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F. Bartolucci and S. Pilipovic and N. Teofanov

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CH-8092 Zürich
Switzerland

Continuity Properties of the Shearlet Transform and the Shearlet Synthesis Operator on the Lizorkin type Spaces

Francesca Bartolucci [*] Stevan Pilipović [†] Nenad Teofanov [†]

Abstract

We develop a distributional framework for the shearlet transform $\mathcal{S}_\psi \colon \mathcal{S}_0(\mathbb{R}^2) \to \mathcal{S}(\mathbb{S})$ and the shearlet synthesis operator $\mathcal{S}_\psi^t \colon \mathcal{S}(\mathbb{S}) \to \mathcal{S}_0(\mathbb{R}^2)$, where $\mathcal{S}_0(\mathbb{R}^2)$ is the Lizorkin test function space and $\mathcal{S}(\mathbb{S})$ is the space of highly localized test functions on the standard shearlet group $\mathbb{S}$. These spaces and their duals $\mathcal{S}_0'(\mathbb{R}^2)$, $\mathcal{S}'(\mathbb{S})$ are called Lizorkin type spaces of test functions and distributions. We analyze the continuity properties of these transforms when the admissible vector ψ belongs to $\mathcal{S}_0(\mathbb{R}^2)$. Then, we define the shearlet transform and the shearlet synthesis operator of Lizorkin type distributions as transpose mappings of the shearlet synthesis operator and the shearlet transform, respectively. They yield continuous mappings from $\mathcal{S}_0'(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{S})$ and from $\mathcal{S}'(\mathbb{S})$ to $\mathcal{S}_0'(\mathbb{R}^2)$. Furthermore, we show the consistency of our definition with the shearlet transform defined by direct evaluation of a distribution on the shearlets. The same can be done for the shearlet synthesis operator. Finally, we give a reconstruction formula for Lizorkin type distributions, from which follows that the action of such generalized functions can be written as an absolutely convergent integral over the standard shearlet group.

Key words. shearlet and shearlet synthesis transforms; distributions of slow growth; Lizorkin type spaces of test functions and their duals;
2010 Mathematics Subject Classification. 46F12, 42C40

1 Introduction

In this paper we provide a sound mathematical background for the shearlet transform and the corresponding shearlet synthesis operator when acting on test function and distribution spaces. This is done by studying the continuity properties of these two integral transforms on the Lizorkin type spaces of test functions and, by duality, on the corresponding distribution spaces. Our results are consistent with the ones given in [6, 13, 18] for L^2-functions. Actually, concerning the continuity results, we extend the corresponding ones of quoted papers.

[*]Department of Mathematics, ETH Zurich, Raemistrasse 101, 8092 Zurich, Switzerland (francesca.bartolucci@sam.math.ethz.ch).

[†]Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia (stevan.pilipovic@gmail.com, nenad.teofanov@dmi.uns.ac.rs).

The theory of shearlets emerged from the research activities aimed to create a new generation of analysis and processing tools for massive and higher dimensional data which could go beyond traditional Fourier and wavelet systems, [3, 19]. In signal analysis it is customary to transform a signal, modeled as an element of a Hilbert space $\mathcal{H}$, into a new function or distribution over a convenient parameter space in order to extract the most relevant information in the most efficient way, [4]. For example, the wavefront set carries both the information on the location and the geometry of the singularity set of a signal and it can be resolved by means of the shearlet transform, [12, 13].

In a certain sense, shearlets behave for high-dimensional signals as wavelets do for one-dimensional ones. We refer to [1] for the link between these two classical transforms in the framework of square-integrable group representations theory, which offers a unified approach to treat different relevant representations in signal analysis when the parameter space forms a group. In such a case, signals are mapped via the so-called voice transform in a new function on the group, which is square-integrable with respect to a Haar measure under some technical assumptions.

An important issue in harmonic analysis is the extension of an integral transform from a Hilbert space to a more general framework of generalized function spaces, and there are two basic approaches to deal with this problem. The first approach is to define the generalized transform through the action of a distribution on a family of test functions in a suitable test function space. In this respect, the coorbit space theory introduced by Feichtinger and Gröchenig in [8, 9] is a powerful tool which applies when the integral transform is the voice transform associated to a square-integrable representation of a locally compact group. This is the case of the shearlet transform, and we refer to [6] for its extension based on the coorbit space theory, which yields completely new regularity spaces, the shearlet coorbit spaces, and the corresponding dual spaces of distributions. The second approach, which we employ in the current paper, consists in defining a formal transpose of the integral transform, and then the extended transform between pairs of dual spaces by transposition. The equivalence between these two approaches can be showed by comparing them on the intersection of their domains.

We follow the second approach based on the duality theory related to locally convex spaces, i.e. the distribution theory introduced by Schwartz in [24]. Moreover, we give the necessary comparison with the direct approach. Our work is motivated by the lack of a complete and rigorous distributional framework for the shearlet transform in the literature. On the contrary, related integral transforms, such as the Radon and the wavelet transforms, are well understood and deeply studied in the context of test function and distribution spaces in e.g. [14, 15, 16, 20, 23].

It turns out that the Lizorkin type spaces of test functions $\mathcal{S}_0(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{S})$ are the natural domain and range space for the shearlet transform. The same holds for the shearlet synthesis operator but in the opposite direction. Recall, the Lizorkin space $\mathcal{S}_0(\mathbb{R}^d)$ consists of smooth, rapidly decreasing functions with vanishing moments of any order. Kostadinova et al. [20] showed that the domain of the ridgelet transform can be enlarged to its dual space $\mathcal{S}'_0(\mathbb{R}^2)$, known as the space of Lizorkin distributions. The proof exploits the close connection between the Radon, the ridgelet and the wavelet transforms. This result, in combination with [1], yields a relation formula between the shearlet transform and the ridgelet transform, as it is shown in [2].

The vanishing moments condition for the functions in the test space is not surprising. Indeed, as pointed out in the wavelet [21] and the shearlet analysis [12], as well as in the recent study of the Taylorlet transform [10], vanishing moments are crucial in order to measure the local regularity and to detect anisotropic structures of a signal. Furthermore, we notice that the rectified linear units (ReLUs), which are important examples of unbounded activation functions in the context of deep learning neural networks, belong to the space of Lizorkin distributions, see [25] for details.

Our main results are continuity theorems for the shearlet transform and the shearlet synthesis operator on appropriate test function spaces. Since the image of a signal under the shearlet transform is a function over the shearlet group $\mathbb{S} = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\times$, and our aim is to analyze its regularity and decay properties with respect to all parameters, the corresponding topology is given by a family of norms determined by eight indices. We prove that the shearlet transform $\mathcal{S}_\psi \colon \mathcal{S}_0(\mathbb{R}^2) \to \mathcal{S}(\mathbb{S})$ and the corresponding synthesis operator $\mathcal{S}^t_\psi \colon \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{R}^2)$ are continuous mappings, where $\mathcal{S}(\mathbb{S})$ is endowed with the above mentioned topology. Then, we use these continuity results to extend the shearlet transform and the shearlet synthesis operator to the space of Lizorkin type distributions $\mathcal{S}'_0(\mathbb{R}^2)$ and $\mathcal{S}'(\mathbb{S})$ by the duality approach and following several ideas in [20]. We show the continuity properties of the shearlet transform when acting on $\mathcal{S}'_0(\mathbb{R}^2)$. Observe that many important Schwartz distribution spaces, such as $\mathcal{E}'(\mathbb{R}^d)$, $\mathcal{O}'_C(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$ and $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ are embedded into the space of Lizorkin distibutions $\mathcal{S}'_0(\mathbb{R}^d)$, see e.g. [20] for details.

We complete our analysis by showing that our definition of the shearlet transform of distributions extends the ones considered so far in e.g. [19, 12], and that it is consistent with those for test functions. Moreover, it follows from our analysis that our duality approach is equivalent to the one based on the coorbit space theory presented in [6].

1.1 Notation

We briefly introduce the d-dimensional notation. We set $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. When $x, y \in \mathbb{R}^d$, $xy = x_1y_1 + x_2y_2 + \cdots + x_dy_d$, $|x|$ denotes the Euclidean norm, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $x^m = x_1^{m_1} \dots x_d^{m_d}$, and $\partial^m = \partial^m_x = \partial^{m_1}_{x_1} \dots \partial^{m_d}_{x_d}$, $m \in \mathbb{N}^d$. We write also $\varphi^{(m)} = \partial^m \varphi$, $m \in \mathbb{N}^d$. By a slight abuse of notation, the length of a multi-index $m \in \mathbb{N}^d$ is denoted by $|m| = m_1 + \cdots + m_d$ and the meaning of $|\cdot|$ shall be clear from the context. For every $b \in \mathbb{R}^d$, the translation operator acts on a function $f : \mathbb{R}^d \to \mathbb{C}$ as $T_b f(x) = f(x - b)$ and the dilation operator $D_a \colon L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is defined by $D_a f(x) = |a|^{-\frac{1}{2}} f(x/a)$ for every $a \in \mathbb{R}^\times$. We write $A \lesssim B$ when $A \leq C \cdot B$ for some positive constant C.

As usual, $L^p(\mathbb{R}^d)$, $p \in [1, +\infty)$, is the Banach space of p-integrable functions $f \colon \mathbb{R}^d \to \mathbb{C}$ with respect to the Lebesgue measure $\mathrm{d}x$ and, if $p = 2$, the corresponding scalar product and norm are $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The space $L^\infty(\mathbb{R}^d)$ consists of essentially bounded functions. The Fourier transform $\mathcal{F}$ is given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i\, \xi x} \mathrm{d}x, \qquad f \in L^1(\mathbb{R}^d),$$

and it extends to $L^2(\mathbb{R}^d)$ in the usual way.

The dual pairing between a test function space $\mathcal{A}$ and its dual space of distributions $\mathcal{A}'$ is denoted by ${}_{\mathcal{A}'}(\,\cdot\,,\,\cdot\,)_{\mathcal{A}}$ and we provide all distribution spaces with the strong dual topologies. For simplicity in the notation, we also use dual pairings without explicitly stating the spaces $\mathcal{A}, \mathcal{A}'$ and they will be clear from the context.

The Schwartz space of rapidly decreasing smooth test functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ denotes its dual space of tempered distributions. Concerning the family of norms which defines a projective limit topology on $\mathcal{S}(\mathbb{R}^d)$, we make the choice

$$\rho_\nu(\varphi) = \sup_{x\in\mathbb{R}^d, |m|\le\nu} \langle x\rangle^\nu |\partial^m \varphi(x)|,$$

for every $\nu \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

If G is a locally compact group, we denote by $L^2(G)$ the Hilbert space of square-integrable functions with respect to a left Haar measure on G. If $A \in M_d(\mathbb{R})$, the vector space of square $d\times d$ matrices with real entries, ${}^tA$ denotes its transpose and we denote the (real) general linear group of size $d \times d$ by $\mathrm{GL}(d, \mathbb{R})$.

