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Exponential Convergence of Mixed hp-DGFEM for the incompressible Navier-Stokes equations in $\mathbb{R}^2$

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In a polygon $\Omega \subset \mathbb{R}^2$, we consider mixed hp-discontinuous Galerkin approximations of the stationary, incompressible Navier-Stokes equations, subject to no-slip boundary conditions. We use geometrically corner-refined meshes and hp spaces with linearly increasing polynomial degrees. Based on recent results on analytic regularity of velocity field and pressure of Leray solutions in Ω, we prove exponential rates of convergence of the mixed hp-discontinuous Galerkin finite element method (hp-DGFEM), with respect to the number of degrees of freedom, for small data which is piecewise analytic.

Keywords: Mixed hp-FEM, discontinuous Galerkin methods, exponential convergence, Navier-Stokes equations
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1. Introduction

In a bounded polygon $\Omega \subset \mathbb{R}^2$ with $n \geqslant 3$ vertices and with n straight sides, we consider viscous, incompressible Newtonian flow subject to a body force acting on the fluid. The velocity and pressure fields of the fluid are solutions of the Navier-Stokes equations (NSE) in Ω, subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$ for the components of the velocity field. Due to the numerous models of viscous, incompressible flow in computational modelling for engineering and life-sciences, the numerical analysis of discretization methods for this boundary value problem has spawned a significant body of research during the past decades. Due to its elliptic nature, velocity and pressure fields exhibit elliptic regularity, as was observed rather early in Masuda (1967) and the references there. Analyticity of Leray-Hopf solutions up to analytic boundaries $\partial\Omega$ of bounded domains Ω was established, also for rather general nonlinear elliptic systems, in Morrey (2008, Chap. 5.7). Domains of engineering interest, however, typically contain corners and (in three space dimensions) edges. Also, even for smooth domains, in numerical simulation often changing boundary conditions are of interest. In these cases, analyticity of solutions of elliptic PDEs with otherwise analytic data is well-known to be lost. High regularity, however, is a necessary mathematical ingredient in the convergence analysis of so-called "high order numerical methods", such as higher order Finite Element Methods (FEM for short), Finite Volume and Finite Difference Methods (FVM and FDM for short). In particular,

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hp-FEM and so-called spectral element methods allow in principle arbitrary high, algebraic convergence rates and, in the presence of analytic regularity of solutions, also *exponential convergence rates*. Principal obstructions to realizing these rates in computational practice are twofold: first, due to the incompressibility constraint, *mixed variational formulations* of saddle point type need to be discretized and, second, due to the mentioned appearance of corner singularities in velocity and pressure fields, overall regularity in standard Sobolev or Besov spaces is rather low.

Both obstructions have been overcome. The incompressibility constraint has been handled by development of inf-sup stable velocity / pressure discretizations where the stability is either independent of or only weakly dependent on the discretization order. We refer to Baker *et al.* (1990); Karakashian & Jureidini (1998) and, for high order methods, to Stenberg & Suri (1996); Bernardi & Maday (1999); Schötzau & Schwab (1998); Schötzau & Wihler (2003) for results of this type. Also, *nonconforming* high-order discretizations have been developed which yield weakly or even exactly divergence-free velocity approximations. We refer to Cockburn *et al.* (2002, 2005, 2009, 2004, 2007); Schötzau *et al.* (2003); Baker *et al.* (1990); Waluga (2012); Lederer & Schöberl (2018). This topic continues to attract attention in the numerical analysis community, see, e.g., Cesmelioglu *et al.* (2017) and the references there.

The analytic regularity theory for solutions which was developed in Morrey (2008) and the references there has been extended to linear, elliptic boundary value problems with analytic coefficients and forcing, in domains with point singularities in Bolley *et al.* (1985); Babuška & Guo (1988). For the linear Stokes problem in polygonal domains, the results in Guo & Schwab (2006) prove analytic regularity and the exponential convergence for mixed, hp-DGFEM is shown in Schötzau & Wihler (2003). In recent work (Marcati & Schwab, 2019), corresponding analytic regularity results have been obtained by us for the NSE in polygonal domains.

In this paper, we establish exponential convergence rates of suitable hp-discontinuous Galerkin discretizations of the NSE in plane and polygonal domains, based on the analytic regularity results in Marcati & Schwab (2019). Specifically, we consider the symmetric interior penalty discontinuous Galerkin method (Wheeler, 1978; Arnold, 1982) and construct the hp spaces on geometrically refined meshes and with linearly increasing polynomial degrees. Our results are limited to incompressible flow at small Reynolds number, where the Leray-Hopf solutions of the NSE exist and are unique. They do constitute a basis exponential convergence results for more complex rheological models, which are built around the classical, Newtonian viscous, incompressible models. We refer to Barrett & Boyaval (2018); Schwab & Suri (1999) and to the references there.

1.1 *Notation*

We use standard notation. For vector fields $\boldsymbol{v}, \boldsymbol{w} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ and tensor fields $\underline{\sigma}, \underline{\tau} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, we write $(\nabla \boldsymbol{v})_{ij} = \partial_j v_i$, $(\nabla \cdot \underline{\sigma})_i = \sum_{j=1}^{2} \partial_j \sigma_{ij}$, and $\underline{\sigma} : \underline{\tau} = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}$. Furthermore, we denote by $\boldsymbol{v} \otimes \boldsymbol{w}$ the tensor whose components are given by $(\boldsymbol{v} \otimes \boldsymbol{w})_{ij} = v_i\, w_j$, and use the identity $\boldsymbol{v} \cdot \underline{\sigma} \cdot \boldsymbol{w} = \sum_{i,j=1}^{2} v_i \sigma_{ij} w_j = \underline{\sigma} : (\boldsymbol{v} \otimes \boldsymbol{w})$. For a multi index $\alpha \in \mathbb{N}_0^2$, $\alpha = (\alpha_1, \alpha_2)$, we write $|\alpha| = \alpha_1 + \alpha_2$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$. For $n \in \mathbb{N}$, $\widetilde{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$, we write, for example, $\widetilde{\gamma} > \beta$ if $\gamma_i > \beta$ for all $i = 1, \ldots, n$. We also write $\widetilde{\gamma} + \beta = (\gamma_1 + \beta, \ldots, \gamma_n + \beta)$ and do the same for all arithmetic operators. By $d(\cdot, \cdot)$ we denote the (Euclidean) distance between sets and/or points. For two quantities A and B, we write $A \simeq B$ if there exist constants $C_1, C_2 > 0$ which are independent of any discretization parameter such that $C_1 A \leqslant B \leqslant C_2 A$.

1.2 *Outline*

The structure of this paper is as follows. In Section 2, we present the boundary value problem and function spaces for its variational formulation. We recapitulate known existence and uniqueness results. In Section 3, we introduce a mixed hp-DG interior penalty discretization and alongside with it in particular the notation and basic DG approximation results which will be required in the ensuing hp-error analysis. In Section 4 we present proofs of existence and stability bounds in mesh-dependent norms for the hp-DG discrete solution. Section 5 recapitulates analytic regularity results for the Leray solutions of the NSE from Marcati & Schwab (2019), and some corollaries from the regularity theory developed there, in order to be able to rely on known hp-approximation results in the proof of exponential convergence in Section 6. A short section on the main conclusions completes the paper.

2. The incompressible Navier-Stokes equations

Let Ω be a bounded Lipschitz polygon in $\mathbb{R}^2$. Given the source term $\boldsymbol{f} \in L^2(\Omega)^2$ and the constant kinematic viscosity $\nu > 0$, the stationary, incompressible Navier-Stokes equations with so-called "no-slip" (i.e., homogeneous Dirichlet) boundary conditions on $\boldsymbol{u}$ at $\partial\Omega$ consist in finding a velocity field $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

On the Sobolev spaces

$$\boldsymbol{V} := H_0^1(\Omega)^2, \qquad Q := L_0^2(\Omega) = \{\, q \in L^2(\Omega) : \int_\Omega q\,d\boldsymbol{x} = 0 \,\},$$

we introduce the forms

$$A(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nu\nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,d\boldsymbol{x}, \qquad B(\boldsymbol{u},p) = -\int_\Omega p\,\nabla\cdot\boldsymbol{u}\,d\boldsymbol{x},$$
$$O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_\Omega ((\boldsymbol{w}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}\,d\boldsymbol{x}.$$

Then the variational problem corresponding to (2.1) consists in finding $(\boldsymbol{u},p) \in \boldsymbol{V}\times Q$ such that

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) + O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x},\\ B(\boldsymbol{u},q) &= 0, \end{aligned} \tag{2.2}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in Q$.

Problem (2.2) admits at least one solution $(\boldsymbol{u},p) \in \boldsymbol{V}\times Q$ and the velocity $\boldsymbol{u}$ belongs to the kernel

$$\boldsymbol{Z} := \{\, \boldsymbol{v} \in \boldsymbol{V} \,:\, B(\boldsymbol{v},q) = 0 \ \forall q \in Q \,\} = \{\, \boldsymbol{v} \in \boldsymbol{V} \,:\, \nabla\cdot\boldsymbol{v} = 0 \text{ in } L^2(\Omega)\}. \tag{2.3}$$

Furthermore, this solution satisfies the stability bound

$$\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} \leqslant \frac{C_P\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu}, \tag{2.4}$$

with C_P denoting the Poincaré constant in Ω. It is in addition well known that under the *small data assumption*

$$\frac{C_O C_P\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu^2} < 1, \tag{2.5}$$

with

$$C_O := \sup_{\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}\in\boldsymbol{V}} \frac{O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{1,\Omega}\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{w}\|_{1,\Omega}} < \infty$$

denoting the boundedness constant of the trilinear convective transport form O, problem (2.2) admits a unique, so-called Leray-Hopf solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$; see, e.g. Girault & Raviart (1986, Chap. IV.2), Di Pietro & Ern (2012); Quarteroni & Valli (1994) and the references therein.

3. A mixed interior penalty discretization

We introduce a mixed hp-DG discretization of (2.2). It is based on an interior penalty discretization for the Stokes terms (see Schötzau *et al.*, 2003), combined with a discontinuous version of a skew-symmetrized form for the convection form (Di Pietro & Ern, 2010; Karakashian & Jureidini, 1998).

3.1 *Meshes and finite element spaces*

Let $\mathfrak{T}$ denote a collection of meshes $\mathcal{T}$ on Ω comprising shape-regular and convex quadrilateral or triangular elements. We assume that each element $K \in \mathcal{T}$ is the *affine image* of the reference square $\widehat{Q} = (0,1)^2$ or of the reference triangle $\widehat{T} = \{(x,y) \in \mathbb{R}^2_+ : x + y < 1\}$ under the affine element map F_K.

We denote by h_K the diameter of the element $K \in \mathcal{T}$. Remark that since the elements are shape-regular, there exist constants $\kappa_1, \kappa_2 > 0$ such that, uniformly in the mesh family $\mathfrak{T}$, holds

$$\forall K \in \mathcal{T}: \quad \|J_F\|_{L^\infty(\widehat{K})} \leqslant \kappa_1 h_K^2 \qquad \|J_{F^{-1}}\|_{L^\infty(K)} \leqslant \kappa_2 h_{\widehat{K}}^{-2},$$

where J_F (resp. $J_{F^{-1}}$) is the Jacobian of F_K (resp. of F_K^{-1}) and $\widehat{K}$ is either $\widehat{Q}$ or $\widehat{T}$, depending on the element K, see Girault & Raviart (1986, Sections A.1 and A.2).

We allow for 1-irregular meshes, i.e., we allow hanging nodes but insist on each edge e being an entire edge of at least one element $K \in \mathcal{T}$ abutting at E. In particular, this allows for geometric corner refinement in meshes $\mathcal{T}$ consisting only of quadrilaterals, as e.g. in Schötzau & Wihler (2003). See Figure 1 in Section 6 for an example of geometric corner refinement with a 1-irregular mesh of quadrilateral elements.

Further, we assign to each element $K \in \mathcal{T}$ an elemental polynomial degree $k_K \geqslant 1$. The local quantities h_K and k_K are stored in the vectors $\underline{h} = \{h_K\}_{K\in\mathcal{T}}$ and $\underline{k} = \{k_K\}_{K\in\mathcal{T}}$, respectively. We introduce the meshwidth of $\mathcal{T}$ as $h = \max_{K\in\mathcal{T}} h_K$ and the maximum polynomial degree $|\underline{k}| = \max_{K\in\mathcal{T}} k_K$. The mesh sizes are assumed to be of bounded local variation: there exists a constant $\kappa_3 > 0$ such that for all $\mathcal{T} \in \mathfrak{T}$, there holds

$$\kappa_3 h_K \leqslant h_{K'} \leqslant \kappa_3^{-1} h_K, \quad K, K' \in \mathcal{T} \tag{3.1}$$

whenever K and K' share an interior edge.

We also assume a similar property for the polynomial degrees: there is a constant $\kappa_4 > 0$ such that

$$\kappa_4 k_K \leqslant k_{K'} \leqslant \kappa_4^{-1} k_K, \tag{3.2}$$

whenever K and K' share an interior edge, uniformly in the mesh family $\mathfrak{T}$.

An interior edge E of $\mathcal{T}$ is the (non-empty) interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}$. We suppose that E is an entire edge of at least one of the adjacent elements K^+ and K^-, for all edges. Similarly, a boundary edge of $\mathcal{T}$ is the (non-empty) interior of $\partial K \cap \partial\Omega$ which consists of entire faces of ∂K. We denote by $\mathcal{E}_{\mathcal{I}}(\mathcal{T})$ the set of all interior faces of $\mathcal{T}$, by $\mathcal{E}_{\mathcal{D}}(\mathcal{T})$ the set of all boundary faces, and set $\mathcal{E}(\mathcal{T}) = \mathcal{E}_{\mathcal{I}}(\mathcal{T}) \cup \mathcal{E}_{\mathcal{D}}(\mathcal{T})$.

For a given mesh $\mathcal{T}$ on Ω and for a polynomial degree vector $\underline{k} = \{k_K : K \in \mathcal{T}\}$, we introduce on every element $K \in \mathcal{T}$ the local polynomial space $\mathbb{S}_k(K) := \{ q = \widehat{q} \circ F_K^{-1} \,:\, \widehat{q} \in \widehat{\mathbb{S}}_k \}$, where

- $\widehat{\mathbb{S}}_k = \mathbb{Q}_k$, the tensor product polynomials of maximum degree k in each coordinate direction, if K is a quadrilateral;
- $\widehat{\mathbb{S}}_k = \mathbb{P}_k$, the space of polynomials of *total* maximum degree k, if K is a triangle.

