Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

Spurious Resonances in Coupled Domain-Boundary Variational Formulations of Transmission Problems in Electromagnetism and Acoustics

E. Schulz and R. Hiptmair

Research Report No. 2020-22
March 2020

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 200021 184848/1

SPURIOUS RESONANCES IN COUPLED DOMAIN–BOUNDARY VARIATIONAL FORMULATIONS OF TRANSMISSION PROBLEMS IN ELECTROMAGNETISM AND ACOUSTICS

ERICK SCHULZ AND RALF HIPTMAIR

Abstract. We develop a framework abstracting the features shared by a few transmission problems arising in electromagnetic scattering and acoustics. We show that spurious resonances haunting coupled domain-boundary formulations which exploit direct boundary integral equations of the first kind originate from the formal structure of their Calderón identities. Using this observation, the kernel of the coupled problem is characterized explicitly and we show that it completely vanishes under the exterior representation formula.

Keywords. electromagnetic scattering, acoustic scattering, resonant frequencies, coupling

1. Introduction

Transmission problems in electromagnetism and acoustics can model the following typical experiment. An incident wave penetrates an object and travels inside the possibly inhomogeneous medium. Concurrently, it also scatters at its surface and propagates in the outside homogeneous region to eventually decay at infinity. Simulation of the complete phenomenon entails coupling the interior and exterior problems. A vast literature is devoted to the design of such couplings for physical situations involving an increasing amount of intricacies. Notably, the described setting is considered in [15], [19], [8], [18] and [25].

On the one hand, domain based variational methods offer a familiar way of modeling wave propagation where material properties vary. The texts [2], [17], [1] and [22] are thorough analyses for electromagnetism. Standard references such as [26] and [12] introduce the reader to the Helmholtz operator as it appears in acoustic scattering.

On the other hand, boundary integral equations are capable of describing the behavior of the waves in unbounded homogeneous regions, because they provide valid Cauchy data that can be fed to the representation formula. Their complete derivation and properties can be found in [23], [20] and [21]. In the following, we consider in particular the *direct* boundary integral equations of the *first kind* detailed in [24], [7] and [11].

While transmission problems have unique solutions for all frequencies, boundary integral equations obtained for an exterior problem involving an Helmholtz-like second-order operator $\mathsf{P} - \lambda \mathrm{Id}$ are haunted by the existence of "spurious frequencies": the kernel of the Dirichlet-to-Neumann map supplied by the first exterior Calderón identity corresponds to the space of interior Dirichlet λ-eigenfunctions of P. Similarly, the related Neumann eigenspace corresponds to the kernel of the Neumann-to-Dirichlet map supplied by the second exterior Calderón identity. This issue was investigated for the electric field integral in [10]. Eigenvalues of the Laplacian were studied in [14] and [24] from the perspective of resonant frequencies.

Unsurprisingly, this deficiency of the boundary integral equations carries over to the the coupled domain-boundary variational formulations. Its consequences for the symmetric approach to the coupling problem

Date: March 26, 2020.

Funding. The work of Erick Schulz was supported by SNF as part of the grant 200021_184848/1.

Seminar in Applied Mathematics, Eidgenössische Technische Hochschule Zurich (ETH), Switzerland (e-mails: erick.schulz@sam.math.ethz.ch, hiptmair@sam.math.ethz.ch).

in the context of electromagnetism (classical **E**–**H** formulation) and acoustics (Helmholtz equation) were stated without proof in [7] and [19], respectively.

In this essay, we unite a few symmetric domain-boundary variational formulations for the time-harmonic solutions of transmission problems in electromagnetism and acoustics under a common framework. Three particular problems are discussed in section 2.2. Costabel's symmetric approach detailed in [13] is generalized to allow for the mixed formulation of the interior problem. The lack of uniqueness due to resonant frequencies is shown to result from the formal structure of the Calderón identities. The phenomenon is thus shared by all three couplings under consideration. The kernel of the astract coupled problem is fully characterized in section 4.

We point out that from a practical/algorithmic point of view, the post-processing required to recover the scattered waves in the exterior region rescue uniqueness of the solutions. Indeed, the kernel of the Dirichlet-to-Neumann map vanishes under the representation formula. Therefore, while these so-called "spurious resonant frequencies" generally cause instabilities after discretization that may bring about the need for regularization strategies, their mere existence is harmless to the physical validity of the domain-boundary coupling models. This explains how classical coercive symmetric couplings remain nonetheless valuable pilot formulations for Galerkin discretization. We refer to [24] for an introduction to a classical approach originally suggested by Brakhage and Werner [3] to regularize the *indirect* BIEs for the Helmholtz operator. We also point out that a CFIE-type stabilization procedure for the Helmholtz transmission problem is studied in [16], where a symmetric coupling stable for all positive frequencies is obtained.

2. Formal framework

2.1. **Notation and conventions.** Let $\Omega^- \subset \mathbb{R}^3$ be a bounded simply connected domain with Lipschitz boundary $\Gamma := \partial\Omega^-$. We think of Ω^- as a bounded inhomogeneous volume occupied by an object with a possibly "rough" surface. It is common to take Ω^- to be a curvilinear polyhedron. Throughout this work, we use Ω generically to denote either Ω^- or $\Omega^+ := \overline{\Omega^-}\backslash\mathbb{R}^3$. Physically, Ω^+ often represents an unbounded homogeneous air region around Ω^-. We let $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Gamma)$ denote, respectively, spaces of square-integrable fields over Ω and Γ. We forgo more precise definions, as they will eventually be instantiated in turn as spaces of complex functions, three-dimensional vector fields and direct products of vector fields. Whenever it is possible, we try in particular problems to use bold letters to differentiate vector quantities from scalars. Capitals are often used to denote fields defined over a volume, while small characters usually refer to boundary variables. The space of smooth fields compactly supported in Ω is written $\mathscr{D}(\Omega)$. The subscript 'loc' is used to extend a given space V to the larger space V_{loc} comprising all functions u such that $u\psi \in V$ for all $\psi \in \mathscr{D}(\Omega)$. A prime will be used to indicate the dual of a space, e.g. V'. Duality paring is written with angular brackets, e.g. $\langle\cdot,\cdot\rangle$, but we also often allow ourselves to substitute integrals for these angular brackets when we want to emphasize $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Gamma)$ as pivot spaces or highlight the analogy between the identities introduced in this formal framework and Green's classical formulas.

We call *weak differential operator matrices* the various linear operators that can be represented by a matrix of partial derivatives. We understand their arrangement in weak sense. If no particular structure is recognized, then we must accept to define them on the Sobolev space $H^1(\Omega)$. However, in the models we consider in this work, the partial derivatives often sum up to form divergence and curl operators respectively defined on

$$\mathbf{H}(\mathrm{div},\Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}(\mathbf{curl},\Omega) := \{\mathbf{U} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}.$$

They are quantities weakening the Green's identities

$$\pm\int_{\Omega^\mp} \mathrm{div}(\mathbf{U})\,P + \mathbf{U}\cdot\nabla P \,\mathrm{d}\mathbf{x} = \int_\Gamma P\,(\mathbf{U}\cdot\mathbf{n})\,\mathrm{d}\sigma, \tag{1a}$$

$$\pm\int_{\Omega} \mathbf{U}\cdot\mathbf{curl}\,(\mathbf{V}) - \mathbf{curl}\,(\mathbf{U})\cdot\mathbf{V}\,\mathrm{d}\mathbf{x} = \int_\Gamma \mathbf{V}\cdot(\mathbf{U}\times\mathbf{n})\,\mathrm{d}\sigma, \tag{1b}$$

where $\mathbf{n}(x)$ stands for the unit normal boundary vector field oriented outward from Ω^-. The same notation is kept throughout this section.

2.2. **"Helmholtz-like" operators.** We consider a formally self-adjoint linear second-order weak differential operator matrix

$$\mathsf{P} : \mathbf{X}_{\mathrm{loc}}(\Omega) \to \mathbf{L}^2(\Omega).$$

In accordance with this definition, we assume that $\mathbf{X}_{\mathrm{loc}}(\mathsf{P};\Omega) \subset \mathbf{L}^2_{\mathrm{loc}}(\Omega)$. Ultimately, our goal is to develop variational transmission equations in which exterior problems of the form

$$(\mathsf{P} - \lambda\,\mathrm{Id})\, U^{\mathrm{ext}} = 0 \quad \text{in } \Omega^+ \tag{2}$$

are formulated using BIEs. When $\mathfrak{Re}(\lambda) > 0$ and $\mathfrak{Im}(\lambda) \geq 0$, we say that the operator on the left hand side of (2) is *Helmholtz-like*.

