \to \mathbb{C}$ as

$$T_b f(x) = f(x-b), \qquad D_a f(x) = a^{-1} f\left(a^{-1}x\right),$$

respectively, for every $b \in \mathbb{R}$ and $a \in \mathbb{R}_+$. Both operators map each $L^p(\mathbb{R})$ onto itself and D_a is normalized to be an isometry on $L^1(\mathbb{R})$. For every $a \in \mathbb{R}_+$, we will use the notation $f_a = D_a f$. Furthermore, let us denote $f^{\#}(x) = \overline{f(-x)}$, $x \in \mathbb{R}$.

Definition 1. Let $1 \leq p \leq \infty$. The wavelet transform of $f \in L^p(\mathbb{R})$ associated to $\psi \in L^1(\mathbb{R})$ is defined by

$$\mathcal{W}_\psi f(b,a) = (f * \psi_a^{\#})(b) = \frac{1}{a}\int_{\mathbb{R}} f(x)\overline{\psi\left(\frac{x-b}{a}\right)}\mathrm{d}x,$$

for every $b \in \mathbb{R}^d$ and $a \in \mathbb{R}_+$.

We observe that, by Young's inequality, $\mathcal{W}_\psi f(\cdot, a) \in L^p(\mathbb{R})$ for every $a \in \mathbb{R}_+$. Let $\Omega > 0$ and denote by PW_Ω the space of bandlimited functions

$$\mathrm{PW}_\Omega = \{f \in L^2(\mathbb{R}) : \mathrm{supp}\widehat{f} \subseteq [-\Omega, \Omega]\},$$

which is a closed subspace of $L^2(\mathbb{R})$. By the Paley–Wiener theorem, every $f \in \mathrm{PW}_\Omega$ has an analytic extension to an entire function of exponential type, which we also denote by f. More precisely,

$$|f(z)| \leq \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_1 \mathrm{e}^{|\mathrm{Im} z|\Omega}, \quad z \in \mathbb{C}.$$

We can therefore consider PW_Ω as a Hilbert space of entire functions. Furthermore, the space of bandlimited functions PW_Ω is a reproducing kernel Hilbert space (RKHS), see for example [3, Chapter 2]. This means that, for every $x \in \mathbb{R}$, the evaluation operator $L_x \colon \mathrm{PW}_\Omega \to \mathbb{C}$ defined by

$$L_x(f) = f(x), \qquad f \in \mathrm{PW}_\Omega,$$

is bounded. Therefore, if f_n is a sequence in PW_Ω which converges to f in $L^2(\mathbb{R})$ as $n \to \infty$, then

$$f_n(x) \to f(x), \quad n \to \infty,$$

for every $x \in \mathbb{R}$. The next lemma will play a crucial role in the proof of our main results. Its proof follows from the sampling theorem in [9] and we also refer to [2] for an alternative proof.

Lemma 2. *Let $\Omega > 0$ and $f \in \mathrm{PW}_\Omega$ be real-valued on the real line. Then, f is uniquely determined by $\{|f(x)| : x \in \mathbb{R}\}$ up to global sign.*

It is worth observing that if $f \in \mathrm{PW}_\Omega$ and $\psi \in L^1(\mathbb{R})$, then $\mathcal{W}_\psi f(\cdot, a)$ is also a bandlimited function for every $a \in \mathbb{R}_+$:

Lemma 3. *Let $\Omega > 0$. If $f \in PW_\Omega$ and $\psi \in L^1(\mathbb{R})$, then $\mathcal{W}_\psi f(\cdot, a) \in PW_\Omega$ for every $a \in \mathbb{R}_+$.*

Lemma 3 follows immediately by the convolution theorem and by the relation

$$(\psi_a^{\#})^\wedge(\xi) = \overline{\widehat{\psi}(a\xi)}, \quad a \in \mathbb{R}_+,\, \xi \in \mathbb{R}.$$

A first insight into wavelet phase retrieval comes from approximation theory.

Definition 4. An approximate identity is a family $\{\phi_\epsilon\}_{\epsilon \in \mathbb{R}_+}$ of functions in $L^1(\mathbb{R})$ such that

i) $\int_{\mathbb{R}} \phi_\epsilon(x)\mathrm{d}x = 1$ for every $\epsilon > 0$,

ii) $\sup_{\epsilon > 0} \|\phi_\epsilon\|_1 < +\infty$,

iii) for every $\delta > 0$,

$$\lim_{\epsilon \to 0} \int_{|x| \geq \delta} |\phi_\epsilon(x)|\mathrm{d}x = 0.$$

Example 1. Let $\phi \in L^1(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} \phi(x)\mathrm{d}x = 1,$$

or equivalently $\widehat{\phi}(0) = 1$, and let $\phi_a(x) = a^{-1}\phi(a^{-1}x)$. Then, the family of functions $\{\phi_a\}_{a \in \mathbb{R}_+}$ forms an approximate identity.