We define the generic hp-version discontinuous Galerkin space

$$S^{\underline{k}}(\mathcal{T}) := \left\{ v \in L^2(\Omega) \,:\, v|_K \in \mathbb{S}_{k_K}(K),\ K \in \mathcal{T} \right\}. \tag{3.3}$$

We wish to approximate the velocities and pressures in the discontinuous finite element spaces $\boldsymbol{V}_{\mathrm{DG}}$ and Q_{DG} given by

$$\boldsymbol{V}_{\mathrm{DG}} := \left[S^{\underline{k}}(\mathcal{T})\right]^2, \qquad Q_{\mathrm{DG}} = Q \cap S^{\underline{k}-1}(\mathcal{T}). \tag{3.4}$$

where the degree vector $\underline{k} - 1$ is given by $\{k_K - 1\}_{K\in\mathcal{T}}$.

For the derivation and analysis of the DG discretizations we will make use of the auxiliary space $\underline{\Sigma}_{\mathrm{DG}}$ defined by

$$\underline{\Sigma}_{\mathrm{DG}} := \left[S^{\underline{k}}(\mathcal{T})\right]^{2\times 2}. \tag{3.5}$$

Note that $\nabla_h \boldsymbol{V}_{\mathrm{DG}} \subset \underline{\Sigma}_{\mathrm{DG}}$, where ∇_h is the broken gradient taken element by element.

3.2 *Trace operators*

In this section, we define the trace operators needed in our discontinuous Galerkin discretizations. To this end, for a partition $\mathcal{T}$ of Ω we introduce the broken Sobolev space

$$H^1(\mathcal{T}) := \{ v \in L^2(\Omega) \,:\, v|_K \in H^1(K),\ K \in \mathcal{T} \}. \tag{3.6}$$

Let $\boldsymbol{v}$, q, and $\underline{\tau}$ be piecewise smooth functions in $H^1(\mathcal{T})^2$, $H^1(\mathcal{T})$, and $H^1(\mathcal{T})^{2\times 2}$, respectively. Let $E \subset \mathcal{E}_{\mathcal{I}}(\mathcal{T})$ be an interior face shared by K^+ and K^-. Let us denote by $\boldsymbol{n}^\pm$ the unit outward normals on $\partial K^\pm$, and by $(\boldsymbol{v}^\pm, q^\pm, \underline{\tau}^\pm)$ the traces of $(\boldsymbol{v}, q, \underline{\tau})$ on E from the interior of $K^\pm$. Then, we define the mean values $\{\!\{\cdot\}\!\}$ at $\boldsymbol{x} \in E$ as

$$\{\!\{\boldsymbol{v}\}\!\} := (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2, \qquad \{\!\{q\}\!\} := (q^+ + q^-)/2, \qquad \{\!\{\underline{\tau}\}\!\} := (\underline{\tau}^+ + \underline{\tau}^-)/2.$$

Furthermore, we introduce the following jumps at $\boldsymbol{x} \in E$:

$$[\![q]\!] := q^+\, \boldsymbol{n}^+ + q^-\, \boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] := \boldsymbol{v}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{v}^- \cdot \boldsymbol{n}^-, \quad \underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-.$$

On a boundary face $E \subset \mathcal{E}_{\mathcal{D}}(\mathcal{T})$ given by $E = \partial K \cap \partial\Omega$, we set accordingly $\{\!\{\boldsymbol{v}\}\!\} := \boldsymbol{v}$, $\{\!\{q\}\!\} := q$, $\{\!\{\underline{\tau}\}\!\} := \underline{\tau}$, as well as $[\![q]\!] := q\boldsymbol{n}$, $[\![\boldsymbol{v}]\!] := \boldsymbol{v}\cdot\boldsymbol{n}$, $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v} \otimes \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$.

3.3 *Discretization*

Given forms A_{DG}, B_{DG}, and O_{DG}, chosen to discretize the vector Laplacian, the divergence operator, and the convection term, respectively, we consider mixed methods of the form: find $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ such that

$$\begin{aligned} A_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}; \boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) + B_{\mathrm{DG}}(\boldsymbol{v}, p_{\mathrm{DG}}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x}, \\ B_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}, q) &= 0, \end{aligned} \tag{3.7}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$.

Let us now specify the forms A_{DG}, B_{DG}, and O_{DG} involved in (3.7). In what follows, we shall use the notations $\int_{\mathcal{F}} g\,ds := \sum_{E\in\mathcal{F}} \int_E g\,ds$ and $\|g\|^p_{L^p(\mathcal{F})} := \sum_{E\in\mathcal{F}} \|g\|^p_{L^p(E)}$ for any subset $\mathcal{F} \subseteq \mathcal{E}(\mathcal{T})$.

3.3.1 *The diffusion form.* To discretize the diffusive terms, we take the symmetric interior penalty term written in terms of lifting operators (see Arnold *et al.*, 2001; Schötzau *et al.*, 2003). It is obtained by first defining the stabilization form $I^{\mathrm{j}}_{\mathrm{DG}}$ as

$$I^{\mathrm{j}}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) := \nu \int_{\mathcal{E}(\mathcal{T})} \mathrm{j}\, \underline{[\![\boldsymbol{u}]\!]} : \underline{[\![\boldsymbol{v}]\!]}\, ds, \qquad \boldsymbol{u}, \boldsymbol{v} \in H^1(\mathcal{T})^2, \tag{3.8}$$

where j is the interior penalty stabilization function. It is defined edgewise as

$$\mathrm{j}|_E = \mathrm{j}_E := \mathrm{j}_0 k_E^2 h_E^{-1}, \qquad E \in \mathcal{E}(\mathcal{T}), \tag{3.9}$$

with $\mathrm{j}_0 > 0$ sufficiently large, independently of $\underline{h}$, $\underline{k}$, and ν, and with h_E and k_E defined by

$$h_E := \begin{cases} \min\{h_K, h_{K'}\} & \text{if } E = \partial K \cap \partial K' \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}), \\ h_K & \text{if } E = \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{D}}(\mathcal{T}), \end{cases} \tag{3.10}$$

respectively,

$$k_E := \begin{cases} \max\{k_K, k_{K'}\} & \text{if } E = \partial K \cap \partial K' \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}), \\ k_K & \text{if } E = \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{D}}(\mathcal{T}). \end{cases} \tag{3.11}$$

Then, the form A_{DG} is chosen as

$$A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nu \big[\nabla_h \boldsymbol{u} : \nabla_h \boldsymbol{v} - \underline{\mathcal{L}}(\boldsymbol{u}) : \nabla_h \boldsymbol{v} - \underline{\mathcal{L}}(\boldsymbol{v}) : \nabla_h \boldsymbol{u}\big]\, d\boldsymbol{x} + I^{\mathrm{j}}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}), \tag{3.12}$$

for $\boldsymbol{u}, \boldsymbol{v} \in H^1(\mathcal{T})^2$. Here, $\underline{\mathcal{L}}$ is the lifting operator $\underline{\mathcal{L}} : H^1(\mathcal{T})^2 \to \underline{\Sigma}_{\mathrm{DG}}$ defined by

$$\int_\Omega \underline{\mathcal{L}}(\boldsymbol{v}) : \underline{\tau}\, d\boldsymbol{x} = \int_{\mathcal{E}(\mathcal{T})} \underline{[\![\boldsymbol{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds \qquad \forall \underline{\tau} \in \underline{\Sigma}_{\mathrm{DG}}, \tag{3.13}$$

see Schötzau *et al.* (2003). Notice that restricted to discrete functions $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$, we have

$$\begin{aligned} A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) = &\int_\Omega \nu \nabla_h \boldsymbol{u} : \nabla_h \boldsymbol{v}\, d\boldsymbol{x} - \int_{\mathcal{E}(\mathcal{T})} \{\!\{\nu \nabla_h \boldsymbol{v}\}\!\} : \underline{[\![\boldsymbol{u}]\!]}\, ds \\ &- \int_{\mathcal{E}(\mathcal{T})} \{\!\{\nu \nabla_h \boldsymbol{u}\}\!\} : \underline{[\![\boldsymbol{v}]\!]}\big)\, ds + I^{\mathrm{j}}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{3.14}$$

which corresponds to the symmetric interior penalty discretization of the vector Laplacian.

3.3.2 *The divergence form.* Following Schötzau *et al.* (2003), the divergence form B_{DG} will be taken as

$$B_{\mathrm{DG}}(\boldsymbol{v}, q) = -\int_\Omega q\, [\nabla_h \cdot \boldsymbol{v} - \mathcal{M}(\boldsymbol{v})]\, d\boldsymbol{x}, \qquad \boldsymbol{v} \in H^1(\mathcal{T})^2,\ q \in Q, \tag{3.15}$$

where the lifting $\mathcal{M} : H^1(\mathcal{T})^2 \to Q_{\mathrm{DG}}$ is given by

$$\int_\Omega \mathcal{M}(\boldsymbol{v}) q\, d\boldsymbol{x} = \int_{\mathcal{E}(\mathcal{T})} [\![\boldsymbol{v}]\!] \{\!\{q\}\!\}\, ds \qquad \forall q \in Q_{\mathrm{DG}}. \tag{3.16}$$

For discrete functions $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$, we have

$$B_{\mathrm{DG}}(\boldsymbol{v}, q) = -\int_{\Omega} q\, \nabla_h \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\mathcal{E}(\mathcal{T})} \{\!\{q\}\!\} [\![\boldsymbol{v}]\!]\, ds. \tag{3.17}$$

We notice that, if $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ is a solution of (2.2), then there holds

$$B_{\mathrm{DG}}(\boldsymbol{u}, q) = 0, \qquad q \in Q_{\mathrm{DG}}. \tag{3.18}$$

Hence, the form B_{DG} is consistent in enforcing the divergence constraint.

3.3.3 *The convective form.* We consider the following discontinuous convection form (cf. Di Pietro & Ern, 2010; Karakashian & Jureidini, 1998):

$$\begin{aligned} O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = &\int_{\Omega} ((\boldsymbol{w} \cdot \nabla_h)\boldsymbol{u}) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \frac{1}{2}\int_{\Omega} (\nabla_h \cdot \boldsymbol{w})\boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ &- \int_{\mathcal{E}_{\mathcal{I}}(\mathcal{T})} \{\!\{\boldsymbol{v}\}\!\} \cdot \underline{[\![\boldsymbol{u}]\!]} \cdot \{\!\{\boldsymbol{w}\}\!\}\, ds - \frac{1}{2}\int_{\mathcal{E}(\mathcal{T})} [\![\boldsymbol{w}]\!] \{\!\{\boldsymbol{u} \cdot \boldsymbol{v}\}\!\}\, ds \end{aligned} \tag{3.19}$$

It is well defined on $H^1(\mathcal{T})^2 \times H^1(\mathcal{T})^2 \times H^1(\mathcal{T})^2$, see Proposition 4.1 below. Clearly, the form is linear in each argument. Moreover, it is consistent in the sense that

$$O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = O(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}), \qquad \boldsymbol{w} \in \boldsymbol{Z},\ \boldsymbol{u} \in \boldsymbol{V},\ \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}. \tag{3.20}$$

4. Stability and existence and uniqueness of discrete solutions

We discuss the hp-version stability properties of the discrete forms involved in (3.7). Consequently, we shall establish the existence and uniqueness of solutions to (3.7) under a discrete version of the small data assumption (2.5).

4.1 *Auxiliary results*

We first show the embedding of the broken space $H^1(\mathcal{T})$ into $L^p(\Omega)$ with constants independent of $\underline{h}$ and $\underline{k}$; see also Girault *et al.* (2005); Karakashian & Jureidini (1998) for related results in the context of h-version approximations. To that end, we introduce the broken norm

$$\|v\|_{1,\mathcal{T}}^2 := \|\nabla_h v\|_{L^2(\Omega)}^2 + \int_{\mathcal{E}(\mathcal{T})} \mathtt{h}^{-1}\, |[\![v]\!]|^2\, ds, \tag{4.1}$$

where we set $\mathtt{h}|_E := h_E$ for $E \in \mathcal{E}(\mathcal{T})$, with h_E defined in (3.10).

The following results follow along the lines of Waluga (2012). We start from an embedding result.

LEMMA 4.1 For any $p \in [1, \infty)$, there is an embedding constant $C > 0$ such that

$$\|v\|_{L^p(\Omega)} \leqslant C\|v\|_{1,\mathcal{T}}, \qquad v \in H^1(\mathcal{T}).$$

The constant $C > 0$ only depends on Ω, p, the shape-regularity of the meshes, and the bounded variation of the local mesh sizes in (3.1).

Proof. We recall the proof from Waluga (2012, Theorem 5.16). Consider first the case $p \geqslant 2$. For $v \in H^1(\mathcal{T})$, let $v_0 := \pi_0 v$ be the L^2-projection of v into the piecewise constants over the partition $\mathcal{T}$. By the triangle inequality, we have

$$\|v\|_{L^p(\Omega)} \leqslant \|v - v_0\|_{L^p(\Omega)} + \|v_0\|_{L^p(\Omega)}.$$

By Di Pietro & Ern (2012, Theorem 5.3, item (ii)) and by adding and subtracting v, we obtain

$$\|v_0\|^2_{L^p(\Omega)} \leqslant C \int_{\mathcal{E}(\mathcal{T})} \mathtt{h}^{-1} |[\![v_0]\!]|^2 \, ds \leqslant C \int_{\mathcal{E}(\mathcal{T})} \mathtt{h}^{-1} |[\![v - v_0]\!]|^2 \, ds + C\|v\|^2_{1,\mathcal{T}}.$$

Then, by the shape-regularity of the meshes, the bounded variation of the local mesh sizes, and standard approximation results for π_0, we obtain

$$\int_{\mathcal{E}(\mathcal{T})} \mathtt{h}^{-1} |[\![v - v_0]\!]|^2 \, ds \leqslant C \sum_{K \in \mathcal{T}} h_K^{-1} \|v - v_0\|^2_{L^2(\partial K)} \leqslant C \sum_{K \in \mathcal{T}} \|\nabla v\|^2_{L^2(K)} \leqslant C\|v\|^2_{1,\mathcal{T}}.$$

These bounds yield $\|v_0\|_{L^p(\Omega)} \leqslant C\|v\|_{1,\mathcal{T}}$.

