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Div-Curl Problems and $\mathbf{H}^1$-regular Stream Functions in 3D Lipschitz Domains

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Abstract

We consider the problem of recovering the divergence-free velocity field $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of a given vorticity $\mathbf{F} = \operatorname{curl}\mathbf{U}$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. To that end, we solve the "div-curl problem" for a given $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$. The solution is expressed in terms of a vector potential (or stream function) $\mathbf{A} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{U} = \operatorname{curl}\mathbf{A}$. After discussing existence and uniqueness of solutions and associated vector potentials, we propose a well-posed construction for the stream function. A numerical method based on this construction is presented, and experiments confirm that the resulting approximations display higher regularity than those of another common approach.

KEYWORDS:
div-curl system, stream function, vector potential, regularity, vorticity

1 | INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Given a vorticity field $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^3$ defined over Ω, we are interested in solving the problem of *velocity recovery*:

$$\begin{cases} \operatorname{curl}\mathbf{U} = \mathbf{F} \\ \operatorname{div}\mathbf{U} = 0 \end{cases} \quad \text{in } \Omega. \tag{1}$$

This problem naturally arises in fluid mechanics when studying the vorticity formulation of the incompressible Navier–Stokes equations. Vortex methods, for example, are based on the vorticity formulation and require a solution of problem (1) at every time-step.[1] While our motivation lies in fluid dynamics, this "div-curl problem" also is interesting in its own right.

On the whole space $\mathbb{R}^3$, this problem is a classical matter. Whenever $\mathbf{F}$ is smooth and compactly supported, the unique solution $\mathbf{U}$ of problem (1) that decays to zero at infinity is given by the Biot–Savart law.[2, Proposition 2.16] However, the case where Ω is a bounded domain is significantly more challenging.

In numerical simulations of the incompressible Navier–Stokes equations, it is common to fulfil the constraint $\operatorname{div}\mathbf{U} = 0$ only approximately, but it has recently been demonstrated that such a violation can cause significant instabilities. The importance for numerical methods to fulfil this constraint *exactly* was stressed by John et al.[3] One way of achieving this requirement is the introduction of a *stream function*, or *vector potential*: instead of solving problem (1) directly, one seeks an approximation $\mathbf{A}_h$ of an auxiliary vector-field $\mathbf{A}$ such that $\mathbf{U} = \operatorname{curl}\mathbf{A}$. Because of the vector calculus identity $\operatorname{div}\circ\operatorname{curl} \equiv 0$, the velocity field $\mathbf{U}_h = \operatorname{curl}\mathbf{A}_h$ is always exactly divergence free.

In particle methods, the particle positions $\mathbf{x}_i \in \Omega$, $i = 1, \dots, N$, are updated by solving $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{U}\big(t, \mathbf{x}_i(t)\big)$, $i = 1, \dots, N$ using a time-stepping scheme. It makes sense to use a *volume-preserving* scheme for this problem. However, most of these schemes require a stream function $\mathbf{A}$ and not the velocity $\mathbf{U}$ as input,[4, Chapter VI.9] and thus arises the desire to have a stream function of maximum regularity at hand. In this work we describe how to compute stream functions that are at least $\mathbf{H}^1(\Omega)$-regular—even on non-smooth domains—thereby improving on the regularity of approximations currently available in related literature.

1.1 | Summary of Results

Our results can be summarised as follows.

1. **Existence of Velocity Fields.** (Theorem 1) Problem (1) has a solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ if and only if $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ and $\langle \mathbf{F}, \mathbf{V} \rangle = 0$ for all $\mathbf{V} \in \mathbf{H}^1_0(\Omega)$ with $\operatorname{curl} \mathbf{V} = \mathbf{0}$. In Lemma 8, we discuss equivalent alternative formulations of the latter condition.

2. **Existence of Stream Functions.** (Theorem 3) Let the velocity $\mathbf{U} \in \mathbf{L}^2(\Omega)$ solve problem (1). Then, $\mathbf{U}$ can be written in terms of a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ as $\mathbf{U} = \operatorname{curl} \mathbf{A}$ if and only if $\mathbf{U}$ fulfils $\int_{\Gamma_i} \mathbf{U} \cdot \mathbf{n} \, \mathrm{d}S = 0$ on each connected component Γ_i of the boundary $\Gamma := \partial\Omega$.

3. **Uniqueness.** (Theorems 2 and 4) If Ω is "handle-free", the solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of problem (1) can be made unique by prescribing its normal trace $\mathbf{U} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$. Moreover, if the prescribed boundary data fulfils $\int_{\Gamma_i} \mathbf{U} \cdot \mathbf{n} \, \mathrm{d}S = 0$ on each connected component $\Gamma_i \subset \Gamma$ of the boundary, there exist conditions that *uniquely* determine a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$.

4. **Construction of Solutions.** (Section 5) The main novelty of this work lies in the explicit construction of $\mathbf{H}^1(\Omega)$-regular stream functions $\mathbf{A}$ directly from the vorticity $\mathbf{F}$. Given a vorticity $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ fulfilling the conditions of item 1 and boundary data $\mathbf{U} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$ fulfilling the conditions of item 2, this construction will yield a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$ solves problem (1). If the domain is handle-free, the obtained solution will be the uniquely defined stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ from item 3. Numerical methods will be described in Section 7.

5. **Well-posedness.** (Theorem 5) From the structure of the construction one can directly infer its well-posedness. The vector-fields $\mathbf{U}$ and $\mathbf{A}$ continuously depend on the given data $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{U} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$.

6. **Regularity.** (Theorem 6) If in addition to the above assumptions the given data fulfils $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\mathbf{U} \cdot \mathbf{n} \in L^2(\Gamma)$, then $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$.

The results concerning existence and uniqueness of velocity fields $\mathbf{U}$ follow from classical functional analytic arguments and are well-known, but covered for completeness. Existence of stream functions $\mathbf{A} \in \mathbf{H}^1(\Omega)$ is due to Girault and Raviart[5, Theorem 3.4]. Their work, however, left unclear how to compute such a potential.

1.2 | Problematic Approaches

A naive approach to the div-curl problem (1) relies on the observation that

$$-\boldsymbol{\Delta}\mathbf{U} = \operatorname{curl}(\underbrace{\operatorname{curl}\mathbf{U}}_{=\mathbf{F}}) - \nabla(\underbrace{\operatorname{div}\mathbf{U}}_{=0}) = \operatorname{curl}\mathbf{F}. \tag{2}$$

Based on this vector-calculus identity, it is tempting to solve three *decoupled* scalar Poisson problems $-\Delta U_i = (\operatorname{curl}\mathbf{F})_i$, $i = 1, 2, 3$, for the components of $\mathbf{U}$, say by prescribing the value of each one on the boundary. However, this approach is problematic: it is our aim to *integrate* $\mathbf{F}$, but instead this strategy asks that we *differentiate* first. Therefore, it needlessly requires to impose more regularity on the right-hand side. Moreover, there is no guarantee that its solution is divergence-free. Finally, since the tangential components of $\mathbf{U}$ allow us to compute $(\operatorname{curl}\mathbf{U}) \cdot \mathbf{n}$ on the boundary, the boundary data must fulfil the compatibility condition $(\operatorname{curl}\mathbf{U}) \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{n}$. We will later see that the solutions of problem (1) are *usually not* $\mathbf{H}^1(\Omega)$-regular, and thus the classic existence and uniqueness results in $H^1(\Omega)$ for the scalar Poisson problems $-\Delta U_i = (\operatorname{curl}\mathbf{F})_i$ are not applicable either.

Another straightforward approach assumes that $\mathbf{F} \in \mathbf{L}^2(\Omega)$. One may then extend $\mathbf{F}$ by zero to the whole space, yielding $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\mathbb{R}^3)$, and apply the Biot–Savart law to this extension. The normal trace $\mathbf{U} \cdot \mathbf{n}$ on Γ can then be prescribed by adding a suitable "potential flow". The main caveat of this strategy is that unless $\mathbf{F} \cdot \mathbf{n} = 0$ on the boundary, the zero extension $\widetilde{\mathbf{F}}$ will *not* be divergence-free, and in this case the Biot–Savart law fails to yield the correct result. We will later see that this approach can in fact be fixed by introducing a suitable correction on the boundary.

1.3 | Our Results in Context

Clearly, for a given velocity field $\mathbf{U}$, the condition $\mathbf{U} = \operatorname{curl}\mathbf{A}$ alone does not uniquely determine $\mathbf{A}$: because of the vector calculus identity $\operatorname{curl} \circ (-\nabla) \equiv \mathbf{0}$, any gradient may be added to $\mathbf{A}$ without changing its curl. It is thus natural to enforce the so-called

Reference	Regularity of Ω	Input	Output	Remarks
Amrouche et al.[6]	Lipschitz	$\mathbf{U} \in \mathbf{L}^2(\Omega)$	$\mathbf{A}_\mathrm{T} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$	tangential potential
Amrouche et al.[6]	Lipschitz	$\mathbf{U} \in \mathbf{L}^2(\Omega)$	$\mathbf{A}_\mathrm{N} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$	normal potential, only if $\mathbf{U} \cdot \mathbf{n} = 0$
Bramble and Pasciak[12]	Lipschitz	$\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$	$\mathbf{U} \in \mathbf{L}^2(\Omega)$	
Alonso and Valli[13]	$C^{1,1}$	$\mathbf{F} \in \mathbf{L}^2(\Omega)$	$\mathbf{U} \in \mathbf{L}^2(\Omega)$	
This work	Lipschitz	$\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$	$\mathbf{A} \in \mathbf{H}^1(\Omega)$	solves both systems simultaneously

TABLE 1 Approaches found in the literature for solving related div-curl problems and the *component-wise* Sobolev regularity of the input and output data. No other approach known to the authors achieves $\mathbf{H}^1(\Omega)$-regularity of $\mathbf{A}$ in non-smooth Lipschitz domains.

Coulomb gauge condition $\operatorname{div} \mathbf{A} = 0$, but this alone still does not ensure uniqueness. For a given $\mathbf{F}$, we are then in fact facing two systems:

$$\begin{aligned} \operatorname{curl} \mathbf{A} &= \mathbf{U}, \\ \operatorname{div} \mathbf{A} &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \operatorname{curl} \mathbf{U} &= \mathbf{F}, \\ \operatorname{div} \mathbf{U} &= 0. \end{aligned} \tag{3}$$

These systems differ in the involved spaces and boundary conditions. For the $\mathbf{U}$-system we would like to prescribe $\mathbf{U} \cdot \mathbf{n}$ on $\partial\Omega$, while for the $\mathbf{A}$-system we actually do not care which boundary conditions are prescribed, as long as they ensure that the solution is unique, and—hopefully—as regular as possible.

Many results in the literature are concerned with only one of these systems, an overview of some results is given in Table 1. For example, the famous work of Amrouche et al.[6] is concerned with the $\mathbf{A}$-system and $\mathbf{U} \in \mathbf{L}^2(\Omega)$. They propose tangential or normal boundary conditions for $\mathbf{A}$ and numerical methods to approximate the resulting stream functions. However, even for perfectly smooth velocity fields $\mathbf{U}$, the resulting potentials will usually only have Sobolev regularity $\mathbf{H}^{\frac{1}{2}}(\Omega)$, unless the domain is assumed to be more regular than just Lipschitz. In particular, functions from $\mathbf{H}(\operatorname{curl};\Omega) \cap \mathbf{H}_0(\operatorname{div};\Omega)$ may develop quite strong singularities near corners of the domain, which makes it difficult to approximate them efficiently.[7, Figure 1.3] Note that these singularities can only occur in non-smooth domains: for $C^{1,1}$-domains one has $\mathbf{A}_\mathrm{T}, \mathbf{A}_\mathrm{N} \in \mathbf{H}^1(\Omega)$, so in this case there is no need to look for more regular potentials. Almost all of the literature concerning numerics for the $\mathbf{A}$-system considers either $\mathbf{A}_\mathrm{T}$ or $\mathbf{A}_\mathrm{N}$.[8, Chapter 6.1] Many authors also consider more general problems involving inhomogeneous material coefficients,[9] or the L^p-setting,[10,11] which on the other hand again often requires higher regularity assumptions on the boundary.

It has long been known that stream functions of regularity $\mathbf{H}^1(\Omega)$ do exist, but so far conditions that uniquely characterise them have not been given. We are unaware of any previous approaches that allow us to efficiently compute $\mathbf{A} \in \mathbf{H}^1(\Omega)$ for the case of general Lipschitz domains. Our work aims to close this gap. It proposes natural conditions which *uniquely* determine a vector potential $\mathbf{A}_1 \in \mathbf{H}^1(\Omega)$ *without* explicitly involving boundary values of $\mathbf{A}_1$. We believe it is because previous approaches do prescribe boundary conditions like $\mathbf{A}_\mathrm{T} \cdot \mathbf{n} = 0$ or $\mathbf{A}_\mathrm{N} \times \mathbf{n} = \mathbf{0}$ that they yield less regular stream functions.

Our algorithm utilises the Newton operator: the bulk of the work lies in the explicit computation of a volume integral. Two companion *scalar* elliptic equations must also be solved on the boundary of the domain, but these are easily tackled using standard boundary element methods. While this construction solves both the $\mathbf{U}$- and $\mathbf{A}$-systems simultaneously, we first discuss existence and uniqueness of solutions for each of them individually.

