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Overcoming the curse of dimensionality in the numerical approximation of high-dimensional semilinear elliptic partial differential equations

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Abstract

Recently, so-called full-history recursive multilevel Picard (MLP) approximation schemes have been introduced and shown to overcome the curse of dimensionality in the numerical approximation of semilinear *parabolic* partial differential equations (PDEs) with Lipschitz nonlinearities. The key contribution of this article is to introduce and analyze a new variant of MLP approximation schemes for certain semilinear *elliptic* PDEs with Lipschitz nonlinearities and to prove that the proposed approximation schemes overcome the curse of dimensionality in the numerical approximation of such semilinear elliptic PDEs.

Contents

1 Introduction

Partial differential equations (PDEs) are widely used as a modelling tool, e.g., in the natural sciences, in the engineering sciences, or in the financial industry. Usually the exact solutions to specific PDE problems in applications can hardly be found. Thus, there is a high demand for approximative solution techniques. In the scientific literature, there are several well-established approximation methods for PDEs like finite difference methods or finite element methods which work well in low dimensions. However, such classical deterministic approximation methods cannot be used in high dimensions as they suffer from the so-called *curse of dimensionality* in the sense that the computational effort for calculating an approximation grows at least exponentially in the PDE dimension $d \in \mathbb{N}$. Probabilistic approximation methods like Monte Carlo averaging, on the other hand, do not suffer from the curse of dimensionality in the numerical approximation of linear second-order parabolic PDEs as well as linear second-order elliptic PDEs.

Recently, several probabilistic approximation methods for high-dimensional nonlinear PDEs have been proposed in hopes of overcoming the curse of dimensionality in the numerical approximation of nonlinear PDEs. We refer, e.g., to [3, 4, 5, 8, 9, 14, 16, 20, 21, 24, 26, 29, 34, 35, 36, 38, 41, 50, 51, 52, 54, 57] for deep learning based approximation methods for possibly nonlinear PDEs, e.g., to [1, 10, 12, 13, 15, 37, 39, 40, 53, 55, 58, 59] for approximation methods for nonlinear second-order parabolic PDEs based on branching diffusions, and, e.g., to [7, 22, 23, 27, 42, 44, 45, 46] for full-history recursive multilevel Picard (MLP) approximation methods for nonlinear second-order parabolic PDEs. Numerical experiments raise hopes that deep learning based approximation methods are able to approximate solutions of high-dimensional nonlinear PDE problems, but at the moment there are only partial explanations for the good performance of deep learning based approximation methods in numerical experiments for high-dimensional PDEs available (cf., e.g., [11, 25, 28, 30, 31, 32, 43, 47, 49, 56]). In contrast to this, there are, however, complete mathematical analyses in the scientific literature showing that branching diffusion approximation methods can efficiently solve high-dimensional semilinear parabolic PDE problems for sufficiently small time horizons. The branching diffusion approximation method, however, breaks down when the time horizon is not sufficiently small anymore. MLP approximation methods are, to the best of our knowledge, up to now the only approximation methods for high-dimensional semilinear parabolic PDEs which are guaranteed by rigorous mathematical proofs to overcome the curse of dimensionality for all time horizons (see [7, 27, 44, 45]). The MLP approximation schemes proposed and studied so far in the scientific literature, however, exclusively deal with parabolic PDE problems and none of these schemes and their analyses can be applied to nonlinear elliptic PDEs. The key contribution of this article is to introduce and analyze MLP approximation schemes for certain semilinear elliptic PDEs and to prove for the first time that approximations of such semilinear elliptic PDEs can be obtained without suffering from the curse of dimensionality. In particular, the main result of this article, Theorem 3.16, proves that the computational effort to obtain an approximation of a desired accuracy $\varepsilon \in (0, \infty)$ grows at most polynomially in the PDE dimension $d \in \mathbb{N}$ as well as in the reciprocal of the desired accuracy. To illustrate the findings of this article in more detail, we present in the following result, Theorem 1.1 below, a special case of Theorem 3.16.

Theorem 1.1. *Let $c, L \in [0,\infty)$, $\lambda \in (L,\infty)$, $M \in \mathbb{N} \cap ((\sqrt{\lambda}+\sqrt{L})^2(\sqrt{\lambda}-\sqrt{L})^{-2},\infty)$, $\Theta = \cup_{n\in\mathbb{N}} \mathbb{Z}^n$, let $u_d \in C^2(\mathbb{R}^d,\mathbb{R})$, $d \in \mathbb{N}$, and $f_d \in C(\mathbb{R}^d \times \mathbb{R},\mathbb{R})$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ that*

$$(\Delta u_d)(x) = f_d(x, u_d(x)), \tag{1}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W^{d,\theta}\colon [0,\infty)\times\Omega \to \mathbb{R}^d$, $\theta \in \Theta$, $d \in \mathbb{N}$, be i.i.d. standard Brownian motions, let $R^\theta\colon \Omega \to [0,\infty)$, $\theta \in \Theta$, be i.i.d. random variables, assume that $(R^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, assume for all $d \in \mathbb{N}$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, $v, w \in \mathbb{R}$, $\varepsilon \in (0,\infty)$ that $|f_d(x,v) - f_d(x,w) - \lambda(v-w)| \leqslant L|v-w|$, $|f_d(x,0)| \leqslant cd^c[1+\sum_{j=1}^d |x_j|]^c$, $\sup_{y=(y_1,\ldots,y_d)\in\mathbb{R}^d}[|u_d(y)|\exp(-\varepsilon\sum_{j=1}^d |y_j|)] < \infty$, and $\mathbb{P}(R^0 \geqslant \varepsilon) = e^{-\lambda\varepsilon}$, let $U_n^{d,\theta} = (U_n^{d,\theta}(x))_{x\in\mathbb{R}^d}\colon \mathbb{R}^d\times\Omega \to \mathbb{R}$, $\theta \in \Theta$, $d, n \in \mathbb{N}_0$, satisfy for all $d, n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that $U_0^{d,\theta}(x) = 0$ and

$$\begin{aligned} U_n^{d,\theta}(x) = \frac{-1}{\lambda M^n}\left[\sum_{m=1}^{M^n} f_d\big(x + \sqrt{2}\, W^{d,(\theta,0,m)}_{R^{(\theta,0,m)}}, 0\big)\right] \\ + \sum_{k=1}^{n-1} \frac{1}{\lambda M^{(n-k)}}\left[\sum_{m=1}^{M^{(n-k)}} \left(\lambda\Big[U_k^{d,(\theta,k,m)}\big(x+\sqrt{2}\, W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}\big) - U_{k-1}^{d,(\theta,k,-m)}\big(x+\sqrt{2}\, W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}\big)\Big]\right.\right. \\ - \Big[f_d\Big(x+\sqrt{2}\, W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{d,(\theta,k,m)}\big(x+\sqrt{2}\, W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}\big)\Big) \\ \left.\left. - f_d\Big(x+\sqrt{2}\, W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{d,(\theta,k,-m)}\big(x+\sqrt{2}\, W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}\big)\Big)\Big]\right)\right], \end{aligned} \tag{2}$$

and let $\mathfrak{C}_{d,n} \in \mathbb{R}$, $d, n \in \mathbb{N}_0$, satisfy for all $d, n \in \mathbb{N}_0$ that $\mathfrak{C}_{d,n} \leqslant (d+1)M^n + \sum_{k=1}^{n-1} M^{(n-k)}(d+1+\mathfrak{C}_{d,k}+\mathfrak{C}_{d,k-1})$. Then there exist $\kappa \in \mathbb{R}$ and $\mathfrak{N}\colon (0,1]\times\mathbb{N} \to \mathbb{N}$ such that for all $\varepsilon \in (0,1]$, $d \in \mathbb{N}$ it holds that

$$\mathfrak{C}_{d,\mathfrak{N}_{\varepsilon,d}} \leqslant \kappa d^\kappa \varepsilon^{-\kappa} \qquad \text{and} \qquad \sup_{x\in[-c,c]^d}\big(\mathbb{E}\big[|u_d(x) - U^{d,0}_{\mathfrak{N}_{\varepsilon,d}}(x)|^2\big]\big)^{1/2} \leqslant \varepsilon. \tag{3}$$

Theorem 1.1 is an immediate consequence of Corollary 3.17. Corollary 3.17, in turn, follows from Theorem 3.16, the main result of this article. In the following we add comments on some of the mathematical objects appearing in Theorem 1.1. The functions $u_d\colon \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, in Theorem 1.1 above describe the solutions of the elliptic PDEs which we intend to solve approximately; see (1) above. The functions $f_d\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $d \in \mathbb{N}$, represent the nonlinearities in the elliptic PDEs in (1). The nonlinearities are assumed to satisfy certain Lipschitz conditions which are formulated with the help of the real numbers $L \in [0,\infty)$ and $\lambda \in (L,\infty)$ in Theorem 1.1 above. The random fields $U_n^{d,\theta}\colon \mathbb{R}^d\times\Omega \to \mathbb{R}$, $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, in (2) above describe the approximation algorithm which we employ in this work to approximately calculate the PDE solutions $u_d\colon \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$. The motivation for the specific form of the random fields $U_n^{d,\theta}\colon \mathbb{R}^d\times\Omega \to \mathbb{R}$, $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, in (2) stems from multilevel Monte Carlo approximations of Picard approximations of certain stochastic fixed point equations (SFPEs) which are satisfied by the solutions $u_d\colon \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, of the elliptic PDEs in (1). The different numbers of Monte Carlo samples in (2) are determined by the constant $M \in \mathbb{N}$ which is restricted by the real numbers $L \in [0,\infty)$ and $\lambda \in (L,\infty)$ used to formulate the Lipschitz assumptions for the nonlinearities $f_d\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $d \in \mathbb{N}$, in the PDEs in (1) above. For every $d, n \in \mathbb{N}$, $x \in \mathbb{R}^d$ we think of the quantity $\mathfrak{C}_{d,n} \in \mathbb{R}$ in (3) in Theorem 1.1 as the computational cost to sample one realization of $U_n^{d,0}(x) \in \mathbb{R}$ (cf. Section 3.6 below). Theorem 1.1 thus, roughly speaking, proves that the random fields $U_n^{d,0}\colon \mathbb{R}^d\times\Omega \to \mathbb{R}$, $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, (see (2) in Theorem 1.1) can achieve an approximation accuracy of size $\varepsilon \in (0,\infty)$ with a computational cost which is bounded by a polynomial in the PDE dimension $d \in \mathbb{N}$ and in the reciprocal of the desired approximation accuracy $\varepsilon \in (0,\infty)$ (see (3) in Theorem 1.1). The real number $c \in [0,\infty)$ is an arbitrarily large real constant which we use to express the growth assumption in Theorem 1.1 that for all $d \in \mathbb{N}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ it holds that $|f_d(x, 0)| \leqslant cd^c[1 + \sum_{j=1}^{d} |x_j|]^c$ as well as to specify the regions $[-c, c]^d \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, on which we measure the L^2-error between the exact solutions $u_d \colon \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, of the PDEs in (1) and the random approximations $U^{d,0}_{\mathfrak{N}_{\varepsilon,d}} \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d \in \mathbb{N}$, $\varepsilon \in (0, 1]$, in (3) in Theorem 1.1 above.

The remainder of this article is organized as follows. Section 2 is devoted to the study of stochastic representations for suitable viscosity solutions of certain semilinear elliptic PDEs. In particular, we establish in Section 2 a one-to-one correspondence between suitable viscosity solutions of certain semilinear elliptic PDEs and solutions of SFPEs associated with such semilinear elliptic PDEs. Section 3 focusses on the introduction and the analysis of MLP schemes for the numerical approximation of solutions of certain SFPEs. Owing to the results in Section 2 these MLP schemes are thus apt to numerically approximate suitable viscosity solutions of certain semilinear elliptic PDEs. The main error estimates for the MLP approximation schemes can be found in Sections 3.4 and 3.5 and a computational cost analysis for the MLP approximation schemes is carried out in Section 3.6. An overall complexity analysis for the MLP approximation schemes is obtained in Section 3.7 by combining the error estimates from Sections 3.4 and 3.5 with the computational cost analysis in Section 3.6.

2 Stochastic representations for elliptic partial differential equations (PDEs)

In the main result of this section, Proposition 2.13 in Section 2.4 below, we establish a stochastic representation formula of the Feynman–Kac type for suitable viscosity solutions of certain semilinear elliptic PDEs (cf. also Corollary 2.14 and Corollary 2.16 in Section 2.4). The established Feynman–Kac type formula will be essential in our error analysis for MLP approximations in Section 3 below. Our proof of Proposition 2.13 is particularly based on the following three tools: (i) an existence and uniqueness result for SFPEs, see Corollary 2.7 in Section 2.2 below, (ii) a result which identifies solutions of certain SFPEs as viscosity solutions of certain semilinear elliptic PDEs, see Corollary 2.11 in Section 2.2, and (iii) a uniqueness result for suitable viscosity solutions of certain semilinear elliptic PDEs, see Proposition 2.12 in Section 2.3 below. Our proof of the existence and uniqueness result in Corollary 2.7 in Section 2.2 (see (i) above) is based on an elementary application of Banach's fixed point theorem which is performed in Section 2.1 (cf. also [6]). The identification result for certain semilinear elliptic PDEs in Corollary 2.11 in Section 2.2 (see (ii) above) follows from an approximation argument which combines a well-known convergence result for viscosity solutions (cf., e.g., Hairer et al. [33, Lemma 4.8] or Barles & Perthame [2, Theorem A.2]) with the well-known construction result for classical PDE solutions in Lemma 2.8 below (cf., e.g., Da Prato & Zabczyk [19, Theorems 7.4.5 and 7.5.1]). Our proof of the uniqueness result for suitable viscosity solutions of certain semilinear elliptic PDEs in Proposition 2.12 in Section 2.3 (see (iii) above) is inspired by Crandall et al. [17, Theorem 2.1].

2.1 Stochastic fixed point equations (SFPEs)

Lemma 2.1. *Let $d \in \mathbb{N}$, $c, \gamma, \rho \in \mathbb{R}$, $\lambda \in (\rho, \infty)$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X = (X_t)_{t \in [0,\infty)} \colon [0, \infty) \times \Omega \to \mathcal{O}$ be $(\mathcal{B}([0, \infty)) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable, let $h \colon \mathcal{O} \to \mathbb{R}$ be $\mathcal{B}(\mathcal{O})/\mathcal{B}(\mathbb{R})$-measurable, let $V \colon \mathcal{O} \to (0, \infty)$ be $\mathcal{B}(\mathcal{O})/\mathcal{B}((0, \infty))$-measurable, and assume for all $t \in [0, \infty)$, $x \in \mathcal{O}$ that $\mathbb{E}[e^{-\rho t} V(X_t)] \leqslant \gamma$ and $|h(x)| \leqslant cV(x)$. Then it holds that*

$$\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} |h(X_t)| \, dt\bigg] \leqslant \frac{c\gamma}{\lambda - \rho}. \tag{4}$$

Proof of Lemma 2.1. First, observe that the hypothesis that $h\colon \mathcal{O} \to \mathbb{R}$ is $\mathcal{B}(\mathcal{O})/\mathcal{B}(\mathbb{R})$-measurable and the hypothesis that $X\colon [0,\infty) \times \Omega \to \mathcal{O}$ is $(\mathcal{B}([0,\infty)) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable prove that $[0,\infty) \times \Omega \ni (t,\omega) \mapsto e^{-\lambda t} h(X_t(\omega)) \in \mathbb{R}$ is $(\mathcal{B}([0,\infty)) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable. This, the hypothesis that for all $x \in \mathcal{O}$ it holds that $|h(x)| \leqslant cV(x)$, the hypothesis that for all $t \in [0,\infty)$ it holds that $\mathbb{E}[e^{-\rho t} V(X_t)] \leqslant \gamma$, and Fubini's theorem ensure that

$$\begin{aligned}\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} |h(X_t)|\, dt\bigg] &\leqslant \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} cV(X_t)\, dt\bigg] = c\int_0^\infty e^{-\lambda t}\, \mathbb{E}[V(X_t)]\, dt \\ &\leqslant c\gamma \int_0^\infty e^{-(\lambda-\rho)t}\, dt = \frac{c\gamma}{\lambda-\rho}.\end{aligned} \tag{5}$$

This establishes (4). The proof of Lemma 2.1 is thus completed. ☐

Lemma 2.2. *Let $d \in \mathbb{N}$, $c, \gamma, \rho \in \mathbb{R}$, $\lambda \in (\rho, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $n \in \mathbb{N}_0$ let $X_n = (X_{n,t})_{t\in[0,\infty)}\colon [0,\infty)\times\Omega \to \mathcal{O}$ be $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable, assume for all $\varepsilon, t \in (0,\infty)$ that $\limsup_{n\to\infty} \mathbb{P}(\|X_{n,t} - X_{0,t}\| \geqslant \varepsilon) = 0$, let $h \in C(\mathcal{O}, \mathbb{R})$ be bounded, let $V\colon \mathcal{O} \to (0,\infty)$ be $\mathcal{B}(\mathcal{O})/\mathcal{B}((0,\infty))$-measurable, and assume for all $n \in \mathbb{N}_0$, $t \in [0,\infty)$, $x \in \mathcal{O}$ that $|h(x)| \leqslant cV(x)$ and $\mathbb{E}[e^{-\rho t} V(X_{n,t})] \leqslant \gamma$. Then*

(i) it holds for all $n \in \mathbb{N}_0$ that $\mathbb{E}\big[\int_0^\infty e^{-\lambda t} |h(X_{n,t})|\, dt\big] < \infty$ and

(ii) it holds that

$$\limsup_{n\to\infty} \bigg|\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} h(X_{n,t})\, dt\bigg] - \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} h(X_{0,t})\, dt\bigg]\bigg| = 0. \tag{6}$$

Proof of Lemma 2.2. First, observe that Lemma 2.1 (with $d \leftarrow d$, $c \leftarrow c$, $\gamma \leftarrow \gamma$, $\rho \leftarrow \rho$, $\mathcal{O} \leftarrow \mathcal{O}$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $X \leftarrow X_n$, $h \leftarrow h$, $V \leftarrow V$ for $n \in \mathbb{N}_0$ in the notation of Lemma 2.1) establishes Item (i). Next note that Kallenberg [48, Lemma 4.3], the assumption that for all $\varepsilon, t \in (0,\infty)$ it holds that $\limsup_{n\to\infty} \mathbb{P}(\|X_{n,t} - X_{0,t}\| \geqslant \varepsilon) = 0$, and the assumption that $h \in C(\mathcal{O}, \mathbb{R})$ assure that for all $\varepsilon, t \in (0,\infty)$ it holds that

$$\limsup_{n\to\infty} \left[\mathbb{P}(|h(X_{n,t}) - h(X_{0,t})| \geqslant \varepsilon)\right] = 0. \tag{7}$$

Combining this with the assumption that h is bounded and Vitali's convergence theorem implies for all $t \in (0,\infty)$ that

$$\limsup_{n\to\infty} \mathbb{E}[|h(X_{n,t}) - h(X_{0,t})|] = 0. \tag{8}$$

Moreover, note that for all $n \in \mathbb{N}_0$, $t \in (0,\infty)$ it holds that

$$e^{-\lambda t}\, \mathbb{E}[|h(X_{n,t})|] \leqslant c e^{-\lambda t}\, \mathbb{E}[V(X_{n,t})] \leqslant c\gamma e^{-(\lambda-\rho)t}. \tag{9}$$

Lebesgue's dominated convergence theorem and (8) hence establish Item (ii). The proof of Lemma 2.2 is thus completed. ☐

Lemma 2.3. *Let $d \in \mathbb{N}$, $\rho \in \mathbb{R}$, $\lambda \in (\rho,\infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0,\infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}\colon \|x\| \leqslant r \text{ and } \{y \in \mathbb{R}^d\colon \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $x \in \mathcal{O}$ let $X^x = (X^x_t)_{t\in[0,\infty)}\colon [0,\infty)\times\Omega \to \mathcal{O}$ be $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable, assume for all $\varepsilon, t \in (0,\infty)$ and all $x_n \in \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n\to\infty} \|x_n - x_0\| = 0$ that $\limsup_{n\to\infty} \mathbb{P}(\|X^{x_n}_t - X^{x_0}_t\| \geqslant \varepsilon) = 0$,* *let $h \in C(\mathcal{O}, \mathbb{R})$, $V \in C(\mathcal{O}, (0, \infty))$ satisfy for all $t \in [0, \infty)$, $x \in \mathcal{O}$ that $\mathbb{E}[e^{-\rho t} V(X_t^x)] \leqslant V(x)$ and $\inf_{r \in (0,\infty)}[\sup_{y \in \mathcal{O}\backslash O_r}(\frac{|h(y)|}{V(y)})] = 0$, and let $u\colon \mathcal{O} \to \mathbb{R}$ satisfy for all $x \in \mathcal{O}$ that*

$$u(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} h(X_t^x)\, dt\bigg] \tag{10}$$

(cf. Lemma 2.1). Then

(i) it holds that $u \in C(\mathcal{O}, \mathbb{R})$ and

(ii) it holds in the case of $\sup_{r \in (0,\infty)}[\inf_{x \in \mathcal{O}\backslash O_r} V(x)] = \infty$ that

$$\limsup_{r \to \infty}\left[\sup_{x \in \mathcal{O}\backslash O_r}\left(\frac{|u(x)|}{V(x)}\right)\right] = 0. \tag{11}$$

Proof of Lemma 2.3. Throughout this proof let $\mathfrak{h}_n\colon \mathcal{O} \to \mathbb{R}$, $n \in \mathbb{N}$, be compactly supported continuous functions which satisfy

$$\limsup_{n \to \infty}\left[\sup_{x \in \mathcal{O}}\left(\frac{|\mathfrak{h}_n(x) - h(x)|}{V(x)}\right)\right] = 0 \tag{12}$$

(cf. [6, Corollary 2.4]) and let $\mathfrak{u}_n\colon \mathcal{O} \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $x \in \mathcal{O}$ that

$$\mathfrak{u}_n(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} \mathfrak{h}_n(X_t^x)\, dt\bigg] \tag{13}$$

(cf. Lemma 2.1). Note that the assumption that V is continuous implies that V is locally bounded. Lemma 2.2 (with $d \leftarrow d$, $c \leftarrow \sup_{y \in \mathcal{O}}(\frac{|\mathfrak{h}_n(y)|}{V(y)})$, $\gamma \leftarrow \sup_{k \in \mathbb{N}_0} V(x_k)$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\mathcal{O} \leftarrow \mathcal{O}$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(X_k)_{k \in \mathbb{N}_0} \leftarrow (X^{x_k})_{k \in \mathbb{N}_0}$, $h \leftarrow \mathfrak{h}_n$, $V \leftarrow V$ for $n \in \mathbb{N}$, $(x_k)_{k \in \mathbb{N}_0} \subseteq \mathcal{O}$ with $\limsup_{k \to \infty} \|x_k - x_0\| = 0$ in the notation of Lemma 2.2) hence ensures for all $n \in \mathbb{N}$, $x_k \in \mathcal{O}$, $k \in \mathbb{N}_0$, with $\limsup_{k \to \infty} \|x_k - x_0\| = 0$ that

$$\limsup_{k \to \infty} |\mathfrak{u}_n(x_k) - \mathfrak{u}_n(x_0)| = \limsup_{k \to \infty}\left|\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} \mathfrak{h}_n(X_t^{x_k})\, dt\bigg] - \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} \mathfrak{h}_n(X_t^{x_0})\, dt\bigg]\right| = 0. \tag{14}$$

Hence, we obtain for all $n \in \mathbb{N}$ that $\mathfrak{u}_n$ is continuous. Next observe that for all $n \in \mathbb{N}$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned}
|\mathfrak{u}_n(x) - u(x)| &= \left|\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} \mathfrak{h}_n(X_t^x)\, dt\bigg] - \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} h(X_t^x)\, dt\bigg]\right| \\
&= \left|\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} (\mathfrak{h}_n(X_t^x) - h(X_t^x))\, dt\bigg]\right| \leqslant \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} |\mathfrak{h}_n(X_t^x) - h(X_t^x)|\, dt\bigg] \\
&= \int_0^\infty e^{-\lambda t}\, \mathbb{E}[|\mathfrak{h}_n(X_t^x) - h(X_t^x)|]\, dt = \int_0^\infty e^{-\lambda t}\, \mathbb{E}\bigg[\frac{|\mathfrak{h}_n(X_t^x) - h(X_t^x)|}{V(X_t^x)} V(X_t^x)\bigg]\, dt \\
&\leqslant \int_0^\infty e^{-\lambda t}\left[\sup_{y \in \mathcal{O}}\left(\frac{|\mathfrak{h}_n(y) - h(y)|}{V(y)}\right)\right] e^{\rho t} V(x)\, dt = \frac{V(x)}{\lambda - \rho}\left[\sup_{y \in \mathcal{O}}\left(\frac{|\mathfrak{h}_n(y) - h(y)|}{V(y)}\right)\right].
\end{aligned} \tag{15}$$

This, (12), the assumption that $V \in C(\mathcal{O}, (0, \infty))$, and the fact that $(\mathfrak{u}_n)_{n \in \mathbb{N}} \subseteq C(\mathcal{O}, \mathbb{R})$ imply that u is continuous. This establishes Item (i). Next we prove Item (ii). For this we assume that $\sup_{r \in (0,\infty)}[\inf_{x \in \mathcal{O}\backslash O_r} V(x)] = \infty$. The assumption that $V \in C(\mathcal{O}, (0, \infty))$ hence assures that $\{x \in \mathcal{O}\colon V(x) = \inf_{y \in \mathcal{O}} V(y)\} \neq \emptyset$. The assumption that for all $t \in [0, \infty)$, $x \in \mathcal{O}$ it holds that

$e^{-\rho t}\mathbb{E}[V(X_t^x)] \leqslant V(x)$ therefore implies for all $t \in (0,\infty)$ that $0 < e^{-\rho t}\min_{y\in\mathcal{O}} V(y) \leqslant \min_{y\in\mathcal{O}} V(y)$. Hence, we obtain that $0 \leqslant \rho < \lambda$. This, the fact that for every $n \in \mathbb{N}$ it holds that $\mathfrak{h}_n \colon \mathcal{O} \to \mathbb{R}$ is bounded, the assumption that $\sup_{r\in(0,\infty)}[\inf_{x\in\mathcal{O}\backslash O_r} V(x)] = \infty$, and (13) demonstrate for all $n \in \mathbb{N}$ that

$$\begin{aligned}\limsup_{r\to\infty}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{|\mathfrak{u}_n(x)|}{V(x)}\right)\right] &\leqslant \left[\sup_{x\in\mathcal{O}}|\mathfrak{u}_n(x)|\right]\limsup_{r\to\infty}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{1}{V(x)}\right)\right] \\ &\leqslant \frac{1}{\lambda}\left(\sup_{x\in\mathcal{O}}|\mathfrak{h}_n(x)|\right)\left(\limsup_{r\to\infty}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{1}{V(x)}\right)\right]\right) = 0.\end{aligned} \tag{16}$$

Combining this with (15) and (12) ensures that

$$\limsup_{r\to\infty}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{|u(x)|}{V(x)}\right)\right] = 0. \tag{17}$$

This establishes Item (ii). The proof of Lemma 2.3 is thus completed. □

Corollary 2.4. *Let $d \in \mathbb{N}$, $L, \rho \in \mathbb{R}$, $\lambda \in (\rho,\infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0,\infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}\colon \|x\| \leqslant r$ and $\{y \in \mathbb{R}^d\colon \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $x \in \mathcal{O}$ let $X^x = (X_t^x)_{t\in[0,\infty)} \colon [0,\infty)\times\Omega \to \mathcal{O}$ be $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable, assume for all $\varepsilon, t \in (0,\infty)$ and all $x_n \in \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n\to\infty}\|x_n - x_0\| = 0$ that $\limsup_{n\to\infty}\mathbb{P}(\|X_t^{x_n} - X_t^{x_0}\| \geqslant \varepsilon) = 0$, let $f \in C(\mathcal{O}\times\mathbb{R},\mathbb{R})$, $u \in C(\mathcal{O},\mathbb{R})$, $V \in C(\mathcal{O},(0,\infty))$ satisfy for all $t \in [0,\infty)$, $x \in \mathcal{O}$ that $\mathbb{E}[e^{-\rho t}V(X_t^x)] \leqslant V(x)$, and assume for all $x \in \mathcal{O}$, $v, w \in \mathbb{R}$ that $\inf_{r\in(0,\infty)}[\sup_{y\in\mathcal{O}\backslash O_r}(\frac{|f(y,0)|+|u(y)|}{V(y)})] = 0$ and $|f(x,v) - f(x,w)| \leqslant L|v - w|$. Then*

(i) it holds for all $x \in \mathcal{O}$ that $\mathbb{E}[\int_0^\infty e^{-\lambda t}|f(X_t^x, u(X_t^x))|\,dt] < \infty$,

(ii) it holds that

$$\mathcal{O} \ni x \mapsto \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(X_t^x, u(X_t^x))\,dt\right] \in \mathbb{R} \tag{18}$$

is continuous, and

(iii) it holds in the case of $\sup_{r\in(0,\infty)}[\inf_{x\in\mathcal{O}\backslash O_r} V(x)] = \infty$ that

$$\lim_{r\to\infty}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{|\mathbb{E}[\int_0^\infty e^{-\lambda t} f(X_t^x, u(X_t^x))\,dt]\,|}{V(x)}\right)\right] = 0. \tag{19}$$

Proof of Corollary 2.4. First, observe that $\mathcal{O} \ni x \mapsto f(x,u(x)) \in \mathbb{R}$ is a continuous function which satisfies for all $x \in \mathcal{O}$ that

$$|f(x,u(x))| \leqslant |f(x,0)| + |f(x,u(x)) - f(x,0)| \leqslant |f(x,0)| + L|u(x)|. \tag{20}$$