To bound the second term $\|v - v_0\|_{L^p(\Omega)}$, by Sobolev's embedding, the Poincaré inequality, and a scaling argument, we conclude that

$$\begin{aligned} \|v - v_0\|_{L^p(\Omega)} &= \Big(\sum_{K \in \mathcal{T}} \|v - v_0\|^p_{L^p(K)} \Big)^{1/p} \\ &\leqslant C \Big(\sum_{K \in \mathcal{T}} h_K^2 \|\nabla v\|^p_{L^2(K)} \Big)^{1/p} \leqslant C \Big(\sum_{K \in \mathcal{T}} \|\nabla v\|^p_{L^2(K)} \Big)^{1/p}. \end{aligned}$$

Since $\|\underline{x}\|_{l^p} \leqslant \|\underline{x}\|_{l^2}$ for any $p \geqslant 2$ and any sequence $\underline{x} \in \mathbb{R}^n_+$, it follows that

$$\|v - v_0\|_{L^p(\Omega)} \leqslant C \Big(\sum_{K \in \mathcal{T}} \|\nabla v\|^2_{L^2(K)} \Big)^{1/2} \leqslant C\|v\|_{1,\mathcal{T}}.$$

This yields the assertion for $p \in [2, \infty)$.

If now $1 \leqslant p < 2$, we have $\|v\|_{L^p(\Omega)} \leqslant C\|v\|_{L^2(\Omega)} \leqslant C\|v\|_{1,\mathcal{T}}$, due to the boundedness of Ω, and the above result for $p = 2$. This completes the proof. □

The trace estimates of Karakashian & Jureidini (1998) allow us to establish the following bounds.

LEMMA 4.2 1. There is a constant $C > 0$ independent of $K \in \mathcal{T}$ such that

$$h_K^{1/4} \|v\|_{L^4(\partial K)} \leqslant C \left(\|v\|_{L^4(K)} + \|\nabla v\|_{L^2(K)} \right), \tag{4.2}$$

for any $v \in H^1(K)$.

2. There is a constant $C > 0$ such that

$$\Big(\sum_{K \in \mathcal{T}} h_K \|v\|^4_{L^4(\partial K)} \Big)^{1/4} \leqslant C\|v\|_{1,\mathcal{T}}, \tag{4.3}$$

for any $v \in H^1(\mathcal{T})$.

Proof. The trace estimate (4.2) has been proven in Karakashian & Jureidini (1998, Equation (7.7)) for $v \in C^\infty(\overline{K})$. By the density of $C^\infty(\overline{K})$ in $H^1(K)$, the fact that the trace operator is continuous from $H^1(K)$ onto $H^{1/2}(\partial K)$, and the continuous embeddings of $H^1(K)$ in $L^4(K)$ and of $H^{1/2}(\partial K)$ in $L^q(\partial K)$ for all $1 \leqslant q < \infty$, respectively, it is also valid for $v \in H^1(K)$, which establishes the first item.

To prove (4.3), we employ (4.2) to obtain

$$\begin{aligned}\Big(\sum_{K\in\mathcal{T}} h_K\|v\|^4_{L^4(\partial K)}\Big)^{1/4} &\leqslant C\Big(\sum_{K\in\mathcal{T}}(\|v\|^4_{L^4(K)}+\|\nabla v\|^4_{L^2(K)})\Big)^{1/4}\\ &\leqslant C\|v\|_{L^4(\Omega)}+C\Big(\sum_{K\in\mathcal{T}}\|\nabla v\|^4_{L^2(K)}\Big)^{1/4}\\ &\leqslant C\left(\|v\|_{L^4(\Omega)}+\|\nabla_h v\|_{L^2(\Omega)}\right),\end{aligned}$$

where in the last step we have used that $\|\underline{x}\|^2_{l^2}\leqslant\|\underline{x}\|^2_{l^1}$ for any sequence $\underline{x}\in\mathbb{R}^n_+$. The embedding in Lemma 4.1 with $p=4$ yields the assertion. □

4.2 *Stability*

We introduce the broken hp-version DG norm

$$\|\boldsymbol{v}\|^2_{\mathrm{DG}}=\|\nabla_h\boldsymbol{v}\|^2_{L^2(\Omega)}+\int_{\mathcal{E}(\mathcal{T})}\mathrm{j}\,|[\![\boldsymbol{v}]\!]|^2\,ds, \tag{4.4}$$

where j is the edgewise constant interior penalty function defined in (3.9). From Lemma 4.1 and supposing that j_0 in (3.9) is big enough so that $\mathrm{j}_0(\min\underline{k})^2\geqslant 1$, there exist constants $C_P^{\mathrm{DG}}, C_{\mathrm{emb}}>0$ such that for any $\boldsymbol{v}\in H^1(\mathcal{T})^2$ there holds

$$\|\boldsymbol{v}\|_{L^2(\Omega)}\leqslant C_P^{\mathrm{DG}}\|\boldsymbol{v}\|_{1,\mathcal{T}}\leqslant C_P^{\mathrm{DG}}\|\boldsymbol{v}\|_{\mathrm{DG}}, \tag{4.5}$$

$$\|\boldsymbol{v}\|_{L^4(\Omega)}\leqslant C_{\mathrm{emb}}\|\boldsymbol{v}\|_{1,\mathcal{T}}\leqslant C_{\mathrm{emb}}\|\boldsymbol{v}\|_{\mathrm{DG}}. \tag{4.6}$$

The constants depend on the shape regularity of the triangulation but they are independent of $\underline{h}$, $\underline{k}$ and ν.

4.2.1 *The elliptic forms.*

In Schötzau *et al.* (2003), the elliptic forms A_{DG} and B_{DG} have been thoroughly studied in the context of the Stokes problem. First, we have the following continuity properties: there are constants $C_{A_{\mathrm{DG}}}>0$ and $C_{B_{\mathrm{DG}}}>0$ independent of $\underline{h}$, $\underline{k}$, and ν such that

$$|A_{\mathrm{DG}}(\boldsymbol{v},\boldsymbol{w})|\leqslant C_{A_{\mathrm{DG}}}\nu^{1/2}\|\boldsymbol{v}\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{w}\|_{\mathrm{DG}},\qquad \boldsymbol{v},\ \boldsymbol{w}\in H^1(\mathcal{T})^2, \tag{4.7}$$

$$|B_{\mathrm{DG}}(\boldsymbol{v},q)|\leqslant C_{B_{\mathrm{DG}}}\|\boldsymbol{v}\|_{\mathrm{DG}}\|q\|_{L^2(\Omega)},\qquad \boldsymbol{v}\in H^1(\mathcal{T})^2,\ q\in Q. \tag{4.8}$$

Then, the form A_{DG} is coercive over the discrete space $\boldsymbol{V}_{\mathrm{DG}}$: there exists a parameter $\mathrm{j}_{0,\min}>0$ independent of $\underline{h}$, $\underline{k}$, and ν such that for any $\mathrm{j}_0\geqslant\mathrm{j}_{0,\min}$ there exists a coercivity constant $C_{\mathrm{coer}}>0$ independent of $\underline{h}$, $\underline{k}$, and ν with

$$A_{\mathrm{DG}}(\boldsymbol{v},\boldsymbol{v})\geqslant C_{\mathrm{coer}}\nu\|\boldsymbol{v}\|^2_{\mathrm{DG}},\qquad \boldsymbol{v}\in\boldsymbol{V}_{\mathrm{DG}}. \tag{4.9}$$

Throughout, we shall assume that $\mathrm{j}_0\geqslant\mathrm{j}_{0,\min}$.

Finally, we establish an hp-version inf-sup condition for the divergence form B_{DG}. As usual, we do so by combining a global low-order condition with a local high-order one, which follows from the results in Schötzau *et al.* (2003) and Lederer & Schöberl (2018). Specifically, we shall assume the meshes $\mathcal{T}$ to be inf-sup stable for discontinuous $[\mathbb{S}_2]^2-\mathbb{S}_0$ elements:

$$\inf_{0\neq q\in S^{\underline{0}}(\mathcal{T})\cap Q}\ \sup_{\boldsymbol{0}\neq\boldsymbol{v}\in[S^{\underline{2}}(\mathcal{T})]^2}\frac{B_{\mathrm{DG}}(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\mathrm{DG}}\|q\|_{L^2(\Omega)}}\geqslant C_{\mathrm{is},0}>0, \tag{4.10}$$

with a constant $C_{\mathrm{is},0}$ independent of $\underline{h}$, $\underline{k}$ and ν.

REMARK 4.1 We are not aware of a systematic treatment of the condition (4.10) on generic, irregular meshes with hanging nodes. In Stenberg & Suri (1996) it was shown that (4.10) holds true for *conforming* $[\mathbb{S}_2]^2 - \mathbb{S}_0$ elements (i.e., for the spaces $[S^{\underline{2}}(\mathcal{T})]^2 \cap \boldsymbol{V} \times S^{\underline{0}}(\mathcal{T}) \cap Q$) on 1-irregularly refined geometric meshes of affinely mapped quadrilaterals with hanging nodes, as also considered in Section 6.3.1 ahead.

On the other hand, for *regular* meshes with no hanging nodes consisting of affinely mapped quadrilaterals and triangles, condition (4.10) can in fact be shown to hold for (the divergence-conforming subspace of) $[\mathbb{S}_1]^2 - \mathbb{S}_0$ elements (see Becker *et al.*, 2003; Cockburn *et al.*, 2007). Indeed, for $\boldsymbol{v} \in \boldsymbol{V}$, we define the divergence-conforming projector $\boldsymbol{\pi v}$ by

$$\int_e \boldsymbol{\pi v} \cdot \boldsymbol{n}_K \, ds = \int_e \boldsymbol{v} \cdot \boldsymbol{n}_K \, ds. \tag{4.11}$$

for all $K \in \mathcal{T}$ and e an elemental edge of ∂K, with $\boldsymbol{n}_K$ the unit outward normal on ∂K. Notice that the moments in (4.11) correspond to the lowest-order Raviart-Thomas (RT) degrees of freedom if K is a quadrilateral and to the lowest-order Brezzi-Douglas-Marini (BDM) degrees if K is a triangle (see Brezzi & Fortin, 1991). In the case that the mesh $\mathcal{T}$ is regular, i.e., without any hanging nodes, the definition (4.11) implies the Fortin property

$$B_{\mathrm{DG}}(\boldsymbol{v}, q) = B_{\mathrm{DG}}(\boldsymbol{\pi v}, q)$$

for any piecewise costant pressure $q \in S^{\underline{0}}(\mathcal{T})$. This then allows one to prove (4.10).

Next, we recall high-order stability results from Schötzau *et al.* (2003) (for quadrilaterals) and Lederer & Schöberl (2018) (for triangles), respectively. To state them, we introduce for $K \in \mathcal{T}$ the local finite element spaces

$$\boldsymbol{V}^K = \left\{ \boldsymbol{v} \in [\mathbb{S}_{k_K}(K)]^2 : \boldsymbol{v} \cdot \boldsymbol{n}_K = 0 \text{ on } \partial K \right\}, \qquad Q^K = \mathbb{S}_{k_K - 1} \cap L^2_0(K). \tag{4.12}$$

The spaces $\boldsymbol{V}^K$ and Q^K are inf-sup stable in the sense that

$$\inf_{q \in Q^K} \sup_{\boldsymbol{v} \in \boldsymbol{V}^K} \frac{B^K_{\mathrm{DG}}(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathrm{DG}} \|q\|_{L^2(K)}} = \inf_{q \in Q^K} \sup_{\boldsymbol{v} \in \boldsymbol{V}^K} \frac{B^K(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathrm{DG,K}} \|q\|_{L^2(K)}} \geqslant C_{\mathrm{is},1} k_K^{-\mathfrak{a}_K} > 0, \tag{4.13}$$

with a constant $C_{\mathrm{is},1} > 0$ independent of K, $\underline{h}$, $\underline{k}$ and ν. Here, $B^K_{\mathrm{DG}}(\boldsymbol{v}, q)$, $B^K(\boldsymbol{v}, q)$ and $\|\boldsymbol{v}\|_{\mathrm{DG,K}}$ are restrictions of $B_{\mathrm{DG}}(\boldsymbol{v}, q)$, $B(\boldsymbol{v}, q)$ and $\|\boldsymbol{v}\|_{\mathrm{DG}}$ to $K \in \mathcal{T}$ (by extension of $\boldsymbol{v}$ and q by zero). The exponent $\mathfrak{a}_K$ quantifies the algebraic dependence of the inf-sup constant on the polynomial degree k_K. For an affinely mapped quadrilateral $K \in \mathcal{T}$, the local condition (4.13) was shown in Schötzau *et al.* (2003, Section 6) with $\mathfrak{a}_K = 1$. For a triangle $K \in \mathcal{T}$, it was proved in Lederer & Schöberl (2018, Corollary 3.3) to be valid with $\mathfrak{a}_K = 0$.

The combination of (4.10) and (4.13) implies the following result (see Schötzau *et al.*, 2003; Girault & Raviart, 1986).

LEMMA 4.3 Assume the low-order condition (4.10), suppose that $\min_{K \in \mathcal{T}} k_K \geqslant 2$ and let

$$\mathfrak{a} = \begin{cases} 1 & \text{if the mesh } \mathcal{T} \text{ contains at least one affinely mapped quadrilateral,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the following discrete inf-sup condition for the finite element spaces $\boldsymbol{V}_{\mathrm{DG}}$ and Q_{DG} in (3.4) holds true:

$$\inf_{0 \neq q \in Q_{\mathrm{DG}}} \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}} \frac{B_{\mathrm{DG}}(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathrm{DG}} \|q\|_{L^2(\Omega)}} \geqslant C_{\mathrm{is}} |\underline{k}|^{-\mathfrak{a}} > 0, \tag{4.14}$$

with a constant $C_{\mathrm{is}} > 0$ independent of $\underline{h}$, $\underline{k}$, and ν.

Proof. We present the proof for the sake of completeness and follow Schötzau *et al.* (2003). To this end, we fix $0 \neq q \in Q_{\mathrm{DG}}$ and decompose it into

$$q = q_0 + \bar{q}\,, \tag{4.15}$$

where q_0 is the L^2-projection of q into the subspace of $L^2_0(\Omega)$ consisting of piecewise constant pressures. Then, $\|q\|^2_{L^2(\Omega)} = \|q_0\|^2_{L^2(\Omega)} + \|\bar{q}\|^2_{L^2(\Omega)}$.