We have in mind to encompass a practical range of well-kown operators. The following examples are important in the study of acoustic and electromagnetic scattering.

2.2.1. *Helmholtz acoustics.* The simplest ones are obtained from elliptic operators acting on scalar real-valued functions, of which the Laplacian

$$-\Delta := -\operatorname{div}\circ\nabla = -\sum_{i=1}^{3}\partial_i^2$$

is the most famous example. It is found differentiating a suitably scaled pressure amplitude U in the scalar Helmholtz equation

$$-\operatorname{div}(\nabla U) - \kappa^2 r(\mathbf{x}) U = 0 \tag{3}$$

that models the travel of plane time harmonic sound waves with real positive wave number $\kappa > 0$. While the bounded refractive index $r(\mathbf{x})$ may vary inside the inhomogeneous body Ω^-, it stays constant in the unbounded air region Ω^+. BIEs offer the most flexible way of tackling the exterior problem, but a domain formulation is best suited to deal with the interior inhomogeneity. Because of the simplicity of its toy-examples, acoustic scattering thus presents itself as a canonical example to illustrate the relevance of coupled domain–boundary variational formulations.

In this framework, the domain of the Laplace operator is easily seen to be

$$H_{\mathrm{loc}}(\Delta,\Omega) := \{U \in H^1_{\mathrm{loc}}(\Omega) |\ \nabla U \in \mathbf{H}_{\mathrm{loc}}(\operatorname{div},\Omega)\}.$$

Boundary value problems are stated using the classical Dirichlet and Neumann traces

$$\gamma^{\mp} U(x) = \lim_{\Omega^{\mp}\ni y\to x} U(y),$$
$$\gamma_n^{\mp} U(x) = -\lim_{\Omega^{\mp}\ni y\to x} \mathbf{n}(x)\cdot\nabla U(y),$$

which enter Green's identity (1a). These traces are well-defined on smooth scalar fields and extend by continuity to the Sobolev spaces

$$\gamma^{\mp} : H^1_{\mathrm{loc}}\left(\Omega^{\mp}\right) \to H^{1/2}(\Gamma), \tag{4a}$$

$$\gamma_n^{\mp} : H_{\mathrm{loc}}(\Delta,\Omega) \to H^{-1/2}(\Gamma). \tag{4b}$$

The classical symmetric coupling for (3) derived in [19] fits the abstract framework of the next sections.

2.2.2. **E**–**H** *electromagnetism.* As explained in the introduction, we also consider the non-elliptic linear operators arising in the simulation of electromagnetic scattering phenomena. Prevalent in the literature and widely used in engineering applications is the **curl curl** operator

$$\mathbf{E} \mapsto \mathbf{curl}\left(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E}\right) \tag{5}$$

entering the frequency domain formulation of the electric wave equation

$$\mathbf{curl}\left(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E}\right) - \omega^2\epsilon(\mathbf{x})\mathbf{E} = 0, \tag{6}$$

in which $\epsilon(\mathbf{x})$ and $\mu(\mathbf{x})$ are material properties known respectively as the dielectric and permeability tensors. Again, these quantities are assumed constant outside the scatterer. This is the most standard time-harmonic model for the propagation of an electromagnetic wave with angular frequency ω. As opposed to the Helmholtz equation of acoustic scattering, the unknown is a vector-valued function.

We note that the **curl curl** operator (5) with constant coefficient $\mu(\mathbf{x}) = \mu_0$ could also be represented by the operator matrix

$$\mathsf{P} := \mu_0^{-1}\begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}^2. \tag{7}$$

Its domain of definition is

$$\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2,\Omega) := \{\mathbf{E} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\Omega)\,|\,\mathbf{curl}(\mathbf{E}) \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\Omega)\}.$$

Well-posed boundary value problems are established for the electric wave equations by weakening the tangential traces

$$\gamma_t^{\mp}\,\mathbf{E}(x) := \mathbf{n}(x) \times \gamma_\tau\left(\mathbf{E}(\mathbf{x})\right), \tag{8a}$$

$$\gamma_R^{\mp}\,\mathbf{E}(x) := -\gamma_\tau^{\mp}\mathbf{curl}\,\mathbf{E}(x), \tag{8b}$$

where $\gamma_\tau^{\mp}\mathbf{E} := \mathbf{E} \times \mathbf{n}$ enters Green's identity (1b). The "magnetic trace" $\gamma_R^{\mp}\,\mathbf{E}$ plays a role akin to the Neumann trace (4b). The relatively recent development of tangential traces theory for Lipschitz domains can be followed in [4], [5] and [6]. A symmetric domain-boundary variational coupling falling under the framework of the next sections is performed in [18].

2.2.3. **A**-ϕ *electromagnetism.* Equation (6) is obtained upon combining the dynamical equations

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu(x)\mathbf{H}, \qquad \mathbf{curl}\,\mathbf{H} = i\omega\epsilon(\mathbf{x})\mathbf{E},$$

that are part of the **E**–**H** formulation of Maxwell's equations. When the magnetic and electric fields are expressed in terms of the vector and scalar electromagnetic potentials, which satisfy $\mathbf{H} = \mu^{-1}(x)\,\mathbf{curl}\,\mathbf{A}$ and $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\phi$, these two equations instead combine to form

$$\mathbf{curl}\left(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{A}\right) + i\omega\epsilon(\mathbf{x})\nabla\phi - \omega^2\epsilon(\mathbf{x})\mathbf{A} = 0.$$

Elimination of ϕ using the Lorentz gauge

$$\operatorname{div}\left(\epsilon(\mathbf{x})\mathbf{A}\right) + i\omega\phi = 0 \tag{9}$$

leads to the Hodge-Helmholtz equation

$$\mathbf{curl}\left(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{A}\right) - \epsilon(\mathbf{x})\nabla\operatorname{div}\left(\epsilon(\mathbf{x})\mathbf{A}\right) - \omega^2\epsilon(\mathbf{x})\mathbf{A} = 0. \tag{10}$$

The link behind electromagnetism and geometry through the Hodge-Laplace operator is the subject of a vast literature. Because (10) is robust in the low-frequency limit $\omega \to 0$, its extension to inhomogeneous material through the generalized Lorentz gauge (9) has resurfaced relatively recently as an interesting alternative to the standard electric wave equation for the simulation of some contemporary physical experiments in quantum optics [9].

When the material properties $\epsilon(\mathbf{x}) = \epsilon_0$ and $\mu(\mathbf{x}) = \mu_0$ are assumed constant, equation (10) reduces to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{A} - \eta\,\nabla\,\mathrm{div}\,\mathbf{A} - \kappa^2\mathbf{A} = 0, \tag{11}$$

where $\eta = \mu_0\epsilon_0^2$ and $\kappa^2 = \mu_0\epsilon_0\omega^2$. The domain of the Hodge-Helmholtz operator on the left hand side is the intersection space $\mathbf{H}_{\mathrm{loc}}\left(\mathbf{curl}^2, \Omega\right) \cap \mathbf{H}_{\mathrm{loc}}(\nabla\mathrm{div}, \Omega)$, where

$$\mathbf{H}_{\mathrm{loc}}(\nabla\mathrm{div}, \Omega) := \{\mathbf{U} \in \mathbf{H}_{\mathrm{loc}}(\mathrm{div}, \Omega) \mid \mathrm{div}\,\mathbf{U} \in H^1_{\mathrm{loc}}(\Omega)\}.$$

A pair of suitable traces for the formulation of boundary value problems is given by

$$\mathcal{T}^{\mp}_{\mathrm{mg}}\,\mathbf{A}(\mathbf{x}) := \begin{pmatrix} \gamma^{\mp}_R\,\mathbf{A}(\mathbf{x}) \\ \gamma^{\mp}_n\,\mathbf{A}(\mathbf{x}) \end{pmatrix}, \tag{12a}$$

$$\mathcal{T}^{\mp}_{\mathrm{el}}\,\mathbf{A}(\mathbf{x}) := \begin{pmatrix} \gamma^{\mp}_t\,\mathbf{A}(\mathbf{x}) \\ \eta\,\gamma^{\mp}\mathrm{div}\mathbf{A}(\mathbf{x}) \end{pmatrix}. \tag{12b}$$

Notice that their ranges are product spaces. This is partly due to the fact that the $\mathbf{A}$–ϕ potential formulation of Maxwell's equations initially introduced two unknowns in the wave equation. Going back to the Lorentz gauge (9), we see in the context of transmission problems that the second component of the "electric trace" (12b) is in hiding a continuity condition for the scalar potential. Once again, it is the "magnetic trace" $\mathcal{T}^{\mp}_{\mathrm{mg}}\,\mathbf{A}$ that resemble the Neumann trace.