The hypothesis that $\inf_{r\in(0,\infty)}[\sup_{x\in\mathcal{O}\backslash O_r}(\frac{|f(x,0)|+|u(x)|}{V(x)})] = 0$ hence ensures that

$$\inf_{r\in(0,\infty)}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{|f(x,u(x))|}{V(x)}\right)\right] \leqslant \inf_{r\in(0,\infty)}\left[\sup_{x\in\mathcal{O}\backslash O_r}\left(\frac{|f(x,0)|}{V(x)} + L\frac{|u(x)|}{V(x)}\right)\right] = 0. \tag{21}$$

This, the hypothesis that $f \in C(\mathcal{O}\times\mathbb{R},\mathbb{R})$, the hypothesis that $u \in C(\mathcal{O},\mathbb{R})$, and the hypothesis that $V \in C(\mathcal{O},(0,\infty))$ imply that $\sup_{x\in\mathcal{O}}(\frac{|f(x,u(x))|}{V(x)}) < \infty$. Lemma 2.1 (with $d \leftarrow d$,

$c \leftarrow \sup_{y\in\mathcal{O}}(\frac{|f(y,u(y))|}{V(y)})$, $\gamma \leftarrow V(x)$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\mathcal{O} \leftarrow \mathcal{O}$, $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $X \leftarrow X^x$, $h \leftarrow (\mathcal{O} \ni y \mapsto f(y,u(y)) \in \mathbb{R})$ for $x \in \mathcal{O}$ in the notation of Lemma 2.1) therefore establishes Item (i). In addition, note that (21) and Lemma 2.3 (with $d \leftarrow d$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\|\cdot\| \leftarrow \|\cdot\|$, $\mathcal{O} \leftarrow \mathcal{O}$, $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $(X^x)_{x\in\mathcal{O}} \leftarrow (X^x)_{x\in\mathcal{O}}$, $h \leftarrow (\mathcal{O} \ni y \mapsto f(y,u(y)) \in \mathbb{R})$, $V \leftarrow V$ in the notation of Lemma 2.3) establish Items (ii) and (iii). The proof of Corollary 2.4 is thus completed. ☐

Lemma 2.5. *Let $d \in \mathbb{N}$, $L,\rho \in \mathbb{R}$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, for every $x \in \mathcal{O}$ let $X^x = (X^x_t)_{t\in[0,\infty)} \colon [0,\infty)\times\Omega \to \mathcal{O}$ be $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable, let $V\colon \mathcal{O} \to (0,\infty)$ be $\mathcal{B}(\mathcal{O})/\mathcal{B}((0,\infty))$-measurable, assume for all $t \in [0,\infty)$, $x \in \mathcal{O}$ that $\mathbb{E}[e^{-\rho t}V(X^x_t)] \leqslant V(x)$, let $f\colon \mathcal{O}\times\mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathcal{O}\times\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, assume for all $x \in \mathcal{O}$, $v,w \in \mathbb{R}$ that $|f(x,v)-f(x,w)| \leqslant L|v-w|$, let $v,w\colon \mathcal{O} \to \mathbb{R}$ be $\mathcal{B}(\mathcal{O})/\mathcal{B}(\mathbb{R})$-measurable, and assume that*

$$\sup_{x\in\mathcal{O}}\left[\frac{|v(x)|+|w(x)|}{V(x)}\right] < \infty. \tag{22}$$

Then it holds for all $\lambda \in (\rho,\infty)$, $x \in \mathcal{O}$ that

$$\frac{1}{V(x)}\mathbb{E}\!\left[\int_0^\infty e^{-\lambda s}\left|f\big(X^x_s, v(X^x_s)\big) - f\big(X^x_s, w(X^x_s)\big)\right| ds\right] \leqslant \frac{L}{\lambda-\rho}\left[\sup_{y\in\mathcal{O}}\left(\frac{|v(y)-w(y)|}{V(y)}\right)\right]. \tag{23}$$

Proof of Lemma 2.5. First, note that the fact that $f\colon \mathcal{O}\times\mathbb{R} \to \mathbb{R}$ is $\mathcal{B}(\mathcal{O}\times\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, the fact that $v,w\colon \mathcal{O} \to \mathbb{R}$ are $\mathcal{B}(\mathcal{O})/\mathcal{B}(\mathbb{R})$-measurable, and the fact that for all $x \in \mathcal{O}$ it holds that $X^x\colon [0,\infty)\times\Omega \to \mathcal{O}$ is $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable ensure that for all $x \in \mathcal{O}$ it holds that

$$[0,\infty)\times\Omega \ni (t,\omega) \mapsto \left|f\big(X^x_t(\omega), v(X^x_t(\omega))\big) - f\big(X^x_t(\omega), w(X^x_t(\omega))\big)\right| \in \mathbb{R} \tag{24}$$

is $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable. Next observe that the hypothesis that for all $t \in [0,\infty)$, $x \in \mathcal{O}$ it holds that $\mathbb{E}[e^{-\rho t}V(X^x_t)] \leqslant V(x)$, the hypothesis that for all $x \in \mathcal{O}$, $a,b \in \mathbb{R}$ it holds that $|f(x,a)-f(x,b)| \leqslant L|a-b|$, and Fubini's theorem ensure that for all $\lambda \in (\rho,\infty)$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned}
&\frac{1}{V(x)}\mathbb{E}\!\left[\int_0^\infty e^{-\lambda t}\left|f\big(X^x_t, v(X^x_t)\big) - f\big(X^x_t, w(X^x_t)\big)\right| dt\right]\\
&\leqslant \mathbb{E}\!\left[\int_0^\infty L e^{-\lambda t}\frac{|v(X^x_t)-w(X^x_t)|}{V(x)}\,dt\right] = L\int_0^\infty e^{-\lambda t}\,\mathbb{E}\!\left[\left(\frac{|v(X^x_t)-w(X^x_t)|}{V(X^x_t)}\right)\left(\frac{V(X^x_t)}{V(x)}\right)\right] dt\\
&\leqslant L\int_0^\infty e^{-\lambda t}\left[\sup_{y\in\mathcal{O}}\left(\frac{|v(y)-w(y)|}{V(y)}\right)\right]\left[\frac{\mathbb{E}[V(X^x_t)]}{V(x)}\right] dt\\
&\leqslant L\left[\sup_{y\in\mathcal{O}}\left(\frac{|v(y)-w(y)|}{V(y)}\right)\right]\left[\int_0^\infty e^{-(\lambda-\rho)t}\,dt\right] = \frac{L}{\lambda-\rho}\left[\sup_{y\in\mathcal{O}}\left(\frac{|v(y)-w(y)|}{V(y)}\right)\right].
\end{aligned} \tag{25}$$

This establishes (23). The proof of Lemma 2.5 is thus completed. ☐

Proposition 2.6. *Let $d \in \mathbb{N}$, $L,\rho \in \mathbb{R}$, $\lambda \in (L+\rho,\infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0,\infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}\colon \|x\| \leqslant r \text{ and } \{y \in \mathbb{R}^d\colon \|y-x\| < 1/r\} \subseteq \mathcal{O}\}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, for every $x \in \mathcal{O}$ let $X^x = (X^x_t)_{t\in[0,\infty)}\colon [0,\infty)\times\Omega \to \mathcal{O}$ be $(\mathcal{B}([0,\infty))\otimes\mathcal{F})/\mathcal{B}(\mathcal{O})$-measurable, assume for all $\varepsilon,t \in (0,\infty)$ and all $x_n \in \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n\to\infty}\|x_n-x_0\| = 0$ that $\limsup_{n\to\infty}\mathbb{P}(\|X^{x_n}_t - X^{x_0}_t\| \geqslant \varepsilon) = 0$, let $V \in C(\mathcal{O},(0,\infty))$, $f \in C(\mathcal{O}\times\mathbb{R},\mathbb{R})$ satisfy for all $t \in [0,\infty)$, $x \in \mathcal{O}$, $v,w \in \mathbb{R}$ that $\mathbb{E}[e^{-\rho t}V(X^x_t)] \leqslant V(x)$ and $|f(x,v)-f(x,w)| \leqslant L|v-w|$, and assume that $\inf_{r\in(0,\infty)}[\sup_{x\in\mathcal{O}\setminus O_r}(\frac{|f(x,0)|}{V(x)})] = 0$ and $\sup_{r\in(0,\infty)}[\inf_{x\in\mathcal{O}\setminus O_r}V(x)] = \infty$. Then there exists a unique $u \in C(\mathcal{O},\mathbb{R})$ such that* *(i) it holds that*

$$\limsup_{r\to\infty}\left[\sup_{x\in\mathcal{O}\setminus O_r}\left(\frac{|u(x)|}{V(x)}\right)\right]=0 \tag{26}$$

and

(ii) it holds for all $x \in \mathcal{O}$ that $\mathbb{E}\big[\int_0^\infty e^{-\lambda t}|f(X_t^x, u(X_t^x))|\,dt\big] < \infty$ and

$$u(x)=\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} f(X_t^x, u(X_t^x))\,dt\bigg]. \tag{27}$$

Proof of Proposition 2.6. Throughout this proof let $\mathcal{V}$ be the set given by

$$\mathcal{V}=\left\{u\in C(\mathcal{O},\mathbb{R})\colon \limsup_{r\to\infty}\left[\sup_{x\in\mathcal{O}\setminus O_r}\left(\frac{|u(x)|}{V(x)}\right)\right]=0\right\} \tag{28}$$

and let $\|\cdot\|_{\mathcal{V}} : \mathcal{V} \to [0,\infty)$ satisfy for all $v \in \mathcal{V}$ that

$$\|v\|_{\mathcal{V}}=\sup_{x\in\mathcal{O}}\left(\frac{|v(x)|}{V(x)}\right). \tag{29}$$

Note that $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is an $\mathbb{R}$-Banach space. Moreover, observe that Corollary 2.4 (with $d \leftarrow d$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\|\cdot\| \leftarrow \|\cdot\|$, $\mathcal{O} \leftarrow \mathcal{O}$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(X^x)_{x\in\mathcal{O}} \leftarrow (X^x)_{x\in\mathcal{O}}$, $f \leftarrow f$, $V \leftarrow V$, $u \leftarrow v$ for $v \in \mathcal{V}$ in the notation of Corollary 2.4) demonstrates that there exists a unique function $\Phi\colon \mathcal{V} \to \mathcal{V}$ which satisfies for all $x \in \mathcal{O}$, $v \in \mathcal{V}$ that

$$[\Phi(v)](x)=\mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} f(X_t^x, v(X_t^x))\,dt\bigg]. \tag{30}$$

Moreover, note that Lemma 2.5 (with $d \leftarrow d$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\mathcal{O} \leftarrow \mathcal{O}$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(X^x)_{x\in\mathcal{O}} \leftarrow (X^x)_{x\in\mathcal{O}}$, $V \leftarrow V$, $f \leftarrow f$ in the notation of Lemma 2.5) ensures for all $v, w \in \mathcal{V}$ that

$$\|\Phi(v)-\Phi(w)\|_{\mathcal{V}} \leqslant \frac{L}{\lambda-\rho}\,\|v-w\|_{\mathcal{V}}\,. \tag{31}$$

This, the hypothesis that $\lambda > L + \rho$, the fact that $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is an $\mathbb{R}$-Banach space, and Banach's fixed point theorem demonstrate that there exists a unique $u \in \mathcal{V}$ which satisfies $\Phi(u) = u$. This establishes Items (i) and (ii). The proof of Proposition 2.6 is thus completed. □

2.2 On the Feynman–Kac connection between SFPEs and semilinear elliptic PDEs

Corollary 2.7. *Let $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $L, \rho \in \mathbb{R}$, $\lambda \in (\rho + L, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $V \in C^2(\mathbb{R}^d, (0,\infty))$ satisfy for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x, v) - f(x, w)| \leqslant L|v - w|$ and*

$$\tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess}V)(x)) \leqslant \rho V(x), \tag{32}$$

assume that $\limsup_{r\to\infty}[\sup_{\|x\|>r}(\frac{|f(x,0)|}{V(x)})] = 0$ and $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V(x)] = \infty$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W\colon [0,\infty) \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion. Then there exists a unique $u \in C(\mathbb{R}^d, \mathbb{R})$ such that

(i) *it holds that*

$$\limsup_{r\to\infty}\left[\sup_{\|x\|>r}\left(\frac{|u(x)|}{V(x)}\right)\right]=0 \tag{33}$$

and

(ii) *for all* $x\in\mathbb{R}^d$ *it holds that* $\mathbb{E}\big[\int_0^\infty e^{-\lambda s}|f(x+BW_s,u(x+BW_s))|\,ds\big]<\infty$ *and*

$$u(x)=\mathbb{E}\left[\int_0^\infty e^{-\lambda s}f(x+BW_s,u(x+BW_s))\,ds\right]. \tag{34}$$

Proof of Corollary 2.7. First, observe that combining [6, Lemmas 3.1 and 3.2] with (32) ensures that for all $t\in[0,\infty)$, $x\in\mathbb{R}^d$ it holds that

$$\mathbb{E}\big[e^{-\rho t}V(x+BW_t)\big]\leqslant V(x). \tag{35}$$

Moreover, note that for all $\varepsilon,t\in(0,\infty)$ and all $x_n\in\mathbb{R}^d$, $n\in\mathbb{N}_0$, with $\limsup_{n\to\infty}\|x_n-x_0\|=0$ it holds that $\limsup_{n\to\infty}\mathbb{P}(\|(x_n+BW_t)-(x_0+BW_t)\|\geqslant\varepsilon)=0$. Proposition 2.6 (with $d\leftarrow d$, $L\leftarrow L$, $\rho\leftarrow\rho$, $\lambda\leftarrow\lambda$, $\|\cdot\|\leftarrow\|\cdot\|$, $\mathcal{O}\leftarrow\mathbb{R}^d$, $(\Omega,\mathcal{F},\mathbb{P})\leftarrow(\Omega,\mathcal{F},\mathbb{P})$, $(X^x)_{x\in\mathbb{R}^d}\leftarrow([0,\infty)\times\Omega\ni(t,\omega)\mapsto x+BW_t(\omega)\in\mathbb{R}^d)_{x\in\mathbb{R}^d}$, $V\leftarrow V$, $f\leftarrow f$ in the notation of Proposition 2.6) and (35) hence assure that there exists a unique $u\in C(\mathbb{R}^d,\mathbb{R})$ such that

(I) it holds that

$$\limsup_{r\to\infty}\left[\sup_{\|x\|>r}\left(\frac{|u(x)|}{V(x)}\right)\right]=0 \tag{36}$$

and

(II) for all $x\in\mathbb{R}^d$ it holds that $\mathbb{E}\big[\int_0^\infty e^{-\lambda s}|f(x+BW_s,u(x+BW_s))|\,ds\big]<\infty$ and

$$u(x)=\mathbb{E}\left[\int_0^\infty e^{-\lambda t}f(x+BW_t,u(x+BW_t))\,dt\right]. \tag{37}$$

This establishes Items (i) and (ii). The proof of Corollary 2.7 is thus completed. $\square$

Lemma 2.8. *Let* $T\in(0,\infty)$, $d,m\in\mathbb{N}$, $B\in\mathbb{R}^{d\times m}$, $\varphi\in C^2(\mathbb{R}^d,\mathbb{R})$ *satisfy* $\sup_{x\in\mathbb{R}^d}[\sum_{i,j=1}^d(|\varphi(x)|+|(\frac{\partial}{\partial x_i}\varphi)(x)|+|(\frac{\partial^2}{\partial x_i\partial x_j}\varphi)(x)|)]<\infty$, *let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space, let* $Z\colon\Omega\to\mathbb{R}^m$ *be a standard normal random variable, and let* $u\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$ *satisfy for all* $t\in[0,T]$, $x\in\mathbb{R}^d$ *that*

$$u(t,x)=\mathbb{E}\big[\varphi(x+\sqrt{t}BZ)\big]. \tag{38}$$

Then

(i) *it holds that* $u\in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ *and*

(ii) *it holds for all* $t\in[0,T]$, $x\in\mathbb{R}^d$ *that*

$$(\tfrac{\partial u}{\partial t})(t,x)=\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(t,x)\big). \tag{39}$$

Proof of Lemma 2.8. Throughout this proof let $e_1=(1,0,\dots,0),e_2=(0,1,\dots,0),\dots,e_m=(0,\dots,0,1)\in\mathbb{R}^m$, let $\langle\cdot,\cdot\rangle\colon(\cup_{k\in\mathbb{N}}(\mathbb{R}^k\times\mathbb{R}^k))\to\mathbb{R}$ satisfy for all $k\in\mathbb{N}$, $x=(x_1,x_2,\dots,x_k),y=(y_1,y_2,\dots,y_k)\in\mathbb{R}^k$ that $\langle x,y\rangle=\sum_{i=1}^k x_iy_i$, let $\|\cdot\|\colon\mathbb{R}^m\to[0,\infty)$ be the standard norm on $\mathbb{R}^m$, and let $\psi_{t,x}=(\psi_{t,x}(y))_{y\in\mathbb{R}^m}\colon\mathbb{R}^m\to\mathbb{R}$, $t\in[0,T]$, $x\in\mathbb{R}^d$, satisfy for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^m$ that $\psi_{t,x}(y)=\varphi(x+\sqrt{t}By)$. Note that the assumption that $\varphi\in C^2(\mathbb{R}^d,\mathbb{R})$, the assumption that $\sup_{x\in\mathbb{R}^d}[\sum_{i,j=1}^d(|\varphi(x)|+|(\frac{\partial}{\partial x_i}\varphi)(x)|+|(\frac{\partial^2}{\partial x_i\partial x_j}\varphi)(x)|)]<\infty$, the chain rule, and Lebesgue's dominated convergence theorem ensure that

(I) for all $x \in \mathbb{R}^d$ it holds that $(0,T] \ni t \mapsto u(t,x) \in \mathbb{R}$ is differentiable,

(II) for all $t \in [0,T]$ it holds that $\mathbb{R}^d \ni x \mapsto u(t,x) \in \mathbb{R}$ is twice differentiable,

(III) for all $t \in (0,T]$, $x \in \mathbb{R}^d$ it holds that

$$(\tfrac{\partial u}{\partial t})(t,x) = \mathbb{E}\big[\langle (\nabla \varphi)(x + \sqrt{t}BZ), \tfrac{1}{2\sqrt{t}} BZ \rangle\big], \tag{40}$$

and

(IV) for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$(\operatorname{Hess}_x u)(t,x) = \mathbb{E}\big[(\operatorname{Hess} \varphi)(x + \sqrt{t}BZ)\big]. \tag{41}$$

Observe that Items (III) and (IV), the assumption that $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$, the assumption that $\sup_{x \in \mathbb{R}^d} [\sum_{i,j=1}^d (|\varphi(x)| + |(\frac{\partial}{\partial x_i} \varphi)(x)| + |(\frac{\partial^2}{\partial x_i \partial x_j} \varphi)(x)|)] < \infty$, the fact that $\mathbb{E}[\|Z\|] < \infty$, and Lebesgue's dominated convergence theorem prove that $(0,T] \times \mathbb{R}^d \ni (t,x) \mapsto (\frac{\partial u}{\partial t})(t,x) \in \mathbb{R}$ and $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto (\operatorname{Hess}_x u)(t,x) \in \mathbb{R}^{d \times d}$ are continuous. Next note that Item (IV) and the fact that for all $X \in \mathbb{R}^{m \times d}$, $Y \in \mathbb{R}^{d \times m}$ it holds that $\operatorname{Trace}(XY) = \operatorname{Trace}(YX)$ imply that for all $t \in (0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\tfrac{1}{2} \operatorname{Trace}\big(BB^* (\operatorname{Hess}_x u)(t,x)\big) = \mathbb{E}\Big[\tfrac{1}{2} \operatorname{Trace}\big(BB^* (\operatorname{Hess} \varphi)(x + \sqrt{t}BZ)\big)\Big] \\
&= \tfrac{1}{2} \mathbb{E}\Big[\operatorname{Trace}\big(B^* (\operatorname{Hess} \varphi)(x + \sqrt{t}BZ) B\big)\Big] = \tfrac{1}{2} \mathbb{E}\Bigg[\sum_{k=1}^m \langle e_k, B^* (\operatorname{Hess} \varphi)(x + \sqrt{t}BZ) B e_k \rangle\Bigg] \\
&= \tfrac{1}{2} \mathbb{E}\Bigg[\sum_{k=1}^m \langle B e_k, (\operatorname{Hess} \varphi)(x + \sqrt{t}BZ) B e_k \rangle\Bigg] = \tfrac{1}{2} \mathbb{E}\Bigg[\sum_{k=1}^m \varphi''(x + \sqrt{t}BZ)(Be_k, Be_k)\Bigg] \\
&= \tfrac{1}{2t} \mathbb{E}\Bigg[\sum_{k=1}^m (\psi_{t,x})''(Z)(e_k, e_k)\Bigg] = \tfrac{1}{2t} \mathbb{E}\Bigg[\sum_{k=1}^m (\tfrac{\partial^2}{\partial y_k^2} \psi_{t,x})(Z)\Bigg] = \tfrac{1}{2t} \mathbb{E}[(\Delta \psi_{t,x})(Z)].
\end{aligned} \tag{42}$$

The assumption that $Z \colon \Omega \to \mathbb{R}^m$ is a standard normal random variable and integration by parts hence ensure that for all $t \in (0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\tfrac{1}{2} \operatorname{Trace}\big(BB^* (\operatorname{Hess}_x u)(t,x)\big) \\
&= \frac{1}{2t} \int_{\mathbb{R}^m} (\Delta \psi_{t,x})(y) \left[\frac{\exp(-\frac{\langle y,y \rangle}{2})}{(2\pi)^{m/2}}\right] dy = \frac{1}{2t} \int_{\mathbb{R}^m} \langle (\nabla \psi_{t,x})(y), y \rangle \left[\frac{\exp(-\frac{\langle y,y \rangle}{2})}{(2\pi)^{m/2}}\right] dy \\
&= \frac{1}{2\sqrt{t}} \int_{\mathbb{R}^m} \Big\langle B^* (\nabla \varphi)(x + \sqrt{t}By), y \Big\rangle \left[\frac{\exp(-\frac{\langle y,y \rangle}{2})}{(2\pi)^{m/2}}\right] dy \\
&= \frac{1}{2\sqrt{t}} \mathbb{E}\big[\langle B^* (\nabla \varphi)(x + \sqrt{t}BZ), Z \rangle\big] = \mathbb{E}\big[\langle (\nabla \varphi)(x + \sqrt{t}BZ), \tfrac{1}{2\sqrt{t}} BZ \rangle\big].
\end{aligned} \tag{43}$$

Item (III) therefore proves for all $t \in (0,T]$, $x \in \mathbb{R}^d$ that

$$(\tfrac{\partial u}{\partial t})(t,x) = \tfrac{1}{2} \operatorname{Trace}\big(BB^* (\operatorname{Hess}_x u)(t,x)\big). \tag{44}$$

The fundamental theorem of calculus hence implies that for all $t, s \in (0,T]$, $x \in \mathbb{R}^d$ it holds that

$$u(t,x) - u(s,x) = \int_s^t (\tfrac{\partial u}{\partial t})(r,x)\, dr = \int_s^t \tfrac{1}{2} \operatorname{Trace}\big(BB^* (\operatorname{Hess}_x u)(r,x)\big)\, dr. \tag{45}$$

The fact that $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto (\operatorname{Hess}_x u)(t,x) \in \mathbb{R}^{d\times d}$ is continuous therefore ensures for all $t \in (0,T]$, $x \in \mathbb{R}^d$ that

$$\frac{u(t,x)-u(0,x)}{t} = \lim_{s\searrow 0}\left[\frac{u(t,x)-u(s,x)}{t}\right] = \frac{1}{t}\int_0^t \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(r,x)\big)\,dr. \tag{46}$$

This and again the fact that $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto (\operatorname{Hess}_x u)(t,x) \in \mathbb{R}^{d\times d}$ is continuous imply for all $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\limsup_{t\searrow 0}\left|\frac{u(t,x)-u(0,x)}{t} - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(0,x)\big)\right| \\
&\leqslant \limsup_{t\searrow 0}\left[\frac{1}{t}\int_0^t \left|\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(s,x)\big) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(0,x)\big)\right| ds\right] \\
&\leqslant \limsup_{t\searrow 0}\left[\sup_{s\in[0,t]}\left|\tfrac{1}{2}\operatorname{Trace}\Big(BB^*\big((\operatorname{Hess}_x u)(s,x) - (\operatorname{Hess}_x u)(0,x)\big)\Big)\right|\right] = 0.
\end{aligned} \tag{47}$$

Item (I) hence establishes that for all $x \in \mathbb{R}^d$ it holds that $[0,T] \ni t \mapsto u(t,x) \in \mathbb{R}$ is differentiable. Combining this with (47) and (44) ensures that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$(\tfrac{\partial u}{\partial t})(t,x) = \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(t,x)\big). \tag{48}$$

This and the fact that $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto (\operatorname{Hess}_x u)(t,x) \in \mathbb{R}^{d\times d}$ is continuous establish Item (i). In addition, note that (48) establishes Item (ii). The proof of Lemma 2.8 is thus completed. □

Corollary 2.9. *Let $T \in (0,\infty)$, $d,m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $\varphi \in C^2(\mathbb{R}^d,\mathbb{R})$ satisfy $\sup_{x\in\mathbb{R}^d}[\sum_{i,j=1}^d(|\varphi(x)| + |(\frac{\partial}{\partial x_i}\varphi)(x)| + |(\frac{\partial^2}{\partial x_i \partial x_j}\varphi)(x)|)] < \infty$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion, and let $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ satisfy for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that*

$$u(t,x) = \mathbb{E}\big[\varphi(x + BW_t)\big]. \tag{49}$$

Then

(i) it holds that $u \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ and

(ii) it holds for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$(\tfrac{\partial u}{\partial t})(t,x) = \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}_x u)(t,x)\big). \tag{50}$$

Proof of Corollary 2.9. First, observe that the assumption that $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ is a standard Brownian motion ensures for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$u(t,x) = \mathbb{E}[\varphi(x + BW_t)] = \mathbb{E}\left[\varphi(x + \sqrt{t}B\frac{W_T}{\sqrt{T}})\right]. \tag{51}$$

The fact that $\frac{W_T}{\sqrt{T}}\colon \Omega \to \mathbb{R}^m$ is standard normally distributed and Lemma 2.8 hence establish Items (i) and (ii). The proof of Corollary 2.9 is thus completed. □

Lemma 2.10. *Let $d,m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $\lambda \in (0,\infty)$, $\rho \in (-\infty,\lambda)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $h \in C(\mathbb{R}^d,\mathbb{R})$, $V \in C^2(\mathbb{R}^d,(0,\infty))$, assume for all $x \in \mathbb{R}^d$ that*

$$\tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} V)(x)) \leqslant \rho V(x), \tag{52}$$

assume that $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V(x)] = \infty$ *and* $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|h(x)|}{V(x)})] = 0$*, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ *be a standard Brownian motion, and let* $u\colon \mathbb{R}^d \to \mathbb{R}$ *satisfy for all* $x \in \mathbb{R}^d$ *that*

$$u(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda s} h(x + BW_s)\, ds\bigg] \tag{53}$$

(cf. Item (ii) *in Corollary 2.7). Then it holds that* u *is a viscosity solution of*

$$\lambda u(x) - \tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} u)(x)) = h(x) \tag{54}$$

for $x \in \mathbb{R}^d$*.*

Proof of Lemma 2.10. Throughout this proof let $\mathfrak{h}_n \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $n \in \mathbb{N}$, be compactly supported functions which satisfy

$$\limsup_{n\to\infty}\left[\sup_{x\in\mathbb{R}^d}\left(\frac{|\mathfrak{h}_n(x) - h(x)|}{V(x)}\right)\right] = 0, \tag{55}$$

let $F_n\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}$, $x, p \in \mathbb{R}^d$, $r \in \mathbb{R}$, $A \in \mathbb{S}_d$ that

$$F_n(x,r,p,A) = \lambda r - \tfrac{1}{2}\operatorname{Trace}(BB^*A) - \mathfrak{h}_n(x) \text{ and } F_0(x,r,p,A) = \lambda r - \tfrac{1}{2}\operatorname{Trace}(BB^*A) - h(x), \tag{56}$$

let $\mathfrak{u}_n\colon \mathbb{R}^d \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\mathfrak{u}_n(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda s}\, \mathfrak{h}_n(x + BW_s)\, ds\bigg], \tag{57}$$

and let $\mathfrak{v}_n\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$ that $\mathfrak{v}_n(t,x) = \mathbb{E}[\mathfrak{h}_n(x + BW_t)]$. Observe that Corollary 2.9 ensures for all $n \in \mathbb{N}$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$ that $\mathfrak{v}_n \in C^{1,2}([0,\infty)\times\mathbb{R}^d, \mathbb{R})$ and

$$(\tfrac{\partial}{\partial t}\mathfrak{v}_n)(t,x) = \tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess}_x \mathfrak{v}_n)(t,x)). \tag{58}$$

This, (57), the fact that for all $n \in \mathbb{N}$ it holds that $\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d}[\sum_{i,j=1}^d(|\mathfrak{v}_n(t,x)| + |(\frac{\partial}{\partial x_i}\mathfrak{v}_n)(t,x)| + |(\frac{\partial^2}{\partial x_i \partial x_j}\mathfrak{v}_n)(t,x)|)] < \infty$, integration by parts, and Lebesgue's dominated convergence theorem guarantee that for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
\mathfrak{u}_n(x) &= \int_0^\infty e^{-\lambda t}\, \mathfrak{v}_n(t,x)\, dt = \lim_{R\to\infty}\bigg[\int_0^R e^{-\lambda t}\, \mathfrak{v}_n(t,x)\, dt\bigg] \\
&= \lim_{R\to\infty}\bigg[\frac{1}{\lambda}\mathfrak{v}_n(0,x) - \frac{e^{-\lambda R}}{\lambda}\mathfrak{v}_n(R,x) + \frac{1}{\lambda}\int_0^R e^{-\lambda t}(\tfrac{\partial}{\partial t}\mathfrak{v}_n)(t,x)\, dt\bigg] \\
&= \frac{1}{\lambda}\bigg[\mathfrak{v}_n(0,x) + \int_0^\infty e^{-\lambda t}(\tfrac{\partial}{\partial t}\mathfrak{v}_n)(t,x)\, dt\bigg] \\
&= \frac{1}{\lambda}\bigg[\mathfrak{h}_n(x) + \int_0^\infty e^{-\lambda t}\tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess}_x \mathfrak{v}_n)(t,x))\, dt\bigg] \\
&= \frac{1}{\lambda}\big[\mathfrak{h}_n(x) + \tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess}\mathfrak{u}_n)(x))\big].
\end{aligned} \tag{59}$$