2 Preliminaries

In this section we first introduce the shearlet transform. We refer to [11] as a classical reference for the theory of group representations of locally compact groups and to [19] for a complete overview of shearlet analysis. Then, we introduce the Lizorkin space of test functions, a closed subspace of $\mathcal{S}(\mathbb{R}^d)$ which plays a crucial role in our analysis, and the space $\mathcal{S}(\mathbb{S})$, which contains the range of the shearlet transform when acting on the Lizorkin space. As already mentioned, both spaces and their duals are called Lizorkin type spaces of test functions and distributions.

2.1 The Shearlet transform

Although we will give the definitions of multi-dimensional Lizorkin test and distribution type spaces, when dealing with the shearlet transform and the shearlet synthesis operator we consider $\mathbb{R}^2$ and the standard shearlet group $\mathbb{S} = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\times$ endowed with the group operation

$$(b, s, a)(b', s', a') = (b + N_s A_a b', s + |a|^{1/2}s', aa'),$$

where

$$N_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}, \qquad A_a = a\begin{bmatrix} 1 & 0 \\ 0 & |a|^{-1/2} \end{bmatrix}, \quad s \in \mathbb{R}, a \in \mathbb{R}^\times.$$

A left Haar measure on $\mathbb{S}$ is given by

$$\mathrm{d}\mu(b, s, a) = |a|^{-3}\mathrm{d}b\mathrm{d}s\mathrm{d}a,$$

where $\mathrm{d}b$, $\mathrm{d}s$ and $\mathrm{d}a$ are the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}^\times$, respectively. The group $\mathbb{S}$ acts on $L^2(\mathbb{R}^2)$ via the square-integrable representation

$$\pi_{b,s,a} f(x) = |a|^{-3/4} f(A_a^{-1} N_s^{-1}(x - b)), \tag{1}$$

or, equivalently, in the frequency domain

$$\mathcal{F}\pi_{b,s,a} f(\xi) = |a|^{3/4} e^{-2\pi i b\xi} \mathcal{F}f(A_a {}^tN_s \xi). \tag{2}$$

We denote by $C(\mathbb{S})$ the space of continuous functions on $\mathbb{S}$ and by $L^\infty(\mathbb{S})$ the space of essentially bounded functions on $\mathbb{S}$ with respect to the Haar measure $\mathrm{d}\mu$.

Definition 1. Let $\psi \in L^2(\mathbb{R}^2)$. The shearlet transform associated to ψ is the map $\mathcal{S}_\psi \colon L^2(\mathbb{R}^2) \to C(\mathbb{S}) \cap L^\infty(\mathbb{S})$ defined by

$$\mathcal{S}_\psi f(b, s, a) = \langle f, \pi_{b,s,a}\psi\rangle = |a|^{-\frac{3}{4}} \int_{\mathbb{R}^2} f(x)\overline{\psi(A_a^{-1}N_s^{-1}(x - b))}\mathrm{d}x,$$

for every $(b, s, a) \in \mathbb{S}$.

It is well-known that the shearlet transform $\mathcal{S}_\psi$ is a non-trivial multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{S}, \mathrm{d}\mu)$ provided that $\psi \in L^2(\mathbb{R}^2)$ satisfies the admissibility condition

$$0 < C_\psi = \int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\xi)|^2}{|\xi_1|^2} \mathrm{d}\xi < +\infty, \tag{3}$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, or equivalently

$$C_\psi = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |\mathcal{F}\psi(A_a{}^tN_s\xi)|^2 \mathrm{d}s \frac{\mathrm{d}a}{|a|^{\frac{3}{2}}}, \qquad \text{for a.e. } \xi \in \mathbb{R}^2 \setminus \{0\}, \tag{4}$$

see e.g. [5]. Functions $\psi \in L^2(\mathbb{R}^2)$ which satisfy (3) are called admissible shearlets. Furthermore, if $\psi \in L^2(\mathbb{R}^2)$ is an admissible shearlet then for every $f \in L^2(\mathbb{R}^2)$ we have the reconstruction formula

$$f = \frac{1}{C_\psi} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \mathcal{S}_\psi f(b, s, a)\, \pi_{b,s,a}\psi \; \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \tag{5}$$

where the equality holds in the weak-sense, i.e.

$$\langle f, g\rangle = \frac{1}{C_\psi} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \mathcal{S}_\psi f(b, s, a)\, \langle \pi_{b,s,a}\psi, g\rangle \; \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \qquad g \in L^2(\mathbb{R}^2).$$

We can also consider a reconstruction formula when the analyzing and the synthesis vectors do not coincide. Precisely, for every pair of functions $\psi, \phi \in L^2(\mathbb{R}^2)$ which satisfy the admissible condition

$$0 < C_{\psi,\phi} = \int_{\mathbb{R}^2} \frac{\overline{\mathcal{F}\psi(\xi)}\mathcal{F}\phi(\xi)}{|\xi_1|^2} \mathrm{d}\xi < +\infty \tag{6}$$

(or equivalently

$$C_{\psi,\phi} = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \overline{\mathcal{F}\psi(A_a{}^tN_s\xi)}\mathcal{F}\phi(A_a{}^tN_s\xi) \; \frac{\mathrm{d}s\mathrm{d}a}{|a|^{\frac{3}{2}}}, \qquad \text{for a.e. } \xi \in \mathbb{R}^2 \setminus \{0\}), \tag{7}$$

we have the reconstruction formula

$$f = \frac{1}{C_{\psi,\phi}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \mathcal{S}_\psi f(b, s, a)\, \pi_{b,s,a}\phi \; \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \qquad f \in L^2(\mathbb{R}^2), \tag{8}$$

where again the equality has to be interpreted in the weak-sense (L^2-weak convergence). It is immediate to see that if $\psi = \phi$ we recover (5). Reconstruction formula (8) is a consequence of the orthogonality relations for square integrable representations proved by Duflo and Moore in [7]. We refer also to [27, Chapter 9]. In Section 4 we show that under suitable choices of the vectors ψ, ϕ and of the function f formula (8) holds pointwise, see Proposition 7.

2.2 The spaces

In this subsection we introduce the functional analytic background for our investigations, and state some auxiliary results which will be used in the proofs of our main results.

The Lizorkin space $\mathcal{S}_0(\mathbb{R}^d)$ consists of rapidly decreasing functions with vanishing moments of any order, i.e.

$$\mathcal{S}_0(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^d) : \ \mu_m(\varphi) = 0, \ \forall m \in \mathbb{N}^d \right\},$$

where $\mu_m(\varphi) = \int_{\mathbb{R}^d} x^m \varphi(x) dx$, $m \in \mathbb{N}^d$. It is a closed subspace of $\mathcal{S}(\mathbb{R}^d)$ equipped with the relative topology inhered from $\mathcal{S}(\mathbb{R}^d)$ and its dual space of Lizorkin distributions $\mathcal{S}'_0(\mathbb{R}^d)$ is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^d)$ by the space of polynomials (cf. [16, 20]).

We are also interested in the Fourier Lizorkin space $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ which consists of rapidly decreasing functions that vanish in zero together with all of their partial derivatives, i.e.

$$\hat{\mathcal{S}}_0(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^d) : \ \partial^m \varphi(0) = 0, \ \forall m \in \mathbb{N}^d \right\}.$$

This is also a closed subspace of $\mathcal{S}(\mathbb{R}^d)$ endowed with the relative topology inhered from $\mathcal{S}(\mathbb{R}^d)$.

We observe that, since $\mathcal{S}_0(\mathbb{R}^d)$ and $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ are closed subspaces of the nuclear space $\mathcal{S}(\mathbb{R}^d)$, they are nuclear as well. We denote by $X \hat{\otimes} Y$ the topological tensor product space obtained as the completion of the tensor product $X \otimes Y$ in the inductive tensor product topology ε or the projective tensor product topology π. Then, we have the following result.

Lemma 2. *The spaces $\mathcal{S}_0(\mathbb{R}^d)$ and $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ are closed under translations, dilations, differentiations and multiplications by a polynomial. Moreover, the Fourier transform is an isomorphism between $\mathcal{S}_0(\mathbb{R}^d)$ and $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ and we have the following canonical isomorphisms*

$$\mathcal{S}_0(\mathbb{R}^d) \cong \mathcal{S}_0(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}_0(\mathbb{R}^{d_2}),$$
$$\hat{\mathcal{S}}_0(\mathbb{R}^d) \cong \hat{\mathcal{S}}_0(\mathbb{R}^{d_1}) \hat{\otimes} \hat{\mathcal{S}}_0(\mathbb{R}^{d_2}),$$

where $d = d_1 + d_2 \in \mathbb{Z}_+$.

Proof. The proof is based on classical arguments and we omit it (cf. [26, Theorem 51.6] for the canonical isomorphisms). □

The next Lemma is a reformulation of [21, Theorem 6.2].

Lemma 3. *Let $f \in \mathcal{S}_0(\mathbb{R}^d)$. Then, for any $m \in \mathbb{N}^d$ there exists $g \in \mathcal{S}_0(\mathbb{R}^d)$ such that*

$$\mathcal{F}f(\xi) = \xi^m \mathcal{F}g(\xi), \qquad \xi \in \mathbb{R}^d,$$

and vice versa. Furthermore, for such f and g and for every $\nu_1 \in \mathbb{N}$, there exists $\nu_2 \in \mathbb{N}$ and a constant $C > 0$ such that

$$\rho_{\nu_1}(g) \leq C \rho_{\nu_2}(f). \tag{9}$$

Proof. We start proving the first part of the statement for $d = 1$ and $m = 1$. Let $f \in \mathcal{S}_0(\mathbb{R})$ and consider

$$g(x) = \int_{-\infty}^{x} f(t) \mathrm{d}t = -\int_{x}^{+\infty} f(t) \mathrm{d}t.$$

It is easy to show that g is a well defined function and

$$\sup_{x \in \mathbb{R}} \langle x \rangle^k |g(x)| \lesssim \rho_{2k+4}(f) < +\infty, \tag{10}$$

for every $k \in \mathbb{N}$ (cf. [2]). Furthermore, $g'(x) = f(x)$, and then

$$\sup_{x\in\mathbb{R}} \langle x\rangle^k |g^{(l)}(x)| = \sup_{x\in\mathbb{R}} \langle x\rangle^k |f^{(l-1)}(x)| < +\infty, \tag{11}$$

for every $k \in \mathbb{N}$ and $l \geq 1$. Moreover, since $f \in \mathcal{S}_0(\mathbb{R})$, for any $n \in \mathbb{N}$ we compute

$$\int_{-\infty}^{+\infty} x^n g(x)\mathrm{d}x = -\int_{-\infty}^{+\infty} x^{n+1} g'(x)\mathrm{d}x = -\int_{-\infty}^{+\infty} x^{n+1} f(x)\mathrm{d}x = 0.$$

Then, $g \in \mathcal{S}_0(\mathbb{R})$ and by the definition of g we have

$$\mathcal{F}f(\xi) = \mathcal{F}g'(\xi) = (2\pi i)\xi\mathcal{F}g(\xi), \qquad \xi \in \mathbb{R}.$$

The opposite direction is obviously true since the space $\mathcal{S}_0(\mathbb{R})$ is closed under multiplication by a polynomial, and this concludes the proof of the first part of the statement for $d = 1$ and $m = 1$. The analogous statement holds true for $|m| > 1$ by iterating the above proof $|m|$-times. The case $d > 1$ follows by analogous but cumbersome computations, and it is therefore omitted.