2 | BASIC DEFINITIONS AND NOTATION

2.1 | Spaces Defined on Volumes

We denote by $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ the space of smooth compactly supported functions in Ω, and write $\mathcal{D}'(\Omega)$ for the space of distributions. Their vector-valued analogues $\boldsymbol{\mathcal{D}}(\Omega) := \left(C_0^\infty(\Omega)\right)^3$ and $\boldsymbol{\mathcal{D}}'(\Omega)$ are distinguished by a **bold** font. In the whole space $\mathbb{R}^3$, we will make use of the space of smooth functions $\mathcal{E}(\mathbb{R}^3) := C^\infty(\mathbb{R}^3)$ and its dual $\mathcal{E}'(\mathbb{R}^3)$—the space of compactly supported distributions. Their vector-valued analogues will be denoted by $\boldsymbol{\mathcal{E}}(\mathbb{R}^3)$ and $\boldsymbol{\mathcal{E}}'(\mathbb{R}^3)$, respectively.

We write $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ for the Hilbert spaces of square integrable scalar and vector-valued functions defined over Ω. $H^s(\Omega)$ and $\mathbf{H}^s(\Omega)$, $s > 0$, refer to the corresponding Sobolev spaces. The spaces $H_0^s(\Omega)$ and $\mathbf{H}_0^s(\Omega)$ are defined as the closures of $\mathcal{D}(\Omega)$ and $\boldsymbol{\mathcal{D}}(\Omega)$ in $H^s(\Omega)$ and $\mathbf{H}^s(\Omega)$, respectively. For $s < 0$ we set $H^s(\Omega) := H_0^{-s}(\Omega)'$ and $\mathbf{H}^s(\Omega) := \mathbf{H}_0^{-s}(\Omega)'$.

We will always identify $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ with their duals, i. e., $L^2(\Omega)' = L^2(\Omega)$ and $\mathbf{L}^2(\Omega)' = \mathbf{L}^2(\Omega)$. The Hilbert spaces $\mathbf{H}(\mathrm{div};\Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\,\mathbf{U} \in L^2(\Omega)\}$ and $\mathbf{H}(\mathrm{curl};\Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) \mid \mathrm{curl}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}$ are equipped with the obvious graph norms. The related "homogeneous spaces" are defined as $\mathbf{H}_0(\mathrm{curl};\Omega) := \overline{\mathcal{D}(\Omega)}^{\mathbf{H}(\mathrm{curl};\Omega)}$ and $\mathbf{H}_0(\mathrm{div};\Omega) := \overline{\mathcal{D}(\Omega)}^{\mathbf{H}(\mathrm{div};\Omega)}$. Accordingly, all differential operators are to be understood in the distributional sense. We refer to Amrouche et al. for a detailed exposition of the regularity and compactness properties of these spaces.[6, Sections 2.2 and 2.3] Evidently, these definitions can also be used with Ω replaced by $\mathbb{R}^3$ and vice-versa.

2.2 | Trace Spaces

We refer to McLean[14, Chapter 3], Sauter and Schwab[15, Chapter 2] and Buffa et al.[16] for theory concerning extension of the traces

$$\gamma V := V|_\Gamma, \quad \nu\mathbf{u} := \mathbf{u}|_\Gamma \cdot \mathbf{n}, \quad \boldsymbol{\tau}\mathbf{u} := \mathbf{u}|_\Gamma - (\mathbf{u}|_\Gamma \cdot \mathbf{n})\mathbf{n} = \mathbf{n} \times (\mathbf{u}|_\Gamma \times \mathbf{n}), \quad \text{and} \quad \boldsymbol{\rho}\mathbf{u} := \mathbf{u}|_\Gamma \times \mathbf{n}, \tag{4}$$

to continuous and surjective mappings

$$\begin{aligned}
\gamma &: & H^1(\Omega) &\to H^{\frac{1}{2}}(\Gamma), & \ker(\gamma) &= H^1_0(\Omega),\\
\nu &: & \mathbf{H}(\mathrm{div};\Omega) &\to H^{-\frac{1}{2}}(\Gamma), & \ker(\nu) &= \mathbf{H}_0(\mathrm{div};\Omega),\\
\boldsymbol{\tau} &: & \mathbf{H}(\mathrm{curl};\Omega) &\to \mathbf{H}_\mathrm{T}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma), & \ker(\boldsymbol{\tau}) &= \mathbf{H}_0(\mathrm{curl};\Omega),\\
\boldsymbol{\rho} &: & \mathbf{H}(\mathrm{curl};\Omega) &\to \mathbf{H}_\mathrm{R}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), & \ker(\boldsymbol{\rho}) &= \mathbf{H}_0(\mathrm{curl};\Omega),
\end{aligned} \tag{5}$$

having continuous right-inverses (so-called *lifting maps*). The fact that the operators $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$ are surjective is one of the main results of Buffa et al.[16] It allows for the derivation of Hodge decompositions, which we will also use later on. The surface differential operators:

$$\mathrm{div}_\Gamma : \mathbf{H}_\mathrm{R}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \qquad \text{and} \qquad \mathrm{curl}_\Gamma : \mathbf{H}_\mathrm{T}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \tag{6}$$

used in those definitions are defined by duality for all $v \in H^{\frac{1}{2}}(\Omega)$ by $\langle \mathrm{div}_\Gamma\,\mathbf{u}, v\rangle_\Gamma := \langle \mathbf{u}, -\nabla_\Gamma v\rangle_\Gamma$ and $\langle \mathrm{curl}_\Gamma\,\mathbf{u}, v\rangle_\Gamma := \langle \mathbf{u}, \mathbf{curl}_\Gamma v\rangle_\Gamma$. Here $-\nabla_\Gamma$ and $\mathbf{curl}_\Gamma$ are suitable extensions of the ordinary trace gradient and trace curl to operators with mappings $-\nabla_\Gamma : H^{\frac{1}{2}}(\Gamma) \to \mathbf{H}_\mathrm{T}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma)$ and $\mathbf{curl}_\Gamma : H^{\frac{1}{2}}(\Gamma) \to \mathbf{H}_\mathrm{R}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$. These operators also give rise to a duality between $\mathbf{H}_\mathrm{R}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ and $\mathbf{H}_\mathrm{T}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma)$.

Finally, we will also make use of the Laplace–Beltrami operator $-\Delta_\Gamma := \mathrm{div}_\Gamma \circ (-\nabla_\Gamma) \equiv \mathrm{curl}_\Gamma \circ \mathbf{curl}_\Gamma$. This operator is known to be coercive on $H^1(\Gamma)/\mathbb{R}$, the space of $H^1(\Gamma)$-regular traces with zero average on each Γ_i, $i = 0, \dots, \beta_2$.

2.3 | Geometry of the Domain

Throughout this article, we suppose that the Lipschitz domain $\Omega \subset \mathbb{R}^3$ of interest is bounded and connected, and we will sometimes refer to the Betti numbers β_1 and β_2. These numbers are related to the topological properties of the domain, see Figure 1. β_1 is the genus of the domain, in other words the number of "handles". β_2 is the number of 'holes' Θ_i, $i = 1, \dots, \beta_2$ in the domain. We define the *exterior domain* Θ_0 via:

$$\Theta_0 := \mathbb{R}^3 \setminus \overline{\Omega \cup \left(\bigcup_{i=1}^{\beta_2} \Theta_i\right)}. \tag{7}$$

The domain's boundary thus always has $\beta_2 + 1$ connected components $\Gamma_i := \partial\Theta_i$, $i = 0, \dots, \beta_2$. A domain with $\beta_1 = 0$ is called *handle-free*,[1] *hole-free* if $\beta_2 = 0$, and we say that the topology of Ω is trivial or simple if $\beta_1 = \beta_2 = 0$. We refer to Arnold and al. for more details.[17, Section 2]

The geometric interpretation of the Betti numbers is best illustrated through an example. In the domain depicted in Figure 1, $\beta_2 = 3$: three cube-like holes Θ_1, Θ_2, and Θ_3 were cut out of the toroidal volume. Their boundaries Γ_1, Γ_2, and Γ_3 are labelled in the figure. Together with the exterior boundary Γ_0, the boundary $\Gamma := \partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ thus has *four* $= \beta_2 + 1$ connected components.

[1] We avoid the term "simply-connected" as it usually refers to *homotopy* as opposed to *homology*.

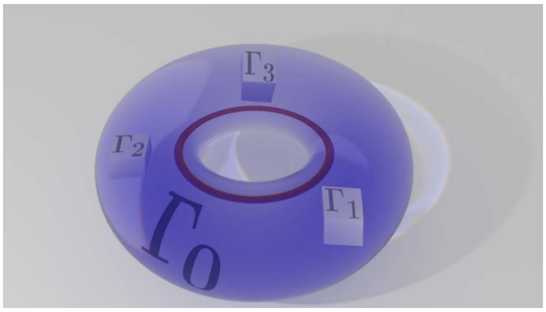

FIGURE 1 A ring-shaped domain with three cubical holes is an example of a domain having non-trivial topology.[6, Section 3] There is one “handle” through it: $\beta_1 = 1$. The red line is a representative of the equivalence class of non-bounding cycles. The three cubical inclusions (“holes”) are not part of the domain: $\beta_2 = 3$. The boundary Γ has four connected components: $\Gamma_1, \Gamma_2, \Gamma_3$, and the domain’s exterior boundary Γ_0.

On the one hand, the value of the second Betti number β_2 is relevant to questions regarding the *existence* results stated in item 1 and item 2 of Section 1. These existence theorems will make use of arbitrary but fixed functions $T_i \in C_0^\infty(\mathbb{R}^3)$, $i = 0, \dots, \beta_2$, that act as indicators for the the connected components of the boundary:

$$T_i = \begin{cases} 1 & \text{in a neighbourhood of } \Gamma_i, \\ 0 & \text{in a neighbourhood of } \Gamma_j,\ j \in \{0, \dots, \beta_2\} \setminus \{i\}. \end{cases} \tag{8}$$

On the other hand, the value of β_1 is crucial to the *uniqueness* results of item 3. For simplicity, we will restrict our attention to handle-free domains ($\beta_1 = 0$), that is domains for which every loop inside Ω is the boundary of a surface within Ω. The domain in Figure 1 is *not* handle-free, as the red loop is a representative of the equivalence class of non-bounding cycles. In that example, $\beta_1 = 1$. Nevertheless, we will make some remarks on what changes in the following results need to be anticipated in order to recover uniqueness of solutions when $\beta_1 > 0$.

2.4 | Laplace and Newton Operator

The scalar Laplacian $-\Delta := \operatorname{div} \circ (-\nabla)$ is central to potential theory, and we assume that the reader is well aware of the classical existence and uniqueness results for boundary value problems related to this operator. Its vector-valued analogue is defined component-wise: $\boldsymbol{-\Delta} \mathbf{V} := (-\Delta V_1, -\Delta V_2, -\Delta V_3)^\top$ for all $\mathbf{V} \in \mathcal{D}'(\Omega)$. This operator is also known as Hodge–Laplacian and can equivalently be written as:

$$\boldsymbol{-\Delta} \mathbf{V} = \operatorname{curl}\operatorname{curl} \mathbf{V} - \nabla \operatorname{div} \mathbf{V}, \qquad \forall \mathbf{V} \in \mathcal{D}'(\Omega). \tag{9}$$

Let us denote by $G(\mathbf{x}) := (4\pi|\mathbf{x}|)^{-1}$ the fundamental solution of the Laplacian $-\Delta$. The Newton operator is defined on the space of compactly supported distributions $\mathcal{E}'(\mathbb{R}^3)$ via convolution as $\mathcal{N} : \mathcal{E}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3)$, $U \mapsto G \star U$. In other words:

$$\forall U \in \mathcal{D}(\mathbb{R}^3): \quad \big(\mathcal{N} U\big)(\mathbf{x}) := \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{U(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}\mathbf{y} \qquad \in \mathcal{E}(\mathbb{R}^3), \tag{10}$$

while $\langle \mathcal{N} U, V \rangle := \langle U, \mathcal{N} V \rangle$ for all $U \in \mathcal{E}'(\mathbb{R}^3)$ and $V \in \mathcal{D}(\mathbb{R}^3)$. For a given $U \in \mathcal{E}'(\mathbb{R}^3)$, the distribution $\mathcal{N} U$ is called the *Newton potential* of U.

This operator is an inverse for the Laplacian:[18, Equations (4.4.2) and (4.4.3)]

$$\forall U \in \mathcal{E}'(\mathbb{R}^3): \qquad -\Delta \mathcal{N} U = \mathcal{N}(-\Delta U) = U. \tag{11}$$

Moreover, because it is an operator of convolutional type, it commutes with differentiation.[18, Equation (4.2.5)] Application of this operator always increases the Sobolev regularity of a distribution $U \in H^s(\mathbb{R}^3) \cap \mathcal{E}'(\mathbb{R}^3)$ by two, i. e., $\mathcal{N}$ has the following mapping property and is continuous:[15, Theorem 3.1.2]

$$\mathcal{N} : H^s(\mathbb{R}^3) \cap \mathcal{E}'(\mathbb{R}^3) \to H^{s+2}_{\mathrm{loc}}(\mathbb{R}^3) \qquad s \in \mathbb{R}. \tag{12}$$

The Newton potential $\mathcal{N}U$ of a compactly supported distribution $U \in \mathcal{E}'(\mathbb{R}^3)$ is regular outside $\operatorname{supp} U$ and decays to zero at infinity:[19, Chapter II, §3.1, Proposition 2]

$$\forall U \in \mathcal{E}'(\mathbb{R}^3): \ \big(\mathcal{N}U\big)(\mathbf{x}) = \mathcal{O}\left(|\mathbf{x}|^{-1}\right) \qquad |\mathbf{x}| \to \infty. \tag{13}$$

Even more importantly, the following result characterises the Newton potential.[19, Chapter II, §3.1, Proposition 3]

Lemma 1. Let $U \in \mathcal{D}'(\mathbb{R}^3)$ and $F \in \mathcal{E}'(\mathbb{R}^3)$. Then U is the Newton potential of F, that is $U = \mathcal{N}F$, if and only if:

$$\begin{cases} -\Delta U = F & \text{on } \mathbb{R}^3, \\ U(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{14}$$

This characterisation allows for the derivation of representation formulæ for solutions of the Laplace equation on bounded domains, leading to boundary integral equations. This will appear at the end of this section. All of these results analogously hold for the vector-valued Newton operator $\boldsymbol{\mathcal{N}}$, which is defined component-wise and distinguished by a bold-face font.