We first consider the piecewise constant part q_0 in (4.15). Owing to (4.10), there exists a low-order velocity field $\boldsymbol{v}_0 \in \boldsymbol{V}_{\mathrm{DG}}$ such that

$$B_{\mathrm{DG}}(\boldsymbol{v}_0, q_0) \geqslant \|q_0\|^2_{L^2(\Omega)}, \qquad \|\boldsymbol{v}_0\|^2_{\mathrm{DG}} \leqslant C_0 \|q_0\|^2_{L^2(\Omega)}. \tag{4.16}$$

Next, we treat $\bar{q}$ in (4.15). For $K \in \mathcal{T}$, we set $\bar{q}_K := \bar{q}|_K$ and note that, by L^2-orthogonality, there holds $\bar{q}_K \in Q^K$, with Q^K in (4.12). Then, due to local inf-sup condition (4.13), there is a velocity field $\bar{\boldsymbol{v}}_K \in \boldsymbol{V}^K$ such that

$$B^K_{\mathrm{DG}}(\bar{\boldsymbol{v}}_K, \bar{q}_K) = -\int_K \bar{q}_K \nabla \cdot \bar{\boldsymbol{v}}_K \, d\boldsymbol{x} \geqslant \|\bar{q}_K\|^2_{L^2(K)}, \qquad \|\bar{\boldsymbol{v}}_K\|^2_{\mathrm{DG,K}} \leqslant C_1 |\underline{k}|^{2\mathfrak{a}} \|\bar{q}_K\|^2_{L^2(K)}, \tag{4.17}$$

with $C_1 > 0$ solely depending on the shape-regularity of the meshes $\mathcal{T}$. We define $\bar{\boldsymbol{v}} \in \boldsymbol{V}_{\mathrm{DG}}$ by setting $\bar{\boldsymbol{v}}|_K := \bar{\boldsymbol{v}}_K$ for $K \in \mathcal{T}$.

Now, to show (4.14), we introduce

$$\boldsymbol{v} = \boldsymbol{v}_0 + \delta \bar{\boldsymbol{v}} \in \boldsymbol{V}_{\mathrm{DG}},$$

with $\delta > 0$ still to be selected. First, we note that from (3.17)

$$B_{\mathrm{DG}}(\bar{\boldsymbol{v}}, q_0) = -\sum_{K\in\mathcal{T}} q_0|_K \int_K \nabla \cdot \bar{\boldsymbol{v}}_K \, d\boldsymbol{x} = -\sum_{K\in\mathcal{T}_h} q_0|_K \int_{\partial K} \bar{\boldsymbol{v}}_K \cdot \boldsymbol{n}_K \, ds = 0,$$

since q_0 is piecewise constant and $\bar{\boldsymbol{v}}_K \in \boldsymbol{V}^K$. In addition, we obtain from (4.8) and (4.16)

$$|B_{\mathrm{DG}}(\boldsymbol{v}_0, \bar{q})| \leqslant C_{B_{\mathrm{DG}}} \|\boldsymbol{v}_0\|_{\mathrm{DG}} \|\bar{q}\|_{L^2(\Omega)} \leqslant \frac{C_{B_{\mathrm{DG}}} C_0}{\varepsilon} \|q_0\|^2_{L^2(\Omega)} + \varepsilon C_{B_{\mathrm{DG}}} \|\bar{q}\|^2_{L^2(\Omega)},$$

with another parameter $\varepsilon > 0$ to be chosen. Therefore, the above results with (4.16) and (4.17) yield

$$\begin{aligned} B_{\mathrm{DG}}(\boldsymbol{v}, q) &= B_{\mathrm{DG}}(\boldsymbol{v}_0, q_0) + B_{\mathrm{DG}}(\boldsymbol{v}_0, \bar{q}) + \delta B_{\mathrm{DG}}(\bar{\boldsymbol{v}}, \bar{q}) \\ &\geqslant (1 - \frac{C_{B_{\mathrm{DG}}} C_0}{\varepsilon}) \|q_0\|^2_{L^2(\Omega)} + (\delta - \varepsilon C_{B_{\mathrm{DG}}}) \|\bar{q}\|^2_{L^2(\Omega)}. \end{aligned}$$

It is then clear that we can choose δ and ε independently of $\underline{h}$ and $\underline{k}$, such that that

$$B_{\mathrm{DG}}(\boldsymbol{v}, q) \geqslant c_1 \|q\|^2_{L^2(\Omega)}, \tag{4.18}$$

for a constant $c_1 > 0$ independent of $\underline{h}$ and $\underline{k}$. Furthermore, from (4.16) and (4.17), we conclude that

$$\|\boldsymbol{v}\|^2_{\mathrm{DG}} \leqslant c\|\boldsymbol{v}_0\|^2_{\mathrm{DG}} + c\delta^2 \sum_{K\in\mathcal{T}} \|\bar{\boldsymbol{v}}_K\|^2_{\mathrm{DG,K}} \leqslant c\|q_0\|^2_{L^2(\Omega)} + c|\underline{k}|^{2\mathfrak{a}} \|\bar{q}\|^2_{L^2(\Omega)} \leqslant c_2 |\underline{k}|^{2\mathfrak{a}} \|q\|^2_{L^2(\Omega)}. \tag{4.19}$$

with $c_2 > 0$ independent of $\underline{h}$ and $\underline{k}$. The inequalities (4.18) and (4.19) imply (4.14). □

4.2.2 *The convection form.* The next result shows two crucial properties of the convection form.

PROPOSITION 4.1 There holds:

1. For $\boldsymbol{w}, \boldsymbol{u} \in \boldsymbol{V}_{\mathrm{DG}}$, we have $O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u}) = 0$.
2. There is a constant $C_{O_{\mathrm{DG}}}$ independent of $\underline{h}$, $\underline{k}$, and ν such that

$$|O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})| \leqslant C_{O_{\mathrm{DG}}} \|\boldsymbol{w}\|_{\mathrm{DG}} \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}} \tag{4.20}$$

for all $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in H^1(\mathcal{T})^2$.

Proof. Item 1: To verify the first item, we note that, by integration by parts, there holds

$$\sum_{K \in \mathcal{T}} \int_K ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{u} \, d\boldsymbol{x} = -\frac{1}{2} \sum_{K \in \mathcal{T}} \int_K (\nabla \cdot \boldsymbol{w}) |\boldsymbol{u}|^2 \, d\boldsymbol{x} + \frac{1}{2} \sum_{K \in \mathcal{T}} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}_K |\boldsymbol{u}|^2 \, ds.$$

Then, by employing the formula in Arnold *et al.* (2001, eq. (3.3)) and since $[\![|\boldsymbol{u}|^2]\!]_j = 2 \sum_{i=1}^2 \{\!\{\boldsymbol{u}\}\!\}_i \underline{[\![\boldsymbol{u}]\!]}_{ij}$ for $j = 1, 2$, we find that

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}_K |\boldsymbol{u}|^2 \, ds &= \sum_{E \in \mathcal{E}(\mathcal{T})} \int_E [\![\boldsymbol{w}]\!] \{\!\{|\boldsymbol{u}|^2\}\!\} \, ds + \sum_{E \in \mathcal{E}_I(\mathcal{T})} \int_E \{\!\{\boldsymbol{w}\}\!\} \cdot [\![|\boldsymbol{u}|^2]\!] \, ds \\ &= \sum_{E \in \mathcal{E}(\mathcal{T})} \int_E [\![\boldsymbol{w}]\!] \{\!\{|\boldsymbol{u}|^2\}\!\} \, ds + 2 \sum_{E \in \mathcal{E}_I(\mathcal{T})} \int_E \{\!\{\boldsymbol{u}\}\!\} \cdot \underline{[\![\boldsymbol{u}]\!]} \cdot \{\!\{\boldsymbol{w}\}\!\} \, ds. \end{aligned}$$

Using these auxiliary calculations in the expression for $O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u})$, the assertion readily follows.

Item 2: We write $O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = T_1 + T_2 + T_3$, where

$$\begin{aligned} T_1 &= \int_\Omega ((\boldsymbol{w} \cdot \nabla_h)\boldsymbol{u}) \cdot \boldsymbol{v} \, d\boldsymbol{x} + \frac{1}{2} \int_\Omega (\nabla_h \cdot \boldsymbol{w}) \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x}, \\ T_2 &= -\int_{\mathcal{E}_\mathcal{I}(\mathcal{T})} \{\!\{\boldsymbol{v}\}\!\} \cdot \underline{[\![\boldsymbol{u}]\!]} \cdot \{\!\{\boldsymbol{w}\}\!\} \, ds, \\ T_3 &= -\frac{1}{2} \int_{\mathcal{E}(\mathcal{T})} [\![\boldsymbol{w}]\!] \{\!\{\boldsymbol{u} \cdot \boldsymbol{v}\}\!\} \, ds. \end{aligned}$$

The volume terms in T_1 can be readily bounded by employing Hölder's inequality and the embeddings in (4.5), (4.6). This results in the existence of a constant $C > 0$ (depending only on the shape regularity of $\mathcal{T}$) such that for every $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in H^1(\mathcal{T})^2$

$$|T_1| \leqslant C \|\boldsymbol{w}\|_{\mathrm{DG}} \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}}.$$

To bound T_2, we apply Hölder's inequality over $\mathcal{E}_\mathcal{I}(\mathcal{T})$. Since $k_K \geqslant 2$, we find that

$$\begin{aligned} |T_2| &\leqslant \|\mathsf{j}^{1/2} |\underline{[\![\boldsymbol{u}]\!]}| \|_{L^2(\mathcal{E}_\mathcal{I}(\mathcal{T}))} \|\mathsf{j}^{-1/4} |\{\!\{\boldsymbol{v}\}\!\}| \|_{L^4(\mathcal{E}_\mathcal{I}(\mathcal{T}))} \|\mathsf{j}^{-1/4} |\{\!\{\boldsymbol{w}\}\!\}| \|_{L^4(\mathcal{E}_\mathcal{I}(\mathcal{T}))} \\ &\leqslant C \|\boldsymbol{u}\|_{\mathrm{DG}} \Big(\sum_{K \in \mathcal{T}} h_K \|\boldsymbol{v}\|^4_{L^4(\partial K)} \Big)^{1/4} \Big(\sum_{K \in \mathcal{T}} h_K \|\boldsymbol{w}\|^4_{L^4(\partial K)} \Big)^{1/4}. \end{aligned}$$

Hence, by Lemma 4.2 we obtain

$$|T_2| \leqslant C \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}} \|\boldsymbol{w}\|_{\mathrm{DG}}.$$

Similarly, since $|[\![\boldsymbol{w}]\!]| \leqslant |\underline{[\![\boldsymbol{w}]\!]}|$, a repeated application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
|T_3| &\leqslant \frac{1}{2}\|\mathrm{j}^{1/2}\underline{[\![\boldsymbol{w}]\!]}\|_{L^2(\mathcal{E}(\mathcal{T})}\|\mathrm{j}^{-1/2}|\{\!\{\boldsymbol{u}\cdot\boldsymbol{v}\}\!\}\|_{L^2(\mathcal{E}(\mathcal{T})} \\
&\leqslant C\|\boldsymbol{w}\|_{\mathrm{DG}}\Big(\sum_{K\in\mathcal{T}} h_K\||\boldsymbol{u}\cdot\boldsymbol{v}|\|^2_{L^2(\partial K)}\Big)^{1/2} \\
&\leqslant C\|\boldsymbol{w}\|_{\mathrm{DG}}\Big(\sum_{K\in\mathcal{T}} h_K\|\boldsymbol{u}\|^4_{L^4(\partial K)}\Big)^{1/4}\Big(\sum_{K\in\mathcal{T}} h_K\|\boldsymbol{v}\|^4_{L^4(\partial K)}\Big)^{1/4}.
\end{aligned}
$$

Again with Lemma 4.2, we conclude that

$$
|T_3| \leqslant \|\boldsymbol{w}\|_{\mathrm{DG}}\|\boldsymbol{u}\|_{\mathrm{DG}}\|\boldsymbol{v}\|_{\mathrm{DG}}.
$$

This implies the desired continuity bound. □

REMARK 4.2 From (4.20) there exists a constant $C_{O_{\mathrm{DG}}} > 0$ which is independent of the polynomial degree or of the level of geometric mesh refinement such that for all $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$ holds

$$
|O_{\mathrm{DG}}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})| \leqslant C_{O_{\mathrm{DG}}}\|\boldsymbol{w}\|_{\mathrm{DG}}\|\boldsymbol{u}\|_{\mathrm{DG}}\|\boldsymbol{v}\|_{\mathrm{DG}} \ . \tag{4.21}
$$

4.3 *Existence and uniqueness of discrete solutions*

We introduce the discrete kernel

$$
\boldsymbol{Z}_{\mathrm{DG}} := \{\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}} \ : \ B_{\mathrm{DG}}(\boldsymbol{v}, q) = 0 \;\forall\, q \in Q_{\mathrm{DG}} \,\}. \tag{4.22}
$$

With the stability results from Section 4.2, the following result is standard. It follows by proceeding as in the continuous case (see Di Pietro & Ern, 2012; Girault & Raviart, 1986; Quarteroni & Valli, 1994).

PROPOSITION 4.2 There exists a solution $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ to (3.7) such that $\boldsymbol{u}_{\mathrm{DG}} \in \boldsymbol{Z}_{\mathrm{DG}}$, and

$$
\|\boldsymbol{u}_{\mathrm{DG}}\|_{\mathrm{DG}} \leqslant \frac{C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{\mathrm{coer}}}. \tag{4.23}
$$

Moreover, under the small data assumption

$$
\frac{C_{O_{\mathrm{DG}}} C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{C_{\mathrm{coer}}^2\nu^2} < 1 \tag{4.24}
$$

the discrete problem (3.7) has a unique solution $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$.