The natural trial and test spaces readily obtained upon establishing domain based variational formulations for (11) using (1a) and (1b) are unfortunately not viable for discretization by finite elements. This is the reason why in [25] the mixed formulation

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} + \nabla P - \kappa^2\mathbf{A} &= 0, \\ \eta\,\mathrm{div}\,(\mathbf{A}) + P &= 0, \end{aligned} \tag{13}$$

is generalized to accommodate variable coefficients. The weak differential operator matrix

$$\mathsf{L} := \begin{pmatrix} \mathbf{curl}\,\mathbf{curl} - \kappa^2\mathrm{Id} & \nabla \\ \eta\,\mathrm{div} & -\mathrm{Id} \end{pmatrix}$$

is well defined over $\mathbf{H}_{\mathrm{loc}}\left(\mathbf{curl}^2, \Omega\right) \times H^1_{\mathrm{loc}}(\Omega)$ and therefore more convenient to model the interior problem. This subtlety justifies generalizing Green's first formula in Assumption 6, because integration by parts yields

$$\int_{\Omega^-} \left\langle \mathsf{L}\begin{pmatrix}\mathbf{A}\\P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\Q\end{pmatrix} \right\rangle \mathrm{d}\mathbf{x} = \mathbf{\Phi}_\kappa\left(\begin{pmatrix}\mathbf{A}\\P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right) + \left\langle \mathcal{T}^+_{\mathrm{mg}}(\mathbf{A}), \begin{pmatrix}\gamma^-_t\mathbf{V}\\ \gamma^- Q\end{pmatrix} \right\rangle, \tag{14}$$

where the left hand side sports the bilinear form

$$\begin{aligned} \mathbf{\Phi}_\kappa\left(\begin{pmatrix}\mathbf{A}\\P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right) := &\int_{\Omega_s} \mu^{-1}\,\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\mathbf{V}\,\mathrm{d}\mathbf{x} + \int_{\Omega_s} \epsilon\,\nabla P\cdot\mathbf{V}\,\mathrm{d}\mathbf{x} + \int_{\Omega_s} P\,Q\,\mathrm{d}\mathbf{x} \\ &- \int_{\Omega_s} \epsilon\,\mathbf{A}\cdot\nabla Q\,\mathrm{d}\mathbf{x} - \omega^2\int_{\Omega_s} \epsilon\,\mathbf{A}\cdot\mathbf{V}\,\mathrm{d}\mathbf{x}. \end{aligned}$$

The observation that for solutions of (13) we have

$$\begin{pmatrix}\gamma^-_t\mathbf{A}\\ \gamma^- P\end{pmatrix} = \begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\mathrm{Id}\end{pmatrix}\mathcal{T}^-_{\mathrm{el}}\,\mathbf{A}(\mathbf{x})$$

ultimately leads to important symmetries that were extremely useful in [25] for the derivation of coercivity estimates.

2.3. **Boundary value problems.** The first step in the formulation of BVPs for P is to establish a definition of boundary data.

Assumption 1 (Green's second formula)**.** *There exist two non-trivial Hilbert spaces of distributions* $\mathbf{H}_N$ *and* $\mathbf{H}_D$*, supported on* Γ*, that are dual under a pairing* $\langle \cdot, \cdot \rangle_\Gamma$*, together with continous and* surjective *linear operators*

$$\mathsf{T}_D^{\mp} : \mathbf{X}_{loc}^D \left(\Omega^{\mp} \right) \rightarrow \mathbf{H}_D \left(\Gamma \right),$$
$$\mathsf{T}_N^{\mp} : \mathbf{X}_{loc}^N \left(\Omega^{\mp} \right) \rightarrow \mathbf{H}_N \left(\Gamma \right),$$

admitting continuous right inverses *and satisfying Green's second formula*

$$\int_{\Omega^{\mp}} \mathsf{P}U \cdot V - U \cdot \mathsf{P}V \, \mathrm{d}\mathbf{x} = \pm \big\langle \mathsf{T}_N^{\mp} U, \mathsf{T}_D^{\mp} V \big\rangle_\Gamma \mp \big\langle \mathsf{T}_N^{\mp} V, \mathsf{T}_D^{\mp} U_\Gamma \big\rangle \tag{15}$$

for all $u, v \in \mathbf{X}_{loc} \left(\Omega^{\mp} \right)$*. We evidently assume that the domain of* P *lies in the intersection of* $\mathbf{X}_{loc}^D (\Omega)$ *and* $\mathbf{X}_{loc}^N (\mathsf{P}; \Omega)$.

By analogy with the boundary value problems arising from the operators introduced in section 2.2, it is reasonable to suppose that $\mathscr{D}(\Omega) \subset \ker \left(\mathsf{T}_D^{\mp} \right) \cap \ker \left(\mathsf{T}_N^{\mp} \right)$*. Roughly speaking, this simply asks for the traces of fields vanishing on the boundary to vanish. Moreover, we take for granted that* $\left[\mathsf{T}_D (\phi) \right] = \left[\mathsf{T}_D (\phi) \right] = 0$ *whenever* ϕ *is continuously differentiable across* Γ*. The square brackets indicate the jump* $[\mathsf{T}_\bullet] := \mathsf{T}_\bullet^- - \mathsf{T}_\bullet^+$ *of a trace, specified by* $\bullet = D$ *or* N*, over the boundary* Γ*.*

The archetypes behind these operators are the Dirichlet and Neumann traces (4a)-(4b), but (8a)-(8b) and (12b)-(12a) also satisfy the assumption.

Given boundary data $g \in \mathbf{H}_D (\Gamma)$ and $\eta \in \mathbf{H}_N (\Gamma)$, we use the traces supplied in Assumption 1 to impose boundary conditions in the statement of interior and exterior BVPs:

$$\begin{cases} \mathsf{P}U - \lambda U = 0, & \text{in } \Omega^{\mp} \\ \mathsf{T}_D^{\mp} U = g, & \text{on } \Gamma, \\ \text{radiation conditions at } \infty, & \text{if } \Omega = \Omega^+ \end{cases} \tag{DP$_\lambda^{\mp}$}$$

$$\begin{cases} \mathsf{P}U - \lambda U = 0, & \text{in } \Omega^{\mp} \\ \mathsf{T}_N^{\mp} U = \eta, & \text{on } \Gamma, \\ \text{radiation conditions at } \infty, & \text{if } \Omega = \Omega^+. \end{cases} \tag{NP$_\lambda^{\mp}$}$$

Assumption 2 (Uniqueness for exterior BVPs)**.** *The solutions to the exterior BVPs (*DP_λ^+*) and (*NP_λ^+*) are* unique *in* $\mathbf{X} \left(\Omega^+ \right)$.

2.4. **Representation by boundary potentials.** Given a formally self-adjoint weak differential operator matrix L and a locally integrable source term F, we say that $\mathsf{L}U = F$ holds in Ω *in the sense of distributions* if

$$\big\langle \mathsf{L}U, V \big\rangle := \int_\Omega U \cdot \mathsf{L}V \, \mathrm{d}\mathbf{x} = \int_\Omega F \cdot \mathsf{L}V \, \mathrm{d}\mathbf{x} \tag{16}$$

for all $V \in \mathscr{D}(\Omega)$. From this point of view, we have $U, F \in \mathscr{D}(\Omega)'$ and the action of L is extended by the left hand side of (16) to be also defined on the space of distributions. That is to say, the solution U is interpreted as a bounded linear functional over the space of test functions.