Let $1 \leq p < \infty$. It is a well-known fact that the convolution $f * \phi_\epsilon$ converges to f in the L^p-norm for every $f \in L^p(\mathbb{R})$:

Proposition 5. *Let $\{\phi_\epsilon\}_{\epsilon \in \mathbb{R}_+}$ be an approximate identity and $1 \leq p < \infty$. Then, $f * \phi_\epsilon \in L^p(\mathbb{R})$ for every $f \in L^p(\mathbb{R})$ and $\epsilon \in \mathbb{R}_+$. Moreover,*

$$\lim_{\epsilon \to 0^+} \|f - f * \phi_\epsilon\|_p = 0.$$

Proposition 5 together with Lemma 2 implies that, given an approximate identity $\{\phi_\epsilon\}_{\epsilon \in \mathbb{R}_+}$, any real-valued $f \in \mathrm{PW}_\Omega$ can be uniquely recovered (up to a global sign factor) from the measurements $\{|f * \phi_\epsilon|\}_{\epsilon \in \mathbb{R}_+}$:

Theorem 6. *Let $\{\phi_\epsilon\}_{\epsilon \in \mathbb{R}_+}$ be an approximate identity. Then, the following are equivalent for $f, g \in \mathrm{PW}_\Omega$ real-valued on the real line:*

i) $|f * \phi_\epsilon| = |g * \phi_\epsilon|, \quad \epsilon \in \mathbb{R}_+$;

ii) $f = \pm g$.

Proof. Let $f, g \in \mathrm{PW}_\Omega$. It is clear that if $f = \pm g$, then *i)* holds true. Conversely, we suppose that $|f * \phi_\epsilon| = |g * \phi_\epsilon|$ for every $\epsilon \in \mathbb{R}_+$. By Proposition 5, we have that $f * \phi_\epsilon$ and $g * \phi_\epsilon$ converge to f and g in $L^2(\mathbb{R})$ as $\epsilon \to 0^+$, respectively. Since by the convolution theorem $f * \phi_\epsilon$ and $g * \phi_\epsilon$ belong to PW_Ω for every $\epsilon \in \mathbb{R}_+$ and PW_Ω is a RKHS, we have that $f * \phi_\epsilon$ and $g * \phi_\epsilon$ converge to f and g pointwise as $\epsilon \to 0^+$. Furthermore, since the modulus is a continuous function, $|f * \phi_\epsilon|$ and $|g * \phi_\epsilon|$ converge pointwise to $|f|$ and $|g|$ as $\epsilon \to 0^+$. Therefore, our assumption implies $|f(x)| = |g(x)|$ for every $x \in \mathbb{R}$. Hence, by Lemma 2, we can conclude that $f = \pm g$. □

By the definition of the wavelet transform (cf. Definition 1) and by Example 1, if we fix $\psi \in L^1(\mathbb{R})$ such that $\widehat{\psi}(0) = 1$, then by Theorem 6 any real-valued $f \in \mathrm{PW}_\Omega$ can be uniquely recovered (up to a global sign factor) from the magnitude of its wavelet transform $\{|f * \psi_a^\#|\}_{a\in\mathbb{R}_+}$. Unfortunately, we cannot apply Theorem 6 when ψ is a classical wavelet since wavelets are always assumed to have zero mean. It is therefore natural to ask if it possible to recover the same uniqueness result when $\widehat{\psi}(0) = 0$. The next section is devoted to answering this question.

3 Main results

We say that a function $\psi \in L^1(\mathbb{R})$ has n vanishing moments, for $n \in \mathbb{N}$, if it satisfies

$$\int_{\mathbb{R}} x^k \psi(x)\mathrm{d}x = 0, \qquad k = 0, \dots, n. \tag{2}$$

By the definition of the Fourier transform, condition (2) with $n = 0$ is equivalent to $\widehat{\psi}(0) = 0$. In general, we have the following result:

Proposition 7 ([5, Lemma 6.0.4])**.** *Let $n \in \mathbb{N}$ and $f \in L^1(\mathbb{R})$ be such that $x^n f \in L^1(\mathbb{R})$. Then, f has n vanishing moments if and only if*

$$\lim_{\xi\to 0} \frac{\widehat{f}(\xi)}{\xi^n} = 0.$$

The next proposition is a classical result, see e.g. [5, Chapter 4, §2]. Here, we give an alternative proof for the sake of completeness.

