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COUPLED DOMAIN–BOUNDARY VARIATIONAL FORMULATIONS FOR HODGE-HELMHOLTZ OPERATORS*

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Abstract. We couple the mixed variational problem for the generalized Hodge-Helmholtz or Hodge-Laplace equation posed on a bounded three-dimensional Lipschitz domain with the first-kind boundary integral equation arising from the latter when constant coefficients are assumed in the unbounded complement. Recently developed Calderón projectors for the relevant boundary integral operators are used to perform a symmetric coupling. We prove stability of the coupled problem away from resonant frequencies by establishing a generalized Gårding inequality (T-coercivity). The resulting system of equations describes the scattering of monochromatic electromagnetic waves at a bounded inhomogeneous isotropic body possibly having a "rough" surface. The low-frequency robustness of the potential formulation of Maxwell's equations makes this model a promising starting point for Galerkin discretization.

Key words. Maxwell, electromagnetism, scattering, Hodge-laplace, Hodge-Helmholtz, Hodge decomposition, Helmholtz decomposition, Calderón projector, symmetric coupling, T-coercivity

AMS subject classifications. 35Q61, 35Q60, 65N30, 65N38, 78A45, 78M10, 78M15

1. Introduction. Inside a bounded inhomogeneous isotropic physical body Ω_s, the potential formulation of Maxwell's equations in frequency domain driven by a source current $\mathbf{J}$ with angular frequency $\omega > 0$ reads [11]

$$\begin{aligned} &\mathbf{curl}\,\Big(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}\Big) + i\omega\epsilon(\mathbf{x})\nabla V - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J} \\ &\mathrm{div}\,\Big(\epsilon(\mathbf{x})\mathbf{U}\Big) + i\omega V = 0, \end{aligned} \tag{1.1}$$

where the Lorentz gauge relates the scalar potential V to the vector potential $\mathbf{U}$ in (1.1). Elimination of V using this relation leads to the Hodge-Helmholtz equation

$$\mathbf{curl}\,\Big(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}\Big) - \epsilon(\mathbf{x})\,\nabla\,\mathrm{div}\,\Big(\epsilon(\mathbf{x})\mathbf{U}\Big) - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J}.$$

Away from the source current, in the unbounded region $\Omega' := \mathbb{R}^3\backslash\overline{\Omega}_s$ outside the scatterer, where we assume a homogeneous material with scalar constant permeability μ_0 and dielectric permittivity ϵ_0, equation (1) reduces to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \eta\,\nabla\,\mathrm{div}\,\mathbf{U} - \kappa^2\mathbf{U} = 0,$$

with constant coefficients $\eta = \mu_0\epsilon_0^2$ and $\kappa^2 = \mu_0\epsilon_0\omega^2$.

The material coefficients are assumed to be bounded in $\mathbb{R}^3$, i.e. $\mu, \epsilon \in L^\infty(\mathbb{R}^3)$. In a non-dissipative medium, the functions μ and ϵ are real-valued and uniformly positive. Dissipative effects are captured by adding non-negative imaginary parts to the coefficients [4, Sec. 1.1.3]. We follow [21] and explicitly suppose that

$$\begin{aligned} &0 < \mu_{\min} \leq \mathfrak{Re}(\mu) \leq \mu_{\max}, && 0 \leq \mathfrak{Im}(\mu), \\ &0 < \epsilon_{\min} \leq \mathfrak{Re}(\epsilon) \leq \epsilon_{\max}, && 0 \leq \mathfrak{Im}(\epsilon). \\ &0 \leq \mathfrak{Re}\left(\kappa^2\right), && 0 \leq \mathfrak{Im}\left(\kappa^2\right). \end{aligned}$$

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Let $\Omega_s \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\Gamma =: \partial\Omega$ [36, Def. 2.1]. We suppose for simplicity that its de Rham cohomology is trivial. For given data $\mathbf{J} \in \mathbf{L}^2(\Omega_s)$, $\mathbf{g}_R \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma)$, $g_n \in H^{-1/2}(\Gamma)$, $\zeta_D \in H^{1/2}(\Gamma)$ and $\boldsymbol{\zeta}_t \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma)$, we are interested in the following PDEs in $\mathbb{R}^3$:

Transmission Problem

Volume equations

$$\mathbf{curl}\left(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{U}\right) - \epsilon(\mathbf{x})\,\nabla\,\mathrm{div}\left(\epsilon(\mathbf{x})\,\mathbf{U}\right) - \omega^2\epsilon(\mathbf{x})\,\mathbf{U} = \mathbf{J}, \quad \text{in } \Omega_s, \tag{1.2a}$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^{\mathrm{ext}} - \eta\,\nabla\mathrm{div}\,\mathbf{U}^{\mathrm{ext}} - \kappa^2\,\mathbf{U}^{\mathrm{ext}} = 0, \quad \text{in } \Omega', \tag{1.2b}$$

Transmission conditions

$$\gamma^-_{R,\mu}(\mathbf{U}) = \gamma^+_R\mathbf{U}^{\mathrm{ext}} + \mathbf{g}_R, \quad \gamma^-_{n,\epsilon}(\mathbf{U}) = \gamma^+_n(\mathbf{U}^{\mathrm{ext}}) + g_n, \quad \text{on } \Gamma, \tag{1.3a}$$

$$\gamma^-_{D,\epsilon}(\mathbf{U}) = \eta\,\gamma^+_D\mathbf{U}^{\mathrm{ext}} + \zeta_D, \quad [\gamma_t\mathbf{U}]_\Gamma = \boldsymbol{\zeta}_t, \quad \text{on } \Gamma, \tag{1.3b}$$

where the traces are defined for a smooth vector-field $\mathbf{U}$ by

$$\gamma^-_{R,\mu}(\mathbf{U}) := -\gamma^-_\tau\left(\mu^{-1}(\mathbf{x})\ \mathbf{curl}(\mathbf{U})\right), \qquad \gamma^+_R(\mathbf{U}^{\mathrm{ext}}) := -\gamma^+_\tau\left(\mathbf{curl}\left(\mathbf{U}^{\mathrm{ext}}\right)\right),$$
$$\gamma^-_{D,\epsilon}(\mathbf{U}) := \gamma^-\left(\mathrm{div}\left(\epsilon(\mathbf{x})\ \mathbf{U}\right)\right), \qquad \gamma^+_D(\mathbf{U}^{\mathrm{ext}}) := \gamma^+\left(\mathrm{div}\left(\mathbf{U}^{\mathrm{ext}}\right)\right),$$
$$\gamma^-_{n,\epsilon}(\mathbf{U}) := \gamma^-_n(\epsilon(\mathbf{x})\ \mathbf{U}) \qquad \gamma^\pm_t(\mathbf{U}) := \mathbf{n} \times \left(\gamma^\pm_\tau(\mathbf{U})\right).$$

We used in those definitions the classical traces

$$\gamma(\mathbf{U}) := \mathbf{U}\big|_\Gamma, \qquad \gamma_n(\mathbf{U}) := \gamma(\mathbf{U})\cdot\mathbf{n}, \qquad \gamma_\tau(\mathbf{U}) := \gamma(\mathbf{U})\times\mathbf{n}.$$

Each of these traces can be extended by continuity to larger Sobolev spaces. The more detailed functional analysis setting in which they must be considered will be reviewed in the next section.

The equations (1.3a) and (1.3b) are transmission conditions for Hodge–Helmholtz and Hodge–Laplace problems [21, Sec. 2.1.2]. In literature, condition (1.3a) is labeled as "magnetic", while (1.3b) is referred to as "electric" (simply because one recovers the magnetic field by taking the curl of the potential $\mathbf{U}$). It is very well possible to "guess" these transmission conditions either by glancing at equation (1.2a) or by translating the classical boundary conditions for the electric and magnetic fields to the potential formulation [11]. However, it is emphasized in [14] and [15] that the traces also appear formally in Green's formula

$$\int_{\Omega'} \mathbf{U}\cdot\left(-\Delta_\eta\mathbf{V}\right)\mathrm{d}\mathbf{x} - \mathbf{V}\cdot\left(-\Delta_\eta\mathbf{U}\right)\mathrm{d}\mathbf{x} = -\eta\langle\gamma^+_n\mathbf{U},\gamma^+_D\mathbf{V}\rangle + \eta\langle\gamma^+_D\mathbf{U},\gamma^+_n\mathbf{V}\rangle$$
$$+\langle\gamma^+_R\mathbf{V},\gamma^+_t\mathbf{U}\rangle - \langle\gamma^+_R\mathbf{U},\gamma^+_t\mathbf{V}\rangle,$$

where $-\Delta_\eta := \mathbf{curl}\,\mathbf{curl} - \eta\,\mathrm{div}\,\nabla$ is the Hodge-Laplace operator.

For positive frequencies $\omega > 0$, we supplement (1.2a)-(1.3b) with the variants of the Silver-Muller's radiation condition imposed at infinity provided in [21]. In the static case where $\kappa = \omega = 0$, we seek a solution in an appropriate weighted Sobolev space that accounts for decay conditions [35, Sec. 2.5].

REMARK 1. *When derived from Maxwell's equations stated in terms of the magnetic and electric fields, the classical wave equation for an electric field* $\mathbf{E}$ *reads*

$$\mathbf{curl}\left(\mu^{-1}(\mathbf{x})\,\mathbf{curl}\,\mathbf{E}\right) - \kappa^2\,\epsilon(\mathbf{x})\,\mathbf{E} = \mathbf{J}. \tag{1.4}$$

The regularizing term $\epsilon\nabla\,div(\epsilon\mathbf{U})$ *which appears in* (1.2a)*, but not in* (1.4)*, makes for a significant structural difference [21]. For suitable boundary conditions, the zero-order term* $\omega^2\epsilon\mathbf{U}$ *in* (1.2) *is a compact perturbation in the weak formulation of the Hodge-Helmholtz equation. Ergo, coercivity of the associated boundary value problem is preserved in the low frequency limit* $\omega \to 0$*. This is not the case for the "Maxwell operator" found on the left hand side of* (1.4)*, whose associated scattering equation is characterized by an "incessant conversion" between electric and magnetic energies that play symmetric roles [10]. Functionally, the infinite dimensional kernel of the curl operator thwart compactness of the embedding* $\mathbf{H}(\mathbf{curl},\Omega_s) \hookrightarrow \mathbf{L}^2(\Omega_s)$*. This is different from the weak variational formulation of the scalar Helmholtz equation* $-\Delta u - \kappa^2 u = f$*. In that model of acoustic scattering, potential energy turns out to be a compact perturbation of the kinetic energy due to by Rellich's compact embedding* $H^1(\Omega_s) \hookrightarrow L^2(\Omega_s)$*.*

REMARK 2. *It is stressed in [11] that from the rapid development in quantum optics emerged the need for electromagnetic models valid in both classical and quantum regimes. Robustness of the potential formulation of Maxwell's equations in the low frequency limit makes it a promising candidate for bridging physical scales.*

REMARK 3. *The terminology used above is rooted in geometry. The equations* (1.2a)-(1.2b) *contain generalized instances of the Hodge–Helmholtz operator* $-\Delta - \kappa^2\mathrm{Id} = \delta\,\mathrm{d} + \mathrm{d}\,\delta - \kappa^2\mathrm{Id}$ *as it applies to differential 1-forms defined over 3D differentiable manifolds. When* $\omega = \kappa = 0$*, the left hand sides reduce to applications of the Hodge-Laplace operator. We refer to [25] and [23] for a thorough introduction to the formulation of Maxwell's equations in terms of differential/integral forms.*

REMARK 4. *Boundary integral operators of the* second-kind *were extensively studied in the literature devoted to the Hodge-Laplace and Hodge-Helmholtz operators acting on differential forms over smooth manifolds (e.g. [30], [31], [35] and [28]). However, little attention was paid to the formulation of Hodge-Helmholtz/Laplace boundary value problems as* first-kind *boundary integral equations. Only recently, a boundary integral representation formula for Hodge-Helmholtz/Laplace equation in three-dimensional Lipschitz domains was derived in [14] which leads to boundary integral operators of the first-kind inducing bounded and coercive sesquilinear forms in the natural energy spaces for that equation. These innovative investigations are particularly relevant to the numerical analysis community. Operators admitting natural variational formulations in well-known energy trace spaces via duality are appealing for the development and numerical analysis of new Galerkin discretizations. For the case* $\kappa^2 = 0$ *of the Hodge–Laplace operator in 3D, a thorough* a priori *analysis of a Galerkin BEM was already proposed in [15] with additional experimental evidence.*

In the following, we couple the mixed formulation of the weak variational problem associated to (1.2a) with the first-kind boundary integral equation arising from (1.2b) using these recently developed Calderón projectors for the Hodge–Helmholtz and Hodge-Laplace operators. This paves the way for the design of finite element methods discretizing Hodge-Laplace and Hodge-Helmholtz transmission problems. The proof of the well-posedness of the coupled problem relies on T-coercivity (c.f. [13]).

2. Preliminaries. In the subsequent analysis, we will make use of various spaces that have become classical in the literature concerned with electromagnetism. Development of the trace-related theory for Lipschitz domains can be followed in [6], [7] and [9]. We summarize its details to fix notation and recall important results. In the next subsections, we slightly generalize the traces to account for the varying coefficients of (1.2a) and adapt them to the system of equations at hand. In Section 2.4, we extend the analysis performed in [24] for the classical electric wave equation to the boundary integral operators arising from Hodge-Helmholtz and Hodge-Laplace problems. In this section, Ω can denote either Ω_s or Ω'.

2.1. Volume function spaces. As usual, $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ denote the Hilbert spaces of square integrable scalar and vector-valued functions defined over Ω. We denote their inner products using round brackets, e.g. $(\cdot,\cdot)_\Omega$. Similarly, for $k \in \mathbb{N}$, $H^k(\Omega)$ and $\mathbf{H}^k(\Omega)$ refer to the corresponding Sobolev spaces. We write $C_0^\infty(\Omega)$ for the space of smooth compactly supported function in Ω, but denote by $\mathscr{D}(\Omega)^3$ the analogous space of vector fields to simplify notation. Their closures $H_0^1(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ in the norms of $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively, are the kernels of the scalar and vector-valued Dirichlet traces, which we both denote γ alike. $C^\infty(\overline{\Omega})$ is defined as the space of uniformly continuous functions over Ω that have uniformly continuous derivatives of all order. The Banach spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{div},\Omega) &:= \{\mathbf{U} \in L^2(\Omega) \,|\, \mathrm{div}(\mathbf{U}) \in L^2(\Omega)\},\\
\mathbf{H}(\epsilon;\mathrm{div},\Omega) &:= \{\mathbf{U} \in L^2(\Omega) \,|\, \epsilon(\mathbf{x})\,\mathbf{U} \in \mathbf{H}(\mathrm{div},\Omega)\},\\
\mathbf{H}(\mathbf{curl},\Omega) &:= \{\mathbf{U} \in L^2(\Omega) \,|\, \mathbf{curl}\,(\mathbf{U}) \in L^2(\Omega)\},\\
\mathbf{H}\left(\nabla\mathrm{div},\Omega\right) &:= \{\mathbf{U} \in \mathbf{H}\left(\mathrm{div},\Omega\right) \,|\, \mathrm{div}(\mathbf{U}) \in H^1(\Omega)\},\\
\mathbf{H}\left(\epsilon;\nabla\mathrm{div},\Omega_s\right) &:= \{\mathbf{U} \in \mathbf{L}^2(\Omega) \,|\, \epsilon(\mathbf{x})\,\mathbf{U} \in \mathbf{H}\left(\nabla\mathrm{div},\Omega\right)\},\\
\mathbf{H}(\mathbf{curl}^2,\Omega) &:= \{\mathbf{U} \in \mathbf{H}(\mathbf{curl},\Omega) \,|\, \mathbf{curl}(\mathbf{U}) \in \mathbf{H}(\mathbf{curl},\Omega)\},\\
\mathbf{H}(\mu^{-1};\mathbf{curl}^2,\Omega) &:= \{\mathbf{U} \in \mathbf{H}(\mathbf{curl},\Omega) \,|\, \mu^{-1}\,\mathbf{curl}(\mathbf{U}) \in \mathbf{H}(\mathbf{curl},\Omega)\},
\end{aligned}$$

equipped with the obvious graph norms will prove to be important.

The variational space for the primal variational formulation of the classical and generalized Hodge–Helmholtz/Laplace operator is given by

$$\mathbf{X}(\Delta,\Omega) := \mathbf{H}(\mathbf{curl}^2,\Omega) \cap \mathbf{H}\left(\nabla\mathrm{div},\Omega\right).$$

A subscript is used to identify spaces of locally integrable functions/vector fields, e.g. $U \in L^2_{\mathrm{loc}}(\Omega)$ if and only if ϕU is square-integrable for all $\phi \in C_0^\infty(\mathbb{R}^3)$. We denote with an asterisk the spaces of functions with zero mean, e.g. $H^1_*(\Omega)$.

2.2. Trace spaces. Rademacher's theorem [19, Thm. 3.1.6] guarantees that the boundary $\Gamma =: \partial\Omega$ of a Lipschitz domain Ω admits a surface measure σ and an essentially bounded unit normal vector field $\mathbf{n} \in \mathbf{L}^\infty(\Gamma)$ directed toward the exterior of Ω. These ingredients warrant Gauß' formulae [29, Thm. 3.34].