Proof. We show existence of a discrete solution. To this end, let $\Phi : \boldsymbol{Z}_{\mathrm{DG}} \to \boldsymbol{Z}_{\mathrm{DG}}$ be the mapping defined by

$$
\Phi(\boldsymbol{v}) \in \boldsymbol{Z}_{\mathrm{DG}} : \quad \int_\Omega \Phi(\boldsymbol{v})\cdot\boldsymbol{w} d\boldsymbol{x} = A_{\mathrm{DG}}(\boldsymbol{v},\boldsymbol{w}) + O_{\mathrm{DG}}(\boldsymbol{v};\boldsymbol{v},\boldsymbol{w}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w} d\boldsymbol{x}
$$

for all $\boldsymbol{w} \in \boldsymbol{Z}_{\mathrm{DG}}$. Then, by the coercivity of A_{DG}, (4.9), Item 1 of Proposition 4.1, the Cauchy-Schwarz inequality, and inequality (4.5), we have

$$
\begin{aligned}
\int_\Omega \Phi(\boldsymbol{v})\cdot\boldsymbol{v}\, d\boldsymbol{x} &\geqslant \nu C_{\mathrm{coer}}\|\boldsymbol{v}\|^2_{\mathrm{DG}} - \|\boldsymbol{f}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)} \\
&\geqslant \big(\nu C_{\mathrm{coer}}\|\boldsymbol{v}\|_{\mathrm{DG}} - C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}\big)\,\|\boldsymbol{v}\|_{\mathrm{DG}}.
\end{aligned}
$$

Hence, for all $\boldsymbol{v} \in \boldsymbol{Z}_{\mathrm{DG}}$ such that $\|\boldsymbol{v}\|_{\mathrm{DG}} \geqslant \dfrac{C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{\mathrm{coer}}}$,

$$\int_\Omega \Phi(\boldsymbol{v}) \cdot \boldsymbol{v}\, d\boldsymbol{x} \geqslant 0.$$

Furthermore, by (4.7) and (4.21), $\Phi : \boldsymbol{Z}_{DG} \to \boldsymbol{Z}_{DG}$ is continuous. Then, by the Brouwer's fixed point argument given e.g. in Lions (1969, Chapter I, Lemma 4.3) —see also Girault & Raviart (1986, Chapter IV, Corollary 1.1)— there exists $\boldsymbol{u}_{\mathrm{DG}} \in \boldsymbol{Z}_{\mathrm{DG}}$ such that (4.23) holds, and $\Phi(\boldsymbol{u}_{\mathrm{DG}}) = 0$, i.e.,

$$A_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}; \boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{Z}_{\mathrm{DG}}.$$

Given a DG velocity solution $\boldsymbol{u}_{\mathrm{DG}}$, due to the discrete inf-sup condition (4.14), and with the continuity of the forms A_{DG} and O_{DG}, the linear problem of finding $p_{\mathrm{DG}} \in Q_{\mathrm{DG}}$ such that

$$B_{\mathrm{DG}}(\boldsymbol{v}, p_{\mathrm{DG}}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} - A_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) - O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}; \boldsymbol{u}_{\mathrm{DG}}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}} \tag{4.25}$$

is uniquely solvable. Therefore, there exists $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{Z}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ solution to (3.7).

To show the uniqueness of the solution under hypothesis (4.24), we introduce the map $T : \boldsymbol{Z}_{\mathrm{DG}} \to \boldsymbol{Z}_{\mathrm{DG}}$ such that $\boldsymbol{u} = T(\boldsymbol{w})$ is the solution of the linear Oseen problem

$$\boldsymbol{u} \in \boldsymbol{Z}_{\mathrm{DG}} : \quad A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{Z}_{\mathrm{DG}}. \tag{4.26}$$

Then, as above, from the coercivity (4.9) of A_{DG}, Item 1 of Proposition 4.1, the Cauchy-Schwarz inequality, and inequality (4.5), we have

$$\begin{aligned} \nu C_{\mathrm{coer}} \|\boldsymbol{u}\|_{\mathrm{DG}}^2 &\leqslant A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{u}) = A_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{u}) + O_{\mathrm{DG}}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) \\ &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, d\boldsymbol{x} \leqslant \|\boldsymbol{f}\|_{L^2(\Omega)} \|\boldsymbol{u}\|_{L^2(\Omega)} \leqslant C_P^{\mathrm{DG}} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\boldsymbol{u}\|_{\mathrm{DG}}, \end{aligned}$$

i.e., $\|\boldsymbol{u}\|_{\mathrm{DG}} \leqslant \dfrac{C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{\mathrm{coer}}}$. We now show that T is, under (4.24), a contraction on the ball

$$\boldsymbol{B}_{\mathrm{DG}} = \left\{ \boldsymbol{v} \in \boldsymbol{Z}_{\mathrm{DG}} : \|\boldsymbol{v}\|_{\mathrm{DG}} \leqslant \frac{C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{\mathrm{coer}}} \right\}. \tag{4.27}$$

Let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{B}_{\mathrm{DG}}$ and $\boldsymbol{u}_i = T(\boldsymbol{w}_i)$, $i = 1, 2$. Then, from (4.26) and using Item 1 of Proposition 4.1 and the boundedness of O_{DG}, we obtain

$$\begin{aligned} A_{\mathrm{DG}}(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) &= O_{\mathrm{DG}}(\boldsymbol{w}_2; \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) - O_{\mathrm{DG}}(\boldsymbol{w}_1; \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ &= O_{\mathrm{DG}}(\boldsymbol{w}_2 - \boldsymbol{w}_1; \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ &\leqslant C_{O_{\mathrm{DG}}} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\mathrm{DG}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathrm{DG}} \|\boldsymbol{u}_2\|_{\mathrm{DG}}. \end{aligned}$$

Therefore, from (4.9) and (4.27),

$$\nu C_{\mathrm{coer}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathrm{DG}} \leqslant C_{O_{\mathrm{DG}}} \frac{C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{\mathrm{coer}}} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\mathrm{DG}}.$$

Under assumption (4.24), then, the map T is a contraction on $\boldsymbol{B}_{DG}$ and admits a unique fixed point $\boldsymbol{u}_{DG} \in \boldsymbol{B}_{\mathrm{DG}}$. Since, due to the discrete inf-sup condition (4.14), the problem (4.25) is uniquely solvable for p_{DG}, it follows that the solution $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ is unique in the ball $\boldsymbol{B}_{\mathrm{DG}}$. □

REMARK 4.3 Assumption (4.24) (as well as its continuous counterpart in (2.5)) is a contraction property. Banach's fixed point theorem implies that, for *arbitrary* initial guess $\boldsymbol{u}_{\mathrm{DG}}^{(0)} \in \boldsymbol{V}_{\mathrm{DG}}$ (an initial guess for the discrete pressure p_{DG} is not required), the Picard iteration: given $\boldsymbol{u}_{\mathrm{DG}}^{(m-1)} \in \boldsymbol{V}_{\mathrm{DG}}$, find the next iterate $(\boldsymbol{u}_{\mathrm{DG}}^{(m)}, p_{\mathrm{DG}}^{(m)}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ solving the linear Oseen problem

$$\begin{aligned} A_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}^{(m)}, \boldsymbol{v}) + O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}^{(m-1)}; \boldsymbol{u}_{\mathrm{DG}}^{(m)}, \boldsymbol{v}) + B_{\mathrm{DG}}(\boldsymbol{v}, p_{\mathrm{DG}}^{(m)}) &= \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \\ B_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}}^{(m)}, q) &= 0, \end{aligned} \tag{4.28}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$, converges linearly to the unique solution $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}})$ of the non-linear problem (3.7). As $\boldsymbol{u}_{DG}^{(1)}$ already belongs to $\boldsymbol{B}_{DG}$, all subsequent iterates will also be in $\boldsymbol{B}_{DG}$. In particular, no smallness condition on the initial iterate is required.

5. Analytic regularity of Leray-Hopf solutions to the Navier Stokes equations

5.1 *Weighted Sobolev spaces in the polygon*

For $n \in \mathbb{N}$, we suppose that the polygon Ω has n vertices $\mathfrak{c}_i$, internal angles ϕ_i such that $\phi_i \in (0,\pi) \cup (\pi, 2\pi)$, and edges e_i, $i = 1, \ldots, n$. We write $r_i = |x - \mathfrak{c}_i|$, $\widetilde{\gamma} = \{\gamma_i\}_{i=1,\ldots,n}$ and

$$r^{\widetilde{\gamma}} = \prod_{i=1,\ldots,n} r_i^{\gamma_i}.$$

For $\ell \in \mathbb{N}_0$, $1 \leqslant s < \infty$, and for $\widetilde{\gamma} \in \mathbb{R}^n$, we introduce the homogeneous, corner-weighted seminorm

$$|v|_{\mathcal{K}^{\ell,s}_{\widetilde{\gamma}}(\Omega)} = \left(\sum_{|\alpha|=\ell} \|r^{|\alpha|-\widetilde{\gamma}} \partial^{\alpha} v\|^s_{L^s(\Omega)} \right)^{1/s}$$

and the weighted Sobolev space $\mathcal{K}^{\ell,s}_{\widetilde{\gamma}}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the corner-weighted norm

$$\|v\|_{\mathcal{K}^{\ell,s}_{\widetilde{\gamma}}(\Omega)} = \left(\sum_{j=0}^{\ell} |v|^s_{\mathcal{K}^{j,s}_{\widetilde{\gamma}}(\Omega)} \right)^{1/s}.$$

We also introduce countably normed, weighted spaces of locally analytic functions

$$\mathcal{K}^{\varpi,s}_{\widetilde{\gamma}}(\Omega) = \left\{ v \in \mathcal{K}^{\infty,s}_{\widetilde{\gamma}}(\Omega) : \text{ex. } A, C > 0 \text{ s.t. } \forall \ell \in \mathbb{N} : \quad \|v\|_{\mathcal{K}^{\ell,s}_{\widetilde{\gamma}}(\Omega)} \leqslant C A^{\ell} \ell! \right\}. \tag{5.1}$$

We write $\mathcal{K}^{\ell}_{\widetilde{\gamma}}(\Omega) = \mathcal{K}^{\ell,2}_{\widetilde{\gamma}}(\Omega)$ and $\mathcal{K}^{\varpi}_{\widetilde{\gamma}}(\Omega) = \mathcal{K}^{\varpi,2}_{\widetilde{\gamma}}(\Omega)$.

5.2 *Finite order regularity shift in corner-weighted spaces*

When the source term $\boldsymbol{f}$ in (2.1) belongs to $L^2(\Omega)^2$, there holds the following basic regularity result.

PROPOSITION 5.1 In the polygon Ω, consider the Navier-Stokes equations (2.1) with $\boldsymbol{f} \in L^2(\Omega)^2$ under the small data assumption (2.5). Let furthermore $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ denote the unique (Leray-Hopf) solution to the Navier-Stokes equations (2.1). Then, there exists $\widetilde{\gamma} > 3/2$ (depending on the opening angles of the corners of Ω) such that

$$\boldsymbol{u} \in \mathcal{K}^2_{\widetilde{\gamma}}(\Omega)^2, \qquad p \in \mathcal{K}^1_{\widetilde{\gamma}-1}(\Omega).$$

Proof. From Marcati & Schwab (2019, Lemma 2.5), there exists $\gamma_{\max} > 1/2$ such that $\boldsymbol{u} \in \mathcal{K}^{1,s}_{\widetilde{\gamma}}(\Omega)^2$ and $p \in \mathcal{K}^{0,s}_{\widetilde{\gamma}-1}(\Omega)$ for all $1 < s < \infty$ and $\widetilde{\gamma} \in \mathbb{R}^n$ such that $\widetilde{\gamma} - 2/s \leqslant \gamma_{\max}$. This implies, in particular, that there exists $t > 2$ such that $\boldsymbol{u} \in W^{1,t}(\Omega)^2$ and, by Sobolev's embedding, $\boldsymbol{u} \in L^\infty(\Omega)^2$. Therefore, $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^2(\Omega)^2$ and, writing (2.1) as

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \text{ in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \text{ on } \partial\Omega, \end{aligned}$$

we obtain, by Marcati & Schwab (2019, Theorem 1.1) applied to every corner sector of the polygon, that $\boldsymbol{u} \in \mathcal{K}^2_{\widetilde{\gamma}}(\Omega)^2$ and $p \in \mathcal{K}^1_{\widetilde{\gamma}-1}(\Omega)$, for all $\widetilde{\gamma} - 1 \leqslant \gamma_{\max}$. □

REMARK 5.1 In Schötzau & Wihler (2003), for the linear (Stokes) problem in a polygon $\Omega \subset \mathbb{R}^2$, the solution is considered as an element of the Hilbertian weighted spaces $H^{k,\ell}_{\widetilde{\beta}}(\Omega)$, for integer $\ell \leqslant k$ and with $\widetilde{\beta} \in (0,1)^n$. The spaces $H^{k,\ell}_{\widetilde{\beta}}(\Omega)$ are *non homogeneous* weighted Sobolev spaces, while the spaces $\mathcal{K}^k_{\widetilde{\gamma}}(\Omega)$ we use here are *homogeneous* weighted Sobolev spaces. When dealing with homogeneous Dirichlet boundary conditions, the solutions to the (Navier-)Stokes equations belong to corner-weighted Sobolev spaces with homogeneous norms. These spaces are smaller than their non-homogeneous counterparts: specifically, we have for all $k \in \mathbb{N}$ and for all integer $0 \leqslant \ell \leqslant k$, the continous embeddings

$$\mathcal{K}^k_{\ell-\widetilde{\beta}}(\Omega) \subset H^{k,\ell}_{\widetilde{\beta}}(\Omega). \tag{5.2}$$

This follows from the definitions of $\mathcal{K}^k_{\ell-\widetilde{\beta}}(\Omega)$ and $H^{k,\ell}_{\widetilde{\beta}}(\Omega)$. Specifically, we have

$$\|v\|^2_{\mathcal{K}^k_{\ell-\widetilde{\beta}}(\Omega)} = \sum_{|\alpha|\leqslant k} \|r^{|\alpha|-\ell+\widetilde{\beta}} \partial^\alpha v\|^2_{L^2(\Omega)}$$

and

$$\|v\|^2_{H^{k,\ell}_{\widetilde{\beta}}(\Omega)} = \sum_{|\alpha|\leqslant \ell-1} \|\partial^\alpha v\|^2_{L^2(\Omega)} + \sum_{\ell\leqslant|\alpha|\leqslant k} \|r^{|\alpha|-\ell+\widetilde{\beta}} \partial^\alpha v\|^2_{L^2(\Omega)}.$$

Since for any $\widetilde{\beta} \in (0,1)^n$ and $|\alpha| \leqslant \ell - 1$ there holds $|\alpha| - \ell + \widetilde{\beta} < 0$, there exists a constant κ depending only on $\widetilde{\beta}$, ℓ, and Ω such that $\kappa r^{|\alpha|-\ell+\widetilde{\beta}} \geqslant 1$ in Ω, for all $|\alpha| \leqslant \ell - 1$ and $\widetilde{\beta} \in (0,1)^n$. Hence,

$$\sum_{|\alpha|\leqslant \ell-1} \|\partial^\alpha v\|_{L^2(\Omega)} \leqslant \kappa \sum_{|\alpha|\leqslant \ell-1} \|r^{|\alpha|-\ell+\widetilde{\beta}} \partial^\alpha v\|_{L^2(\Omega)}$$

and (5.2) follows. As a consequence, if we consider the analytic-type classes $B^\ell_{\widetilde{\beta}}(\Omega)$ defined, among others, in Schötzau & Wihler (2003), we also have

$$\mathcal{K}^\varpi_{-\widetilde{\beta}}(\Omega) = B^0_{\widetilde{\beta}}(\Omega) \quad \text{and} \quad \mathcal{K}^\varpi_{\ell-\widetilde{\beta}}(\Omega) \subset B^\ell_{\widetilde{\beta}}(\Omega), \forall\, \ell \in \mathbb{N}.$$

Furthermore, given an opening angle $\phi \in (0, 2\pi)$, we consider the corresponding sector with vertex at 0, i.e. $S = \{x \in \mathbb{R}^2 : r(x) \in (0,\infty), \theta(x) \in (0,\phi)\}$, where $r(x), \theta(x)$ are the polar coordinates at a point $x \in \mathbb{R}^2$. If the weight exponent β is such that $\beta \in (0,1)$, then

$$H^{2,2}_\beta(S) = H^{2,1}_{\beta-1}(S) = \mathcal{K}^2_{2-\beta}(S) \oplus \mathbb{R} \quad \text{and} \quad H^{1,1}_\beta(S) = \mathcal{K}^1_{1-\beta}(S),$$

see Kozlov *et al.* (1997, Theorem 7.1.1) or Costabel *et al.* (2010, Theorem 3.23).