Proposition 8. *Let $\Omega > 0$ and let $\psi \in L^1(\mathbb{R})$ be such that*

$$\lim_{\xi\to 0} \xi^{-\ell}\widehat{\psi}(\xi) = (-1)^\ell (2\pi i)^\ell,$$

for some $\ell \in \mathbb{N}$. Then, for every $f \in PW_\Omega$

$$\lim_{a\to 0^+} \|f^{(\ell)} - a^{-\ell}\mathcal{W}_\psi f(\cdot, a)\|_2 = 0.$$

Proof. By the definition of the wavelet transform and the Plancherel theorem we have

$$\begin{aligned}\|f^{(\ell)} - a^{-\ell}\mathcal{W}_\psi f(\cdot, a)\|_2^2 &= \|f^{(\ell)} - a^{-\ell} f * \psi_a^\#\|_2^2 \\ &= \int_{\mathbb{R}} |\xi^\ell \widehat{f}(\xi)|^2 |(2\pi i)^\ell - (a\xi)^{-\ell}\overline{\widehat{\psi}(a\xi)}|^2 \mathrm{d}\xi.\end{aligned}$$

By the Riemann–Lebesgue lemma, $\widehat{\psi}$ is a continuous function which goes to zero at infinity and by hypothesis

$$\lim_{\xi\to 0} \frac{\widehat{\psi}(\xi)}{\xi^\ell} = (-1)^\ell (2\pi i)^\ell.$$

Therefore, we have the estimate

$$|\xi^\ell \widehat{f}(\xi)|^2 |(2\pi i)^\ell - (a\xi)^{-\ell}\overline{\widehat{\psi}(a\xi)}|^2 \le M|\xi^\ell \widehat{f}(\xi)|^2,$$

where $M = \sup_{\xi\in\mathbb{R}} |(2\pi i)^\ell - (a\xi)^{-\ell}\overline{\widehat{\psi}(a\xi)}|^2$ is finite and independent of a. Furthermore, for almost every $\xi \in \mathbb{R}$

$$\lim_{a\to 0^+} |\xi^\ell \widehat{f}(\xi)|^2 |(2\pi i)^\ell - (a\xi)^{-\ell}\overline{\widehat{\psi}(a\xi)}|^2 = 0.$$

Hence, by the dominated convergence theorem

$$\lim_{a\to 0^+} \|f^{(\ell)} - a^{-\ell}\mathcal{W}_\psi f(\cdot,a)\|_2 = 0$$

and this concludes the proof. □

We are now in a position to state our first result, establishing uniqueness of wavelet phase retrieval for real-valued bandlimited signals when the wavelet has finitely many vanishing moments.

Theorem 9. *Let $\Omega > 0$ and let $\psi \in L^1(\mathbb{R})$ be such that*

$$\lim_{\xi\to 0} \xi^{-\ell}\widehat{\psi}(\xi) = c \in \mathbb{C}\setminus\{0\},$$

for some $\ell \in \mathbb{N}$. Then, the following are equivalent for $f, g \in \mathrm{PW}_\Omega$ real-valued on the real line:

i) $|\mathcal{W}_\psi f(b,a)| = |\mathcal{W}_\psi g(b,a)|, \quad b \in \mathbb{R},\ a \in \mathbb{R}_+$*;*

ii) $f = \pm g$.

Proof. Let $f, g \in \mathrm{PW}_\Omega$. It is clear that if $f = \pm g$, then i) holds. Conversely, we suppose that ii) holds. Let us define

$$\phi := \frac{(-1)^\ell (2\pi i)^\ell}{c}\psi.$$

Then, it is clear that

$$\lim_{\xi\to 0} \xi^{-\ell}\widehat{\phi}(\xi) = (-1)^\ell (2\pi i)^\ell$$

and that by i)