2.5 | Decompositions of Helmholtz–Hodge Type

Decomposition theorems will play a central role in our analysis, so we collect the most important results here.

Lemma 2 (Helmholtz Decomposition)**.** Every compactly supported distribution $\mathbf{U} \in \boldsymbol{\mathcal{E}}'(\mathbb{R}^3)$ can be decomposed into a divergence-free and a curl-free part:

$$\mathbf{U} = \operatorname{curl} \mathbf{A} - \nabla P, \tag{15}$$

where $\mathbf{A} := \boldsymbol{\mathcal{N}} \operatorname{curl} \mathbf{U} \in \boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$ and $P := \mathcal{N} \operatorname{div} \mathbf{U} \in \mathcal{D}'(\mathbb{R}^3)$.

Proof. $\mathbf{U} = -\boldsymbol{\Delta}\boldsymbol{\mathcal{N}}\mathbf{U} = \operatorname{curl}(\operatorname{curl}\boldsymbol{\mathcal{N}}\mathbf{U}) - \nabla(\operatorname{div}\boldsymbol{\mathcal{N}}\mathbf{U}) = \operatorname{curl}(\boldsymbol{\mathcal{N}}\operatorname{curl}\mathbf{U}) - \nabla(\mathcal{N}\operatorname{div}\mathbf{U})$. □

Note that in this decomposition neither $\operatorname{curl} \mathbf{A}$ nor $-\nabla p$ are necessarily compactly supported. This makes the following result useful, for which we refer to the works of Bramble and Pasciak[12, Proposition 3.2], Pasciak and Zhao[20, Lemma 2.2], as well as Hiptmair and Pechstein[21, Remark 3].

Lemma 3. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then any $\mathbf{U} \in \mathbf{H}_0(\operatorname{curl};\Omega)$ can be decomposed as:

$$\mathbf{U} = \mathbf{W} - \nabla P, \tag{16}$$

for some $P \in H^1(\Omega)$, where $\mathbf{W} \in \mathbf{H}^1_0(\Omega)$ satisfies: $\|\mathbf{W}\|_{\mathbf{H}^1(\Omega)} \leq C\|\operatorname{curl}\mathbf{U}\|_{\mathbf{L}^2(\Omega)}$.

Finally, the tangential trace spaces also allow similar decompositions. We only require the result for $\mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\operatorname{div}_\Gamma;\Gamma)$.

Lemma 4 (Hodge Decomposition[16, Theorem 5.5])**.** Let $\Gamma = \partial\Omega$ be the boundary of a handle-free, bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. Then any $\mathbf{s} \in \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\operatorname{div}_\Gamma;\Gamma)$ can be uniquely decomposed as:

$$\mathbf{s} = \mathbf{curl}_\Gamma\, p - \nabla_\Gamma\, q, \tag{17}$$

where $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ and $q \in H^1(\Gamma)/\mathbb{R}$ with $-\Delta_\Gamma q \in H^{-\frac{1}{2}}(\Gamma)$ are uniquely determined up to a constant on each connected component Γ_i, $i = 0, \dots, \beta_2$ of the boundary.

2.6 | Trace Jumps and a Representation Formula

The trace operators introduced in the previous sections were all defined with respect to the domain Ω. One can instead also consider the corresponding traces with respect to the complementary domain $\Omega^C := \mathbb{R}^3 \setminus \overline{\Omega}$. These *one-sided* traces exist whenever the restriction of a vector field $\mathbf{U} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ to the domains Ω and Ω^C is sufficiently smooth. If $\mathbf{U}$ is smooth *across* Γ, then the one-sided traces coincide. Otherwise the difference of these traces is denoted by the jump operator $[\![\cdot]\!]$. For example, for $\boldsymbol{\rho}$ one writes: $[\![\boldsymbol{\rho}\mathbf{U}]\!] := \boldsymbol{\rho}\mathbf{U}|_\Omega - \boldsymbol{\rho}^C\mathbf{U}|_{\Omega^C}$. The importance of the jump operator lies in the following representation formula.

Lemma 5 (Representation Formula[22, Sec. 4.2]). Let $\mathbf{U} \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ fulfil:

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{U} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \Gamma, \\ \operatorname{div} \mathbf{U} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{U}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{18}$$

Then $\mathbf{U} = \mathcal{N}(-\boldsymbol{\Delta}\mathbf{U})$ and $-\boldsymbol{\Delta}\mathbf{U} = \boldsymbol{\tau}' [\![\boldsymbol{\rho} \operatorname{curl} \mathbf{U}]\!]$.

Proof. The fact that $\mathbf{U} = \mathcal{N}(-\boldsymbol{\Delta}\mathbf{U})$ directly follows from Lemma 1. Because $\operatorname{div} \mathbf{U} = 0$ globally, we have $-\boldsymbol{\Delta}\mathbf{U} = \operatorname{curl}(\operatorname{curl} \mathbf{U})$ and $\operatorname{curl}(\operatorname{curl} \mathbf{U})|_{\mathbb{R}^3 \setminus \Gamma} = \mathbf{0}$. We thus obtain for all $\mathbf{V} \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle -\boldsymbol{\Delta}\mathbf{U}, \mathbf{V} \rangle = \langle \operatorname{curl} \mathbf{U}, \operatorname{curl} \mathbf{V} \rangle = \int_\Omega \operatorname{curl} \mathbf{U} \cdot \operatorname{curl} \mathbf{V} \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3 \setminus \overline{\Omega}} \operatorname{curl} \mathbf{U} \cdot \operatorname{curl} \mathbf{V} \,\mathrm{d}\mathbf{x}, \tag{19}$$

where we used that $\operatorname{curl} \mathbf{U} \in \mathbf{L}^2_{\text{loc}}(\mathbb{R}^3)$ since $\mathbf{U} \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$. Now, by definition of the rotated tangential trace:

$$\langle \boldsymbol{\rho} \operatorname{curl} \mathbf{U}, \boldsymbol{\tau} \mathbf{V} \rangle_\Gamma = \int_\Omega \operatorname{curl} \mathbf{U} \cdot \operatorname{curl} \mathbf{V} - \underbrace{\operatorname{curl}(\operatorname{curl} \mathbf{U})}_{=\mathbf{0}} \cdot \mathbf{V} \,\mathrm{d}\mathbf{x} = \int_\Omega \operatorname{curl} \mathbf{U} \cdot \operatorname{curl} \mathbf{V} \,\mathrm{d}\mathbf{x}. \tag{20}$$

Applying the same methodology to the integral over $\mathbb{R}^3 \setminus \overline{\Omega}$ and using the definition of $\boldsymbol{\rho}^C$ yields the desired result, because the fact that $\mathbf{V}$ is smooth across Γ guarantees that $[\![\boldsymbol{\tau} \mathbf{V}]\!] = \mathbf{0}$. □

3 | VELOCITY FIELDS

In this section we prove the existence of velocity fields solving (1) as claimed in Item 1. The abstract integrability condition is reformulated in Lemma 8. The uniqueness result for velocity fields presented in Item 3 is also covered.

3.1 | Existence of Velocity Fields

Theorem 1. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. The div-curl system

$$\begin{cases} \operatorname{curl} \mathbf{U} = \mathbf{F} \\ \operatorname{div} \mathbf{U} = 0 \end{cases} \quad \text{in } \Omega \tag{21}$$

has a solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ if and only if $\mathbf{F}$ lies in the space $\mathbf{H}^{-1}(\Omega)$ and fulfils the following integrability condition:

$$\langle \mathbf{F}, \mathbf{V} \rangle = 0 \qquad \forall \mathbf{V} \in \ker \operatorname{curl} \Big|_{\mathbf{H}^1_0(\Omega)}. \tag{22}$$

Remark 1. A more general result was proven by Bramble and Pasciak,[12, Theorem 4.1] using entirely different techniques. We follow another route that sheds new light on the classical conditions imposed on $\mathbf{F}$ and that helps in the construction of vector potentials later.

Remark 2. The condition $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ might seem unnatural, because for an arbitrary vector-field $\mathbf{U} \in \mathbf{L}^2(\Omega)$ it holds that

$$\forall \mathbf{V} \in \mathcal{D}(\Omega): \quad \langle \operatorname{curl} \mathbf{U}, \mathbf{V} \rangle = \int_\Omega \mathbf{U} \cdot \operatorname{curl} \mathbf{V} \,\mathrm{d}\mathbf{x} \leq \|\mathbf{U}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{V}\|_{\mathbf{H}(\operatorname{curl};\Omega)}. \tag{23}$$

The distribution $\operatorname{curl} \mathbf{U} \in \mathcal{D}'(\Omega)$ thus admits a unique continuous extension to $\overline{\mathcal{D}(\Omega)}^{\mathbf{H}(\operatorname{curl};\Omega)} = \mathbf{H}_0(\operatorname{curl};\Omega)$ and the associated operator

$$\operatorname{curl} \Big|_{\mathbf{L}^2(\Omega)} : \mathbf{L}^2(\Omega) \to \mathbf{H}_0(\operatorname{curl};\Omega)' \tag{24}$$

is continuous. On the one hand, any solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of (21) must therefore also necessarily fulfil $\mathbf{F} = \operatorname{curl} \mathbf{U} \in \mathbf{H}_0(\operatorname{curl};\Omega)'$. On the other hand, we have the *proper* inclusion $\mathbf{H}_0(\operatorname{curl};\Omega)' \subsetneq \mathbf{H}^{-1}(\Omega)$. The following result resolves this issue.

Lemma 6. The two spaces:

$$\left\{ \mathbf{F} \in \mathbf{H}_0(\mathrm{curl};\Omega)' \,\middle|\, \langle \mathbf{F}, \mathbf{V} \rangle = 0 \quad \forall \mathbf{V} \in \ker \mathrm{curl}\big|_{\mathbf{H}_0(\mathrm{curl};\Omega)} \right\}$$

and

$$\left\{ \mathbf{F} \in \mathbf{H}^{-1}(\Omega) \,\middle|\, \langle \mathbf{F}, \mathbf{V} \rangle = 0 \quad \forall \mathbf{V} \in \ker \mathrm{curl}\big|_{\mathbf{H}_0^1(\Omega)} \right\}$$

coincide with equivalent norms: $\|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} \leq \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'} \leq C\|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$.

Proof. "⊂" The first inclusion is trivial because of the continuous embedding $\mathbf{H}_0(\mathrm{curl};\Omega)' \hookrightarrow \mathbf{H}^{-1}(\Omega)$. We thus also immediately obtain the first inequality $\|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} \leq \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}$.

"⊃" Let $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ be as above, and let $\mathbf{V} \in \mathcal{D}(\Omega)$ be arbitrary. We use the decomposition from Lemma 3 and write $\mathbf{V} = \mathbf{W} - \nabla P$ for some $P \in H^1(\Omega)$ and $\mathbf{W} \in \mathbf{H}_0^1(\Omega)$ that satisfies $\|\mathbf{W}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathrm{curl}\,\mathbf{V}\|_{\mathbf{L}^2(\Omega)}$. As $\mathbf{W} \in \mathbf{H}_0^1(\Omega)$, we necessarily also have $-\nabla P \in \mathbf{H}_0^1(\Omega)$ and we may write:

$$\langle \mathbf{F}, \mathbf{V} \rangle = \langle \mathbf{F}, \mathbf{W} \rangle - \underbrace{\langle \mathbf{F}, \nabla P \rangle}_{=0} \leq \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}\|\mathbf{W}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}\|\mathbf{V}\|_{\mathbf{H}(\mathrm{curl};\Omega)}. \tag{25}$$

The distribution $\mathbf{F} \in \mathcal{D}'(\Omega)$ thus admits a unique continuous extension to $\overline{\mathcal{D}(\Omega)}^{\mathbf{H}(\mathrm{curl};\Omega)} = \mathbf{H}_0(\mathrm{curl};\Omega)$ and thus $\mathbf{F} \in \mathbf{H}_0(\mathrm{curl};\Omega)'$ with $\|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'} \leq C\|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$. □

Lemma 7. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain.The equation

$$\mathrm{curl}\,\mathbf{W} = \mathbf{F} \tag{26}$$

has a solution $\mathbf{W} \in \mathbf{L}^2(\Omega)$ if and only if $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{F}$ fulfils the integrability condition (22).