This shows for every $n \in \mathbb{N}$ that $\mathfrak{u}_n$ is a viscosity solution of

$$\lambda\mathfrak{u}_n(x) - \tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess}\mathfrak{u}_n)(x)) = \mathfrak{h}_n(x) \tag{60}$$

for $x \in \mathbb{R}^d$. Next note that (55) and (56) ensure for every non-empty compact set $K \subseteq \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$ that

$$\begin{aligned}
&\limsup_{n\to\infty} \left[\sup_{(x,r,p,A)\in K} |F_n(x,r,p,A) - F_0(x,r,p,A)| \right] = \limsup_{n\to\infty} \left[\sup_{(x,r,p,A)\in K} |\mathfrak{h}_n(x) - h(x)| \right] \\
&\leqslant \limsup_{n\to\infty} \left(\left[\sup_{y\in\mathbb{R}^d} \left(\frac{|\mathfrak{h}_n(y) - h(y)|}{V(y)} \right) \right] \left[\sup_{(x,r,p,A)\in K} V(x) \right] \right) = 0.
\end{aligned} \tag{61}$$

Moreover, note that (52), (53), and (57) guarantee for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\frac{|\mathfrak{u}_n(x) - u(x)|}{V(x)} = \frac{1}{V(x)} \left| \mathbb{E}\!\left[\int_0^\infty e^{-\lambda s} \left(h(x + BW_s) - \mathfrak{h}_n(x + BW_s) \right) ds \right] \right| \\
&\leqslant \int_0^\infty e^{-\lambda s} \left[\sup_{y\in\mathbb{R}^d} \left(\frac{|h(y) - \mathfrak{h}_n(y)|}{V(y)} \right) \right] \frac{\mathbb{E}[V(x + BW_s)]}{V(x)}\, ds \leqslant \frac{1}{\lambda - \rho} \left[\sup_{y\in\mathbb{R}^d} \left(\frac{|h(y) - \mathfrak{h}_n(y)|}{V(y)} \right) \right].
\end{aligned} \tag{62}$$

This and (55) imply for every non-empty compact set $K \subseteq \mathbb{R}^d$ that

$$\limsup_{n\to\infty} \left[\sup_{x\in K} |\mathfrak{u}_n(x) - u(x)| \right] = 0. \tag{63}$$

This, (61), the fact that for all $n \in \mathbb{N}$ it holds that $\mathfrak{u}_n$ is a viscosity solution of

$$F_n(x, \mathfrak{u}_n(x), (\nabla \mathfrak{u}_n)(x), (\operatorname{Hess} \mathfrak{u}_n)(x)) = 0 \tag{64}$$

for $x \in \mathbb{R}^d$ (cf. (56) and (60)), and Hairer et al. [33, Lemma 4.8] (see also Barles & Perthame [2]) imply that u is a viscosity solution of

$$F_0(x, u(x), (\nabla u)(x), (\operatorname{Hess} u)(x)) = 0 \tag{65}$$

for $x \in \mathbb{R}^d$. This establishes (54). The proof of Lemma 2.10 is thus completed. □

Corollary 2.11. *Let $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $L, \rho \in \mathbb{R}$, $\lambda \in (L+\rho, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $u \in C(\mathbb{R}^d, \mathbb{R})$, $V \in C^2(\mathbb{R}^d, (0,\infty))$ satisfy $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V(x)] = \infty$ and $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|f(x,0)|+|u(x)|}{V(x)})] = 0$, assume for all $x \in \mathbb{R}^d$ that*

$$\tfrac{1}{2} \operatorname{Trace}\big(BB^*(\operatorname{Hess} V)(x)\big) \leqslant \rho V(x), \tag{66}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W : [0,\infty) \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion, and assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w)| \leqslant L|v - w|$ and

$$u(x) = \mathbb{E}\!\left[\int_0^\infty e^{-\lambda t} f(x + BW_t, u(x + BW_t))\, dt \right] \tag{67}$$

(cf. Corollary 2.7). Then it holds that u is a viscosity solution of

$$\lambda u(x) - \tfrac{1}{2} \operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \tag{68}$$

for $x \in \mathbb{R}^d$.

Proof of Corollary 2.11. Throughout this proof let $h\colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that $h(x) = f(x, u(x))$. Observe that the assumptions that $V \in C(\mathbb{R}^d, (0,\infty))$ and $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V(x)] = \infty$ imply that $\{x \in \mathbb{R}^d\colon V(x) = \inf_{y\in\mathbb{R}^d} V(y)\} \neq \emptyset$. This, the fact that for all $x \in \{y \in \mathbb{R}^d\colon V(y) = \inf_{z\in\mathbb{R}^d} V(z)\}$ it holds that $(\operatorname{Hess} V)(x) \geqslant 0$, and (66) ensure that $\rho \geqslant 0$. Hence, we obtain that $\lambda \in (0,\infty)$. Next note that (67) and Lemma 2.10 prove that u is a viscosity solution of

$$\lambda u(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = h(x) \tag{69}$$

for $x \in \mathbb{R}^d$. This implies for all $x \in \mathbb{R}^d$, $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$ with $\varphi \geqslant u$ and $\varphi(x) = u(x)$ that

$$\begin{aligned} 0 &\geqslant \lambda\varphi(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}\varphi)(x)\big) - h(x) \\ &= \lambda\varphi(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}\varphi)(x)\big) - f(x,\varphi(x)). \end{aligned} \tag{70}$$

Moreover, note that (69) implies for all $x \in \mathbb{R}^d$, $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$ with $\varphi \leqslant u$ and $\varphi(x) = u(x)$ that

$$\begin{aligned} 0 &\leqslant \lambda\varphi(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}\varphi)(x)\big) - h(x) \\ &= \lambda\varphi(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess}\varphi)(x)\big) - f(x,\varphi(x)). \end{aligned} \tag{71}$$

This and (70) demonstrate (68). The proof of Corollary 2.11 is thus completed. ☐

2.3 On a comparison principle for viscosity solutions of semilinear elliptic PDEs

Proposition 2.12 (Comparison principle)**.** *Let $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $L, \rho \in \mathbb{R}$, $\lambda \in (\rho + L, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $g, h, u, v \in C(\mathbb{R}^d, \mathbb{R})$, $V \in C^2(\mathbb{R}^d, (0,\infty))$ satisfy for all $x \in \mathbb{R}^d$, $a, b \in \mathbb{R}$ that $[f(x,a) - f(x,b) - L(a-b)](a-b) \leqslant 0$ and*

$$\tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} V)(x)) \leqslant \rho V(x), \tag{72}$$

assume that $\limsup_{r\to\infty}[\sup_{\|x\|>r}(\frac{|f(x,0)|+|g(x)|+|h(x)|+|u(x)|+|v(x)|}{V(x)})] = 0$, assume that u is viscosity supersolution of

$$\lambda u(x) - \tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} u)(x)) = f(x, u(x)) + g(x) \tag{73}$$

for $x \in \mathbb{R}^d$, and assume that v is a viscosity subsolution of

$$\lambda v(x) - \tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} v)(x)) = f(x, v(x)) + h(x) \tag{74}$$

for $x \in \mathbb{R}^d$. Then it holds that

$$\sup_{x\in\mathbb{R}^d}\left[\max\left\{\frac{v(x)-u(x)}{V(x)}, 0\right\}\right] \leqslant \frac{1}{\lambda - (L+\rho)}\left[\sup_{x\in\mathbb{R}^d}\left(\max\left\{\frac{h(x)-g(x)}{V(x)}, 0\right\}\right)\right]. \tag{75}$$

Proof of Proposition 2.12. Throughout this proof let $\langle\cdot,\cdot\rangle\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the standard scalar product on $\mathbb{R}^d$, let $w_1, w_2 \in C(\mathbb{R}^d, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^d$ that $w_1(x) = \frac{u(x)}{V(x)}$ and $w_2(x) = \frac{v(x)}{V(x)}$, and let $\eta_\alpha\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\alpha \in (0,\infty)$, satisfy for all $\alpha \in (0,\infty)$, $x, y \in \mathbb{R}^d$ that $\eta_\alpha(x,y) = w_2(x) - w_1(y) - \frac{\alpha}{2}\|x-y\|^2$. Note that (75) is clear in the case of $v \leqslant u$. Therefore, we assume in the following that there exists $x \in \mathbb{R}^d$ such that $v(x) - u(x) > 0$. This implies that there exists $x \in \mathbb{R}^d$ such that $w_2(x) - w_1(x) > 0$. Next note that the hypothesis that $\limsup_{r\to\infty}[\sup_{\|x\|>r}(\frac{|u(x)|+|v(x)|}{V(x)})] = 0$ demonstrates that

$$\limsup_{r\to\infty}\left[\sup_{\|x\|>r}\Big(|w_1(x)| + |w_2(x)|\Big)\right] = 0. \tag{76}$$

Combining this with the fact that $w_1, w_2 \in C(\mathbb{R}^d, \mathbb{R})$ and the fact that $\sup_{x\in\mathbb{R}^d}(\max\{w_2(x) - w_1(x), 0\}) \in (0, \infty]$ implies that there exists $x_0 \in \mathbb{R}^d$ which satisfies that

$$w_2(x_0) - w_1(x_0) = \sup_{y\in\mathbb{R}^d} \Big(w_2(y) - w_1(y) \Big) > 0. \tag{77}$$

In addition, note that (76) and the fact that $w_1, w_2 \in C(\mathbb{R}^d, \mathbb{R})$ ensure that $\sup_{x\in\mathbb{R}^d}(|w_1(x)| + |w_2(x)|) < \infty$. This and (77) imply for all $\alpha \in (0, \infty)$, $\beta \in (\alpha, \infty)$ that

$$\begin{aligned} \infty > \left[\sup_{x\in\mathbb{R}^d} |w_1(x)|\right] + \left[\sup_{x\in\mathbb{R}^d} |w_2(x)|\right] &\geqslant \sup_{x,y\in\mathbb{R}^d} (w_2(x) - w_1(y)) \geqslant \sup_{z\in\mathbb{R}^d\times\mathbb{R}^d} \eta_\alpha(z) \\ \geqslant \sup_{z\in\mathbb{R}^d\times\mathbb{R}^d} \eta_\beta(z) \geqslant \sup_{x\in\mathbb{R}^d} \eta_\beta(x, x) &= \sup_{x\in\mathbb{R}^d} (w_2(x) - w_1(x)) > 0. \end{aligned} \tag{78}$$

Next let $r_\alpha \in (0, \infty)$, $\alpha \in (0, \infty)$, satisfy for all $\alpha \in (0, \infty)$ that $r_\alpha = [\frac{2}{\alpha}([\sup_{x\in\mathbb{R}^d} |w_1(x)|] + [\sup_{x\in\mathbb{R}^d} |w_2(x)|])]^{1/2}$ and let $R \in (0, \infty)$ satisfy that for all $x \in \mathbb{R}^d$ with $\|x\| > R$ it holds that $|w_1(x)| + |w_2(x)| < \frac{1}{4} \sup_{y\in\mathbb{R}^d}(w_2(y) - w_1(y))$. Furthermore, observe that for all $\alpha \in (0, \infty)$, $x, y \in \mathbb{R}^d$ with $\|x - y\| > r_\alpha$ it holds that $w_2(x) - w_1(y) - \frac{\alpha}{2} \|x - y\|^2 \leqslant 0$. Hence, we obtain for all $\alpha \in (0, \infty)$, $x, y \in \mathbb{R}^d$ with $\max\{\|x\|, \|y\|\} > R + r_\alpha$ that

$$\eta_\alpha(x, y) \begin{cases} \leqslant 0 & : \|x - y\| > r_\alpha, \\ \leqslant \frac{1}{2} \sup_{z\in\mathbb{R}^d}(w_2(z) - w_1(z)) & : \|x - y\| \leqslant r_\alpha. \end{cases} \tag{79}$$

Combining this with (78) demonstrates for all $\alpha \in (0, \infty)$ that

$$\sup_{z\in\mathbb{R}^d\times\mathbb{R}^d} \eta_\alpha(z) = \sup_{\max\{\|x\|,\|y\|\}\leqslant R+r_\alpha} \eta_\alpha(x, y). \tag{80}$$

Hence, we obtain that for every $\alpha \in (0, \infty)$ there exist $x_\alpha, y_\alpha \in \mathbb{R}^d$ which satisfy that $\max\{\|x_\alpha\|, \|y_\alpha\|\} \leqslant R + r_\alpha$ and

$$w_2(x_\alpha) - w_1(y_\alpha) - \frac{\alpha}{2} \|x_\alpha - y_\alpha\|^2 = \sup_{z\in\mathbb{R}^d\times\mathbb{R}^d} \eta_\alpha(z) > 0. \tag{81}$$

Crandall et al. [18, Theorem 3.2] (with $k \leftarrow 2$, $N_1 \leftarrow d$, $N_2 \leftarrow d$, $\mathcal{O}_1 \leftarrow \mathbb{R}^d$, $\mathcal{O}_2 \leftarrow \mathbb{R}^d$, $u_1 \leftarrow w_2$, $u_2 \leftarrow -w_1$, $\varphi \leftarrow (\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \frac{\alpha}{2} \|x - y\|^2 \in \mathbb{R})$, $\hat{x} \leftarrow (x_\alpha, y_\alpha)$ for $\alpha \in (0, \infty)$ in the notation of [18, Theorem 3.2]) therefore guarantees that there exist $X_\alpha, Y_\alpha \in \mathbb{S}_d$, $\alpha \in (0, \infty)$, which satisfy for all $\alpha \in (0, \infty)$ that $(\alpha(x_\alpha - y_\alpha), X_\alpha) \in (\hat{J}^2_+ w_2)(x_\alpha)$, $(\alpha(x_\alpha - y_\alpha), Y_\alpha) \in (\hat{J}^2_- w_1)(y_\alpha)$, and

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{82}$$

(see Hairer et al. [33, Definition 4.3] for definitions of $\hat{J}^2_+ w_2$ and $\hat{J}^2_- w_1$). Next observe that (73) implies that w_1 is a viscosity supersolution of

$$\begin{aligned} \lambda w_1(x) - \bigg[&\tfrac{1}{2} \operatorname{Trace}\left(BB^*(\operatorname{Hess} w_1)(x)\right) + \left\langle BB^* \tfrac{(\nabla V)(x)}{V(x)}, (\nabla w_1)(x) \right\rangle \\ &+ \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(x)}{V(x)}\Big) w_1(x) + \tfrac{1}{V(x)} f\big(x, V(x) w_1(x)\big) + \tfrac{g(x)}{V(x)} \bigg] = 0 \end{aligned} \tag{83}$$

for $x \in \mathbb{R}^d$. Combining this and (82) assures for all $\alpha \in (0, \infty)$ that

$$\begin{aligned} \lambda w_1(y_\alpha) - \bigg[&\tfrac{1}{2} \operatorname{Trace}\left(BB^* Y_\alpha\right) + \left\langle BB^* \tfrac{(\nabla V)(y_\alpha)}{V(y_\alpha)}, \alpha(x_\alpha - y_\alpha) \right\rangle \\ &+ \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(y_\alpha)}{V(y_\alpha)}\Big) w_1(y_\alpha) + \tfrac{1}{V(y_\alpha)} f\big(y_\alpha, V(y_\alpha) w_1(y_\alpha)\big) + \tfrac{g(y_\alpha)}{V(y_\alpha)} \bigg] \geqslant 0. \end{aligned} \tag{84}$$

In addition, note that (74) ensures that w_2 is a viscosity subsolution of

$$\begin{aligned}\lambda w_2(x) - \Big[&\tfrac{1}{2} \operatorname{Trace}\big(BB^*(\operatorname{Hess} w_2)(x)\big) + \Big\langle BB^* \tfrac{(\nabla V)(x)}{V(x)}, (\nabla w_2)(x) \Big\rangle \\ &+ \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(x)}{V(x)} \Big) w_2(x) + \tfrac{1}{V(x)} f\big(x, V(x) w_2(x)\big) + \tfrac{h(x)}{V(x)} \Big] = 0\end{aligned} \tag{85}$$

for $x \in \mathbb{R}^d$. Combining this and (82) implies for all $\alpha \in (0,\infty)$ that

$$\begin{aligned}\lambda w_2(x_\alpha) - \Big[&\tfrac{1}{2} \operatorname{Trace}\big(BB^* X_\alpha\big) + \Big\langle BB^* \tfrac{(\nabla V)(x_\alpha)}{V(x_\alpha)}, \alpha(x_\alpha - y_\alpha) \Big\rangle \\ &+ \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(x_\alpha)}{V(x_\alpha)} \Big) w_2(x_\alpha) + \tfrac{1}{V(x_\alpha)} f\big(x_\alpha, V(x_\alpha) w_2(x_\alpha)\big) + \tfrac{h(x_\alpha)}{V(x_\alpha)} \Big] \leqslant 0.\end{aligned} \tag{86}$$

This and (84) assure for all $\alpha \in (0,\infty)$ that

$$\begin{aligned}\lambda(w_2(x_\alpha) - w_1(y_\alpha)) \leqslant{} &\tfrac{1}{2} \operatorname{Trace}(BB^*(X_\alpha - Y_\alpha)) + \Big\langle BB^* \Big(\tfrac{(\nabla V)(x_\alpha)}{V(x_\alpha)} - \tfrac{(\nabla V)(y_\alpha)}{V(y_\alpha)} \Big), \alpha(x_\alpha - y_\alpha) \Big\rangle \\ &+ \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \Big(\tfrac{(\operatorname{Hess} V)(x_\alpha)}{V(x_\alpha)} w_2(x_\alpha) - \tfrac{(\operatorname{Hess} V)(y_\alpha)}{V(y_\alpha)} w_1(y_\alpha) \Big) \Big) \\ &+ \tfrac{1}{V(x_\alpha)} f\big(x_\alpha, V(x_\alpha) w_2(x_\alpha)\big) - \tfrac{1}{V(y_\alpha)} f\big(y_\alpha, V(y_\alpha) w_1(y_\alpha)\big) + \tfrac{h(x_\alpha)}{V(x_\alpha)} - \tfrac{g(y_\alpha)}{V(y_\alpha)}.\end{aligned} \tag{87}$$

Next note that (82) ensures for all $\alpha \in (0,\infty)$ that $X_\alpha \leqslant Y_\alpha$. This and (87) imply for all $\alpha \in (0,\infty)$ that

$$\begin{aligned}\lambda(w_2(x_\alpha) - w_1(y_\alpha)) \leqslant{} &\Big\langle BB^* \Big(\tfrac{(\nabla V)(x_\alpha)}{V(x_\alpha)} - \tfrac{(\nabla V)(y_\alpha)}{V(y_\alpha)} \Big), \alpha(x_\alpha - y_\alpha) \Big\rangle \\ &+ \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \Big(\tfrac{(\operatorname{Hess} V)(x_\alpha)}{V(x_\alpha)} w_2(x_\alpha) - \tfrac{(\operatorname{Hess} V)(y_\alpha)}{V(y_\alpha)} w_1(y_\alpha) \Big) \Big) \\ &+ \tfrac{1}{V(x_\alpha)} f\big(x_\alpha, V(x_\alpha) w_2(x_\alpha)\big) - \tfrac{1}{V(y_\alpha)} f\big(y_\alpha, V(y_\alpha) w_1(y_\alpha)\big) + \tfrac{h(x_\alpha)}{V(x_\alpha)} - \tfrac{g(y_\alpha)}{V(y_\alpha)}.\end{aligned} \tag{88}$$

Moreover, observe that (78) implies that $\lim_{\alpha\to\infty}[\sup_{z\in\mathbb{R}^d\times\mathbb{R}^d} \eta_\alpha(z)] \in \mathbb{R}$ exists. Hairer et al. [33, Lemma 4.9] (with $d \leftarrow 2d$, $O \leftarrow \mathbb{R}^d \times \mathbb{R}^d$, $\eta \leftarrow (\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto w_2(x) - w_1(y) \in \mathbb{R})$, $\phi \leftarrow (\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto \frac{1}{2}\|x-y\|^2 \in [0,\infty))$, $x \leftarrow ((0,\infty) \ni \alpha \mapsto (x_\alpha, y_\alpha) \in \mathbb{R}^d \times \mathbb{R}^d)$ in the notation of Hairer et al. [33, Lemma 4.9]) and (81) therefore ensure that $\limsup_{\alpha\to\infty}[\alpha \|x_\alpha - y_\alpha\|^2] = 0$. This, the fact that $\limsup_{\alpha\to\infty} r_\alpha = 0$, the fact that $\frac{\nabla V}{V} \colon \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous, and (81) imply that

$$\limsup_{\alpha\to\infty} \Big| \Big\langle BB^* \Big(\tfrac{(\nabla V)(x_\alpha)}{V(x_\alpha)} - \tfrac{(\nabla V)(y_\alpha)}{V(y_\alpha)} \Big), \alpha(x_\alpha - y_\alpha) \Big\rangle \Big| = 0. \tag{89}$$

In addition, note that the fact that $\limsup_{\alpha\to\infty} r_\alpha = 0$ and (81) assure that there exist $\hat{x} \in \mathbb{R}^d$ and $\alpha_n \in (0,\infty)$, $n \in \mathbb{N}$, which satisfy that $\liminf_{n\to\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \|x_{\alpha_n} - \hat{x}\| = 0$. This, the fact that $(\operatorname{Hess} V) \in C(\mathbb{R}^d, \mathbb{S}_d)$, the fact that $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, the fact that $V \in C(\mathbb{R}^d, (0,\infty))$, the fact that $u, v, g, h \in C(\mathbb{R}^d, \mathbb{R})$, and the fact that $\limsup_{\alpha\to\infty}[\frac{\alpha}{2}\|x_\alpha - y_\alpha\|^2] = 0$ prove that

(i) it holds that

$$\begin{aligned}\limsup_{n\to\infty} \Big[&\Big| \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(x_{\alpha_n})}{V(x_{\alpha_n})} w_2(x_{\alpha_n}) \Big) - \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(\hat{x})}{V(\hat{x})} w_2(\hat{x}) \Big) \Big| \\ &+ \Big| \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(y_{\alpha_n})}{V(y_{\alpha_n})} w_1(y_{\alpha_n}) \Big) - \tfrac{1}{2} \operatorname{Trace}\Big(BB^* \tfrac{(\operatorname{Hess} V)(\hat{x})}{V(\hat{x})} w_1(\hat{x}) \Big) \Big| \Big] = 0,\end{aligned} \tag{90}$$

(ii) it holds that

$$\limsup_{n\to\infty}\Bigg[\Big|\tfrac{1}{V(x_{\alpha_n})}f\big(x_{\alpha_n},V(x_{\alpha_n})w_2(x_{\alpha_n})\big)-\tfrac{1}{V(\hat{x})}f\big(\hat{x},V(\hat{x})w_2(\hat{x})\big)\Big| + \Big|\tfrac{1}{V(y_{\alpha_n})}f\big(y_{\alpha_n},V(y_{\alpha_n})w_1(y_{\alpha_n})\big)-\tfrac{1}{V(\hat{x})}f\big(\hat{x},V(\hat{x})w_1(\hat{x})\big)\Big|\Bigg]=0, \tag{91}$$

(iii) and it holds that

$$\limsup_{n\to\infty}\left[\left|\tfrac{h(x_{\alpha_n})}{V(x_{\alpha_n})}-\tfrac{h(\hat{x})}{V(\hat{x})}\right|+\left|\tfrac{g(y_{\alpha_n})}{V(y_{\alpha_n})}-\tfrac{g(\hat{x})}{V(\hat{x})}\right|\right]=0. \tag{92}$$

Combining this with (88) and (89) shows that

$$\begin{aligned}&\lambda(w_2(\hat{x})-w_1(\hat{x}))\\&\leqslant \tfrac{1}{2}\operatorname{Trace}\Big(BB^*\tfrac{(\operatorname{Hess}V)(\hat{x})}{V(\hat{x})}\Big)(w_2(\hat{x})-w_1(\hat{x}))+\tfrac{f(\hat{x},V(\hat{x})w_2(\hat{x}))-f(\hat{x},V(\hat{x})w_1(\hat{x}))}{V(\hat{x})}+\tfrac{h(\hat{x})}{V(\hat{x})}-\tfrac{g(\hat{x})}{V(\hat{x})}.\end{aligned} \tag{93}$$

Next note that the second part of the statement of Hairer et al. [33, Lemma 4.9] (with $d\leftarrow 2d$, $O\leftarrow \mathbb{R}^d\times\mathbb{R}^d$, $\eta\leftarrow(\mathbb{R}^d\times\mathbb{R}^d\ni(x,y)\mapsto w_2(x)-w_1(y)\in\mathbb{R})$, $\phi\leftarrow(\mathbb{R}^d\times\mathbb{R}^d\ni(x,y)\mapsto\frac{1}{2}\|x-y\|^2\in[0,\infty))$, $x\leftarrow((0,\infty)\ni\alpha\mapsto(x_\alpha,y_\alpha)\in\mathbb{R}^d\times\mathbb{R}^d)$, $(\alpha_n)_{n\in\mathbb{N}}\leftarrow(\alpha_n)_{n\in\mathbb{N}}$, $x_0\leftarrow(\hat{x},\hat{x})$ in the notation of Hairer et al. [33, Lemma 4.9]) demonstrates that $w_2(\hat{x})-w_1(\hat{x})=\sup_{x\in\mathbb{R}^d}(w_2(x)-w_1(x))>0$. This, (93), (72), and the assumption that for all $x\in\mathbb{R}^d$, $a,b\in\mathbb{R}$ it holds that $[f(x,a)-f(x,b)](a-b)\leqslant L|a-b|^2$ ensure that

$$\begin{aligned}&\lambda\left[\sup_{z\in\mathbb{R}^d}(\max\{w_2(z)-w_1(z),0\})\right]\\&=\lambda\left(w_2(\hat{x})-w_1(\hat{x})\right)\leqslant\rho\left(w_2(\hat{x})-w_1(\hat{x})\right)+\tfrac{L[V(\hat{x})w_2(\hat{x})-V(\hat{x})w_1(\hat{x})]}{V(\hat{x})}+\tfrac{h(\hat{x})}{V(\hat{x})}-\tfrac{g(\hat{x})}{V(\hat{x})}\\&\leqslant(\rho+L)(w_2(\hat{x})-w_1(\hat{x}))+\sup_{z\in\mathbb{R}^d}\left[\max\left\{\tfrac{h(z)-g(z)}{V(z)},0\right\}\right]\\&=(\rho+L)\left[\sup_{z\in\mathbb{R}^d}(\max\{w_2(z)-w_1(z),0\})\right]+\sup_{z\in\mathbb{R}^d}\left[\max\left\{\tfrac{h(z)-g(z)}{V(z)},0\right\}\right].\end{aligned} \tag{94}$$

Hence, we obtain that

$$\Big[\lambda-(L+\rho)\Big]\left[\sup_{x\in\mathbb{R}^d}\left(\max\left\{\tfrac{v(x)-u(x)}{V(x)},0\right\}\right)\right]\leqslant\sup_{x\in\mathbb{R}^d}\left[\max\left\{\tfrac{h(x)-g(x)}{V(x)},0\right\}\right]. \tag{95}$$

This establishes (75). The proof of Proposition 2.12 is thus completed. □

2.4 Existence and uniqueness results for viscosity solutions of semilinear elliptic PDEs

Proposition 2.13 (Existence and uniqueness of viscosity solutions)**.** *Let $d,m\in\mathbb{N}$, $B\in\mathbb{R}^{d\times m}$, $L,\rho\in\mathbb{R}$, $\lambda\in(\rho+L,\infty)$, let $\|\cdot\|\colon\mathbb{R}^d\to[0,\infty)$ be a norm on $\mathbb{R}^d$, let $f\in C(\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $V\in C^2(\mathbb{R}^d,(0,\infty))$ satisfy for all $x\in\mathbb{R}^d$, $v,w\in\mathbb{R}$ that $|f(x,v)-f(x,w)|\leqslant L|v-w|$ and*

$$\tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess}V)(x))\leqslant\rho V(x), \tag{96}$$

assume that $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|f(x,0)|}{V(x)})]=0$ and $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r}V(x)]=\infty$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $W\colon[0,\infty)\times\Omega\to\mathbb{R}^m$ be a standard Brownian motion. Then *(i) there exists a unique* $u \in \{v \in C(\mathbb{R}^d, \mathbb{R})\colon \inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|v(x)|}{V(x)})] = 0\}$ *which satisfies that* u *is a viscosity solution of*

$$\lambda u(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \tag{97}$$

for $x \in \mathbb{R}^d$ *and*

(ii) it holds for all $x \in \mathbb{R}^d$ *that* $\mathbb{E}\big[\int_0^\infty e^{-\lambda t}|f(x + BW_t, u(x + BW_t))|\,dt\big] < \infty$ *and*

$$u(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} f(x + BW_t, u(x + BW_t))\,dt\bigg]. \tag{98}$$