Finally, (9) follows immediately by (10) and (11). More precisely, if $f \in \mathcal{S}_0(\mathbb{R})$ and

$$\mathcal{F}f(\xi) = \xi^m \mathcal{F}g(\xi), \quad \xi \in \mathbb{R},$$

for some $g \in \mathcal{S}_0(\mathbb{R})$ and $m \in \mathbb{N}$, then by (10) and (11) for every $\nu \in \mathbb{N}$ we have

$$\rho_\nu(g) = \sup_{x\in\mathbb{R}, l\leq\nu} \langle x\rangle^\nu |g^{(l)}(x)| \lesssim \rho_{2\nu+4}(f).$$

and this concludes the second part of the statement for $d = 1$. The analogous result holds true for $d > 1$ by using similar arguments. □

By Lemma 3, if $f \in \mathcal{S}_0(\mathbb{R}^2)$, then for any given $k, l \in \mathbb{N}$ there exists $g \in \mathcal{S}_0(\mathbb{R}^2)$ such that

$$\mathcal{F}f(\xi_1, \xi_2) = \xi_1^k \xi_2^l \mathcal{F}g(\xi_1, \xi_2), \quad (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Moreover, it is worth observing that by Lemmas 2 and 3, $f \in \mathcal{S}_0(\mathbb{R}^d)$ if and only if it satisfies the directional vanishing moments

$$\int_{\mathbb{R}} x_j^m f(x_1, x_2, \ldots, x_d) dx_j = 0,$$

for all $m \in \mathbb{N}$ and for every $j \in \{1, \ldots, d\}$. As noted in [13], a sufficient number of directional vanishing moments imposes the crucial frequency localization property needed for the detection of anisotropic structures.

In order to study qualitative properties of the range of the shearlet transform, we introduce appropriate norms which measure regularity and decay properties with respect to all of the parameters. To this aim it is necessary to involve eight indices. More precisely, we denote by $\mathcal{S}(\mathbb{S})$ the space of functions $\Phi \in C^\infty(\mathbb{S})$ such that the norms

$$p_{k_1,k_2,l,m}^{\alpha_1,\alpha_2,\beta,\gamma}(\Phi) = \sup_{((b_1,b_2),s,a)\in\mathbb{S}} \langle b_1\rangle^{k_1} \langle b_2\rangle^{k_2} \langle s\rangle^l \left(|a|^m + \frac{1}{|a|^m}\right) \left|\partial_a^\gamma \partial_s^\beta \partial_{b_2}^{\alpha_2} \partial_{b_1}^{\alpha_1} \Phi((b_1, b_2), s, a)\right| \tag{12}$$

are finite for all $k_1, k_2, l, m, \alpha_1, \alpha_2, \beta, \gamma \in \mathbb{N}$. The topology of $\mathcal{S}(\mathbb{S})$ is defined by means of the norms (12). Its dual $\mathcal{S}'(\mathbb{S})$ will play a crucial role in the definition of the shearlet transform of Lizorkin distributions since it contains the range of this transform.

If F is a function of at most polynomial growth on $\mathbb{S}$, i.e., if there exist positive constants ν_1, ν_2, ν_3 and C such that

$$|F(b,s,a)| \leq C\langle b\rangle^{\nu_1}\langle s\rangle^{\nu_2}\left(|a|^{\nu_3} + \frac{1}{|a|^{\nu_3}}\right), \qquad (b,s,a) \in \mathbb{S},$$

then we identify F with an element of $\mathcal{S}'(\mathbb{S})$ as follows

$$ {}_{\mathcal{S}'(\mathbb{S})}(F,\Phi)_{\mathcal{S}(\mathbb{S})} = \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} F(b,s,a)\Phi(b,s,a)\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \tag{13}$$

for every $\Phi \in \mathcal{S}(\mathbb{S})$.

3 The continuity of the Shearlet Transform

We are now ready to state our first main result. Let $f, \psi \in L^2(\mathbb{R}^2)$. By Definition 1, the Plancherel theorem, and formula (2) we obtain

$$\begin{aligned} \mathcal{S}_\psi f(b,s,a) &= \langle \mathcal{F}f, \mathcal{F}\pi_{b,s,a},\psi\rangle \\ &= |a|^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}f(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\overline{\mathcal{F}\psi(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2, \end{aligned} \tag{14}$$

for every $(b,s,a) \in \mathbb{S}$. Moreover, we can consider this transform as a sesquilinear mapping $\mathcal{S}\colon (f,\psi) \mapsto \mathcal{S}_\psi f$.

Theorem 4. *The operator $\mathcal{S}\colon (f,\psi) \mapsto \mathcal{S}_\psi f$ is a continuous sesquilinear mapping from $\mathcal{S}_0(\mathbb{R}^2) \times \mathcal{S}_0(\mathbb{R}^2)$ into $\mathcal{S}(\mathbb{S})$. In particular, for every $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ the shearlet transform $\mathcal{S}_\psi\colon L^2(\mathbb{R}^2) \to C(\mathbb{S}) \cap L^\infty(\mathbb{S})$ restricts to a continuous operator from $\mathcal{S}_0(\mathbb{R}^2)$ into $\mathcal{S}(\mathbb{S})$.*

Proof. We must prove that for every $f, \psi \in \mathcal{S}_0(\mathbb{R}^2)$, given $k_1, k_2, l, m, \alpha_1, \alpha_2, \beta, \gamma \in \mathbb{N}$, there exist $\nu_1, \nu_2 \in \mathbb{N}$ such that

$$\rho^{\alpha_1,\alpha_2,\beta,\gamma}_{k_1,k_2,l,m}(\mathcal{S}_\psi f) \lesssim \rho_{\nu_1}(f)\rho_{\nu_2}(\psi). \tag{15}$$

First, in three steps, we show that without lost on generality we can assume $\alpha_1 = \alpha_2 = \beta = \gamma = k_1 = k_2 = l = 0$. Then, in the last step we finish the proof by taking an arbitrary $m \in \mathbb{N}$ and proving that

$$\rho^{0,0,0,0}_{0,0,0,m}(\mathcal{S}_\psi f) \lesssim C\rho_{\nu_1}(f)\rho_{\nu_2}(\psi),$$

for some $\nu_1, \nu_2 \in \mathbb{N}$. The idea of the proof is similar to the one given in [20] which concerns the continuity of the ridgelet transform on $\mathcal{S}_0(\mathbb{R}^2)$. We note that formula (14) and Lemma 3 play a crucial role in the proof.

1. We first show that the derivatives of $\mathcal{S}_\psi f$ are bounded by appropriate norms of the family (12) with $\alpha_1 = \alpha_2 = \beta = \gamma = 0$. By formula (14) we have that

$$\begin{aligned}
&|\partial_s^\beta \partial_{b_2}^{\alpha_2} \partial_{b_1}^{\alpha_1} \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= |a|^{\frac{3}{4}} |(2\pi i)^{\alpha_1+\alpha_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1^{\alpha_1}\xi_2^{\alpha_2} \mathcal{F}f(\xi_1,\xi_2) e^{2\pi i(b_1\xi_1+b_2\xi_2)} \times \\
&\times \overline{\partial_s^\beta(\mathcal{F}\psi(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1)))} \mathrm{d}\xi_1 \mathrm{d}\xi_2| \\
&= |a|^{\frac{3}{4}+\frac{\beta}{2}} |(2\pi i)^{\alpha_1+\alpha_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1^{\alpha_1+\beta}\xi_2^{\alpha_2} \mathcal{F}f(\xi_1,\xi_2) e^{2\pi i(b_1\xi_1+b_2\xi_2)} \times \\
&\times \overline{\partial_2^\beta \mathcal{F}\psi(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))} \mathrm{d}\xi_1 \mathrm{d}\xi_2|. \qquad (16)
\end{aligned}$$

Then, by formula (16) and the Liebniz formula, we compute

$$\begin{aligned}
&|\partial_a^\gamma \partial_s^\beta \partial_{b_2}^{\alpha_2} \partial_{b_1}^{\alpha_1} \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= |(2\pi i)^{\alpha_1+\alpha_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1^{\alpha_1+\beta}\xi_2^{\alpha_2} \mathcal{F}f(\xi_1,\xi_2) e^{2\pi i(b_1\xi_1+b_2\xi_2)} \times \\
&\times \partial_a^\gamma (|a|^{\frac{3}{4}+\frac{\beta}{2}} \overline{\partial_2^\beta \mathcal{F}\psi(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))}) \mathrm{d}\xi_1 \mathrm{d}\xi_2| \\
&= |\sum_{j+l=\gamma} c_{j,l} (2\pi i)^{\alpha_1+\alpha_2} |a|^{\frac{3}{4}+\frac{\beta}{2}-j} \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1^{\alpha_1+\beta}\xi_2^{\alpha_2} \mathcal{F}f(\xi_1,\xi_2) e^{2\pi i(b_1\xi_1+b_2\xi_2)} \times \\
&\times \partial_a^l (\overline{\partial_2^\beta \mathcal{F}\psi(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))}) \mathrm{d}\xi_1 \mathrm{d}\xi_2|, \qquad (17)
\end{aligned}$$

where the constants $c_{j,l}$ depend only on j, l and β. The last expression in (17) is bounded from above by a finite sum of terms of the form

$$\begin{aligned}
&|a|^{\frac{5}{4}+\frac{\beta}{2}-j-3\frac{r_2}{2}-r_1} |c_{j,l} (2\pi i)^{\alpha_1+\alpha_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1^{\alpha_1+\beta}\xi_2^{\alpha_2} \mathcal{F}f(\xi_1,\xi_2) e^{2\pi i(b_1\xi_1+b_2\xi_2)} \times \\
&\times a^{r_1+r_2} |a|^{-\frac{r_2}{2}} \overline{\xi_1^{r_1}(\xi_2 - s\xi_1)^{r_2} (\partial_1^{r_1}\partial_2^{\beta+r_2} \mathcal{F}\psi(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1)))} \mathrm{d}\xi_1 \mathrm{d}\xi_2| \\
&\lesssim \rho^{0,0,0,0}_{0,0,0,|\frac{1}{2}+\frac{\beta}{2}-j-3\frac{r_2}{2}-r_1|} (\mathcal{S}_{\psi_{r_1,r_2}}(\partial_1^{\alpha_1+\beta}\partial_2^{\alpha_2} f)),
\end{aligned}$$

with ψ_{r_1,r_2} given by $\mathcal{F}\psi_{r_1,r_2}(\xi_1,\xi_2) = \xi_1^{r_1}\xi_2^{r_2}\partial_1^{r_1}\partial_2^{\beta+r_2}\mathcal{F}\psi(\xi_1,\xi_2)$ for every r_1, $r_2 \in \mathbb{N}$, $r_1 + r_2 \leq l$. Since differentiation and multiplication by polynomials are continuous operators from $\mathcal{S}_0(\mathbb{R}^2)$ into itself, we can assume $\alpha_1 = \alpha_2 = \beta = \gamma = 0$.