Let $U \in \mathbf{L}^2_{\text{loc}}\left(\mathbb{R}^3\right)$ be such that $U|_{\Omega^-} \in \mathbf{X}\left(\Omega^-\right)$ and $U|_{\Omega^+} \in \mathbf{X}_{\text{loc}}\left(\mathsf{P};\Omega^+\right)$ with $(\mathsf{P}-\lambda\text{Id})\, U|_{\Omega^\mp} = 0$, where the restrictions are to be understood in the sense of distributions. Using Green's second formula (15), we obtain

$$\langle \mathsf{P}U - \lambda U, \psi\rangle = \left\langle \left[\mathsf{T}_D U\right], \mathsf{T}_N \psi\right\rangle_\Gamma - \left\langle \mathsf{T}_D\psi, \left[\mathsf{T}_N U\right]\right\rangle_\Gamma$$

for all smooth compactly supported fields ψ defined over $\mathbb{R}^3$. Therefore, in the sense of distributions,

$$\mathsf{P}U - \lambda U = \left(\mathsf{T}_N^-\right)^* \left[\mathsf{T}_D u\right] - \left(\mathsf{T}_D^-\right)^* \left[\mathsf{T}_N u\right], \tag{17}$$

where the mappings $\left(\mathsf{T}_N^-\right)^*$ and $\left(\mathsf{T}_D^-\right)^*$ are adjoint to T_N^- and T_D^-, respectively.

Assumption 3 (Fundamental solution)**.** *There exists a smooth (possibly matrix-valued) complex Green tensor G_λ defined over $\mathbb{R}^3\backslash\{0\}$ satisfying*

$$(\mathsf{P} - \lambda\, Id) \circ G_\lambda = \delta_0\, Id$$

as a distribution and whose components respect the decay conditions at infinity stated in ($\text{DP}_\lambda^\mp$) *and* ($\text{NP}_\lambda^\mp$)*. On the right hand side, δ_0 is standard notation for the Dirac distribution centered at 0, i.e. $\langle\delta_0,\psi\rangle = \psi(0)$ for all $\psi \in \mathscr{D}(\mathbb{R}^3)$.*

Convoluting with G_λ on both sides of (17) yields the representation formula

$$U = \mathcal{SL}_\lambda\left(\left[\mathsf{T}_N U\right]\right) + \mathcal{DL}_\lambda\left(\left[\mathsf{T}_D U\right]\right), \tag{18}$$

where we have defined for all $g \in \mathbf{H}_D(\Gamma)$ and $\eta \in \mathbf{H}_N(\Gamma)$ the layer potentials

$$\mathcal{SL}_\lambda(g) := -G_\lambda \star \left(\left(\mathsf{T}_D^-\right)^* g\right),$$
$$\mathcal{DL}_\lambda(\eta) := G_\lambda \star \left(\left(\mathsf{T}_N^-\right)^* \eta\right).$$

Here, $\star$ indicates the convolution operation.

2.5. **Boundary integral operators.**

Boundary integral equations for the Dirichlet problems ($\text{DP}_\lambda^\mp$) and ($\text{DP}_\lambda^\mp$) are obtained by establishing the famous Caldéron identities.

Assumption 4 (Jump identities)**.** *The boundary potentials $\mathcal{SL}_\lambda : \mathbf{H}_N(\mathsf{P};\Gamma) \to \mathbf{X}_{loc}(\mathsf{P};\Omega)$ and $\mathcal{DL}_\lambda : \mathbf{H}_D(\mathsf{P};\Gamma) \to \mathbf{X}_{loc}(\mathsf{P};\Omega)$ are continuous and satisfy the jump relations*

$$[\mathsf{T}_D]\,\mathcal{DL}_\lambda = Id, \qquad [\mathsf{T}_D]\,\mathcal{DL}_\lambda = 0,$$
$$[\mathsf{T}_D]\,\mathcal{SL}_\lambda = 0, \qquad [\mathsf{T}_N]\,\mathcal{SL}_\lambda = Id.$$

Applying averaged traces $\{\mathsf{T}_\bullet\} := 1/2\left(\mathsf{T}_\bullet^+ + \mathsf{T}_\bullet^-\right)$ specified with $\bullet = D$ and N to $\mathcal{SL}_\lambda$ and $\mathcal{DL}_\lambda$ yields four continuous boundary integral operators:

$$\mathcal{A}_\lambda := \{\mathsf{T}_D\}\,\mathcal{SL}_\lambda : \mathbf{H}_N(\Gamma) \to \mathbf{H}_D(\Gamma),$$
$$\mathcal{B}_\lambda := \{\mathsf{T}_N\}\,\mathcal{SL}_\lambda : \mathbf{H}_N(\Gamma) \to \mathbf{H}_N(\Gamma),$$
$$\mathcal{C}_\lambda := \{\mathsf{T}_D\}\,\mathcal{DL}_\lambda : \mathbf{H}_D(\Gamma) \to \mathbf{H}_D(\Gamma),$$
$$\mathcal{D}_\lambda := \{\mathsf{T}_N\}\,\mathcal{DL}_\lambda : \mathbf{H}_D(\Gamma) \to \mathbf{H}_N(\Gamma).$$

Taking the traces on both sides of the representation formula (18) and using the jump relations of Assumption 4, we obtain the interior and exterior Caldéron identities

$$\underbrace{\begin{pmatrix} \mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{A}_\lambda \\ \mathcal{D}_\lambda & \mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathbb{P}_\lambda^-} \begin{pmatrix} \mathsf{T}_D^- U \\ \mathsf{T}_N^- U \end{pmatrix} = \begin{pmatrix} \mathsf{T}_D^- U \\ \mathsf{T}_N^- U \end{pmatrix},$$

$$\underbrace{\begin{pmatrix} -\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & -\mathcal{A}_\lambda \\ -\mathcal{D}_\lambda & -\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathbb{P}_\lambda^+} \begin{pmatrix} \mathsf{T}_D^+ U \\ \mathsf{T}_N^+ U \end{pmatrix} = \begin{pmatrix} \mathsf{T}_D^+ U \\ \mathsf{T}_N^+ U \end{pmatrix},$$

respectively. Note that $\mathbb{P}_\lambda^+ + \mathbb{P}_\lambda^- = \mathrm{Id}$ so that the range of $\mathbb{P}_\lambda^+$ coincides with the kernel of $\mathbb{P}_\lambda^-$ and vice-versa. The next theorem is a consequence of the existence of continuous right inverses for the traces stated in Assumption 1. It promotes the Caldéron projectors to a pivotal role in domain–boundary formulations of transmission problems.

Lemma 1. *A pair of boundary data $(g,\eta) \in \mathbf{H}_D(\Gamma) \times \mathbf{H}_N(\Gamma)$ is valid interior or exterior Cauchy data of some distribution $u \in \mathbf{X}(\Omega)$ that solves the homogeneous equation $\mathsf{P}u - \lambda u = 0$ in $\Omega^\mp$ if and only it lies in the kernel of $\mathbb{P}_\lambda^+$ or $\mathbb{P}_\lambda^-$, respectively.*

2.6. **Boundary integral equations.** The rows of the exterior Caldéron identities give rise to the following two direct variational BIEs of the first kind for the exterior Dirichlet ($\mathrm{DP}_\lambda^\mp$) and Neumann (NP_λ^+) problems respectively:

$$(\mathrm{DBP}_\lambda) \qquad \begin{cases} \text{Seek } \xi \in \mathbf{H}_N(\Gamma) \text{ satisfying} \\ \displaystyle\int_\Gamma \mathcal{A}_\lambda \xi \cdot \zeta \,\mathrm{d}\sigma = -\int_\Gamma \left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right) g \cdot \zeta \,\mathrm{d}\sigma \\ \text{for all } \zeta \in \mathbf{H}_N(\Gamma). \end{cases}$$

$$(\mathrm{NBP}_\lambda) \qquad \begin{cases} \text{Seek } a \in \mathbf{H}_D(\Gamma) \text{ satisfying} \\ \displaystyle\int_\Gamma \mathcal{D}_\lambda \xi \cdot \zeta \,\mathrm{d}\sigma = -\int_\Gamma \left(\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right) \eta \cdot \zeta \,\mathrm{d}\sigma \\ \text{for all } \zeta \in \mathbf{H}_D(\Gamma). \end{cases}$$

3. Coupled domain–boundary variational formulations

Now, let L be a weak linear differential operator defined on $\mathbf{X}\left(\mathsf{L}, \Omega^-\right)$ satisfying Assumption 1 with traces $\mathsf{T}_{\mathsf{L},D}^-$ and $\mathsf{T}_{\mathsf{L},N}^-$. Simlarly, allow P to be defined on $\mathbf{X}\left(\mathsf{P}, \Omega^\mp\right)$ such that it satisfies assumptions 1 to 4 with traces $\mathsf{T}_{\mathsf{P},D}^\mp$ and $\mathsf{T}_{\mathsf{P},N}^\mp$.