2.2.1. Classical traces. For $\mathbf{U},\mathbf{V} \in \mathbf{C}^\infty(\overline{\Omega})$ and $P \in C^\infty(\overline{\Omega})$, the two identities $\mathrm{div}(\mathbf{U}P) = \mathrm{div}(\mathbf{U})\,P + \mathbf{U}\cdot\nabla P$ and $\mathrm{div}(\mathbf{U}\times\mathbf{V}) = \mathbf{curl}(\mathbf{U})\cdot\mathbf{V} - \mathbf{curl}(\mathbf{V})\cdot\mathbf{U}$ hold; therefore, the divergence theorem yields —whenever the integrals are defined—

Green's formulae (+ for $\Omega = \Omega_s$)

$$\pm \int_\Omega \operatorname{div}(\mathbf{U})\, P + \mathbf{U} \cdot \nabla P \,\mathrm{d}\mathbf{x} = \int_\Gamma \gamma\,(P)\, \gamma_n\,(\mathbf{U})\,\mathrm{d}\sigma, \tag{2.1a}$$

$$\pm \int_\Omega \mathbf{U} \cdot \mathbf{curl}\,(\mathbf{V}) - \mathbf{curl}\,(\mathbf{U}) \cdot \mathbf{V}\,\mathrm{d}\mathbf{x} = \int_\Gamma \gamma\,(\mathbf{V}) \cdot \gamma_\tau\,(\mathbf{U})\,\mathrm{d}\sigma, \tag{2.1b}$$

where (2.1b) is valid since $\gamma\,(\mathbf{U} \times \mathbf{V}) \cdot \mathbf{n} = -(\gamma\,(\mathbf{U}) \times \mathbf{n}) \cdot \gamma\,(\mathbf{V})$ is defined almost everywhere. Since the unique extension $\gamma : H^1_{\text{loc}}(\Omega) \to H^{1/2}(\Gamma)$ of the Dirichlet trace is a bounded operator with a continuous right-inverse $\mathcal{E}$ [29, Thm. 3.37], these traces can be extended by continuity to bounded operators $\gamma_n : \mathbf{H}_{\text{loc}}\,(\operatorname{div}, \Omega) \to H^{-1/2}(\Gamma)$ and $\gamma_\tau : \mathbf{H}_{\text{loc}}\,(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\Gamma)$ with null spaces $\ker(\gamma_n) = \mathbf{H}_0(\operatorname{div}, \Omega) := \overline{\mathscr{D}(\Omega)^3}^{\mathbf{H}(\operatorname{div},\Omega)}$ and $\ker(\gamma_\tau) = \mathbf{H}_0(\mathbf{curl}, \Omega) := \overline{\mathscr{D}(\Omega)^3}^{\mathbf{H}(\mathbf{curl},\Omega)}$ [20, Chap. 2]. Evidently, these extensions generalize (2.1a) to functions $\mathbf{U} \in \mathbf{H}(\operatorname{div}, \Omega)$, $P \in H^1(\Omega)$ [32, Thm. 3.24] and (2.1b) to $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{V} \in \mathbf{H}^1(\Omega)$ [32, Thm. 3.29], where the boundary terms are to be understood as duality parings $\langle \cdot, \cdot \rangle_\Gamma$ with pivot spaces $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$, respectively.

While the normal component trace γ_n described thus is seen to be surjective [20, Cor. 2.8], it is evident from $(\mathbf{v} \times \mathbf{n}) \cdot \mathbf{n} = 0\ \forall \mathbf{v} \in \mathbf{L}^2(\Gamma)$ that the image of the tangential trace γ_τ acting on $\mathbf{H}_{\text{loc}}\,(\mathbf{curl}, \Omega)$ is a tangential *proper* subspace of $\mathbf{H}^{-1/2}(\Gamma)$. Naturally, the same holds true for the extension $\gamma_t : \mathbf{H}_{\text{loc}}\,(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\Gamma)$ of the tangential components trace $\gamma_t(\mathbf{U}) := \mathbf{n} \times \gamma_\tau(\mathbf{U})$.

Tangential differential operators are required to remedy this problem. For $\boldsymbol{\xi} \in \mathbf{H}^{1/2}(\Gamma)$, let $\operatorname{curl}_\Gamma(\boldsymbol{\xi}) \in H^{-1/2}(\Omega)$ be uniquely determined by

$$\langle \gamma(V), \operatorname{curl}_\Gamma(\boldsymbol{\xi}) \rangle_\Gamma = \langle \gamma_\tau(\nabla V), \boldsymbol{\xi} \rangle_\Gamma, \quad \forall V \in C^\infty(\overline{\Omega}).$$

As $\mathbf{curl} \circ \nabla = 0$, $\nabla \left(\mathbf{H}^1_{\text{loc}}(\Omega)\right) \subset \mathbf{H}_{\text{loc}}\,(\mathbf{curl}, \Omega)$, and the operator $\gamma_\tau \circ \nabla : H^1_{\text{loc}}(\Omega) \to \mathbf{H}^{-1/2}(\Omega)$ is bounded accordingly. In that sense, $\operatorname{curl}_\Gamma : \mathbf{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is adjoint to the vectorial tangential curl operator $\mathbf{curl}_\Gamma := \gamma_\tau \circ \nabla \circ \mathcal{E} : H^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma)$, whose definition is independent of the choice of right-inverse since $\mathbf{H}^1_0(\Omega) \subset \ker(\gamma_\tau \circ \nabla)$. Concretely, Green's formulae show that $\operatorname{curl}_\Gamma = \gamma_n \circ \mathbf{curl} \circ \mathcal{E}$. Independence of this expression from the choice of lifting $\mathcal{E}$ is guaranteed by the inclusion of $\mathbf{H}^1_0(\Omega)$ in $\ker\,(\gamma_n \circ \mathbf{curl})$. Similarly, the tangential divergence $\operatorname{div}_\Gamma : \mathbf{H}^{1/2}(\Gamma) \to H^{-1/2}(\Omega)$, defined as the rotated operator $\operatorname{div}_\Gamma(\mathbf{p}) := \operatorname{curl}_\Gamma\,(\mathbf{n} \times \mathbf{p})$, is adjoint to the negative surface gradient $\nabla_\Gamma := \gamma_t \circ \nabla \circ \mathcal{E}$, that is $\langle \nabla_\Gamma\, q, \mathbf{p} \rangle_\tau = -\langle\, q, \operatorname{div}_\Gamma\,(\mathbf{p}) \rangle$.

The space of traces $\mathbf{H}^{1/2}_T(\Gamma) := \gamma_t(\mathbf{H}^1_{\text{loc}}(\Omega))$ and of rotated traces $\mathbf{H}^{1/2}_R(\Gamma) := \mathbf{n} \times \mathbf{H}^{1/2}_T(\Gamma)$ are complete when equipped with the norms

$$\|\mathbf{v}\|_{\mathbf{H}^{1/2}_T(\Gamma)} := \inf\{\mathbf{U} \in \mathbf{H}^1_{\text{loc}}(\Omega) \,|\, \gamma_t(\mathbf{U}) = \mathbf{v}\}, \qquad \|\mathbf{u}\|_{\mathbf{H}^{1/2}_R(\Gamma)} := \|\mathbf{n} \times \mathbf{u}\|_{\mathbf{H}^{1/2}_T(\Gamma)}\,,$$

that enforce continuity of the traces [9, def. 2.2].

Lemma 2.1 (See [24, Lem. 3.2]). *The embeddings $H^{1/2}_T(\Gamma)$, $H^{1/2}_R(\Gamma) \hookrightarrow \mathbf{L}^2_t(\Gamma)$ are compact.*

Based on these intermediate spaces, we define subspaces

$$\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma) := \{\boldsymbol{\xi} \in \mathbf{H}^{-1/2}_T(\Omega) \,|\, \operatorname{curl}_\Gamma(\boldsymbol{\xi}) \in H^{-1/2}(\Gamma)\}, \tag{2.2a}$$

$$\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) := \{\mathbf{p} \in \mathbf{H}^{-1/2}_R(\Omega) \,|\, \operatorname{div}_\Gamma(\mathbf{p}) \in H^{-1/2}(\Gamma)\}, \tag{2.2b}$$

onto which γ_t and γ_τ are continuous and surjective [9, Thm. 4.1], respectively. These subspaces can be put in duality with a pairing $\langle\cdot,\cdot\rangle_\tau$ for which $\mathbf{L}^2_t(\Gamma)$, the space of square integrable tangent vector fields on Γ, act as pivot space [9, Sec. 5]. Then, the extension of (2.1b) to any $\mathbf{U},\mathbf{V}\in\mathbf{H}(\mathbf{curl},\Omega)$ reads [32, Thm. 3.31]

$$\pm\int_\Omega \mathbf{U}\cdot\mathbf{curl}\,(\mathbf{V})-\mathbf{curl}\,(\mathbf{U})\cdot\mathbf{V}\,\mathrm{d}\mathbf{x}=\langle\gamma_t\,(\mathbf{V})\,,\gamma_\tau\,(\mathbf{U})\rangle_\tau. \tag{2.3}$$

Finally, upon defining $\gamma_R:=-\gamma_\tau\circ\mathbf{curl}:\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2,\Omega)\to\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, which satisfies for all $\mathbf{V}\in\mathbf{H}(\mathbf{curl},\Omega)$ the crucial integral identity

$$\pm\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{V}-\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\mathbf{V}\,\mathrm{d}\mathbf{x}=\langle\gamma_t(\mathbf{V}),\gamma_R(\mathbf{U})\rangle_\tau, \tag{2.4}$$

and $\gamma_D:=\gamma\circ\mathrm{div}:\mathbf{H}_{\mathrm{loc}}(\nabla\mathrm{div},\Omega)\to H^{1/2}(\Gamma)$, we are equipped with a full set of traces to tackle Hodge–Laplace and Hodge–Helmholtz problems.

We indicate with curly brackets the average

$$\{\gamma_\bullet\}:=\frac{1}{2}\left(\gamma_\bullet^++\gamma_\bullet^-\right)$$

of a trace and with square brackets its jump

$$[\gamma_\bullet]:=\gamma_\bullet^--\gamma_\bullet^+$$

over the interface Γ, $\bullet=R$, D, t, τ, or n. Analogous notation will be used for the compounded traces introduced in the next section.

WARNING. *Notice the sign in the jump* $[\gamma]=\gamma^--\gamma^+$*, which is often taken to be the opposite in the literature!*

LEMMA 2.2 (See [14, Lem. 6.4]). *The surface divergence extends to a continuous surjection* $\mathrm{div}_\Gamma:\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\to H_*^{-1/2}(\Gamma)$*, while* $\mathbf{curl}_\Gamma:H_*^{1/2}\to\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *is a bounded injection with closed range such that* $\mathbf{curl}_\Gamma(\xi)=\nabla_\Gamma(\xi)\times\mathbf{n}$ *for all* $\xi\in H^{1/2}(\Gamma)$*. They satisfy* $\mathrm{div}_\Gamma\circ\mathbf{curl}_\Gamma=0$.

2.2.2. Compounded trace spaces. As explained in [14, Sec. 3], a theory of differential equations for the Hodge–Helmholtz/Laplace problem in three dimensions entails partitioning our collection of traces into two "dual" pairs. Accordingly, we now introduce new mappings $\mathcal{T}^-_{D,\epsilon}:\mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\Omega_s)\cap\mathbf{H}_{\mathrm{loc}}(\epsilon;\nabla\mathrm{div},\Omega_s)\to\mathcal{H}_D(\Gamma)$ and $\mathcal{T}^-_{N,\mu}:\mathbf{H}_{\mathrm{loc}}(\mu^{-1};\mathbf{curl}^2,\Omega_s)\cap\mathbf{H}_{\mathrm{loc}}(\epsilon;\mathrm{div},\Omega_s)\to\mathcal{H}_N(\Gamma)$ which we define by

$$\mathcal{T}^-_{D,\epsilon}(\mathbf{U}):=\begin{pmatrix}\gamma_t^-(\mathbf{U})\\ \gamma_{D,\epsilon}^-(\mathbf{U})\end{pmatrix},\qquad\qquad \mathcal{T}^-_{N,\mu}(\mathbf{U}):=\begin{pmatrix}\gamma_{R,\mu}(\mathbf{U})\\ \gamma_{n,\epsilon}^-(\mathbf{U})\end{pmatrix},$$

respectively, where

$$\mathcal{H}_D:=\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)\times H^{1/2}(\Gamma),\qquad \mathcal{H}_N:=\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\Gamma).$$

The related classical compounded traces are defined similarly by

$$\mathcal{T}_D^+(\mathbf{U}) := \begin{pmatrix} \gamma_t^+(\mathbf{U}) \\ \gamma_{D,\eta}^+(\mathbf{U}) \end{pmatrix}, \qquad \mathcal{T}_N^+(\mathbf{U}) := \begin{pmatrix} \gamma_R(\mathbf{U}) \\ \gamma_n(\mathbf{U}) \end{pmatrix}.$$

The choice of subscripts is motivated by the analogy between this pair of traces and the classical Dirichlet and Neumann boundary conditions for second-order elliptic BVPs. The trace spaces $\mathcal{H}_D$ and $\mathcal{H}_N$ are put in duality using the sum of the inherited component-wise duality parings. That is, for $\vec{\mathbf{p}} = (\mathbf{p}, q) \in \mathcal{H}_N$ and $\vec{\boldsymbol{\eta}} = (\boldsymbol{\eta}, \zeta) \in \mathcal{H}_D$, we define

$$\langle \vec{\mathbf{p}}, \vec{\boldsymbol{\eta}} \rangle := \langle \mathbf{p}, \boldsymbol{\eta} \rangle_\tau + \langle q, \zeta \rangle_\Gamma. \tag{2.5}$$

The importance of the spaces (2.2a)-(2.2b) is highlighted by this next result.

Lemma 2.3 (See [14, Lem. 3.2]). *The compound traces $\mathcal{T}_D$ and $\mathcal{T}_N$ have continuous right inverses, i.e. lifting maps $\mathcal{R}_D : \mathcal{H}_D \to \mathbf{X}(\Delta, \Omega)$ and $\mathcal{R}_N \to \mathbf{X}(\Delta, \Omega)$, respectively.*

2.3. Boundary potentials. By exploiting the radiating fundamental solution

$$G_\nu(\mathbf{x}) := \exp\big(i\nu|\mathbf{x}|\big)/4\pi|\mathbf{x}|$$

for the scalar Helmholtz operator $\Delta - \nu^2\mathrm{Id}$, it is shown in [14, Sec. 4.2] that a distributional solution $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$ such that $\mathbf{U}|_{\Omega_s} \in \mathbf{X}(\Delta, \Omega_s)$ and $\mathbf{U}|_{\Omega'} \in \mathbf{X}_{\mathrm{loc}}(\Delta, \Omega')$ of the homogeneous (scaled) Hodge–Helmholtz/Laplace equation (1.2b) with constant coefficients $\eta > 0$, $\kappa \geq 0$, stated in the whole of $\mathbb{R}^3$ with radiation conditions at infinity as considered in Section 1, affords a representation formula.

Representation Formula

$$\mathbf{U} = \mathcal{SL}_\kappa \cdot [\mathcal{T}_N(\mathbf{U})] + \mathcal{DL}_\kappa \cdot [\mathcal{T}_D(\mathbf{U})]. \tag{2.6}$$

Letting $\tilde{\kappa} = \kappa/\sqrt{n}$, the Hodge-Helmholtz single layer potential is explicitly given by

$$\mathcal{SL}_\kappa\left(\begin{pmatrix} \mathbf{p} \\ q \end{pmatrix}\right) = -\boldsymbol{\Psi}_\kappa(\mathbf{p}) - \nabla\tilde{\psi}_k\left(\mathrm{div}_\Gamma(\mathbf{p})\right) + \nabla\psi_{\tilde{\kappa}}(q), \tag{2.7}$$

where the Helmholtz scalar single-layer, vector single-layer and the regular potentials are written individually for $\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $q \in H^{-1/2}(\Gamma)$ as

$$\psi_{\tilde{\kappa}}(q)(\mathbf{x}) := \int_\Gamma q(\mathbf{y}) G_{\tilde{\kappa}}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma, \tag{2.8a}$$

$$\mathbf{\Psi}_\kappa(\mathbf{p})(\mathbf{x}) := \int_\gamma \mathbf{p}(\mathbf{y}) G_\kappa(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma, \tag{2.8b}$$

$$\tilde{\psi}_\kappa(q)(\mathbf{x}) := \int_\Gamma q(\mathbf{y}) \frac{G_\kappa - G_{\tilde{\kappa}}}{\kappa^2}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma, \tag{2.8c}$$

respectively. The expression (2.7) is derived with (2.8a)-(2.8c) understood as duality pairings. However, if the essential supremum of $\mathbf{p}$, q and $\mathrm{div}_\Gamma(\mathbf{p})$ is bounded, then they can safely be computed as improper integrals [14, Rmk. 4.2].

Lemma 2.4 (See [24, Lem. 4.1]). *The single layer potentials* (2.8a) *and* (2.8b) *are continuous mappings* $\psi_\nu : H^{-1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $\mathbf{\Psi}_\nu : \mathbf{H}_T^{-1/2}(\Gamma) \to \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$.

The classical potentials solve the equations

$$-\,\mathrm{div}\,\nabla\psi_{\tilde{\kappa}}(q) = \tilde{\kappa}^2\psi_{\tilde{\kappa}}(q), \tag{2.9a}$$

$$-\,\Delta\mathbf{\Psi}_\kappa(\mathbf{p}) = \kappa^2\mathbf{\Psi}_\kappa(\mathbf{p}), \tag{2.9b}$$

$$-\mathrm{div}\,\nabla\tilde{\psi}_\kappa(q) = \psi_\kappa(q) + \frac{1}{\eta}\psi_{\tilde{\kappa}}(q), \tag{2.9c}$$

and satisfy the identity [27, Lem. 2.3]

$$\mathrm{div}\,\mathbf{\Psi}_\nu(\mathbf{p}) = \psi_\nu\left(\mathrm{div}_\Gamma\mathbf{p}\right), \qquad \forall\,\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma). \tag{2.10}$$

These observations are used along with Lemma 2.4 in the proof the following lemma.

Lemma 2.5 (See [14, Sec. 5]). *The potentials* $\nabla\psi_{\tilde{\kappa}}$, $\mathbf{\Psi}_\kappa$ *and* $\nabla\tilde{\psi}_\kappa$ *are continuous mappings from* $H^{-1/2}(\Gamma)$ *and* $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *into* $\mathbf{X}(\Delta,\Omega_s)$ *or* $\mathbf{X}_{\mathrm{loc}}(\Delta,\Omega')$.