Proof of Proposition 2.13. First, observe that Corollary 2.7 (with $d \leftarrow d$, $m \leftarrow m$, $B \leftarrow B$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\|\cdot\| \leftarrow \|\cdot\|$, $f \leftarrow f$, $V \leftarrow V$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ in the notation of Corollary 2.7) guarantees that there exists $u \in C(\mathbb{R}^d, \mathbb{R})$ which satisfies for all $x \in \mathbb{R}^d$ that $\limsup_{r\to\infty}[\sup_{\|y\|>r}(\frac{|u(y)|}{V(y)})] = 0$, $\mathbb{E}\big[\int_0^\infty e^{-\lambda t}|f(x + BW_t, u(x + BW_t))|\,dt\big] < \infty$, and

$$u(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t} f(x + BW_t, u(x + BW_t))\,dt\bigg]. \tag{99}$$

Corollary 2.11 (with $d \leftarrow d$, $m \leftarrow m$, $B \leftarrow B$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\|\cdot\| \leftarrow \|\cdot\|$, $f \leftarrow f$, $u \leftarrow u$, $V \leftarrow V$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ in the notation of Corollary 2.11) therefore implies that u is a viscosity solution of

$$\lambda u(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \tag{100}$$

for $x \in \mathbb{R}^d$. Furthermore, observe that Proposition 2.12 (with $d \leftarrow d$, $m \leftarrow m$, $B \leftarrow B$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $f \leftarrow f$, $g \leftarrow (\mathbb{R}^d \ni x \mapsto 0 \in \mathbb{R})$, $h \leftarrow (\mathbb{R}^d \ni x \mapsto 0 \in \mathbb{R})$, $V \leftarrow V$ in the notation of Proposition 2.12) demonstrates that for every $v \in \{w \in C(\mathbb{R}^d, \mathbb{R})\colon \inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|w(x)|}{V(x)})] = 0\}$ which satisfies that v is a viscosity solution of

$$\lambda v(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} v)(x)\big) = f(x, v(x)) \tag{101}$$

for $x \in \mathbb{R}^d$ it holds that $u = v$. Combining this with (99) and (100) establishes Item (i). In addition, note that (99), (100), and (101) establish Item (ii). The proof of Proposition 2.13 is thus completed. ☐

Corollary 2.14. *Let* $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $L, \rho \in \mathbb{R}$, $\lambda \in (\rho + L, \infty)$, *let* $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ *be a norm on* $\mathbb{R}^d$, *let* $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $V \in C^2(\mathbb{R}^d, (0, \infty))$ *satisfy for all* $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ *that* $|f(x, v) - f(x, w) - \lambda(v - w)| \leqslant L|v - w|$ *and* $\frac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} V)(x)) \leqslant \rho V(x)$, *assume that* $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|f(x,0)|}{V(x)})] = 0$ *and* $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V(x)] = \infty$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* $W\colon [0, \infty) \times \Omega \to \mathbb{R}^m$ *be a standard Brownian motion. Then*

(i) there exists a unique $u \in \{v \in C(\mathbb{R}^d, \mathbb{R})\colon \inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|v(x)|}{V(x)})] = 0\}$ *which satisfies that* u *is a viscosity solution of*

$$\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \tag{102}$$

for $x \in \mathbb{R}^d$ *and*

(ii) it holds for all $x \in \mathbb{R}^d$ *that* $\mathbb{E}\big[\int_0^\infty e^{-\lambda t}|\lambda u(x + BW_t) - f(x + BW_t, u(x + BW_t))|\,dt\big] < \infty$ *and*

$$u(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t}(\lambda u(x + BW_t) - f(x + BW_t, u(x + BW_t)))\,dt\bigg]. \tag{103}$$

Proof of Corollary 2.14. Throughout this proof let $g\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ that $g(x, v) = \lambda v - f(x, v)$. Note that the assumption that for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ it holds that $|f(x, v) - f(x, w) - \lambda(v - w)| \leqslant L|v - w|$ ensures that for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ it holds that $|g(x, v) - g(x, w)| \leqslant L|v - w|$. Moreover, observe that the assumption that $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|f(x,0)|}{V(x)})] = 0$ implies that $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|g(x,0)|}{V(x)})] = 0$. In addition, note that for all $u \in C(\mathbb{R}^d, \mathbb{R})$ it holds that

$$\begin{pmatrix} u \text{ is a viscosity solution of} \\ \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \\ \text{for } x \in \mathbb{R}^d \end{pmatrix} \Leftrightarrow \begin{pmatrix} u \text{ is a viscosity solution of} \\ \lambda u(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = g(x, u(x)) \\ \text{for } x \in \mathbb{R}^d \end{pmatrix}. \tag{104}$$

Proposition 2.13 (with $d \leftarrow d$, $m \leftarrow m$, $B \leftarrow B$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\|\cdot\| \leftarrow \|\cdot\|$, $f \leftarrow g$, $V \leftarrow V$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ in the notation of Proposition 2.13) therefore establishes Items (i) and (ii). The proof of Corollary 2.14 is thus completed. ∎

Lemma 2.15. *Let $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $\varepsilon \in (0, \infty)$, $\|\cdot\| : (\cup_{k\in\mathbb{N}}\mathbb{R}^k) \to [0, \infty)$ satisfy for all $k \in \mathbb{N}$, $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ that $\|x\| = [\sum_{i=1}^k |x_i|^2]^{1/2}$ and let $V\colon \mathbb{R}^d \to (0, \infty)$ satisfy for all $x \in \mathbb{R}^d$ that*

$$V(x) = \exp\big(\varepsilon(1 + \|x\|^2)^{1/2}\big). \tag{105}$$

Then

(i) it holds for all $x \in \mathbb{R}^d$ that

$$\frac{\|B^*(\nabla V)(x)\|^2}{V(x)} \leqslant \varepsilon^2 \left[\sup_{y\in\mathbb{R}^m\setminus\{0\}} \left(\frac{\|By\|}{\|y\|}\right)\right]^2 V(x) \tag{106}$$

and

(ii) it holds for all $x \in \mathbb{R}^d$ that

$$\operatorname{Trace}\big(BB^*(\operatorname{Hess} V)(x)\big) \leqslant (\varepsilon^2 + \varepsilon d) \left[\sup_{y\in\mathbb{R}^m\setminus\{0\}} \left(\frac{\|By\|}{\|y\|}\right)\right]^2 V(x). \tag{107}$$

Proof of Lemma 2.15. Throughout this proof let $|||\cdot||| : \mathbb{R}^{d\times m} \to [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d\times m}$. Observe that for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} (\nabla V)(x) &= \frac{\varepsilon x}{(1 + \|x\|^2)^{1/2}} V(x) \qquad \text{and} \\ (\operatorname{Hess} V)(x) &= \left[\frac{\varepsilon^2 x \otimes x}{1 + \|x\|^2} + \frac{\varepsilon \operatorname{Id}_{\mathbb{R}^d}}{(1 + \|x\|^2)^{1/2}} - \frac{\varepsilon x \otimes x}{(1 + \|x\|^2)^{3/2}}\right] V(x). \end{aligned} \tag{108}$$

Hence, we obtain for all $x \in \mathbb{R}^d$ that

$$\frac{\|B^*(\nabla V)(x)\|^2}{V(x)} = \frac{\varepsilon^2 \|B^* x\|^2}{1 + \|x\|^2} V(x) \leqslant \varepsilon^2 \left[\sup_{y\in\mathbb{R}^m\setminus\{0\}} \left(\frac{\|By\|}{\|y\|}\right)\right]^2 V(x). \tag{109}$$

This establishes Item (i). Moreover, note that (108) demonstrates for all $x \in \mathbb{R}^d$ that

$$\begin{aligned}\operatorname{Trace}\big(BB^*(\operatorname{Hess} V)(x)\big) &= \left[\frac{\varepsilon^2 \|B^*x\|^2}{1+\|x\|^2} + \frac{\varepsilon \,|\!|\!|B|\!|\!|^2}{(1+\|x\|^2)^{1/2}} - \frac{\varepsilon \|B^*x\|^2}{(1+\|x\|^2)^{3/2}}\right] V(x) \\ &\leqslant (\varepsilon^2+\varepsilon d)\left[\sup_{y\in\mathbb{R}^m\setminus\{0\}}\left(\frac{\|By\|}{\|y\|}\right)\right]^2 V(x).\end{aligned} \tag{110}$$

This establishes Item (ii). The proof of Lemma 2.15 is thus completed. ☐

Corollary 2.16. *Let $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $L \in \mathbb{R}$, $\lambda \in (L,\infty)$, $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v)-f(x,w)-\lambda(v-w)| \leqslant L|v-w|$, assume that f is at most polynomially growing, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion. Then*

(i) there exists a unique $u \in \{v \in C(\mathbb{R}^d,\mathbb{R})\colon (\forall\, \varepsilon \in (0,\infty)\colon [\sup_{x=(x_1,\dots,x_d)\in\mathbb{R}^d}(\frac{|v(x)|}{\exp(\varepsilon\sum_{i=1}^d |x_i|)})] < \infty)\}$ which satisfies that u is a viscosity solution of

$$\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x,u(x)) \tag{111}$$

for $x \in \mathbb{R}^d$ and

(ii) it holds for all $x \in \mathbb{R}^d$ that $\mathbb{E}\big[\int_0^\infty e^{-\lambda t}|\lambda u(x+BW_t) - f(x+BW_t, u(x+BW_t))|\,dt\big] < \infty$ and

$$u(x) = \mathbb{E}\left[\int_0^\infty e^{-\lambda t}(\lambda u(x+BW_t) - f(x+BW_t,u(x+BW_t)))\,dt\right]. \tag{112}$$

Proof of Corollary 2.16. Throughout this proof let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $\beta, p \in [0,\infty)$ satisfy

$$\sup_{x\in\mathbb{R}^d}\left(\frac{|f(x,0)|}{1+\|x\|^p}\right) < \infty \qquad \text{and} \qquad \beta = \tfrac{1}{2}\left[\sup_{x=(x_1,\dots,x_m)\in\mathbb{R}^m\setminus\{0\}}\left(\frac{\|Bx\|^2}{\sum_{i=1}^m |x_i|^2}\right)\right], \tag{113}$$

and let $V_\varepsilon\colon \mathbb{R}^d \to (0,\infty)$, $\varepsilon \in (0,\infty)$, satisfy for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ that

$$V_\varepsilon(x) = \exp\big(\varepsilon(1+\|x\|^2)^{1/2}\big). \tag{114}$$

Observe that Item (ii) in Lemma 2.15 (with $d \leftarrow d$, $m \leftarrow m$, $B \leftarrow B$, $\varepsilon \leftarrow \varepsilon$ for $\varepsilon \in (0,\infty)$ in the notation of Lemma 2.15) demonstrates for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ that

$$\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} V_\varepsilon)(x)\big) \leqslant (\varepsilon^2+\varepsilon d)\beta V_\varepsilon(x). \tag{115}$$

In addition, note that for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ it holds that

$$\exp(\varepsilon\|x\|) \leqslant V_\varepsilon(x) \leqslant e^\varepsilon \exp(\varepsilon\|x\|). \tag{116}$$

Moreover, note that for all $\varepsilon \in (0,\infty)$ it holds that $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V_\varepsilon(x)] = \infty$ and

$$\begin{aligned}\limsup_{r\to\infty}\left[\sup_{\|x\|>r}\left(\frac{|f(x,0)|}{V_\varepsilon(x)}\right)\right] &= \limsup_{r\to\infty}\left[\sup_{\|x\|>r}\left(\frac{|f(x,0)|}{1+\|x\|^p}\frac{1+\|x\|^p}{V_\varepsilon(x)}\right)\right] \\ &\leqslant \limsup_{r\to\infty}\left(\left[\sup_{x\in\mathbb{R}^d}\left(\frac{|f(x,0)|}{1+\|x\|^p}\right)\right]\left[\sup_{\|x\|>r}\left(\frac{1+\|x\|^p}{V_\varepsilon(x)}\right)\right]\right) \\ &\leqslant \left[\sup_{x\in\mathbb{R}^d}\left(\frac{|f(x,0)|}{1+\|x\|^p}\right)\right]\left[\limsup_{r\to\infty}\left(\sup_{\|x\|>r}\left(\frac{1+\|x\|^p}{\exp(\varepsilon\|x\|)}\right)\right)\right] = 0.\end{aligned} \tag{117}$$

This, (115), (116), and Corollary 2.14 (with $d \leftarrow d$, $m \leftarrow m$, $B \leftarrow B$, $L \leftarrow L$, $\rho \leftarrow (\varepsilon^2 + \varepsilon d)\beta$, $\lambda \leftarrow \lambda$, $f \leftarrow f$, $V \leftarrow V_\varepsilon$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ for $\varepsilon \in (0,\infty)$ in the notation of Corollary 2.14) ensure for every $\varepsilon \in (0,\infty)$ with $(\varepsilon^2 + \varepsilon d)\beta < \lambda - L$ that there exists a unique $u_\varepsilon \in \{v \in C(\mathbb{R}^d, \mathbb{R})\colon \inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|v(x)|}{V_\varepsilon(x)})] = 0\}$ which satisfies that u_ε is a viscosity solution of

$$\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u_\varepsilon)(x)\big) = f(x, u_\varepsilon(x)) \tag{118}$$

for $x \in \mathbb{R}^d$. Corollary 2.14 hence assures for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ with $(\varepsilon^2 + \varepsilon d)\beta < \lambda - L$ that

$$u_\varepsilon(x) = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda t}(\lambda u_\varepsilon(x + BW_t) - f(x + BW_t, u_\varepsilon(x + BW_t)))\, dt\bigg]. \tag{119}$$

Next note that the fact that for all $\varepsilon \in (0,\infty)$, $\eta \in (0,\varepsilon)$, $x \in \mathbb{R}^d$ it holds that $V_\eta(x) \leqslant V_\varepsilon(x)$ and (118) ensure that for all $\varepsilon \in (0,\infty)$, $\eta \in (0,\varepsilon)$, $x \in \mathbb{R}^d$ with $(\varepsilon^2 + \varepsilon d)\beta < \lambda - L$ it holds that $u_\varepsilon(x) = u_\eta(x)$. Hence, we obtain that there exists $\mathfrak{u}\colon \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ with $(\varepsilon^2 + \varepsilon d)\beta < \lambda - L$ that $\mathfrak{u}(x) = u_\varepsilon(x)$. This and (118) ensure that for every $v \in \{w \in C(\mathbb{R}^d, \mathbb{R})\colon (\forall\, \varepsilon \in (0,\infty)\colon \sup_{x\in\mathbb{R}^d}[(\frac{|w(x)|}{V_\varepsilon(x)})] < \infty)\}$ which satisfies that v is a viscosity solution of

$$\tfrac{1}{2}\operatorname{Trace}(BB^*(\operatorname{Hess} v)(x)) = f(x, v(x)) \tag{120}$$

for $x \in \mathbb{R}^d$ it holds that $v = \mathfrak{u}$. This and (118) establish Item (i). Next note that (119) and the fact that there exist $\varepsilon_n \in (0,\infty)$, $n \in \mathbb{N}$, with $\limsup_{n\to\infty} \varepsilon_n = 0$ such that $\mathfrak{u} = u_{\varepsilon_n}$ establish Item (ii). The proof of Corollary 2.16 is thus completed. □

2.5 A priori estimates for solutions of SFPEs

Proposition 2.17 (A priori estimate). *Let $d, m \in \mathbb{N}$, $B \in \mathbb{R}^{d\times m}$, $L \in \mathbb{R}$, $\lambda \in (L,\infty)$, $\varepsilon \in (L^2/\lambda, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,\infty) \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w)| \leqslant L|v - w|$ and $\int_0^\infty e^{-\lambda t}\,\mathbb{E}[|f(x + BW_t, 0)|]\, dt < \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, and assume for all $x \in \mathbb{R}^d$ that $\int_0^\infty e^{(\varepsilon-\lambda)t}\,\mathbb{E}[|u(x + BW_t)|^2]\, dt < \infty$ and*

$$u(x) = \int_0^\infty e^{-\lambda t}\,\mathbb{E}[f(x + BW_t, u(x + BW_t))]\, dt. \tag{121}$$

Then

(i) it holds for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$ that

$$e^{-\lambda t}\,\mathbb{E}\big[|u(x + BW_t)|^2\big] \leqslant \frac{1}{\lambda}\int_t^\infty e^{-\lambda s}\,\mathbb{E}\big[|f(x + BW_s, u(x + BW_s))|^2\big]\, ds, \tag{122}$$

(ii) it holds for all $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\bigg[\int_0^\infty e^{(\varepsilon-\lambda)t}\,\mathbb{E}\big[|u(x + BW_t)|^2\big]\, dt\bigg]^{1/2} \\ &\leqslant \frac{1}{\sqrt{\varepsilon\lambda} - L}\bigg[\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}\big[|f(x + BW_s, 0)|^2\big]\, ds\bigg]^{1/2}, \end{aligned} \tag{123}$$

and

(iii) it holds for all $x \in \mathbb{R}^d$ that

$$|u(x)| \leqslant \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon\lambda} - L} \left[\int_0^\infty e^{(\varepsilon-\lambda)t} \, \mathbb{E}\big[|f(x + BW_t, 0)|^2 \big] \, dt \right]^{1/2}. \tag{124}$$

Proof of Proposition 2.17. Throughout this proof let $\nu_{t,x} \colon \mathcal{B}(\mathbb{R}^d) \to [0,1]$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$, satisfy for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_{t,x}(A) = \mathbb{P}(x + BW_t \in A)$. Observe that (121) and the Cauchy-Schwarz inequality ensure for all $x \in \mathbb{R}^d$ that

$$\begin{aligned} |u(x)|^2 &\leqslant \left[\int_0^\infty e^{-\lambda t} \, dt \right] \left[\int_0^\infty e^{-\lambda t} \, |\mathbb{E}[f(x + BW_t, u(x + BW_t))]|^2 \, dt \right] \\ &\leqslant \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} \, \mathbb{E}\big[|f(x + BW_t, u(x + BW_t))|^2 \big] \, dt. \end{aligned} \tag{125}$$

This, Fubini's theorem, and the fact that for all $t, s \in [0,\infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that $\nu_{t+s,x}(A) = \int_{\mathbb{R}^d} \nu_{s,y}(A) \, \nu_{t,x}(dy)$ ensure for all $t \in [0,\infty)$ $x \in \mathbb{R}^d$ that

$$\begin{aligned} \mathbb{E}\big[|u(x + BW_t)|^2 \big] &= \int_{\mathbb{R}^d} |u(y)|^2 \, \nu_{t,x}(dy) \\ &\leqslant \frac{1}{\lambda} \int_{\mathbb{R}^d} \int_0^\infty e^{-\lambda s} \, \mathbb{E}\big[|f(y + BW_s, u(y + BW_s))|^2 \big] \, ds \, \nu_{t,x}(dy) \\ &= \frac{1}{\lambda} \int_0^\infty e^{-\lambda s} \, \mathbb{E}\big[|f(x + BW_{t+s}, u(x + BW_{t+s}))|^2 \big] \, ds \\ &= \frac{e^{\lambda t}}{\lambda} \int_t^\infty e^{-\lambda s} \, \mathbb{E}\big[|f(x + BW_s, u(x + BW_s))|^2 \big] \, ds. \end{aligned} \tag{126}$$

This establishes Item (i). Combining Item (i) with Fubini's theorem ensures for all $x \in \mathbb{R}^d$ that

$$\begin{aligned} \int_0^\infty e^{(\varepsilon-\lambda)t} \, \mathbb{E}\big[|u(x + BW_t)|^2 \big] \, dt &\leqslant \int_0^\infty \frac{e^{\varepsilon t}}{\lambda} \int_t^\infty e^{-\lambda s} \, \mathbb{E}\big[|f(x + BW_s, u(x + BW_s))|^2 \big] \, ds \, dt \\ &= \int_0^\infty \left[\int_0^s \frac{e^{\varepsilon t}}{\lambda} \, dt \right] e^{-\lambda s} \, \mathbb{E}\big[|f(x + BW_s, u(x + BW_s))|^2 \big] \, ds \\ &\leqslant \frac{1}{\varepsilon\lambda} \int_0^\infty e^{(\varepsilon-\lambda)s} \, \mathbb{E}\big[|f(x + BW_s, u(x + BW_s))|^2 \big] \, ds. \end{aligned} \tag{127}$$

Minkowski's inequality hence shows for all $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\left[\int_0^\infty e^{(\varepsilon-\lambda)t} \, \mathbb{E}\big[|u(x + BW_t)|^2 \big] \, dt \right]^{1/2} \\ &\leqslant \frac{1}{\sqrt{\varepsilon\lambda}} \left[\int_0^\infty e^{(\varepsilon-\lambda)s} \, \mathbb{E}\big[|f(x + BW_s, u(x + BW_s))|^2 \big] \, ds \right]^{1/2} \\ &\leqslant \frac{1}{\sqrt{\varepsilon\lambda}} \left[\int_0^\infty e^{(\varepsilon-\lambda)s} \, \mathbb{E}\big[|f(x + BW_s, 0)|^2 \big] \, ds \right]^{1/2} + \frac{L}{\sqrt{\varepsilon\lambda}} \left[\int_0^\infty e^{(\varepsilon-\lambda)s} \, \mathbb{E}\big[|u(x + BW_s)|^2 \big] \, ds \right]^{1/2}. \end{aligned} \tag{128}$$

This implies for all $x \in \mathbb{R}^d$ that

$$\left[\int_0^\infty e^{(\varepsilon-\lambda)t} \, \mathbb{E}\big[|u(x + BW_t)|^2 \big] \, dt \right]^{1/2} \leqslant \frac{1}{\sqrt{\varepsilon\lambda} - L} \left[\int_0^\infty e^{(\varepsilon-\lambda)t} \, \mathbb{E}\big[|f(x + BW_t, 0)|^2 \big] \, dt \right]^{1/2}. \tag{129}$$

This establishes Item (ii). Next observe that the triangle inequality, Fubini's theorem, the assumption that for all $x \in \mathbb{R}^d$, $a, b \in \mathbb{R}$ it holds that $|f(x,a) - f(x,b)| \leqslant L|a-b|$, and (121) ensure for all $x \in \mathbb{R}^d$ that

$$|u(x)| \leqslant \int_0^\infty e^{-\lambda t}\, \mathbb{E}[|f(x + BW_t, 0)|]\, dt + L \int_0^\infty e^{-\lambda t}\, \mathbb{E}[|u(x + BW_t)|]\, dt. \tag{130}$$

The Cauchy-Schwarz inequality and Item (ii) hence prove for all $x \in \mathbb{R}^d$ that

$$\begin{aligned}
|u(x)| &\leqslant \frac{1}{\sqrt{\lambda}} \left[\int_0^\infty e^{-\lambda t}\, \mathbb{E}\big[|f(x + BW_t, 0)|^2\big]\, dt\right]^{1/2} + \frac{L}{\sqrt{\lambda}} \left[\int_0^\infty e^{-\lambda t}\, \mathbb{E}\big[|u(x + BW_t)|^2\big]\, dt\right]^{1/2} \\
&\leqslant \frac{1}{\sqrt{\lambda}} \left[1 + \frac{L}{\sqrt{\varepsilon\lambda} - L}\right] \left[\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|f(x + BW_t, 0)|^2\big]\, dt\right]^{1/2} \\
&= \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon\lambda} - L} \left[\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|f(x + BW_t, 0)|^2\big]\, dt\right]^{1/2}.
\end{aligned} \tag{131}$$

This establishes Item (iii). The proof of Proposition 2.17 is thus completed. □

3 Full-history recursive multilevel Picard (MLP) approximations

In this section we introduce and analyze MLP approximation schemes for SFPEs related to semilinear elliptic PDEs (see Setting 3.1 in Section 3.1 below). In Sections 3.2–3.3 we establish some rather technical results concerning measurability, distributional, and integrability properties of MLP approximations. Section 3.4 contains fundamental estimates for the biases and variances of MLP approximations (see Lemmas 3.6 and 3.7 below). These fundamental estimates for the biases and variances of MLP approximations are merged in Proposition 3.9 and Corollary 3.10 in Section 3.5 below to obtain a full error analysis for MLP approximations in Corollary 3.11 in Section 3.5 below. In Section 3.6 we provide an upper bound for the computational cost to sample realizations of MLP approximations. In Section 3.7 we finally relate the error analysis for MLP approximations established in Corollary 3.11 in Section 3.5 with the computational cost analysis for MLP approximations established in Section 3.6 to obtain in Theorem 3.16 in Section 3.7 an overall complexity analysis for the proposed MLP approximation schemes.

3.1 MLP approximations

Setting 3.1. *Let $d, M \in \mathbb{N}$, $\lambda \in (0,\infty)$, $\Theta = \cup_{n\in\mathbb{N}} \mathbb{Z}^n$, $B \in \mathbb{R}^{d\times d}$, $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W^\theta \colon [0,\infty) \times \Omega \to \mathbb{R}^d$, $\theta \in \Theta$, be i.i.d. standard Brownian motions, let $R^\theta \colon \Omega \to [0,\infty)$, $\theta \in \Theta$, be i.i.d. random variables which satisfy for all $t \in [0,\infty)$ that $\mathbb{P}(R^0 \geqslant t) = e^{-\lambda t}$, assume that $(R^\theta)_{\theta\in\Theta}$ and $(W^\theta)_{\theta\in\Theta}$ are independent, and let $U_n^\theta = (U_n^\theta(x))_{x\in\mathbb{R}^d} \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, satisfy for all $\theta \in \Theta$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $U_0^\theta(x) = 0$ and*

$$\begin{aligned}
U_n^\theta(x) = \sum_{k=1}^{n-1} \frac{1}{\lambda M^{(n-k)}} \Bigg[\sum_{m=1}^{M^{(n-k)}} \Big(& f\big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{(\theta,k,m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\big) \\
& - f\big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{(\theta,k,-m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\big)\Big)\Bigg] + \frac{1}{\lambda M^n} \Bigg[\sum_{m=1}^{M^n} f\big(x + BW^{(\theta,0,m)}_{R^{(\theta,0,m)}}, 0\big)\Bigg].
\end{aligned} \tag{132}$$

3.2 Measurability properties of MLP approximations

Lemma 3.2. *Assume Setting 3.1. Then*

(i) it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$ that $U_n^\theta \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a continuous random field,

(ii) it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$ that $\sigma_\Omega(U_n^\theta) \subseteq \sigma_\Omega((R^{(\theta,\vartheta)})_{\vartheta\in\Theta}, (W^{(\theta,\vartheta)})_{\vartheta\in\Theta})$,

(iii) it holds that $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto x + BW^\theta_{R^\theta(\omega)}(\omega) \in \mathbb{R}^d$, $\theta \in \Theta$, are identically distributed random fields, and

(iv) it holds for all $n \in \mathbb{N}_0$ that U_n^θ, $\theta \in \Theta$, are identically distributed random fields.