2. In the second step we estimate multiplications by $b = (b_1, b_2) \in \mathbb{R}^2$, and show that it is enough to consider $k_1 = k_2 = 0$ when proving (15). Observe that we can assume $|b_1| \geq 1$, $|b_2| \geq 1$. By formula (14), and since

$$e^{2\pi i\xi_1 b_1} = \frac{(1 - \partial_{\xi_1}^2)^N e^{2\pi i\xi_1 b_1}}{\langle 2\pi b_1\rangle^{2N}}$$

for every $N \in \mathbb{N}$ and ξ_1, $b_1 \in \mathbb{R}$, it follows that

$$
\begin{aligned}
&\langle b_1\rangle^{k_1}|\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= \langle b_1\rangle^{k_1}|a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{F}f(\xi_1,\xi_2)\frac{(1-\partial_{\xi_1}^2)^N e^{2\pi i\xi_1 b_1}}{\langle 2\pi b_1\rangle^{2N}}e^{2\pi i b_2\xi_2}\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&= \frac{\langle b_1\rangle^{k_1}}{\langle 2\pi b_1\rangle^{2N}}|a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}}(1-\partial_{\xi_1}^2)^N(\mathcal{F}f(\xi_1,\xi_2)\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))})e^{2\pi i b\xi}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&= \frac{\langle b_1\rangle^{k_1}}{\langle 2\pi b_1\rangle^{2N}}|\sum_{j,r_1,r_2\leq 2N}|a|^{\frac{3}{4}+r_1+\frac{r_2}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}P_{j,r_1,r_2}(s)\partial_1^j\mathcal{F}f(\xi_1,\xi_2)\times \\
&\times\partial_1^{r_1}\partial_2^{r_2}\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}e^{2\pi i b\xi}\mathrm{d}\xi_1\mathrm{d}\xi_2|,
\end{aligned}
$$

for some polynomials $P_{j,r_1,r_2}(t)=\sum_{m=0}^{p_{j,r_1,r_2}}c_m t^m$, and $b\xi = b_1\xi_1+b_2\xi_2$. Then, we obtain

$$
\begin{aligned}
&\langle b_1\rangle^{k_1}|\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\leq \frac{\langle b_1\rangle^{k_1}|a|^N\langle s\rangle^{2N}}{\langle 2\pi b_1\rangle^{2N}}|\sum_{j,r_1,r_2\leq 2N}\sum_{m=0}^{p_{j,r_1,r_2}}c_m|a|^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_1^j\mathcal{F}f(\xi_1,\xi_2)\times \\
&\times\partial_1^{r_1}\partial_2^{r_2}\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}e^{2\pi i b\xi}\mathrm{d}\xi_1\mathrm{d}\xi_2|,
\end{aligned}
$$

where the last inequality follows by the fact that we can assume $|a|\geq 1$ and $|s|\geq 1$. The above estimate yields

$$
\langle b_1\rangle^{k_1}|\mathcal{S}_\psi f((b_1,b_2),s,a)| \leq \frac{\langle b_1\rangle^{k_1}}{\langle 2\pi b_1\rangle^{2N}}\sum_{j,r_1,r_2\leq 2N}\sum_{m=0}^{p_{j,r_1,r_2}}|c_m|\rho^{0,0,0,0}_{0,0,2N,N}(\mathcal{S}_{\psi_{r_1,r_2}}f_j),
$$

with $\mathcal{F}f_j(\xi_1,\xi_2)=\partial_1^j\mathcal{F}f(\xi_1,\xi_2)$ for every j and ψ_{r_1,r_2} given by $\mathcal{F}\psi_{r_1,r_2}(\xi_1,\xi_2)=\partial_1^{r_1}\partial_2^{r_2}\mathcal{F}\psi(\xi_1,\xi_2)$ for every pair $r_1,r_2\in\mathbb{N}$, $r_1,r_2\leq 2N$. Analogously, for every $M\in\mathbb{N}$ we have that

$$
\begin{aligned}
&\langle b_2\rangle^{k_2}|\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= \langle b_2\rangle^{k_2}|a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{F}f(\xi_1,\xi_2)\frac{(1-\partial_{\xi_2}^2)^M e^{2\pi i\xi_2 b_2}}{\langle 2\pi b_2\rangle^{2M}}e^{2\pi i b_1\xi_1}\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&= \frac{\langle b_2\rangle^{k_2}}{\langle 2\pi b_2\rangle^{2M}}|a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}}(1-\partial_{\xi_2}^2)^M(\mathcal{F}f(\xi_1,\xi_2)\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))})e^{2\pi i b\xi}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&\lesssim \frac{\langle b_2\rangle^{k_2}}{\langle 2\pi b_2\rangle^{2M}}|\sum_{j,r\leq 2M}|a|^{\frac{3}{4}+\frac{r}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_2^j\mathcal{F}f(\xi_1,\xi_2)\partial_2^r\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}e^{2\pi i b\xi}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&\lesssim \frac{\langle b_2\rangle^{k_2}|a|^M}{\langle 2\pi b_2\rangle^{2M}}|\sum_{j,r\leq 2M}|a|^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_2^j\mathcal{F}f(\xi_1,\xi_2)\partial_2^r\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}e^{2\pi i b\xi}\mathrm{d}\xi_1\mathrm{d}\xi_2|
\end{aligned}
$$

where the last inequality follows by assuming $|a|\geq 1$. Hence, we obtain

$$
\langle b_2\rangle^{k_2}|\mathcal{S}_\psi f((b_1,b_2),s,a)| \leq \frac{\langle b_2\rangle^{k_2}}{\langle 2\pi b_2\rangle^{2M}}\sum_{j,r\leq 2M}\rho^{0,0,0,0}_{0,0,0,M}(\mathcal{S}_{\psi_r}\tilde{f}_j),
$$

with $\tilde{f}_j$ given by $\mathcal{F}\tilde{f}_j(\xi_1,\xi_2) = \partial_2^j \mathcal{F}f(\xi_1,\xi_2)$, and ψ_r given by $\mathcal{F}\psi_r(\xi_1,\xi_2) = \partial_2^r \mathcal{F}\psi(\xi_1,\xi_2)$ for every $j, r \in \mathbb{N}$, $j, r \le 2M$.

Therefore, recalling that we have assumed $|b_1| \ge 1$ and $|b_2| \ge 1$, if we choose $N = k_1$ and $M = k_2$, we obtain

$$
\begin{aligned}
\langle b_1\rangle^{k_1}\langle b_2\rangle^{k_2}|\mathcal{S}_\psi f((b_1,b_2),s,a)|^2 &\lesssim \frac{\langle b_1\rangle^{k_1}\langle b_2\rangle^{k_2}}{\langle b_1\rangle^{2k_1}\langle b_2\rangle^{2k_2}}\times \\
&\times [\sum_{j,k_1,k_2\le 2N}\sum_{m=0}^{p_{j,k_1,k_2}} |c_m|\rho^{0,0,0,0}_{0,0,2N,N}(\mathcal{S}_{\psi_{r_1,r_2}} f_j)][\sum_{j,r\le 2M}\rho^{0,0,0,0}_{0,0,0,M}(\mathcal{S}_{\psi_r}\tilde{f}_j)] \\
&\lesssim [\sum_{j,k_1,k_2\le 2N}\sum_{m=0}^{p_{j,k_1,k_2}} |c_m|\rho^{0,0,0,0}_{0,0,2N,N}(\mathcal{S}_{\psi_{r_1,r_2}} f_j)][\sum_{j,r\le 2M}\rho^{0,0,0,0}_{0,0,0,M}(\mathcal{S}_{\psi_r}\tilde{f}_j)].
\end{aligned}
$$

and we conclude that we can assume $k_1 = k_2 = 0$ since differentiation and multiplication by polynomials are continuous operators from $\mathcal{S}_0(\mathbb{R}^2)$ into itself.

3. The third step consists in showing that when considering multiplications by powers of s, we can assume $l = 0$ in (15). By formula (14) and

$$\langle s\rangle \le \langle s\xi_1\rangle\left(\frac{1+\xi_1^2}{\xi_1^2}\right), \qquad s,\, \xi_1 \in \mathbb{R},$$

it follows that

$$
\begin{aligned}
&\langle s\rangle^l|\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= |a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}f(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\langle s\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&\le |a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}} \xi_1^{-2l}\langle\xi_1\rangle^{2l}\mathcal{F}f(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\langle s\xi_1\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|.
\end{aligned}
$$

By Lemma 3 there exists $f_l \in \mathcal{S}_0(\mathbb{R}^2)$ such that

$$\mathcal{F}f(\xi_1,\xi_2) = \xi_1^{2l}\mathcal{F}f_l(\xi_1,\xi_2),$$

for every $(\xi_1,\xi_2)\in\mathbb{R}^2$, and then

$$
\begin{aligned}
&\langle s\rangle^l|\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\le |a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle\xi_1\rangle^{2l}\mathcal{F}f_l(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\langle s\xi_1\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|.
\end{aligned}
$$

Without loss of generality, we may assume that l is even. We divide the proof in the two cases $|a| \le 1$ and $|a| > 1$. If $|a| \ge 1$, by Peetre's inequality we have that

$$\langle s\xi_1\rangle^l \le \langle a|a|^{-\frac{1}{2}}s\xi_1\rangle^l \le 2^l\langle a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)\rangle^l\langle a|a|^{-\frac{1}{2}}\xi_2\rangle^l, \qquad s,\,\xi_1,\,\xi_2\in\mathbb{R}.$$