$$|\mathcal{W}_\phi f(b,a)| = |\mathcal{W}_\phi g(b,a)| \tag{3}$$

for every $b \in \mathbb{R}$ and $a \in \mathbb{R}_+$. By Proposition 8, it follows that $a^{-\ell}\mathcal{W}_\phi f(\cdot,a)$ and $a^{-\ell}\mathcal{W}_\phi g(\cdot,a)$ converge to $f^{(\ell)}$ and $g^{(\ell)}$ in $L^2(\mathbb{R})$ as $a \to 0^+$, respectively. Furthermore, by Lemma 3, we know that $a^{-\ell}\mathcal{W}_\phi f(\cdot,a)$ and $a^{-\ell}\mathcal{W}_\phi g(\cdot,a)$ belong to PW_Ω for every $a \in \mathbb{R}_+$ and PW_Ω is a RKHS. Thus, $a^{-\ell}\mathcal{W}_\phi f(\cdot,a)$ and $a^{-\ell}\mathcal{W}_\phi g(\cdot,a)$ converge pointwise to $f^{(\ell)}$ and $g^{(\ell)}$ as $a \to 0^+$. Since the modulus is a continuous function, $a^{-\ell}|\mathcal{W}_\phi f(\cdot,a)|$ and $a^{-\ell}|\mathcal{W}_\phi g(\cdot,a)|$ converge pointwise to $|f^{(\ell)}|$ and $|g^{(\ell)}|$ as $a \to 0^+$. Combining this with equation (3) implies that $|f^{(\ell)}(b)| = |g^{(\ell)}(b)|$, for every $b \in \mathbb{R}$, and employing Lemma 2 we can conclude that $f^{(\ell)} = \pm g^{(\ell)}$.

Together with the analyticity of f and g this implies

$$f(x) \mp g(x) = P(x), \quad x \in \mathbb{R},$$

where P is a polynomial of degree $\ell - 1$. Now, if P is not the null polynomial, then $f \mp g$ is not in $L^2(\mathbb{R})$ and we have a contradiction. Therefore, $P \equiv 0$ and $f = \pm g$. □

Remark 10. It is worth observing that Theorem 9 does not hold for progressive wavelets, that is for wavelets with only positive frequencies. Indeed, the hypothesis of Theorem 9 will always be violated since for all progressive wavelets ψ and any $\ell \in \mathbb{N}$

$$\lim_{\xi \to 0^-} \xi^{-\ell} \widehat{\psi}(\xi) = 0.$$

Actually, real-valued signals cannot be determined up to global phase by the magnitude of their wavelet transform with respect to any progressive wavelet. By the definition of the wavelet transform, it is immediate to observe that if $f, g \in L^2(\mathbb{R})$ are such that $f_+ = e^{i\alpha} g_+$ and ψ is a progressive wavelet, then

$$|\mathcal{W}_\psi f(b,a)| = |\mathcal{W}_\psi g(b,a)|, \tag{4}$$

for any $b \in \mathbb{R}$ and $a \in \mathbb{R}_+$. We show that it is actually possible to construct real-valued signals that do not agree up to global phase even though their analytic representations do, and thus (4) is satisfied. We consider $f, g \in L^2(\mathbb{R})$ as well as $\alpha \in \mathbb{R}$ and we suppose that

$$f_+ = e^{i\alpha} g_+,$$

or equivalently

$$\operatorname{Re} f_+ = \operatorname{Re}(e^{i\alpha} g_+), \quad \operatorname{Im} f_+ = \operatorname{Im}(e^{i\alpha} g_+).$$

We recall that

$$f_+(x) = f(x) + i(\mathcal{H}f)(x), \tag{5}$$

where $\mathcal{H}f$ denotes the Hilbert transform of f. By equation (5), $\operatorname{Re} f_+ = \operatorname{Re}(e^{i\alpha} g_+)$ is equivalent to

$$f = \cos\alpha \cdot g - \sin\alpha \cdot \mathcal{H}g \tag{6}$$

and, analogously, $\operatorname{Im} f_+ = \operatorname{Im}(e^{i\alpha} g_+)$ is equivalent to

$$\mathcal{H}f = \cos\alpha \cdot \mathcal{H}g + \sin\alpha \cdot g. \tag{7}$$

Furthermore, the property $\mathcal{H}(\mathcal{H}f) = -f$ implies that equations (6) and (7) are equivalent and thus, $f_+ = e^{i\alpha} g_+$ if and only if f takes the form (6). Therefore, if we take a real-valued signal $g \in L^2(\mathbb{R})$ as well as $\alpha \in \mathbb{R}$ and we define f by (6), then f is real-valued and $f_+ = e^{i\alpha} g_+$. However, if α is not a multiple of π, then f and g will in general not agree up to global phase.