Proof. The continuous operator

$$\mathrm{curl}\big|_{\mathbf{H}_0(\mathrm{curl};\Omega)} : \mathbf{H}_0(\mathrm{curl};\Omega) \to \mathbf{L}^2(\Omega) \tag{27}$$

has closed range.[7, Box 3.1] The curl operator is symmetric and the dual of the mapping (27) is the operator $\mathrm{curl}\big|_{\mathbf{L}^2(\Omega)}$ given in (24). Hence, Banach's closed range theorem yields

$$\mathrm{Range}\left(\mathrm{curl}\big|_{\mathbf{L}^2(\Omega)}\right) = \left(\ker \mathrm{curl}\big|_{\mathbf{H}_0(\mathrm{curl};\Omega)}\right)^0 \tag{28}$$

That is,

$$\mathrm{Range}\left(\mathrm{curl}\big|_{\mathbf{L}^2(\Omega)}\right) = \left\{ \mathbf{F} \in \mathbf{H}_0(\mathrm{curl};\Omega)' \,\middle|\, \langle \mathbf{F}, \mathbf{V} \rangle = 0 \quad \forall \mathbf{V} \in \ker \mathrm{curl}\big|_{\mathbf{H}_0(\mathrm{curl};\Omega)} \right\}. \tag{29}$$

Evidently, problem (26) has a solution if and only if $\mathbf{F} \in \mathrm{Range}\left(\mathrm{curl}\big|_{\mathbf{L}^2(\Omega)}\right)$. Thus, together with Lemma 6, the claim follows. □

Proof of Theorem 1. Lemma 7 guarantees the existence of a $\mathbf{W} \in \mathbf{L}^2(\Omega)$ such that $\mathrm{curl}\,\mathbf{W} = \mathbf{F}$. This function does not necessarily fulfil $\mathrm{div}\,\mathbf{W} = 0$. But in this case we let $P \in H_0^1(\Omega)$ denote the unique solution to the Poisson problem

$$-\Delta P = \mathrm{div}\,\mathbf{W} \qquad \text{in } \Omega, \tag{30}$$

and note that $\mathbf{U} := \mathbf{W} + \nabla P$ solves the div-curl system (21). □

The integrability condition (22) is most natural for the chosen method of proof. However, it is hard to verify in practice. For this reason, it is worthwhile considering equivalent alternative conditions.

Lemma 8. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and let $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$. Together, the following conditions are equivalent to the integrability condition (22):

$$\mathrm{div}\,\mathbf{F} = 0 \qquad \text{in } \Omega, \tag{31a}$$

$$\langle \mathbf{F}, -\nabla T_i\big|_{\Omega} \rangle = 0 \qquad i = 1, \dots, \beta_2. \tag{31b}$$

If in particular $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div} \mathbf{F} = 0$, condition (31b) is equivalent to:

$$\int_{\Gamma_i} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = 0 \qquad i = 1, \dots, \beta_2. \tag{32}$$

Remark 3. Notice that definition (8) guarantees that $-\nabla T_i\big|_\Omega \in \mathcal{D}(\Omega)$.

Remark 4. Together (31a) and (31b) also imply that $\langle \mathbf{F}, -\nabla T_0|_\Omega \rangle = 0$ holds. If in particular $\beta_2 = 0$, it suffices to demand $\operatorname{div} \mathbf{F} = 0$.

Proof. "$\Rightarrow$" Since $\operatorname{curl} \circ (-\nabla) \equiv \mathbf{0}$, conditions (31a) and (31b) are immediately seen to be necessary from the definitions.

"$\Leftarrow$" To see that they also are sufficient, let $\mathbf{V} \in \ker \operatorname{curl}\big|_{\mathbf{H}_0^1(\Omega)}$ be arbitrary. We may extend this function by zero outside Ω:

$$\widetilde{\mathbf{V}} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{x} \mapsto \begin{cases} \mathbf{V}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \mathbf{0} & \text{else.} \end{cases} \tag{33}$$

Since $\boldsymbol{\rho}(\mathbf{V}) = \mathbf{0}$, we have $\operatorname{curl} \widetilde{\mathbf{V}} = \mathbf{0}$ on all of $\mathbb{R}^3$. Since its support is compact, we may use the Helmholtz decomposition (15) to rewrite this extension in terms of

$$\widetilde{\mathbf{V}} = \operatorname{curl} \mathcal{N} \underbrace{\operatorname{curl} \widetilde{\mathbf{V}}}_{=\mathbf{0}} - \nabla \underbrace{\mathcal{N} \operatorname{div} \widetilde{\mathbf{V}}}_{=:\widetilde{P}} = -\nabla \widetilde{P}. \tag{34}$$

The restriction $P := \widetilde{P}\big|_\Omega$ belongs to $H^1(\Omega)$, because $-\nabla P = \mathbf{V} \in \mathbf{L}^2(\Omega)$. Moreover, we see from $\boldsymbol{\tau}(\mathbf{V}) = \mathbf{0}$ that $P = C_i$ for some constant $C_i \in \mathbb{R}$ on each connected component Γ_i of the boundary, $i = 0, 1, \dots, \beta_2$. Because $\widetilde{P} \to 0$ at infinity, $-\nabla \widetilde{P} = \mathbf{0}$ outside Ω, and $\widetilde{P} \in H^1_{\text{loc}}(\mathbb{R}^3)$ we have $C_0 = 0$. From the decomposition

$$P = \underbrace{\left(P - \sum_{i=1}^{\beta_2} C_i T_i\Big|_\Omega \right)}_{=:P_0 \in H_0^1(\Omega)} + \sum_{i=1}^{\beta_2} C_i T_i\Big|_\Omega, \tag{35}$$

we obtain

$$\langle \mathbf{F}, \mathbf{V} \rangle = \langle \mathbf{F}, -\nabla P \rangle = \underbrace{\langle \mathbf{F}, -\nabla P_0 \rangle}_{=0,\ (31a)} + \sum_{i=1}^{\beta_2} C_i \underbrace{\langle \mathbf{F}, -\nabla T_i\big|_\Omega \rangle}_{=0,\ (31b)} = 0. \tag{36}$$

Thus (22) is equivalent to the combination of (31a) and (31b).

Finally, the equivalence of (31b) and (32) directly follows from the definition of the normal trace: if $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div} \mathbf{F} = 0$, we also have $\mathbf{F} \in \mathbf{H}(\operatorname{div}; \Omega)$. Thus $\mathbf{F}$ has a well-defined normal trace and by definition:

$$\int_{\Gamma_i} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \langle \nu \mathbf{F}, \gamma T_i \rangle_\Gamma = \int_\Omega \mathbf{F} \cdot \nabla T_i \, \mathrm{d}\mathbf{x} + \int_\Omega \underbrace{\operatorname{div} \mathbf{F}}_{=0} T_i \, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{F} \cdot \nabla T_i \, \mathrm{d}\mathbf{x}. \tag{37}$$

□

3.2 | Uniqueness of Velocity Fields

Theorem 2. Let $\Omega \subset \mathbb{R}^3$ be a bounded, handle-free Lipschitz domain and let $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ fulfil the integrability condition (22). Additionally, let $g \in H^{-\frac{1}{2}}(\Gamma)$ be given such that $\langle g, 1 \rangle_\Gamma = 0$. Then the div-curl system (21) has exactly one solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ with $\mathbf{U} \cdot \mathbf{n} = g$ on Γ.

Remark 5. The normal trace $\mathbf{U} \cdot \mathbf{n}$ is well-defined because the solution of the div-curl system satisfies $\operatorname{div} \mathbf{U} = 0$.

Proof. Let us first remark that the condition $\langle g, 1 \rangle_\Gamma = 0$ is necessary. To see this, note that because $\operatorname{div} \mathbf{U} = 0$, any solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of the div-curl system (21) must fulfil:

$$\int_\Gamma \mathbf{U} \cdot \mathbf{n} \, \mathrm{d}S = \int_\Omega \operatorname{div} \mathbf{U} \, \mathrm{d}\mathbf{x} = 0. \tag{38}$$

Now let $\mathbf{W} \in \mathbf{L}^2(\Omega)$ denote any solution of the div-curl system, whose existence is guaranteed by Theorem 1. Let $P \in H^1(\Omega)/\mathbb{R}$ be the unique solution of the Neumann problem:

$$\begin{cases} -\Delta P = 0 & \text{in } \Omega, \\ -\nabla P \cdot \mathbf{n} = g - \mathbf{W} \cdot \mathbf{n} & \text{on } \Gamma. \end{cases} \tag{39}$$

Then the function $\mathbf{U} := \mathbf{W} - \nabla P$ fulfils the conditions of the theorem.

To see that it is unique, let $\mathbf{U}_1, \mathbf{U}_2 \in \mathbf{L}^2(\Omega)$ denote two solutions of the div-curl system (21) that fulfil $\mathbf{U}_1 \cdot \mathbf{n} = \mathbf{U}_2 \cdot \mathbf{n} = g$ on the boundary Γ. Then their difference $\mathbf{D} := \mathbf{U}_1 - \mathbf{U}_2$ solves

$$\begin{cases} \operatorname{div} \mathbf{D} = 0 & \text{in } \Omega, \\ \operatorname{curl} \mathbf{D} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{D} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{40}$$

In other words, $\mathbf{D}$ is a so-called Neumann harmonic field. These functions form a space of dimension β_1.[6, Proposition 3.14 7, Section 4.3] By hypothesis $\beta_1 = 0$, and thus $\mathbf{D} = \mathbf{0}$. □

The above proof hints at what needs to be done in order to recover uniqueness in the case $\beta_1 \neq 0$. One needs to prescribe β_1 functionals that determine the Neumann harmonic components of $\mathbf{U}$. A construction of these fields and corresponding functionals can be found in the work of Amrouche et al.[6]

4 | STREAM FUNCTIONS

We prove the existence result for stream functions of Item 2 in this section. The related uniqueness statement of Item 3 is proven in Theorem 4.

4.1 | Existence of Stream Functions

The following theorem is a variant of a result by Girault and Raviart.[5, Theorem 3.4] We give a different proof, which uses the Newton operator instead of Fourier transforms.

Theorem 3. Let $\Omega \subset \mathbb{R}^3$ denote a bounded Lipschitz domain. Then $\mathbf{U} \in \mathbf{L}^2(\Omega)$ satisfies

$$\operatorname{div} \mathbf{U} = 0, \qquad \text{in } \Omega, \tag{41a}$$

$$\int_{\Gamma_i} \mathbf{U} \cdot \mathbf{n} \, \mathrm{d}S = 0, \qquad i = 1, \dots, \beta_2, \tag{41b}$$

if and only if there exists a vector-field $\mathbf{A} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{curl} \mathbf{A} = \mathbf{U} \quad \text{in } \Omega, & -\boldsymbol{\Delta} \mathbf{A} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \operatorname{div} \mathbf{A} = 0 \quad \text{in } \mathbb{R}^3, & \mathbf{A}(\mathbf{x}) \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{42}$$

Proof. "⇐" Note that the conditions (41a) and (41b) are exactly the integrability conditions (31a) and (32). Because of Lemma 7 and Lemma 8, these conditions are necessary to ensure the existence of a vector-field $\mathbf{A} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$ in Ω.

"⇒" In order to show sufficiency, the idea is to extend $\mathbf{U}$ to $\mathbb{R}^3$ by "potential flows" matching $\mathbf{U} \cdot \mathbf{n}$ on Γ, then use the Newton operator.

We want to exploit the following scalar functions. For $i = 0$, we let $P_0 \in H^1_{\mathrm{loc}}(\Theta_0)$ denote the solution of the problem:

$$\begin{cases} -\Delta P_0 = 0 & \text{in } \Theta_0, \\ -\nabla P_0 \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n} & \text{on } \Gamma_0, \\ P_0(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{43}$$

whereas for $i = 1, \dots, \beta_2$ we define $P_i \in H^1(\Omega_i)/\mathbb{R}$ as the solution of:

$$\begin{cases} -\Delta P_i = 0 & \text{in } \Theta_i, \\ -\nabla P_i \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n} & \text{on } \Gamma_i. \end{cases} \tag{44}$$

Because of condition (41b), it is well-known that these problems are well-posed.

We are now ready to extend $\mathbf{U}$ to the whole space as

$$\widetilde{\mathbf{U}} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{x} \mapsto \begin{cases} \mathbf{U}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ -\nabla P_i(\mathbf{x}) & \mathbf{x} \in \Theta_i,\ i = 0, ..., \beta_2. \end{cases} \tag{45}$$

Because $[\![\widetilde{\mathbf{U}} \cdot \mathbf{n}]\!] = 0$, we have $\operatorname{div} \widetilde{\mathbf{U}} = 0$ on $\mathbb{R}^3$. Since $\operatorname{supp}\left(\operatorname{curl} \widetilde{\mathbf{U}}\right) \subset \overline{\Omega}$, $\operatorname{curl} \widetilde{\mathbf{U}} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ is compactly supported, and we may define $\mathbf{A} := \mathcal{N} \operatorname{curl} \widetilde{\mathbf{U}}$.