Then

$$
\begin{aligned}
&\langle s\rangle^l|\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\le |a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}} 2^l\langle\xi_1\rangle^{2l}\langle a|a|^{-\frac{1}{2}}\xi_2\rangle^l\mathcal{F}f_l(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\times \\
&\times\langle a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|,
\end{aligned}
$$

which is less than or equal to a finite sum of terms of the form

$$|a|^{\frac{3}{4}+\frac{r_2}{2}}|\int_{\mathbb{R}}\int_{\mathbb{R}}2^l\xi_1^{r_1}\xi_2^{r_2}\mathcal{F}f_l(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\times$$
$$\times\langle a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|\lesssim\rho^{0,0,0,0}_{0,0,0,\frac{r_2}{2}}(\mathcal{S}_{\tilde{\psi}}(\partial_1^{r_1}\partial_2^{r_2}f_l)),$$

where $0\le r_1,\, r_2\le 3l$ and with $\tilde{\psi}$ given by $\mathcal{F}\tilde{\psi}(\xi_1,\xi_2)=\langle\xi_2\rangle^l\mathcal{F}\psi(\xi_1,\xi_2)$. Analogously, if $|a|\le 1$, by Peetre's inequality we have

$$\langle s\xi_1\rangle^l\le|a|^{-\frac{l}{2}}\langle a|a|^{-\frac{1}{2}}s\xi_1\rangle^l\le|a|^{-\frac{l}{2}}2^l\langle a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)\rangle^l\langle a|a|^{-\frac{1}{2}}\xi_2\rangle^l,\qquad s,\,\xi_1,\,\xi_2\in\mathbb{R},$$

and then

$$\begin{aligned}&\langle s\rangle^l|\mathcal{S}_\psi f((b_1,b_2),s,a)|\\&\le|a|^{\frac{3}{4}-\frac{l}{2}}|\int_{\mathbb{R}}\int_{\mathbb{R}}2^l\langle\xi_1\rangle^{2l}\langle a|a|^{-\frac{1}{2}}\xi_2\rangle^l\mathcal{F}f_l(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\times\\&\times\langle a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|,\end{aligned}$$

which is less than or equal to a finite sum of terms of the form

$$|a|^{\frac{3}{4}-\frac{l}{2}+\frac{r_4}{2}}|\int_{\mathbb{R}}\int_{\mathbb{R}}2^l\xi_1^{r_3}\xi_2^{r_4}\mathcal{F}f_l(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}\times$$
$$\times\langle a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)\rangle^l\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|\lesssim\rho^{0,0,0,0}_{0,0,0,|\frac{r_4}{2}-\frac{l}{2}|}(\mathcal{S}_{\tilde{\psi}}(\partial_1^{r_3}\partial_2^{r_4}f_l)),$$

where $0\le r_3,\, r_4\le 3l$. Therefore, by Lemma 3, and since differentiation and multiplication by polynomials are continuous operators from $\mathcal{S}_0(\mathbb{R}^2)$ into itself, we can assume $l=0$.

4. Finally, we consider multiplication by positive and negative powers of $|a|$, $a\in\mathbb{R}^\times$. Let $m\in\mathbb{N}$. By Lemma 3 there exists $g_m\in\mathcal{S}_0(\mathbb{R}^2)$ such that

$$\mathcal{F}f(\xi_1,\xi_2)=\xi_1^m\mathcal{F}g_m(\xi_1,\xi_2),$$

for every $(\xi_1,\xi_2)\in\mathbb{R}^2$. Then, assuming $|a|\ge 1$, we have

$$\begin{aligned}&|a|^m|\mathcal{S}_\psi f((b_1,b_2),s,a)|\\&=|a|^{\frac{3}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{F}g_m(\xi_1,\xi_2)e^{2\pi i(b_1\xi_1+b_2\xi_2)}a^m\xi_1^m\overline{\mathcal{F}\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1))}\mathrm{d}\xi_1\mathrm{d}\xi_2|\\&\lesssim|a|^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\mathcal{F}g_m(\xi_1,\xi_2)||\mathcal{F}(\partial_1^m\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)))|\mathrm{d}\xi_1\mathrm{d}\xi_2\\&\lesssim\|g_m\|_1|a|^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\mathcal{F}(\partial_1^m\psi(a\xi_1,a|a|^{-\frac{1}{2}}(\xi_2-s\xi_1)))|\mathrm{d}\xi_1\mathrm{d}\xi_2\\&\lesssim\|g_m\|_1|a|^{-\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\mathcal{F}(\partial_1^m\psi(\xi_1,\xi_2))|\mathrm{d}\xi_1\mathrm{d}\xi_2\le C\|g_m\|_1|a|^{-\frac{3}{4}}\|\mathcal{F}\partial_1^m\psi\|_1\\&\lesssim\rho_2(g_m)\rho_2(\mathcal{F}\partial_1^m\psi).\end{aligned}$$

Therefore, by Lemma 3, and since the Fourier transform and the differentiation are continuous operators from $\mathcal{S}_0(\mathbb{R}^2)$ into itself, we conclude that there exist $\nu_1,\,\nu_2\in\mathbb{N}$ such that

$$|a|^m|\mathcal{S}_\psi f((b_1,b_2),s,a)|\lesssim\rho_{\nu_1}(f)\rho_{\nu_2}(\psi)\tag{18}$$

for every $(b_1, b_2) \in \mathbb{R}^2$, $s \in \mathbb{R}$ and $|a| \geq 1$.

We finish the proof by considering multiplications by negative powers of $|a|$. Let $m \in \mathbb{N}$. By Lemma 3 there exists $\psi_m \in \mathcal{S}_0(\mathbb{R}^2)$ such that

$$\mathcal{F}\psi(\xi_1, \xi_2) = \xi_1^{m+1} \mathcal{F}\psi_m(\xi_1, \xi_2),$$

for every $(\xi_1, \xi_2) \in \mathbb{R}^2$. Assuming $|a| \leq 1$, we compute

$$\begin{aligned}
&|a|^{-m}|\mathcal{S}_\psi f((b_1, b_2), s, a)| \\
&= |a|^{\frac{7}{4}} \Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \xi_1^{m+1} \mathcal{F}f(\xi_1, \xi_2) e^{2\pi i (b_1\xi_1 + b_2\xi_2)} \overline{\mathcal{F}\psi_m(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \Big| \\
&\lesssim |a|^{\frac{7}{4}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}(\partial_1^{m+1} f)(\xi_1, \xi_2)| |\mathcal{F}\psi_m(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\lesssim |a|^{\frac{7}{4}} \|\partial_1^{m+1} f\|_1 \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}\psi_m(a\xi_1, a|a|^{-\frac{1}{2}}(\xi_2 - s\xi_1))| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\lesssim |a|^{\frac{1}{4}} \|\partial_1^{m+1} f\|_1 \|\mathcal{F}\psi_m\|_1 \lesssim \rho_{m+1+2}(f) \rho_2(\mathcal{F}\psi_m).
\end{aligned}$$

By Lemma 3, and since the Fourier transform and the differentiation are continuous operators from $\mathcal{S}_0(\mathbb{R}^2)$ into itself, we obtain

$$|a|^{-m}|\mathcal{S}_\psi f((b_1, b_2), s, a)| \lesssim \rho_{\nu_1}(f) \rho_{\nu_2}(\psi) \tag{19}$$

for some ν_1, $\nu_2 \in \mathbb{N}$ and for every $(b_1, b_2) \in \mathbb{R}^2$, $s \in \mathbb{R}$ and $|a| \leq 1$.

Equation (18) together with (19) show that for every $f, \psi \in \mathcal{S}_0(\mathbb{R}^2)$, given $m \in \mathbb{N}$, there exist $\nu_1, \nu_2 \in \mathbb{N}$ such that

$$\rho^{0,0,0,0}_{0,0,0,m}(\mathcal{S}_\psi f) \lesssim \rho_{\nu_1}(f) \rho_{\nu_2}(\psi)$$

and this concludes the proof. □

Theorem 4 shows how the regularity and decay properties in the range of the shearlet transform are controlled by the corresponding properties of both the shearlet ψ and the analyzed signal f.

4 The Shearlet Synthesis Operator

Reconstruction formula (5) suggests to define a linear operator which maps functions over $\mathbb{S}$ to functions over the Euclidean plane $\mathbb{R}^2$. The operator that we are after is analogous to the wavelet synthesis operator considered by Holschneider [16].

Definition 5. Let $\psi \in L^1(\mathbb{R}^2)$ and let $F \in \mathcal{S}(\mathbb{S})$. We define the shearlet synthesis operator $\mathcal{S}^t_\psi F$ of F with respect to ψ as the function

$$\mathcal{S}^t_\psi F(x) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{\mathbb{R}^2} F(b, s, a)\ \pi_{b,s,a}\psi(x)\ \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \qquad x \in \mathbb{R}^2. \tag{20}$$

It is immediate to verify that for every $F \in \mathcal{S}(\mathbb{S})$ the integral in (20) is absolutely convergent for all $x \in \mathbb{R}^2$. Moreover, $\mathcal{S}^t : (F, \psi) \mapsto \mathcal{S}^t_\psi F$ is a continuous bilinear mapping from $\mathcal{S}(\mathbb{S}) \times L^1(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, which is proved in what follows. Let

$F \in \mathcal{S}(\mathbb{S})$, $x \in \mathbb{R}^2$, and $\epsilon > 0$. Indeed, by the Fubini theorem and the definition of the shearlet representation (1) it follows that

$$
\begin{aligned}
|\mathcal{S}^t_\psi F(x)| &\leq \int_{|a|<\epsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |a|^{-\frac{3}{4}}|F(b,s,a)|\ |\psi(A_a^{-1}N_s^{-1}(x-b))|\ \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}\\
&+ \int_{|a|>\epsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |a|^{-\frac{3}{4}}|F(b,s,a)|\ |\psi(A_a^{-1}N_s^{-1}(x-b))|\ \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}\\
&\leq \rho^{0,0,0,0}_{0,0,2,\frac{9}{4}}(F)\int_{|a|<\epsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |a|^{-\frac{3}{2}}|\psi(A_a^{-1}N_s^{-1}(x-b))|\ \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{(1+s^2)}\\
&+ \rho^{0,0,0,0}_{0,0,2,0}(F)\int_{|a|>\epsilon}|a|^{-\frac{9}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |a|^{-\frac{3}{2}}|\psi(A_a^{-1}N_s^{-1}(x-b))|\ \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{(1+s^2)}\\
&\leq \rho^{0,0,0,0}_{0,0,2,\frac{9}{4}}(F)\|\psi\|_1\int_{|a|<\epsilon}\int_{\mathbb{R}}\frac{\mathrm{d}s\mathrm{d}a}{(1+s^2)} + \rho^{0,0,0,0}_{0,0,2,0}(F)\|\psi\|_1\int_{|a|>\epsilon}|a|^{-\frac{9}{4}}\int_{\mathbb{R}}\frac{\mathrm{d}s\mathrm{d}a}{(1+s^2)}\\
&\lesssim \|\psi\|_1(\rho^{0,0,0,0}_{0,0,2,\frac{9}{4}}(F) + \rho^{0,0,0,0}_{0,0,2,0}(F)) < +\infty.
\end{aligned}
$$

The terminology shearlet synthesis operator follows by an analogy with the frame theory. Indeed, an admissible shearlet ψ for the shearlet representation (3) gives rise to the continuous frame $\{\pi_{b,s,a}\psi\}_{(b,s,a)\in\mathbb{S}}$, see e.g. [12], and the frame synthesis operator associated to such a frame is actually the shearlet synthesis operator $\mathcal{S}^t_\psi$ given by (20).