We now introduce the Paley–Wiener space of bandlimited functions

$$\mathrm{PW}^1_\Omega = \{f \in L^1(\mathbb{R}) : \operatorname{supp} \widehat{f} \subseteq [-\Omega, \Omega]\}$$

and we observe that $\mathrm{PW}^1_\Omega \subseteq \mathrm{PW}_\Omega$ for every $\Omega > 0$. Furthermore, we have the following result:

Proposition 11. *Let $\Omega > 0$ and $f \in \mathrm{PW}_\Omega$. Then, $|f|^2 \in \mathrm{PW}^1_{2\Omega}$.*

Proof. We first note that $|f|^2 = f\overline{f} \in L^1(\mathbb{R})$ and by the convolution theorem

$$(|f|^2)^\wedge = \widehat{f} * \widehat{\overline{f}} = \widehat{f} * \widehat{f}^\#.$$

Then, since $\operatorname{supp}(|f|^2)^\wedge \subseteq \operatorname{supp} \widehat{f} + \operatorname{supp} \widehat{f}^\# \subseteq [-2\Omega, 2\Omega]$, we conclude that $|f|^2 \in \mathrm{PW}^1_{2\Omega}$. □

Finally, before stating our main theorem, we recall a classical result in the theory of Paley–Wiener spaces known as the Whittaker–Shannon–Kotelnikov (WSK) sampling theorem.

Theorem 12 (WSK sampling theorem)**.** *Let* $\Omega > 0$ *and* $f \in \mathrm{PW}_\Omega$*. Then, for every* $x \in \mathbb{R}$

$$f(x) = \sum_{m\in\mathbb{Z}} f\left(\frac{m}{2\Omega}\right) \operatorname{sinc}(2\Omega x - m).$$

We can now state and prove our main result on uniqueness of wavelet phase retrieval from samples:

Theorem 13. *Let* $\Omega > 0$ *and let* $\psi \in L^1(\mathbb{R})$ *be such that*

$$\lim_{\xi\to 0} \xi^{-\ell}\widehat{\psi}(\xi) = c \in \mathbb{C} \setminus \{0\},$$

for some $\ell \in \mathbb{N}$*. Furthermore, let* $(a_k)_{k\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}_+$ *such that* $a_k \to 0$ *as* $k \to \infty$*. Then, the following are equivalent for* $f, g \in \mathrm{PW}_\Omega$ *real-valued on the real line:*

i) $\left|\mathcal{W}_\psi f(\frac{m}{4\Omega}, a_k)\right| = \left|\mathcal{W}_\psi g(\frac{m}{4\Omega}, a_k)\right|, \quad m, k \in \mathbb{N}$*;*

ii) $f = \pm g$*.*

Proof. Let $f, g \in \mathrm{PW}_\Omega$. It is clear that if $f = \pm g$, then $i)$ holds. Conversely, assume that $i)$ is true. Setting

$$\phi = \frac{(-1)^\ell (2\pi i)^\ell}{c} \psi$$

and following the same argument as in the proof of Theorem 9, we can establish that $a_k^{-2\ell} |\mathcal{W}_\phi f(\frac{m}{4\Omega}, a_k)|^2$ and $a_k^{-2\ell} |\mathcal{W}_\phi g(\frac{m}{4\Omega}, a_k)|^2$ converge to $|f^{(\ell)}(\frac{m}{4\Omega})|^2$ and $|g^{(\ell)}(\frac{m}{4\Omega})|^2$, respectively, for every $m \in \mathbb{N}$ as $k \to \infty$. Then, since $i)$ holds, we have that

$$\left|f^{(\ell)}\left(\frac{m}{4\Omega}\right)\right|^2 = \left|g^{(\ell)}\left(\frac{m}{4\Omega}\right)\right|^2, \quad \text{for all } m \in \mathbb{N}.$$

Furthermore, by Proposition 11 we know that $|f^{(\ell)}|^2$ and $|g^{(\ell)}|^2$ belong to $\mathrm{PW}^1_{2\Omega} \subseteq \mathrm{PW}_{2\Omega}$. Thus, by the WSK sampling theorem it follows that

$$\left|f^{(\ell)}(x)\right|^2 = \left|g^{(\ell)}(x)\right|^2, \quad \text{for all } x \in \mathbb{R},$$

and consequently $|f^{(\ell)}(x)| = |g^{(\ell)}(x)|$ for all $x \in \mathbb{R}$. Finally, as in the proof of Theorem 9, we can conclude that $f = \pm g$. □