We now claim that $\operatorname{curl} \mathbf{A} = \widetilde{\mathbf{U}}$ on $\mathbb{R}^3$. From the properties of $\mathcal{N}$ it follows that $\mathbf{A} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\mathbf{A}(\mathbf{x}) \to \mathbf{0}$ at infinity. Because $\mathcal{N}$ commutes with differentiation, we have $\operatorname{div} \mathbf{A} = \mathcal{N} \operatorname{div} \operatorname{curl} \widetilde{\mathbf{U}} = 0$ on $\mathbb{R}^3$, and thus:

$$\begin{cases} \operatorname{curl}(\operatorname{curl} \mathbf{A}) = & -\Delta \mathbf{A} & = \operatorname{curl} \widetilde{\mathbf{U}} \\ \operatorname{div}(\operatorname{curl} \mathbf{A}) = & 0 & = \operatorname{div} \widetilde{\mathbf{U}} \end{cases} \quad \text{on } \mathbb{R}^3. \tag{46}$$

The difference $\mathbf{D} := \operatorname{curl} \mathbf{A} - \widetilde{\mathbf{U}}$ therefore fulfils:

$$\begin{cases} -\Delta \mathbf{D} = \mathbf{0} & \text{on } \mathbb{R}^3, \\ \mathbf{D}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{47}$$

so that from Lemma 1 we conclude $\mathbf{D} = \mathcal{N}\mathbf{0} = \mathbf{0}$, that is $\operatorname{curl} \mathbf{A} = \widetilde{\mathbf{U}}$. □

4.2 | Uniqueness of Stream Functions

For a given velocity field $\mathbf{U}$, a stream function $\mathbf{A}$ can be constructed as in the proof of Theorem 3. In the following section, however, we construct a stream function $\mathbf{A}$ directly from $\mathbf{F}$, so there the extended velocity field is *a-priori unknown*. The following result allows us to establish that the stream functions from Theorem 3 and Section 5 coincide if $\beta_1 = 0$. This in turn will also allow us to establish higher regularity in Theorem 6. Note that one obtains uniqueness of $\mathbf{A}$, while *neither* explicitly referring to the extension of $\mathbf{U}$ outside Ω, *nor* to the boundary values of $\mathbf{A}$ on Γ. The proof also gives motivation for the constructions presented later.

Theorem 4. Let $\Omega \subset \mathbb{R}^3$ be a handle-free, bounded Lipschitz domain ($\beta_1 = 0$) and let $\mathbf{U} \in \mathbf{L}^2(\Omega)$ fulfil the conditions of Theorem 3. Then there exists exactly one vector-field $\mathbf{A} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ satisfying (42).

Proof. Suppose that $\mathbf{A}_1$ and $\mathbf{A}_2$ are two vector-fields in $\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ satisfying (42). Then their difference $\mathbf{D} := \mathbf{A}_1 - \mathbf{A}_2 \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ fulfils:

$$\begin{cases} \operatorname{curl} \mathbf{D} = \mathbf{0} & \text{in } \Omega, & -\Delta \mathbf{D} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \Gamma, \\ \operatorname{div} \mathbf{D} = 0 & \text{in } \mathbb{R}^3, & \mathbf{D}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{48}$$

Thus, Lemma 5 is applicable, yielding $\mathbf{D} = \mathcal{N}(-\Delta \mathbf{D})$ and $-\Delta \mathbf{D} = \boldsymbol{\tau}'[\![\boldsymbol{\rho} \operatorname{curl} \mathbf{D}]\!]$. From the mapping properties of $\boldsymbol{\rho}$ it follows that $\mathbf{s} := [\![\boldsymbol{\rho} \operatorname{curl} \mathbf{D}]\!] \in \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\operatorname{div}_\Gamma; \Gamma)$. The Hodge decomposition from Lemma 4 furthermore yields

$$\mathbf{s} := [\![\boldsymbol{\rho} \operatorname{curl} \mathbf{D}]\!] = \mathbf{curl}_\Gamma\, p - \nabla_\Gamma\, q \tag{49}$$

for some functions $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$, $q \in H^1(\Gamma)/\mathbb{R}$ that are uniquely determined up to a constant on each connected part Γ_i, $i = 0, \dots, \beta_2$ of the boundary. It thus suffices to establish that $p = q = 0$.

We first consider q. The fact that $\operatorname{div} \mathbf{D} = 0$ on $\mathbb{R}^3$ implies that for all $V \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle \operatorname{div}_\Gamma \mathbf{s}, \gamma V \rangle_\Gamma = \langle \mathbf{s}, -\nabla_\Gamma \gamma V \rangle_\Gamma = \langle \mathbf{s}, \boldsymbol{\tau}(-\nabla V) \rangle_\Gamma = \langle -\Delta \mathbf{D}, -\nabla V \rangle = \langle \operatorname{div}(-\Delta \mathbf{D}), V \rangle = \langle -\Delta \operatorname{div} \mathbf{D}, V \rangle = 0, \tag{50}$$

that is $\operatorname{div}_\Gamma \mathbf{s} = 0$. This in turn implies $-\Delta_\Gamma\, q = \operatorname{div}_\Gamma \mathbf{s} = 0$, and by the coercivity of the Laplace–Beltrami operator on $H^1(\Gamma)/\mathbb{R}$ we conclude that $q = 0$.

Considering p, we note that because $\operatorname{curl} \mathbf{D} = \mathbf{0}$ in Ω, we have $0 = (\operatorname{curl} \mathbf{D}) \cdot \mathbf{n} = \operatorname{curl}_\Gamma \boldsymbol{\tau}\mathbf{D}$ on Γ. Thus for all $v \in H^{\frac{1}{2}}(\Gamma)$:

$$0 = \langle \operatorname{curl}_\Gamma \boldsymbol{\tau}\mathbf{D}, v \rangle_\Gamma = \langle \boldsymbol{\tau}\mathbf{D}, \mathbf{curl}_\Gamma\, v \rangle_\Gamma = \langle \boldsymbol{\tau}\mathcal{N}\boldsymbol{\tau}'\mathbf{s}, \mathbf{curl}_\Gamma\, v \rangle_\Gamma = \langle \boldsymbol{\tau}\mathcal{N}\boldsymbol{\tau}'\, \mathbf{curl}_\Gamma\, p, \mathbf{curl}_\Gamma\, v \rangle_\Gamma. \tag{51}$$

The last expression can be enlightened using a more explicit representation. Following Claeys and Hiptmair,[22, Equations (41) and (42)] under the additional assumption that $\mathbf{curl}_\Gamma\, p, \mathbf{curl}_\Gamma\, v \in \mathbf{L}^\infty(\Gamma)$, we have:

$$\langle \boldsymbol{\tau}\mathcal{N}\boldsymbol{\tau}'\, \mathbf{curl}_\Gamma\, p, \mathbf{curl}_\Gamma\, v \rangle_\Gamma = \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{\mathbf{curl}_\Gamma\, p(\mathbf{y}) \cdot \mathbf{curl}_\Gamma\, v(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}S(\mathbf{y})\, \mathrm{d}S(\mathbf{x}). \tag{52}$$

Here one clearly recognises the hypersingular boundary integral operator for the scalar Laplace equation.[15, Section 3.3.4] This operator is known to be coercive on $H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$,[15, Theorem 3.5.3] and we conclude that $p = 0$. □

Let us now make some remarks on the case $\beta_1 \neq 0$. We define $\Omega^C := \mathbb{R}^3 \setminus \overline{\Omega}$ as the complementary domain of Ω, and $\mathbf{B} := \operatorname{curl} \mathbf{D}|_{\Omega}$ and $\mathbf{B}^C := \operatorname{curl} \mathbf{D}|_{\Omega^C}$. These functions are Neumann harmonic fields:

$$\begin{aligned} \operatorname{div} \mathbf{B} &= 0, & \operatorname{curl} \mathbf{B} &= \mathbf{0} \quad \text{in } \Omega, & \mathbf{B} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma, \\ \operatorname{div} \mathbf{B}^C &= 0, & \operatorname{curl} \mathbf{B}^C &= \mathbf{0} \quad \text{in } \Omega^C, & \mathbf{B}^C \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{53}$$

Ultimately, the idea is to rely on the fact that in handle-free domains the space of Neumann harmonic fields only contains the zero element, and thus $\mathbf{B} = \mathbf{0}$ and $\mathbf{B}^C = \mathbf{0}$. In the case $\beta_1 \neq 0$, however, neither Ω nor Ω^C are handle-free, and in fact we have $\beta_1^C = \beta_1$. The spaces of Neumann harmonic fields on Ω and Ω^C then each have dimension β_1.

Buffa has derived the analogue of Lemma 4 for the case of Lipschitz polyhedra with $\beta_1 \neq 0$.[23] Because $\beta_1 = \beta_1^C$, it contains an additional term from the $2\beta_1$-dimensional space of harmonic tangential fields. Half of these components are fixed because of the condition $\operatorname{curl} \mathbf{A}|_{\Omega} = \mathbf{U}$, the other half concerns the external harmonic fields. To ensure uniqueness of $\mathbf{A}$, one additionally needs to prescribe the Neumann harmonic components of $\mathbf{U}^C := \operatorname{curl} \mathbf{A}|_{\Omega^C}$.

5 | CONSTRUCTION OF SOLUTIONS AND WELL-POSEDNESS

In this section, we provide a construction for a stream function $\mathbf{A} \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ for the general case of a given vorticity field $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$. This construction may also be considered an alternative proof of the existence results of Theorems 1 and 3.

The idea is to first find a suitable extension $\widetilde{\mathbf{F}} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ of $\mathbf{F}$, that additionally satisfies $\operatorname{div} \widetilde{\mathbf{F}} \in H^{-1}(\mathbb{R}^3)$. The spurious divergence of $\widetilde{\mathbf{F}}$ can then be cancelled out using a surface functional, and the problem can be solved by applying the Newton operator. In order for this approach to work it is crucial to make use of Lemma 6 and to interpret $\mathbf{F}$ as a member of $\mathbf{H}_0(\operatorname{curl}; \Omega)'$. Otherwise, one will usually only obtain $\operatorname{div} \widetilde{\mathbf{F}} \in H^{-2}(\mathbb{R}^3)$ and the construction will fail.

In computational practice, one will often have $\mathbf{F} \in \mathbf{L}^2(\Omega)$. Under this assumption we can simplify the construction, yielding an algorithm that is more easily implementable.

5.1 | General Vorticity Fields

Let $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ be given and suppose that it fulfils the integrability condition (22), or the equivalent conditions (31a) and (31b). Furthermore, let $g \in H^{-\frac{1}{2}}(\Gamma)$ be given boundary data such that $\langle g, 1 \rangle_{\Gamma_i} = 0$, $i = 0, \dots, \beta_2$. Because of Lemma 6, we also have $\mathbf{F} \in \mathbf{H}_0(\operatorname{curl}; \Omega)'$. Let $\mathbf{R} \in \mathbf{H}_0(\operatorname{curl}; \Omega)$ denote the Riesz representative of $\mathbf{F}$, i. e., the uniquely determined function $\mathbf{R}$ such that for all $\mathbf{V} \in \mathbf{H}_0(\operatorname{curl}; \Omega)$:

$$\underbrace{\int_{\Omega} \mathbf{R} \cdot \mathbf{V} + \operatorname{curl} \mathbf{R} \cdot \operatorname{curl} \mathbf{V} \, \mathrm{d}\mathbf{x}}_{=:\mathfrak{B}(\mathbf{R},\mathbf{V})} = \langle \mathbf{F}, \mathbf{V} \rangle. \tag{54}$$

The expression $\mathfrak{B}(\mathbf{R}, \mathbf{V})$ is not only well-defined for $\mathbf{V} \in \mathbf{H}_0(\operatorname{curl}; \Omega)$, but also for any $\mathbf{V} \in \mathbf{H}_0(\operatorname{curl}; \mathbb{R}^3)$. We thus define $\widetilde{\mathbf{F}}$ as follows:

$$\forall \mathbf{V} \in \mathbf{H}_0(\operatorname{curl}; \mathbb{R}^3) : \ \langle \widetilde{\mathbf{F}}, \mathbf{V} \rangle := \mathfrak{B}(\mathbf{R}, \mathbf{V}). \tag{55}$$

Obviously $\widetilde{\mathbf{F}}$ extends $\mathbf{F}$ and is compactly supported with $\operatorname{supp} \widetilde{\mathbf{F}} \subset \overline{\Omega}$. Additionally we also immediately obtain that $\|\widetilde{\mathbf{F}}\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$. This extension does not necessarily fulfil $\operatorname{div} \widetilde{\mathbf{F}} = 0$ on all of $\mathbb{R}^3$. However, the following result is useful.

Lemma 9. One has $\operatorname{div} \widetilde{\mathbf{F}} \in H^{-1}(\mathbb{R}^3)$. Moreover, there exists a uniquely determined surface functional $f \in H^{-\frac{1}{2}}(\Gamma)$ such that:

$$\langle \operatorname{div} \widetilde{\mathbf{F}}, V \rangle = -\langle f, \gamma V \rangle_{\Gamma} \qquad \forall V \in H^1(\mathbb{R}^3), \tag{56}$$

and

$$\begin{cases} \langle f, 1 \rangle_{\Gamma_i} = 0, & i = 0, \dots, \beta_2, \\ \|f\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}. \end{cases} \tag{57}$$

Proof. First note that $\forall V \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle \operatorname{div} \widetilde{\mathbf{F}}, V \rangle = \langle \widetilde{\mathbf{F}}, -\nabla V \rangle = \mathfrak{B}(\mathbf{R}, -\nabla V) = \int_\Omega \mathbf{R} \cdot (-\nabla V)\,\mathrm{d}\mathbf{x} \le \|\mathbf{R}\|_{\mathbf{L}^2(\Omega)} \|-\nabla V\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'} \|V\|_{H^1(\Omega)}. \tag{58}$$

The distribution $\operatorname{div} \widetilde{\mathbf{F}} \in \mathcal{D}'(\mathbb{R}^3)$ thus admits a unique continuous extension to $\overline{\mathcal{D}(\mathbb{R}^3)}^{\|\cdot\|_{H^1(\mathbb{R}^3)}} = H^1(\mathbb{R}^3)$, and we may write $\operatorname{div} \widetilde{\mathbf{F}} \in H^{-1}(\mathbb{R}^3)$ with $\|\operatorname{div} \widetilde{\mathbf{F}}\|_{H^{-1}(\mathbb{R}^3)} \le \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$.