Proof of Lemma 3.2. First, we prove Item (i). For this let $\mathbb{A}$ and $\mathbb{B}$ be the sets given by

$$\mathbb{A} = \left\{ n \in \mathbb{N}_0 \colon \left(\begin{array}{c} \text{For all } \theta \in \Theta \text{ it holds that} \\ U_n^\theta \colon \mathbb{R}^d \times \Omega \to \mathbb{R} \text{ is a continuous random field} \end{array} \right) \right\} \tag{133}$$

and $\mathbb{B} = \{n \in \mathbb{N} \colon \{0,1,\ldots,n-1\} \subseteq \mathbb{A}\}$. Observe that the assumption that for all $\theta \in \Theta$, $x \in \mathbb{R}^d$ it holds that $U_0^\theta(x) = 0$ shows that $0 \in \mathbb{A}$. Hence, we obtain that $1 \in \mathbb{B}$. Next note that the assumption that W^θ, $\theta \in \Theta$, are Brownian motions, the assumption that $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, and (132) demonstrate for all $n \in \mathbb{B}$, $\theta \in \Theta$ that $U_n^\theta \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a continuous random field. Hence, we obtain for every $n \in \mathbb{B}$ that $n \in \mathbb{A}$. This shows for all $n \in \mathbb{N}$ that $(n \in \mathbb{B} \Rightarrow n+1 \in \mathbb{B})$. Combining this with the fact that $1 \in \mathbb{B}$ and induction demonstrates that $\mathbb{N} \subseteq \mathbb{B}$. Hence, we obtain that $\mathbb{N}_0 \subseteq \mathbb{A}$. This establishes Item (i). Next we prove Item (ii). For this let $\mathbb{A}$ and $\mathbb{B}$ be the sets given by

$$\mathbb{A} = \left\{ n \in \mathbb{N}_0 \colon \left(\begin{array}{c} \text{For all } \theta \in \Theta \text{ it holds that} \\ \sigma_\Omega(U_n^\theta) \subseteq \sigma_\Omega((R^{(\theta,\vartheta)})_{\vartheta\in\Theta}, (W^{(\theta,\vartheta)})_{\vartheta\in\Theta}) \end{array} \right) \right\} \tag{134}$$

and $\mathbb{B} = \{n \in \mathbb{N} \colon \{0,1,\ldots,n-1\} \subseteq \mathbb{A}\}$. Observe that the assumption that for all $\theta \in \Theta$, $x \in \mathbb{R}^d$ it holds that $U_0^\theta(x) = 0$ shows that $0 \in \mathbb{A}$. This implies that $1 \in \mathbb{B}$. Next note that Item (i) ensures for all $n \in \mathbb{N}_0$, $\theta \in \Theta$ that $U_n^\theta \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is $(\mathcal{B}(\mathbb{R}^d) \otimes \sigma_\Omega(U_n^\theta))/\mathcal{B}(\mathbb{R})$-measurable. Combining this with the fact that for every $\theta \in \Theta$ it holds that $W^\theta \colon [0,\infty) \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}([0,\infty)) \otimes \sigma_\Omega(W^\theta))/\mathcal{B}(\mathbb{R}^d)$-measurable, the fact that for every $\theta \in \Theta$ it holds that $R^\theta \colon \Omega \to [0,\infty)$ is $\sigma_\Omega(R^\theta)/\mathcal{B}([0,\infty))$-measurable, the assumption that $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, and (132) demonstrates for all $n \in \mathbb{B}$, $\theta \in \Theta$ that $U_n^\theta \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is $(\mathcal{B}(\mathbb{R}^d) \otimes \sigma_\Omega((R^{(\theta,\vartheta)})_{\vartheta\in\Theta}, (W^{(\theta,\vartheta)})_{\vartheta\in\Theta}))/\mathcal{B}(\mathbb{R})$-measurable. This shows that for every $n \in \mathbb{B}$ it holds that $n \in \mathbb{A}$. Hence, we obtain for all $n \in \mathbb{N}$ that $(n \in \mathbb{B} \Rightarrow n+1 \in \mathbb{B})$. This, the fact that $1 \in \mathbb{B}$, and induction demonstrate that $\mathbb{N} \subseteq \mathbb{B}$. Therefore, we obtain that $\mathbb{N}_0 \subseteq \mathbb{A}$. This establishes Item (ii). Next we prove Item (iii). For this note that Hutzenthaler et al. [44, Corollary 2.5] (with $U_1 \leftarrow ([0,\infty)\times\Omega \ni (t,\omega) \mapsto \varphi(W_t^\theta(\omega)) \in \mathbb{R})$, $U_2 \leftarrow ([0,\infty)\times\Omega \ni (t,\omega) \mapsto \varphi(W_t^0(\omega)) \in \mathbb{R})$, $Y_1 \leftarrow (\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto R^\theta(\omega) \in [0,\infty))$, $Y_2 \leftarrow (\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto R^0(\omega) \in [0,\infty))$ for $\theta \in \Theta$, $\varphi = (\mathbb{R}^d \ni (x_1,x_2,\ldots,x_d) \mapsto \sum_{i=1}^d a_i x_i \in \mathbb{R})$ for $a_1,a_2,\ldots,a_d \in \mathbb{R}$ in the notation of Hutzenthaler et al. [44, Corollary 2.5]) ensures for all $\theta \in \Theta$ that $W^\theta_{R^\theta}$ and $W^0_{R^0}$ are identically distributed random variables. Hence, we obtain for all $\theta \in \Theta$ that $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto x + BW^\theta_{R^\theta(\omega)}(\omega) \in \mathbb{R}^d$ and $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto x + BW^0_{R^0(\omega)}(\omega) \in \mathbb{R}^d$ are identically distributed random fields. This establishes Item (iii). Next we prove Item (iv). For this let $\mathbb{A}$ and $\mathbb{B}$ be the sets given by

$$\mathbb{A} = \left\{ n \in \mathbb{N}_0 \colon \left(\begin{array}{c} U_n^\theta \colon \mathbb{R}^d \times \Omega \to \mathbb{R}, \theta \in \Theta, \\ \text{are identically distributed random fields} \end{array} \right) \right\} \tag{135}$$

and $\mathbb{B} = \{n \in \mathbb{N} \colon \{0,1,\ldots,n-1\} \subseteq \mathbb{A}\}$. Observe that the assumption that for all $\theta \in \Theta$, $x \in \mathbb{R}^d$ it holds that $U_0^\theta(x) = 0$ shows that $0 \in \mathbb{A}$. This implies that $1 \in \mathbb{B}$. Next note that Item (i), Item (ii), Item (iii), and Hutzenthaler et al. [44, Corollary 2.5] ensure for all $n \in \mathbb{B}$, $\theta \in \Theta$ that U_n^θ and U_n^0 are identically distributed random fields. This demonstrates that for all $n \in \mathbb{B}$ it holds that $n \in \mathbb{A}$. Hence, we obtain that for all $n \in \mathbb{N}$ it holds that $(n \in \mathbb{B} \Rightarrow n+1 \in \mathbb{B})$. This, the fact that $1 \in \mathbb{B}$, and induction establish Item (iv). The proof of Lemma 3.2 is thus completed. □

3.3 Integrability properties of MLP approximations

Lemma 3.3. *Assume Setting 3.1, let $p \in [1,\infty)$, $k \in \mathbb{N}_0$, $\theta, \vartheta \in \Theta$, assume that $\theta \notin \{(\vartheta,\eta) \colon \eta \in \Theta\}$, and let $\nu_{t,x} \colon \mathcal{B}(\mathbb{R}^d) \to [0,\infty)$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$, satisfy for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_{t,x}(A) = \mathbb{P}(x + BW_t^0 \in A)$. Then*

(i) it holds for all $x \in \mathbb{R}^d$ that

$$\begin{aligned} \mathbb{E}\big[|U_k^\vartheta(x + BW_{R^\theta}^\theta)|^p\big] &= \int_0^\infty \mathbb{E}\big[|U_k^\vartheta(x + BW_s^\theta)|^p\big] \big(R^\theta(\mathbb{P})\big)(ds) \\ &= \int_0^\infty \lambda e^{-\lambda s}\, \mathbb{E}\big[|U_k^\vartheta(x + BW_s^\theta)|^p\big]\, ds \end{aligned} \tag{136}$$

and

(ii) it holds for all $x \in \mathbb{R}^d$, $t \in [0,\infty)$ that

$$\mathbb{E}\big[|U_k^\vartheta(x + BW_t^\theta)|^p\big] = \int_{\mathbb{R}^d} \mathbb{E}\big[|U_k^0(y)|^p\big]\, \nu_{t,x}(dy). \tag{137}$$

Proof of Lemma 3.3. First, observe that Item (ii) in Lemma 3.2 and the fact that $W^\theta \colon [0,\infty) \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}([0,\infty)) \otimes \sigma_\Omega(W^\theta))/\mathcal{B}(\mathbb{R}^d)$-measurable ensure for all $x \in \mathbb{R}^d$ that

$$[0,\infty) \times \Omega \ni (t,\omega) \mapsto |U_k^\vartheta(x + BW_t^\theta(\omega), \omega)|^p \in [0,\infty) \tag{138}$$

is $(\mathcal{B}([0,\infty)) \otimes \sigma_\Omega(W^\theta, (W^{(\vartheta,\eta)})_{\eta\in\Theta}, (R^{(\vartheta,\eta)})_{\eta\in\Theta}))/\mathcal{B}([0,\infty))$-measurable. In addition, note that the assumption that $\theta \notin \{(\vartheta,\eta) \colon \eta \in \Theta\}$ proves that R^θ is independent of $\sigma_\Omega(W^\theta, (W^{(\vartheta,\eta)})_{\eta\in\Theta}, (R^{(\vartheta,\eta)})_{\eta\in\Theta})$. Hutzenthaler et al. [44, Lemma 2.2] (with $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{G} \leftarrow \sigma_\Omega(W^\theta, (W^{(\vartheta,\eta)})_{\eta\in\Theta}, (R^{(\vartheta,\eta)})_{\eta\in\Theta})$, $(S,\mathcal{S}) \leftarrow ([0,\infty), \mathcal{B}([0,\infty)))$, $U \leftarrow ([0,\infty) \times \Omega \ni (t,\omega) \mapsto |U_k^\vartheta(x + BW_t^\theta(\omega),\omega)|^p \in [0,\infty))$, $Y \leftarrow R^\theta$ for $x \in \mathbb{R}^d$ in the notation of Hutzenthaler et al. [44, Lemma 2.2]) therefore demonstrates for all $x \in \mathbb{R}^d$ that

$$\mathbb{E}\big[\big|U_k^\vartheta(x + BW_{R^\theta}^\theta)\big|^p\big] = \int_0^\infty \mathbb{E}\big[\big|U_k^\vartheta(x + BW_t^\theta)\big|^p\big] \big(R^\theta(\mathbb{P})\big)(dt). \tag{139}$$

This and the assumption that for all $t \in (0,\infty)$ it holds that $\mathbb{P}(R^\theta \geqslant t) = e^{-\lambda t}$ establish Item (i). Next observe that the assumption that $\theta \notin \{(\vartheta,\eta) \colon \eta \in \Theta\}$ ensures that W^θ and $\sigma_\Omega((W^{(\vartheta,\eta)})_{\eta\in\Theta}, (R^{(\vartheta,\eta)})_{\eta\in\Theta})$ are independent. Combining this with Item (ii) in Lemma 3.2 and Hutzenthaler et al. [44, Lemma 2.2] (with $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{G} \leftarrow \sigma_\Omega((W^{(\vartheta,\eta)})_{\eta\in\Theta}, (R^{(\vartheta,\eta)})_{\eta\in\Theta})$, $(S,\mathcal{S}) \leftarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $U \leftarrow (\mathbb{R}^d \times \Omega \ni (y,\omega) \mapsto |U_k^\vartheta(y,\omega)|^p \in [0,\infty))$, $Y \leftarrow (\Omega \ni \omega \mapsto x + BW_t^\theta(\omega) \in \mathbb{R}^d)$ for $t \in [0,\infty)$, $x \in \mathbb{R}^d$ in the notation of Hutzenthaler et al. [44, Lemma 2.2]) demonstrates for all $x \in \mathbb{R}^d$, $t \in [0,\infty)$ that

$$\mathbb{E}\big[\big|U_k^\vartheta(x + BW_t^\theta)\big|^p\big] = \int_{\mathbb{R}^d} \mathbb{E}\big[|U_k^\vartheta(y)|^p\big]\, \nu_{t,x}(dy). \tag{140}$$

Item (iv) in Lemma 3.2 hence establishes Item (ii). The proof of Lemma 3.3 is thus completed. □

Lemma 3.4. *Assume Setting 3.1, let $\varepsilon \in (0,\infty)$, $p \in [1,\infty)$, $L \in \mathbb{R}$, and assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w)| \leqslant L|v-w|$ and $\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}[|f(x+BW_s^0,0)|^p]\,ds < \infty$. Then it holds for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that*

$$\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}\big[|U_n^0(x+BW_s^0)|^p\big]\,ds < \infty. \tag{141}$$

Proof of Lemma 3.4. Throughout this proof let $x \in \mathbb{R}^d$, let $\mathbb{A} \subseteq \mathbb{N}$ be the set given by

$$\mathbb{A} = \left\{ n \in \mathbb{N} \colon \left(\forall\, k \in \{0,1,\ldots,n-1\} \colon \int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}\big[|U_k^0(x+BW_s^0)|^p\big]\,ds < \infty \right) \right\}, \tag{142}$$

and let $\nu_{t,y} \colon \mathcal{B}(\mathbb{R}^d) \to [0,\infty)$, $t \in [0,\infty)$, $y \in \mathbb{R}^d$, satisfy for all $t \in [0,\infty)$, $y \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_{t,y}(A) = \mathbb{P}(y + BW_t^0 \in A)$. Observe that the assumption that for all $y \in \mathbb{R}^d$ it holds that $U_0^0(y) = 0$ ensures that $1 \in \mathbb{A}$. Moreover, note that (132), the fact that for all $a, b \in \mathbb{R}$ it holds that $|a+b|^p \leqslant 2^{p-1}|a|^p + 2^{p-1}|b|^p$, and the assumption that for all $y \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ it holds that $|f(y,v) - f(y,w)| \leqslant L|v-w|$ ensure for all $n \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\begin{aligned}
|U_n^0(y)|^p &\leqslant 2^{p-1}\Bigg[\sum_{k=1}^{n-1}\frac{1}{\lambda M^{(n-k)}}\sum_{m=1}^{M^{(n-k)}}\Big|f\big(y+BW^{(0,k,m)}_{R^{(0,k,m)}}, U_k^{(0,k,m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\big) \\
&\quad - f\big(y+BW^{(0,k,m)}_{R^{(0,k,m)}}, U_{k-1}^{(0,k,-m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\big)\Big|\Bigg]^p + 2^{p-1}\Bigg[\frac{1}{\lambda M^n}\sum_{m=1}^{M^n}\Big|f(y+BW^{(0,0,m)}_{R^{(0,0,m)}},0)\Big|\Bigg]^p \\
&\leqslant 2^{p-1}\Bigg[\sum_{k=1}^{n-1}\frac{1}{\lambda M^{(n-k)}}\sum_{m=1}^{M^{(n-k)}} L\,\Big|U_k^{(0,k,m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}}) - U_{k-1}^{(0,k,-m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\Big|\Bigg]^p \\
&\quad + 2^{p-1}\Bigg[\frac{1}{\lambda M^n}\sum_{m=1}^{M^n}\Big|f(y+BW^{(0,0,m)}_{R^{(0,0,m)}},0)\Big|\Bigg]^p.
\end{aligned} \tag{143}$$

This and the fact that $[0,\infty) \ni t \mapsto t^p \in [0,\infty)$ is convex guarantee for all $n \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\begin{aligned}
|U_n^0(y)|^p &\leqslant \frac{2^{p-1}}{\lambda^p M^n}\sum_{m=1}^{M^n}\Big|f(y+BW^{(0,0,m)}_{R^{(0,0,m)}},0)\Big|^p \\
&\quad + \sum_{k=1}^{n-1}\frac{2^{p-1}n^pL^p}{\lambda^p M^{(n-k)}}\sum_{m=1}^{M^{(n-k)}}\Big|U_k^{(0,k,m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}}) - U_{k-1}^{(0,k,-m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\Big|^p \\
&\leqslant \frac{2^{p-1}}{\lambda^p M^n}\sum_{m=1}^{M^n}\Big|f(y+BW^{(0,0,m)}_{R^{(0,0,m)}},0)\Big|^p \\
&\quad + \sum_{k=1}^{n-1}\frac{4^{p-1}n^pL^p}{\lambda^p M^{(n-k)}}\sum_{m=1}^{M^{(n-k)}}\Big[\Big|U_k^{(0,k,m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\Big|^p + \Big|U_{k-1}^{(0,k,-m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\Big|^p\Big].
\end{aligned} \tag{144}$$

Item (iii) in Lemma 3.2 hence ensures for all $n \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\begin{aligned}
\mathbb{E}\big[|U_n^0(y)|^p\big] &\leqslant \frac{2^{p-1}}{\lambda^p}\mathbb{E}\big[\big|f(y+BW^0_{R^0},0)\big|^p\big] \\
&\quad + \sum_{k=1}^{n-1}\frac{4^{p-1}n^pL^p}{\lambda^p M^{(n-k)}}\sum_{m=1}^{M^{(n-k)}}\mathbb{E}\Big[\Big|U_k^{(0,k,m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\Big|^p + \Big|U_{k-1}^{(0,k,-m)}(y+BW^{(0,k,m)}_{R^{(0,k,m)}})\Big|^p\Big].
\end{aligned} \tag{145}$$

Next note that Item (i) in Lemma 3.3 (with $k \leftarrow k$, $p \leftarrow p$, $\theta \leftarrow (0,k,m)$, $\vartheta \leftarrow (0,k,m)$ for $k,m \in \mathbb{N}$ in the notation of Lemma 3.3) demonstrates for all $k,m \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\mathbb{E}\Big[|U_k^{(0,k,m)}(y + BW^{(0,k,m)}_{R^{(0,k,m)}})|^p\Big] = \int_0^\infty \lambda e^{-\lambda s}\, \mathbb{E}\Big[|U_k^{(0,k,m)}(y + BW^{(0,k,m)}_s)|^p\Big]\, ds. \tag{146}$$

Moreover, Item (i) in Lemma 3.3 (with $k \leftarrow k-1$, $p \leftarrow p$, $\theta \leftarrow (0,k,m)$, $\vartheta \leftarrow (0,k,-m)$ for $k,m \in \mathbb{N}$ in the notation of Lemma 3.3) proves for all $k,m \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\mathbb{E}\Big[|U_{k-1}^{(0,k,-m)}(y + BW^{(0,k,m)}_{R^{(0,k,m)}})|^p\Big] = \int_0^\infty \lambda e^{-\lambda s}\, \mathbb{E}\Big[|U_{k-1}^{(0,k,-m)}(y + BW^{(0,k,m)}_s)|^p\Big]\, ds. \tag{147}$$

This, (145), and (146) ensure for all $n \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\begin{aligned}\mathbb{E}\big[|U_n^0(y)|^p\big] &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(y + BW_s^0, 0)|^p\big]\, ds \\ &+ \sum_{k=1}^{n-1} \frac{4^{p-1} n^p L^p}{\lambda^{p-1} M^{(n-k)}} \sum_{m=1}^{M^{(n-k)}} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\Big[\Big|U_k^{(0,k,m)}(y + BW_s^{(0,k,m)})\Big|^p + \Big|U_{k-1}^{(0,k,-m)}(y + BW_s^{(0,k,m)})\Big|^p\Big]\, ds.\end{aligned} \tag{148}$$

Item (ii) in Lemma 3.3 (with $k \leftarrow k$, $p \leftarrow p$, $\theta \leftarrow (0,k,m)$, $\vartheta \leftarrow (0,k,m)$ for $k,m \in \mathbb{N}$ in the notation of Lemma 3.3) and Item (ii) in Lemma 3.3 (with $k \leftarrow k-1$, $p \leftarrow p$, $\theta \leftarrow (0,k,m)$, $\vartheta \leftarrow (0,k,-m)$ for $k,m \in \mathbb{N}$ in the notation of Lemma 3.3) hence guarantee for all $n \in \mathbb{N}$, $y \in \mathbb{R}^d$ that

$$\begin{aligned}\mathbb{E}\big[|U_n^0(y)|^p\big] &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(y + BW_s^0, 0)|^p\big]\, ds \\ &+ \sum_{k=1}^{n-1} \frac{4^{p-1} n^p L^p}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|U_k^0(y + BW_s^0)|^p + |U_{k-1}^0(y + BW_s^0)|^p\big]\, ds \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s} \left[\int_{\mathbb{R}^d} \left(|f(z,0)|^p + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(z)|^p\big]\right) \nu_{s,y}(dz)\right] ds.\end{aligned} \tag{149}$$

Item (ii) in Lemma 3.3 (with $k \leftarrow n$, $p \leftarrow p$, $\theta \leftarrow 0$, $\vartheta \leftarrow 0$ for $n \in \mathbb{N}$ in the notation of Lemma 3.3) therefore implies for all $n \in \mathbb{N}$, $t \in [0,\infty)$ that

$$\begin{aligned}\mathbb{E}\big[|U_n^0(x + BW_t^0)|^p\big] &= \int_{\mathbb{R}^d} \mathbb{E}\big[|U_n^0(y)|^p\big]\, \nu_{t,x}(dy) \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_{\mathbb{R}^d} \int_0^\infty e^{-\lambda s} \left[\int_{\mathbb{R}^d} \left(|f(z,0)|^p + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(z)|^p\big]\right) \nu_{s,y}(dz)\right] ds\, \nu_{t,x}(dy).\end{aligned} \tag{150}$$

Fubini's theorem hence proves for all $n \in \mathbb{N}$, $t \in [0,\infty)$ that

$$\begin{aligned}&\mathbb{E}\big[|U_n^0(x + BW_t^0)|^p\big] \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \left(|f(z,0)|^p + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(z)|^p\big]\right) \nu_{s,y}(dz)\right] \nu_{t,x}(dy)\, ds.\end{aligned} \tag{151}$$

The fact that for all $t,s \in [0,\infty)$, $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that $\nu_{t+s,x}(A) = \int_{\mathbb{R}^d} \nu_{s,y}(A)\, \nu_{t,x}(dy)$ therefore ensures for all $n \in \mathbb{N}$, $t \in [0,\infty)$ that

$$\begin{aligned}&\mathbb{E}\big[|U_n^0(x + BW_t^0)|^p\big] \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s} \left[\int_{\mathbb{R}^d} \left(|f(z,0)|^p + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(z)|^p\big]\right) \nu_{t+s,x}(dz)\right] ds.\end{aligned} \tag{152}$$

Combining this with Item (ii) in Lemma 3.3 (with $k \leftarrow k$, $p \leftarrow p$, $\theta \leftarrow 0$, $\vartheta \leftarrow 0$ for $k \in \mathbb{N}$ in the notation of Lemma 3.3) demonstrates for all $n \in \mathbb{N}$, $t \in [0, \infty)$ that

$$\begin{aligned} &\mathbb{E}\big[|U_n^0(x + BW_t^0)|^p\big] \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s} \left[\mathbb{E}\big[|f(x + BW_{t+s}^0, 0)|^p\big] + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(x + BW_{t+s}^0)|^p\big]\right] ds \\ &= e^{\lambda t} \frac{2^{p-1}}{\lambda^{p-1}} \int_t^\infty e^{-\lambda s} \left(\mathbb{E}\big[|f(x + BW_s^0, 0)|^p\big] + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(x + BW_s^0)|^p\big]\right) ds. \end{aligned} \tag{153}$$

Hence, we obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} &\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|U_n^0(x + BW_t^0)|^p\big]\, dt \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{\varepsilon t} \int_t^\infty e^{-\lambda s} \left(\mathbb{E}\big[|f(x + BW_s^0, 0)|^p\big] + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(x + BW_s^0)|^p\big]\right) ds\, dt \\ &= \frac{2^{p-1}}{\lambda^{p-1}} \int_0^\infty e^{-\lambda s} \left(\mathbb{E}\big[|f(x + BW_s^0, 0)|^p\big] + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(x + BW_s^0)|^p\big]\right) \left[\int_0^s e^{\varepsilon t}\, dt\right] ds \\ &\leqslant \frac{2^{p-1}}{\lambda^{p-1}\varepsilon} \int_0^\infty e^{(\varepsilon-\lambda)s} \left(\mathbb{E}\big[|f(x + BW_s^0, 0)|^p\big] + 2^p n^p L^p \sum_{k=0}^{n-1} \mathbb{E}\big[|U_k^0(x + BW_s^0)|^p\big]\right) ds. \end{aligned} \tag{154}$$

This implies for all $n \in \mathbb{A}$ that $\int_0^\infty e^{(\varepsilon-\lambda)s}\, \mathbb{E}[|U_n^0(x + BW_s^0)|^p]\, ds < \infty$. Therefore, we obtain for all $n \in \mathbb{N}$ that ($n \in \mathbb{A} \Rightarrow n + 1 \in \mathbb{A}$). Combining this with the fact that $1 \in \mathbb{A}$ and induction demonstrates that $\mathbb{N} \subseteq \mathbb{A}$. This establishes (141). The proof of Lemma 3.4 is thus completed. ☐

3.4 Bias and variance estimates for MLP approximations

Lemma 3.5 (Mean)**.** *Assume Setting 3.1, let $L \in \mathbb{R}$, $\varepsilon \in (0, \infty)$, and assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x, v) - f(x, w)| \leqslant L|v - w|$ and $\int_0^\infty e^{(\varepsilon-\lambda)s}\, \mathbb{E}[|f(x + BW_s^0, 0)|]\, ds < \infty$. Then*

(i) it holds for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $\int_0^\infty e^{(\varepsilon-\lambda)s}\, \mathbb{E}[|f(x + BW_s^0, U_{n-1}^0(x + BW_s^0))|]\, ds < \infty$ and

(ii) it holds for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\mathbb{E}\big[U_n^\theta(x)\big] = \frac{1}{\lambda} \mathbb{E}\big[f\big(x + BW_{R^0}^0, U_{n-1}^0(x + BW_{R^0}^0)\big)\big]. \tag{155}$$

Proof of Lemma 3.5. First, observe that Lemma 3.4 (with $\varepsilon \leftarrow \varepsilon$, $p \leftarrow 1$, $L \leftarrow L$ in the notation of Lemma 3.4) assures for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}[|U_n^0(x + BW_t^0)|]\, dt < \infty. \tag{156}$$

Combining this with the assumption that for all $x \in \mathbb{R}^d$ it holds that $\int_0^\infty e^{(\varepsilon-\lambda)s}\, \mathbb{E}[|f(x+BW_s^0, 0)|]\, ds < \infty$ and the assumption that for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ it holds that $|f(x, v) - f(x, w)| \leqslant L|v - w|$ ensures for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|f(x + BW_t^0, U_{n-1}^0(x + BW_t^0))|\big]\, dt \\ &\leqslant \int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|f(x + BW_t^0, 0)|\big]\, dt + L \int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|U_{n-1}^0(x + BW_t^0)|\big]\, dt < \infty. \end{aligned} \tag{157}$$

This establishes Item (i). Next note that Items (i)–(iv) in Lemma 3.2 and Hutzenthaler et al. [44, Corollary 2.5] ensure

- for all $\theta \in \Theta$, $k, m \in \mathbb{N}$ that $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto f(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega), U^{(\theta,k,m)}_k(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega),\omega)) \in \mathbb{R}$ and $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto f(x + BW^0_{R^0(\omega)}(\omega), U^0_k(x + BW^0_{R^0(\omega)}(\omega),\omega)) \in \mathbb{R}$ are identically distributed random fields and
- for all $\theta \in \Theta$, $k, m \in \mathbb{N}$ that $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto f(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega), U^{(\theta,k,-m)}_{k-1}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega),\omega)) \in \mathbb{R}$ and $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto f(x + BW^0_{R^0(\omega)}(\omega), U^0_{k-1}(x + BW^0_{R^0(\omega)}(\omega),\omega)) \in \mathbb{R}$ are identically distributed random fields.

Combining this with Item (i) shows for all $\theta \in \Theta$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\mathbb{E}\big[U^\theta_n(x)\big] = \sum_{k=1}^{n-1} \frac{1}{\lambda M^{(n-k)}} \sum_{m=1}^{M^{(n-k)}} \bigg(\mathbb{E}\Big[f\Big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U^{(\theta,k,m)}_k\big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}\big)\Big)\Big] \\
&- \mathbb{E}\Big[f\Big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U^{(\theta,k,-m)}_{k-1}\big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}\big)\Big)\Big]\bigg) + \frac{1}{\lambda M^n} \sum_{m=1}^{M^n} \mathbb{E}\Big[f\big(x + BW^{(\theta,0,m)}_{R^{(\theta,0,m)}}, 0\big)\Big] \\
&= \sum_{k=1}^{n-1} \frac{1}{\lambda}\bigg(\mathbb{E}\big[f\big(x + BW^0_{R^0}, U^0_k(x + BW^0_{R^0})\big)\big] - \mathbb{E}\big[f\big(x + BW^0_{R^0}, U^0_{k-1}(x + BW^0_{R^0})\big)\big]\bigg) \\
&+ \frac{1}{\lambda}\mathbb{E}\big[f\big(x + BW^0_{R^0}, 0\big)\big].
\end{aligned} \tag{158}$$

The assumption that for all $x \in \mathbb{R}^d$ it holds that $U^0_0(x) = 0$ hence proves for all $\theta \in \Theta$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
\mathbb{E}\big[U^\theta_n(x)\big] &= \frac{1}{\lambda}\Big(\mathbb{E}\big[f\big(x + BW^0_{R^0}, 0\big)\big] \\
&\quad + \mathbb{E}\big[f\big(x + BW^0_{R^0}, U^0_{n-1}(x + BW^0_{R^0})\big)\big] - \mathbb{E}\big[f\big(x + BW^0_{R^0}, U^0_0(x + BW^0_{R^0})\big)\big]\Big) \\
&= \frac{1}{\lambda}\mathbb{E}\big[f\big(x + BW^0_{R^0}, U^0_{n-1}(x + BW^0_{R^0})\big)\big].
\end{aligned} \tag{159}$$

This establishes Item (ii). The proof of Lemma 3.5 is thus completed. ☐

Lemma 3.6 (Bias)**.** *Assume Setting 3.1, let $L \in \mathbb{R}$, $\varepsilon \in (0,\infty)$, $p \in [1,\infty)$, assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w)| \leqslant L|v-w|$ and $\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}[|f(x+BW^0_s, 0)|]\,ds < \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, and assume for all $x \in \mathbb{R}^d$ that $\int_0^\infty e^{-\lambda s}\,\mathbb{E}[|u(x + BW^0_s)|]\,ds < \infty$ and*

$$u(x) = \frac{1}{\lambda}\mathbb{E}\big[f(x + BW^0_{R^0}, u(x + BW^0_{R^0}))\big] = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda s} f(x + BW^0_s, u(x + BW^0_s))\,ds\bigg]. \tag{160}$$

Then it holds for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\big|\mathbb{E}\big[U^\theta_n(x)\big] - u(x)\big|^p \leqslant \frac{L^p}{\lambda^p}\mathbb{E}\big[|U^0_{n-1}(x + BW^0_{R^0}) - u(x + BW^0_{R^0})|^p\big]. \tag{161}$$

Proof of Lemma 3.6. First, observe that Lemma 3.5 (with $\varepsilon \leftarrow \varepsilon$, $L \leftarrow L$ in the notation of Lemma 3.5) and (160) guarantee for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\mathbb{E}\big[U^\theta_n(x)\big] - u(x) = \frac{1}{\lambda}\mathbb{E}\big[f(x + BW^0_{R^0}, U^0_{n-1}(x + BW^0_{R^0})) - f(x + BW^0_{R^0}, u(x + BW^0_{R^0}))\big]. \tag{162}$$

The assumption that for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ it holds that $|f(x,v) - f(x,w)| \leqslant L|v-w|$ therefore implies for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\big|\mathbb{E}\big[U_n^\theta(x)\big] - u(x)\big| \leqslant \frac{L}{\lambda}\mathbb{E}\big[|U_{n-1}^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|\big]. \tag{163}$$

Jensen's inequality hence establishes (161). The proof of Lemma 3.6 is thus completed. □

Lemma 3.7 (Variance)**.** *Assume Setting 3.1, let $L \in \mathbb{R}$, $\varepsilon \in (0,\infty)$, and assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w)| \leqslant L|v-w|$ and $\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}[|f(x + BW_s^0, 0)|]\,ds < \infty$. Then it holds for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that*

$$\begin{aligned}&\mathrm{Var}[U_n^\theta(x)]\\&\leqslant \frac{1}{\lambda^2}\left[\frac{1}{M^n}\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big] + \sum_{k=1}^{n-1}\frac{L^2}{M^{(n-k)}}\mathbb{E}\big[|U_k^0(x + BW_{R^0}^0) - U_{k-1}^1(x + BW_{R^0}^0)|^2\big]\right].\end{aligned} \tag{164}$$