We are interested in well-posed transmission problems

$$(\mathrm{TP}_\lambda) \qquad \begin{cases} \mathsf{L}U = f, & \text{in } \Omega^- \\ \mathsf{P}U^{\mathrm{ext}} + \lambda U^{\mathrm{ext}} = 0, & \text{in } \Omega^+ \\ \mathsf{T}_{\mathsf{L},D}^- U = \mathsf{T}_{\mathsf{P},D}^+ U^{\mathrm{ext}} + g, & \text{on } \Gamma, \\ \mathsf{T}_{\mathsf{L},N}^- U = \mathsf{T}_{\mathsf{P},N}^+ U^{\mathrm{ext}} + \eta, & \text{on } \Gamma, \\ \text{radiation conditions at } \infty. \end{cases}$$

This presupposes that the images of the traces are such that

$$\mathbf{H}_D := \mathbf{H}_D(\mathsf{P};\Gamma) = \mathbf{H}_D(\mathsf{L};\Gamma),$$
$$\mathbf{H}_N := \mathbf{H}_N(\mathsf{P};\Gamma) = \mathbf{H}_N(\mathsf{L};\Gamma).$$

Assumption 5. *Given a source term $f \in \mathbf{L}^2\left(\Omega^-\right)$ and boundary data $(g,\eta) \in \mathbf{H}_D \times \mathbf{H}_\mathbf{N}$, the transmission problem* (TP$_\lambda$) *is uniquely solvable in $\mathbf{L}^2\left(\mathbb{R}^3\right)$.*

The idea behind the so-called symmetric approach to marrying domain and boundary variational formulations originally developed in [13] for problems involving linear strongly elliptic differential operators is to introduce a particularly clever choice of Dirichlet-to-Neumann map

$$\mathsf{DtN} : \mathbf{H}_D \to \mathbf{H}_N$$

into Green's first formula—the validity of which, following Costabel, we must also require here.

Assumption 6 (Green's first formula)**.** *There exist a non-trivial subspace $\mathbf{V}\left(\Omega^-\right) \subset \mathbf{X}^D\left(\mathsf{L};\Omega^-\right)$ and a bilinear form $\mathbf{\Phi} : \mathbf{V}\left(\Omega^-\right) \times \mathbf{V}\left(\Omega^-\right) \to \mathbb{R}$ such that*

$$\int_{\Omega^-} \mathsf{L}U \cdot V \,\mathrm{d}\mathbf{x} = \mathbf{\Phi}(U,V) + \left\langle \mathsf{T}^-_{\mathsf{L},N}U, \mathsf{T}_{\mathsf{L},\mathbf{\Phi}}V \right\rangle_\Gamma \tag{19}$$

for every $U \in \mathbf{X}\left(\mathsf{L};\Omega^-\right)$ and $V \in \mathbf{V}\left(\Omega^-\right)$. We assume that $\mathsf{T}_{\mathsf{L},\mathbf{\Phi}} : \mathbf{V}\left(\Omega^-\right) \to \mathbf{H}_D$ is surjective and continuous.

We have introduced yet another linear trace mapping $\mathsf{T}_{\mathsf{L},\Psi}$ in place of the Dirichlet trace that would justifiably be expected by analogy with the classic Green's first formula. It is true that this map happens to be the Dirichlet trace when L is the scalar Helmholtz operator or the classical electric wave operator. The reason behind this generalization becomes apparent from equation (14). Its purpose is to generalize Green's first formula to account for mixed formulations.

Assumption 6 states that for $g \in \mathbf{H}_D$,

$$\text{(IVP)} \qquad \begin{cases} \text{Seek } U \in \mathbf{V}\left(\Omega^-\right) \cap \{\mathsf{T}^-_{\mathsf{L},D}U = g\} \text{ satisfying} \\ \qquad \mathbf{\Phi}(U,V) = 0 \\ \text{for all } V \in \mathbf{V}\left(\Omega^-\right) \cap \ker\left(\mathsf{T}_{\mathsf{L},\mathbf{\Phi}}\right). \end{cases}$$

is a weak variational formulation for the interior Dirichlet problem

$$\text{(IP)} \qquad \begin{cases} \mathsf{L}U = 0, & \text{in } \Omega^- \\ \mathsf{T}^{\mp}_{\mathsf{L},D}U = g, & \text{on } \Gamma. \end{cases}$$

By testing with $V \in \mathscr{D}\left(\Omega^-\right)$, we immediately find that a solution $U \in \mathbf{V}\left(\Omega^-\right)$ solves $\mathsf{L}U = 0$ in the sense of distributions. Therefore, it also solves (IP) in $\mathbf{L}^2(\Omega)$ if it is regular enough. It is necessary and sufficient that $U \in \mathbf{X}\left(\mathsf{L};\Omega^-\right)$. It is thus reasonable to assume the following regularity result.

Assumption 7 (Regularity)**.** *A distribution $U \in \mathbf{V}\left(\Omega^-\right)$ which solves $\mathsf{L}U = 0$ in the sense of distributions also belongs to $\mathbf{X}\left(\mathsf{L};\Omega^-\right)$.*

Either row of the exterior Caldéron projection $\mathbb{P}_\lambda^+$ realizes a Dirichlet-to-Neumann map [24, Sec. 3.7]:

$$\mathcal{A}_\lambda^{-1}\left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right) : \mathbf{H}_D \to \mathbf{H}_N,$$
$$\left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)^{-1}\mathcal{D}_\lambda : \mathbf{H}_D \to \mathbf{H}_N.$$

Costabel's insight was to combine both rows into the expression

$$\mathsf{DtN} := -\mathcal{D}_\lambda - \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\mathcal{A}_\lambda^{-1}\left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right).$$

Introducing the transmission conditions into (19), the scattering problem (TP_λ) can be cast into the operator equation

$$\boldsymbol{\Psi}U - (\mathsf{T}_{\mathsf{L},\boldsymbol{\Phi}})^*\left(\mathsf{DtN}^+ \circ \mathsf{T}_D^- U\right) = \mathbf{r.h.s.},$$

where $(\mathsf{T}_{\mathsf{L},\boldsymbol{\Phi}})^* : \mathbf{H}_N \to \mathbf{X}^N\left(\mathsf{L};\Omega^-\right)$ denotes the adjoint of $\mathsf{T}_{\mathsf{L},\boldsymbol{\Phi}}$ and $\boldsymbol{\Psi}U(V) := \boldsymbol{\Phi}(U,V)$.

Of course, we dispense with the explicit inverse of $\mathcal{A}_\lambda$ by introducing an auxiliary unknown $\xi \in \mathbf{H}_N$ and seeking a solution pair to the following variational problem.

$$(\mathrm{CP}_\lambda)\qquad \begin{cases} \text{Seek } (U,\xi) \in \mathbf{V}(\Omega) \times \mathbf{H}_N \text{ satisfying} \\ \boldsymbol{\Phi}(U,V) + \Big\langle \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi, \mathsf{T}_{\mathsf{L},\boldsymbol{\Phi}}V\Big\rangle \\ \qquad\qquad + \Big\langle -\mathcal{D}_\lambda \mathsf{T}_{\mathsf{L},D}^- U, \mathsf{T}_{\mathsf{L},\boldsymbol{\Phi}}V\Big\rangle = \mathsf{R}_{\mathbf{V}}(v), \\ \quad \Big\langle \left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right)\mathsf{T}_{\mathsf{L},D}^- U, \zeta\Big\rangle + \big\langle \mathcal{A}_\lambda \xi, \zeta\big\rangle = \mathsf{R}_{\mathsf{T}}(\zeta), \\ \quad \text{for all } (V,\zeta) \in \mathbf{V}(\Omega) \times \mathbf{H}_N. \end{cases}$$

A few terms were moved to the continuous functionals on the right hand sides. In particular,

$$\mathsf{R}_{\mathbf{V}}(V) := \int_{\Omega^-} f \cdot V \,\mathrm{d}\mathbf{x} - \big\langle \eta, \mathsf{T}_{\mathsf{L}\boldsymbol{\Phi}}V\big\rangle - \big\langle \mathcal{D}_\lambda g, \mathsf{T}_{\mathsf{L},\boldsymbol{\Phi}}V\big\rangle,$$
$$\mathsf{R}_{\mathsf{T}}(\zeta) := \Big\langle \left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right) g, \zeta\Big\rangle.$$