Next, we find that the value $\langle \operatorname{div} \widetilde{\mathbf{F}}, V \rangle$ only depends on the Dirichlet trace $\gamma V \in H^{\frac{1}{2}}(\Gamma)$ of the trial function $V \in H^1(\mathbb{R}^3)$. To see this, let $V_1, V_2 \in H^1(\mathbb{R}^3)$ have the same Dirichlet trace, $\gamma V_1 = \gamma V_2$. Because $\gamma(V_1 - V_2) = 0$, one finds that $-\nabla(V_1 - V_2)|_\Omega \in \mathbf{H}_0(\mathrm{curl};\Omega)$, and thus:

$$\langle \operatorname{div} \widetilde{\mathbf{F}}, V_1 \rangle - \langle \operatorname{div} \widetilde{\mathbf{F}}, V_2 \rangle = \mathfrak{B}\big(\mathbf{R}, -\nabla(V_1 - V_2)\big) = \langle \mathbf{F}, -\nabla(V_1 - V_2)|_\Omega \rangle \overset{(22)}{=} 0. \tag{59}$$

We may thus define $f \in H^{-\frac{1}{2}}(\Gamma)$ as follows:

$$\forall v \in H^{\frac{1}{2}}(\Gamma): \qquad \langle f, v \rangle_\Gamma := -\langle \operatorname{div} \widetilde{\mathbf{F}}, \gamma^{-1} v \rangle, \tag{60}$$

where $\gamma^{-1} : H^{\frac{1}{2}}(\Gamma) \to H^1(\mathbb{R}^3)$ is fixed, but may be any linear and bounded lifting operator. Clearly, we have $\|f\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$, because

$$\langle f, v \rangle_\Gamma = -\langle \operatorname{div} \widetilde{\mathbf{F}}, \gamma^{-1} v \rangle \le \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} \|\gamma^{-1} v\|_{H^1(\mathbb{R}^3)} \le \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} \|\gamma^{-1}\|_{H^{\frac{1}{2}}(\Gamma) \to H^1(\mathbb{R}^3)} \|v\|_{H^{\frac{1}{2}}(\Gamma)} \tag{61}$$

for all $v \in H^{\frac{1}{2}}(\Gamma)$.

Finally, for $i = 0, \dots, \beta_2$, we have

$$\langle f, 1 \rangle_{\Gamma_i} = \langle f, \gamma T_i \rangle_\Gamma = -\langle \operatorname{div} \widetilde{\mathbf{F}}, \gamma^{-1} T_i \rangle = \langle \mathbf{F}, \nabla \gamma^{-1} T_i|_\Omega \rangle \overset{(22)}{=} 0. \tag{62}$$

□

As a consequence of the preceding lemma we may define $q \in H^1(\Gamma)/\mathbb{R}$, uniquely up to a constant on each connected component of the boundary Γ_i, $i = 0, \dots, \beta_2$, as the solution to the Laplace–Beltrami equation:

$$-\Delta_\Gamma q = f \qquad \text{on } \Gamma, \tag{63}$$

and furthermore define $\widehat{\mathbf{F}} := \widetilde{\mathbf{F}} - \boldsymbol{\tau}' \nabla_\Gamma q$, and $\widehat{\mathbf{A}} := \boldsymbol{\mathcal{N}} \widehat{\mathbf{F}}$.

Lemma 10. One has $\widehat{\mathbf{F}} \in \mathbf{H}^{-1}(\mathbb{R}^3) \cap \mathcal{E}'(\mathbb{R}^3)$, $\widehat{\mathbf{A}} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ decaying to zero at infinity, and moreover:

$$\begin{cases} \|\widehat{\mathbf{A}}\|_{\mathbf{H}^1(\Omega)} \lesssim \|\widehat{\mathbf{F}}\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}, & -\boldsymbol{\Delta} \widehat{\mathbf{A}} = \widehat{\mathbf{F}} = \mathbf{F} \quad \text{in } \Omega, \\ \quad \operatorname{div} \widehat{\mathbf{A}} = \operatorname{div} \widehat{\mathbf{F}} = 0 \quad \text{in } \mathbb{R}^3, & -\boldsymbol{\Delta} \widehat{\mathbf{A}} = \widehat{\mathbf{F}} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{64}$$

Proof. It suffices to establish the properties of $\widehat{\mathbf{F}}$; the results for $\widehat{\mathbf{A}}$ then immediately follow from the properties of $\boldsymbol{\mathcal{N}}$. We already established that $\widetilde{\mathbf{F}} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ and $\|\widetilde{\mathbf{F}}\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$. For the surface functional,

$$\|\nabla_\Gamma q\|_{\mathbf{L}^2(\Gamma)} \le \|q\|_{H^1(\Gamma)} \lesssim \|f\|_{H^{-1}(\Gamma)} \lesssim \|f\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}, \tag{65}$$

so that $\forall \mathbf{V} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$: $\langle \nabla_\Gamma q, \boldsymbol{\tau} \mathbf{V} \rangle_\Gamma \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{V}\|_{\mathbf{H}^1(\mathbb{R}^3)}$. Thus $-\boldsymbol{\tau}' \nabla_\Gamma q \in \mathbf{H}^{-1}(\mathbb{R}^3)$ with $\| - \boldsymbol{\tau}' \nabla_\Gamma q\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}$.

The fact that $\widehat{\mathbf{F}} = \mathbf{0}$ on $\mathbb{R}^3 \setminus \overline{\Omega}$ and $\widehat{\mathbf{F}} = \mathbf{F}$ on Ω is obvious. For the divergence, we note that $\forall V \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle \nabla_\Gamma q, \boldsymbol{\tau} \nabla V \rangle_\Gamma = \langle \nabla_\Gamma q, \nabla_\Gamma \gamma V \rangle_\Gamma = \langle -\Delta_\Gamma q, \gamma V \rangle_\Gamma = \langle f, \gamma V \rangle_\Gamma = -\langle \operatorname{div} \widetilde{\mathbf{F}}, V \rangle, \tag{66}$$

and therefore $\operatorname{div} \widehat{\mathbf{F}} = 0$. □

With these properties in place, it immediately follows that $\widehat{\mathbf{U}} := \operatorname{curl} \widehat{\mathbf{A}}$ solves the div-curl system (21), but does not necessarily fulfil $\widehat{\mathbf{U}} \cdot \mathbf{n} = g$ on Γ. To fix its normal component, it then suffices to solve the hypersingular boundary integral equation:

$$\forall v \in H^{\frac{1}{2}}(\Gamma): \quad \langle \boldsymbol{\tau} \boldsymbol{\mathcal{N}} \boldsymbol{\tau}' \, \mathbf{curl}_\Gamma \, p, \mathbf{curl}_\Gamma \, v \rangle_\Gamma = \langle g - \operatorname{curl} \widehat{\mathbf{A}} \cdot \mathbf{n}, v \rangle_\Gamma, \tag{67}$$

for the unknown $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$. This problem is known to be well-posed, and its solution continuously depends on $\widehat{\mathbf{U}} \cdot \mathbf{n}$ and g:[15, Theorem 3.5.3]

$$\|p\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim \|\widehat{\mathbf{U}} \cdot \mathbf{n} - g\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\widehat{\mathbf{U}}\|_{\mathbf{H}(\mathrm{div};\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\widehat{\mathbf{A}}\|_{\mathbf{H}^1(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{68}$$

We now finally define:

$$\begin{cases} \mathbf{s} := \mathbf{curl}_\Gamma\, p - \nabla_\Gamma q \in \mathbf{H}_\mathrm{R}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \\ \mathbf{A} := \mathcal{N}\big(\widetilde{\mathbf{F}} + \boldsymbol{\tau}'\mathbf{s}\big) \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3), \\ \mathbf{U} := \operatorname{curl}\mathbf{A} \in \mathbf{L}^2(\mathbb{R}^3). \end{cases} \tag{69}$$

Then $\mathbf{U}$ solves the div-curl system and $\mathbf{U}\cdot\mathbf{n} = g$ on Γ, and $\mathbf{A}$ is a stream function for $\mathbf{U}$. In the case of a handle-free domain, Theorems 2 and 4 guarantee that these functions are unique. Moreover, the solution continuously depends on $\mathbf{F}$ and g. In total we have therefore proven the following theorem.

Theorem 5. Let $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ be given and fulfil the integrability condition (22), or the equivalent conditions (31a) and (31b). Furthermore, let $g \in H^{-\frac{1}{2}}(\Gamma)$ be given, such that $\langle g, 1\rangle_{\Gamma_i} = 0$ for all $i = 0,\dots,\beta_2$.

Then a solution to the div-curl system (21) with $\mathbf{U}\cdot\mathbf{n} = g$ on Γ, and its associated stream function $\mathbf{A}$ are given by (69). In case of a handle-free domain $\mathbf{U}$ and $\mathbf{A}$ are the uniquely determined functions from Theorems 2 and 4.

These functions linearly and continuously depend on the data $\mathbf{F}$ and g, and we have:

$$\|\mathbf{U}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{A}\|_{\mathbf{H}^1(\Omega)} \lesssim \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{70}$$

5.2 | Square-integrable Vorticity Fields

In case we actually have $\mathbf{F} \in \mathbf{L}^2(\Omega)$, the construction can be simplified. Thus, let $\mathbf{F} \in \mathbf{L}^2(\Omega)$ fulfil the integrability conditions (31a) and (32). We first note that $\mathbf{F}$ now possesses a natural extension by zero:

$$\widetilde{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto \begin{cases} \mathbf{F}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \mathbf{0} & \text{else.} \end{cases} \tag{71}$$

Next, we note that because $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div}\mathbf{F} = 0$ in Ω, we also have $\mathbf{F} \in \mathbf{H}(\mathrm{div};\Omega)$. Thus $\mathbf{F}$ has a normal trace $\mathbf{F}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$, that by condition (32) satisfies $\langle \mathbf{F}\cdot\mathbf{n}, 1\rangle_{\Gamma_i} = 0$, $i = 0,\dots,\beta_2$. One then has $\operatorname{div}\widetilde{\mathbf{F}} = -\gamma'(\mathbf{F}\cdot\mathbf{n})$, so we may instead define $q \in H^1(\Gamma)/\mathbb{R}$ as the solution to the Laplace–Beltrami equation

$$-\Delta_\Gamma q = \mathbf{F}\cdot\mathbf{n} \qquad \text{on } \Gamma. \tag{72}$$

We then let $\widehat{\mathbf{F}} := \widetilde{\mathbf{F}} - \boldsymbol{\tau}'\nabla_\Gamma q$ as before and note that Lemma 10 holds. The term $\boldsymbol{\tau}'(-\nabla_\Gamma q)$ is the correction to the Biot–Savart law mentioned section 1.2. From this point the construction proceeds as before.

6 | REGULARITY

We begin this section by recalling a result of Costabel.[24]

Lemma 11. Let $\Omega \subset \mathbb{R}^3$ denote a handle-free, bounded Lipschitz domain and let $\mathbf{U} \in \mathbf{L}^2(\Omega)$ fulfil:

$$\operatorname{div}\mathbf{U} \in L^2(\Omega), \ \operatorname{curl}\mathbf{U} \in \mathbf{L}^2(\Omega). \tag{73}$$

Then $\mathbf{U}$ satisfies $\mathbf{U}\cdot\mathbf{n} \in L^2(\Gamma)$ on Γ if and only if $\mathbf{U}\times\mathbf{n} \in \mathbf{L}^2(\Gamma)$; and in this case $\mathbf{U}$ fulfils $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$.

Remark 6. There are extensions of this result to domains with $\beta_1 \neq 0$, for example in Monk's book.[25, Theorem 3.47] However, this extension is lacking the statement $\mathbf{U}\cdot\mathbf{n} \in L^2(\Gamma) \iff \mathbf{U}\times\mathbf{n} \in \mathbf{L}^2(\Gamma)$ which we need for our proof below. We thus refer to the original work of Costabel for $\beta_1 = 0$, but one can expect that this equivalence also generalises to the case $\beta_1 \neq 0$. Under this assumption the following regularity result remains true if zero Neumann harmonic components are prescribed for $\mathbf{U}^C = \operatorname{curl}\mathbf{A}|_{\mathbb{R}^3\setminus\overline{\Omega}}$.

This result can directly be applied to velocity fields $\mathbf{U}$ solving the div-curl system (21). In the following, we show that it also implies higher regularity of the associated stream functions $\mathbf{A}$.

Theorem 6. Let $\Omega \subset \mathbb{R}^3$ be a bounded, handle-free Lipschitz domain. Let $\mathbf{F} \in \mathbf{L}^2(\Omega)$ be given and fulfil the integrability condition (22), or the equivalent conditions (31a) and (32). Furthermore, let $g \in L^2(\Gamma)$ be given, such that $\langle g, 1\rangle_{\Gamma_i} = 0$ for all $i = 0,\dots,\beta_2$.

Then, the unique solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of the div-curl system (21) with $\mathbf{U} \cdot \mathbf{n} = g$ on Γ fulfils $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$, and its uniquely determined stream function $\mathbf{A}$ from Theorem 4 fulfils $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$.