Proof of Lemma 3.7. First, observe that Item (ii) in Lemma 3.2 and Setting 3.1 ensure for all $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&\mathbb{N}_0 \times \mathbb{N} \ni (k,m) \mapsto\\&\begin{cases}\left(\begin{array}{l}\Omega \ni \omega \mapsto f(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega), U_k^{(\theta,k,m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega), \omega))\\ -f(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega), U_{k-1}^{(\theta,k,-m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}(\omega)}(\omega), \omega)) \in \mathbb{R}\end{array}\right) & : k \geqslant 1\\ \Omega \ni \omega \mapsto f(x + BW^{(\theta,0,m)}_{R^{(\theta,0,m)}(\omega)}(\omega), 0) \in \mathbb{R} & : k = 0\end{cases}\end{aligned} \tag{165}$$

is an independent family of random variables. This, Item (i) in Lemma 3.5, and (132) imply for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}\mathrm{Var}[U_n^\theta(x)] = \frac{1}{\lambda^2}\Bigg(&\sum_{m=1}^{M^n}\mathrm{Var}\left[\frac{f(x + BW^{(\theta,0,m)}_{R^{(\theta,0,m)}}, 0)}{M^n}\right]\\&+\sum_{k=1}^{n-1}\sum_{m=1}^{M^{(n-k)}}\mathrm{Var}\Bigg[\frac{f(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{(\theta,k,m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}))}{M^{(n-k)}}\dots\\&\qquad\dots - \frac{f(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{(\theta,k,-m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}))}{M^{(n-k)}}\Bigg]\Bigg).\end{aligned} \tag{166}$$

Items (i)–(iv) in Lemma 3.2 and Hutzenthaler et al. [44, Corollary 2.5] hence imply for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&\mathrm{Var}\big[U_n^\theta(x)\big] = \frac{1}{\lambda^2}\Bigg[\frac{1}{M^n}\,\mathrm{Var}[f(x + BW_{R^0}^0, 0)]\\&+\sum_{k=1}^{n-1}\frac{1}{M^{(n-k)}}\,\mathrm{Var}\big[f(x + BW_{R^0}^0, U_k^0(x + BW_{R^0}^0)) - f(x + BW_{R^0}^0, U_{k-1}^1(x + BW_{R^0}^0))\big]\Bigg].\end{aligned} \tag{167}$$

Combining this with the fact that for every random variable $Y\colon \Omega \to \mathbb{R}$ with $\mathbb{E}[|Y|] < \infty$ it holds that $\mathrm{Var}\big[Y\big] \leqslant \mathbb{E}[Y^2] \in [0,\infty]$ guarantees that for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathrm{Var}\big[U_n^\theta(x)\big] \leqslant \frac{1}{\lambda^2}\Bigg[\frac{1}{M^n}\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big]\\&+\sum_{k=1}^{n-1}\frac{1}{M^{(n-k)}}\mathbb{E}\big[|f(x + BW_{R^0}^0, U_k^0(x + BW_{R^0}^0)) - f(x + BW_{R^0}^0, U_{k-1}^1(x + BW_{R^0}^0))|^2\big]\Bigg].\end{aligned} \tag{168}$$

The assumption that for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ it holds that $|f(x,v) - f(x,w)| \leqslant L|v-w|$ therefore ensures for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\mathrm{Var}\big[U_n^\theta(x)\big] \\ &\leqslant \frac{1}{\lambda^2}\Bigg[\frac{1}{M^n}\mathbb{E}\big[|f(x+BW^0_{R^0},0)|^2\big] + \sum_{k=1}^{n-1}\frac{L^2}{M^{(n-k)}}\mathbb{E}\big[|U_k^0(x+BW^0_{R^0}) - U^1_{k-1}(x+BW^0_{R^0})|^2\big]\Bigg]. \end{aligned} \tag{169}$$

This establishes (164). The proof of Lemma 3.7 is thus completed. ☐

3.5 Error analysis for MLP approximations

Lemma 3.8 (Recursive overall error estimate)**.** *Assume Setting 3.1, let $L \in \mathbb{R}$, $\varepsilon \in (0,\infty)$, assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v)-f(x,w)| \leqslant L|v-w|$ and $\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}[|f(x+BW^0_s,0)|]\,ds < \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, assume for all $x \in \mathbb{R}^d$ that $\int_0^\infty e^{-\lambda s}\,\mathbb{E}[|u(x+BW^0_s)|]\,ds < \infty$ and*

$$u(x) = \frac{1}{\lambda}\mathbb{E}\big[f(x+BW^0_{R^0}, u(x+BW^0_{R^0}))\big] = \mathbb{E}\bigg[\int_0^\infty e^{-\lambda s} f(x+BW^0_s, u(x+BW^0_s))\,ds\bigg], \tag{170}$$

and let $\alpha, \beta\colon \mathcal{B}([0,\infty)) \to [0,\infty)$ be measures which satisfy for all $A \in \mathcal{B}([0,\infty))$ that $\beta(A) \geqslant \int_A e^{-\lambda s}(\int_{[0,s]} e^{\lambda t}\,\alpha(dt))\,ds$. Then it holds for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\bigg[\int_{[0,\infty)} \mathbb{E}\big[|U_n^\theta(x+BW_t^\theta) - u(x+BW_t^\theta)|^2\big]\,\alpha(dt)\bigg]^{1/2} \\ &\quad\leqslant \frac{1}{\sqrt{\lambda M^n}}\bigg[\int_{[0,\infty)}\mathbb{E}\big[|f(x+BW^0_s,0)|^2\big]\,\beta(ds)\bigg]^{1/2} \\ &\qquad + \sum_{k=0}^{n-1}\frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}}\bigg[\int_{[0,\infty)}\mathbb{E}\big[|U_k^0(x+BW^0_s) - u(x+BW^0_s)|^2\big]\,\beta(ds)\bigg]^{1/2}. \end{aligned} \tag{171}$$

Proof of Lemma 3.8. Throughout this proof let $\nu_{t,x}\colon \mathcal{B}(\mathbb{R}^d) \to [0,\infty)$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$, satisfy for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that

$$\nu_{t,x}(A) = \mathbb{P}(x + BW_t^0 \in A). \tag{172}$$

Observe that the triangle inequality ensures for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \leqslant \big(\mathbb{E}\big[|U_n^\theta(x) - \mathbb{E}\big[U_n^\theta(x)\big]\,|^2\big]\big)^{1/2} + \big(\mathbb{E}\big[|\mathbb{E}\big[U_n^\theta(x)\big] - u(x)|^2\big]\big)^{1/2} \\ &= \big(\mathbb{E}\big[|U_n^\theta(x) - \mathbb{E}\big[U_n^\theta(x)\big]\,|^2\big]\big)^{1/2} + \big|\mathbb{E}\big[U_n^\theta(x)\big] - u(x)\big| = \sqrt{\mathrm{Var}[U_n^\theta(x)]} + \big|\mathbb{E}\big[U_n^\theta(x)\big] - u(x)\big|. \end{aligned} \tag{173}$$

Lemma 3.6 (with $L \leftarrow L$, $\varepsilon \leftarrow \varepsilon$, $p \leftarrow 2$, $u \leftarrow u$ in the notation of Lemma 3.6) and Lemma 3.7 (with $L \leftarrow L$, $\varepsilon \leftarrow \varepsilon$ in the notation of Lemma 3.7) hence prove for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \leqslant \frac{L}{\lambda}\big(\mathbb{E}\big[|U^0_{n-1}(x+BW^0_{R^0}) - u(x+BW^0_{R^0})|^2\big]\big)^{1/2} \\ &+ \frac{1}{\lambda}\Bigg[\frac{1}{M^n}\mathbb{E}\big[|f(x+BW^0_{R^0},0)|^2\big] + \sum_{k=1}^{n-1}\frac{L^2}{M^{(n-k)}}\mathbb{E}\big[|U_k^0(x+BW^0_{R^0}) - U^1_{k-1}(x+BW^0_{R^0})|^2\big]\Bigg]^{1/2}. \end{aligned} \tag{174}$$

Combining this with the fact that for all $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in [0, \infty)$ it holds that $\sqrt{\sum_{k=1}^n a_k} \leqslant \sum_{k=1}^n \sqrt{a_k}$ shows for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\
&\leqslant \frac{L}{\lambda} \big(\mathbb{E}\big[|U_{n-1}^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2} + \frac{1}{\lambda\sqrt{M^n}} \big(\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big]\big)^{1/2} \\
&+ \sum_{k=1}^{n-1} \frac{L}{\lambda\sqrt{M^{(n-k)}}} \big(\mathbb{E}\big[|U_k^0(x + BW_{R^0}^0) - U_{k-1}^1(x + BW_{R^0}^0)|^2\big]\big)^{1/2}.
\end{aligned} \tag{175}
$$

The triangle inequality hence ensures for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\
&\leqslant \frac{L}{\lambda} \big(\mathbb{E}\big[|U_{n-1}^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2} + \frac{1}{\lambda\sqrt{M^n}} \big(\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big]\big)^{1/2} \\
&+ \sum_{k=1}^{n-1} \frac{L}{\lambda\sqrt{M^{(n-k)}}} \bigg[\big(\mathbb{E}\big[|U_k^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2} \\
&\qquad + \big(\mathbb{E}\big[|U_{k-1}^1(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2} \bigg].
\end{aligned} \tag{176}
$$

This, Items (ii) and (iv) in Lemma 3.2, and Hutzenthaler et al. [44, Lemma 2.2] (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \leftarrow \sigma_\Omega((W^{(1,\vartheta)})_{\vartheta \in \Theta}, (R^{(1,\vartheta)})_{\vartheta \in \Theta})$, $(S, \mathcal{S}) \leftarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $U \leftarrow (\mathbb{R}^d \times \Omega \ni (y, \omega) \mapsto |U_{k-1}^1(y, \omega) - u(y)|^2 \in [0, \infty))$, $Y \leftarrow (\Omega \ni \omega \mapsto x + BW_{R^0(\omega)}^0(\omega) \in \mathbb{R}^d)$ for $x \in \mathbb{R}^d$, $k \in \mathbb{N}$ in the notation of Hutzenthaler et al. [44, Lemma 2.2]) demonstrate for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$
\begin{aligned}
\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} &\leqslant \frac{1}{\lambda\sqrt{M^n}} \big(\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big]\big)^{1/2} \\
&+ \sum_{k=1}^{n-1} \frac{L}{\lambda\sqrt{M^{(n-k)}}} \big(\mathbb{E}\big[|U_k^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2} \\
&+ \sum_{k=1}^{n} \frac{L}{\lambda\sqrt{M^{(n-k)}}} \big(\mathbb{E}\big[|U_{k-1}^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2}.
\end{aligned} \tag{177}
$$

Hence, we obtain for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$
\begin{aligned}
\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} &\leqslant \frac{1}{\lambda\sqrt{M^n}} \big(\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big]\big)^{1/2} \\
&+ \sum_{k=0}^{n-1} \frac{L\mathbb{1}_{\mathbb{N}}(k) + L\sqrt{M}}{\lambda\sqrt{M^{(n-k)}}} \big(\mathbb{E}\big[|U_k^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2} \\
&\leqslant \frac{1}{\lambda\sqrt{M^n}} \big(\mathbb{E}\big[|f(x + BW_{R^0}^0, 0)|^2\big]\big)^{1/2} \\
&+ \sum_{k=0}^{n-1} \frac{L(1 + \sqrt{M})}{\lambda\sqrt{M^{(n-k)}}} \big(\mathbb{E}\big[|U_k^0(x + BW_{R^0}^0) - u(x + BW_{R^0}^0)|^2\big]\big)^{1/2}.
\end{aligned} \tag{178}
$$

Item (ii) in Lemma 3.2 and Hutzenthaler et al. [44, Lemma 2.2] hence ensure for all $n \in \mathbb{N}$, $\theta \in \Theta$,

$x \in \mathbb{R}^d$ that

$$\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \leqslant \frac{1}{\sqrt{\lambda M^n}} \bigg(\int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\bigg)^{1/2} + \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}} \bigg(\int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|U_k^0(x + BW_s^0) - u(x + BW_s^0)|^2\big]\, ds\bigg)^{1/2}. \tag{179}$$

Moreover, note that Items (ii) and (iv) in Lemma 3.2, Hutzenthaler et al. [44, Lemma 2.2] (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \leftarrow \sigma_\Omega((W^{(\theta,\vartheta)})_{\vartheta\in\Theta}, (R^{(\theta,\vartheta)})_{\vartheta\in\Theta})$, $(S, \mathcal{S}) \leftarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $U \leftarrow (\mathbb{R}^d \times \Omega \ni (y, \omega) \mapsto |U_n^\theta(y, \omega) - u(y)|^2 \in [0, \infty))$, $Y \leftarrow (\Omega \ni \omega \mapsto x + BW_t^\theta(\omega) \in \mathbb{R}^d)$ for $\theta \in \Theta$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ in the notation of Hutzenthaler et al. [44, Lemma 2.2]), and (172) prove for all $n \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\mathbb{E}\big[|U_n^\theta(x + BW_t^\theta) - u(x + BW_t^\theta)|^2\big]\big)^{1/2} &= \bigg[\int_{\mathbb{R}^d} \mathbb{E}\big[|U_n^\theta(y) - u(y)|^2\big]\, \nu_{t,x}(dy)\bigg]^{1/2}. \\ &= \bigg[\int_{\mathbb{R}^d} \mathbb{E}\big[|U_n^0(y) - u(y)|^2\big]\, \nu_{t,x}(dy)\bigg]^{1/2}.\end{aligned} \tag{180}$$

The triangle inequality and (179) hence show for all $n \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&\big(\mathbb{E}\big[|U_n^\theta(x + BW_t^\theta) - u(x + BW_t^\theta)|^2\big]\big)^{1/2} \\ &\leqslant \frac{1}{\sqrt{\lambda M^n}} \bigg[\int_{\mathbb{R}^d} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(y + BW_s^0, 0)|^2\big]\, ds\, \nu_{t,x}(dy)\bigg]^{1/2} \\ &+ \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}} \bigg[\int_{\mathbb{R}^d} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|U_k^0(y + BW_s^0) - u(y + BW_s^0)|^2\big]\, ds\, \nu_{t,x}(dy)\bigg]^{1/2}.\end{aligned} \tag{181}$$

Combining this with the fact that for all $t, s \in [0, \infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that $\nu_{t+s,x}(A) = \int_{\mathbb{R}^d} \nu_{s,y}(A)\, \nu_{t,x}(dy)$ and Fubini's theorem proves for all $n \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&\big(\mathbb{E}\big[|U_n^\theta(x + BW_t^\theta) - u(x + BW_t^\theta)|^2\big]\big)^{1/2} \\ &\leqslant \frac{1}{\sqrt{\lambda M^n}} \bigg[\int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_{t+s}^0, 0)|^2\big]\, ds\bigg]^{1/2} \\ &+ \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}} \bigg[\int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|U_k^0(x + BW_{t+s}^0) - u(x + BW_{t+s}^0)|^2\big]\, ds\bigg]^{1/2}.\end{aligned} \tag{182}$$

The triangle inequality hence ensures for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&\bigg[\int_{[0,\infty)} \mathbb{E}\big[|U_n^\theta(x + BW_t^\theta) - u(x + BW_t^\theta)|^2\big]\, \alpha(dt)\bigg]^{1/2} \\ &\leqslant \frac{1}{\sqrt{\lambda M^n}} \bigg[\int_{[0,\infty)} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_{t+s}^0, 0)|^2\big]\, ds\, \alpha(dt)\bigg]^{1/2} \\ &+ \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}} \bigg[\int_{[0,\infty)} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|U_k^0(x + BW_{t+s}^0) - u(x + BW_{t+s}^0)|^2\big]\, ds\, \alpha(dt)\bigg]^{1/2} \\ &= \frac{1}{\sqrt{\lambda M^n}} \bigg[\int_{[0,\infty)} e^{\lambda t} \int_t^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\, \alpha(dt)\bigg]^{1/2} \\ &+ \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}} \bigg[\int_{[0,\infty)} e^{\lambda t} \int_t^\infty e^{-\lambda s}\, \mathbb{E}\big[|U_k^0(x + BW_s^0) - u(x + BW_s^0)|^2\big]\, ds\, \alpha(dt)\bigg]^{1/2}.\end{aligned} \tag{183}$$

Fubini's theorem therefore implies for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\left[\int_{[0,\infty)} \mathbb{E}\big[|U_n^\theta(x + BW_t^\theta) - u(x + BW_t^\theta)|^2\big]\,\alpha(dt)\right]^{1/2} \\
&\leqslant \frac{1}{\sqrt{\lambda M^n}}\left[\int_0^\infty \left(\int_{[0,s]} e^{\lambda t}\,\alpha(dt)\right) e^{-\lambda s}\,\mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\,ds\right]^{1/2} \\
&+ \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}}\left[\int_0^\infty \left(\int_{[0,s]} e^{\lambda t}\,\alpha(dt)\right) e^{-\lambda s}\,\mathbb{E}\big[|U_k^0(x + BW_s^0) - u(x + BW_s^0)|^2\big]\,ds\right]^{1/2}.
\end{aligned} \tag{184}$$

The assumption that for all $A \in \mathcal{B}([0,\infty))$ it holds that $\beta(A) \geqslant \int_A e^{-\lambda s}(\int_{[0,s]} e^{\lambda t}\,\alpha(dt))\,ds$ hence ensures for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\left[\int_{[0,\infty)} \mathbb{E}\big[|U_n^\theta(x + BW_t^\theta) - u(x + BW_t^\theta)|^2\big]\,\alpha(dt)\right]^{1/2} \\
&\quad \leqslant \frac{1}{\sqrt{\lambda M^n}}\left[\int_{[0,\infty)} \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\,\beta(ds)\right]^{1/2} \\
&\quad + \sum_{k=0}^{n-1} \frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}}\left[\int_{[0,\infty)} \mathbb{E}\big[|U_k^0(x + BW_s^0) - u(x + BW_s^0)|^2\big]\,\beta(ds)\right]^{1/2}.
\end{aligned} \tag{185}$$

This establishes (171). The proof of Lemma 3.8 is thus completed. ☐

Proposition 3.9 (Overall error estimate)**.** *Assume Setting 3.1, let $L \in (0,\lambda)$, assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w)| \leqslant L|v - w|$ and $\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}[|f(x + BW_s^0, 0)|]\,ds < \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, and assume for all $x \in \mathbb{R}^d$ that $\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}[|u(x + BW_s^0)|^2]\,ds < \infty$ and*

$$u(x) = \frac{1}{\lambda}\mathbb{E}\big[f(x + BW_{R^0}^0, u(x + BW_{R^0}^0))\big] = \mathbb{E}\left[\int_0^\infty e^{-\lambda s} f(x + BW_s^0, u(x + BW_s^0))\,ds\right]. \tag{186}$$

Then it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\
&\leqslant \frac{1}{\sqrt{\lambda} - \sqrt{L}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\,ds\right)^{1/2} \frac{1}{\sqrt{M^n}}\left[1 + (1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n.
\end{aligned} \tag{187}$$

Proof of Proposition 3.9. Throughout this proof let $\delta_0\colon \mathcal{B}([0,\infty)) \to [0,\infty)$ satisfy for all $A \in \mathcal{B}([0,\infty))$ that

$$\delta_0(A) = \begin{cases} 1 & : 0 \in A \\ 0 & : 0 \notin A. \end{cases} \tag{188}$$

Observe that

(i) for all $A \in \mathcal{B}([0,\infty))$ it holds that

$$\int_A e^{-\lambda s} \int_{[0,s]} e^{\lambda t}\,\delta_0(dt)\,ds = \int_A e^{-\lambda s}\,ds \leqslant \int_A e^{(L-\lambda)s}\,ds \tag{189}$$

and

(ii) for all $A \in \mathcal{B}([0,\infty))$ it holds that

$$\int_A e^{-\lambda s} \int_{[0,s]} e^{\lambda t}\, e^{(L-\lambda)t}\, dt\, ds = \int_A e^{-\lambda s} \frac{e^{Ls}-1}{L}\, ds \leqslant \frac{1}{L} \int_A e^{(L-\lambda)s}\, ds. \tag{190}$$

Note that Item (ii) and Lemma 3.8 (with $\varepsilon \leftarrow L$, $L \leftarrow L$, $u \leftarrow u$, $\alpha \leftarrow (\mathcal{B}([0,\infty)) \ni A \mapsto \int_A e^{(L-\lambda)t}\, dt \in [0,\infty))$, $\beta \leftarrow (\mathcal{B}([0,\infty)) \ni A \mapsto \frac{1}{L}\int_A e^{(L-\lambda)t}\, dt \in [0,\infty))$ in the notation of Lemma 3.8) ensure for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|U_n^0(x+BW_t^0) - u(x+BW_t^0)|^2\big]\, dt\right)^{1/2} \\ &\leqslant \frac{1}{\sqrt{L\lambda M^n}} \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|f(x+BW_t^0,0)|^2\big]\, dt\right)^{1/2} \\ &\quad + \sum_{k=0}^{n-1} \sqrt{\frac{L}{\lambda}} \frac{1+\sqrt{M}}{\sqrt{M^{(n-k)}}} \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|U_k^0(x+BW_t^0) - u(x+BW_t^0)|^2\big]\, dt\right)^{1/2}. \end{aligned} \tag{191}$$

For the next step let $\eta_n^{(x)} \in [0,\infty]$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$, satisfy for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\eta_n^{(x)} = \sqrt{M^n} \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|U_n^0(x+BW_t^0) - u(x+BW_t^0)|^2\big]\, dt\right)^{1/2} \tag{192}$$

and let $a_1^{(x)}, a_2^{(x)} \in [0,\infty]$, $x \in \mathbb{R}^d$, satisfy for all $x \in \mathbb{R}^d$ that

$$a_1^{(x)} = \frac{1}{\sqrt{L\lambda}} \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|f(x+BW_t^0,0)|^2\big]\, dt\right)^{1/2} \qquad \text{and} \qquad a_2^{(x)} = (1+\sqrt{M})\sqrt{\frac{L}{\lambda}}. \tag{193}$$

Note that (191) ensures for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\eta_n^{(x)} \leqslant a_1^{(x)} + a_2^{(x)} \sum_{k=0}^{n-1} \eta_k^{(x)}. \tag{194}$$

The discrete Gronwall inequality therefore proves for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\eta_n^{(x)} \leqslant (a_1^{(x)} + a_2^{(x)}\eta_0^{(x)})(1+a_2^{(x)})^{n-1}. \tag{195}$$

Next observe that Item (ii) in Proposition 2.17 (with $d \leftarrow d$, $m \leftarrow d$, $B \leftarrow B$, $L \leftarrow L$, $\lambda \leftarrow \lambda$, $\varepsilon \leftarrow L$, $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $W \leftarrow W^0$, $f \leftarrow f$, $u \leftarrow u$ in the notation of Proposition 2.17) demonstrates for all $x \in \mathbb{R}^d$ that

$$\begin{aligned} \eta_0^{(x)} &= \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|u(x+BW_t^0)|^2\big]\, dt\right)^{1/2} \\ &\leqslant \frac{1}{\sqrt{L\lambda}-L} \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|f(x+BW_t^0,0)|^2\big]\, dt\right)^{1/2} = \frac{a_1^{(x)}}{1-\sqrt{\frac{L}{\lambda}}}. \end{aligned} \tag{196}$$

This and (195) ensure for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that $\eta_n^{(x)} \leqslant [1-\sqrt{L/\lambda}]^{-1} a_1^{(x)} (1+a_2^{(x)})^n$. Hence, we obtain for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|U_n^0(x+BW_t^0) - u(x+BW_t^0)|^2\big]\, dt\right)^{1/2} \\ &\leqslant \frac{1}{\sqrt{M^n}} \left[1 + (1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n \frac{1}{\sqrt{L\lambda}-L} \left(\int_0^\infty e^{(L-\lambda)t}\, \mathbb{E}\big[|f(x+BW_t^0,0)|^2\big]\, dt\right)^{1/2}. \end{aligned} \tag{197}$$

Next note that Item (i) and Lemma 3.8 (with $\varepsilon \leftarrow L$, $L \leftarrow L$, $u \leftarrow u$, $\alpha \leftarrow \delta_0$, $\beta \leftarrow (\mathcal{B}([0,\infty)) \ni A \mapsto \int_A e^{(L-\lambda)s}\,ds \in [0,\infty))$ in the notation of Lemma 3.8) imply for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} &\leqslant \frac{1}{\sqrt{\lambda M^n}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2}\\ &+ \sum_{k=0}^{n-1}\frac{L(1+\sqrt{M})}{\sqrt{\lambda M^{(n-k)}}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|U_k^0(x+BW_s^0) - u(x+BW_s^0)|^2\big]\,ds\right)^{1/2}.\end{aligned} \tag{198}$$

This and (197) demonstrate for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} &\leqslant \frac{1}{\sqrt{\lambda M^n}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2} \\ &+ \sum_{k=0}^{n-1}\frac{L(1+\sqrt{M})}{\sqrt{\lambda M^n}}\frac{1}{\sqrt{L\lambda}-L}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2}\left[1+(1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^k.\end{aligned} \tag{199}$$

Hence, we obtain for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} &\leqslant \frac{1}{\sqrt{\lambda M^n}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2}\\ &\cdot\left[1+\sum_{k=0}^{n-1}\frac{L(1+\sqrt{M})}{\sqrt{L\lambda}-L}\left(1+(1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right)^k\right].\end{aligned} \tag{200}$$

The fact that for all $q \in \mathbb{R}$, $n \in \mathbb{N}$ it holds that $(q-1)\sum_{k=0}^{n-1} q^k = q^n - 1$ therefore implies for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} &\leqslant \frac{1}{\sqrt{\lambda M^n}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2}\\ &\cdot\left[1+\frac{\sqrt{L\lambda}}{\sqrt{L\lambda}-L}\left(1+(1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right)^n - \frac{\sqrt{L\lambda}}{\sqrt{L\lambda}-L}\right].\end{aligned} \tag{201}$$

The fact that $\frac{\sqrt{L\lambda}}{\sqrt{L\lambda}-L} > 1$ hence implies for all $n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2}\\ &\leqslant \frac{1}{\sqrt{\lambda M^n}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2}\frac{\sqrt{L\lambda}}{\sqrt{L\lambda}-L}\left[1+(1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n\\ &= \frac{1}{\sqrt{\lambda}-\sqrt{L}}\left(\int_0^\infty e^{(L-\lambda)s}\,\mathbb{E}\big[|f(x+BW_s^0,0)|^2\big]\,ds\right)^{1/2}\frac{1}{\sqrt{M^n}}\left[1+(1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n.\end{aligned} \tag{202}$$

Combining this with the fact that for all $\theta \in \Theta$, $x \in \mathbb{R}^d$ it holds that $U_0^\theta(x) = 0$ and Item (iii) in Proposition 2.17 establishes (187). The proof of Proposition 3.9 is thus completed. □

Corollary 3.10. *Assume Setting 3.1, let $\varepsilon \in (0,\infty)$, assume for all $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ that $f(x,v) = f(x,0)$ and $\int_0^\infty e^{(\varepsilon-\lambda)s}\,\mathbb{E}[|f(x+BW_s^0,0)|^2]\,ds < \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, and assume for all $x \in \mathbb{R}^d$ that*

$$u(x) = \mathbb{E}\left[\int_0^\infty e^{-\lambda s} f(x+BW_s^0, u(x+BW_s^0))\,ds\right] = \mathbb{E}\left[\int_0^\infty e^{-\lambda s} f(x+BW_s^0,0)\,ds\right]. \tag{203}$$

Then it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \leqslant \frac{1}{\sqrt{\lambda M^n}} \left(\int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\right)^{1/2}. \tag{204}$$

Proof of Corollary 3.10. Throughout this proof let $\nu_{t,x}\colon \mathcal{B}(\mathbb{R}^d) \to [0,\infty)$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$, satisfy for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ that $\nu_{t,x}(A) = \mathbb{P}(x + BW_t^0 \in A)$. Observe that the Cauchy-Schwarz inequality and (203) ensure for all $y \in \mathbb{R}^d$ that

$$|u(y)|^2 \leqslant \frac{1}{\lambda} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(y + BW_s^0, 0)|^2\big]\, ds. \tag{205}$$

Fubini's theorem therefore shows for all $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|u(x + BW_t^0)|^2\big]\, dt = \int_0^\infty e^{(\varepsilon-\lambda)t} \int_{\mathbb{R}^d} |u(y)|^2\, \nu_{t,x}(dy)\, dt \\
&\leqslant \int_0^\infty e^{(\varepsilon-\lambda)t} \int_{\mathbb{R}^d} \frac{1}{\lambda} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(y + BW_s^0, 0)|^2\big]\, ds\, \nu_{t,x}(dy)\, dt \\
&= \frac{1}{\lambda} \int_0^\infty e^{(\varepsilon-\lambda)t} \int_0^\infty e^{-\lambda s} \int_{\mathbb{R}^d} \mathbb{E}\big[|f(y + BW_s^0, 0)|^2\big]\, \nu_{t,x}(dy)\, ds\, dt.
\end{aligned} \tag{206}$$

The fact that for all $t, s \in [0,\infty)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that $\nu_{t+s,x}(A) = \int_{\mathbb{R}^d} \nu_{s,y}(A)\, \nu_{t,x}(dy)$ and Fubini's theorem hence ensure that for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
\int_0^\infty e^{(\varepsilon-\lambda)t}\, \mathbb{E}\big[|u(x + BW_t^0)|^2\big]\, dt &\leqslant \frac{1}{\lambda} \int_0^\infty e^{(\varepsilon-\lambda)t} \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_{t+s}^0, 0)|^2\big]\, ds\, dt \\
&= \frac{1}{\lambda} \int_0^\infty e^{\varepsilon t} \int_t^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\, dt \\
&= \frac{1}{\lambda} \int_0^\infty \left[\int_0^s e^{\varepsilon t}\, dt\right] e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds \\
&\leqslant \frac{1}{\varepsilon\lambda} \int_0^\infty e^{(\varepsilon-\lambda)s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds < \infty.
\end{aligned} \tag{207}$$