4. Resonant frequencies

We call *Dirichlet or Neumann resonant frequency* any eigenvalue in the Dirichlet or Neumann spectrum

$$\Lambda_D\left(\mathsf{P},\Omega^-\right) := \{\lambda \in \mathbb{C} |\, \exists U \in \mathbf{X}\left(\mathsf{P};\Omega^-\right), 0 \neq U \text{ solving}(\mathrm{DP}_\lambda^-) \text{ with } g = 0\},$$
$$\Lambda_N\left(\mathsf{P},\Omega^-\right) := \{\lambda \in \mathbb{C} |\, \exists U \in \mathbf{X}\left(\mathsf{P};\Omega^-\right), o \neq U \text{ solving}(\mathrm{NP}_\lambda^-) \text{ with } \eta = 0\},$$

respectively. Given a frequency $\lambda \in \Lambda_D$ or Λ_N, we denote the λ-eigenspaces by

$$E_D^\lambda\left(\mathsf{P},\Omega^-\right) := \{U \in \mathbf{X}\left(\mathsf{P};\Omega^-\right) |\, U \text{ solving } (\mathrm{DP}_\lambda^-) \text{ with } g = 0\},$$
$$E_N^\lambda\left(\mathsf{P},\Omega^-\right) := \{U \in \mathbf{X}\left(\mathsf{P};\Omega^-\right) |\, U \text{ solving } (\mathrm{NP}_\lambda^-) \text{ with } \eta = 0\},$$

respectively.

4.1. **Kernels of first-kind direct boundary integral equations.** The eigenfunctions in $E_D^\lambda\left(\mathsf{P},\Omega^-\right)$ and $E_N^\lambda\left(\mathsf{P},\Omega^-\right)$ foil uniqueness of solutions of the boundary integral problems (DBP$_\lambda$) and (NBP$_\lambda$). The next lemmas completely characterize the kernels of the operators $\mathcal{A}_\lambda$ and $\mathcal{D}_\lambda$.

Lemma 2. $\ker(\mathcal{A}_\lambda) = \mathsf{T}^-_{\mathsf{P},N}\left(E_D^\lambda\left(\mathsf{P},\Omega^-\right)\right)$

Proof. ($\Leftarrow$) Suppose that $\lambda \in \Lambda_D$ and let $0 \neq U \in E_D^\lambda\left(\mathsf{P},\Omega^-\right)$. By Lemma 1, the valid Cauchy data $\left(0, \mathsf{T}^-_{\mathsf{P},N}U\right) \in \mathbf{H}_D \times \mathbf{H}_N$ for the interior problem is in the kernel of the exterior Caldéron projection. The first row of the matrix equation

$$\begin{pmatrix} -\mathcal{A}_\lambda \mathsf{T}^-_{\mathsf{P},N}U \\ \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\mathsf{T}^-_{\mathsf{P},N}U \end{pmatrix} = \begin{pmatrix} -\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & -\mathcal{A}_\lambda \\ -\mathcal{D}_\lambda & -\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix}\begin{pmatrix} 0 \\ \mathsf{T}^-_{\mathsf{P},N}U \end{pmatrix} = \mathbb{P}^+_\lambda \begin{pmatrix} 0 \\ \mathsf{T}^-_{\mathsf{P},N}U \end{pmatrix} = 0 \tag{20}$$

reads $\mathsf{T}^-_{\mathsf{P},N}U \in \ker(\mathcal{A}_\lambda)$.

($\Rightarrow$) If $\xi \in \mathbf{H}_N$ is such that $\mathcal{A}_\lambda\xi = 0$, then

$$\mathbb{P}^+_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi \end{pmatrix}.$$

Lemma 1 then guarantees that $\left(0, \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi\right)^\top$ is valid Cauchy data for the exterior boundary value problem (DP$^+_\lambda$). By Assumption 2, the unique solution to (DP$^+_\lambda$) with $g = 0$ is trivial, so it must be that $\mathcal{B}_\lambda\xi = \frac{1}{2}\xi$. Therefore, we find that

$$\mathbb{P}^-_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} \mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{A}_\lambda \\ \mathcal{D}_\lambda & \mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix}\begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$$

We conclude relying on Lemma 1 again that there exists $0 \neq U \in E_D^\lambda\left(\mathsf{P},\Omega^-\right)$ with $\mathsf{T}^-_{\mathsf{P},N}U = \xi$. ☐

Because of the formal symmetry in the structure of the Caldéron identities, we also conclude from the above demonstration that the kernel of $\mathcal{D}_\lambda$ is spanned by the Dirichlet traces of the interior Neumann eigenfunctions of P.

Lemma 3. $\ker(\mathcal{D}_\lambda) = \mathsf{T}^-_{\mathsf{P},D}\left(E_N^\lambda\left(\mathsf{P},\Omega^-\right)\right)$

The operators on the right hand sides of (DBP$_\lambda$) and (NBP$_\lambda$) display similar properties.

Lemma 4. $\ker\left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right) = \mathsf{T}^-_{\mathsf{P},N}\left(E_D^\lambda\left(\mathsf{P},\Omega^-\right)\right)$

Proof. ($\Leftarrow$) Suppose that $\lambda \in \Lambda_D$ and let $U \in E_D^\lambda\left(\mathsf{P},\Omega^-\right)$. Using Theorem 1, the valid Cauchy data $\left(0, \mathsf{T}^-_{\mathsf{P},N}U\right) \in \mathbf{H}_D \times \mathbf{H}_N$ belongs to the kernel of $\mathbb{P}^+_\lambda$. We read from (20) that $\mathsf{T}^-_{\mathsf{P},N}U \in \ker\left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)$.

($\Rightarrow$) If $\left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi = 0$, then similarly as in the proof of Lemma 2,

$$\mathbb{P}^+_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} -\mathcal{A}_\lambda\xi \\ 0 \end{pmatrix},$$

which by Lemma 1 shows that $(\mathcal{A}_\lambda\xi, 0)$ is valid Cauchy data for the exterior boundary value problem. By Assumption 2, the unique solution to (DP$^+_\lambda$) with $\eta = 0$ is trivial, so it must be that $\mathcal{A}_\lambda\xi = 0$. The conclusion follows from Lemma 2. ☐

The following result shouldn't come as a surprise now.

Lemma 5. $\ker\left(-\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right) = \mathsf{T}^-_{\mathsf{P},D}\left(E^\lambda_N\left(\mathsf{P},\Omega^-\right)\right)$

Corollary 1. *A solution to the Dirichlet variational problem* (DBP$_\lambda$) *is unique if and only if* $\lambda \notin \Lambda_D$.

Corollary 2. *A solution to the Neumann variational problem* (NBP$_\lambda$) *is unique if and only if* $\lambda \notin \Lambda_N$.

4.2. **Kernel of the domain-boundary symmetric coupling.** At this point, we are well equipped to study the kernel of the operator

$$\mathscr{P} := \begin{pmatrix} \mathbf{\Psi} - \left(\mathsf{T}^-_{\mathsf{L},\mathbf{\Phi}}\right)^* \mathcal{D}_\lambda \circ \mathsf{T}^-_{\mathsf{L},D} & \left(\mathsf{T}^-_{\mathsf{L},\mathbf{\Phi}}\right)^* \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right) \\ \left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right) & \mathcal{A}_\lambda \end{pmatrix}$$

shown here in "variational arrangement".

Proposition 1. *The following are equivalent.*

(1) $(U,\xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ *is in the kernel of* $\mathscr{P}$.

(2) The pair $(U,\xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ *is such that*

- $\mathsf{L}U = 0$ *in the sense of distribution,*
- $\left(\mathsf{T}^-_{\mathsf{L},N}U - \xi\right) \in \mathsf{T}^-_{\mathsf{P},N}E^\lambda_D\left(\mathsf{P},\Omega^-\right)$,
- $\left(\mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}^-_{\mathsf{L},N}U\right)$ *is valid Cauchy data for* (DP$^+_\lambda$).

Proof. $(1 \Rightarrow 2)$ Suppose that $(U,\xi) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ is such that

$$\begin{aligned} \langle \mathbf{\Psi}U, V\rangle + \left\langle \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi, \mathsf{T}_{\mathsf{L},\mathbf{\Phi}}V\right\rangle + \left\langle -\mathcal{D}_\lambda \mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}_{\mathsf{L},\mathbf{\Phi}}V\right\rangle &= 0, \\ \left\langle \left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right)\mathsf{T}^-_{\mathsf{L},D}U, \zeta\right\rangle + \langle \mathcal{A}_\lambda\,\xi, \zeta\rangle &= 0. \end{aligned}$$

There are three elements that we need to check.