4 Examples

4.1 The Morlet wavelet

Let $\xi_0 \in \mathbb{R}\setminus\{0\}$. The Morlet wavelet, also known as the Gabor wavelet, is at the origin of the development of wavelet analysis. It was introduced by Grossmann and Morlet in [4]. It is defined on the frequency side by the function

$$\widehat{\psi}(\xi) = \pi^{-\frac{1}{4}}[e^{-(\xi-\xi_0)^2/2} - e^{-\xi^2/2} e^{-\xi_0^2/2}], \quad \xi \in \mathbb{R}.$$

Its Fourier transform is a shifted Gaussian adjusted with a corrective term in order to have $\widehat{\psi}(0) = 0$. The Morlet wavelet is complex-valued:

$$\psi(x) = \pi^{-\frac{1}{4}}[e^{-i\xi_0 x} - e^{-\xi_0^2/2}]e^{-x^2/2}, \quad x \in \mathbb{R},$$

even though most applications in which it is used involve only real-valued signals. By a direct computation, the Fourier transform $\widehat{\psi}(\xi)$ goes to zero as $\xi \to 0$ with infinitesimal order 1. Indeed, using a Taylor expansion yields

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\widehat{\psi}(\xi)}{\xi} &= \lim_{\xi\to 0} \frac{\pi^{-\frac{1}{4}}[e^{-(\xi-\xi_0)^2/2} - e^{-\xi^2/2}e^{-\xi_0^2/2}]}{\xi} \\
&= \lim_{\xi\to 0} \frac{\pi^{-\frac{1}{4}}[e^{-\xi_0^2/2} + \xi_0 e^{-\xi_0^2/2}\xi - e^{-\xi_0^2/2} + e^{-\xi_0^2/2}\xi^2/2 + o(\xi)]}{\xi} \\
&= \lim_{\xi\to 0} \frac{\pi^{-\frac{1}{4}}[\xi_0 e^{-\xi_0^2/2}\xi + o(\xi)]}{\xi} = \pi^{-\frac{1}{4}}\xi_0 e^{-\xi_0^2/2}.
\end{aligned}$$

Thus, ψ satisfies the hypothesis of Theorem 9 with $\ell = 1$. Therefore, all real-valued $f \in \mathrm{PW}_\Omega$ can be recovered up to global sign from the measurements $|\mathcal{W}_\psi f(b,a)|$, for $b \in \mathbb{R}$, $a \in \mathbb{R}_+$. Furthermore, by Theorem 13 we know that it is enough to know the magnitude of the wavelet transform $\mathcal{W}_\psi f$ for the samples $\{(m/4\Omega, a_k) : m \in \mathbb{N},\ k \in \mathbb{N}\}$, where $(a_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}_+$ which goes to zero as $k \to \infty$.

4.2 The linear-chirp wavelet

Another example of a complex-valued wavelet that satisfies our hypothesis is the linear-chirp wavelet. The idea to use chirps as wavelets, also called chirplets, was introduce in [7, 8]. We refer also to [5] for a concise presentation. Let ξ_0, $\beta \in \mathbb{R}$. It is defined by windowing a linear chirp with a Gaussian:

$$\psi(x) = e^{i(\xi_0 + \beta x/2)x}e^{-x^2/2} + \eta(x).$$

Again, the corrective term η is added in order to have zero mean. Its Fourier transform is given by

$$\widehat{\psi}(\xi) = \sqrt{\frac{2\pi}{1-i\beta}}e^{-(\xi-\xi_0)^2/2(1-i\beta)} + \widehat{\eta}(\xi).$$

For instance, we may set

$$\widehat{\eta}(\xi) = -\sqrt{\frac{2\pi}{1-i\beta}}e^{-\xi_0^2/2(1-i\beta)}e^{-\xi^2/2}$$

and, with Taylor expansion, we obtain

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\widehat{\psi}(\xi)}{\xi} &= \lim_{\xi\to 0} \sqrt{\frac{2\pi}{1-i\beta}} \frac{e^{-(\xi-\xi_0)^2/2(1-i\beta)} - e^{-\xi_0^2/2(1-i\beta)}e^{-\xi^2/2}}{\xi} \\
&= \lim_{\xi\to 0} \sqrt{\frac{2\pi}{1-i\beta}} \frac{e^{-\xi_0^2/2(1-i\beta)}\xi_0\xi/(1-i\beta) + o(\xi)}{\xi} = \frac{\sqrt{2\pi}}{(1-i\beta)^{3/2}}e^{-\xi_0^2/2(1-i\beta)}\xi_0.
\end{aligned}$$

This shows that ψ satisfies the hypothesis of Theorem 9 and Theorem 13 with $\ell = 1$.