Proposition 3.9 (with $L \leftarrow L$, $u \leftarrow u$ for $L \in (0, \min\{\varepsilon, \lambda\})$ in the notation of Proposition 3.9) therefore proves for all $L \in (0, \min\{\varepsilon, \lambda\})$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\
&\leqslant \frac{1}{\sqrt{\lambda} - \sqrt{L}} \left(\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\right)^{1/2} \frac{1}{\sqrt{M^n}} \left[1 + (1 + \sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n.
\end{aligned} \tag{208}$$

Lebesgue's dominated convergence theorem hence ensures for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\
&\leqslant \lim_{L \searrow 0} \left(\frac{1}{\sqrt{\lambda} - \sqrt{L}} \left(\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\right)^{1/2} \frac{1}{\sqrt{M^n}} \left[1 + (1 + \sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n\right) \\
&= \frac{1}{\sqrt{\lambda M^n}} \left(\int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\right)^{1/2}.
\end{aligned} \tag{209}$$

This establishes (204). The proof of Corollary 3.10 is thus completed. ☐

Corollary 3.11. *Assume Setting 3.1, let $L \in [0, \lambda)$, $\eta \in (0, \infty)$, assume for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x, v) - f(x, w)| \leqslant L|v - w|$ and $\int_0^\infty e^{(\max\{L,\eta\}-\lambda)s}\, \mathbb{E}[|f(x + BW_s^0, 0)|^2]\, ds < \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, and assume for all $x \in \mathbb{R}^d$ that $\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}[|u(x + BW_s^0)|^2]\, ds < \infty$ and*

$$u(x) = \frac{1}{\lambda}\mathbb{E}\big[f(x + BW_{R^0}^0, u(x + BW_{R^0}^0))\big] = \int_0^\infty e^{-\lambda s}\, \mathbb{E}\big[f(x + BW_s^0, u(x + BW_s^0))\big]\, ds. \tag{210}$$

Then it holds for all $\theta \in \Theta$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\ &\leqslant \frac{1}{\sqrt{\lambda} - \sqrt{L}} \left(\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\right)^{1/2} \frac{1}{\sqrt{M^n}} \left[1 + (1 + \sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n. \end{aligned} \tag{211}$$

Proof of Corollary 3.11. First, observe that the assumption that for all $x \in \mathbb{R}^d$ it holds that $\int_0^\infty e^{(\max\{L,\eta\}-\lambda)s}\, \mathbb{E}[|f(x + BW_s^0, 0)|^2]\, ds < \infty$ guarantees for all $x \in \mathbb{R}^d$ that $\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}[|f(x + BW_s^0, 0)|^2]\, ds < \infty$ and $\int_0^\infty e^{(\eta-\lambda)s}\, \mathbb{E}[|f(x + BW_s^0, 0)|^2]\, ds < \infty$. Therefore, we obtain that Proposition 3.9 (with $L \leftarrow L$, $u \leftarrow u$ in the notation of Proposition 3.9) establishes (211) in the case $L \in (0, \lambda)$ and Corollary 3.10 (with $\varepsilon \leftarrow \eta$, $u \leftarrow u$ in the notation of Corollary 3.10) establishes (211) in the case $L = 0$. The proof of Corollary 3.11 is thus completed. ☐

Corollary 3.12. *Assume Setting 3.1, let $L, \rho \in [0, \infty)$, assume that $\lambda \in (L + 2\rho, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $u \in C(\mathbb{R}^d, \mathbb{R})$, $V \in C^2(\mathbb{R}^d, (0, \infty))$ satisfy for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x, v) - f(x, w)| \leqslant L|v - w|$ and*

$$\operatorname{Trace}(BB^*(\operatorname{Hess} V)(x)) + \tfrac{\|B^*(\nabla V)(x)\|^2}{V(x)} \leqslant 2\rho V(x), \tag{212}$$

assume that $\sup_{r\in(0,\infty)}[\inf_{\|x\|>r} V(x)] = \infty$ and $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|f(x,0)|+|u(x)|}{V(x)})] = 0$, and assume that u is a viscosity solution of

$$\lambda u(x) - \tfrac{1}{2} \operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \tag{213}$$

for $x \in \mathbb{R}^d$ (cf. Proposition 2.13). Then it holds for all $\theta \in \Theta$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\left(\mathbb{E}\left[\left|\frac{U_n^\theta(x) - u(x)}{V(x)}\right|^2\right]\right)^{1/2} \\ &\leqslant \frac{1}{(\sqrt{\lambda} - \sqrt{L})\sqrt{\lambda - (L + 2\rho)}} \left[\sup_{y\in\mathbb{R}^d} \left(\frac{|f(y, 0)|}{V(y)}\right)\right] \left[\frac{1}{\sqrt{M}} \left(1 + (1 + \sqrt{M})\sqrt{\frac{L}{\lambda}}\right)\right]^n. \end{aligned} \tag{214}$$

Proof of Corollary 3.12. Throughout this proof let $c \in \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that $|u(x)| \leqslant cV(x)$. Note that (212) ensures that

(I) for all $x \in \mathbb{R}^d$ it holds that $\frac{1}{2} \operatorname{Trace}(BB^*(\operatorname{Hess} V)(x)) \leqslant \rho V(x)$ and

(II) for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\tfrac{1}{2} \operatorname{Trace}\big(BB^*(\operatorname{Hess} V^2)(x)\big) \\ &= \tfrac{1}{2} \operatorname{Trace}\big(BB^*(2V(x)(\operatorname{Hess} V)(x) + 2(\nabla V)(x) \otimes (\nabla V)(x))\big) \\ &= V(x) \Big[\operatorname{Trace}\big(BB^*(\operatorname{Hess} V)(x)\big) + \tfrac{\|B^*(\nabla V)(x)\|^2}{V(x)}\Big] \leqslant 2\rho[V(x)]^2. \end{aligned} \tag{215}$$

Item (I) and Proposition 2.13 (with $d \leftarrow d$, $m \leftarrow d$, $B \leftarrow B$, $L \leftarrow L$, $\rho \leftarrow \rho$, $\lambda \leftarrow \lambda$, $\|\cdot\| \leftarrow \|\cdot\|$, $f \leftarrow f$, $V \leftarrow V$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W^0$ in the notation of Proposition 2.13) demonstrate for all $x \in \mathbb{R}^d$ that

$$u(x) = \mathbb{E}\left[\int_0^\infty e^{-\lambda s} f(x + BW_s^0, u(x + BW_s^0))\, ds\right]. \tag{216}$$

Next note that [6, Lemmas 3.1 and 3.2] and Item (II) guarantee for all $t \in [0, \infty)$, $x \in \mathbb{R}^d$ that

$$\mathbb{E}\big[|V(x + BW_t^0)|^2\big] \leqslant e^{2\rho t}[V(x)]^2. \tag{217}$$

The fact that $\sup_{y\in\mathbb{R}^d}(\frac{|f(y,0)|}{V(y)}) < \infty$ hence implies for all $x \in \mathbb{R}^d$, $\eta \in [0, \infty)$ with $L + 2\rho + \eta < \lambda$ that

$$\begin{aligned}
&\int_0^\infty e^{(L+\eta-\lambda)s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds \\
&\leqslant \int_0^\infty e^{(L+\eta-\lambda)s} \left[\sup_{y\in\mathbb{R}^d}\left(\frac{|f(y,0)|}{V(y)}\right)\right]^2 \mathbb{E}\big[|V(x + BW_s^0)|^2\big]\, ds \\
&\leqslant \left[\sup_{y\in\mathbb{R}^d}\left(\frac{|f(y,0)|}{V(y)}\right)\right]^2 \left[\int_0^\infty e^{(L+\eta+2\rho-\lambda)s}\, ds\right] |V(x)|^2 \\
&= \frac{|V(x)|^2}{\lambda - (L + \eta + 2\rho)} \left[\sup_{y\in\mathbb{R}^d}\left(\frac{|f(y,0)|}{V(y)}\right)\right]^2 < \infty.
\end{aligned} \tag{218}$$

In addition, note that the fact that for all $x \in \mathbb{R}^d$ it holds that $|u(x)| \leqslant cV(x)$ and (217) prove for all $x \in \mathbb{R}^d$ that

$$\begin{aligned}
\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}\big[|u(x + BW_s^0)|^2\big]\, ds &\leqslant \int_0^\infty e^{(L-\lambda)s}\, c^2\, \mathbb{E}\big[|V(x + BW_s^0)|^2\big]\, ds \\
&\leqslant c^2|V(x)|^2 \int_0^\infty e^{(L+2\rho-\lambda)s}\, ds = \frac{[cV(x)]^2}{\lambda - (L + 2\rho)} < \infty.
\end{aligned} \tag{219}$$

This, (216), (218), and Corollary 3.11 assure for all $\theta \in \Theta$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\big(\mathbb{E}\big[|U_n^\theta(x) - u(x)|^2\big]\big)^{1/2} \\
&\leqslant \frac{1}{\sqrt{\lambda} - \sqrt{L}} \left(\int_0^\infty e^{(L-\lambda)s}\, \mathbb{E}\big[|f(x + BW_s^0, 0)|^2\big]\, ds\right)^{1/2} \frac{1}{\sqrt{M^n}} \left[1 + (1 + \sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n \\
&\leqslant \frac{1}{\sqrt{\lambda} - \sqrt{L}} \frac{V(x)}{\sqrt{\lambda - (L + 2\rho)}} \left[\sup_{y\in\mathbb{R}^d}\left(\frac{|f(y,0)|}{V(y)}\right)\right] \frac{1}{\sqrt{M^n}} \left[1 + (1 + \sqrt{M})\sqrt{\frac{L}{\lambda}}\right]^n.
\end{aligned} \tag{220}$$

This establishes (214). The proof of Corollary 3.12 is thus completed. □

Corollary 3.13. *Let $d, M \in \mathbb{N}$, $B \in \mathbb{R}^{d\times d}$, $L \in \mathbb{R}$, $\lambda \in (L, \infty)$, $\Theta = \cup_{n\in\mathbb{N}}\mathbb{Z}^n$, $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(x,v) - f(x,w) - \lambda(v - w)| \leqslant L|v - w|$, assume that f is at most polynomially growing, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W^\theta \colon [0, \infty) \times \Omega \to \mathbb{R}^d$, $\theta \in \Theta$, be i.i.d. standard Brownian motions, let $R^\theta : \Omega \to [0, \infty)$, $\theta \in \Theta$, be i.i.d. random variables, assume for all $t \in [0, \infty)$ that $\mathbb{P}(R^0 \geqslant t) = e^{-\lambda t}$, assume that $(R^\theta)_{\theta\in\Theta}$ and $(W^\theta)_{\theta\in\Theta}$ are independent, let $U_n^\theta = (U_n^\theta(x))_{x\in\mathbb{R}^d} \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$, $\theta \in \Theta$,*

$n \in \mathbb{N}_0$, satisfy for all $\theta \in \Theta$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $U_0^\theta(x) = 0$ and

$$\begin{aligned} U_n^\theta(x) = \frac{-1}{\lambda M^n}\left[\sum_{m=1}^{M^n} f(x + BW^{(\theta,0,m)}_{R^{(\theta,0,m)}}, 0)\right] \\ + \sum_{k=1}^{n-1}\frac{1}{\lambda M^{(n-k)}}\left[\sum_{m=1}^{M^{(n-k)}}\left(\lambda\Big[U_k^{(\theta,k,m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}) - U_{k-1}^{(\theta,k,-m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\Big]\right.\right. \\ - \Big[f\Big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{(\theta,k,m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\Big) \\ \left.\left. - f\Big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{(\theta,k,-m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\Big)\Big]\right)\right], \end{aligned} \tag{221}$$

and let $u \in \{v \in C(\mathbb{R}^d, \mathbb{R}) \colon (\forall\, \varepsilon \in (0,\infty) \colon [\sup_{x=(x_1,\ldots,x_d)\in\mathbb{R}^d}(|v(x)|\exp(-\varepsilon\sum_{i=1}^d |x_i|))] < \infty)\}$ be a viscosity solution of

$$\tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = f(x, u(x)) \tag{222}$$

for $x \in \mathbb{R}^d$ (cf. Corollary 2.16). Then it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $x \in \mathbb{R}^d$, $\varepsilon \in (0,\infty)$ with $2(\varepsilon^2 + \varepsilon d)[\sup_{y\in\mathbb{R}^d\backslash\{0\}}(\|By\|\,\|y\|^{-1})]^2 < \lambda - L$ that

$$\begin{aligned} \Big(\mathbb{E}\Big[\big|U_n^\theta(x) - u(x)\big|^2\Big]\Big)^{1/2} \leqslant \left[\frac{1}{\sqrt{M}}\left(1 + (1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right)\right]^n \left[\sup_{y\in\mathbb{R}^d}\left(\frac{|f(y,0)|}{\exp(\varepsilon(1+\|y\|^2)^{1/2})}\right)\right] \\ \cdot \frac{\exp\big(\varepsilon(1+\|x\|^2)^{1/2}\big)}{(\sqrt{\lambda}-\sqrt{L})\sqrt{\lambda - (L + 2(\varepsilon^2+\varepsilon d)[\sup_{y\in\mathbb{R}^d\backslash\{0\}}(\|By\|\,\|y\|^{-1})]^2)}}. \end{aligned} \tag{223}$$

Proof of Corollary 3.13. Throughout this proof let $g \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ that $g(x,v) = \lambda v - f(x,v)$, let $\beta \in [0,\infty)$ satisfy $\beta = [\sup_{y\in\mathbb{R}^d\backslash\{0\}}(\frac{\|By\|}{\|y\|})]^2$, and let $V_\varepsilon \colon \mathbb{R}^d \to (0,\infty)$, $\varepsilon \in (0,\infty)$, satisfy for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ that

$$V_\varepsilon(x) = \exp\big(\varepsilon(1+\|x\|^2)^{1/2}\big). \tag{224}$$

Observe that Lemma 2.15 implies for all $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ that

$$\operatorname{Trace}\big(BB^*(\operatorname{Hess} V_\varepsilon)(x)\big) + \frac{\|B^*(\nabla V_\varepsilon)(x)\|^2}{V_\varepsilon(x)} \leqslant 2(\varepsilon^2 + \varepsilon d)\beta V_\varepsilon(x). \tag{225}$$

In addition, note that the fact that for all $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ it holds that $g(x,v) = \lambda v - f(x,v)$ and (221) ensure for all $\theta \in \Theta$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} U_n^\theta(x) = \sum_{k=1}^{n-1}\frac{1}{\lambda M^{(n-k)}}\left[\sum_{m=1}^{M^{(n-k)}}\left(g\Big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{(\theta,k,m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\Big)\right.\right. \\ \left.\left. - g\Big(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{(\theta,k,-m)}(x + BW^{(\theta,k,m)}_{R^{(\theta,k,m)}})\Big)\right)\right] + \frac{1}{\lambda M^n}\left[\sum_{m=1}^{M^n} g(x + BW^{(\theta,0,m)}_{R^{(\theta,0,m)}}, 0)\right]. \end{aligned} \tag{226}$$

Moreover, observe that the fact that for all $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ it holds that $g(x,v) = \lambda v - f(x,v)$ and (222) ensure that u is a viscosity solution of

$$\lambda u(x) - \tfrac{1}{2}\operatorname{Trace}\big(BB^*(\operatorname{Hess} u)(x)\big) = g(x, u(x)) \tag{227}$$

for $x \in \mathbb{R}^d$. In addition, note that the fact that for all $\varepsilon \in (0,\infty)$ it holds that $\sup_{x\in\mathbb{R}^d}(\frac{|g(x,0)|+|u(x)|}{V_\varepsilon(x)}) < \infty$ guarantees for all $\varepsilon \in (0,\infty)$ that $\inf_{r\in(0,\infty)}[\sup_{\|x\|>r}(\frac{|g(x,0)|+|u(x)|}{V_\varepsilon(x)})] = 0$. Corollary 3.12 (with $d \leftarrow d$, $M \leftarrow M$, $\lambda \leftarrow \lambda$, $\Theta \leftarrow \Theta$, $B \leftarrow B$, $f \leftarrow g$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(W^\theta)_{\theta\in\Theta} \leftarrow (W^\theta)_{\theta\in\Theta}$, $(R^\theta)_{\theta\in\Theta} \leftarrow (R^\theta)_{\theta\in\Theta}$, $L \leftarrow L$, $\rho \leftarrow (\varepsilon^2 + \varepsilon d)\beta$, $u \leftarrow u$, $V \leftarrow V_\varepsilon$ for $\varepsilon \in (0,\infty)$ in the notation of Corollary 3.12), (225), (226), and (227) therefore demonstrate for all $\theta \in \Theta$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$, $\varepsilon \in (0,\infty)$ with $2(\varepsilon^2 + \varepsilon d)\beta < \lambda - L$ that

$$\begin{aligned} &\frac{\left(\mathbb{E}\left[|U_n^\theta(x) - u(x)|^2\right]\right)^{1/2}}{V_\varepsilon(x)} \\ &\leqslant \frac{\left[\sup_{y\in\mathbb{R}^d}\left(\frac{|g(y,0)|}{V_\varepsilon(y)}\right)\right]}{(\sqrt{\lambda} - \sqrt{L})\sqrt{\lambda - (L + 2(\varepsilon^2 + \varepsilon d)[\sup_{y\in\mathbb{R}^d\setminus\{0\}}(\frac{\|By\|}{\|y\|})]^2)}} \left[\frac{1}{\sqrt{M}}\left(1 + (1+\sqrt{M})\sqrt{\frac{L}{\lambda}}\right)\right]^n. \end{aligned} \tag{228}$$

This and the fact that for all $y \in \mathbb{R}^d$ it holds that $-f(y,0) = g(y,0)$ establish (223). The proof of Corollary 3.13 is thus completed. ☐

3.6 Computational cost analysis for MLP approximations

Lemma 3.14. *Let $\alpha, \beta \in [0,\infty)$, $M \in [1,\infty)$, $(C_n)_{n\in\mathbb{N}_0} \subseteq [0,\infty)$ satisfy for all $n \in \mathbb{N}$ that*

$$C_n \leqslant \alpha M^n + \sum_{k=1}^{n-1} M^{(n-k)}(\beta + C_k + C_{k-1}). \tag{229}$$

Then it holds for all $n \in \mathbb{N}$ that

$$C_n \leqslant \left[\frac{\alpha + \beta + C_0}{2}\right](2M+1)^n \leqslant \left[\frac{\alpha + \beta + C_0}{2}\right](3M)^n. \tag{230}$$

Proof of Lemma 3.14. Throughout this proof let $c_n \in [0,\infty)$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}_0$ that $c_n = \frac{C_n}{M^n}$ and let $S_n \in [0,\infty)$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}_0$ that

$$S_n = \frac{\beta n}{1 + 1/M} + \frac{\beta}{(1 + 1/M)^2} + \frac{\alpha - c_0}{1 + 1/M} + \sum_{k=0}^{n} c_k. \tag{231}$$

Note that (229), the fact that $(c_n)_{n\in\mathbb{N}_0} \subseteq [0,\infty)$, and the assumption that $M \geqslant 1$ ensure for all $n \in \mathbb{N}$ that

$$\begin{aligned} c_n &\leqslant \alpha + \sum_{k=1}^{n-1} M^{-k}(\beta + C_k + C_{k-1}) \leqslant \alpha + (n-1)\beta + \sum_{k=1}^{n-1}(c_k + \frac{c_{k-1}}{M}) \\ &= \alpha + (n-1)\beta + \sum_{k=1}^{n-1} c_k + \frac{1}{M}\sum_{k=0}^{n-2} c_k \leqslant \alpha - c_0 + (n-1)\beta + (1 + \frac{1}{M})\sum_{k=0}^{n-1} c_k \\ &= (1 + \frac{1}{M})S_{n-1} - \frac{\beta}{1 + 1/M}. \end{aligned} \tag{232}$$

Combining this with (231) proves for all $n \in \mathbb{N} \cap [2,\infty)$ that

$$S_n = S_{n-1} + \frac{\beta}{1 + 1/M} + c_n \leqslant (2 + \frac{1}{M})S_{n-1}. \tag{233}$$

This implies for all $n \in \mathbb{N}$ that $S_n \leqslant (2+\frac{1}{M})^{n-1} S_1$. Combining this with (232) and the assumption that $\beta \geqslant 0$ demonstrates for all $n \in \mathbb{N} \cap [2,\infty)$ that

$$c_n \leqslant (1 + {}^1\!/\!_M) S_{n-1} - \frac{\beta}{1 + {}^1\!/\!_M} \leqslant (1 + {}^1\!/\!_M) S_{n-1} \leqslant (1 + {}^1\!/\!_M)(2 + {}^1\!/\!_M)^{n-2} S_1. \tag{234}$$

Next observe that (232) implies that $c_1 \leqslant \alpha$. Combining this with (231) ensures that

$$\begin{aligned} S_1 &= \frac{\beta}{1 + {}^1\!/\!_M} + \frac{\beta}{(1 + {}^1\!/\!_M)^2} + \frac{\alpha - c_0}{1 + {}^1\!/\!_M} + c_0 + c_1 \\ &\leqslant \frac{\beta}{1 + {}^1\!/\!_M} + \frac{\beta}{(1 + {}^1\!/\!_M)^2} + \frac{\alpha - c_0}{1 + {}^1\!/\!_M} + c_0 + \alpha \\ &= \frac{2 + {}^1\!/\!_M}{1 + {}^1\!/\!_M} \left[\frac{\beta}{1 + {}^1\!/\!_M} + \alpha \right] + \frac{{}^1\!/\!_M}{1 + {}^1\!/\!_M} C_0 \leqslant \frac{2 + {}^1\!/\!_M}{1 + {}^1\!/\!_M} (\alpha + \beta + c_0). \end{aligned} \tag{235}$$

This and (234) show for all $n \in \mathbb{N} \cap [2,\infty)$ that

$$\begin{aligned} c_n &\leqslant (2 + {}^1\!/\!_M)^{n-2} (1 + {}^1\!/\!_M) \left[\frac{2 + {}^1\!/\!_M}{1 + {}^1\!/\!_M} \right] (\alpha + \beta + c_0) = (2 + {}^1\!/\!_M)^{n-1} (\alpha + \beta + c_0) \\ &\leqslant (2 + {}^1\!/\!_M)^n \left[\frac{\alpha + \beta + c_0}{2} \right]. \end{aligned} \tag{236}$$

Combining this with the fact that $c_1 \leqslant \alpha \leqslant (2 + {}^1\!/\!_M)\frac{\alpha+\beta+c_0}{2}$ and the fact that for all $n \in \mathbb{N}_0$ it holds that $C_n = c_n M^n$ establishes (230). The proof of Lemma 3.14 is thus completed. ☐

Lemma 3.15. *Let $m \in \mathbb{N}_0$, $\alpha, \beta, \kappa_1, \kappa_2 \in \mathbb{R}$ satisfy $0 < \alpha < 1 < \beta$, let $c_n, e_n \in [0,\infty)$, $n \in \mathbb{N}_0 \cap [m,\infty)$, assume for all $n \in \mathbb{N}_0 \cap [m,\infty)$ that $e_n \leqslant \kappa_1 \alpha^n$ and $c_n \leqslant \kappa_2 \beta^n$, and let $N \colon (0,1] \to \mathbb{N}_0 \cap [m,\infty)$ satisfy for all $\varepsilon \in (0,1]$ that*

$$N_\varepsilon = \min\{n \in \mathbb{N}_0 \cap [m,\infty) \colon \kappa_1 \alpha^n \leqslant \varepsilon\}. \tag{237}$$

Then it holds for all $\varepsilon \in (0,1]$ that $e_{N_\varepsilon} \leqslant \varepsilon$ and

$$c_{N_\varepsilon} \leqslant \kappa_2 \max\left\{\beta^m, \beta \kappa_1^{(\frac{\ln(\beta)}{\ln(1/\alpha)})}\right\} \left[\frac{1}{\varepsilon}\right]^{(\frac{\ln(\beta)}{\ln(1/\alpha)})}. \tag{238}$$

Proof of Lemma 3.15. First, observe that (237) ensures for all $\varepsilon \in (0,1]$ that $e_{N_\varepsilon} \leqslant \varepsilon$. Moreover, note that (237) guarantees for all $\varepsilon \in (0,1]$ with $N_\varepsilon \geqslant m+1$ that $\kappa_1 \alpha^{N_\varepsilon - 1} > \varepsilon$. Hence, we obtain for all $\varepsilon \in (0,1]$ with $N_\varepsilon \geqslant m+1$ that

$$(N_\varepsilon - 1)\ln({}^1\!/\!_\alpha) < \ln(\kappa_1) - \ln(\varepsilon). \tag{239}$$

This implies for all $\varepsilon \in (0,1]$ with $N_\varepsilon \geqslant m+1$ that

$$\begin{aligned} c_{N_\varepsilon} &\leqslant \kappa_2 \beta^{N_\varepsilon} = \kappa_2 \beta \exp((N_\varepsilon - 1)\ln(\beta)) \leqslant \kappa_2 \beta \exp\!\left((\ln(\kappa_1) - \ln(\varepsilon)) \frac{\ln(\beta)}{\ln({}^1\!/\!_\alpha)} \right) \\ &= \left[\kappa_2 \beta \kappa_1^{(\frac{\ln(\beta)}{\ln(1/\alpha)})} \right] \left[\frac{1}{\varepsilon} \right]^{(\frac{\ln(\beta)}{\ln(1/\alpha)})}. \end{aligned} \tag{240}$$

Next note that the assumption that for all $n \in \mathbb{N}_0 \cap [m,\infty)$ it holds that $c_n \leqslant \kappa_2 \beta^n$ and the fact that $\frac{\ln(\beta)}{\ln({}^1\!/\!_\alpha)} \in (0,\infty)$ ensure that for all $\varepsilon \in (0,1]$ with $N_\varepsilon = m$ it holds that

$$c_{N_\varepsilon} = c_m \leqslant \kappa_2 \beta^m \leqslant \kappa_2 \beta^m \left[\frac{1}{\varepsilon}\right]^{(\frac{\ln(\beta)}{\ln(1/\alpha)})}. \tag{241}$$

This and (240) establish (238). The proof of Lemma 3.15 is thus completed. ☐

3.7 Overall complexity analysis for MLP approximations

Theorem 3.16. *Let* $\kappa, L, \mathfrak{n}, p, q, r, s \in [0,\infty)$, $\lambda \in (L,\infty)$, $M \in \mathbb{N} \cap ((\sqrt{\lambda}+\sqrt{L})^2(\sqrt{\lambda}-\sqrt{L})^{-2}, \infty)$, $\alpha = -\ln(3M)[\ln(1/\sqrt{M} + (1+1/\sqrt{M})\sqrt{L/\lambda})]^{-1}$, $\Theta = \cup_{n\in\mathbb{N}}\mathbb{Z}^n$, *let* $u_d \in C(\mathbb{R}^d,\mathbb{R})$, $d \in \mathbb{N}$, *let* $B_d \in \mathbb{R}^{d\times d}$, $d \in \mathbb{N}$, *let* $f_d \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $d \in \mathbb{N}$, *assume for every* $d \in \mathbb{N}$ *that* u_d *is a viscosity solution of*

$$\tfrac{1}{2}\operatorname{Trace}(B_d(B_d)^*(\operatorname{Hess} u_d)(x)) = f_d(x, u_d(x)) \tag{242}$$

for $x \in \mathbb{R}^d$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W^{d,\theta}\colon [0,\infty)\times\Omega \to \mathbb{R}^d$, $\theta \in \Theta$, $d \in \mathbb{N}$, *be i.i.d. standard Brownian motions, let* $R^\theta\colon \Omega \to [0,\infty)$, $\theta \in \Theta$, *be i.i.d. random variables, assume that* $(R^\theta)_{\theta\in\Theta}$ *and* $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ *are independent, let* $\|\cdot\|\colon (\cup_{d\in\mathbb{N}}\mathbb{R}^d) \to [0,\infty)$ *satisfy for all* $d \in \mathbb{N}$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ *that* $\|x\| = [\sum_{i=1}^d |x_i|^2]^{1/2}$, *assume for all* $d \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ *that* $\|B_d x\| \leqslant \kappa d^r \|x\|$, $|f_d(x,0)| \leqslant \kappa d^s(1+\|x\|^p)$, $|f_d(x,v) - f_d(x,w) - \lambda(v-w)| \leqslant L|v-w|$, $\sup_{y\in\mathbb{R}^d}[|u_d(y)|\exp(-\varepsilon\|y\|)] < \infty$, *and* $\mathbb{P}(R^0 \geqslant \varepsilon) = e^{-\lambda\varepsilon}$, *let* $U_n^{d,\theta} = (U_n^{d,\theta}(x))_{x\in\mathbb{R}^d}\colon \mathbb{R}^d\times\Omega \to \mathbb{R}$, $\theta \in \Theta$, $d, n \in \mathbb{N}_0$, *satisfy for all* $d, n \in \mathbb{N}$, $\theta \in \Theta$, $x \in \mathbb{R}^d$ *that* $U_0^{d,\theta}(x) = 0$ *and*

$$\begin{aligned} U_n^{d,\theta}(x) &= \frac{-1}{\lambda M^n}\left[\sum_{m=1}^{M^n} f_d(x + B_d W^{d,(\theta,0,m)}_{R^{(\theta,0,m)}}, 0)\right] \\ &+ \sum_{k=1}^{n-1}\frac{1}{\lambda M^{(n-k)}}\left[\sum_{m=1}^{M^{(n-k)}}\left(\lambda\Big[U_k^{d,(\theta,k,m)}(x + B_d W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}) - U_{k-1}^{d,(\theta,k,-m)}(x + B_d W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}})\Big]\right.\right. \\ &\qquad - \Big[f_d\Big(x + B_d W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{d,(\theta,k,m)}(x + B_d W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}})\Big) \\ &\qquad\quad \left.\left. - f_d\Big(x + B_d W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{d,(\theta,k,-m)}(x + B_d W^{d,(\theta,k,m)}_{R^{(\theta,k,m)}})\Big)\Big]\right)\right], \end{aligned} \tag{243}$$

and let $\mathfrak{C}_{d,n} \in \mathbb{R}$, $d, n \in \mathbb{N}_0$, *satisfy for all* $d, n \in \mathbb{N}_0$ *that* $\mathfrak{C}_{d,n} \leqslant (d+1)M^n + \sum_{k=1}^{n-1} M^{(n-k)}(d+1+\mathfrak{C}_{d,k} + \mathfrak{C}_{d,k-1})$. *Then there exist* $c \in \mathbb{R}$ *and* $\mathfrak{N}\colon (0,1]\times\mathbb{N} \to (\mathbb{Z}\cap[\mathfrak{n},\infty))$ *such that for all* $\varepsilon \in (0,1]$, $d \in \mathbb{N}$ *it holds that*

$$\mathfrak{C}_{d,\mathfrak{N}_{\varepsilon,d}} \leqslant c d^{1+\alpha(s+p\max\{q,2r+1\})}\varepsilon^{-\alpha} \quad \text{and} \quad \sup\nolimits_{x\in\mathbb{R}^d, \|x\|\leqslant\kappa d^q}\big(\mathbb{E}\big[|u_d(x) - U^{d,0}_{\mathfrak{N}_{\varepsilon,d}}(x)|^2\big]\big)^{1/2} \leqslant \varepsilon. \tag{244}$$