We consider the shearlet synthesis operator as the bilinear map $\mathcal{S}^t\colon (F,\psi)\mapsto \mathcal{S}^t_\psi F$ for a wide class of pairs (F,ψ) for which Definition 5 makes sense, and prove its continuity from $\mathcal{S}(\mathbb{S})\times\mathcal{S}_0(\mathbb{R}^2)$ into $\mathcal{S}_0(\mathbb{R}^2)$. We refer to [16, Theorem 19.0.1] for the analogous statement for the wavelet synthesis operator.

Theorem 6. *The operator $\mathcal{S}^t\colon (F,\psi)\mapsto\mathcal{S}_\psi F$ is a continuous bilinear mapping from $\mathcal{S}(\mathbb{S})\times\mathcal{S}_0(\mathbb{R}^2)$ into $\mathcal{S}_0(\mathbb{R}^2)$. In particular, the shearlet synthesis operator S^t_ψ is a continuous operator from $\mathcal{S}(\mathbb{S})$ into $\mathcal{S}_0(\mathbb{R}^2)$ for every $\psi\in\mathcal{S}_0(\mathbb{R}^2)$.*

Proof. We start by proving the continuity, and we refer to [2] for the analogous proof under stronger conditions on the admissible vectors. Indeed, compared to [2], here we consider general admissible shearlets $\psi\in\mathcal{S}_0(\mathbb{R}^2)$.

We need to show that for every $F\in\mathcal{S}(\mathbb{S})$, $\psi\in\mathcal{S}_0(\mathbb{R}^2)$ and $\nu_1\in\mathbb{N}$, there exist $k_1,k_2,l,m,\alpha_1,\alpha_2,\beta,\gamma,\nu_2\in\mathbb{N}$ such that

$$\rho_{\nu_1}(\mathcal{S}^t_\psi F)\leq C\rho^{\alpha_1,\alpha_2,\beta,\gamma}_{k_1,k_2,l,m}(F)\rho_{\nu_2}(\psi).$$

We use the fact that the families $\hat{\rho}_\nu(f)=\rho_\nu(\mathcal{F}f)$ and $\hat{\rho}^{\alpha_1,\alpha_2,\beta,\gamma}_{k_1,k_2,l,m}(F)=\rho^{\alpha_1,\alpha_2,\beta,\gamma}_{k_1,k_2,l,m}(\mathcal{F}F)$, where $\mathcal{F}F$ denotes the Fourier transform of F with respect to the variable b, are bases of norms for the topologies of $\mathcal{S}_0(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{S})$, respectively (cf. [20]). Furthermore, by the Plancherel theorem, equation (2) and the Fubini theorem we have that

$$
\begin{aligned}
\mathcal{S}^t_\psi F(x) &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}|a|^{3/4}\int_{\mathbb{R}^2}\mathcal{F}F(\xi,s,a)\ e^{2\pi i x\xi}\overline{\mathcal{F}\psi(-a\xi_1,a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))}\ \mathrm{d}\xi\frac{\mathrm{d}s\mathrm{d}a}{|a|^3}\\
&= \int_{\mathbb{R}^2}e^{2\pi i x\xi}\int_{\mathbb{R}^\times}\int_{\mathbb{R}}|a|^{3/4}\mathcal{F}F(\xi,s,a)\ \overline{\mathcal{F}\psi(-a\xi_1,a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))}\ \frac{\mathrm{d}s\mathrm{d}a}{|a|^3}\mathrm{d}\xi, \qquad (21)
\end{aligned}
$$

for every $F\in\mathcal{S}(\mathbb{S})$ and $\psi\in\mathcal{S}_0(\mathbb{R}^2)$.

We first consider the derivatives of $\mathcal{S}^t_\psi F$. Let $\epsilon > 0$ and $N \in \mathbb{N}$, $N > 2$. By formula (21) for every $\alpha \in \mathbb{N}$ we have that

$$
\begin{aligned}
&|\partial^\alpha_{x_1}(\mathcal{S}^t_\psi F)(x_1,x_2)| \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{\frac{3}{4}}|\xi_1|^\alpha |\mathcal{F}F(\xi,s,a)|\ |\mathcal{F}\psi(-A_a{}^tN_s\xi)|\ \frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{|a|^3} \\
&\lesssim \int_{\mathbb{R}^2}\int_{|a|<\epsilon}\int_{\mathbb{R}} \langle\xi\rangle^N\langle s\rangle^2 |a|^{-\frac{9}{4}}|\xi_1|^\alpha |\mathcal{F}F(\xi,s,a)|\ |\mathcal{F}\psi(-A_a{}^tN_s\xi)| \frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{\langle s\rangle^2\langle\xi\rangle^N} \\
&+ \int_{\mathbb{R}^2}\int_{|a|>\epsilon}\int_{\mathbb{R}} \langle\xi\rangle^N\langle s\rangle^2 |a|^{-\frac{9}{4}}|\xi_1|^\alpha |\mathcal{F}F(\xi,s,a)|\ |\mathcal{F}\psi(-A_a{}^tN_s\xi)| \frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{\langle s\rangle^2\langle\xi\rangle^N} \\
&\lesssim \rho^{0,0,0,0}_{N+\alpha,N,2,\frac{9}{4}}(\mathcal{F}F)\rho_0(\psi) + \rho^{0,0,0,0}_{N+\alpha,N,2,0}(\mathcal{F}F)\rho_0(\psi),
\end{aligned}
$$

which is dominated by a single product of norms. The terms $|\partial^\beta_{x_2}(S^t_\psi\Phi)(x_1,x_2)|$, $\beta \in \mathbb{N}$, can be estimated in a similar fashion.

Next we consider multiplications by monomials x_1^k, $k \in \mathbb{Z}_+$. By formula (21) we have

$$
\begin{aligned}
&|x_1^k(\mathcal{S}^t_\psi F)(x_1,x_2)| \\
&= |\int_{\mathbb{R}^2} x_1^k e^{2\pi i x\xi}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{\frac{3}{4}}\mathcal{F}F(\xi,s,a)\ \overline{\mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))}\ \frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{|a|^3}| \\
&= |\int_{\mathbb{R}^2} \frac{e^{2\pi i x\xi}}{(2\pi i)^k}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{\frac{3}{4}}\partial^k_{\xi_1}[\mathcal{F}F(\xi,s,a)\ \overline{\mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))}]\ \frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi}{|a|^3}| \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{\frac{3}{4}}|\partial^k_{\xi_1}[\mathcal{F}F(\xi,s,a)\ \mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))]|\ \frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi}{|a|^3},
\end{aligned}
$$

which is less than or equal to a finite sum of addends of the form

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{\frac{3}{4}+k_2+\frac{k_3}{2}}|s|^{k_3}|\partial^{k_1}_{\xi_1}\mathcal{F}F(\xi,s,a)|\times \\
&\times|\partial^{k_2}_1\partial^{k_3}_2\mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))|\frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{|a|^3} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{-\frac{9}{4}+k_2+\frac{k_3}{2}}|s|^{k_3}(1+|\xi|^2)^{N/2}(1+s^2)^{\frac{M}{2}}|\partial^{k_1}_{\xi_1}\mathcal{F}F(\xi,s,a)|\times \\
&\times|\partial^{k_2}_1\partial^{k_3}_2\mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))|\frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{(1+s^2)^{\frac{M}{2}}(1+|\xi|^2)^{\frac{N}{2}}},
\end{aligned}
$$

where k_1, k_2, $k_3 \in \mathbb{N}$ are less than k, $N > 2$ and $M > k_3 + 1$. Then, splitting the integral over $\mathbb{R}^\times$ into integrals over $|a| < \epsilon$ and $|a| > \epsilon$, with $\epsilon > 0$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} |a|^{\frac{3}{4}+k_2+\frac{k_3}{2}}|s|^{k_3}|\partial^{k_1}_{\xi_1}\mathcal{F}F(\xi,s,a)|\times \\
&\times|\partial^{k_2}_1\partial^{k_3}_2\mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2+s\xi_1))|\frac{\mathrm{d}s\mathrm{d}a\mathrm{d}\xi_1\mathrm{d}\xi_2}{|a|^3} \\
&\lesssim \rho^{k_1,0,0,0}_{N,N,M,|-\frac{9}{4}+k_2+\frac{k_3}{2}|}(\mathcal{F}F)\rho_{k_2+k_3}(\psi) + \rho^{k_1,0,0,0}_{N,N,M,k_2+\frac{k_3}{2}}(\mathcal{F}F)\rho_{k_2+k_3}(\psi),
\end{aligned}
$$

which is dominated by a single product of norms.

Obviously, we can treat $|{x_2}^k(\mathcal{S}^t_\psi F)(x_1,x_2)|$, $k \in \mathbb{Z}_+$, in the same manner, and we conclude that the operator $\mathcal{S}^t$ is continuous from $\mathcal{S}(\mathbb{S}) \times \mathcal{S}_0(\mathbb{R}^2)$ into $\mathcal{S}(\mathbb{R}^2)$.

Finally, we have to show that $\mathcal{S}^t_\psi F \in \mathcal{S}_0(\mathbb{R}^2)$ for every $F \in \mathcal{S}(\mathbb{S})$ and $\psi \in \mathcal{S}_0(\mathbb{R}^2)$. The idea is to prove the equivalent condition

$$\lim_{\xi \to 0} \frac{\mathcal{F}\mathcal{S}^t_\psi F(\xi)}{|\xi|^k} = 0, \tag{22}$$

for every $k \in \mathbb{N}$, see [16, Lemma 6.0.4]. We first observe that by formula (21) and the Fourier inversion formula we have

$$\mathcal{F}\mathcal{S}^t_\psi F(\xi_1, \xi_2) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |a|^{3/4} \mathcal{F}F(\xi, s, a)\, \overline{\mathcal{F}\psi(-a\xi_1, a|a|^{-\frac{1}{2}}(-\xi_2 + s\xi_1))}\, \frac{\mathrm{d}s\mathrm{d}a}{|a|^3}, \tag{23}$$

where we recall that $\mathcal{F}F$ denotes the Fourier transform of F with respect to the variable b. We will prove that for every $k \in \mathbb{N}$ there exists $N_k \in \mathbb{Z}_+$ and a constant $C > 0$ such that

$$\frac{|\mathcal{F}\mathcal{S}^t_\psi F(r\cos\theta, r\sin\theta)|}{r^k} \leq C r^{N_k},$$

for every $r \in \mathbb{R}_+$ and for every $\theta \in [0, 2\pi)$. Indeed, by equation (23), we have

$$\begin{aligned} |\mathcal{F}S^t_\psi F(r\cos\theta, r\sin\theta)| \leq \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |a|^{3/4} |\mathcal{F}F((r\cos\theta, r\sin\theta), s, a)| \times \\ \times |\mathcal{F}\psi(-ar\cos\theta, a|a|^{-\frac{1}{2}} r(-\sin\theta + s\cos\theta))|\, \frac{\mathrm{d}s\mathrm{d}a}{|a|^3}. \end{aligned}$$