5.3 *Analytic regularity*

If the right hand side of the Navier-Stokes equations is analytic in weighted Sobolev spaces, so is the solution. To show this, we introduce three auxiliary lemmas.

LEMMA 5.1 Let $1 < t < s \leqslant \infty$ and $\widetilde{\gamma}, \widetilde{\eta} \in \mathbb{R}^n$ such that $\widetilde{\gamma} - 2/s > \widetilde{\eta} - 2/t$. Let $v \in \mathcal{K}^{\infty,s}_{\widetilde{\gamma}}(\Omega)$. Then, for all $\ell \in \mathbb{N}_0$ there exists $C > 0$ independent of ℓ such that

$$|v|_{\mathcal{K}^{\ell,t}_{\widetilde{\eta}}(\Omega)} \leqslant C|v|_{\mathcal{K}^{\ell,s}_{\widetilde{\gamma}}(\Omega)}. \tag{5.3}$$

Proof. Let $1/q = 1/t - 1/s$. Then, by the Hölder inequality,

$$\|r^{|\alpha|-\widetilde{\eta}}\partial^\alpha u\|_{L^t(\Omega)} \leqslant C\|r^{|\alpha|-\widetilde{\gamma}}\partial^\alpha u\|_{L^s(\Omega)}\|r^{\widetilde{\gamma}-\widetilde{\eta}}\|_{L^q(\Omega)}. \tag{5.4}$$

We consider the corner sectors $S_i = \{x \in \Omega : |x - \mathfrak{c}_i| \leqslant R\}$, $i = 1, \dots, n$, choosing a sufficiently small $R > 0$ so that they do not overlap. Then, denoting by χ_{S_i} the characteristic function of S_i,

$$\|r^{\widetilde{\gamma}-\widetilde{\eta}}\chi_{S_i}\|^q_{L^q(\Omega)} = \phi_i \int_0^R r^{(\widetilde{\gamma}-\widetilde{\eta})q+1}.$$

If $(\widetilde{\gamma} - \widetilde{\eta})q > -2$, i.e., $\widetilde{\gamma} - 2/s > \widetilde{\eta} - 2/t$, then, the second norm at the right hand side of (5.4) is bounded by a constant that depends only on $\widetilde{\gamma}$, $\widetilde{\eta}$, s, t and on the domain Ω. □

LEMMA 5.2 Let $\widetilde{\gamma} \in \mathbb{R}^n$ such that $\widetilde{\gamma} - 1 \geqslant 0$, and $\ell \in \mathbb{N}_0$. Let $v \in \mathcal{K}^{\infty,2}_{\widetilde{\gamma}}(\Omega)$. Then, there exists C independent of ℓ such that

$$\|v\|_{\mathcal{K}^{\ell,\infty}_{\widetilde{\gamma}-1}(\Omega)} \leqslant C(\ell+1)^2\|v\|_{\mathcal{K}^{\ell+2,2}_{\widetilde{\gamma}}(\Omega)}. \tag{5.5}$$

Proof. We consider a unit plane sector with vertex at 0 and opening angle $\theta_{\max} \in (0, 2\pi)$, i.e.

$$S = \{x \in \mathbb{R}^2 : r(x) \in (0,1), \theta(x) \in (0, \theta_{\max})\},$$

where $r(x), \theta(x)$ are the polar coordinates at a point $x \in \mathbb{R}^2$. Let then $\gamma \in \mathbb{R}$, consider the annuli

$$\Gamma_j = \left\{x \in S : 2^{-j-1} < |x| < 2^{-j}\right\},\ j \in \mathbb{N}_0$$

and let $\widehat{\Gamma} = \Gamma_0$. For all $j \in \mathbb{N}$, let φ_j be the homothety from $\widehat{\Gamma}$ to Γ_j and denote by $\hat{v} = v \circ \varphi$ the quantities rescaled on $\widehat{\Gamma}$. Then, by a scaling argument and since $1/2 < \hat{r}_{|\widehat{\Gamma}} < 1$, we have

$$\max_{|\alpha|\leqslant\ell}\|r^{|\alpha|-\gamma+1}\partial^\alpha v\|_{L^\infty(\Gamma_j)} \leqslant 2^{|\gamma-1|}2^{j(\gamma-1)}\max_{|\alpha|\leqslant\ell}\|\hat{r}^{|\alpha|}\widehat{\partial}^\alpha\hat{v}\|_{L^\infty(\widehat{\Gamma})}.$$

By the embedding of $H^2(\widehat{\Gamma})$ in $L^\infty(\widehat{\Gamma})$, then, there exists $C > 0$ independent of ℓ and j such that

$$\max_{|\alpha|\leqslant\ell}\|r^{|\alpha|-\gamma+1}\partial^\alpha v\|_{L^\infty(\Gamma_j)} \leqslant C2^{j(\gamma-1)}\max_{|\alpha|\leqslant\ell}\|\hat{r}^{|\alpha|}\widehat{\partial}^\alpha\hat{v}\|_{H^2(\widehat{\Gamma})}.$$

Hence, by a simple differentiation, inserting the necessary weight, using again that $1/2 < \hat{r}_{|\widehat{\Gamma}} < 1$, and bounding the maximum over $|\alpha| \leqslant \ell$ with the respective sum, we arrive at

$$\max_{|\alpha|\leqslant\ell}\|r^{|\alpha|-\gamma+1}\partial^\alpha v\|_{L^\infty(\Gamma_j)} \leqslant C2^{j(\gamma-1)}(\ell+1)^2\left(\sum_{|\alpha|\leqslant\ell+2}\|\hat{r}^{|\alpha|-\gamma}\widehat{\partial}^\alpha\hat{v}\|^2_{L^2(\widehat{\Gamma})}\right)^{1/2}.$$

Scaling back to the original annulus Γ_j, we obtain the existence of $C > 0$ independent of ℓ and j such that

$$\max_{|\alpha|\leqslant\ell} \|r^{|\alpha|-\gamma+1}\partial^\alpha v\|_{L^\infty(\Gamma_j)} \leqslant C(\ell+1)^2\|v\|_{\mathcal{K}^{\ell+2,2}_{\gamma}(\Gamma_j)}.$$

Hence there exists $C > 0$ such that for all $\ell \in \mathbb{N}$ holds

$$\|v\|_{\mathcal{K}^{\ell,\infty}_{\gamma-1}(S)} = \sup_{j\in\mathbb{N}_0} \|v\|_{\mathcal{K}^{\ell,\infty}_{\gamma-1}(\Gamma_j)} \leqslant C(\ell+1)^2\|v\|_{\mathcal{K}^{\ell+2}_{\gamma}(S)}.$$

This argument extends easily to the whole polygon Ω by localizing to a sector around each corner and using the classical theory of Sobolev spaces in the remaining (smooth) part of the domain. □

LEMMA 5.3 Let $\widetilde{\gamma} \in \mathbb{R}^n$ and $2 < s < \infty$. In the case that $v \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}}(\Omega)$, there holds $v \in \mathcal{K}^{\varpi,s}_{\widetilde{\eta}}(\Omega)$ for all $\widetilde{\eta} - 2/s < \widetilde{\gamma} - 1$.

Proof. By Lemma 5.2 and by the hypothesis on v, there exist $C_1, A > 0$ such that, for all $\ell \in \mathbb{N}_0$,

$$\|v\|_{\mathcal{K}^{\ell,\infty}_{\widetilde{\gamma}-1}(\Omega)} \leqslant C(\ell+1)^2\|v\|_{\mathcal{K}^{\ell+2,2}_{\widetilde{\gamma}}(\Omega)} \leqslant C_1(\ell+1)^2A^{\ell+2}(\ell+2)!. \tag{5.6}$$

Furthermore, from Lemma 5.1, for $2 < s < \infty$ and $\widetilde{\eta} - 2/s < \widetilde{\gamma} - 1$, there exists $C_2 > 0$ such that for $\ell \geqslant 1$ holds

$$\|v\|_{\mathcal{K}^{\ell,s}_{\widetilde{\eta}}(\Omega)} \leqslant C_2\|v\|_{\mathcal{K}^{\ell,\infty}_{\widetilde{\gamma}-1}(\Omega)}. \tag{5.7}$$

Therefore, by (5.6) and (5.7),

$$\|v\|_{\mathcal{K}^{\ell,s}_{\widetilde{\eta}}(\Omega)} \leqslant C_1C_2(\ell+2)^4A^{\ell+2}\ell!\ .$$

We now introduce $\widetilde{A}$ and $\widetilde{C}$ such that $(\ell+2)^4A^\ell \leqslant \widetilde{A}^\ell$ for all $\ell \in \mathbb{N}$ and $2^4C_1C_2A^2 \leqslant \widetilde{C}$; hence, there holds for all $\widetilde{\eta} - 2/s < \widetilde{\gamma} - 1$

$$\|v\|_{\mathcal{K}^{\ell,s}_{\widetilde{\eta}}(\Omega)} \leqslant \widetilde{C}\widetilde{A}^\ell\ell!\ , \quad \ell = 1, 2, ...$$

□

The following statement is, then, our analytic regularity assertion.

PROPOSITION 5.2 Under the small data assumption (2.5), let $(\boldsymbol{u}, p)$ denote the unique solution in $\boldsymbol{V} \times Q$ to the Navier-Stokes equations (2.1). Suppose furthermore that there exists $\widetilde{\gamma}_f \in \mathbb{R}^n$ with $\widetilde{\gamma}_f > -1$ such that $\boldsymbol{f} \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}_f}(\Omega)^2$.

Then, there exists $\widetilde{\gamma} > 1$ such that

$$\boldsymbol{u} \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}}(\Omega)^2, \qquad p \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}-1}(\Omega). \tag{5.8}$$

Proof. By Lemma 5.3, for all $2 < s < \infty$, $\boldsymbol{f} \in \mathcal{K}^{\varpi,s}_{\widetilde{\gamma}_1}(\Omega)$ for $\widetilde{\gamma}_1 - 2/s < \widetilde{\gamma}_f - 1$. Therefore, by Marcati & Schwab (2019, Theorem 2.6), there exists $\widetilde{\gamma}_2$ with $\widetilde{\gamma}_2 - 2/s > 0$ such that $\boldsymbol{u} \in \mathcal{K}^{\varpi,s}_{\widetilde{\gamma}_2}(\Omega)^2$ and $p \in \mathcal{K}^{\varpi,s}_{\widetilde{\gamma}_2-1}(\Omega)$. Then, by Lemma 5.1, we conclude that there exists $\widetilde{\gamma} > 1$ such that $\boldsymbol{u} \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}}(\Omega)^2$ and $p \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}-1}(\Omega)$. □

6. Error analysis

We are now in position to provide the proof of exponential convergence of the hp-DGFEM in a polygon Ω. The present section is structured in several parts. First, in Section 6.1, we provide in Theorem 6.1 an abstract consistency error bound of the hp-DGFEM.

6.1 *Abstract error estimates*

As is well-known, due to the use of the lifting operators, the DG forms $A_{\rm DG}$ and $B_{\rm DG}$ are not fully consistent (cf. Schötzau *et al.*, 2003). As a measure for the inconsistency of a solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ of (2.2), we introduce its weak residual as

$$R_{\rm DG}(\boldsymbol{u}, p; \boldsymbol{v}) := A_{\rm DG}(\boldsymbol{u}, \boldsymbol{v}) + O_{\rm DG}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + B_{\rm DG}(\boldsymbol{v}, p) - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \tag{6.1}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_{\rm DG}$. We point out that, due to the consistency of the convection form (3.20), the residual has the same form as in the Stokes case considered in Schötzau & Wihler (2003), apart from the inclusion of the nonlinear convection term in (6.1). Our abstract error estimates will then be expressed in terms of the weak residual $\mathcal{R}_{\rm DG}(\boldsymbol{u}, p)$ given by

$$\mathcal{R}_{\rm DG}(\boldsymbol{u}, p) := \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \boldsymbol{V}_{\rm DG}} \frac{|R_{\rm DG}(\boldsymbol{u}, p; \boldsymbol{v})|}{\nu^{1/2} \|\boldsymbol{v}\|_{\rm DG}}. \tag{6.2}$$

We define

$$C_{\rm sm} := \frac{\max\{C_O, C_{O_{\rm DG}}\} \max\{C_P, C_P^{\rm DG}\}}{\min\{1, C_{\rm coer}^2\}}\ . \tag{6.3}$$

Hence, under the small data hypothesis $C_{\rm sm}\nu^{-2}\|\boldsymbol{f}\|_{L^2(\Omega)} < 1$, both the continuous and discrete solutions in (2.2) and (3.7) exist and are unique. For simplicity, we further set

$$|||(\boldsymbol{u}, p)|||^2 := \nu\|\boldsymbol{u}\|_{\rm DG}^2 + \nu^{-1}\|p\|_{L^2(\Omega)}^2. \tag{6.4}$$

THEOREM 6.1 Assume that in (2.1) the volume force $\boldsymbol{f}$ is such that there exists a constant $C_{\rm sm} > 0$ so that for $0 < \nu \leqslant 1$ holds

$$C_{\rm sm}\nu^{-2}\|\boldsymbol{f}\|_{L^2(\Omega)} \leqslant \frac{1}{2}. \tag{6.5}$$

Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ be the solution of (2.2), and let $(\boldsymbol{u}_{\rm DG}, p_{\rm DG}) \in \boldsymbol{V}_{\rm DG} \times Q_{\rm DG}$ denote the DG approximation in (3.7) obtained with $\mathfrak{j}_0 \geqslant \mathfrak{j}_{0,\min}$.