Corollary 2.6. *The Hodge–Laplace/Helmholtz single layer potential is a continuous map from* $\mathcal{H}_N$ *into* $\mathbf{X}(\Delta,\Omega_s)$ *or* $\mathbf{X}_{\mathrm{loc}}(\Delta,\Omega')$.

Ultimately, we will resort to a Fredholm alternative argument to prove well-posedness of the coupled system. It is therefore evident that the compactness properties of the boundary integral operators introduced in the next Lemma will be extensively used both explicitly and implicitly —notably through exploiting the results found in [14, Sec. 6].

Lemma 2.7 (see [33, Lem. 3.9.8] and [10, Lem. 7]). *For any* $\nu \geq 0$, *the following operators are compact:*

$$\begin{aligned}
\gamma^{\pm}\left(\psi_\nu - \psi_0\right) &: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\\
\gamma_n^{\pm}\left(\nabla\psi_\nu - \nabla\psi_0\right) &: H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)\\
\gamma_t^{\pm}\left(\mathbf{\Psi}_\nu - \mathbf{\Psi}_0\right) &: \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)\\
\gamma_n^{\pm}\nabla\tilde{\psi}_\nu &: H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)
\end{aligned}$$

Proof. Compactness of the second boundary integral operator listed in the statement of Lemma 2.7 immediately entails compactness of

$$\nu^2\gamma_n^{\pm}\nabla\tilde{\psi}_\nu = \gamma_n^{\pm}\left(\nabla\psi_\nu - \nabla\psi_{\tilde{\nu}}\right) = \gamma_n^{\pm}\left(\nabla\psi_\nu - \nabla\psi_0\right) - \left(\gamma_n^{\pm}\left(\nabla\psi_{\tilde{\nu}} - \nabla\psi_0\right)\right)$$

by linearity. While it seems that blow-up occurs in $\tilde{\psi}_\nu$ as $\nu \to 0$, $\nabla\tilde{\psi}_\nu$ happens to be an entire function of ν that vanishes at $\nu = 0$ [14, Sec. 4.1]. ◻

The Hodge-Helmholtz double layer potential is given for $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\xi \in H^{1/2}(\Gamma)$ by

$$\mathcal{DL}_\kappa\left(\begin{pmatrix}\boldsymbol{\eta}\\ \xi\end{pmatrix}\right) := \mathbf{curl}\,\boldsymbol{\Psi}_\kappa(\boldsymbol{\eta}\times\mathbf{n}) + \Upsilon_\kappa(\xi). \tag{2.11}$$

We recognize in (2.11) the (electric) Maxwell double layer potential (c.f. [24, Sec. 4], [10, Eq. 28]) and the normal vector single-layer potential

$$\Upsilon_\kappa(\xi) := \int_\Gamma \xi(\mathbf{y})\mathbf{G}_\kappa(\mathbf{x}-\mathbf{y})\mathbf{n}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x}\in\mathbb{R}^3\backslash\Gamma, \tag{2.12}$$

in which appears the matrix-valued fundamental solution

$$\mathbf{G}_\kappa := G_\kappa\mathrm{Id} + \frac{1}{\kappa^2}\nabla^2\left(G_\kappa - G_{\tilde{\kappa}}\right)$$

satisfying $-\Delta_\eta\mathbf{G}_\kappa - \kappa^2\mathbf{G}_\kappa = \delta_0\mathrm{Id}$ exploited in [14] and detailed in [21, App. A].

Lemma 2.8 (See [14, Sec. 5.4]). *The normal vector single layer potential Υ_κ is a continuous mapping $\Upsilon_\kappa : H^{1/2}(\Gamma) \to \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$.*

This surface potential solves the equation

$$-\Delta_\eta\Upsilon_\kappa(\xi) = \kappa^2\Upsilon_\kappa(\xi) \tag{2.13}$$

and satisfies the identity [14, Sec.5.4]

$$\mathbf{curl}\,\Upsilon_\kappa(\xi) = \mathbf{curl}\boldsymbol{\Psi}_\kappa(\xi\mathbf{n}) \tag{2.14}$$

These results can be used in proving the following lemma.

Lemma 2.9 (See [14, Sec. 5]). *The potentials $\mathbf{curl}\boldsymbol{\Psi}_\kappa(\cdot\times\mathbf{n})$ and Υ_κ are continuous mappings from $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $H^{1/2}(\Gamma)$ respectively, into $\mathbf{X}(\Delta,\Omega_s)$ and $\mathbf{X}_{\mathrm{loc}}(\Delta,\Omega')$.*

Corollary 2.10. *The Hodge-Helmholtz double layer potential is a continuous map from $\mathcal{H}_D$ into $\mathbf{X}(\Delta,\Omega_s)$ or $\mathbf{X}_{\mathrm{loc}}(\Delta,\Omega')$.*

2.4. Integral operators. The well-known Caldéron identities are obtained from (2.6) upon taking the classical compounded traces on both sides and utilizing the jump relations

Jump Relations

$$[\mathcal{T}_D]\cdot\mathcal{DL}_\kappa(\vec{\boldsymbol{\eta}})=\vec{\boldsymbol{\eta}},\qquad [\mathcal{T}_N]\cdot\mathcal{DL}_\kappa(\vec{\boldsymbol{\eta}})=0,\qquad \vec{\boldsymbol{\eta}}\in\mathcal{H}_D, \tag{2.15a}$$

$$[\mathcal{T}_D]\cdot\mathcal{SL}_\kappa(\vec{\mathbf{p}})=0,\qquad [\mathcal{T}_N]\cdot\mathcal{SL}_\kappa(\vec{\mathbf{p}})=\vec{\mathbf{p}},\qquad \vec{\mathbf{p}}\in\mathcal{H}_N, \tag{2.15b}$$

given in [14, Thm. 5.1]. The operator forms of the interior and exterior Caldéron projectors defined on $\mathcal{H}_D\times\mathcal{H}_N$, which we denote $\mathbb{P}_\kappa^-$ and $\mathbb{P}_\kappa^+$ respectively, enter the Caldéron identites:

$$\underbrace{\begin{pmatrix}\{\mathcal{T}_D\}\cdot\mathcal{DL}_k+\frac{1}{2}\mathrm{Id} & \{\mathcal{T}_D\}\cdot\mathcal{SL}_k\\ \{\mathcal{T}_N\}\cdot\mathcal{DL}_k & \{\mathcal{T}_N\}\cdot\mathcal{SL}_k+\frac{1}{2}\mathrm{Id}\end{pmatrix}}_{=:\,\mathbb{P}_\kappa^-}\begin{pmatrix}\mathcal{T}_D^-\mathbf{U}\\ \mathcal{T}_N^-\mathbf{U}\end{pmatrix}=\begin{pmatrix}\mathcal{T}_D^-\mathbf{U}\\ \mathcal{T}_N^-\mathbf{U}\end{pmatrix}, \tag{2.16a}$$

$$\underbrace{\begin{pmatrix}-\{\mathcal{T}_D\}\cdot\mathcal{DL}_k+\frac{1}{2}\mathrm{Id} & -\{\mathcal{T}_D\}\cdot\mathcal{SL}_k\\ -\{\mathcal{T}_N\}\cdot\mathcal{DL}_k & -\{\mathcal{T}_N\}\cdot\mathcal{SL}_k+\frac{1}{2}\mathrm{Id}\end{pmatrix}}_{=:\,\mathbb{P}_\kappa^+}\begin{pmatrix}\mathcal{T}_D^+\mathbf{U}^{\mathrm{ext}}\\ \mathcal{T}_N^+\mathbf{U}^{\mathrm{ext}}\end{pmatrix}=\begin{pmatrix}\mathcal{T}_D^+\mathbf{U}^{\mathrm{ext}}\\ \mathcal{T}_N^+\mathbf{U}^{\mathrm{ext}}\end{pmatrix}, \tag{2.16b}$$

Note that $\mathbb{P}_\kappa^-+\mathbb{P}_\kappa^+=\mathrm{Id}$ and that the range of $\mathbb{P}_\kappa^+$ coincides with the kernel of $\mathbb{P}_\kappa^-$ and vice-versa [10, Sec. 5]. As a consequence of the jump relations (2.15a)-(2.15b), the representation formula (2.6) and the Lemma 2.3, we obtain the next proposition.

Lemma 2.11 (See [36, Lem. 6.18], [10, Thm. 8] and [14, Prop. 5.2]). *The pair of "magnetic" and "electric" traces $(\vec{\boldsymbol{\eta}}\ \vec{\mathbf{p}})^\top\in\mathcal{H}_D\times\mathcal{H}_N$ is valid interior or exterior Cauchy data, if and only if it lies in the kernel of $\mathbb{P}_\kappa^+$ or $\mathbb{P}_\kappa^-$ respectively, i.e.*

$$\begin{aligned}\ker\left(\mathbb{P}_\kappa^+\right)=\{(\vec{\boldsymbol{\eta}}\ \vec{\mathbf{p}})^\top:=\left(\mathcal{T}_D^-(\mathbf{U}),\mathcal{T}_N^-(\mathbf{U})\right)^\top\in\mathcal{H}_D\times\mathcal{H}_N\,\big|\,\mathbf{U}\in\mathbf{X}(\Delta,\Omega_s),\\ \Delta_\eta\mathbf{U}+\kappa^2\mathbf{U}=0\text{ in }\Omega_s\},\\ \ker\left(\mathbb{P}_\kappa^-\right)=\{(\vec{\boldsymbol{\eta}}\ \vec{\mathbf{p}})^\top:=\left(\mathcal{T}_D^-(\mathbf{U}),\mathcal{T}_N^-(\mathbf{U})\right)^\top\in\mathcal{H}_D\times\mathcal{H}_N\,\big|\,\mathbf{U}\in\mathbf{X}_{\mathrm{loc}}(\Delta,\Omega'),\\ \Delta_\eta\mathbf{U}+\kappa^2\mathbf{U}=0\text{ in }\Omega',\mathbf{U}\text{ satisfying radiation conditions at infinity as in [21]}\}.\end{aligned}$$

Inspecting equations (2.16a)-(2.16b) reveals that the Caldéron projectors share a common structure. They can be written as

$$\mathbb{P}_\kappa^-=\frac{1}{2}\mathrm{Id}+\mathbb{A}_\kappa\qquad\text{and}\qquad\mathbb{P}_\kappa^+=\frac{1}{2}\mathrm{Id}-\mathbb{A}_\kappa,$$

where the Caldéron operator is given by

$$\mathbb{A}_\kappa=\begin{pmatrix}\mathbb{A}_\kappa^{DD} & \mathbb{A}_\kappa^{ND}\\ \mathbb{A}_\kappa^{DN} & \mathbb{A}_\kappa^{NN}\end{pmatrix}:=\begin{pmatrix}\{\mathcal{T}_D\}\cdot\mathcal{DL}_\kappa & \{\mathcal{T}_D\}\cdot\mathcal{SL}_\kappa\\ \{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa & \{\mathcal{T}_N\}\cdot\mathcal{SL}_\kappa\end{pmatrix}:\mathcal{H}_{\mathcal{D}}\times\mathcal{H}_{\mathcal{N}}\to\mathcal{H}_{\mathcal{D}}\times\mathcal{H}_{\mathcal{N}}.$$

An analog of the operator matrix $\mathbb{A}_k$ was found convenient in the study of the boundary integral equations of electromagnetic scattering problems [10, Sec. 6]. It is known from [14] that the off-diagonal blocks of $\mathbb{A}_k$ independently satisfy generalized Gårding inequalities making them of Fredholm type with index 0. Injectivity holds when κ^2 lies outside a discrete set of "forbidden resonant frequencies" accumulating at infinity [14, Sec. 3]. In the static case $\kappa = 0$, the dimensions of $\ker\left(\{\mathcal{T}_N\}\cdot\mathcal{SL}_0\right)$ and $\ker\left(\{\mathcal{T}_D\}\cdot\mathcal{DL}_0\right)$ are finite and agree with the zeroth and first Betti number of Γ, respectively. [14, Sec. 7]

In the case of the classical electric wave equation, the boundary integral operators involved in the Caldéron projectors enjoy a hidden symmetry:

Lemma 2.12 (See [24, Lem. 5.4] and [10, Lem. 6]). *There exists a compact linear operator* $\mathbf{C}_k : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *such that*

$$\langle\{\gamma_R\}\mathbf{\Psi}_k(\mathbf{p}),\boldsymbol{\eta}\rangle_\tau = \langle\mathbf{p},\{\gamma_t\}\mathbf{\Psi}_\kappa\mathbf{curl}(\boldsymbol{\eta}\times\mathbf{n})\rangle_\tau + \langle\mathbf{C}_k\mathbf{p},\boldsymbol{\eta}\rangle_\tau$$

for all $\mathbf{p}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *and* $\boldsymbol{\eta}\in\mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma,\Gamma)$.

We will extend Lemma 2.12 to the integral operators defined for the scaled Hodge-Helmholtz/Laplace equation to better characterize the structure of (2.4). The symmetry we are about to reveal in the diagonal blocks of the Caldéron projectors will be crucial in the derivation of the main T-coercivity estimate of this work. It will be exploited for complete cancellation, up to compact terms, of the operators lying on the off-diagonal of the block operator matrix associated to the coupled variational system introduced in Section 3. The following lemmas are required.

Lemma 2.13. *There exists a compact linear operator* $C_k : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *such that*

$$\langle\{\gamma_n\}\nabla\psi_{\tilde{\kappa}}(q),\xi\rangle_\Gamma = -\langle q,\{\eta\,\gamma_D\}\Upsilon_\kappa(\xi)\rangle_\Gamma + \langle C_k q,\xi\rangle,$$

for all $q\in H^{-1/2}(\Gamma)$, $\xi\in H^{1/2}(\Gamma)$.

Proof. This proof utilizes a strategy found in [24, Lem. 5.4] and [8, Thm. 3.9]. Let $\rho>0$ be such that B_ρ is an open ball containing $\overline{\Omega}_s$. We will indicate with a hat (e.g. $\widehat{\gamma}$) the traces taken over the boundary ∂B_ρ of that ball. Due to the lemmas 2.5 and 2.9, we can use the extension of formula (2.1a) to compare the following terms.

On the one hand, using the scalar Helmholtz equation (2.9a) and recalling that $\tilde{\kappa}=\kappa/\sqrt{\eta}$, we have

$$\begin{aligned}
&\langle\eta\;\gamma_D^-\,\nabla\psi_{\tilde{\kappa}}(q),\,\gamma_n^-\,\Upsilon_\kappa(\xi)\rangle_\Gamma\\
&\qquad=\int_{\Omega_s}\eta\,\mathrm{div}\left(\nabla\psi_{\tilde{\kappa}}(q)\right)\mathrm{div}\Upsilon_k(\xi)+\eta\nabla\mathrm{div}\left(\nabla\psi_{\tilde{\kappa}}(q)\right)\cdot\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x}\\
&\qquad=-\int_{\Omega_s}\kappa^2\psi_{\tilde{\kappa}}(q)\mathrm{div}\Upsilon_k(\xi)\,\mathrm{d}\mathbf{x}-\int_{\Omega_s}\kappa^2\nabla\psi_{\tilde{\kappa}}(q)\cdot\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x},
\end{aligned}\tag{2.17}$$

and similarly,

$$\langle \eta\, \gamma_D^+ \nabla\psi_{\tilde\kappa}(q), \gamma_n^+ \Upsilon_\kappa(\xi)\rangle_\Gamma = \int_{\Omega'\cap B_\rho} \kappa^2 \psi_{\tilde\kappa}(q)\, \mathrm{div}\Upsilon_k(\xi) + \nabla\psi_{\tilde\kappa}(q)\cdot \Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x}$$
$$+\langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}.$$

On the other hand, using (2.9a) together with the scaled Hodge-Helmholtz equation (2.13), we also have

$$\begin{aligned}
&\langle \gamma_n^- \nabla\psi_{\tilde\kappa}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi)\rangle_\Gamma \\
&\quad = \int_{\Omega_s} \eta\, \mathrm{div}\big(\nabla\psi_{\tilde\kappa}(q)\big)\, \mathrm{div}\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x} + \int_{\Omega_s} \eta\, \nabla\psi_{\tilde\kappa}(q)\cdot \nabla \mathrm{div}\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x} \\
&\quad = -\int_{\Omega_s} \kappa^2 \psi_{\tilde\kappa}(q)\, \mathrm{div}\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x} + \int_{\Omega_s} \nabla\psi_{\tilde\kappa}(q)\cdot \mathbf{curl}\,\mathbf{curl}\,\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x} \\
&\qquad -\int_{\Omega_s} \kappa^2 \nabla\psi_{\tilde\kappa}(q)\cdot \Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{2.18}$$

Equations (2.17) and (2.18) together yield

$$\langle \gamma_n^- \nabla\psi_{\tilde\kappa}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi)\rangle_\Gamma = \langle \eta\, \gamma_D^- \nabla\psi_\kappa(q), \gamma_n^- \Upsilon_\kappa(\xi)\rangle_\Gamma + \int_{\Omega_s} \nabla\psi_{\tilde\kappa}(q)\cdot \mathbf{curl}\,\mathbf{curl}\,\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x}.$$

Similarly, the terms involving the exterior traces satisfy

$$\begin{aligned}
\langle \gamma_n^+ \nabla\psi_{\tilde\kappa}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma &= \langle \eta\, \gamma_D^+ \nabla\psi_\kappa(q), \gamma_n^+ \Upsilon_\kappa(\xi)\rangle_\Gamma - \langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} \\
&\quad - \int_{\Omega'\cap B_\rho} \nabla\psi_{\tilde\kappa}(q)\cdot \mathbf{curl}\,\mathbf{curl}\,\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x} + \langle \widehat{\gamma}_n^+ \nabla\psi_{\tilde\kappa}(q), \eta\, \widehat{\gamma}_D^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}.
\end{aligned}$$