Proof. The regularity of $\mathbf{U}$ is exactly Costabel's result Lemma 11. For the regularity of $\mathbf{A}$, we first consider the case $g = 0$. Thus, let $\mathbf{U}_0$ denote the unique solution of the div-curl system (21) that satisfies $\mathbf{U}_0 \cdot \mathbf{n} = 0$ on Γ, and let $\mathbf{A}_0 \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ denote its associated stream function. An application of the representation formula for the vector Laplacian then yields:

$$\begin{cases} \mathbf{A}_0 = \boldsymbol{\mathcal{N}}(\widetilde{\mathbf{F}} + \boldsymbol{\tau}'\mathbf{s}_0) & \text{on } \mathbb{R}^3, \\ \mathbf{s}_0 = [\![\operatorname{curl} \mathbf{A}_0 \times \mathbf{n}]\!] & \text{on } \Gamma, \end{cases} \tag{74}$$

where $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\mathbb{R}^3)$ is $\mathbf{F}$'s zero extension as defined in (71). Clearly, from the mapping properties of the Newton operator, it follows that $\boldsymbol{\mathcal{N}}\widetilde{\mathbf{F}} \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$. For the boundary term we note that from the construction of $\mathbf{A}_0$ in the proof of Theorem 3 it is clear that $\operatorname{curl} \mathbf{A}_0 = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. This implies that

$$\mathbf{s}_0 = \mathbf{U}_0 \times \mathbf{n} \qquad \text{on } \Gamma, \tag{75}$$

and because of Lemma 11 this yields $\mathbf{s}_0 \in \mathbf{L}^2(\Gamma)$. The boundary term $\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\mathbf{s}_0$ may thus alternatively be interpreted as a component-wise application of the scalar single layer potential operator $\mathcal{N}\gamma'$ to the components of $\mathbf{s}_0$. For this operator the following mapping property is known:[15, Remark 3.1.18b]

$$\mathcal{N}\gamma' : L^2(\Gamma) \to H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma), \tag{76}$$

and thus $\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\mathbf{s}_0 \in \mathbf{H}^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$.

For general boundary data $\mathbf{U} \cdot \mathbf{n} = g \in L^2(\Gamma)$, one then needs to solve the hypersingular boundary integral equation:

$$\forall v \in H^{\frac{1}{2}}(\Gamma) : \qquad \langle \boldsymbol{\tau}\boldsymbol{\mathcal{N}}\boldsymbol{\tau}' \, \mathbf{curl}_\Gamma \, p, \mathbf{curl}_\Gamma \, v\rangle_\Gamma = \langle g, v\rangle_\Gamma, \tag{77}$$

for the unknown $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ and set $\mathbf{s} := \mathbf{s}_0 + \operatorname{curl}_\Gamma p$, $\mathbf{A} := \boldsymbol{\mathcal{N}}(\widetilde{\mathbf{F}} + \boldsymbol{\tau}'\mathbf{s})$. For the integral equation the following regularity result is known:[15, Theorem 3.2.3b]

$$g \in L^2(\Gamma) \Rightarrow p \in H^1(\Gamma). \tag{78}$$

Thus $\operatorname{curl}_\Gamma p \in \mathbf{L}^2(\Gamma)$ and by the same arguments as above one obtains that $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$. ☐

7 | NUMERICAL APPROXIMATIONS

The case where $\mathbf{F} \in \mathbf{L}^r(\Omega)$ for some $r > 2$, leads to a particularly simple discretisation using Raviart–Thomas elements. In practice this condition on $\mathbf{F}$ is often not a real restriction. Here, we will only give a brief sketch of this scheme and its analysis. The case $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ is technically more involved, and we restrict ourselves to giving some remarks on possible numerical realisations. At the end of this section we give a numerical example with $\mathbf{F} \in \mathbf{C}^\infty(\Omega)$ and $\mathbf{U} \in \mathbf{C}^\infty(\Omega)$. Even for such smooth data, the tangential $\mathbf{A}_\mathrm{T} \in \mathbf{H}(\mathrm{curl}; \Omega) \cap \mathbf{H}_0(\mathrm{div}; \Omega)$ shows quite strong singularities. The newly proposed stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ is not smooth either, but displays increased regularity compared to $\mathbf{A}_\mathrm{T}$.

7.1 | Meshes and Spaces

For our numerical approximations we will assume that the domain Ω is polyhedral, handle-free, and that a family $\{\mathcal{T}_h\}_{h>0}$ of shape-regular, quasi-uniform, tetrahedral meshes is available. We will write $T \in \mathcal{T}_h$ for the tetrahedra of such a mesh, the parameter $h > 0$ refers to the average diameter of these tetrahedra. On these meshes we respectively define standard Lagrangian elements, Nédélec elements, Raviart–Thomas elements, and discontinuous elements of *order* $n \in \mathbb{N}$:

$$\begin{aligned} S^n_h(\Omega) &:= \{V_h \in H^1(\Omega) \,|\, \forall T \in \mathcal{T}_h : V_h|_T \in \mathbb{P}_{n-1}\}, \\ \mathbf{NED}^n_h(\Omega) &:= \{\mathbf{V}_h \in \mathbf{H}(\mathrm{curl}; \Omega) \,|\, \forall T \in \mathcal{T}_h : \mathbf{V}_h|_T = \mathbf{a} + \mathbf{b},\ \mathbf{a} \in \mathbb{P}^3_{n-1},\ \mathbf{b} \in \overline{\mathbb{P}}^3_n,\ \mathbf{b}(\mathbf{x}) \cdot \mathbf{x} \equiv 0\}, \\ \mathbf{RT}^n_h(\Omega) &:= \{\mathbf{V}_h \in \mathbf{H}(\mathrm{div}; \Omega) \,|\, \forall T \in \mathcal{T}_h : \mathbf{V}_h|_T = \mathbf{a} + \mathbf{x}b, \mathbf{a} \in \mathbb{P}^3_{n-1}, b \in \overline{\mathbb{P}}_{n-1}\}, \\ S^n_{h,\mathrm{disc}}(\Omega) &:= \{V_h \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_h : V_h|_T \in \mathbb{P}_{n-1}\}. \end{aligned} \tag{79}$$

Here, $\mathbb{P}_{n-1}$ refers to the space of polynomials of total *degree* $n-1$ or less, and $\overline{\mathbb{P}}_{n-1}$ to the space homogeneous polynomials of total degree exactly $n-1$. We also make use of sub-spaces with zero boundary conditions:

$$\begin{aligned} S^n_{h,0}(\Omega) &:= \{\, V_h \in S^n_h(\Omega) \quad \mid \gamma V_h = 0\}, \\ \mathbf{NED}^n_{h,0}(\Omega) &:= \{\mathbf{V}_h \in \mathbf{NED}^n_h(\Omega) \mid \boldsymbol{\tau}\mathbf{V}_h = \mathbf{0}\}, \\ \mathbf{RT}^n_{h,0}(\Omega) &:= \{\mathbf{V}_h \in \mathbf{RT}^n_h(\Omega) \quad \mid \nu \mathbf{V}_h = 0\}. \end{aligned} \tag{80}$$

A family of tetrahedral meshes $\{\mathcal{T}_h\}_{h>0}$ automatically gives rise to a family of boundary triangulations $\{\partial\mathcal{T}_h\}_{h>0}$, consisting of triangles $t \in \partial\mathcal{T}_h$. On these boundary meshes we will make use of the boundary element spaces $S^n_h(\Gamma)$, $\mathbf{RT}^n_h(\Gamma)$, and $S^n_{h,\mathrm{disc}}(\Gamma)$, which are the natural analogues of their respective counterparts on the domain Ω.

7.2 | The Case $\mathbf{F} \in \mathbf{L}^r(\Omega)$ with $r > 2$

Let $\mathbf{F} \in \mathbf{L}^2(\Omega)$ be given, and let $\mathbf{F}$ fulfil the integrability conditions $\operatorname{div}\mathbf{F} = 0$ and $\langle \mathbf{F}\cdot\mathbf{n}, 1\rangle_{\Gamma_i} = 0$, $i = 1,\dots,\beta_2$. Let us furthermore assume that there exists an $r > 2$ such that $\mathbf{F} \in \mathbf{L}^r(\Omega)$. This condition in particular allows us to define $\mathbf{F}_h \in \mathbf{RT}^n_h(\Omega)$ as the *canonical interpolant* of $\mathbf{F}$.[26, Section III.3] Note that then $\mathbf{F}_h$ *also fulfils the integrability conditions*, and furthermore the standard interpolation error-bound:

$$\|\mathbf{F}-\mathbf{F}_h\|_{\mathbf{L}^2(\Omega)} = \|\mathbf{F}-\mathbf{F}_h\|_{\mathbf{H}(\mathrm{div};\Omega)} \lesssim h^s |\mathbf{F}|_{\mathbf{H}^s(\Omega)} \qquad 1 \le s \le n. \tag{81}$$

Using the boundedness of the canonical interpolator on $\mathbf{L}^r(\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$, we furthermore easily obtain by standard arguments that for $n \ge 2$:

$$\|\mathbf{F}-\mathbf{F}_h\|_{\mathbf{H}^{-1}(\Omega)} \lesssim h^{s+1} \times \begin{cases} \|\mathbf{F}\|_{\mathbf{W}^{s,r}(\Omega)} & \text{if } 0 \le s \le 1, \\ |\mathbf{F}|_{\mathbf{H}^s(\Omega)} & \text{if } 1 \le s \le n. \end{cases} \tag{82}$$

Let us denote by $\widetilde{\mathbf{F}}_h$ the zero extension of $\mathbf{F}_h$ to $\mathbb{R}^3$. The Newton potential $\mathcal{N}\widetilde{\mathbf{F}}_h$ can then be evaluated analytically and component-wise, as $\mathbf{F}_h$ is a piece-wise polynomial on a simplical mesh.

Next we seek the correction $\mathbf{r} := -\nabla_\Gamma q$ on the boundary, to cancel out $\operatorname{div}\widetilde{\mathbf{F}}_h$. It is possible to solve the Laplace–Beltrami equation $-\Delta_\Gamma q_h = \mathbf{F}_h \cdot \mathbf{n}$ using a standard Galerkin method on the space $S^n_h(\Gamma)/\mathbb{R}$. The convergence rate of such an approach would then depend on the regularity of q, which in turn non-trivially depends on the shape of the boundary Γ.[27, Theorem 8] Note, however, that we have $\mathbf{F}_h \cdot \mathbf{n} \in S^n_{h,\mathrm{disc}}(\Gamma)/\mathbb{R}$, because $\mathbf{n}$ is constant on each triangle $t \in \partial\mathcal{T}_h$. Furthermore we have $\operatorname{div}_\Gamma \mathbf{RT}^n_h(\Gamma) = S^n_{h,\mathrm{disc}}(\Gamma)/\mathbb{R}$, so it makes sense to use the mixed formulation with Raviart–Thomas boundary elements instead. Hence, we seek $(\mathbf{r}_h, q_h) \in \mathbf{RT}^n_h(\Gamma) \times S^n_{h,\mathrm{disc}}(\Gamma)/\mathbb{R}$ such that:

$$\begin{aligned} \forall \mathbf{v}_h \in \mathbf{RT}^n_h(\Gamma) :& \quad \int_\Gamma \mathbf{r}_h \cdot \mathbf{v}_h \,\mathrm{d}S - \int_\Gamma q_h (\operatorname{div}_\Gamma \mathbf{v}_h)\,\mathrm{d}S = 0, \\ \forall l_h \in S^n_{h,\mathrm{disc}}(\Gamma)/\mathbb{R} :& \quad \int_\Gamma (\operatorname{div}_\Gamma \mathbf{r}_h) l_h \,\mathrm{d}S = \int_\Gamma (\mathbf{F}_h\cdot\mathbf{n}) l_h \,\mathrm{d}S. \end{aligned} \tag{83}$$

It is well-known that this formulation is well-posed, and that moreover we have $\operatorname{div}_\Gamma \mathbf{r}_h = \mathbf{F}_h \cdot \mathbf{n}$ exactly. Thus, abbreviating $\widehat{\mathbf{F}}_h := \widetilde{\mathbf{F}}_h + \boldsymbol{\tau}'\mathbf{r}_h$, we have $\operatorname{div}\widehat{\mathbf{F}}_h = 0$ on $\mathbb{R}^3$ *exactly*. The Newton potential of $\boldsymbol{\tau}'\mathbf{r}_h$ is the component-wise application of the standard, scalar single layer potential operator $\mathcal{N}\gamma'$ to the components of $\mathbf{r}_h$ and can also be computed efficiently analytically.

It remains to compute p_h. For this we abbreviate $\widehat{\mathbf{A}}_h := \mathcal{N}(\widetilde{\mathbf{F}}_h + \boldsymbol{\tau}'\mathbf{r}_h)$ and seek $p_h \in S^n_h(\Gamma)/\mathbb{R}$ such that:

$$\forall v_h \in S^n_h(\Gamma)/\mathbb{R} : \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{\mathbf{curl}_\Gamma\, p_h(\mathbf{y}) \cdot \mathbf{curl}_\Gamma\, v_h(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|} \,\mathrm{d}S(\mathbf{y})\,\mathrm{d}S(\mathbf{x}) = \int_\Gamma \left(g - \operatorname{curl}\widehat{\mathbf{A}}_h \cdot \mathbf{n}\right) v_h \,\mathrm{d}S. \tag{84}$$

This is a standard Galerkin boundary element method, analysis and efficient implementation techniques can for example be found in the book of Sauter and Schwab.[15] The numerical approximation is then defined as $\mathbf{s}_h := \mathbf{curl}_\Gamma\, p_h + \mathbf{r}_h$, and $\mathbf{A}_h := \mathcal{N}\widetilde{\mathbf{F}}_h + \boldsymbol{\tau}'\mathbf{s}_h$.