Testing with $V \in \mathscr{D}(\Omega)$, we immediately find that $\mathsf{L}U = 0$ holds in the sense of distributions ✓.

Therefore, we can rely on Assumption 7 and use the generalized version (19) of Green's first formula to obtain

$$\left\langle \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi, \mathsf{T}^-_{\mathsf{L},\mathbf{\Phi}}V\right\rangle + \left\langle -\mathcal{D}_\lambda \mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}^-_{\mathsf{L},\mathbf{\Phi}}V\right\rangle = \langle \mathsf{T}^-_{\mathsf{L},N}U, \mathsf{T}^-_{\mathsf{L},\mathbf{\Phi}}V\rangle, \tag{21}$$

$$\left\langle \left(\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id}\right)\mathsf{T}^-_{\mathsf{L},D}U, \zeta\right\rangle + \langle \mathcal{A}_\lambda\,\xi, \zeta\rangle = 0. \tag{22}$$

This allows us to evaluate

$$\mathbb{P}^+_\lambda \begin{pmatrix} \mathsf{T}^-_{\mathsf{L},D}U \\ \xi \end{pmatrix} = \begin{pmatrix} -\mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & -\mathcal{A}_\lambda \\ -\mathcal{D}_\lambda & -\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}^-_{\mathsf{L},D}U \\ \xi \end{pmatrix} = \begin{pmatrix} \mathsf{T}^-_{\mathsf{L},D}U \\ \mathsf{T}^-_{\mathsf{L},N}U \end{pmatrix},$$

where the last equality was obtained by adding $\mathsf{T}^-_{\mathsf{L},D}U$ on both sides of (22). By Theorem 1, this tells us that the pair $\left(\mathsf{T}^-_{\mathsf{L},D}U, \mathsf{T}^-_{\mathsf{L},N}U\right) \in \mathbf{H}_D \times \mathbf{H}_N$ is valid exterior Cauchy data for (DP$^+_\lambda$) ✓.

As such, it must lie in the kernel of the interior Caldéron projector. Since from (21) and (22) respectively we know that

$$\begin{aligned} \mathcal{A}_\lambda \mathsf{T}_{\mathsf{L},D}U &= -\mathcal{A}_\lambda \xi, \\ \mathcal{D}_\lambda \mathsf{T}_{\mathsf{L},D}U &= \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\xi - \mathsf{T}_{\mathsf{L},N}U, \end{aligned}$$

we find that

$$0 = \mathbb{P}_\lambda^- \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix} = \begin{pmatrix} \mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{A}_\lambda \\ \mathcal{D}_\lambda & \mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix} = \begin{pmatrix} \mathcal{A}_\lambda \left(\mathsf{T}_{\mathsf{L},N}^- U - \xi\right) \\ \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right)\left(\xi - \mathsf{T}_{\mathsf{L},N}^- U\right) \end{pmatrix} \tag{23}$$

and conclude from Lemma 2 that $\mathsf{T}_{\mathsf{L},N}^- U - \xi \in \mathsf{T}_{\mathsf{P},N}^- E_D^\lambda \left(\mathsf{P}, \Omega^-\right)$ ✓.

($2 \Rightarrow 1$) Since $\mathsf{T}_{\mathsf{L},N}^- U - \xi$ is the interior Neumann trace of a Dirichlet λ-eigenfunction of P, it follows from Lemma 2 and Lemma 4 that

$$\mathcal{A}_\lambda \mathsf{T}_N^- U = \mathcal{A}_\lambda \xi,$$

and

$$\left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right) \mathsf{T}_N^- U = \left(-\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id}\right) \xi.$$

Moreover, because $\mathsf{L}U = 0$ in the sense of distributions, then Assumption 7 guarantees that $U \in \mathbf{X}\left(\mathsf{L}, \Omega^-\right)$ with $\left\langle \mathbf{\Psi} U, V \right\rangle = \left\langle -\mathsf{T}_{\mathsf{L},N}^- U, \mathsf{T}_{\mathsf{L},\mathbf{\Phi}} V \right\rangle_\Gamma$ for all $V \in \mathbf{V}\left(\Omega^-\right)$.

Therefore,

$$\begin{aligned} \left\langle \mathscr{P} \begin{pmatrix} U \\ \xi \end{pmatrix}, \begin{pmatrix} V \\ \zeta \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} \mathcal{C}_\lambda + \frac{1}{2}\mathrm{Id} & \mathcal{A}_\lambda \\ -\mathcal{D}_\lambda & -\mathcal{B}_\lambda + \frac{1}{2}\mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix} - \begin{pmatrix} 0 \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix}, \begin{pmatrix} \mathsf{T}_{\mathsf{L},\mathbf{\Phi}} V \\ \zeta \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \mathbb{P}_\lambda^- \begin{pmatrix} \mathsf{T}_{\mathsf{L},D}^- U \\ \mathsf{T}_{\mathsf{L},N}^- U \end{pmatrix}, \begin{pmatrix} \mathsf{T}_{\mathsf{L},\mathbf{\Phi}} V \\ \zeta \end{pmatrix} \right\rangle \end{aligned}$$

vanishes for all $(V, \zeta) \in \mathbf{V}(\Omega) \times \mathbf{H}_N$, since valid exterior Cauchy data for $\mathsf{P} - \lambda\mathrm{Id}$ lies in the kernel of the interior Caldéron projector. This shows $(U, \xi) \in \ker(\mathscr{P})$. □

The previous characterization is technical, but it tells us a lot more than meets the eye. It leads to the following main result.

Theorem 1. *The interior function $U \in \mathbf{V}\left(\Omega^-\right)$ of a solution pair $(U, \xi) \in \mathbf{V}\left(\Omega^-\right) \times \mathbf{H}_N$ solving the coupled variational problem (CP_λ) is always unique. If $\lambda \notin \Lambda_D$, then the boundary data ξ is also unique. It is otherwise only unique up to summation with a boundary function lying in $\mathsf{T}_{\mathsf{P},N}^-\left(E_D^\lambda\left(\mathsf{P}, \Omega^-\right)\right)$. In other words,*

$$\ker(\mathscr{P}) = \{0\} \times \mathsf{T}_{\mathsf{P},N}^- E_D^\lambda\left(\mathsf{P}, \Omega^-\right). \tag{24}$$

Proof. Suppose that the pair $\left(U^{\mathrm{in}}, \xi\right) \in \mathbf{V}(\Omega^-) \times \mathbf{H}_N$ is in the kernel of $\mathscr{P}$. By Proposition 1, $U^{\mathrm{in}} \in \mathbf{X}\left(\mathsf{L}, \Omega^-\right)$ and it solves $\mathsf{L}U^{\mathrm{in}} = 0$ in the sense of distributions. The lemma also guarantees that the boundary field $\left(\mathsf{T}_{\mathsf{L},D}^- U^{\mathrm{in}}, \mathsf{T}_{\mathsf{L},N}^- U^{\mathrm{in}}\right)$ is valid Cauchy data for (DP_λ^+). Thus, $\exists U^{\mathrm{ext}} \in \mathbf{X}\left(\mathsf{P}, \Omega^+\right)$ with $\mathsf{P}U - \lambda U = 0$ satisfying $\mathsf{T}_{\mathsf{L},D}^- U^{\mathrm{in}} = \mathsf{T}_{\mathsf{P},D}^+ U^{\mathrm{ext}}$ and $\mathsf{T}_{\mathsf{L},N}^- U^{\mathrm{in}} = \mathsf{T}_{\mathsf{P},N}^+ U^{\mathrm{ext}}$.

The function $U \in \mathbf{L}^2\left(\mathbb{R}^3\right)$ defined by $U|_{\Omega^-} := U^{\mathrm{in}}$ and $U|_{\Omega^+} := U^{\mathrm{ext}}$, where the restrictions are understood in the sense of distributions, is a solution to the transmission problem (TP_λ) with $g = 0$ and $\eta = 0$. Therefore, by Assumption 5, it can only be the trivial solution.