From the first row of the jump properties [14, Sec. 5]

$$[\gamma_D]\nabla\psi_{\tilde\kappa}(q) = 0, \qquad [\gamma_n]\Upsilon_\kappa(\xi) = 0, \tag{2.19a}$$
$$[\gamma_D]\Upsilon_\kappa(\xi) = \xi/\eta, \qquad [\gamma_n]\nabla\psi_{\tilde\kappa}(q) = q, \tag{2.19b}$$

we obtain, by gathering the above results, integrating by parts again and using the fact that $\mathbf{curl}\circ\nabla \equiv 0$,

$$\begin{aligned}
\langle \gamma_n^- \nabla\psi_{\tilde\kappa}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi)\rangle_\Gamma &= \langle \eta\, \gamma_D^+ \nabla\psi_\kappa(q), \gamma_n^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \int_{\Omega_s} \kappa^2 \nabla\psi_{\tilde\kappa}(q)\cdot \mathbf{\Psi}_\kappa(\xi\mathbf{n})\,\mathrm{d}\mathbf{x} \\
&= \langle \gamma_n^+ \nabla\psi_{\tilde\kappa}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \int_{B_\rho} \nabla\psi_{\tilde\kappa}(q)\cdot \mathbf{curl}\,\mathbf{curl}\,\Upsilon_\kappa(\xi)\,\mathrm{d}\mathbf{x} \\
&\quad + \langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} - \langle \widehat{\gamma}_n^+ \nabla\psi_{\tilde\kappa}(q), \eta\, \widehat{\gamma}_D^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}. \\
&= \langle \gamma_n^+ \nabla\psi_{\tilde\kappa}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \langle \gamma_t \nabla\psi_{\tilde\kappa}(q), \gamma_R \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} \\
&\quad + \langle \eta\, \widehat{\gamma}_D^+ \nabla\psi_\kappa(q), \widehat{\gamma}_n^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho} - \langle \widehat{\gamma}_n^+ \nabla\psi_{\tilde\kappa}(q), \eta\, \widehat{\gamma}_D^+ \Upsilon_\kappa(\xi)\rangle_{\partial B_\rho}.
\end{aligned} \tag{2.20}$$

Fortunately, when restricted to domains away from Γ, the potentials are C^∞-smoothing. Hence, their evaluation on ∂B_ρ, the highlighted terms in (2.20), induce compact operators. This shows that for some compact operator $C_k : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$,

$$\langle \gamma_n^- \nabla \psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^- \Upsilon_\kappa(\xi)\rangle_\Gamma = \langle \gamma_n^+ \nabla \psi_{\tilde{\kappa}}(q), \eta\, \gamma_D^+ \Upsilon_\kappa(\xi)\rangle_\Gamma + \langle C_k q, \xi\rangle_\Gamma. \tag{2.21}$$

The jump identities (2.19b) for the potentials yield formulas of the form $\{\gamma_*\}K = \gamma_*^\pm K \pm (1/2)\mathrm{Id}$, where $* = n$, D and $K = \nabla\psi_{\tilde{\kappa}}$, Υ_κ accordingly. Substituting each one-sided trace involved in the two leftmost duality pairings of (2.21) for the integral operators using these equations completes the proof. ∎

Lemma 2.14. *For all* $\mathbf{p} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\xi \in H^{1/2}(\Gamma)$*, we have*

$$\langle \mathbf{p}, \gamma_t^\pm \Upsilon_\kappa(\xi)\rangle_\tau = \langle \gamma_n^\pm \mathbf{\Psi}_\kappa(\mathbf{p}), \xi\rangle_\Gamma + \langle \gamma_n^\pm \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma(\mathbf{p})), \xi\rangle_\Gamma.$$

Proof. In the following calculations, the boundary integrals are to be understood as duality pairings. Since $\mathbf{p} \in \mathbf{L}_t^2(\Gamma)$ is a tangent vector field lying in the image of γ_t, the tangential trace operator can safely be dropped in expanding these integrals using the definitions of Section 2.3. On the one hand, this leads to

$$\begin{aligned}\langle \mathbf{p}, \gamma_t^\pm \Upsilon_\kappa(\xi)\rangle_\tau &= \int_\Gamma \int_\Gamma \xi(\mathbf{y})\mathbf{p}(\mathbf{x}) \cdot \big(\mathbf{G}_\kappa(\mathbf{x}-\mathbf{y})\mathbf{n}(\mathbf{y})\big)\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}) \\ &= \int_\Gamma \int_\Gamma \xi(\mathbf{y}) G_\kappa(\mathbf{x}-\mathbf{y})\mathbf{p}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}) \\ &\qquad + \int_\Gamma \int_\Gamma \xi(\mathbf{y})\mathbf{p}(\mathbf{x}) \cdot \Big(\nabla^2 \tilde{G}_\kappa(\mathbf{x}-\mathbf{y})\mathbf{n}(\mathbf{y})\Big)\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}),\end{aligned} \tag{2.22}$$

where $\tilde{G}_\kappa := (G_\kappa - G_{\tilde{\kappa}})/\kappa^2$.

On the other hand, the same observation implies that $\langle \mathbf{p}, \nabla_\Gamma \gamma \mathbf{V})\rangle_\tau = \langle \mathbf{p}, \gamma \nabla \mathbf{V})\rangle_\tau$ for any $\mathbf{V} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$, and thus that

$$\begin{aligned}\langle \gamma_n^\pm \nabla \tilde{\psi}_\kappa(\mathrm{div}_\Gamma(\mathbf{p})), \xi\rangle_\Gamma &= \int_\gamma \int_\gamma \xi(\mathbf{y})\mathbf{n}(\mathbf{y}) \cdot \nabla \tilde{G}_\kappa\,(\mathbf{y}-\mathbf{x})\, \mathrm{div}_\Gamma\,\big(\mathbf{p}(\mathbf{x})\big)\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}) \\ &= -\int_\gamma \int_\gamma \xi(\mathbf{y})\mathbf{p}(x) \nabla_{\mathbf{x}} \Big(\mathbf{n}(\mathbf{y}) \cdot \nabla \tilde{G}_\kappa\,(\mathbf{y}-\mathbf{x})\Big)\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}) \\ &= \int_\gamma \int_\gamma \xi(\mathbf{y})\mathbf{p}(x) \Big(\nabla^2 \tilde{G}_\kappa(\mathbf{x}-\mathbf{y})\mathbf{n}(\mathbf{y})\Big)\, \mathrm{d}\sigma(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{x}),\end{aligned}$$

where we have remembered that the tangential divergence defined in Section 2.2.1 was adjoint to the negative surface gradient. Recognizing the Helmholtz vector single-layer potential in the first expression on the right hand side of (2.22) concludes the proof. ∎

Symmetry of Calderón Projector Up to Compact Perturbations

Proposition 2.15. *There exists a compact operator* $\mathcal{C}_k : \mathcal{H}_N \to \mathcal{H}_N$ *such that*

$$\langle \mathbb{A}_\kappa^{NN}(\vec{\mathbf{p}}), \vec{\boldsymbol{\eta}}\rangle = -\langle \vec{\mathbf{p}}, \mathbb{A}_\kappa^{DD}(\vec{\boldsymbol{\eta}})\rangle + \langle \mathcal{C}_k \vec{\mathbf{p}}, \vec{\boldsymbol{\eta}}\rangle$$

for all $\vec{\boldsymbol{\eta}} := (\boldsymbol{\eta},\ \xi)^\top \in \mathcal{H}_D$ *and* $\vec{\mathbf{p}} := (\mathbf{p},\ q)^\top \in \mathcal{H}_N$.

Proof. Recall that $\mathbb{A}_\kappa^{NN} = \{\mathcal{T}_N\}\cdot\mathcal{SL}_\kappa$ and $\mathbb{A}_\kappa^{DD} = \{\mathcal{T}_D\}\cdot\mathcal{DL}_\kappa$. Since $\mathbf{curl}\circ\nabla = 0$, $\langle\{\gamma_R\}\nabla\psi_{\tilde{k}}(q),\boldsymbol{\eta}\rangle_\tau = 0$ and $\langle\{\gamma_R\}\nabla\tilde{\psi}_k(\mathrm{div}_\Gamma(\mathbf{p})),\boldsymbol{\eta}\rangle_\tau = 0$; therefore,

$$\begin{aligned}\langle\{\mathcal{T}_N\}\cdot\mathcal{SL}_k(\vec{\mathbf{p}}),\vec{\boldsymbol{\eta}}\rangle &= \langle-\{\gamma_R\}\boldsymbol{\Psi}_\kappa(\mathbf{p}),\boldsymbol{\eta}\rangle_\tau + \langle\{\gamma_n\}\nabla\psi_{\tilde{\kappa}}(q),\xi\rangle_\Gamma \\ &\quad - \langle\{\gamma_n\}\boldsymbol{\Psi}_\kappa(\mathbf{p}),\xi\rangle_\Gamma - \langle\{\gamma_n\}\nabla\tilde{\psi}_\kappa(\mathrm{div}_\Gamma(\mathbf{p})),\xi\rangle_\Gamma.\end{aligned} \tag{2.23}$$

Since $\mathrm{div}\circ\mathbf{curl} = 0$, we also have $\{\gamma_D\}\,\mathbf{curl}\boldsymbol{\Psi}_{\boldsymbol{\kappa}} = 0$. Hence, we need to compare (2.23) with

$$\langle\vec{\mathbf{p}},\{\mathcal{T}_D\}\cdot\mathcal{DL}_k(\vec{\boldsymbol{\eta}})\rangle = \langle\mathbf{p},\{\gamma_t\}\mathbf{curl}\boldsymbol{\Psi}_k(\boldsymbol{\eta}\times\mathbf{n})\rangle_\tau + \langle q,\{\eta\,\gamma_D\}\Upsilon_\kappa(\xi)\rangle_\Gamma + \langle\mathbf{p},\{\gamma_t\}\Upsilon_\kappa(\xi)\rangle_\tau.$$

The desired result follows by combining Lemma 2.12, Lemma 2.13 and Lemma 2.14. ∎

As consequence of Proposition 2.15, we have

$$\left(\mathbb{P}_\kappa^+\right)_{11}^* \mathrel{\hat{=}} \left(\mathbb{P}_\kappa^-\right)_{22}, \tag{2.24}$$

where $\hat{=}$ is used to indicate equality up to compact terms.

3. Coupled problem. In this section, we derive a variational formulation for the system (1.2a)-(1.3b) which couples a mixed variational formulation defined in the interior domain to a boundary integral equation of the first kind that arises in the exterior domain.

As proposed in [3], we introduce a new variable $P = -\mathrm{div}\left(\epsilon(\mathbf{x})\mathbf{U}\right)$ into equation (1.2a). Applying Green's formulae (2.4) in Ω_s, we obtain

$$\begin{aligned}&\int_{\Omega_s}\mu^{-1}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\mathbf{V}\,\mathrm{d}\mathbf{x} + \int_{\Omega_s}\epsilon\,\nabla P\cdot\mathbf{V}\,\mathrm{d}\mathbf{x} \\ &\qquad -\omega^2\int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\mathbf{V}\,\mathrm{d}\mathbf{x} + \langle\gamma_{R,\mu}^-\mathbf{U},\gamma_t^-\mathbf{V}\rangle_\tau = (\mathbf{J},\mathbf{V})_{\Omega_s}, \\ &\int_{\Omega_s}P\,Q\,\mathrm{d}\mathbf{x} - \int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\nabla Q\,\mathrm{d}\mathbf{x} + \langle\gamma_{n,\epsilon}^-\mathbf{U},\gamma^-Q\rangle_\Gamma = 0\end{aligned} \tag{3.1}$$

for all $\mathbf{V}\in\mathbf{H}(\mathbf{curl},\Omega_s)$, $Q\in H^1(\Omega_s)$. The volume integrals in these equations enter the interior symmetric bi-linear form

$$\begin{aligned}\mathfrak{B}_\kappa\left(\begin{pmatrix}\mathbf{U}\\P\end{pmatrix},\begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right) &:= \int_{\Omega_s}\mu^{-1}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\mathbf{V}\,\mathrm{d}\mathbf{x} + \int_{\Omega_s}\epsilon\,\nabla P\cdot\mathbf{V}\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\Omega_s}P\,Q\,\mathrm{d}\mathbf{x} - \int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\nabla Q\,\mathrm{d}\mathbf{x} - \omega^2\int_{\Omega_s}\epsilon\,\mathbf{U}\cdot\mathbf{V}\,\mathrm{d}\mathbf{x}\end{aligned} \tag{3.2}$$

related to the one supplied for the Hodge-Laplace operator in [2, Sec. 3.2]. We aim to couple (3.2) with the BIEs replacing the PDEs in Ω'. We may utilize the transmission conditions (1.3a)-(1.3b) to amend (3.1) to the variational equation

$$\mathfrak{B}_\kappa\left(\begin{pmatrix}\mathbf{U}\\P\end{pmatrix},\begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right) + \left\langle\mathcal{T}_N^+(\mathbf{U}^{\mathrm{ext}}),\begin{pmatrix}\gamma_t^-\mathbf{V}\\\gamma^-Q\end{pmatrix}\right\rangle = \mathscr{G}\left(\begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right),$$

which sports a functional $\mathscr{G}\left((\mathbf{V}\,Q)^\top\right) := (\mathbf{J}, \mathbf{V})_{\Omega_s} - \langle(\mathbf{g}_R\, g_n)^\top, (\gamma_t^-\mathbf{V}\,\gamma^- Q)^\top\rangle$ bounded over the test space. The exterior Calderón projector can be used to express the so-called Dirichlet-to-Neumann operator in different ways.

Introducing the jump conditions into the first exterior Calderón identity given on the first line of (2.16b) along with a new unknown $\vec{\mathbf{p}} = \mathcal{T}_N^+(\mathbf{U}^{\text{ext}})$ yields a variational system

$$\begin{aligned} &\mathfrak{B}_\kappa\left(\begin{pmatrix}\mathbf{U}\\P\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right) + \left\langle \vec{\mathbf{p}}, \begin{pmatrix}\gamma_t^-\mathbf{V}\\ \gamma^- Q\end{pmatrix}\right\rangle = \mathscr{G}\left(\begin{pmatrix}\mathbf{V}\\Q\end{pmatrix}\right),\\ &\left\langle \left(\{\mathcal{T}_\mathcal{D}\}\cdot\mathcal{DL}_\kappa + \frac{1}{2}\mathrm{Id}\right)\mathcal{T}^-_{D,\epsilon}(\mathbf{U}), \vec{\mathbf{a}}\right\rangle + \left\langle\{\mathcal{T}_D\}\cdot\mathcal{SL}_\kappa(\vec{\mathbf{p}}), \vec{\mathbf{a}}\right\rangle = \mathscr{R}(\vec{\mathbf{a}}), \end{aligned} \tag{3.3}$$

for all $(\mathbf{V}\,Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s)\times H^1(\Omega_s)$ and $\vec{\mathbf{a}} \in \mathcal{H}_N$, resembling the original Johnson-Nedélec coupling [5]. The new functional appearing on the right hand side of (3.3) is defined as $\mathscr{R}(\vec{\mathbf{a}}) := \langle\left(\{\mathcal{T}_D\}\cdot\mathcal{DL}_\kappa + \frac{1}{2}\mathrm{Id}\right)(\boldsymbol{\zeta}_t,\, \zeta_D)^\top, \vec{\mathbf{a}}\rangle$.

Following the exposition of Costabel in [17], we also retain the second exterior Calderón identity —in which we again introduce the jump conditions to eliminate the dependence on the exterior solution— and insert the resulting equation in (3.3) to obtain the symmetrically coupled problem: find $\vec{\mathbf{U}} := (\mathbf{U},\ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s)\times H^1(\Omega_s)$ and $\vec{\mathbf{p}} \in \mathcal{H}_N$ such that

$$\begin{aligned} &\mathfrak{B}_\kappa\left(\vec{\mathbf{U}}, \vec{\mathbf{V}}\right) + \left\langle\left(-\{\mathcal{T}_N\}\cdot\mathcal{SL}_\kappa + \frac{1}{2}\mathrm{Id}\right)\vec{\mathbf{p}}, \begin{pmatrix}\gamma_t^-\mathbf{V}\\ \gamma^- Q\end{pmatrix}\right\rangle\\ &\qquad + \left\langle -\{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix}, \begin{pmatrix}\gamma_t^-\mathbf{V}\\ \gamma^- Q\end{pmatrix}\right\rangle = \mathscr{F}\left(\vec{\mathbf{V}}\right)\\ &\left\langle\left(\{\mathcal{T}_\mathcal{D}\}\cdot\mathcal{DL}_\kappa + \frac{1}{2}\mathrm{Id}\right)\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix}, \vec{\mathbf{a}}\right\rangle + \left\langle\{\mathcal{T}_D\}\cdot\mathcal{SL}_\kappa(\vec{\mathbf{p}}), \vec{\mathbf{a}}\right\rangle = \mathscr{R}(\vec{\mathbf{a}}), \end{aligned} \tag{3.4}$$

for all $\vec{\mathbf{V}} := (\mathbf{V},\ Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s)\times H^1(\Omega_s)$, $\vec{\mathbf{a}} \in \mathcal{H}_N$. Yet again, the right hand side of our system of equations has been modified to bear a new bounded linear functional $\mathscr{F}(\vec{\mathbf{V}}) := \mathscr{G}(V) + \langle -\{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa(\boldsymbol{\zeta}_t\,\zeta_D)^\top, (\gamma_t^-\mathbf{V},\, \gamma^- Q)^\top\rangle$.