5 Sampling Cauchy wavelet transform magnitudes

5.1 Introduction

Our main uniqueness result for phase retrieval from wavelet magnitude samples (Theorem 13) is not applicable to so-called *progressive wavelets* which are wavelets that only have positive frequencies.

This observation is not surprising in light of the fact that real-valued signals f are not uniquely determined (up to global phase) by wavelet transform magnitude measurements $|\mathcal{W}_\psi f|$ for progressive wavelets ψ (see Remark 10 in Section 3). It does, however, raise the following question:

(Q) Is there a class of signals which can be recovered (up to global phase) from wavelet transform magnitude measurements with progressive mother wavelets?

In general, this question is hard to answer. An elegant partial answer is, however, given in [6].

The authors of [6] consider the so-called *Cauchy wavelet* given by

$$\widehat{\psi}(\xi) = \rho(\xi)\xi^p \mathrm{e}^{-\xi} \mathbf{1}_{\xi>0}, \qquad \xi \in \mathbb{R}, \tag{8}$$

where $p > 0$ and $\rho \in L^\infty(\mathbb{R})$ is such that $\rho(a\xi) = \rho(\xi)$, for a.e. $\xi \in \mathbb{R}$, and $\rho(\xi) \neq 0$, for all $\xi \in \mathbb{R}$. Using tools from the theory of entire functions, they show that the class of *analytic signals* may be recovered uniquely (up to global phase) from the magnitude of the Cauchy wavelet transform. Analytic signals are functions $f \in L^2(\mathbb{R})$ which have no negative frequencies. To be precise, they show the following theorem.

Theorem 14 (Corollary 2.2 in [6], p. 1259)**.** *Let $a > 1$ and let ψ be the Cauchy wavelet defined as in equation* (8)*. Let, moreover, $f, g \in L^2(\mathbb{R})$ be such that for some $j, k \in \mathbb{Z}$, with $j \neq k$,*

$$\left|\mathcal{W}_\psi f\left(\cdot, a^j\right)\right| = \left|\mathcal{W}_\psi g\left(\cdot, a^j\right)\right| \qquad \textit{and} \qquad \left|\mathcal{W}_\psi f\left(\cdot, a^k\right)\right| = \left|\mathcal{W}_\psi g\left(\cdot, a^k\right)\right|.$$

We denote by f_+ and g_+ the analytic representations *of f and g which are defined through the equations*

$$\widehat{f_+}(\xi) = 2\widehat{f}(\xi)\mathbf{1}_{\xi>0} \qquad \textit{and} \qquad \widehat{g_+}(\xi) = 2\widehat{g}(\xi)\mathbf{1}_{\xi>0},$$

for $\xi \in \mathbb{R}$. Then, there exists an $\alpha \in \mathbb{R}$ such that

$$f_+ = \mathrm{e}^{\mathrm{i}\alpha} g_+.$$

Note that the above is more than a simple uniqueness theorem for phase retrieval from Cauchy wavelet transform magnitude measurements of analytic signals. It is, in fact, a uniqueness result for the semi-discrete wavelet frame. Even more, the Cauchy wavelet magnitudes are assumed to agree on two scales only. It does therefore stand to reason that further restricting the signal class to analytic bandlimited signals should allow us to come up with a full sampling result for the Cauchy wavelet transform.

In the following, we will assume that the function $\rho \in L^\infty(\mathbb{R})$ used in the definition of the Cauchy wavelet ψ is such that $\psi \in L^1(\mathbb{R})$. This is a natural assumption as mother wavelets are usually assumed to be in $L^1(\mathbb{R})$. Moreover, there is a wide variety of $\rho \in L^\infty(\mathbb{R})$ which satisfy this assumption (see Remark 15). We want to stress, however, that this assumption is not necessary for our arguments to work and is made purely to simplify the mathematical exposition. Indeed, by the definition of the Cauchy wavelet (8), one can see immediately that $\psi \in L^2 \cap L^\infty(\mathbb{R})$. Therefore, we may replace our subsequent use of the WSK sampling theorem by the use of classical sampling theory in the Bernstein space $B_{2\Omega}$ to obtain a sampling result for more general $\rho \in L^\infty(\mathbb{R})$ at a slightly finer sampling density in frequency.