Proof of Theorem 3.16. Throughout this proof let $\lceil\cdot\rceil\colon \mathbb{R}\to\mathbb{Z}$ satisfy for all $x \in \mathbb{R}$ that $\lceil x\rceil = \min(\mathbb{Z}\cap[x,\infty))$, let $c_1, c_2, c_3 \in \mathbb{R}$ satisfy for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $\|B_d x\| \leqslant c_1 d^r\|x\|$, $|f_d(x,0)| \leqslant c_2 d^s(1+\|x\|^p)$, and

$$c_3 = \frac{4c_2((\lceil p\rceil)!)}{(\sqrt{\lambda}-\sqrt{L})^2}\max\left\{1, \left[\frac{8|c_1|^2}{\lambda - L}\right]^p\right\}\exp(1+\kappa), \tag{245}$$

and let $\eta_d \in (0,\infty)$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$ that $\eta_d = \sup\{t \in [0,\infty)\colon (8|c_1|^2 d^{2r+1} t \leqslant \lambda - L \text{ and } td^q \leqslant 1)\}$. Observe that the fact that for all $d \in \mathbb{N}$ it holds that $\eta_d \leqslant 1$, the fact that for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that $\|B_d x\| \leqslant c_1 d^r\,\|x\|$, and the fact that for all $d \in \mathbb{N}$ it holds that $8\eta_d|c_1|^2 d^{2r+1} \leqslant \lambda - L$ ensure that for all $d \in \mathbb{N}$ it holds that

$$|\eta_d|^2\left[\sup_{x\in\mathbb{R}^d\setminus\{0\}}\left(\frac{\|B_d x\|}{\|x\|}\right)\right]^2 \leqslant d\eta_d\left[\sup_{x\in\mathbb{R}^d\setminus\{0\}}\left(\frac{\|B_d x\|}{\|x\|}\right)\right]^2 \leqslant \eta_d|c_1|^2 d^{2r+1} \leqslant \frac{\lambda - L}{8}. \tag{246}$$

This implies for all $d \in \mathbb{N}$ that

$$\lambda - L - 2(|\eta_d|^2 + d\eta_d)\left[\sup_{x\in\mathbb{R}^d\setminus\{0\}}\left(\frac{\|B_d x\|}{\|x\|}\right)\right]^2 \geqslant \frac{\lambda - L}{2}. \tag{247}$$

Next note that the fact that for all $d \in \mathbb{N}$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ it holds that $\|x\| \leqslant \sum_{i=1}^{d} |x_i|$ and the assumption that for all $d \in \mathbb{N}$, $\varepsilon \in (0, \infty)$ it holds that $\sup_{x\in\mathbb{R}^d}[|u_d(x)| \exp(-\varepsilon \|x\|)] < \infty$ prove that for all $d \in \mathbb{N}$, $\varepsilon \in (0, \infty)$ it holds that

$$\sup_{x=(x_1,\dots,x_d)\in\mathbb{R}^d} \left[\frac{|u_d(x)|}{\exp(\varepsilon \sum_{i=1}^{d} |x_i|)} \right] \leqslant \sup_{x\in\mathbb{R}^d} \left[\frac{|u_d(x)|}{\exp(\varepsilon\|x\|)} \right] < \infty. \tag{248}$$

Corollary 3.13 (with $d \leftarrow d$, $M \leftarrow M$, $B \leftarrow B_d$, $L \leftarrow L$, $\lambda \leftarrow \lambda$, $\Theta \leftarrow \Theta$, $f \leftarrow f_d$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(W^\theta)_{\theta\in\Theta} \leftarrow (W^{d,\theta})_{\theta\in\Theta}$, $(R^\theta)_{\theta\in\Theta} \leftarrow (R^\theta)_{\theta\in\Theta}$, $u \leftarrow u_d$ for $d \in \mathbb{N}$ in the notation of Corollary 3.13) therefore ensures for all $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&\left(\mathbb{E}\left[|U_n^{d,0}(x) - u_d(x)|^2\right]\right)^{1/2} \\
&\leqslant \left[\frac{1}{\sqrt{M}}\left(1 + (1 + \sqrt{M})\sqrt{L/\lambda}\right)\right]^n \left[\sup_{y\in\mathbb{R}^d}\left(\frac{|f_d(y,0)|}{\exp(\eta_d(1+\|y\|^2)^{1/2})}\right)\right] \frac{\sqrt{2}\exp(\eta_d(1+\|x\|^2)^{1/2})}{(\sqrt{\lambda}-\sqrt{L})\sqrt{\lambda - L}} \\
&\leqslant 2c_2\left[\frac{1}{\sqrt{M}}\left(1 + (1 + \sqrt{M})\sqrt{L/\lambda}\right)\right]^n \left[\sup_{y\in\mathbb{R}^d}\left(\frac{d^s(1+\|y\|^p)}{\exp(\eta_d(1+\|y\|^2)^{1/2})}\right)\right] \frac{\exp(\eta_d(1+\|x\|^2)^{1/2})}{(\sqrt{\lambda}-\sqrt{L})^2} \\
&\leqslant 4c_2\left[\frac{1}{\sqrt{M}}\left(1 + (1 + \sqrt{M})\sqrt{L/\lambda}\right)\right]^n \left[\sup_{y\in\mathbb{R}^d}\left(\frac{d^s(1+\|y\|^2)^{p/2}}{\exp(\eta_d(1+\|y\|^2)^{1/2})}\right)\right] \frac{\exp(\eta_d(1+\|x\|^2)^{1/2})}{(\sqrt{\lambda}-\sqrt{L})^2}.
\end{aligned} \tag{249}$$

The fact that for all $x \in (0, \infty)$ it holds that $x^p \leqslant ((\lceil p \rceil)!) \exp(x)$ and the fact that for all $d \in \mathbb{N}$ it holds that $\eta_d d^q \leqslant 1$ hence imply that for all $d \in \mathbb{N}$, $n \in \mathbb{N}_0$ it holds that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d, \|x\|\leqslant\kappa d^q} \left(\mathbb{E}\left[|U_n^{d,0}(x) - u_d(x)|^2\right]\right)^{1/2} \\
&\leqslant \frac{4c_2((\lceil p \rceil)!)}{(\sqrt{\lambda}-\sqrt{L})^2}\left[\frac{1}{\sqrt{M}}\left(1 + (1 + \sqrt{M})\sqrt{L/\lambda}\right)\right]^n \frac{d^s}{\eta_d^p}\exp(1+\kappa) \\
&\leqslant c_3 \max\left\{d^{s+pq}, d^{s+p(2r+1)}\right\}\left[\frac{1}{\sqrt{M}}\left(1 + (1 + \sqrt{M})\sqrt{L/\lambda}\right)\right]^n.
\end{aligned} \tag{250}$$

Moreover, observe that Lemma 3.14 (with $\alpha \leftarrow d+1$, $\beta \leftarrow d+1$, $M \leftarrow M$, $(C_n)_{n\in\mathbb{N}_0} \leftarrow (\max\{\mathfrak{C}_{d,n}, 0\})_{n\in\mathbb{N}_0}$ for $d \in \mathbb{N}$ in the notation of Lemma 3.14) ensures for all $d, n \in \mathbb{N}$ that $\mathfrak{C}_{d,n} \leqslant 3d(3M)^n$. Combining this with the fact that for all $d \in \mathbb{N}$ it holds that $\mathfrak{C}_{d,0} \leqslant d+1 \leqslant 3d$ implies that for all $d \in \mathbb{N}$, $n \in \mathbb{N}_0$ it holds that $\mathfrak{C}_{d,n} \leqslant 3d(3M)^n$. Next let $N\colon (0,1] \times \mathbb{N} \to \mathbb{N}_0$ satisfy for all $d \in \mathbb{N}$, $\varepsilon \in (0,1]$ that

$$N_{\varepsilon,d} = \min\left\{n \in \mathbb{N}_0 \cap [\mathfrak{n}, \infty) \colon c_3 d^{s+p\max\{q,2r+1\}}\left[\frac{1}{\sqrt{M}}\left(1 + (1 + \sqrt{M})\sqrt{L/\lambda}\right)\right]^n \leqslant \varepsilon\right\}. \tag{251}$$

Lemma 3.15 (with $\alpha \leftarrow \frac{1}{\sqrt{M}}[1 + (1+\sqrt{M})\sqrt{L/\lambda}]$, $\beta \leftarrow 3M$, $\kappa_1 \leftarrow c_3 d^{s+p\max\{q,2r+1\}}$, $\kappa_2 \leftarrow 3d$, $m \leftarrow \lceil \mathfrak{n} \rceil$, $(e_n)_{n\in\mathbb{Z}\cap[\mathfrak{n},\infty)} \leftarrow (\sup_{x\in\mathbb{R}^d,\|x\|\leqslant\kappa d^q}(\mathbb{E}[|U_n^{d,0}(x) - u_d(x)|^2])^{1/2})_{n\in\mathbb{Z}\cap[\mathfrak{n},\infty)}$, $(c_n)_{n\in\mathbb{Z}\cap[\mathfrak{n},\infty)} \leftarrow (\max\{\mathfrak{C}_{d,n}, 0\})_{n\in\mathbb{Z}\cap[\mathfrak{n},\infty)}$ for $d \in \mathbb{N}$ in the notation of Lemma 3.15), the fact that for all $d \in \mathbb{N}$, $n \in \mathbb{N}_0$ it holds that $\mathfrak{C}_{d,n} \leqslant 3d(3M)^n$, and (250) therefore imply that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1]$ it holds that $\sup_{x\in\mathbb{R}^d,\|x\|\leqslant\kappa d^q}(\mathbb{E}[|U_{N_{\varepsilon,d}}^{d,0}(x) - u_d(x)|^2])^{1/2} \leqslant \varepsilon$ and

$$\begin{aligned}
\mathfrak{C}_{d,N_{\varepsilon,d}} &\leqslant 3d\max\left\{(3M)^{\lceil\mathfrak{n}\rceil}, 3M(c_3 d^{s+p\max\{q,2r+1\}})^\alpha\right\}\left[\frac{1}{\varepsilon}\right]^\alpha \\
&\leqslant \max\left\{3(3M)^{\lceil\mathfrak{n}\rceil}, 9Mc_3^\alpha\right\} d^{1+(s+p\max\{q,2r+1\})\alpha}\left[\frac{1}{\varepsilon}\right]^\alpha.
\end{aligned} \tag{252}$$

This establishes (244). The proof of Theorem 3.16 is thus completed. □

Corollary 3.17. *Let $c, L \in [0,\infty)$, $\lambda \in (L,\infty)$, $M \in \mathbb{N} \cap ((\sqrt{\lambda}+\sqrt{L})^2(\sqrt{\lambda}-\sqrt{L})^{-2},\infty)$, $\Theta = \cup_{n\in\mathbb{N}}\mathbb{Z}^n$, let $u_d \in C(\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, let $B_d \in \mathbb{R}^{d\times d}$, $d \in \mathbb{N}$, let $f_d \in C(\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $d\in\mathbb{N}$, assume for every $d\in\mathbb{N}$ that u_d is a viscosity solution of*

$$\tfrac{1}{2}\operatorname{Trace}(B_d(B_d)^*(\operatorname{Hess} u_d)(x)) = f_d(x,u_d(x)) \tag{253}$$

for $x \in \mathbb{R}^d$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W^{d,\theta}\colon [0,\infty)\times\Omega\to\mathbb{R}^d$, $\theta\in\Theta$, $d\in\mathbb{N}$, be i.i.d. standard Brownian motions, let $R^\theta\colon\Omega\to[0,\infty)$, $\theta\in\Theta$, be i.i.d. random variables, assume that $(R^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, let $\|\cdot\|\colon(\cup_{d\in\mathbb{N}}\mathbb{R}^d)\to[0,\infty)$ satisfy for all $d\in\mathbb{N}$, $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ that $\|x\| = [\sum_{i=1}^d |x_i|^2]^{1/2}$, assume for all $d\in\mathbb{N}$, $x\in\mathbb{R}^d$, $v,w\in\mathbb{R}$, $\varepsilon\in(0,\infty)$ that $\|B_d x\| \leqslant cd^c\|x\|$, $|f_d(x,v)-f_d(x,w)-\lambda(v-w)| \leqslant L|v-w|$, $|f_d(x,0)| \leqslant cd^c(1+\|x\|^c)$, $\sup_{y\in\mathbb{R}^d}[|u_d(y)|\exp(-\varepsilon\|y\|)] < \infty$, and $\mathbb{P}(R^0 \geqslant \varepsilon) = e^{-\lambda\varepsilon}$, let $U_n^{d,\theta} = (U_n^{d,\theta}(x))_{x\in\mathbb{R}^d}\colon\mathbb{R}^d\times\Omega\to\mathbb{R}$, $\theta\in\Theta$, $d,n\in\mathbb{N}_0$, satisfy for all $d,n\in\mathbb{N}$, $\theta\in\Theta$, $x\in\mathbb{R}^d$ that $U_0^{d,\theta}(x)=0$ and

$$\begin{aligned} U_n^{d,\theta}(x) &= \frac{-1}{\lambda M^n}\left[\sum_{m=1}^{M^n} f_d(x+B_dW^{d,(\theta,0,m)}_{R^{(\theta,0,m)}},0)\right] \\ &+ \sum_{k=1}^{n-1}\frac{1}{\lambda M^{(n-k)}}\left[\sum_{m=1}^{M^{(n-k)}}\left(\lambda\Big[U_k^{d,(\theta,k,m)}(x+B_dW^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}) - U_{k-1}^{d,(\theta,k,-m)}(x+B_dW^{d,(\theta,k,m)}_{R^{(\theta,k,m)}})\Big]\right.\right. \\ &\qquad - \Big[f_d\Big(x+B_dW^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}, U_k^{d,(\theta,k,m)}(x+B_dW^{d,(\theta,k,m)}_{R^{(\theta,k,m)}})\Big) \\ &\qquad\quad \left.\left. - f_d\Big(x+B_dW^{d,(\theta,k,m)}_{R^{(\theta,k,m)}}, U_{k-1}^{d,(\theta,k,-m)}(x+B_dW^{d,(\theta,k,m)}_{R^{(\theta,k,m)}})\Big)\Big]\right)\right], \end{aligned} \tag{254}$$

and let $\mathfrak{C}_{d,n}\in\mathbb{R}$, $d,n\in\mathbb{N}_0$, satisfy for all $d,n\in\mathbb{N}_0$ that $\mathfrak{C}_{d,n} \leqslant (d+1)M^n + \sum_{k=1}^{n-1} M^{(n-k)}(d+1+\mathfrak{C}_{d,k}+\mathfrak{C}_{d,k-1})$. Then there exist $\kappa\in\mathbb{R}$ and $\mathfrak{N}\colon(0,1]\times\mathbb{N}\to\mathbb{N}$ such that for all $\varepsilon\in(0,1]$, $d\in\mathbb{N}$ it holds that

$$\mathfrak{C}_{d,\mathfrak{N}_{\varepsilon,d}} \leqslant \kappa d^\kappa \varepsilon^{-\kappa} \qquad \text{and} \qquad \sup\nolimits_{x\in[-cd^c,cd^c]^d}\left(\mathbb{E}\big[|u_d(x)-U^{d,0}_{\mathfrak{N}_{\varepsilon,d}}(x)|^2\big]\right)^{1/2} \leqslant \varepsilon. \tag{255}$$

Proof of Corollary 3.17. Throughout this proof let $\alpha\in\mathbb{R}$ satisfy $\alpha = -\ln(3M)[\ln(1/\sqrt{M} + (1+1/\sqrt{M})\sqrt{L/\lambda})]^{-1}$. Note that Theorem 3.16 (with $\kappa \leftarrow c$, $L \leftarrow L$, $\mathfrak{n}\leftarrow 1$, $p\leftarrow c$, $q\leftarrow c+1/2$, $r\leftarrow c$, $s\leftarrow c$, $\lambda\leftarrow\lambda$, $M\leftarrow M$, $\Theta\leftarrow\Theta$, $(u_d)_{d\in\mathbb{N}}\leftarrow(u_d)_{d\in\mathbb{N}}$, $(B_d)_{d\in\mathbb{N}}\leftarrow(B_d)_{d\in\mathbb{N}}$, $(f_d)_{d\in\mathbb{N}}\leftarrow(f_d)_{d\in\mathbb{N}}$, $(\Omega,\mathcal{F},\mathbb{P})\leftarrow(\Omega,\mathcal{F},\mathbb{P})$, $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}\leftarrow(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$, $(R^\theta)_{\theta\in\Theta}\leftarrow(R^\theta)_{\theta\in\Theta}$, $(\mathfrak{C}_{d,n})_{(d,n)\in\mathbb{N}_0\times\mathbb{N}_0}\leftarrow(\mathfrak{C}_{d,n})_{(d,n)\in\mathbb{N}_0\times\mathbb{N}_0}$ in the notation of Theorem 3.16) ensures that there exist $\gamma\in(0,\infty)$ and $\mathfrak{N}\colon(0,1]\times\mathbb{N}\to\mathbb{N}$ which satisfy for all $\varepsilon\in(0,1]$, $d\in\mathbb{N}$ that

$$\mathfrak{C}_{d,\mathfrak{N}_{\varepsilon,d}} \leqslant \gamma d^{1+2(c^2+c)\alpha}\varepsilon^{-\alpha} \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^d,\|x\|\leqslant cd^{(c+1/2)}}\left(\mathbb{E}\Big[\big|U^{d,0}_{\mathfrak{N}_{\varepsilon,d}}(x)-u_d(x)\big|^2\Big]\right)^{1/2} \leqslant \varepsilon. \tag{256}$$

This, the fact that for all $d\in\mathbb{N}$, $x\in[-cd^c,cd^c]^d$ it holds that $\|x\| \leqslant cd^{c+1/2}$, and the fact that for all $d\in\mathbb{N}$, $\varepsilon\in(0,1]$ it holds that $\gamma d^{1+2(c^2+c)\alpha}\varepsilon^{-\alpha} \leqslant \max\{1,\gamma,\alpha,1+2(c^2+c)\alpha\}(d/\varepsilon)^{\max\{1,\gamma,\alpha,1+2(c^2+c)\alpha\}}$ establish (255). The proof of Corollary 3.17 is thus completed. □

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References

[1] Agarwal, A., and Claisse, J. Branching diffusion representation of semi-linear elliptic PDEs and estimation using Monte Carlo method. *arXiv:1704.00328* (2017), 24 pages.

[2] Barles, G., and Perthame, B. Discontinuous solutions of deterministic optimal stopping time problems. *RAIRO Modél. Math. Anal. Numér. 21*, 4 (1987), 557–579.

[3] Beck, C., Becker, S., Cheridito, P., Jentzen, A., and Neufeld, A. Deep splitting method for parabolic PDEs. *arXiv:1907.03452* (2019), 40 pages.

[4] Beck, C., Becker, S., Grohs, P., Jaafari, N., and Jentzen, A. Solving stochastic differential equations and Kolmogorov equations by means of deep learning. *arXiv:1806.00421* (2018), 56 pages.

[5] Beck, C., E, W., and Jentzen, A. Machine learning approximation algorithms for high-dimensional fully nonlinear partial differential equations and second-order backward stochastic differential equations. *Journal of Nonlinear Science* (2017), 1–57.

[6] Beck, C., Gonon, L., Hutzenthaler, M., and Jentzen, A. On existence and uniqueness properties for solutions of stochastic fixed point equations. *arXiv:1908.03382* (2019), 33 pages.

[7] Beck, C., Hornung, F., Hutzenthaler, M., Jentzen, A., and Kruse, T. Overcoming the curse of dimensionality in the numerical approximation of Allen–Cahn partial differential equations via truncated full-history recursive multilevel Picard approximations. *arXiv:1907.06729* (2019), 31 pages.

[8] Becker, S., Cheridito, P., and Jentzen, A. Deep optimal stopping. *J. Mach. Learn. Res. 20* (2019), Paper No. 74, 25.

[9] Becker, S., Cheridito, P., Jentzen, A., and Welti, T. Solving high-dimensional optimal stopping problems using deep learning. *arXiv:1908.01602* (2019), 42 pages.

[10] Belak, C., Hoffmann, D., and Seifried, F. T. Probabilistic representations of nonlocal nonlinear PDEs via branching diffusions with jumps. *Available at SSRN 3451280* (2019), 28 pages.

[11] Berner, J., Grohs, P., and Jentzen, A. Analysis of the generalization error: Empirical risk minimization over deep artificial neural networks overcomes the curse of dimensionality in the numerical approximation of Black–Scholes partial differential equations. *arXiv:1809.03062* (2018), 35 pages.

[12] Bouchard, B., Tan, X., and Warin, X. Numerical approximation of general Lipschitz BSDEs with branching processes. *ESAIM: Proceedings and Surveys 65* (2019), 309–329.

[13] Bouchard, B., Tan, X., Warin, X., and Zou, Y. Numerical approximation of BSDEs using local polynomial drivers and branching processes. *Monte Carlo Methods and Applications 23*, 4 (2017), 241–263.

[14] Chan-Wai-Nam, Q., Mikael, J., and Warin, X. Machine learning for semi linear PDEs. *J. Sci. Comput. 79*, 3 (2019), 1667–1712.

[15] Chang, D., Liu, H., and Xiong, J. A branching particle system approximation for a class of FBSDEs. *Probab. Uncertain. Quant. Risk 1* (2016), Paper No. 9, 34.

[16] Chen, Y., and Wan, J. W. Deep neural network framework based on backward stochastic differential equations for pricing and hedging american options in high dimensions. *arXiv:1909.11532* (2019), 35 pages.

[17] Crandall, M. G., Evans, L. C., and Lions, P.-L. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc. 282*, 2 (1984), 487–502.

[18] Crandall, M. G., Ishii, H., and Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.) 27*, 1 (1992), 1–67.

[19] Da Prato, G., and Zabczyk, J. *Second order partial differential equations in Hilbert spaces*, vol. 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[20] Dockhorn, T. A discussion on solving partial differential equations using neural networks. *arXiv:1904.07200* (2019), 9 pages.

[21] E, W., Han, J., and Jentzen, A. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics 5* (2017), 349–380.

[22] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. *arXiv:1607.03295* (2016), 19 pages.

[23] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. *J. Sci. Comput. 79*, 3 (2019), 1534–1571.

[24] E, W., and Yu, B. The deep Ritz method: A deep learning-based numerical algorithm for solving variational problems. *arXiv:1710.00211* (2017), 14 pages.

[25] Elbrächter, D., Grohs, P., Jentzen, A., and Schwab, C. DNN expression rate analysis of high-dimensional PDEs: Application to option pricing. *arXiv:1809.07669* (2018), 50 pages.

[26] Fujii, M., Takahashi, A., and Takahashi, M. Asymptotic expansion as prior knowledge in deep learning method for high dimensional BSDEs. *arXiv:1710.07030* (2017), 16 pages.

[27] Giles, M. B., Jentzen, A., and Welti, T. Generalised multilevel Picard approximations. *arXiv:1911.03188* (2019), 61 pages.

[28] Gonon, L., Grohs, P., Jentzen, A., Kofler, D., and Šiška, D. Uniform error estimates for artificial neural network approximations for heat equations. *arXiv:1911.09647* (2019), 70 pages.

[29] Goudenege, L., Molent, A., and Zanette, A. Machine Learning for Pricing American Options in High Dimension. *arXiv:1903.11275* (2019), 11 pages.

[30] Grohs, P., Hornung, F., Jentzen, A., and von Wurstemberger, P. A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black–Scholes partial differential equations. *Accepted in Memoirs of the American Mathematical Society, arXiv:1809.02362* (2018), 124 pages.

[31] Grohs, P., Hornung, F., Jentzen, A., and Zimmermann, P. Space-time error estimates for deep neural network approximations for differential equations. *arXiv:1908.03833* (2019), 86 pages.

[32] Grohs, P., Jentzen, A., and Salimova, D. Deep neural network approximations for Monte Carlo algorithms. *arXiv:1908.10828* (2019), 45 pages.

[33] Hairer, M., Hutzenthaler, M., and Jentzen, A. Loss of regularity for Kolmogorov equations. *Ann. Probab. 43*, 2 (2015), 468–527.

[34] Han, J., Jentzen, A., and E, W. Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences 115*, 34 (2018), 8505–8510.

[35] Han, J., and Long, J. Convergence of the deep BSDE method for coupled FBSDEs. *arXiv:1811.01165* (2018), 26 pages.

[36] Han, J., Nica, M., and Stinchcombe, A. R. A derivative-free method for solving elliptic partial differential equations with deep neural networks. *arXiv:2001.06145* (2020), 25 pages.

[37] Henry-Labordère, P. Counterparty risk valuation: a marked branching diffusion approach. *arXiv:1203.2369* (2012), 17 pages.

[38] Henry-Labordere, P. Deep primal-dual algorithm for BSDEs: Applications of machine learning to CVA and IM. *Available at SSRN 3071506* (2017), 16 pages.

[39] Henry-Labordère, P., Oudjane, N., Tan, X., Touzi, N., and Warin, X. Branching diffusion representation of semilinear PDEs and Monte Carlo approximation. *Ann. Inst. Henri Poincaré Probab. Stat. 55*, 1 (2019), 184–210.

[40] Henry-Labordère, P., Tan, X., and Touzi, N. A numerical algorithm for a class of BSDEs via the branching process. *Stochastic Process. Appl. 124*, 2 (2014), 1112–1140.

[41] Huré, C., Pham, H., and Warin, X. Some machine learning schemes for high-dimensional nonlinear PDEs. *arXiv:1902.01599* (2019), 33 pages.

[42] Hutzenthaler, M., Jentzen, A., and Kruse, T. Overcoming the curse of dimensionality in the numerical approximation of parabolic partial differential equations with gradient-dependent nonlinearities. *arXiv:1912.02571* (2019), 33 pages.

[43] Hutzenthaler, M., Jentzen, A., Kruse, T., and Nguyen, T. A. A proof that rectified deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear heat equations. *Accepted in Springer Nat. Part. Diff. Equ. Appl., arXiv:1901.10854* (2019), 29 pages.

[44] Hutzenthaler, M., Jentzen, A., Kruse, T., Nguyen, T. A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *arXiv:1807.01212* (2018), 30 pages.

[45] Hutzenthaler, M., Jentzen, A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the approximative pricing of financial derivatives with default risks. *Accepted in Electronic Journal of Probability, arXiv:1903.05985* (2019), 71 pages.

[46] Hutzenthaler, M., and Kruse, T. Multi-level Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. *arXiv:1711.01080* (2017), 19 pages.

[47] Jentzen, A., Salimova, D., and Welti, T. A proof that deep artificial neural networks overcome the curse of dimensionality in the numerical approximation of Kolmogorov partial differential equations with constant diffusion and nonlinear drift coefficients. *arXiv:1809.07321* (2018), 48 pages.

[48] Kallenberg, O. *Foundations of modern probability*, second ed. Probability and its Applications (New York). Springer-Verlag, New York, 2002.

[49] Kutyniok, G., Petersen, P., Raslan, M., and Schneider, R. A theoretical analysis of deep neural networks and parametric PDEs. *arXiv:1904.00377* (2019), 43 pages.

[50] Long, Z., Lu, Y., Ma, X., and Dong, B. PDE-Net: Learning PDEs from Data. *arXiv:1710.09668* (2017), 15 pages.

[51] Lye, K. O., Mishra, S., and Ray, D. Deep learning observables in computational fluid dynamics. *arXiv:1903.03040* (2019), 57 pages.

[52] Magill, M., Qureshi, F., and de Haan, H. Neural networks trained to solve differential equations learn general representations. *Advances in Neural Information Processing Systems* (2018), 4075–4085.

[53] McKean, H. P. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math. 28*, 3 (1975), 323–331.

[54] Pham, H., Pham, H., and Warin, X. Neural networks-based backward scheme for fully nonlinear PDEs. *arXiv:1908.00412* (2019), 15 pages.

[55] Rasulov, A., Raimova, G., and Mascagni, M. Monte Carlo solution of Cauchy problem for a nonlinear parabolic equation. *Math. Comput. Simulation 80*, 6 (2010), 1118–1123.

[56] Reisinger, C., and Zhang, Y. Rectified deep neural networks overcome the curse of dimensionality for nonsmooth value functions in zero-sum games of nonlinear stiff systems. *arXiv:1903.06652* (2019), 34 pages.

[57] Sirignano, J., and Spiliopoulos, K. DGM: A deep learning algorithm for solving partial differential equations. *Journal of Computational Physics 375* (2018), 1339–1364.

[58] Skorohod, A. V. Branching diffusion processes. *Teor. Verojatnost. i Primenen. 9* (1964), 492–497.

[59] Watanabe, S. On the branching process for Brownian particles with an absorbing boundary. *J. Math. Kyoto Univ. 4* (1965), 385–398.