In terms of the Calderón projector, problem (3.4) can be rewritten more succinctly as

Symmetrically Coupled Problem

Find $\vec{\mathbf{U}} := (\mathbf{U},\ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ and $\vec{\mathbf{p}} \in \mathcal{H}_N$ such that

$$\mathfrak{B}_\kappa\left(\vec{\mathbf{U}}, \vec{\mathbf{V}}\right) + \left\langle \left(-\mathbb{A}_\kappa^{NN} + \frac{1}{2}\mathrm{Id}\right)\vec{\mathbf{p}}, \begin{pmatrix}\gamma_t^- \mathbf{V} \\ \gamma^- Q\end{pmatrix}\right\rangle + \left\langle -\mathbb{A}_\kappa^{DN}\begin{pmatrix}\gamma_t^-\mathbf{U} \\ -\gamma^-(P)\end{pmatrix}, \begin{pmatrix}\gamma_t^-\mathbf{V} \\ \gamma^- Q\end{pmatrix}\right\rangle = \mathscr{F}\left(\vec{\mathbf{V}}\right) \tag{3.5}$$

$$\left\langle \left(\mathbb{A}_\kappa^{DD} + \frac{1}{2}\mathrm{Id}\right)\begin{pmatrix}\gamma_t^-\mathbf{U} \\ -\gamma^-(P)\end{pmatrix}, \vec{\mathbf{a}}\right\rangle + \left\langle \mathbb{A}_\kappa^{DD}(\vec{\mathbf{p}}), \vec{\mathbf{a}}\right\rangle = \mathscr{R}(\vec{\mathbf{a}}),$$

for all $\vec{\mathbf{V}} := (\mathbf{V},\ Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$, $\vec{\mathbf{a}} \in \mathcal{H}_N$.

Remark 5. *Part of the justification for using mixed formulations for problems involving the Hodge–Helmholtz/Laplace operator is the need to move away from trial spaces contained in* $\mathbf{H}(\mathbf{curl}, \Omega_s) \cap \mathbf{H}(div, \Omega_s)$, *because the latter doesn't allow for viable discretizations using finite elements [2]. While from (3.3) the issue seems to reappear after using the Caldéron identities, the benefits of the introduced new unknown* $P \in H^1(\Omega_s)$ *conveniently carries over to the coupled system (3.5) upon substituting* $-\gamma^-(P)$ *in place of* $\gamma_{D,\epsilon}(\mathbf{U})$ *in* $\mathcal{T}_{D,\epsilon}^-(\mathbf{U})$.

In the following proposition, we call *forbidden resonant frequencies* the interior "Dirichlet" eigenvalues of the scaled Hodge-Laplace operator with constant coefficient $\eta = \mu_0\epsilon_0^2$. That is, κ^2 is a forbidden frequency if there exists $0 \neq \mathbf{U} \in \mathbf{X}(\Delta, \Omega)$ with $\Delta_\eta \mathbf{U} = \kappa^2\mathbf{U}$ and $\mathcal{T}_D^-\mathbf{U} = 0$. We refer the reader to [14], where the spectrum of the scaled Hodge-Laplace operator is completely characterized. See for e.g. [34], [33], [12], [18] and [16] for an overview of the issue of spurious resonances in electromagnetic and acoustic scattering models based on integral equations.

Proposition 3.1. *Suppose that* $\kappa^2 \in \mathbb{C}$ *avoids forbidden resonant frequencies. By retaining an interior solution* $U \in \mathbf{H}(\mathbf{curl}, \Omega_s)$ *and producing* $\mathbf{U}^{ext} \in \mathbf{X}_{\mathrm{loc}}(\Delta, \Omega')$ *using the representation formula (2.6) for the Cauchy data* $(\vec{\mathbf{p}}, \mathcal{T}_{D,\epsilon}^- U - (\boldsymbol{\zeta}_t,\ \zeta_D)^\top)$ *with* $\gamma_{D,\epsilon}^-(\mathbf{U}) = -\gamma^-(P)$, *a solution to (3.5) solves the transmission system (1.2a)-(1.3b) in the sense of distribution.*

Proof. The proof follows the approach in [24, Lem. 6.1]. Since $\mathscr{D}(\Omega_s)^3 \times C_0^\infty(\Omega_s)$ is a subset of the volume test space, any solution to the problem (3.5) solves (1.2a) in Ω_s in the sense of distribution. It follows that (3.1) holds for all admissible $\vec{\mathbf{V}}$, which reduces (3.5) to the variational system

$$-\left\langle \vec{\mathbf{q}}, \begin{pmatrix}\gamma_t^-\mathbf{V} \\ \gamma^- Q\end{pmatrix}\right\rangle + \left\langle \left(-\mathbb{A}_\kappa^{NN} + \frac{1}{2}\mathrm{Id}\right)\vec{\mathbf{p}}, \begin{pmatrix}\gamma_t^-\mathbf{V} \\ \gamma^- Q\end{pmatrix}\right\rangle - \left\langle \mathbb{A}_\kappa^{DN}(\vec{\boldsymbol{\xi}}), \begin{pmatrix}\gamma_t^-\mathbf{V} \\ \gamma^- Q\end{pmatrix}\right\rangle = 0$$

$$\left\langle \left(\mathbb{A}_\kappa^{DD} + \frac{1}{2}\mathrm{Id}\right)\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\eta}}\right\rangle + \left\langle \{\mathbb{A}_\kappa^{ND}(\vec{\mathbf{p}}), \vec{\boldsymbol{\eta}}\right\rangle = 0$$

where $\vec{\mathbf{q}} := \mathcal{T}_{N,\mu}^-(\mathbf{U}) - (\mathbf{g}_R,\ g_n)^\top$ and $\vec{\boldsymbol{\xi}} := \mathcal{T}_{D,\epsilon}^-(\mathbf{U}) - (\boldsymbol{\zeta}_t,\ \zeta_D)^\top$.

We recognize in the equivalent operator equation

$$\underbrace{\begin{pmatrix} \mathbb{A}_\kappa^{NN} + \frac{1}{2}\mathrm{Id} & \mathbb{A}_\kappa^{DN} \\ \mathbb{A}_\kappa^{ND} & \mathbb{A}_\kappa^{DD} + \frac{1}{2}\mathrm{Id} \end{pmatrix}}_{\mathbb{P}_\kappa^-} \begin{pmatrix} \vec{\mathbf{p}} \\ \vec{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{p}} - \vec{\mathbf{q}} \\ 0 \end{pmatrix} \tag{3.6}$$

the interior Caldéron projector (2.16a) whose image, by Lemma 2.11, is the space of valid Cauchy data for the homogeneous (scaled) Hodge–Laplace/Helmholtz interior equation with constant coefficient η. In particular, $\vec{\mathbf{p}} - \vec{\mathbf{q}} = \mathcal{T}_N^-\left(\tilde{\mathbf{U}}\right)$ for some vector-field $\tilde{\mathbf{U}} \in \mathbf{X}(\Delta, \Omega_s)$ satisfying

$$\begin{aligned} \Delta_\eta \tilde{\mathbf{U}} - \kappa^2 \tilde{\mathbf{U}} &= 0, && \text{in } \Omega_s \\ \mathcal{T}_D^-\left(\tilde{\mathbf{U}}\right) &= 0, && \text{on } \Gamma. \end{aligned} \tag{3.7}$$

If $\kappa^2 \neq 0$, we rely on the hypothesis that κ^2 doesn't belong to the set of forbidden resonant frequencies to guarantee injectivity of the above boundary value problem [14, Prop. 3.7]. Otherwise, a trivial de Rham cohomology implies that zero is not a Dirichlet eigenvalue. We conclude that $\tilde{\mathbf{U}} = 0$ is the unique trivial solution to (3.7). Therefore, for the right hand side of (3.6) to exhibit valid Neumann data, it must be that $\vec{\mathbf{p}} = \vec{\mathbf{q}}$.

By Lemma 2.11 again, the null space of the interior Caldéron projector $\mathbb{P}_\kappa^-$ coincides with valid Cauchy data for the exterior boundary value problem (1.2b) complemented with the radiation conditions at infinity introduced in Section 1. In particular $(\vec{\mathbf{p}},\ \vec{\boldsymbol{\xi}})^\top$ is valid Cauchy data for that exterior Hodge-Helmholtz/Laplace problem and $\mathbf{U}^{\mathrm{ext}} = \mathcal{SL}_\kappa(\vec{\mathbf{p}}) + \mathcal{DL}_\kappa\left(\vec{\boldsymbol{\xi}}\right)$ solves (1.2b) and (1.3b) by construction. The fact that $\vec{\mathbf{p}} = \mathcal{T}_N^+\left(\mathbf{U}^{\mathrm{ext}}\right)$ solves (1.3a) is confirmed by the earlier observation that $\vec{\mathbf{p}} = \vec{\mathbf{q}}$. ∎

Corollary 3.2. *Suppose that $\kappa^2 \in \mathbb{C}$ avoids forbidden resonant frequencies. A solution pair $\left(\vec{\mathbf{U}}, \vec{\mathbf{p}}\right)$ to the coupled problem* (3.5) *is unique.*

Remark 6. *We show in [34] that when κ^2 happens to be a resonant frequency, the interior solution $\mathbf{U}$ remains unique. This is no longer true for $\vec{\mathbf{p}}$ however, which is in general only unique up to Neumann traces of interior Dirichlet eigenfunctions of Δ_η associated to the eigenvalue κ^2. Fortunately, this kernel vanishes under the exterior representation formula obtained from* (2.6).

Remark 7. *In principle, the CFIE-type stabilization strategy proposed in [26] for the (scalar) Helmholtz transmission problem could also be attempted here to get rid of the spurious resonances haunting the coupled problem* (3.5)*, but such developments lie outside the scope of this work.*

4. Space decompositions. Using the classical Hodge decomposition, a general inf-sup condition for Hodge–Laplace problems posed on closed Hilbert complexes was derived in [2]. However, as orthogonality won't be required, we rather opt for the enhanced regularity of the regular decomposition suggested in [10] and [14].

Lemma 4.1 (See [1, Lem. 3.5]). *There exists a continuous linear lifting L : $\mathbf{H}(\mathrm{div}, \Omega_s) \cap \ker(\mathrm{div}) \to \mathbf{H}^1(\Omega_s)$ such that $\mathrm{div}(\mathsf{L}\mathbf{U}) = 0$ and $\mathbf{curl}(\mathsf{L}\mathbf{U}) = \mathbf{U}$.*

Lemma 4.2. *The operator $\mathsf{Z} : \mathbf{H}(\mathbf{curl}, \Omega_s) \to \mathbf{H}^1(\Omega_s)$ defined by*

$$\mathsf{Z}(\mathbf{U}) := \mathsf{L}\left(\mathbf{curl}(\mathbf{U})\right)$$

is a continuous projection with $\ker(\mathsf{Z}) = \ker(\mathbf{curl}) \cap \mathbf{H}(\mathbf{curl}, \Omega_s)$ *and satisfying* $\mathbf{curl}\left(\mathsf{Z}(\mathbf{U})\right) = \mathbf{curl}(\mathbf{U})$.

The following corollary follows immediately from Rellich theorem.

Corollary 4.3. *The projection operator of Lemma 4.2 is compact as a mapping* $\mathsf{Z} : \mathbf{H}(\mathbf{curl}, \Omega_s) \to \mathbf{L}^2(\Omega_s)$.

The subspaces $\mathbf{X}(\mathbf{curl}, \Omega_s) := \mathsf{Z}\left(\mathbf{H}(\mathbf{curl}, \Omega_s)\right)$ and $\mathbf{N}(\mathbf{curl}, \Omega_s) := \ker(\mathbf{curl}) \cap \mathbf{H}(\mathbf{curl}, \Omega_s)$ provide, by vitrue of the continuity of Z, a stable direct regular decomposition

$$\mathbf{H}(\mathbf{curl}, \Omega_s) = \mathbf{X}(\mathbf{curl}, \Omega_s) \oplus \mathbf{N}(\mathbf{curl}, \Omega_s).$$

A decomposition with similar properties can be designed for $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. We define $\mathsf{J} : H^{-1/2}(\Gamma) \to \mathbf{H}^1(\Omega_s)$ by $\mathsf{J}(g) = \mathsf{L}(\nabla U)$, where $U \in H^1_*(\Omega_s) := \{U \in H^1(\Omega_s) : \int_{\Omega_s} U \,\mathrm{d}\mathbf{x} = 0\}$ solves the Neumann problem

$$\begin{aligned} \Delta U &= 0, && \text{in } \Omega_s, \\ \gamma_n^-(\nabla U) &= g, && \text{on } \Gamma. \end{aligned} \tag{4.1}$$

This map is well-defined, since (4.1) ensures that $\nabla U \in \mathrm{Dom}(\mathsf{L})$.

Lemma 4.4. *The operator* $\mathsf{Z}^\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_R^{1/2}(\Gamma)$ *defined by*

$$\mathsf{Z}^\Gamma := \gamma_\tau \circ \mathsf{J} \circ \mathrm{div}_\Gamma \tag{4.2}$$

is a continuous projection with $\ker(\mathsf{Z}^\Gamma) = \ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and satisfying* $\mathrm{div}_\Gamma\left(\mathsf{Z}^\Gamma(\mathbf{p})\right) = \mathrm{div}_\Gamma(\mathbf{p})$.

As before, the extra regularity of the range, in this case provided by Lemma 2.1, leads to a valuable corollary.

Corollary 4.5. *The projection operator of Lemma 4.4 is compact as a mapping* $\mathsf{Z}^\Gamma : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_R^{-1/2}(\Gamma)$.

The subspaces $\mathbf{X}(\mathrm{div}_\Gamma, \Gamma) := \mathsf{Z}^\Gamma\left(\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)\right)$ and $\mathbf{N}(\mathrm{div}_\Gamma, \Gamma) := \ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ provide a stable direct regular decomposition

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \oplus \mathbf{N}(\mathrm{div}_\Gamma, \Gamma).$$

In the following, we may simplify notation by using $\mathbf{U}^\perp := \mathsf{Z}\mathbf{U}$, $\mathbf{p}^\perp := \mathsf{Z}^\Gamma\mathbf{p}$, $\mathbf{U}^0 := (\mathrm{Id} - \mathsf{Z})\,\mathbf{U}$ and $\mathbf{p}^0 := \left(\mathrm{Id} - \mathsf{Z}^\Gamma\right)\mathbf{p}$.

A very useful property of this pair of decompositions is stated in the next proposition.

Lemma 4.6 (See [24, Lem. 8.1] and [24, Lem. 8.2]). *The operators*

$$\left(\gamma_t^-\right)^* \circ \left(\{\gamma_R\}\mathbf{\Psi}_\kappa + \frac{1}{2}\mathrm{Id}\right) : \mathbf{N}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{N}(\mathbf{curl}, \Omega_s)',$$

and

$$\left(\gamma_t^-\right)^* \circ \left(\{\gamma_R\}\mathbf{\Psi}_\kappa + \frac{1}{2}\mathrm{Id}\right) : \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}(\mathbf{curl}, \Omega_s)'$$

are compact.

Another benefit of this pair of regular decompositions will become explicit in the poof of Lemma 5.9 found in the next section (see equation (5.9)).

It follows from Lemma 2.2 that $\mathrm{div}_\Gamma : \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \to H_*^{-1/2}(\Gamma)$ is a continuous bijection. The bounded inverse theorem guarantees the existence of a continuous inverse $(\mathrm{div}_\Gamma)^\dagger : H_*^{-1/2}(\Gamma) \to \mathbf{X}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$(\mathrm{div}_\Gamma)^\dagger \circ \mathrm{div}_\Gamma = \mathrm{Id}\Big|_{\mathbf{X}(\mathrm{div}_\Gamma,\Gamma)}, \qquad \mathrm{div}_\Gamma \circ (\mathrm{div}_\Gamma)^\dagger = \mathrm{Id}\Big|_{H_*^{-1/2}(\Gamma)}.$$

We denote $Q_* : H^1(\Omega_s) \to H^1_*(\Omega_s)$ the projection onto mean zero functions.

5. Well-posedness of the coupled variational problem. We use the direct decompositions introduced in Section 4 to prove that the bilinear form associated to the coupled system (3.1) of Section 3 satisfies a generalized Gårding inequality.

The coupled variational problem (3.5) translates into the operator equation

$$\mathbb{G}_\kappa \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathscr{F} \\ \mathscr{R} \end{pmatrix} \in \Big(\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega)\Big)' \times (\mathcal{H}_N)'. \tag{5.1}$$

Letting $\mathsf{B}_\kappa : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \to \big(\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)\big)'$ be the operator

$$\langle \mathsf{B}_\kappa \left(\vec{\mathbf{U}}\right) \vec{\mathbf{V}} \rangle := \mathfrak{B}_\kappa\left(\vec{\mathbf{U}}, \vec{\mathbf{V}}\right)$$

associated with the Hodge-Helmholtz/Laplace volume contribution to the system, the operator $\mathbb{G}_\kappa : \big(\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega)\big) \times \mathcal{H}_N \to \big(\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega)\big)' \times (\mathcal{H}_N)'$ can be represented by the block operator matrix

$$\mathbb{G}_\kappa = \left(\begin{array}{c|c} \mathsf{B}_\kappa - \begin{pmatrix} \begin{pmatrix}\gamma_t^-\end{pmatrix}^* \\ \begin{pmatrix}\gamma^-\end{pmatrix}^* \end{pmatrix} \cdot \mathbb{A}_\kappa^{DN} \cdot \begin{pmatrix} \gamma_t^- \\ -\gamma^- \end{pmatrix} & \begin{pmatrix} \begin{pmatrix}\gamma_t^-\end{pmatrix}^* \\ \begin{pmatrix}\gamma^-\end{pmatrix}^* \end{pmatrix} \cdot \left(\mathbb{P}_\kappa^+\right)_{22} \\ \hline \left(\mathbb{P}_\kappa^-\right)_{11} \cdot \begin{pmatrix} \gamma_t^- \\ -\gamma^- \end{pmatrix} & \mathbb{A}_\kappa^{ND} \end{array}\right), \tag{5.2}$$

shown here in "variational arrangement".

The symmetry revealed in (2.24) makes explicit much of the structure of the above operator. We have introduced colors to better highlight the contribution of each individual block in the following sections.