By Lemma 3, for every $m \in \mathbb{N}$ there exists $g \in \mathcal{S}_0(\mathbb{R}^2)$ such that

$$\mathcal{F}\psi(\xi_1, \xi_2) = \xi_1^m \mathcal{F}g(\xi_1, \xi_2), \quad \xi_1,\, \xi_2 \in \mathbb{R}$$

and we can continue the above inequality as follows

$$\begin{aligned} &|\mathcal{F}\mathcal{S}^t_\psi F(r\cos\theta, r\sin\theta)| \\ &\lesssim \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |a|^{-\frac{9}{4}+m} |\mathcal{F}F((r\cos\theta, r\sin\theta), s, a)| r^m |\cos\theta|^m \times \\ &\times |\mathcal{F}g(-ar\cos\theta, a|a|^{-\frac{1}{2}} r(-\sin\theta + s\cos\theta))|(1+s^2) \frac{\mathrm{d}s\mathrm{d}a}{(1+s^2)}. \end{aligned}$$

Next, by splitting the integral over $\mathbb{R}^\times$ into integrals ove $|a| < \epsilon$ and $|a| > \epsilon$ for some $\epsilon > 0$, we obtain

$$|\mathcal{F}\mathcal{S}^t_\psi F(r\cos\theta, r\sin\theta)| \lesssim r^m [\rho^{0,0,0,0}_{0,0,2,\frac{9}{4}}(\mathcal{F}F)\rho_0(g) + \rho^{0,0,0,0}_{0,0,2,m}(\mathcal{F}F)\rho_0(g)] \lesssim r^m,$$

where the hidden constant is independent of $\theta \in [0, 2\pi)$. Therefore, by choosing $m > k$, we obtain

$$\frac{|\mathcal{F}\mathcal{S}^t_\psi F(r\cos\theta, r\sin\theta)|}{r^k} \lesssim r^{m-k},$$

which implies that (22) holds true for every $k \in \mathbb{N}$, and we conclude that $\mathcal{S}^t_\psi F$ belongs to $\mathcal{S}_0(\mathbb{R}^2)$. □

Next we show that the shearlet synthesis operator is in fact the adjoint of the shearlet transform in the following sense. Let $\psi \in \mathcal{S}_0(\mathbb{R}^2)$. If $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and

$F \in \mathcal{S}(\mathbb{S})$, then by Fubini theorem we have that

$$\begin{aligned}
\int_{\mathbb{R}^2} f(x)\mathcal{S}^t_\psi F(x)\mathrm{d}x &= \int_{\mathbb{R}^2} f(x) \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} F(b,s,a)\,\overline{\pi_{b,s,a}\psi(x)}\,\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}\mathrm{d}x \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} F(b,s,a)\int_{\mathbb{R}^2} f(x)\overline{\pi_{b,s,a}\psi(x)}\mathrm{d}x\,\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3} \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{S}_\psi f(b,s,a)F(b,s,a)\,\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}. \qquad (24)
\end{aligned}$$

Therefore, since $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ naturally embeds into $\mathcal{S}_0'(\mathbb{R}^2)$ and by the identification (13), we may write (24) as

$$_{\mathcal{S}'(\mathbb{S})}(\mathcal{S}_\psi f, F)_{\mathcal{S}(\mathbb{S})} = {}_{\mathcal{S}_0'(\mathbb{R}^2)}(f, \mathcal{S}^t_\psi F)_{\mathcal{S}_0(\mathbb{R}^2)}.$$

This duality relation will motivate our definition of the distributional shearlet transform in Section 5.

If $f \in L^2(\mathbb{R}^2)$ then, by using the definition of the shearlet synthesis operator, we can rewrite reconstruction formula (8) as

$$f = \frac{1}{C_{\psi,\phi}}(\mathcal{S}^t_\phi \circ \mathcal{S}_\psi)f, \qquad (25)$$

where the equality holds in the weak-sense. Next we show that if ψ, ϕ, $f \in \mathcal{S}_0(\mathbb{R}^2)$, then the reconstruction formula holds pointwise.

Proposition 7. *Let ψ, $\phi \in \mathcal{S}_0(\mathbb{R}^2)$ be such that (7) holds true. If $f \in \mathcal{S}_0(\mathbb{R}^2)$, then*

$$f(x) = \frac{1}{C_{\psi,\phi}}(\mathcal{S}^t_\phi \circ \mathcal{S}_\psi)f(x), \quad x \in \mathbb{R}^2. \qquad (26)$$

Proof. We recall that by Theorem 4 if ψ and f belong to $\mathcal{S}_0(\mathbb{R}^2)$, then $\mathcal{S}_\psi f \in \mathcal{S}(\mathbb{S})$. By the Plancherel theorem, Fubini theorem and formula (2) for every $x \in \mathbb{R}^2$ we have

$$\begin{aligned}
(\mathcal{S}^t_\phi \circ \mathcal{S}_\psi)f(x) &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{S}_\psi f(b,s,a)\,\pi_{b,s,a}\phi(x)\,\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3} \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{F}f(\xi)e^{2\pi i b\xi}\overline{\mathcal{F}\psi(A_a{}^tN_s\xi)}\,\pi_{b,s,a}\phi(x)\,\frac{\mathrm{d}\xi\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^{\frac{9}{4}}} \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{F}f(\xi)\overline{\mathcal{F}\psi(A_a{}^tN_s\xi)}\int_{\mathbb{R}^2} e^{2\pi i b\xi}\,\pi_{b,s,a}\phi(x)\mathrm{d}b\,\frac{\mathrm{d}\xi\mathrm{d}s\mathrm{d}a}{|a|^{\frac{9}{4}}} \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{F}f(\xi)\overline{\mathcal{F}\psi(A_a{}^tN_s\xi)}\mathcal{F}\pi_{-x,s,a}\phi(\xi)\,\frac{\mathrm{d}\xi\mathrm{d}s\mathrm{d}a}{|a|^{\frac{9}{4}}} \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{F}f(\xi)\overline{\mathcal{F}\psi(A_a{}^tN_s\xi)}\mathcal{F}\phi(A_a{}^tN_s\xi)e^{2\pi i x\xi}\,\frac{\mathrm{d}\xi\mathrm{d}s\mathrm{d}a}{|a|^{\frac{3}{2}}}.
\end{aligned}$$

Hence, by equation (7) for every $x \in \mathbb{R}^2$

$$(\mathcal{S}^t_\phi \circ \mathcal{S}_\psi)f(x) = \int_{\mathbb{R}^2} \mathcal{F}f(\xi)e^{2\pi i x\xi}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \overline{\mathcal{F}\psi(A_a{}^tN_s\xi)}\mathcal{F}\phi(A_a{}^tN_s\xi)\,\frac{\mathrm{d}s\mathrm{d}a}{|a|^{\frac{3}{2}}}\mathrm{d}\xi = C_{\psi,\phi}f(x),$$

and this concludes the proof. □

5 The shearlet transform on $\mathcal{S}_0'(\mathbb{R}^2)$

In this last section, we extend the definition of the shearlet transform to the space of Lizorkin distributions and we prove its consistency with the classical definition for test functions. Then we extend reconstruction formula (25) to $\mathcal{S}_0'(\mathbb{R}^2)$.

Definition 8. Let $f \in \mathcal{S}_0'(\mathbb{R}^2)$ and $\psi \in \mathcal{S}_0(\mathbb{R}^2)$. We define the shearlet transform $\mathscr{S}_\psi f$ of f with respect to ψ as the distribution in $\mathcal{S}'(\mathbb{S})$ whose action on test functions is given by

$$ {}_{\mathcal{S}'(\mathbb{S})}(\mathscr{S}_\psi f, \Phi)_{\mathcal{S}(\mathbb{S})} = {}_{\mathcal{S}_0'(\mathbb{R}^2)}(f, \mathcal{S}^t_{\overline{\psi}}\Phi)_{\mathcal{S}_0(\mathbb{R}^2)}, \quad \Phi \in \mathcal{S}(\mathbb{S}). $$

The validity of this definition follows from Theorem 6. Moreover, Definition 8 and Theorem 4 motivate the next one.

Definition 9. Let $\Psi \in \mathcal{S}'(\mathbb{S})$ and $\psi \in \mathcal{S}_0(\mathbb{R}^2)$. The shearlet synthesis operator $\mathscr{S}^t_\psi \Psi$ of Ψ with respect to ψ is defined as the Lizorkin distribution whose action on test functions is given by

$$ {}_{\mathcal{S}_0'(\mathbb{R}^2)}(\mathscr{S}^t_\psi \Psi, f)_{\mathcal{S}_0(\mathbb{R}^2)} = {}_{\mathcal{S}'(\mathbb{S})}(\Psi, \mathcal{S}_{\overline{\psi}} f)_{\mathcal{S}(\mathbb{S})}, \quad f \in \mathcal{S}_0(\mathbb{R}^2). $$

The consistence of Definitions 8 and 9 is guaranteed by Theorems 6 and 4, respectively. Furthermore, we immediately obtain the following result.

Proposition 10. *The shearlet transform $\mathscr{S}_\psi \colon \mathcal{S}_0'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{S})$ and the shearlet synthesis operator $\mathscr{S}^t_\psi : \mathcal{S}'(\mathbb{S}) \to \mathcal{S}_0'(\mathbb{R}^2)$ are continuous linear operators.*

Proof. The proof follows immediately by Theorem 4 and 6. □

We now extend the reconstruction formula (25) to the space of Lizorkin distributions.