Then we have the error estimates

$$\nu^{1/2}\|\boldsymbol{u} - \boldsymbol{u}_{\rm DG}\|_{\rm DG} \leqslant C|\underline{k}|^{\mathfrak{a}} \Big[\inf_{(\boldsymbol{v},q) \in \boldsymbol{V}_{\rm DG} \times Q_{\rm DG}} |||(\boldsymbol{u} - \boldsymbol{v}, p - q)||| + \mathcal{R}_{\rm DG}(\boldsymbol{u}, p) \Big], \tag{6.6}$$

and

$$\nu^{-1/2}\|p - p_{\rm DG}\|_{\rm DG} \leqslant C|\underline{k}|^{2\mathfrak{a}} \Big[\inf_{(\boldsymbol{v},q) \in \boldsymbol{V}_{\rm DG} \times Q_{\rm DG}} |||(\boldsymbol{u} - \boldsymbol{v}, p - q)||| + \mathcal{R}_{\rm DG}(\boldsymbol{u}, p) \Big], \tag{6.7}$$

where $\mathfrak{a}$ is defined in Lemma 4.3, and where the constant $C > 0$ is independent of $\underline{h}$, $\underline{k}$, and of ν.

Proof. The proof extends the one given in Schötzau *et al.* (2003) for the Stokes equations to the presently considered Navier-Stokes equations.

Step 1: We first claim that

$$\nu^{1/2}\|\boldsymbol{u} - \boldsymbol{u}_{\rm DG}\|_{\rm DG} \leqslant C \left(\inf_{(\boldsymbol{z},q) \in \boldsymbol{Z}_{\rm DG} \times Q_{\rm DG}} |||(\boldsymbol{u} - \boldsymbol{z}, p - q)||| + \mathcal{R}_{\rm DG}(\boldsymbol{u}, p) \right). \tag{6.8}$$

To show (6.8), fix $\boldsymbol{z} \in \boldsymbol{Z}_{\rm DG}$, and $q \in Q_{\rm DG}$. We write the velocity error as

$$\boldsymbol{u} - \boldsymbol{u}_{\rm DG} = (\boldsymbol{u} - \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{u}_{\rm DG}) =: \boldsymbol{\eta}_u + \boldsymbol{\xi}_u. \tag{6.9}$$

By the triangle inequality

$$\nu^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_{\mathrm{DG}}\|_{\mathrm{DG}} \leqslant \nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}} + \nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}. \tag{6.10}$$

Hence, it is sufficient to bound the term $\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}$. To do so, notice that $\boldsymbol{\xi}_u \in \boldsymbol{Z}_{\mathrm{DG}}$. From the coercivity of A_{DG} in (4.9) and from the definition of the residual R_{DG} in (6.1), we readily find that

$$\begin{aligned}\nu C_{\mathrm{coer}}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2 &\leqslant A_{\mathrm{DG}}(\boldsymbol{\xi}_u,\boldsymbol{\xi}_u)\\ &= -A_{\mathrm{DG}}(\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) + A_{\mathrm{DG}}(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{DG}},\boldsymbol{\xi}_u) =: T_1+T_2+T_3+T_4,\end{aligned} \tag{6.11}$$

with

$$\begin{aligned} &T_1 := -A_{\mathrm{DG}}(\boldsymbol{\eta}_u,\boldsymbol{\xi}_u), && T_2 := -O_{\mathrm{DG}}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\xi}_u) + O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{u}_{\mathrm{DG}},\boldsymbol{\xi}_u),\\ &T_3 := -B_{\mathrm{DG}}(\boldsymbol{\xi}_u, p-p_{\mathrm{DG}}), && T_4 := R_{\mathrm{DG}}(\boldsymbol{u},p;\boldsymbol{\xi}_u).\end{aligned}$$

Obviously, by (4.7),

$$|T_1| \leqslant C_{A_{\mathrm{DG}}}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}. \tag{6.12}$$

For the term T_2, we first write

$$\begin{aligned} T_2 &= -O_{\mathrm{DG}}(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{u},\boldsymbol{\xi}_u) + O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{u}_{\mathrm{DG}}-\boldsymbol{u},\boldsymbol{\xi}_u),\\ &= -O_{\mathrm{DG}}(\boldsymbol{\eta}_u;\boldsymbol{u},\boldsymbol{\xi}_u) - O_{\mathrm{DG}}(\boldsymbol{\xi}_u;\boldsymbol{u},\boldsymbol{\xi}_u) - O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) - O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\xi}_u,\boldsymbol{\xi}_u).\end{aligned}$$

Due to the first item in Proposition 4.1, we have $O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\xi}_u,\boldsymbol{\xi}_u) = 0$. Moreover, by the boundedness of O_{DG} in (4.20), the stability bound (4.23), and the small data assumption (6.5),

$$\begin{aligned} |O_{\mathrm{DG}}(\boldsymbol{u}_{\mathrm{DG}};\boldsymbol{\eta}_u,\boldsymbol{\xi}_u)| &\leqslant C_{O_{\mathrm{DG}}}\|\boldsymbol{u}_{\mathrm{DG}}\|_{\mathrm{DG}}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}\\ &\leqslant \frac{C_{O_{\mathrm{DG}}}C_P^{\mathrm{DG}}\|\boldsymbol{f}\|_{L^2(\Omega)}}{C_{\mathrm{coer}}\nu^2}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}\\ &\leqslant \frac{1}{2}C_{\mathrm{coer}}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}.\end{aligned}$$

Similarly, using the continuous stability bound (2.4) and (6.5), we obtain

$$|O_{\mathrm{DG}}(\boldsymbol{\xi}_u;\boldsymbol{u},\boldsymbol{\xi}_u)| \leqslant \frac{1}{2}\min\{1,C_{\mathrm{coer}}^2\}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2 \leqslant \frac{1}{2}C_{\mathrm{coer}}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2,$$

as well as

$$|O_{\mathrm{DG}}(\boldsymbol{\eta}_u;\boldsymbol{u},\boldsymbol{\xi}_u)| \leqslant \frac{1}{2}C_{\mathrm{coer}}\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2.$$

It follows that

$$|T_2| \leqslant C\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\mathrm{DG}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}} + \frac{1}{2}C_{\mathrm{coer}}\nu\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}^2. \tag{6.13}$$

To bound T_3, we notice that, since $\boldsymbol{\xi}_u \in \boldsymbol{Z}_{\mathrm{DG}}$,

$$T_3 = B_{\mathrm{DG}}(\boldsymbol{\xi}_u, p-p_{\mathrm{DG}}) = B_{\mathrm{DG}}(\boldsymbol{\xi}_u, p) = B_{\mathrm{DG}}(\boldsymbol{\xi}_u, p-q).$$

Hence, from the boundedness of B_{DG} in (4.8) we obtain

$$|T_3| \leqslant C_{B_{\mathrm{DG}}}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\mathrm{DG}}\nu^{-1/2}\|p-q\|_{L^2(\Omega)}. \tag{6.14}$$

Finally, by the definition of $\mathcal{R}_{\rm DG}$, the term T_4 is bounded by

$$|T_4| \leqslant \mathcal{R}_{\rm DG}(\boldsymbol{u},p)\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\rm DG}. \tag{6.15}$$

By combining (6.11) with the bounds for T_1 through T_4 in (6.12)–(6.15), respectively, and by bringing the term $\frac{1}{2}C_{\rm coer}\nu\|\boldsymbol{\xi}_u\|^2_{\rm DG}$ in (6.13) to the left-hand side of the resulting inequality, we conclude that

$$\frac{1}{2}C_{\rm coer}\nu^{1/2}\|\boldsymbol{\xi}_u\|_{\rm DG} \leqslant C\left(\nu^{1/2}\|\boldsymbol{\eta}_u\|_{\rm DG} + \nu^{-1/2}\|p-q\|_{L^2(\Omega)} + \mathcal{R}_{\rm DG}(\boldsymbol{u},p)\right),$$

which implies (6.8).

Step 2: We now prove the velocity error bound (6.6). Let $\boldsymbol{v} \in \boldsymbol{V}_{\rm DG}$, and consider the problem of finding $\boldsymbol{z} \in \boldsymbol{V}_{\rm DG}$ such that

$$B_{\rm DG}(\boldsymbol{z},q) = B_{\rm DG}(\boldsymbol{u}-\boldsymbol{v},q) \qquad \forall q \in Q_{\rm DG}.$$

Thanks to the discrete inf-sup condition in (4.14), the continuity of $B_{\rm DG}$ in (4.8) and Brezzi & Fortin (1991, Proposition 1.2, p. 39), there exists a solution $\boldsymbol{z} \in \boldsymbol{V}_{\rm DG}$ to this problem. Furthermore,

$$C_{\rm is}|\underline{k}|^{-\mathfrak{a}}\|\boldsymbol{z}\|_{\rm DG} \leqslant \sup_{0\neq q\in Q_{\rm DG}} \frac{B_{\rm DG}(\boldsymbol{z},q)}{\|q\|_{L^2(\Omega)}} = \sup_{0\neq q\in Q_{\rm DG}} \frac{B_{\rm DG}(\boldsymbol{u}-\boldsymbol{v},q)}{\|q\|_{L^2(\Omega)}} \leqslant C_{B_{\rm DG}}\|\boldsymbol{u}-\boldsymbol{v}\|_{\rm DG},$$

where we have used the continuity of $B_{\rm DG}$ in (4.8). By construction and property (3.18), we have $\boldsymbol{z}+\boldsymbol{v} \in \boldsymbol{Z}_{\rm DG}$. Therefore, By inserting $\boldsymbol{z}+\boldsymbol{v}$ in (6.8), using the triangle inequality, and applying the above bound for $\|\boldsymbol{z}\|_{\rm DG}$ readily yield the error estimate (6.6).

Step 3: Finally, we show the pressure bound (6.7). To that end, fix $q \in Q_{\rm DG}$, and set $p-p_{\rm DG} = (p-q)+(q-p_{\rm DG}) =: \eta_p + \xi_p$.

Due to the inf-sup condition in (4.14), we have

$$C_{\rm is}|\underline{k}|^{-\mathfrak{a}}\nu^{-1/2}\|\xi_p\|_{L^2(\Omega)} \leqslant \sup_{\boldsymbol{0}\neq \boldsymbol{v}\in \boldsymbol{V}_{\rm DG}} \frac{B_{\rm DG}(\boldsymbol{v},q-p_{\rm DG})}{\nu^{1/2}\|\boldsymbol{v}\|_{\rm DG}}.$$

Then, employing the residual expression in (6.1), it can be readily seen that

$$B_{\rm DG}(\boldsymbol{v},\xi_p) =: S_1 + S_2,$$

for any $\boldsymbol{v} \in \boldsymbol{V}_{\rm DG}$, where

$$\begin{aligned} S_1 &:= -B_{\rm DG}(\boldsymbol{v},\eta_p) - A_{DG}(\boldsymbol{u}-\boldsymbol{u}_{\rm DG},\boldsymbol{v}) + R_{\rm DG}(\boldsymbol{u},p;\boldsymbol{v}),\\ S_2 &:= -O_{\rm DG}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + O_{\rm DG}(\boldsymbol{u}_{\rm DG};\boldsymbol{u}_{\rm DG},\boldsymbol{v}). \end{aligned}$$

By the continuity of $B_{\rm DG}$, $A_{\rm DG}$, and the definition of $\mathcal{R}_{\rm DG}$, the first term S_1 can be bounded by

$$|S_1| \leqslant \left(C_{B_{\rm DG}}\nu^{-1/2}\|\eta_p\|_{L^2(\Omega)} + C_{A_{\rm DG}}\nu^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_{\rm DG}\|_{\rm DG} + \mathcal{R}_{\rm DG}(\boldsymbol{u},p)\right)\nu^{1/2}\|\boldsymbol{v}\|_{\rm DG}.$$

To bound the second term S_2, we use elementary manipulations, the boundedness of $O_{\rm DG}$ in Proposition 4.1, the stability bounds in (2.4) and (4.23), respectively, and the small data assumption (6.5). This results in

$$\begin{aligned} |S_2| &\leqslant |O_{\rm DG}(\boldsymbol{u}-\boldsymbol{u}_{\rm DG};\boldsymbol{u},\boldsymbol{v})| + |O_{\rm DG}(\boldsymbol{u}_{\rm DG};\boldsymbol{u}-\boldsymbol{u}_{\rm DG},\boldsymbol{v})|\\ &\leqslant C_{O_{\rm DG}}\left(\|\boldsymbol{u}\|_{\rm DG} + \|\boldsymbol{u}_{\rm DG}\|_{\rm DG}\right)\|\boldsymbol{u}-\boldsymbol{u}_{\rm DG}\|_{\rm DG}\|\boldsymbol{v}\|_{\rm DG}\\ &\leqslant C\nu^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_{\rm DG}\|_{\rm DG}\nu^{1/2}\|\boldsymbol{v}\|_{\rm DG}. \end{aligned}$$

Hence, with the above estimates we obtain

$$\nu^{-1/2}\|\xi_p\|_{L^2(\Omega)} \leqslant C|\underline{k}|^{\mathfrak{a}} \left(\|(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{DG}}, \eta_p)\| + \mathcal{R}_{\mathrm{DG}}(\boldsymbol{u}, p) \right). \tag{6.16}$$

The pressure error bound (6.7) follows now from the triangle inequality, the bound (6.16), and the velocity bound (6.6). □

6.2 *Bounding the weak residual*

Due to (3.20) and the definitions of R_{DG}, A_{DG}, O_{DG}, and B_{DG}, we immediately obtain the following result.

LEMMA 6.1 Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ be a Leray-Hopf solution of (3.7). Then we have

$$R_{\mathrm{DG}}(\boldsymbol{u}, p; \boldsymbol{v}) = \int_\Omega (\nu\nabla\boldsymbol{u} - p\underline{I}) : \nabla_h \boldsymbol{v} d\boldsymbol{x} - \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f})\cdot \boldsymbol{v}\, d\boldsymbol{x} \\ - \int_\Omega \nu\nabla\boldsymbol{u} : \underline{\mathcal{L}}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega p\mathcal{M}(\boldsymbol{v})\, d\boldsymbol{x},$$

for any $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{DG}}$.

Thanks to Proposition 5.1 and Remark 5.1, the following two results can therefore be proven as in the proofs of Lemma 6 and Theorem 2 of Schötzau & Wihler (2003), respectively.