Our goal is to design an isomorphism $\mathbb{X}$ of the test space and resort to compact perturbations of $\mathbb{G}_\kappa \circ \mathbb{X}^{-1}$ to achieve an operator block structure with diagonal blocks that are elliptic over the splittings of Section 4 and off-diagonal blocks that fit a skew-symmetric pattern. Stability of the coupled system can then be obtained from the next theorem.

Theorem 5.1 (See [10, Thm. 4]). *If a bilinear form $a : V \times V \to \mathbb{C}$ on a reflexive Banach space V is T-coercive:*

$$\left|a(u, \mathbb{X}\overline{u}) + c(u, \overline{u})\right| \geq C\|u\|_V^2 \quad \forall u \in V, \tag{5.3}$$

with $C > 0$, $c : V \times V \to \mathbb{C}$ compact and $\mathbb{X} : V \to V$ an isomorphism of V, then the operator $A : V \to V'$ defined by $A : u \mapsto a(u, \cdot)$ is Fredholm with index 0.

The authors of [8] refer to (5.3) as "Generalized Gårding inequality", because

$$\left|a\left(u, \mathbb{X}\overline{u}\right)\right| \geq C\|u\|_V^2 - \left|c\left(u, \overline{u}\right)\right|, \qquad \forall\, u \in V,$$

generalizes the classical Gårding inequality which reads

$$b(u,u) \geq C_1\|u\|^2_{H^\ell(\Omega)} - C_2\|u\|_{L^2(\Omega)}, \qquad \forall\, u \in H_0^\ell(\Omega),$$

for some $C_2 \geq 0$, $C_1 > 0$, where b is a bilinear form associated to a uniformly elliptic operator of even order 2ℓ. Assuming that (5.3) holds with $\mathbb{X} = \mathrm{Id}$, a simple proof of the stability estimate

$$\|u\|_V \leq C\|f\|_{V'},$$

obtained for the unique solution of the operator equation $Au = f$ when A is injective is given in [36, Thm. 3.15]. A proof of the general case can be deduced from [22].

T-coercivity theory is a reformulation of the Banach-Nečas-Babuška theory. The former relies on the construction of explicit inf-sup operators at the discrete and continuous levels, whereas the later develops on an abstract inf-sup condition [13].

In deriving the following results, it will be convenient to denote $\vec{\mathbf{U}} := \left(\overline{\mathbf{U}},\ \overline{P}\right)^\top$ and $\vec{\mathbf{p}} := \left(\overline{\mathbf{p}},\ \overline{q}\right)^\top$.

5.1. Space isomorphisms. In this section, we take up the challenge of finding a suitable isomorphism $\mathbb{X}$. We build it separately for the function spaces in Ω_s and on the boundary Γ. Crucial hints are offered by the construction of the sign-flip isomorphism of [10].

We start with devising an isomorphism Ξ of the volume function spaces and show that the upper-left diagonal block of $\mathbb{G}_\kappa$ satisfy a generalized Guårding inequality.

Under the assumption that Ω_s has trivial de Rham cohomology, there exists a bijective "scalar potential lifting" $\mathsf{S} : \mathbf{N}(\mathbf{curl}, \Omega_s) \to H^1_*(\Omega_s)$ satisfying $\nabla\mathsf{S}(\mathbf{U}) = \mathbf{U}$. The Poincaré-Friedrichs inequality guarantees that this map is continuous.

Notice that since it also follows from the Poincaré-Friedrichs inequality that $\nabla : H^1_*(\Omega_s) \to \mathbf{N}(\mathbf{curl}, \Omega_s)$ is injective, $\mathsf{S}\circ\nabla : H^1(\Omega_s) \to H^1_*(\Omega_s)$ is a bounded projection onto the space of Lebesgue measurable functions having zero mean. It's nullspace consists of the constant functions in Ω_s.

Isomorphism of the Volume Function Spaces

Proposition 5.2. *For any $\theta > 0$ and $\beta > 0$, the bounded linear operator $\Xi : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \to \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ defined by*

$$\Xi\left(\vec{\mathbf{U}}\right) := \begin{pmatrix} \mathbf{U}^\perp - \mathbf{U}^0 + \beta\,\nabla P \\ -\theta\left(J\left(\mathbf{U}^0\right) + \beta\,\mathbf{mean}\,(P)\right)\end{pmatrix}.$$

has a continuous inverse. In other words, Ξ is an isomorphism of Banach spaces.

Proof. By showing that Ξ is a bijection, the theorem follows as a consequence of the bounded inverse theorem.

Let $(\mathbf{V}\ Q)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$. Since $\nabla Q \in \mathbf{N}(\mathbf{curl}, \Omega_s)$, we immediately have $\mathsf{Z}\left(\mathbf{V}^\perp - \theta^{-1}\nabla Q\right) = \mathbf{V}^\perp$ and $(\mathrm{Id} - \mathsf{Z})\left(\mathbf{V}^\perp - \theta^{-1}\nabla Q\right) = -\theta^{-1}\nabla Q$. Hence, relying on the resulting observation that $\nabla \mathsf{S}\left(\left(\mathbf{V}^{\perp} - \theta^{-1}\nabla Q\right)^{0}\right) = -\theta^{-1}\nabla Q$ and exploiting that $\mathbf{mean}\left(H^1_*(\Omega_s)\right) = \{0\}$, we have

$$\Xi\left(\begin{pmatrix} \mathbf{V}^{\perp} - \theta^{-1}\nabla Q \\ \beta^{-1}\left(\mathsf{S}\left(\mathbf{V}^{0}\right) - \theta^{-1}Q\right)\end{pmatrix}\right) = \begin{pmatrix} \mathbf{V} \\ \mathsf{S}\left(\nabla Q\right) + \mathbf{mean}\left(Q\right)\end{pmatrix}. \tag{5.4}$$

Since $H^1(\Omega_s)$ decomposes into the stable direct sum of $H^1_*(\Omega_s)$ and the space of constant functions in Ω_s, (5.4) shows that Ξ is surjective.

Now, suppose that $\Xi\left(\vec{\mathbf{V}}\right) = \Xi\left(\vec{\mathbf{U}}\right)$. Then, we have

$$\mathbf{U}^0 - \mathbf{V}^0 = \nabla \mathsf{S}\left(\mathbf{U}^0 - \mathbf{V}^0\right) = \beta\,\nabla\left(\mathbf{mean}\left(Q-P\right)\right) = 0.$$

Since the considerations of Section 4 readily yield that $\mathbf{V}^{\perp} = \mathbf{U}^{\perp}$, we conclude that $\mathbf{V} = \mathbf{U}$. In turn, it follows that $\nabla P = \nabla Q$ and $\mathbf{mean}(P) = \mathbf{mean}(Q)$. Therefore, Ξ is injective. ∎

We now turn to the design of an isomorphism for the Neumann trace space $\mathcal{H}_N$ and prove that the lower-right block $\mathbb{A}^{ND}_{\kappa}$ of $\mathbb{G}_{\kappa}$ satisfies a generalized Guårding inequality.

Isomorphism of the Trace Spaces

PROPOSITION 5.3. *For any $\tau > 0$ and $\lambda > 0$, the bounded linear operator $\Xi^{\Gamma} : \mathcal{H}_N \to \mathcal{H}_N$ defined by*

$$\Xi^{\Gamma}(\vec{\mathbf{p}}) := \begin{pmatrix} \mathbf{p}^{\perp} - \mathbf{p}^{0} - \lambda\left(\mathrm{div}_{\Gamma}\right)^{\dagger}\mathsf{Q}_{*}q \\ -\tau\left(\mathrm{div}_{\Gamma}\left(\mathbf{p}\right) + \lambda\,\mathbf{mean}\left(q\right)\right)\end{pmatrix}$$

has a continuous inverse. In other words, Ξ^{Γ} is an isomorphism of Banach spaces.

Proof. We proceed as in proposition 5.2. Since $\left(\mathrm{div}_{\Gamma}\right)^{\dagger}\mathsf{Q}_{*}q \in \mathbf{X}(\mathrm{div}_{\Gamma},\Gamma)$, we have $\mathsf{Z}^{\Gamma}\left(\Xi^{\Gamma}_1(\vec{\mathbf{p}})\right) = \mathbf{p}^{\perp} - \left(\mathrm{div}_{\Gamma}\right)^{\dagger}\mathsf{Q}_{*}q$. Using that $\mathbf{mean}\circ\mathrm{div}_{\Gamma} = 0$ and $\left(\mathrm{div}_{\Gamma}\right)^{\dagger}\mathrm{div}_{\Gamma}\mathbf{p} = \mathbf{p}^{\perp}$, we evaluate

$$\Xi^{\Gamma}\left(\begin{pmatrix} -\mathbf{p}^{0} - \tau^{-1}\left(\mathrm{div}_{\Gamma}\right)^{\dagger}\mathsf{Q}_{*}q \\ \lambda^{-1}\left(-\mathrm{div}_{\Gamma}(\mathbf{p}) - \tau^{-1}q\right)\end{pmatrix}\right) = \begin{pmatrix} \mathbf{p}^{0} + \mathbf{p}^{\perp} \\ \mathsf{Q}_{*}q + \mathbf{mean}\left(q\right)\end{pmatrix}.$$

This shows that Ξ^{Γ} is surjective.

Suppose that $X^{\Gamma}(\vec{\mathbf{p}}) = X^{\Gamma}(\vec{\mathbf{a}})$. It is immediate that $\mathbf{p}^0 = \mathbf{a}^0$. On the one hand, we obtain from $X^{\Gamma}_1(\vec{\mathbf{p}}) = X^{\Gamma}_1(\vec{\mathbf{a}})$ that

$$\mathbf{p}^{\perp} - \mathbf{a}^{\perp} = \lambda\left(\mathrm{div}_{\Gamma}\right)^{\dagger}\left(\mathsf{Q}_{*}q - \mathsf{Q}_{*}b\right). \tag{5.5}$$

On the other hand, $X^{\Gamma}_2(\vec{\mathbf{p}}) = X^{\Gamma}_2(\vec{\mathbf{a}})$ implies that

$$\mathrm{div}_{\Gamma}\left(\mathbf{p} - \mathbf{a}\right) = \lambda\,\mathbf{mean}\left(q - b\right). \tag{5.6}$$

Relying on the fact that $\mathrm{div}_{\Gamma} = \mathrm{div}_{\Gamma}\circ\mathsf{Z}^{\Gamma}$ again, combining (5.5) and (5.6) yields

$$\mathsf{Q}_{*}q + \mathbf{mean}(q) = \mathsf{Q}_{*}b + \mathbf{mean}(b).$$

Evidently, (5.5) then also guarantees that $\mathbf{p}^{\perp} = \mathbf{a}^{\perp}$. We can finally conclude that X^{Γ} is injective and thus the result follows from the bounded inverse theorem. ◻

In the following, we will write Ξ_1^{Γ} and Ξ_2^{Γ} for the components of the isomorphism of the trace space.

5.2. Main result. The main result of this work, stated in Theorem 5.6, states that the operator $\mathbb{G}_{\kappa}$ associated with the coupled system (3.5) is well-posed when κ^2 lies outside the discrete set of forbidden frequencies described in [14]. It relies on two main propositions, whose proofs are postponed until the end of section 5.

The first claims that the diagonal of $\mathbb{G}_{\kappa}$ (as a sum of block operators) is T-coercive.

Proposition 5.4. *For any frequency $\omega \geq 0$, there exist a compact operator $\mathsf{K} : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N \to \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N$, a positive constant $C > 0$ and parameters $\theta > 0$ and $\tau > 0$, possibly depending on Ω_s, ϵ, μ, κ and ω, such that*

$$\mathfrak{Re}\left\langle \operatorname{diag}(\mathbb{G}_{\kappa}) \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \Xi\vec{\vec{\mathbf{U}}} \\ \Xi^{\Gamma}\vec{\vec{\mathbf{p}}} \end{pmatrix} \right\rangle + \left\langle \mathsf{K} \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \vec{\vec{\mathbf{U}}} \\ \vec{\vec{\mathbf{p}}} \end{pmatrix} \right\rangle \geq C \left(\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_s)} + \|P\|^2_{H^1(\Omega_s)} + \|\vec{\mathbf{p}}\|^2_{\mathcal{H}_N} \right)$$

for all $\vec{\mathbf{U}} := (\mathbf{U}\ P)^{\top} \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$ and $\vec{\mathbf{p}} \in \mathcal{H}_N$.

The proof of this proposition will rely on several steps: Lemma 5.8, Lemma 5.9 and Lemma 5.10.

The second proposition states that the off-diagonal blocks are compact operators. The proof of that fact relies on definitions and results that belong to the next technical section. It will materialize as the last piece of the puzzle that completes the proof of the T-coercivity of $\mathbb{G}_{\kappa}$.

Proposition 5.5. *For any frequency $\omega \geq 0$, there exists, for a suitable choice of τ, β, θ and λ, a continuous compact endomorphism K of the space $\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N$ such that*

$$\mathfrak{Re}\left\langle \left(\mathbb{G}_{\kappa} - \operatorname{diag}(\mathbb{G}_{\kappa})\right) \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \Xi\vec{\vec{\mathbf{U}}} \\ \Xi^{\Gamma}\vec{\vec{\mathbf{p}}} \end{pmatrix} \right\rangle = \left\langle \mathsf{K} \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \vec{\vec{\mathbf{U}}} \\ \vec{\vec{\mathbf{p}}} \end{pmatrix} \right\rangle. \tag{5.7}$$

The main result seamlessly follows from the two previous propositions.

T-Coercivity of the Coupled System

THEOREM 5.6. *For any $\omega \geq 0$, there exists an isomorphism $\mathbb{X}_\kappa$ of the trial space $\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N$, and compact operator $\mathbb{K} : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N \to \left(\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)\right)' \times \mathcal{H}'_N$ such that*

$$\mathfrak{Re} \left\langle (\mathbb{G}_\kappa + \mathbb{K}) \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \mathbb{X} \begin{pmatrix} \vec{\overline{\mathbf{U}}} \\ \vec{\overline{\mathbf{p}}} \end{pmatrix} \right\rangle \geq C \left(\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl}, \Omega_s)} + \|P\|^2_{H^1(\Omega_s)} + \|\vec{\mathbf{p}}\|^2_{\mathcal{H}_N} \right)$$

for some positive constant $C > 0$.

Proof. The proof will amount to the recognition that the choices of parameters in the previous Proposition 5.4 and Proposition 5.5 are compatible. ∎

The following corollary is immediate upon applying Theorem 5.1.

COROLLARY 5.7. *The system operator $\mathbb{G}_k : \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s) \times \mathcal{H}_N \to \left(\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)\right)' \times \mathcal{H}'_N$ associated with the variational problem (3.5) is Fredholm of index 0.*

Injectivity, guaranteed when κ^2 avoids resonant frequencies by corollary 3.2, yields well-posedness.

5.3. Compactness and coercivity. Equipped with the isomorphism Ξ, let us now study coercivity of the bilinear form $\mathfrak{B}_\kappa$ defined in (3.2) and associated to the Hodge–Helmholtz/Laplace operator.

LEMMA 5.8. *For any frequency $\omega \geq 0$ and parameter $\beta > 0$, there exist a positive constant $C > 0$ and a parameter $\theta > 0$, possibly depending on Ω_s, μ, ϵ and ω, and a compact bounded sesqui-linear form $\mathfrak{K}$ defined over $\mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$, such that*

$$\mathfrak{Re} \left(\mathfrak{B}_\kappa \left(\vec{\mathbf{U}}, \Xi \left(\vec{\overline{\mathbf{U}}} \right) \right) - \mathfrak{K} \left(\vec{\mathbf{U}}, \vec{\mathbf{U}} \right) \right) \geq C \left(\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl}, \Omega_s)} + \|P\|^2_{H^1(\Omega_s)} \right)$$

for all $\vec{\mathbf{U}} := (\mathbf{U},\ P)^\top \in \mathbf{H}(\mathbf{curl}, \Omega_s) \times H^1(\Omega_s)$.