In particular, $U^{\mathrm{in}} = 0$. Going back to Proposition 1 with this new information, we are left with the assertion that $\xi \in \mathsf{T}_{\mathsf{P},N}^- E_D^\lambda\left(\mathsf{P}, \Omega^-\right)$. □

4.3. **Recovery of field solution in Ω^+.** In practice, one is less interested by the solution pair $\left(U^{\text{in}},\, \xi\right)$ of (CP$_\lambda$) than by the actual simulation $\left(U^{\text{in}},\, U^{\text{ext}}\right)$ solving the transmission problem (TP$_\lambda$). To recover the exterior function U^{ext}, we use the exterior representation formula

$$U^{\text{ext}} = -\mathcal{SL}_\lambda\left(\mathsf{T}^+_{\mathsf{P},N}U\right) - \mathcal{DL}_\lambda\left(\mathsf{T}^+_{\mathsf{P},D}U\right) \tag{25}$$

obtained from (18). This step was called *post-processing* in the introduction.

It goes as follows. The right hand side of (25) defines an operator $\mathfrak{R} : \mathbf{H}_D \times \mathbf{H}_N \to \mathbf{X}\left(\mathsf{P}, \Omega^+\right)$ by

$$\mathfrak{R}\begin{pmatrix} h \\ \zeta \end{pmatrix} = -\mathcal{SL}_\lambda\left(\zeta\right) - \mathcal{DL}_\lambda\left(h\right)$$

Therefore, given a solution pair $\left(U^{\text{in}},\, \xi\right)$ solving (CP$_\lambda$), one retrieves the value of the scattered wave at a location $\mathbf{x} \in \Omega^+$ in the exterior region by computing

$$U^{\text{ext}}\left(\mathbf{x}\right) = \mathfrak{R}\begin{pmatrix} \mathcal{T}_{\mathsf{L},D}U^{\text{in}} - g \\ \xi \end{pmatrix}\left(\mathbf{x}\right). \tag{26}$$

Because (24) was established in Theorem 1, we need to verify the following.

Proposition 2. $\{0\} \times \mathsf{T}^-_{\mathsf{P},N}E^\lambda_D\left(\mathsf{P}, \Omega^-\right) \subset \ker\left(\mathfrak{R}\right)$

Proof. Let $\xi \in \mathsf{T}^-_{\mathsf{P},N}E^\lambda_D\left(\mathsf{P}, \Omega^-\right)$. Using the jump identities of Assumption 4, we notice that

$$\begin{aligned} \mathsf{T}^+_{\mathsf{P},D}\,\mathfrak{R}\begin{pmatrix} 0 \\ \xi \end{pmatrix} &= -\left\{\mathsf{T}_{\mathsf{P},D}\right\}\mathcal{SL}_\lambda\left(\xi\right) \\ &= -\mathcal{A}_\lambda\left(\xi\right) \end{aligned}$$

vanishes by Lemma 2. We conclude that $\mathfrak{R}\begin{pmatrix} 0 \\ \xi \end{pmatrix}$ solves (DP$^+_\lambda$) with $g = 0$. By assumption 2, this can only occur for $\mathfrak{R}\left(0\ \xi\right)^\top = 0$ in $\mathbf{L}^2\left(\Omega^+\right)$. ☐

Since $\mathfrak{R}$ is linear, this confirms uniqueness of the pair $\left(U^{\text{in}},\, U^{\text{ext}}\right)$, and along with it validity of the coupled problem (CP$_\lambda$) as a physical model for electromagnetic and acoustic transmission problems.

5. Conclusion

We have abstracted the common characteristics of the three particular problems presented in 2.2. As a consequence, Costabel's original symmetric coupling was generalized to allow for a larger class of operators. The issues raised by spurious resonant frequencies were found to be rooted in the formal structure detailed by the framework of section 2. In section 4, the consequences of their existence were investigated. In doing so, the kernels of the operators entering the problems (DBP$_\lambda$), (NBP$_\lambda$) and (CP$_\lambda$) were completely characterized. It was also shown that the Neumann eigenfunctions which thwart the uniqueness of solutions for the coupled problem vanish under the exterior representation formula, thus showing that the complete field solution U remains unique despite the existence of spurious resonance frequencies. Symmetric couplings therefore remain valuable starting point for Galerkin discretization.

References

[1] Franck Assous, Patrick Ciarlet, and Simon Labrunie. *Mathematical foundations of computational electromagnetism*, volume 198. Springer, 2018.

[2] Alfredo Bermúdez de Castro and Alberto Valli, editors. *Maxwell's Equations: Continuous and Discrete*, pages 1–58. Springer International Publishing, Cham, 2015.

[3] Helmut Brakhage and Peter Werner. Über das dirichletsche aussenraumproblem für die helmholtzsche schwingungsgleichung. *Archiv der Mathematik*, 16(1):325–329, 1965.

[4] Annalisa Buffa and Patrick Ciarlet Jr. On traces for functional spaces related to maxwell's equations part i: An integration by parts formula in lipschitz polyhedra. *Mathematical Methods in the Applied Sciences*, 24(1):9–30, 2001.

[5] Annalisa Buffa and Patrick Ciarlet Jr. On traces for functional spaces related to maxwell's equations part ii: Hodge decompositions on the boundary of lipschitz polyhedra and applications. *Mathematical Methods in the Applied Sciences*, 24(1):31–48, 2001.

[6] Annalisa Buffa, Martin Costabel, and Dongwoo Sheen. On traces for h (curl) in lipschitz domains. *J. Math. Anal. Appl*, 276(2):845–867, 2002.

[7] Annalisa Buffa and Ralf Hiptmair. Galerkin boundary element methods for electromagnetic scattering. In *Topics in computational wave propagation*, pages 83–124. Springer, 2003.

[8] Annalisa Buffa, Ralf Hiptmair, Tobias von Petersdorff, and Christoph Schwab. Boundary element methods for maxwell transmission problems in lipschitz domains. *Numerische Mathematik*, 95(3):459–485, 2003.

[9] Weng Cho Chew. Vector potential electromagnetics with generalized gauge for inhomogeneous media: Formulation. *Progress In Electromagnetics Research*, 149:69–84, 2014.

[10] Snorre Christiansen. Discrete fredholm properties and convergence estimates for the electric field integral equation. *Mathematics of Computation*, 73(245):143–167, 2004.

[11] Xavier Claeys and Ralf Hiptmair. First-kind boundary integral equations for the hodge-helmholtz equation. *SAM Research Report*, 2017, 2017.

[12] David Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93. Springer Science & Business Media, 2012.

[13] Martin Costabel. Symmetric methods for the coupling of finite elements and boundary elements (invited contribution). In *Mathematical and Computational Aspects*, pages 411–420. Springer, 1987.

[14] Leszek Demkowicz. Asymptotic convergence in finite and boundary element methods: Part 1: Theoretical results. *Computers & Mathematics with Applications*, 27(12):69–84, 1994.

[15] Christophe Hazard and Marc Lenoir. On the solution of time-harmonic scattering problems for maxwell's equations. *SIAM Journal on Mathematical Analysis*, 27(6):1597–1630, 1996.

[16] R Hiptmair and P Meury. Stable fem-bem coupling for helmholtz transmission problems. ETH, Seminar für Angewandte Mathematik, 2005.

[17] Ralf Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002.

[18] Ralf Hiptmair. Coupling of finite elements and boundary elements in electromagnetic scattering. *SIAM Journal on Numerical Analysis*, 41(3):919–944, 2003.

[19] Ralf Hiptmair and P Meury. Stabilized fem-bem coupling for helmholtz transmission problems. *SIAM Journal on Numerical Analysis*, 44(5):2107–2130, 2006.

[20] Rainer Kress, V Maz'ya, and V Kozlov. *Linear integral equations*, volume 82. Springer, 1989.

[21] William McLean and William Charles Hector McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge university press, 2000.

[22] Peter Monk et al. *Finite element methods for Maxwell's equations*. Oxford University Press, 2003.

[23] Jean-Claude Nédélec. *Acoustic and electromagnetic equations: integral representations for harmonic problems*. Springer Science & Business Media, 2001.

[24] Stefan A Sauter and Christoph Schwab. Boundary element methods. In *Boundary Element Methods*, pages 183–287. Springer, 2010.

[25] E. Schulz and R. Hiptmair. Coupled domain-boundary variational formulations for hodge-helmholtz operators. Technical Report 2020-21, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2020.

[26] Olaf Steinbach. *Numerical approximation methods for elliptic boundary value problems: finite and boundary elements*. Springer Science & Business Media, 2007.