Remark 15. One can show that if $\rho \in L^\infty(\mathbb{R})$ is continuous and satisfies

$$|\rho'(\xi)| \lesssim \mathrm{e}^{\xi/2} \qquad \text{and} \qquad |\rho''(\xi)| \lesssim \mathrm{e}^{\xi/2},$$

for all $\xi \in \mathbb{R}$, then the Cauchy wavelet ψ defined by (8) is in $L^1(\mathbb{R})$. In particular, if ρ is a constant function, then $\psi \in L^1(\mathbb{R})$.

5.2 The sampling result for analytic signals

We remind the reader of two pertinent results stated earlier in this manuscript: First, the wavelet transform $\mathcal{W}_\psi f(\cdot, a)$ of a bandlimited signal f is bandlimited itself, for $a \in \mathbb{R}_+$ (see Lemma 3). Secondly, bandlimitedness carries over from any function to its absolute value squared. These two insights combined yield the following corollary.

Corollary 16. *Let $\Omega > 0$. If $f \in \mathrm{PW}_\Omega$ and $\psi \in L^1(\mathbb{R})$, then $|\mathcal{W}_\psi f(\cdot, a)|^2 \in \mathrm{PW}^1_{2\Omega} \subset \mathrm{PW}_{2\Omega}$, for all $a \in \mathbb{R}_+$.*

What remains is to combine Theorem 14 with the classical WSK sampling theorem (Theorem 12). Thereby, we obtain the following sampling result for the recovery of analytic signals.

Theorem 17. *Let $\Omega > 0$, $a > 1$ and let $\psi \in L^1(\mathbb{R})$ be as in equation* (8) *with $\rho \in L^\infty(\mathbb{R})$. Then, the following are equivalent for $f, g \in \mathrm{PW}_\Omega$:*

i) For all $k \in \mathbb{Z}$,

$$\left|\mathcal{W}_\psi f\left(\frac{k}{4\Omega}, 1\right)\right| = \left|\mathcal{W}_\psi g\left(\frac{k}{4\Omega}, 1\right)\right| \text{ and } \left|\mathcal{W}_\psi f\left(\frac{k}{4\Omega}, a\right)\right| = \left|\mathcal{W}_\psi g\left(\frac{k}{4\Omega}, a\right)\right|.$$

ii) $f_+ = \mathrm{e}^{\mathrm{i}\alpha} g_+$, for some $\alpha \in \mathbb{R}$.

Proof. It is obvious that *ii*) implies *i*). Now, suppose that *i*) holds. By assumption, we have that $\psi \in L^1(\mathbb{R})$. Therefore, we may apply Corollary 16 to see that $|\mathcal{W}_\psi f(\cdot, a^j)|^2$ as well as $|\mathcal{W}_\psi g(\cdot, a^j)|^2$ are in $\mathrm{PW}_{2\Omega}$, for $j \in \{0, 1\}$. Hence, it follows from *i*) along with the WSK sampling theorem that

$$|\mathcal{W}_\psi f(\cdot, 1)| = |\mathcal{W}_\psi g(\cdot, 1)| \qquad \text{and} \qquad |\mathcal{W}_\psi f(\cdot, a)| = |\mathcal{W}_\psi g(\cdot, a)|.$$

Finally, Theorem 14 implies that the analytic representations f_+ and g_+ of f and g satisfy $f_+ = \mathrm{e}^{\mathrm{i}\alpha} g_+$, for some $\alpha \in \mathbb{R}$. □

Remark 18. Note that the sampling set $\{(k/4\Omega, a^j) \mid j = 0, 1,\ k \in \mathbb{Z}\}$ in Theorem 17 can be replaced by a multitude of different sampling sets:

In scale, we might sample at any two elements of $a^{\mathbb{Z}}$ as is evident from Theorem 14. In addition, one might show that Theorem 14 continues to hold for a^j replaced by a_0 and a^k replaced by a_1, for all $0 < a_0 < a_1$, provided that $\rho(\xi) = \rho(a_0\xi) = \rho(a_1\xi)$, for a.e. $\xi \in \mathbb{R}$.

In time, we can replace the uniform sampling by any separated, uniformly dense sampling sequence with density lower bounded by 4Ω.

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