Proof. As $\mathbf{curl}\left(\mathbf{U}^0\right) = 0$, $\mathbf{curl}\,(\nabla P) = 0$, and $\nabla \circ \mathbf{mean} = 0$, we evaluate

$$\begin{aligned}
&\mathfrak{B}_\kappa \left(\begin{pmatrix} \mathbf{U} \\ P \end{pmatrix}, \begin{pmatrix} \overline{\mathbf{U}}^\perp - \overline{\mathbf{U}}^0 + \beta \overline{\nabla P} \\ -\theta \left(\mathsf{S}\left(\overline{\mathbf{U}}^0\right) + \beta\, \mathbf{mean}\left(\overline{P}\right) \right) \end{pmatrix} \right) \\
&= \left(\mu^{-1} \mathbf{curl}\left(\mathbf{U}^\perp\right), \mathbf{curl}\left(\mathbf{U}^\perp\right) \right)_{\Omega_s} + \left(\epsilon \nabla P, \mathbf{U}^\perp \right)_{\Omega_s} - \left(\epsilon \nabla P, \mathbf{U}^0 \right)_{\Omega_s} + \beta \left(\epsilon \nabla P, \nabla P \right)_{\Omega_s} \\
&\quad + \theta \left(\epsilon \mathbf{U}^\perp, \mathbf{U}^0 \right)_{\Omega_s} + \theta \left(\epsilon \mathbf{U}^0, \mathbf{U}^0 \right)_{\Omega_s} - \omega^2 \left(\epsilon \mathbf{U}^\perp, \mathbf{U}^\perp - \mathbf{U}^0 + \beta \nabla P \right)_{\Omega_s} \\
&\quad - \omega^2 \left(\epsilon \mathbf{U}^0, \mathbf{U}^\perp \right) + \omega^2 \left(\epsilon \mathbf{U}^0, \mathbf{U}^0 \right) - \beta \omega^2 \left(\epsilon \mathbf{U}^0, \nabla P \right) \\
&\quad - \left(P, \theta \mathsf{S}\left(\mathbf{U}^0\right) \right)_{\Omega_s} - \left(P, \theta \beta\, \mathbf{mean}(P) \right)_{\Omega_s}.
\end{aligned}$$

Upon application of the Cauchy-Schwartz inequality, the bounded sesqui-linear form

$$\mathfrak{K}\left(\vec{\mathbf{U}},\vec{\mathbf{U}}\right) := \left(\epsilon\nabla P,\mathbf{U}^{\perp}\right)_{\Omega_s} - \left(P,\theta\mathsf{S}\left(\mathbf{U}^0\right)\right)_{\Omega_s} + \theta\left(\epsilon\mathbf{U}^{\perp},\mathbf{U}^0\right)_{\Omega_s} - \omega^2\left(\epsilon\mathbf{U}^0,\mathbf{U}^{\perp}\right)_{\Omega_s} \\ - \omega^2\left(\epsilon\mathbf{U}^{\perp},\mathbf{U}^{\perp}-\mathbf{U}^0+\beta\nabla P\right)_{\Omega_s} - \left(P,\theta\beta\,\mathbf{mean}(P)\right)_{\Omega_s}$$

is shown to be compact by Proposition 4.3 and the Rellich theorem. Using Young's inequality twice with $\delta > 0$, we estimate

$$\mathfrak{Re}\left(\mathfrak{B}_\kappa\left(\vec{\mathbf{U}},\Xi\left(\vec{\mathbf{U}}\right)\right) - \mathfrak{K}\left(\vec{\mathbf{U}},\vec{\mathbf{U}}\right)\right) \geq \mu_{\max}^{-1}\left\|\mathbf{curl}\,\mathbf{U}^{\perp}\right\|_{\Omega_s}^2 \\ + \left(\epsilon_{\min}\left(\theta+\omega^2\right) - \delta\,\epsilon_{\max}\left(1+\beta\omega^2\right)\right)\left\|\mathbf{U}^0\right\|_{\Omega_s}^2 \\ + \mathfrak{Re}\left(\epsilon_{\min}\,\beta - \frac{1}{\delta}\epsilon_{\max}\left(1+\beta\omega^2\right)\right)\|\nabla P\|_{\Omega_s}^2 .$$

The operator $\mathbf{curl} : \mathsf{Z}\left(\mathbf{H}(\mathbf{curl},\Omega)\right) \to \mathbf{L}^2(\Omega_s)$ is a continuous injection, hence since its image is closed in $\mathbf{L}^2(\Omega_s)$, it is also bounded below. Hence, for any $\beta > 0$, choose $\delta > 0$ large enough, then $\theta > 0$ accordingly large, and the desired inequality follows. ∎

The complex inner products

$$(a,b)_{-1/2} := \int_\Gamma\int_\Gamma G_0(\mathbf{x}-\mathbf{y})\,a(\mathbf{x})\,\overline{b(\mathbf{y})}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}),$$
$$(\mathbf{a},\mathbf{b})_{-1/2} := \int_\Gamma\int_\Gamma G_0(\mathbf{x}-\mathbf{y})\,\mathbf{a}(\mathbf{x})\cdot\overline{\mathbf{b}(\mathbf{y})}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}),$$

defined over $H^{-1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ respectively, are positive definite Hermitian forms and they induce equivalent norms on the trace spaces. Combined with the stability of the decomposition introduced in Section 4, this observation also allows us to conclude that

$$\mathbf{a} \mapsto \left\|\mathrm{div}_\Gamma(\mathbf{a})\right\|_{-1/2} + \left\|\left(\mathrm{Id}-P^\Gamma\right)\mathbf{a}\right\|_{-1/2}$$

also defines an equivalent norm in $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$.

Let us denote the two components of the isomorphism Ξ by

$$\Xi_1(\vec{\mathbf{U}}) := \mathbf{U}^{\perp} - \mathbf{U}^0 + \nabla P, \qquad \text{and} \qquad \Xi_2(\vec{\mathbf{U}}) := -\theta\left(\mathsf{S}\left(\mathbf{U}^0\right) + \mathbf{mean}(P)\right).$$

We now derive an estimate similar to the one found in Lemma 5.8 that completes the proof of the coercivity of the upper-left diagonal block of $\mathbb{G}_\kappa$.

LEMMA 5.9. *For any frequency $\omega \geq 0$ and parameter $\beta > 0$, there exist a positive constant $C > 0$ and a parameter $\theta > 0$, possibly depending on Ω_s, μ, ϵ and κ, and a compact linear operator $\mathcal{K} : \mathbf{H}(\mathbf{curl},\Omega_s) \times H^1(\Omega_s) \to \mathbf{H}(\mathbf{curl},\Omega_s) \times H^1(\Omega_s)$ such* *that*

$$\mathfrak{Re}\Bigg(\Big\langle -\mathbb{A}_\kappa^{DN}\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\vec{\mathbf{U}}\\ \gamma^-\Xi_2\vec{\mathbf{U}}\end{pmatrix}\Big\rangle + \Big\langle\mathcal{K}\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\vec{\mathbf{U}}\\ \gamma^-\Xi_2\vec{\mathbf{U}}\end{pmatrix}\Big\rangle\Bigg) \geq C\left\|\begin{pmatrix}\gamma_t^-\mathbf{U}\\ \gamma^-(P)\end{pmatrix}\right\|^2_{\mathcal{H}_D(\Omega_s)}$$

for all $\vec{\mathbf{U}} := (\mathbf{U}\ P)^\top \in \mathbf{H}(\mathbf{curl},\Omega_s)\times H^1(\Omega_s)$.

Proof. We indicate with a hat equality up to a compact perturbation (e.g. $\hat{=}$). The jump conditions (2.15a) yield $\{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa = \mathcal{T}_N\cdot\mathcal{DL}_\kappa$. We deduce from [14, Sec. 6.4] that,

$$\begin{aligned}
&\Big\langle -\mathcal{T}_N\cdot\mathcal{DL}_\kappa\begin{pmatrix}\gamma_t^-\mathbf{U}\\ -\gamma^-(P)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\vec{\mathbf{U}}\\ \gamma^-\Xi_2\vec{\mathbf{U}}\end{pmatrix}\Big\rangle \\
&\hat{=}\Big(\mathrm{div}_\Gamma\big(\mathbf{n}\times\gamma_t^-\mathbf{U}\big),\mathrm{div}_\Gamma\big(\mathbf{n}\times\gamma_t^-\Xi_1\vec{\mathbf{U}})\big)\Big)_{-1/2} - \kappa^2\big(\mathbf{n}\times\gamma_t^-\mathbf{U},\mathbf{n}\times\gamma_t^-\Xi_1\vec{\mathbf{U}}\big)_{-1/2} \\
&\quad + \Big(\mathbf{n}\times\gamma_t^-\mathbf{U},\mathbf{curl}_\Gamma\big(\gamma^-\Xi_2\vec{\mathbf{U}}\big)\Big)_{-1/2} - \big(\mathbf{n}\times\gamma_t^-\Xi_1\vec{\mathbf{U}},\mathbf{curl}_\Gamma\left(\gamma^-(P)\right)\big)_{-1/2} \\
&\quad = \Big(\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right),\mathrm{div}_\Gamma\big(\gamma_\tau^-\Xi_1\vec{\mathbf{U}})\big)\Big)_{-1/2} - \kappa^2\big(\gamma_\tau^-\mathbf{U},\gamma_\tau^-\Xi_1\vec{\mathbf{U}}\big)_{-1/2} \\
&\qquad - \Big(\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma\big(\gamma^-\Xi_2\vec{\mathbf{U}}\big)\Big)_{-1/2} + \big(\gamma_\tau^-\Xi_1\vec{\mathbf{U}},\mathbf{curl}_\Gamma\left(\gamma^-(P)\right)\big)_{-1/2}
\end{aligned} \tag{5.8}$$

We consider each component of the isomorphim Ξ in turn. Since Lemma 4.2 guarantees that $\mathsf{Z}(\mathbf{U}) \in \mathbf{H}^1(\Omega_s)$, $\gamma_\tau\circ\mathsf{Z}$ is a continuous mapping $\mathbf{H}(\mathbf{curl},\Omega_s)\to\mathbf{H}_R^{1/2}(\Omega_s)$ found compact by Lemma 2.1. Therefore,

$$\begin{aligned}
\gamma_\tau^-\Xi_1\big(\vec{\mathbf{U}}\big) &= \gamma_\tau^-\mathbf{U}^\perp - \gamma_\tau^-\mathbf{U}^0 + \beta\gamma_\tau^-\nabla P \\
&\hat{=}\mathsf{Z}^\Gamma\left(\gamma_\tau^-\mathbf{U}\right) - \big(\mathrm{Id}-\mathsf{Z}^\Gamma\big)\gamma_\tau^-\mathbf{U} + \beta\,\mathbf{curl}_\Gamma\left(\gamma^-P\right).
\end{aligned} \tag{5.9}$$

Let's introduce expression (5.9) in the various terms of (5.8) involving $\Xi_1(\vec{\mathbf{U}})$. Lemma 4.4 yields

$$\begin{aligned}
\Big(\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right),\mathrm{div}_\Gamma\big(\gamma_\tau^-\Xi_1\vec{\mathbf{U}})\big)\Big)_{-1/2} &\hat{=}\Big(\mathrm{div}_\Gamma\left(\gamma_\tau\mathbf{U}\right),\mathrm{div}_\Gamma\big(\mathsf{Z}^\Gamma\left(\gamma_\tau^-\mathbf{U}\right)\big)\Big)_{-1/2} \\
&\quad - \Big(\mathrm{div}_\Gamma\left(\gamma_\tau\mathbf{U}\right),\mathrm{div}_\Gamma\Big(\big(\mathrm{Id}-\mathsf{Z}^\Gamma\big)\gamma_\tau^-\mathbf{U}\Big)\Big)_{-1/2} \\
&\quad + \beta\Big(\mathrm{div}_\Gamma\left(\gamma_\tau\mathbf{U}\right),\mathrm{div}_\Gamma\big(\mathbf{curl}_\Gamma\left(\gamma^-P\right)\big)\Big)_{-1/2} \\
&= \big(\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right),\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right)\big)_{-1/2}.
\end{aligned}$$

Similarly,

$$-\kappa^2\left(\gamma_\tau^-\mathbf{U},\gamma_\tau^-\Xi_1\vec{\mathbf{U}}\right)_{-1/2}\hat{=}\kappa^2\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U}\right)_{-1/2}$$
$$-\beta\kappa^2\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma\left(\gamma^-P\right)\right)_{-1/2}$$

and

$$\left(\gamma_\tau^-\Xi_1\vec{\mathbf{U}},\mathbf{curl}_\Gamma\left(\gamma^-\left(P\right)\right)\right)_{-1/2}\hat{=}-\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma\left(\gamma^-\left(P\right)\right)\right)_{-1/2}$$
$$+\beta\left(\mathbf{curl}_\Gamma\left(\gamma^-P\right),\mathbf{curl}_\Gamma\left(\gamma^-\left(P\right)\right)\right)_{-1/2}.$$

We now want to evaluate the terms involving $\Xi_2(\vec{\mathbf{U}})$. We introduce

$$\mathbf{curl}_\Gamma\left(\gamma^-\Xi_2\vec{\mathbf{U}}\right)=-\theta\gamma_\tau^-\nabla\left(\mathsf{S}(\mathbf{U}^0)+\mathbf{mean}(P)\right)=-\theta\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},$$

in (5.8) to obtain

$$-\left(\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma\left(\gamma^-\Xi_2\vec{\mathbf{U}}\right)\right)_{-1/2}=\theta\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau\mathbf{U}\right)_{-1/2}$$

Using Young's inequality twice with $\delta>0$,

$$\mathfrak{Re}\left(\left\langle-\{\mathcal{T}_N\}\cdot\mathcal{DL}_\kappa\begin{pmatrix}\gamma_t^-\mathbf{U}\\-\gamma^-\left(P\right)\end{pmatrix},\begin{pmatrix}\gamma_t^-\Xi_1\vec{\mathbf{U}}\\\gamma^-\Xi_2\vec{\mathbf{U}}\end{pmatrix}\right\rangle\right)$$
$$\hat{=}\left\|\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right)\right\|_{-1/2}^2+\left(\mathfrak{Re}\left(\kappa^2\right)+\theta\right)\left\|\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U}\right\|_{-1/2}^2$$
$$+\beta\left\|\mathbf{curl}_\Gamma\left(\gamma^-\left(P\right)\right)\right\|^2-\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma\left(\gamma^-\left(P\right)\right)\right)_{-1/2}$$
$$-\beta\,\mathfrak{Re}\left(\kappa^2\right)\left(\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U},\mathbf{curl}_\Gamma\left(\gamma^-P\right)\right)_{-1/2}$$

$$\geq\left\|\mathrm{div}_\Gamma\left(\gamma_\tau^-\mathbf{U}\right)\right\|_{-1/2}^2+\left(\beta-\frac{1}{\delta}\left(1+\beta\,\mathfrak{Re}\left(\kappa^2\right)\right)\right)\left\|\mathbf{curl}_\Gamma\left(\gamma^-\left(P\right)\right)\right\|^2$$
$$+\left(\mathfrak{Re}\left(\kappa^2\right)+\theta-\delta\left(1+\beta\,\mathfrak{Re}\left(\kappa^2\right)\right)\right)\left\|\left(\mathrm{Id}-\mathsf{Z}^\Gamma\right)\gamma_\tau^-\mathbf{U}\right\|_{-1/2}^2.$$

The operator $\mathbf{curl}_\Gamma:H_*^1(\Omega_s)\to\mathbf{H}^{-1/2}\left(\mathrm{div}_\Gamma,\Gamma\right)$ is a continuous injection [14, Lem. 6.4]. It is thus bounded below. Since the mean operator has finite rank, it is compact. Therefore, for any $\beta>0$, choose $\delta>0$ large enough, then $\theta>0$ accordingly large, and the desired inequality follows by equivalence of norms. ∎

LEMMA 5.10. *For any frequency $\omega \geq 0$, there exist a compact linear operator $\mathcal{K} : \mathcal{H}_N \to \mathcal{H}_D$, a positive constants $C > 0$ and parameters $\tau > 0$ and $\lambda > 0$, possibly depending on Ω_s, μ, ϵ and κ, such that*

$$\mathfrak{Re}\left(\left\langle \mathbb{A}_\kappa^{ND}\left(\vec{\mathbf{p}}\right), \Xi^\Gamma \vec{\mathbf{p}}\right\rangle + \left\langle \mathcal{K}\,\vec{\mathbf{p}}, \vec{\mathbf{p}}\right\rangle\right) \geq C\|\vec{\mathbf{p}}\|^2_{\mathcal{H}_N}$$

for all $\vec{\mathbf{p}} \in \mathcal{H}_N$. In particular, for $\mathfrak{Re}\left(k^2\right) \neq 0$, the inequality holds with $\tau = 1/\kappa^2$.

Proof. We indicate with a hat equality up to a compact perturbation (e.g. $\hat{=}$). The jump conditions (2.15b) yield $\{\mathcal{T}_D\} \cdot \mathcal{SL}\left(\vec{\mathbf{p}}\right) = \mathcal{T}_D \cdot \mathcal{SL}\left(\vec{\mathbf{p}}\right)$. We deduce from [14, Sec. 6.3] and the compact embedding of $\mathbf{X}\left(\mathrm{div}_\Gamma, \Gamma\right)$ into $\mathbf{H}_R^{-1/2}(\Gamma)$ that

$$\begin{aligned}
\left\langle \mathcal{T}_D \cdot \mathcal{SL}\left(\vec{\mathbf{p}}\right), \Xi^\Gamma \vec{\mathbf{p}}\right\rangle &\hat{=} -\left(\mathbf{p}^0, \Xi_1^\Gamma(\mathbf{p})\right)_{-1/2} - \left(q, \mathrm{div}_\Gamma\left(\Xi_1^\Gamma(\mathbf{p})\right)\right)_{-1/2} \\
&\quad - \left(\mathrm{div}_\Gamma(\mathbf{p}), \Xi_2^\Gamma \vec{\mathbf{p}}\right)_{-1/2} - \kappa^2\left(q, \Xi_2^\Gamma\left(\vec{\mathbf{p}}\right)\right)_{-1/2} \\
&\hat{=} \left(\mathbf{p}^0, \mathbf{p}^0\right)_{-1/2} - \left(q, \mathrm{div}_\Gamma(\mathbf{p}^\perp)\right)_{-1/2} + \lambda\left(q, \mathsf{Q}_* q\right)_{-1/2} \\
&\quad + \tau\left(\mathrm{div}_\Gamma(\mathbf{p}), \mathrm{div}_\Gamma(\mathbf{p})\right)_{-1/2} + \tau\kappa^2\left(q, \mathrm{div}_\Gamma(\mathbf{p}^\perp)\right)_{-1/2}.
\end{aligned}$$

When $\mathfrak{Re}\left(\kappa^2\right) > 0$, setting $\tau = 1/\kappa^2$ immediately yields the existence of a compact linear operator $\mathcal{K} : \mathcal{H}_N \to \mathcal{H}_D$ such that

$$\left\langle \mathcal{T}_D \cdot \mathcal{SL}\left(\vec{\mathbf{p}}\right), \Xi^\Gamma \vec{\mathbf{p}}\right\rangle + \left\langle \mathcal{K}\vec{\mathbf{p}}, \Xi^\Gamma \vec{\mathbf{p}}\right\rangle \geq C\left(\left\|\mathrm{div}_\Gamma\left(\mathbf{p}\right)\right\|^2_{-1/2} + \left\|\mathbf{p}^0\right\|^2_{-1/2} + \|\mathsf{Q}_* q\|^2_{-1/2}\right).$$

When $\kappa^2 = 0$, the same inequality is obtained for any $\lambda > 0$ by using Young's inequality as in the proof of Lemma 5.9 and choosing τ large enough. The claimed inequality follows by equivalence of norms. ∎

Equipped with the previous three lemmas, we are now ready to prove Lemma 5.4.