Let us define $g_h := \mathbf{n}\cdot\operatorname{curl}\mathbf{A}_h$. Then $\mathbf{A}_h$ is the *exact solution to the perturbed problem* with data $\mathbf{F}_h$ and g_h, so by the well-posedness result of Theorem 5, we immediately obtain:

$$\|\operatorname{curl}\mathbf{A}_h - \mathbf{U}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{A}_h - \mathbf{A}\|_{\mathbf{H}^1(\Omega)} \lesssim \|\mathbf{F}_h - \mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} + \|g_h - g\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{85}$$

For the first part $\|\mathbf{F}-\mathbf{F}_h\|_{\mathbf{H}^{-1}(\Omega)}$ we can use (82). Note that this part of the error neither depends on the regularity of $\mathbf{A}$ nor on that of $\mathbf{U}$! For smooth data $\mathbf{F}$ one may thus use coarse meshes and high order n on Ω. All of the "irregularity" due to the non-smooth boundaries is "concentrated" in the term $\|g - g_h\|$, and for $g \in L^2(\Gamma)$, as a consequence of the regularity result Theorem 6, we may at most expect convergence of order $\mathcal{O}(h^{\frac{1}{2}})$. However, also note that (85) is an *a-posteriori* bound that is easily computable. The term $\|g - g_h\|$, for example, can be made arbitrarily small by increasing the mesh resolution for p_h. Because p_h is the solution to a *scalar* equation on the boundary, adaptive refinement strategies are a comparatively cheap remedy to tackle the irregularity.

7.3 | Remarks on the General Case $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$

It is straightforward to discretise the general construction given in Section 5.1. An obvious choice would be a standard Galerkin method and finding $\mathbf{R}_h \in \mathbf{NED}^n_{h,0}(\Omega)$ such that $\mathfrak{B}(\mathbf{R}_h, \mathbf{V}_h) = \langle \mathbf{F}, \mathbf{V}_h \rangle$ for all $\mathbf{V}_h \in \mathbf{NED}^n_{h,0}(\Omega)$. Ultimately, however, the rate of convergence for such a method would less depend on the regularity of $\mathbf{F}$ itself, but more on that of its Riesz representative $\mathbf{R}$. If $\mathbf{F}$ happens to be smooth, $\mathbf{R}$ can turn out to be much less regular than the function $\mathbf{F}$ it represents.

An alternative approach could be based on the normal potential $\mathbf{A}_\mathrm{N} \in \mathbf{H}_0(\mathrm{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$ for the velocity field $\mathbf{U}_0 \in \mathbf{L}^2(\Omega)$ with $\mathbf{U}_0 \cdot \mathbf{n} = 0$ on Γ. In their work Amrouche et al. describe a finite element method for its approximation. For example, for a hole-free domain ($\beta_2 = 0$), their method reads as follows. Find $(\mathbf{A}_{\mathrm{N},h}, P_h) \in \mathbf{NED}^n_{h,0}(\Omega) \times S^n_{h,0}(\Omega)$ such that:[6, Equation (4.13)]

$$\begin{aligned} \forall \mathbf{V}_h \in \mathbf{NED}^n_{h,0}(\Omega): \quad & \int_\Omega \mathrm{curl}\,\mathbf{A}_{\mathrm{N},h} \cdot \mathrm{curl}\,\mathbf{V}_h \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{V}_h \cdot \nabla P_h \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{U}_0 \cdot \mathrm{curl}\,\mathbf{V}_h \,\mathrm{d}\mathbf{x}, \\ \forall Q_h \in S^n_{h,0}(\Omega): \quad & \int_\Omega \mathbf{A}_{\mathrm{N},h} \cdot \nabla Q_h \,\mathrm{d}\mathbf{x} = 0. \end{aligned} \tag{86}$$

This scheme has the remarkable property that $\|\mathrm{curl}\,\mathbf{A}_{\mathrm{N},h} - \mathbf{U}_0\|_{\mathbf{L}^2(\Omega)} \lesssim h^s \|\mathbf{U}_0\|_{\mathbf{H}^s(\Omega)}$ for all $0 \leq s \leq n$, regardless of the regularity of $\mathbf{A}_\mathrm{N}$. Only minor modifications are necessary for the case $\beta_2 > 0$.

Note that the right side may be immediately replaced with $\langle \mathbf{F}, \mathbf{V}_h \rangle$, as $\mathbf{V}_h \in \mathbf{H}_0(\mathrm{curl};\Omega)$ and $\mathbf{F} \in \mathbf{H}_0(\mathrm{curl};\Omega)'$ by Lemma 6. This is not possible in the corresponding method for the tangential potential $\mathbf{A}_\mathrm{T}$, where the test functions do not have a vanishing tangential trace.

The true solution fulfils $\mathrm{div}\,\mathbf{A}_\mathrm{N} = 0$ in Ω, it then suffices to cancel the spurious divergence of its zero extension $\widetilde{\mathbf{A}}_\mathrm{N}$. We have $\mathrm{div}\,\widetilde{\mathbf{A}}_\mathrm{N} = -\gamma' \mathbf{A}_\mathrm{N} \cdot \mathbf{n}$, so we may solve $-\Delta_\Gamma q = \mathbf{A}_\mathrm{N} \cdot \mathbf{n}$, set $\mathbf{r} := -\nabla_\Gamma q$, and obtain $\widehat{\mathbf{A}} := \widetilde{\mathbf{A}}_\mathrm{N} + \mathcal{N}\boldsymbol{\tau}'\mathbf{r} \in \mathbf{H}^1_\mathrm{loc}(\mathbb{R}^3)$. The numerical approximation $\mathbf{A}_{\mathrm{N},h}$ will usually not be exactly divergence free, here the jumps of the normal traces on the internal faces will also need to be cancelled out in order to achieve $\mathbf{H}^1(\Omega)$-regularity. From here we may proceed analogously as before: for the correction of the normal trace of $\mathrm{curl}\,\widehat{\mathbf{A}}$ we solve a boundary integral equation.

7.4 | An Example Illustrating Higher Regularity

We consider the domain $\Omega := (0,1)^3 \setminus [0.1, 0.8]^3$ and the smooth velocity field $\mathbf{U} \in \mathbf{C}^\infty(\Omega)$ associated to the vorticity $\mathbf{F} \in \mathbf{C}^\infty(\Omega)$ given by:

$$\mathbf{U}(\mathbf{x}) := \frac{1}{2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \qquad \mathbf{F}(\mathbf{x}) := \mathrm{curl}\,\mathbf{U}(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{87}$$

The domain Ω was chosen to be asymmetric, non-smooth, non-convex, and topologically non-trivial ($\beta_2 = 1$), while at the same time being "easy" from the viewpoint of meshing.

Neither for the tangential potential $\mathbf{A}_\mathrm{T}$, nor for the potential introduced in this work explicit expressions are known. Thus, the finite element method by Amrouche et al. has been implemented to compute $\mathbf{A}_\mathrm{T}$.[6, Equation (4.5)] We use order $n = 2$ for $\mathbf{A}_{\mathrm{T},h} \in \mathbf{NED}^n_h(\Omega)$, such that the velocity field is recovered exactly.

For the other stream function, notice that in this case the Newton potential:

$$\mathcal{N}\widetilde{\mathbf{F}}(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \frac{\mathrm{d}\mathbf{y}}{|\mathbf{x} - \mathbf{y}|} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{88}$$

can be evaluated directly, without further discretisation.[28] The Laplace–Beltrami equation for q_h and $\mathbf{r}_h$, as well as the hypersingular boundary integral equation for p_h, are discretised as described above, with order $n = 2$.

We remark that the method by Amrouche et al. requires to use the velocity field $\mathbf{U}$ as input, while the approach discussed in this work only requires $\mathbf{F}$ and the boundary data $\mathbf{U} \cdot \mathbf{n}$.

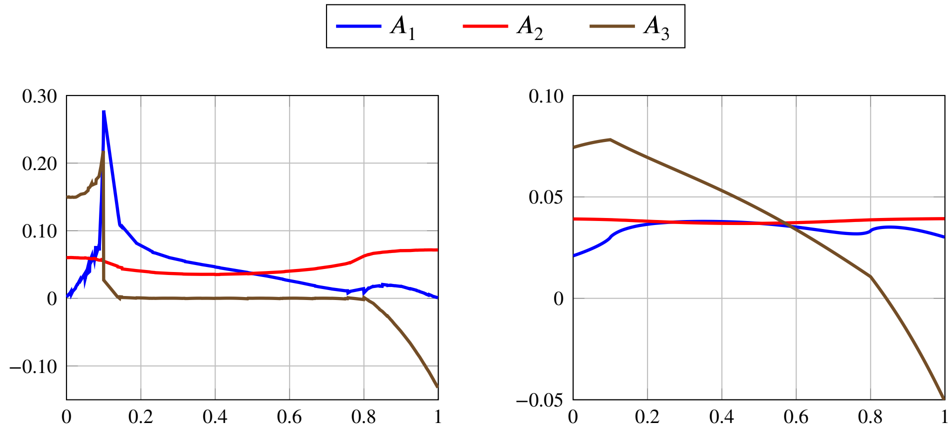

FIGURE 2 Numerical approximations of two different vector potentials for the same velocity field $\mathbf{U} = \frac{1}{2}(-x_2, x_1, 0)^\top$ on the domain $\Omega = (0,1)^3 \setminus [0.1, 0.8]^3$, plotted along the line $(x_1, 0.5, 0.8)^\top$. Left: the tangential vector potential $\mathbf{A}_\mathrm{T} \in \mathbf{H}_0(\mathrm{div};\Omega) \cap \mathbf{H}(\mathrm{curl};\Omega)$ by Amrouche et al.[6] Especially A_1 and A_2 show very steep gradients at $x_1 = 0.1$, and furthermore exhibit jump-type discontinuities. Further discontinuities away from $x_1 \in \{0.1, 0.8\}$ are due to the finite element approximation. Right: the new vector potential $\mathbf{A}$ presented in this work. In this case we have $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$; only a small kink in the components is visible at $x_1 = 0.1$ and $x_1 = 0.8$. Also note the different scales.

The numerical results along the line $(x_1, 0.5, 0.8)^\top$ are shown in Figure 2. In the interval $x_1 \in [0.1, 0.8]$ this line touches the internal boundary Γ_1. One clearly sees that close to the corners at $x_1 = 0.1$ and $x_1 = 0.8$ neither solutions are smooth. But while the tangential potential $\mathbf{A}_\mathrm{T}$ develops very steep gradients and jumps near $x_1 = 0.1$, the new potential $\mathbf{A}$ only exhibits a small kink, which suggests more regularity.

8 | CONCLUSIONS AND OUTLOOK

In this work, we have established precise conditions under which a divergence-free velocity field $\mathbf{U} \in \mathbf{L}^2(\Omega)$ can be recovered from its given curl and boundary data $\mathbf{U} \cdot \mathbf{n}$. Additionally, minor complementary assumptions on the boundary data guarantees that this velocity field can be represented in terms of a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$, which can be explicitly constructed. This stream function is more regular than the tangential vector potential suggested by Amrouche et al.[6]

The regularity result of Theorem 6 is sharp in several ways. Let us for example consider the case of a handle-free domain Ω with $\beta_1 = 0$ and suppose that $\mathbf{F} \equiv \mathbf{0}$. It is then a classical result that the velocity field $\mathbf{U}$ can be written in terms of the gradient of a scalar potential: $\mathbf{U} = -\nabla P$, where $-\Delta P = 0$ in Ω. Even if the given boundary data $\mathbf{U} \cdot \mathbf{n}$ is smooth, it is known from the regularity theory for the scalar Laplace equation that, on general Lipschitz domains, the highest regularity one can expect is $P \in H^{\frac{3}{2}}(\Omega)$, which therefore only leads to $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$. For any $\varepsilon > 0$ there are indeed examples of domains Ω and boundary data $\mathbf{U} \cdot \mathbf{n}$ where $P \notin H^{\frac{3}{2}+\varepsilon}(\Omega)$. In this sense, the vector potential $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$ introduced in this work has the highest possible regularity one can expect for arbitrary Lipschitz domains.

However, an interesting question remains. Suppose that the given data $\mathbf{F}$ and $\mathbf{U} \cdot \mathbf{n}$ are such that the velocity field $\mathbf{U}$ does happen to have higher regularity, say $\mathbf{U} \in \mathbf{H}^s(\Omega)$ for some $s \gg \frac{1}{2}$. McIntosh and Costabel have proven that in this case, another vector potential $\mathbf{A}_s \in \mathbf{H}^{1+s}(\Omega)$ exists.[29, Corollary 4.7] In other words, there always exists a stream function that is more regular than its velocity field by one order. In the numerical example discussed in section 7.4, such a smooth vector potential is given by:

$$\mathbf{A}_\infty(\mathbf{x}) = -\frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ x_1^2 + x_2^2 \end{pmatrix}. \tag{89}$$

However, the numerical experiments indicate that the vector potential proposed in this work is *not smooth*. Therefore, the problem of devising an algorithm to approximate reliably and efficiently the “smoothest possible” stream function remains open.

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