Proposition 11. *Let ψ, $\phi \in \mathcal{S}_0(\mathbb{R}^2)$ be such that (7) holds true. For every $f \in \mathcal{S}_0'(\mathbb{R}^2)$*

$$ f = \frac{1}{C_{\psi,\phi}} (\mathscr{S}^t_\phi \circ \mathscr{S}_\psi) f. $$

Proof. Let $f \in \mathcal{S}_0'(\mathbb{R}^2)$. By Definitions 8, 9 and by Proposition 7 we have that for every $\varphi \in \mathcal{S}_0(\mathbb{R}^2)$ the following equalities hold:

$$ ((\mathscr{S}^t_\phi \circ \mathscr{S}_\psi) f, \varphi) = (\mathscr{S}_\psi f, \mathcal{S}_{\overline{\phi}} \varphi) = (f, (\mathcal{S}^t_{\overline{\psi}} \circ \mathcal{S}_{\overline{\phi}})\varphi) = C_{\overline{\phi},\overline{\psi}}(f, \varphi) = C_{\psi,\phi}(f, \varphi), $$

as claimed. □

The next theorem shows that Definition 8 is in fact consistent with the shearlet transform for test functions introduced in Definition 1. Furthermore, it also shows that our definition is a natural extension of the one considered in [18, 12], where the shearlet transform of a tempered distribution f with respect to an admissible vector $\psi \in \mathcal{S}(\mathbb{R}^2)$ is defined as the function on $\mathbb{S}$ given by

$$ S_\psi f(b, s, a) = {}_{\mathcal{S}'(\mathbb{R}^2)}(f, \pi_{b,s,a}\psi)_{\mathcal{S}(\mathbb{R}^2)}, $$

for every $(b, s, a) \in \mathbb{S}$. Following the coorbit space approach, given a suitable test function space usually denoted by $\mathcal{H}_{1,w}$, where w is a weight function, and its anti-dual $\mathcal{H}^\sim_{1,w}$, the extended shearlet transform of $f \in \mathcal{H}^\sim_{1,w}$ with respect to $\psi \in \mathcal{H}_{1,w}$ is defined by

$$ \mathcal{S}_\psi f(b, s, a) = {}_{\mathcal{H}^\sim_{1,w}}(f, \pi_{b,s,a}\psi)_{\mathcal{H}_{1,w}}, $$

for every $(b,s,a) \in \mathbb{S}$, [6]. Theorem 12 shows the equivalence of our duality approach with the coorbit space one in the sense that it identifies the shearlet transform of any Lizorkin distribution with the function given by

$$(b,s,a) \mapsto {}_{\mathcal{S}_0'(\mathbb{R}^2)}(f, \pi_{b,s,a}\psi)_{\mathcal{S}_0(\mathbb{R}^2)}, \quad (b,s,a) \in \mathbb{S}.$$

Theorem 12. *Let $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ and $f \in \mathcal{S}_0'(\mathbb{R}^2)$. The function $S_\psi f$ defined by*

$$S_\psi f(b,s,a) = {}_{\mathcal{S}_0'(\mathbb{R}^2)}(f, \pi_{b,s,a}\psi)_{\mathcal{S}_0(\mathbb{R}^2)} \tag{27}$$

is a smooth function of at most polynomial growth on $\mathbb{S}$. Furthermore, by identification (13) it follows that

$${}_{\mathcal{S}'(\mathbb{S})}(\mathscr{S}_\psi f, \Phi)_{\mathcal{S}(\mathbb{S})} = {}_{\mathcal{S}'(\mathbb{S})}(S_\psi f, \Phi)_{\mathcal{S}(\mathbb{S})}, \quad \textit{for every} \quad \Phi \in \mathcal{S}(\mathbb{S}). \tag{28}$$

Proof. Consider $f \in \mathcal{S}_0'(\mathbb{R}^2)$ and $\psi \in \mathcal{S}_0(\mathbb{R}^2)$. Since the derivatives of $\pi_{b,s,a}\psi$ are still in $\mathcal{S}_0(\mathbb{R}^2)$ it immediately follows that the function given by (27) is smooth. We prove that $S_\psi f(b,s,a)$ is a function of at most polynomial growth on $\mathbb{S}$ dividing the proof in three steps.

We start by considering $S_\psi f(b,0,1)$ for every $b \in \mathbb{R}^2$. Since $f \in \mathcal{S}_0'(\mathbb{R}^2)$, it follows that there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned}|S_\psi f(b,0,1)| = |(f, \pi_{b,0,1}\psi)| &\lesssim \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu |\partial^m(\pi_{b,0,1}\psi)(x)| \\ &= \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu |\partial^m \psi(x-b)|.\end{aligned}$$

Then, by Peetre's inequality we have that

$$|S_\psi f(b,0,1)| \lesssim \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu \langle x-b\rangle^{-\nu} \lesssim \langle b\rangle^\nu$$

and $S_\psi f(b,0,1)$ is of at most polynomial growth in $b \in \mathbb{R}^2$.

We now consider $S_\psi f(0,s,1)$ for every $s \in \mathbb{R}$. Since $f \in \mathcal{S}_0'(\mathbb{R}^2)$, it follows that there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned}&|S_\psi f(0,s,1)| = |(f, \pi_{0,s,1}\psi)| \lesssim \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu |\partial^m(\pi_{0,s,1}\psi)(x)| \\ &= \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu \Big|\sum_{i=0}^{p_m} c_i s^i (\pi_{0,s,1}\partial^m\psi)(x)\Big| \le \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i |\partial^m\psi(N_s^{-1}x)| \\ &\lesssim \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i (1+|N_s^{-1}x|^2)^{-\frac{\nu}{2}}.\end{aligned}$$

We recall that, for any $x \in \mathbb{R}^2$ and $M \in GL(2,\mathbb{R})$

$$|M^{-1}x| \ge \|M\|^{-1}|x|,$$

where $\|M\|$ denotes the spectral norm of the matrix M. Thus, since $\|N_s\| = (1 + s^2/2 + (s^2+s^2/2)^{1/2})^{1/2}$ for all $s \in \mathbb{R}$, we have that

$$\begin{aligned}|S_\psi f(0,s,1)| &\lesssim \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i \langle \|N_s\|^{-1}x\rangle^{-\nu} \\ &\lesssim \sup_{x\in\mathbb{R}^2,\, |m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i \|N_s\|^\nu \langle x\rangle^{-\nu} = \sup_{|m|\le\nu} \sum_{i=0}^{p_m} |c_i||s|^i \|N_s\|^\nu,\end{aligned}$$

which proves that the function $S_\psi f(0,s,1)$ is of at most polynomial growth in $s \in \mathbb{R}$.

Finally, we consider $S_\psi f(0,0,a)$ for every $a \in \mathbb{R}^\times$. Since $f \in \mathcal{S}_0'(\mathbb{R}^2)$, it follows that there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned}|S_\psi f(0,0,a)| = |(f, \pi_{0,0,a}\psi)| &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |\partial^m(\pi_{0,0,a}\psi)(x)| \\ &= \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |a|^{-m_1-\frac{m_2}{2}-\frac{3}{4}} |\pi_{0,0,a}\partial^m\psi(A_a^{-1}x)| \\ &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |a|^{-m_1-\frac{m_2}{2}-\frac{3}{4}} \langle \|A_a\|^{-1}x\rangle^{-\nu},\end{aligned}$$

where $(m_1,m_2) \in \mathbb{N}^2$ and $m_1+m_2 = m$. Since $\|A_a\| = |a|^{\frac{1}{2}}$ for $|a|<1$, then

$$|S_\psi f(0,0,a)| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |a|^{-m_1-\frac{m_2}{2}-\frac{3}{4}} \langle x\rangle^{-\nu} \lesssim a^{-p},$$

for some $p \in \mathbb{N}$. If $|a| \ge 1$, then $\|A_a\| = |a|$ and we have

$$|S_\psi f(0,0,a)| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |a|^{-m_1-\frac{m_2}{2}-\frac{3}{4}+\nu} \langle x\rangle^{-\nu} \lesssim a^{\nu},$$

which proves that the function $S_\psi f(0,0,a)$ is of at most polynomial growth in a on $\mathbb{R}^\times$.

Therefore, since

$$S_\psi f(b,s,a) = (f, \pi_{b,0,1}\pi_{0,s,1}\pi_{0,0,a}\psi),$$

we conclude that there exist $C, \nu_1, \nu_2, \nu_3 > 0$ such that

$$|S_\psi f(b,s,a)| \le C\langle b\rangle^{\nu_1}\langle s\rangle^{\nu_2}\left(a^{\nu_3} + \frac{1}{a^{\nu_3}}\right),$$

for every $(b,s,a) \in \mathbb{S}$, and we finally identify $S_\psi f$ as an element in $\mathcal{S}'(\mathbb{S})$ by formula (13).

To prove (28) we use the fact that the space of Lizorkin distributions $\mathcal{S}_0'(\mathbb{R}^2)$ is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^2)$ by the space of polynomials. By the Schwartz structural theorem [24, Theorém VI], we can write $f = \partial^\alpha g + p$, where g is a continuous slowly growing function on $\mathbb{R}^2$, $\alpha \in \mathbb{N}^2$ and p is a polynomial. Then it can be easily shown that

$$(\partial^\alpha g, \mathcal{S}_\psi^t \Phi) = \int_{\mathbb{S}} \Phi(b,s,a)\ (\partial^\alpha g, \pi_{b,s,a}\psi) \mathrm{d}\mu(b,s,a), \quad \Phi \in \mathcal{S}(\mathbb{S}),$$

and analogously,

$$(p, \mathcal{S}_\psi^t \Phi) = \int_{\mathbb{S}} \Phi(b,s,a)\ (p, \pi_{b,s,a}\psi) \mathrm{d}\mu(b,s,a), \quad \Phi \in \mathcal{S}(\mathbb{S}).$$

Then, for $f \in \mathcal{S}_0'(\mathbb{R}^2)$ represented by $f = \partial^\alpha g + p$ we obtain

$$\begin{aligned}(f, \mathcal{S}_\psi^t \Phi) = (\partial^\alpha g + p, \mathcal{S}_\psi^t \Phi) &= \int_{\mathbb{S}} \Phi(b,s,a)\ (\partial^\alpha g + p, \pi_{b,s,a}\psi) \mathrm{d}\mu(b,s,a) \\ &= \int_{\mathbb{S}} \Phi(b,s,a)\ (f, \pi_{b,s,a}\psi) \mathrm{d}\mu(b,s,a) \\ &= \int_{\mathbb{S}} S_\psi f(b,s,a)\ \Phi(b,s,a) \mathrm{d}\mu(b,s,a),\end{aligned}$$

which concludes the proof. □

We finish the paper by showing that the shearlet transform may represent the action of a Lizorkin distribution on any test function in the form of an absolutely convergent integral over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\times$. This is sometimes called *desingularization*, we refer to [20] for details, see also [16, Theorem 24.1.4, Chapter I] for analogous statement related to the distributional wavelet transform.

Proposition 13. *Let ψ, $\phi \in \mathcal{S}_0(\mathbb{R}^2)$ be such that (7) holds true and $f \in \mathcal{S}_0'(\mathbb{R}^2)$. Then, for every $\varphi \in \mathcal{S}_0(\mathbb{R}^2)$*

$$(f, \varphi) = \frac{1}{C_{\psi,\phi}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{\mathbb{R}^2} S_\psi f(b, s, a)\ \mathcal{S}_{\overline{\phi}} \varphi(b, s, a) \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{|a|^3}. \tag{29}$$

Proof. Let ψ, $\phi \in \mathcal{S}_0(\mathbb{R}^2)$ satisfy (7) and let $f \in \mathcal{S}_0'(\mathbb{R}^2)$. By Proposition 11, formula (28), and the identification (13), it follows that

$$(f, \varphi) = \frac{1}{C_{\psi,\phi}}((\mathscr{S}_\phi^t \circ \mathscr{S}_\psi) f, \varphi) = \frac{1}{C_{\psi,\phi}}(\mathscr{S}_\psi f, \mathcal{S}_{\overline{\phi}} \varphi) = \frac{1}{C_{\psi,\phi}}(S_\psi f, \mathcal{S}_{\overline{\phi}} \varphi), \quad \varphi \in \mathcal{S}_0(\mathbb{R}^2),$$

where $S_\psi f$ is the function on $\mathbb{S}$ defined by (27) and we conclude that formula (29) holds true. □

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