Proof of Proposition 5.4. For any parameters $\beta > 0$ and $\lambda > 0$, the choices of δ and θ in the proofs of Lemma 5.8 and Lemma 5.9 are not mutually exclusive. The choice of τ in Lemma 5.10 is independent of that choice of θ. ∎

Finally, The off-diagonal blocks remain to be considered. We will show that, up to compact perturbations, a suitable choice of parameters in the isomorphisms Ξ and Ξ^Γ of the test space leads to a skew-symmetric pattern in $\mathbb{G}_\kappa$. In other words, up to compact terms, the volume and boundary parts of the system decouples over the space decompositions introduced in Section 4.

Proof of Proposition 5.5. We indicate with a hat equality up to a compact perturbation (e.g. $\hat{=}$). The isomorphisms Ξ and Ξ^Γ were designed so that favorable cancellations occur in evaluating the left hand side of (5.7).

From the jump properties (2.15b), we have $\{\mathcal{T}_N\}\mathcal{SL}_\kappa = \mathcal{T}_N^- \mathcal{SL}_\kappa - (1/2)\mathrm{Id}$. Therefore, as in (2.23), we evaluate

$$
\begin{aligned}
(5.10)\quad & \Big\langle (\mathbb{P}_\kappa^+)_{22}\,\vec{\mathbf{p}}, \begin{pmatrix} \gamma_t^- \Xi_1 \vec{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}} \end{pmatrix} \Big\rangle = \Big\langle \Big(-\{\mathcal{T}_N\}\cdot \mathcal{SL}_\kappa + \frac{1}{2}\mathrm{Id}\Big)\vec{\mathbf{p}}, \begin{pmatrix} \gamma_t^- \Xi_1 \vec{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}} \end{pmatrix} \Big\rangle \\
&= \Big\langle -\mathcal{T}_N^- \cdot \mathcal{SL}_\kappa(\vec{\mathbf{p}}), \begin{pmatrix} \gamma_t^- \Xi_1 \vec{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}} \end{pmatrix} \Big\rangle + \Big\langle \vec{\mathbf{p}}, \begin{pmatrix} \gamma_t^- \Xi_1 \vec{\overline{\mathbf{U}}} \\ \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}} \end{pmatrix} \Big\rangle \\
&= \langle \gamma_R^- \mathbf{\Psi}_\kappa(\mathbf{p}), \gamma_t^- \Xi_1 \vec{\overline{\mathbf{U}}}\rangle_\tau - \langle \gamma_n^- \nabla \psi_{\tilde\kappa}(q), \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}}\rangle_\Gamma + \langle \gamma_n^- \mathbf{\Psi}_\kappa(\mathbf{p}), \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}}\rangle_\Gamma \\
&\quad + \langle \gamma_n^- \nabla \tilde\psi_\kappa(\mathrm{div}_\Gamma \mathbf{p}), \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}}\rangle_\Gamma + \langle \mathbf{p}, \gamma_t^- \Xi_1 \vec{\overline{\mathbf{U}}}\rangle_\tau + \langle q, \gamma^- \Xi_2 \vec{\overline{\mathbf{U}}}\rangle_\Gamma \\
&\hat{=} \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\mathbf{p}^0\right), \gamma_t \overline{\mathbf{U}}^\perp\rangle_\tau - \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\mathbf{p}^0\right), \gamma_t \overline{\mathbf{U}^0}\rangle_\tau + \beta\,\langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\mathbf{p}^0\right), \gamma_t \nabla \overline{P}\rangle_\tau \\
&\quad + \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\mathbf{p}^\perp\right), \gamma_t \overline{\mathbf{U}}^\perp\rangle_\tau - \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\mathbf{p}^\perp\right), \gamma_t \overline{\mathbf{U}^0}\rangle_\tau + \beta\,\langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\mathbf{p}^\perp\right), \gamma_t \nabla \overline{P}\rangle_\tau \\
&\quad + \theta\,\langle \gamma_n^- \nabla \psi_{\tilde\kappa}(q), \gamma^- \mathsf{S}\left(\overline{\mathbf{U}}^0\right)\rangle_\Gamma - \theta\,\langle \gamma_n^- \mathbf{\Psi}_\kappa(\mathbf{p}), \gamma^- \mathsf{S}\left(\overline{\mathbf{U}}^0\right)\rangle_\Gamma \\
&\quad - \langle \gamma_n^- \nabla \tilde\psi_\kappa(\mathrm{div}_\Gamma \mathbf{p}), \theta\,\gamma^- \mathsf{S}\left(\overline{\mathbf{U}}^0\right)\rangle_\Gamma + \langle \mathbf{p}^0, \gamma_t^- \overline{\mathbf{U}}^\perp\rangle_\tau + \langle \mathbf{p}^\perp, \gamma_t^- \overline{\mathbf{U}}^\perp\rangle_\tau - \langle \mathbf{p}^0, \gamma_t^- \overline{\mathbf{U}}^0\rangle_\tau \\
&\quad - \langle \mathbf{p}^\perp, \gamma_t^- \overline{\mathbf{U}}^0\rangle_\tau + \beta\,\langle \mathbf{p}^0, \gamma_t^- \nabla \overline{P}\rangle_\tau + \beta\,\langle \mathbf{p}^\perp, \gamma_t^- \nabla \overline{P}\rangle_\tau - \theta\,\langle q, \gamma^- \mathsf{S}\left(\mathbf{U}^0\right)\rangle_\Gamma,
\end{aligned}
$$

where we have used that the finite rank of the mean operator implies compactness.

Similarly, using Proposition 2.15, we find

$$
\begin{aligned}
(5.11)\quad & \Big\langle (\mathbb{P}_\kappa^-)_{11} \begin{pmatrix} \gamma_t^- \mathbf{U} \\ -\gamma^-(P) \end{pmatrix}, \Xi^\Gamma \vec{\overline{\mathbf{p}}} \Big\rangle = \Big\langle \begin{pmatrix} \gamma_t^- \mathbf{U} \\ -\gamma^-(P) \end{pmatrix}, (\mathbb{P}_\kappa^+)_{22} \Xi^\Gamma \vec{\overline{\mathbf{p}}} \Big\rangle \\
&\hat{=} \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\overline{\mathbf{p}}^\perp\right), \gamma_t \mathbf{U}^0\rangle_\tau - \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\overline{\mathbf{p}}^0\right), \gamma_t \mathbf{U}^\perp\rangle_\tau - \lambda\,\langle \gamma_R^- \mathbf{\Psi}_\kappa\left((\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\right), \gamma_t^- \mathbf{U}^0\rangle_\tau \\
&\quad + \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\overline{\mathbf{p}}^0\right), \gamma_t \mathbf{U}^\perp\rangle_\tau - \langle \gamma_R^- \mathbf{\Psi}_\kappa\left(\overline{\mathbf{p}}^0\right), \gamma_t \mathbf{U}^\perp\rangle_\tau - \lambda\,\langle \gamma_R^- \mathbf{\Psi}_\kappa\left((\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\right), \gamma_t^- \mathbf{U}^\perp\rangle_\tau \\
&\quad - \tau\,\langle \gamma_n^- \nabla \psi_{\tilde\kappa}\left(\mathrm{div}_\Gamma \overline{\mathbf{p}}^\perp\right), \gamma^- P\rangle - \langle \gamma_n^- \mathbf{\Psi}_\kappa\left(\overline{\mathbf{p}}^\perp\right), \gamma^- P\rangle_\Gamma + \langle \gamma_n^- \mathbf{\Psi}_\kappa\left(\overline{\mathbf{p}}^0\right), \gamma^- P\rangle_\Gamma \\
&\quad + \lambda\,\langle \gamma_n^- \mathbf{\Psi}_\kappa\left((\mathrm{div})^\dagger Q_* \overline{q}\right), \gamma^- P\rangle_\Gamma - \langle \gamma_n^- \nabla \tilde\psi_\kappa\left(\mathrm{div}_\Gamma \overline{\mathbf{p}}^\perp\right), \gamma^- P\rangle_\Gamma \\
&\quad + \lambda\,\langle \gamma_n^- \nabla \tilde\psi_\kappa(Q_* \overline{q}), \gamma^- P\rangle_\Gamma + \langle \gamma_t^- \mathbf{U}^0, \overline{\mathbf{p}}^\perp\rangle_\tau + \langle \gamma_t^- \mathbf{U}^\perp, \overline{\mathbf{p}}^\perp\rangle_\tau - \langle \gamma_t^- \mathbf{U}^\perp, \overline{\mathbf{p}}^0\rangle_\tau - \langle \gamma_t^- \mathbf{U}^0, \overline{\mathbf{p}}^0\rangle_\tau \\
&\quad - \lambda\,\langle \gamma_t^- \mathbf{U}^\perp, (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle_\Gamma - \lambda\,\langle \gamma_t^- \mathbf{U}^0, (\mathrm{div}_\Gamma)^\dagger Q_* \overline{q}\rangle + \tau\,\langle \gamma^- P, \mathrm{div}_\Gamma\left(\overline{\mathbf{p}}^\perp\right)\rangle_\Gamma.
\end{aligned}
$$

Many terms in these equations can be combined and asserted compact by Lemma 4.6. They are indicated in blue. When summing the real parts of (5.10) and (5.11), the terms in red cancel. Relying on Lemma 2.7, some terms amount to compact perturbations so that we may replace κ and $\tilde\kappa$ by 0 in those instances.

We have arrived at the following identity:

(5.12)
$$
\begin{aligned}
&\mathfrak{Re}\left(\left\langle \left(\mathbb{G}_\kappa - \operatorname{diag}\left(\mathbb{G}_\kappa\right)\right)\begin{pmatrix}\vec{\mathbf{U}}\\ \vec{\mathbf{p}}\end{pmatrix}, \begin{pmatrix}\Xi\,\vec{\overline{\mathbf{U}}}\\ \Xi^\Gamma\vec{\overline{\mathbf{p}}}\end{pmatrix}\right\rangle\right)\\
&\quad\hat{=}\ \mathfrak{Re}\Bigg(\beta\left\langle\gamma_R^-\boldsymbol{\Psi}_0\left(\mathbf{p}^\perp\right),\gamma_t\nabla\overline{P}\right\rangle_\tau + \theta\left\langle\gamma_n^-\nabla\psi_0\left(q\right),\gamma^-\mathsf{S}\left(\overline{\mathbf{U}}^0\right)\right\rangle_\Gamma\\
&\qquad -\theta\left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\mathbf{p}\right),\gamma^-\mathsf{S}\left(\overline{\mathbf{U}}^0\right)\right\rangle_\Gamma + \beta\left\langle\mathbf{p}^\perp,\gamma_t^-\nabla\overline{P}\right\rangle_\tau - \theta\left\langle q,\gamma^-\mathsf{S}\left(\mathbf{U}^0\right)\right\rangle_\Gamma\\
&\qquad -\lambda\left\langle\gamma_R^-\boldsymbol{\Psi}_0\left(\left(\operatorname{div}_\Gamma\right)^\dagger Q_*\overline{q}\right),\gamma_t^-\mathbf{U}^0\right\rangle_\tau - \tau\left\langle\gamma_n^-\nabla\psi_0\left(\operatorname{div}_\Gamma\overline{\mathbf{p}}^\perp\right),\gamma^-P\right\rangle_\Gamma\\
&\qquad -\left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\overline{\mathbf{p}}^\perp\right),\gamma^-P\right\rangle_\Gamma + \left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\overline{\mathbf{p}}^0\right),\gamma^-P\right\rangle_\Gamma + \lambda\left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\left(\operatorname{div}\right)^\dagger Q_*\overline{q}\right),\gamma^-P\right\rangle_\Gamma\\
&\qquad -\lambda\left\langle\gamma_t^-\mathbf{U}^0,\left(\operatorname{div}_\Gamma\right)^\dagger Q_*\overline{q}\right\rangle_\tau + \tau\left\langle\gamma^-P,\operatorname{div}_\Gamma\left(\overline{\mathbf{p}}^\perp\right)\right\rangle_\Gamma\Bigg).
\end{aligned}
$$

We claim that the terms colored in green are compact. Indeed, the integral identities of Section 2.2.1 together with equality (2.10) yield

$$
\begin{aligned}
\left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\mathbf{p}\right),\gamma^-\mathsf{S}\left(\overline{\mathbf{U}}^0\right)\right\rangle_\Gamma &\leq \left(\left\|\psi_0\left(\operatorname{div}_\Gamma\mathbf{p}\right)\right\|_{L^2(\Omega_s)} + \left\|\boldsymbol{\Psi}_0\left(\mathbf{p}\right)\right\|_{\mathbf{L}^2(\Omega_s)}\right)\left\|\overline{\mathbf{U}}^0\right\|_{\mathbf{L}^2(\Omega_s)},\\
\left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\overline{\mathbf{p}}\right),\gamma^-P\right\rangle_\Gamma &\leq \left(\left\|\psi_0\left(\operatorname{div}_\Gamma\overline{\mathbf{p}}\right)\right\|_{L^2(\Omega_s)} + \left\|\boldsymbol{\Psi}_0\left(\overline{\mathbf{p}}\right)\right\|_{\mathbf{L}^2(\Omega_s)}\right)\left\|P\right\|_{H^1(\Omega_s)}\\
\left\langle\gamma_n^-\boldsymbol{\Psi}_0\left(\left(\operatorname{div}\right)^\dagger Q_*\overline{q}\right),\gamma^-P\right\rangle_\Gamma &\leq \left(\left\|\psi_0\left(Q_*q\right)\right\|_{L^2(\Omega_s)} + \left\|\boldsymbol{\Psi}_0\left(\operatorname{div}_\Gamma\overline{\mathbf{p}}\right)\right\|_{\mathbf{L}^2(\Omega_s)}\right)\left\|P\right\|_{H^1(\Omega_s)}
\end{aligned}
$$

Since Lemma 2.4 states that $\psi_0 : H^{-1/2}(\Gamma) \to H^1(\Omega_s)$ and $\boldsymbol{\Psi}_0 : \mathbf{H}^{-1/2}(\Gamma) \to \mathbf{H}^1(\Omega_s)$ are continuous, compactness is guaranteed by Rellich Theorem.

To go further, we need to settle for a choice of parameters in the volume and boundary isomorphisms. Choose τ to satisfy the requirements of Lemma 5.10, then set $\beta = \tau$. We are still free to let θ satisfy both Lemma 5.8 and Lemma 5.9, and then choose $\lambda = \theta$.

Under this choice of parameters, the terms in orange vanish, because we have $\langle\mathbf{p}^\perp\gamma_t^-\nabla\overline{P}\rangle_\tau = \langle\mathbf{p}^\perp,\nabla_\Gamma\gamma^-\overline{P}\rangle_\tau = -\langle\operatorname{div}_\Gamma\left(\mathbf{p}^\perp\right),\gamma^-\overline{P}\rangle_\Gamma$, and similarly

$$
\left\langle\gamma_t^-\mathbf{U}^0,\left(\operatorname{div}_\Gamma\right)^\dagger Q_*\overline{q}\right\rangle_\tau = \left\langle\gamma_t^-\nabla\mathsf{S}\left(\mathbf{U}^0\right),\left(\operatorname{div}_\Gamma\right)^\dagger Q_*\overline{q}\right\rangle_\tau = -\left\langle\gamma^-\mathsf{S}\left(\mathbf{U}^0\right),Q_*\overline{q}\right\rangle_\Gamma.
$$

Finally, relying on (2.9a), (2.9b) and (2.10) once more, we observe that

$$
\begin{aligned}
\left\langle\gamma_R^-\boldsymbol{\Psi}_0\left(\mathbf{p}^\perp\right),\gamma_t^-\nabla\overline{P}\right\rangle_\tau &= \left(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_0\left(\mathbf{p}^\perp\right),\nabla P\right)_{\Omega_s}\\
&= \left(\nabla\psi_0\left(\operatorname{div}_\Gamma\mathbf{p}^\perp\right),\nabla P\right)_{\Omega_s} = \left\langle\gamma_n^-\nabla\psi_0\left(\operatorname{div}_\Gamma\mathbf{p}^\perp\right),\gamma^-\overline{P}\right\rangle_\Gamma.
\end{aligned}
$$

A similar derivation shows that

$$
\left\langle\gamma_n^-\nabla\psi_0\left(q\right),\gamma^-\mathsf{S}\left(\overline{\mathbf{U}}^0\right)\right\rangle_\Gamma\ \hat{=}\ \left\langle\gamma_R^-\boldsymbol{\Psi}_0\left(\left(\operatorname{div}_\Gamma\right)^\dagger Q_*\overline{q}\right),\gamma_t^-\mathbf{U}^0\right\rangle_\tau.
$$

We conclude that for such a choice of parameters,

$$\mathfrak{Re}\left(\left\langle \left(\mathbb{G}_\kappa - \operatorname{diag}\left(\mathbb{G}_\kappa\right)\right) \begin{pmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{p}} \end{pmatrix}, \begin{pmatrix} \Xi \vec{\vec{\mathbf{U}}} \\ \Xi^\Gamma \vec{\vec{\mathbf{p}}} \end{pmatrix} \right\rangle\right) \hat{=} 0,$$

which concludes the proof of this proposition. ∎

6. Conclusion. Section 3 offers a system of equations coupling the mixed formulation of the variational form of the Hodge-Helmholtz and Hodge-Laplace equation with *first-kind* boundary integral equations. Well-posedness of the coupled problem was obtained using a T-coercivity argument demonstrating that the operator associated to the coupled variational problem was Fredholm of index 0. When $\kappa^2 \in \mathbb{C}$ avoids resonant frequencies, the operator's injectivity was guaranteed, and thus stability of the problem was obtained along with the existence and uniqueness of the solution. For such κ^2, Proposition 3.1 showed how solution to the coupled variational problem are in one-to-one correspondence with solutions of the transmission system.

The symmetrically coupled system (3.5) offers a variational formulation of the transmission problem (1.2) in well-known energy spaces suited for discretization by finite and boundary elements. It is therefore a promising starting point for Galerkin discretization.

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