LEMMA 6.2 Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ denote the weak solution of (2.1) with $\boldsymbol{f} \in L^2(\Omega)^2$. Then, there exists a constant $C > 0$ depending only on the shape regularity of $\mathcal{T}$ such that for all $\boldsymbol{w} \in \boldsymbol{V}_{\mathrm{DG}}$ holds

$$R_{\mathrm{DG}}(\boldsymbol{u}, p; \boldsymbol{w}) \leqslant C \inf_{(\boldsymbol{v},q)\in \boldsymbol{V}_{\mathrm{DG}}\times Q_{\mathrm{DG}}} \Bigg(\|(\boldsymbol{u}-\boldsymbol{v}, p-q)\|\, \|\boldsymbol{w}\|_{\mathrm{DG}} \\ + \left| \int_{\mathcal{E}} \{\!\{\nabla\boldsymbol{u} - \nabla\boldsymbol{v}\}\!\} : \underline{[\![\boldsymbol{w}]\!]} - \int_{\mathcal{E}} \{\!\{p-q\}\!\} [\![\boldsymbol{w}]\!] \right| \Bigg). \tag{6.17}$$

Remark that, by Proposition 5.1, we have $\nabla\boldsymbol{u} \in \mathcal{K}^1_{\widetilde{\gamma}-1}(\Omega)^{2\times 2}$ and $p \in \mathcal{K}^1_{\widetilde{\gamma}-1}(\Omega)$ for $\widetilde{\gamma} > 3/2$. Hence, the edge integral terms in (6.17) are well defined due to Schötzau & Wihler (2003, Lemma 1) and relationship (5.2).

Given a triangulation $\mathcal{T}$, we collect all elements abutting at one of the corners in the set

$$\mathcal{T}_{\mathrm{vert}} = \left\{ K \in \mathcal{T} : \overline{K} \cap \{c_1, \ldots, c_n\} \neq \emptyset \right\}. \tag{6.18}$$

We then define $\mathcal{T}_{\mathrm{int}}$ so that $\mathcal{T}_{\mathrm{int}} \cup \mathcal{T}_{\mathrm{vert}} = \mathcal{T}$. We also suppose that the mesh is fine enough so that every element $K \in \mathcal{T}_{\mathrm{vert}}$ touches at most one corner. For a weight vector $\widetilde{\gamma} = \{\gamma_i\}_{i=1}^n$ and for an element $K \in \mathcal{T}_{\mathrm{vert}}$ abutting at corner $\mathfrak{c}_j$, we write $\gamma_K = \gamma_j$.

PROPOSITION 6.2 Assume that the small data hypothesis (6.5) holds. Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ denote the solution to (2.1) with $\boldsymbol{f} \in L^2(\Omega)^2$, and denote by $(\boldsymbol{u}_{\mathrm{DG}}, p_{\mathrm{DG}}) \in \boldsymbol{V}_{\mathrm{DG}} \times Q_{\mathrm{DG}}$ the DG approximation in (3.7) obtained with minimum polynomial degree $\min \underline{k} \geqslant 2$.

Then we have the error estimate

$$\|(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{DG}}, p - p_{\mathrm{DG}})\| \leqslant C|\underline{k}|^{2\mathfrak{a}+1} \inf_{(\boldsymbol{v},q)\in \boldsymbol{V}_{\mathrm{DG}}\times Q_{\mathrm{DG}}} (E_1 + E_2 + E_3), \tag{6.19}$$

where $\mathfrak{a}$ is defined in Lemma 4.3 and where the terms E_i are given by

$$E_1^2 = \sum_{K\in\mathcal{T}} \left(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)} + h_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(K)} + \|p-q\|^2_{L^2(K)} \right), \tag{6.20a}$$

$$E_2^2 = \sum_{K\in\mathcal{T}_{\mathrm{int}}} h_K^2 \left(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^2(K)} + |p-q|^2_{H^1(K)} \right), \tag{6.20b}$$

$$E_3^2 = \sum_{K\in\mathcal{T}_{\mathrm{vert}}} h_K^{2(\gamma_K-1)} \left(|\boldsymbol{u}-\boldsymbol{v}|^2_{\mathcal{K}^2_{\gamma_K}(K)} + |p-q|^2_{\mathcal{K}^1_{\gamma_K-1}(K)} \right). \tag{6.20c}$$

6.3 *Exponential convergence*

When the right hand side of the Navier-Stokes equation belongs to analytic-type weighted Sobolev spaces, it was shown in Marcati & Schwab (2019) that the solution $(\boldsymbol{u}, p)$ likewise belongs to such spaces. Here, we prove exponential convergence of suitable mixed hp-DG discretization, which combines sequences of *meshes with geometric refinement* towards the corners of the domain Ω with corresponding order increase. Specifically, we consider discontinuous, piecewise polynomial approximations with elemental polynomial degrees which are increasing linearly away from the corners. We prove that this provides numerical solutions that converge with exponential rate in terms of the total number of degrees of freedom.

6.3.1 *Geometric mesh and discrete space.* We outline here the construction of the hp-DG spaces for the approximation of the velocity field $\boldsymbol{u}$ and the pressure variable p in Ω. The spaces use sequences of partitions of Ω with *isotropic geometric mesh refinement* and *linear polynomial slope*. These ingredients are classically used to obtain exponential convergence of the approximation in problems with point singularities (see, e.g., Schwab (1998); Feischl & Schwab (2018) and references there). We fix a refinement ratio $\sigma \in (0, 1/2)$ and a polynomial slope $\mathfrak{s} > 0$. Let $R > 0$ be such that $|\mathfrak{c}_i - \mathfrak{c}_j|/2 > R$ for all $i, j = 1, \dots, n$; we start by considering the mesh around a corner of the domain.

Mesh in a corner sector Let $\ell \in \mathbb{N}$ and fix $n_L \in \mathbb{N}$. We consider a corner $\mathfrak{c} \in \{\mathfrak{c}_1, \dots, \mathfrak{c}_n\}$ and the elements of the mesh $\mathcal{T}^\ell_{\mathfrak{c}} = \{K \in \mathcal{T} : d(K, \mathfrak{c}) < R\}$. Suppose that the mesh is fine enough so that we can partition $\mathcal{T}^\ell_{\mathfrak{c}}$ into mesh layers

$$\mathcal{T}^\ell_{\mathfrak{c}} = \mathfrak{L}^\ell_0 \,\dot\cup\, \mathfrak{L}^\ell_1 \,\dot\cup \cdots \dot\cup\, \mathfrak{L}^\ell_\ell \tag{6.21}$$

such that

(i) the layer $\mathfrak{L}^\ell_\ell$ contains the elements that touch the corner, i.e., $\overline{K} \cap \mathfrak{c} \neq \emptyset$ for all $K \in \mathfrak{L}^\ell_\ell$,

(ii) $\max_{K\in\mathfrak{L}^\ell_j} d(K, \mathfrak{c}) \simeq \min_{K\in\mathfrak{L}^\ell_j} d(K, \mathfrak{c}) \simeq R\sigma^j$,

(iii) for all $K \in \mathfrak{L}^\ell_j$ there holds $h_K \simeq R\sigma^j$,

(iv) $\#\mathfrak{L}^\ell_j \simeq n_L$,

where all relationships are uniform in ℓ and j. For each mesh element $T \in \mathcal{T}^\ell_{\mathfrak{c}}$ that is a triangle, we assume additionally, as in Feischl & Schwab (2018), the existence of an affine map F_T such that $T = F_T(\widehat{T})$ and that, denoting

$$Q_T = F_T(\widehat{Q}), \tag{6.22}$$

there holds

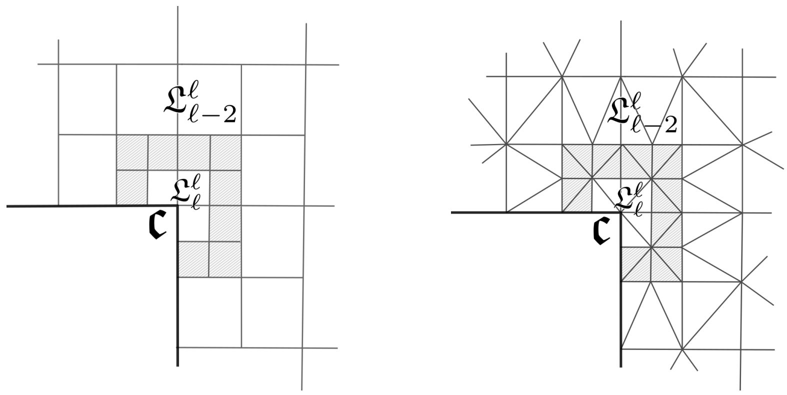

FIG. 1: Example of geometrically refined corner mesh with quadrilateral elements, containing 1-irregular nodes (left panel), and corresponding conforming subdivision of it containing triangular elements (right panel).

(v) $Q_T \subset \Omega$,

(vi) $d(Q_T, \mathfrak{c}) \geqslant Cd(T, \mathfrak{c})$ uniformly with respect to T and ℓ.

Furthermore, we suppose that refinement happens only at the corner, i.e., given (6.21) the refined mesh is obtained by

$$\mathcal{T}_{\mathfrak{c}}^{\ell+1} = \mathfrak{L}_0^{\ell+1} \,\dot{\cup}\, \mathfrak{L}_1^{\ell+1} \,\dot{\cup} \cdots \dot{\cup}\, \mathfrak{L}_{\ell+1}^{\ell+1},$$

with $\mathfrak{L}_j^\ell = \mathfrak{L}_j^{\ell+1}$ for all $j = 0, \ldots, \ell - 1$. This implies also that for all elements K abutting at $\mathfrak{c}$, $h_K \simeq \sigma^\ell$. See Figure 1 for an illustration of the corner mesh and of the mesh layers.

We fix $\widehat{k}_{\min} \geqslant 2$ and, for a slope parameter $\mathfrak{s} > 0$, introduce the *linear polynomial degree vector* $\underline{\widehat{k}}$ such that, for $j \in 0, \ldots, \ell$,

$$\widehat{k}_K = \widehat{k}_{\min} + \lfloor \mathfrak{s}(\ell - j) \rfloor \tag{6.23}$$

is the degree associated with all elements K of the mesh layer $\mathfrak{L}_j^\ell$.

MESH AND DISCRETE SPACE IN THE POLYGON Ω The mesh $\mathcal{T}^\ell$ in the polygon Ω is obtained by collecting the corner meshes $\mathcal{T}_{\mathfrak{c}_i}^\ell$, for $i = 1, \ldots, n$ and by introducing a quasi-uniform partition in the remaining part of the polygon. The polynomial degrees correspond to the degrees (6.23) in the corner meshes, while they are equal to $|\underline{\widehat{k}}| = k_{\min} + \lfloor \mathfrak{s}\ell \rfloor$ in the remaining elements of the mesh $\mathcal{T}^\ell$. See Figure 2 for an illustration of the distribution of polynomial degrees. Recall from (6.18) that we denote by $\mathcal{T}_{\text{vert}}$ the collection of elements abutting at one of the corners, while $\mathcal{T}_{\text{int}}$ contains all the other elements.

6.3.2 *Exponential Convergence.* The exponential convergence of the mixed DG approximation follows from the error decomposition of Proposition 6.2 and the regularity result given in Proposition 5.2. The proof is by now classical in the analysis of exponential convergence for hp FEM.

THEOREM 6.3 Assume that the small data hypothesis (6.5) holds. Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ be the solution to the Navier-Stokes problem (2.1) with $\boldsymbol{f} \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}_f}(\Omega)^2$ for a weight vector $\widetilde{\gamma}_f > -1$. Let $(\boldsymbol{u}_{\text{DG}}, p_{\text{DG}}) \in \boldsymbol{V}_{\text{DG}} \times Q_{\text{DG}}$ denote the DG approximation in (3.7). Let $\boldsymbol{V}_{\text{DG}}$ and Q_{DG} be the spaces defined in (3.4) and constructed in Section 6.3.1, with linearly increasing polynomial

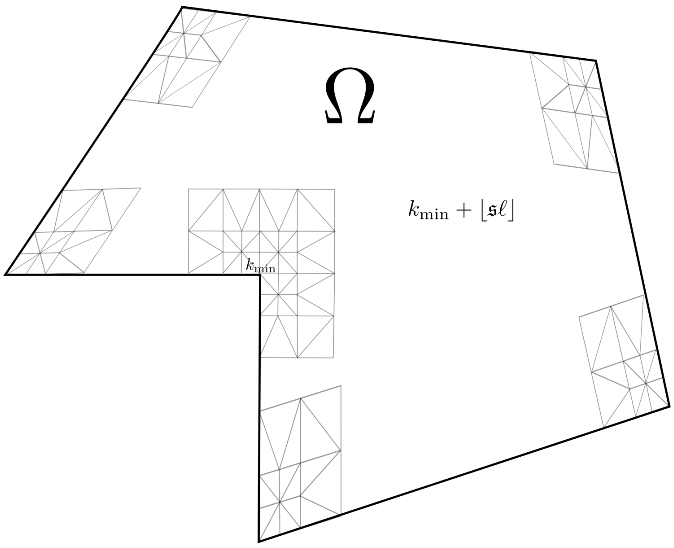

FIG. 2: Sketch of geometrically refined corner patches in the polygon Ω. In hp-DG FEM, the remaining part of the polygon Ω is assumed partitioned in a regular, fixed triangulation whose elements are affine-equivalent to either the reference square or the reference triangle (elements not shown). On all elements of this fixed partition, local polynomial approximations with increasing elemental polynomial degrees $k_{\min} + \lfloor \mathfrak{s}\ell \rfloor$ are considered where ℓ denotes the number of mesh layers in the geometric refinements at the corners.

degree vector $\widehat{\underline{k}}$ with slope $\mathfrak{s} > 0$ and geometrically corner-refined mesh $\mathcal{T}^\ell$ with refinement ratio $\sigma \in (0, 1/2)$. Then, with the norm $|\!|\!|\cdot|\!|\!|$ as defined in (6.4), there exist $C, b > 0$ such that for every $N = \dim(\boldsymbol{V}_{\mathrm{DG}}) \simeq \dim(Q_{\mathrm{DG}})$ there holds

$$|\!|\!|(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{DG}}, p - p_{\mathrm{DG}})|\!|\!| \leqslant C \exp\left(-bN^{1/3}\right).$$

Proof. The statement of the theorem follows directly from an exponential convergence bound on the terms (6.20a) to (6.20c) of Proposition 6.2. This follows along the lines of, e.g., Schötzau & Wihler (2003), which considered the case of geometric meshes of quadrilateral elements. We outline the proof for readers' convenience and to render our exposition self-contained.

Using Proposition 5.2, we fix $\widetilde{\gamma} \in \mathbb{R}^n$ such that $\widetilde{\gamma} > 1$, $\boldsymbol{u} \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}}(\Omega)^2$, and $p \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}-1}(\Omega)$. Given an element $K \in \mathcal{T}_{\mathrm{vert}}$ abutting at a corner $\mathbf{c}_j$, we write $\gamma_K = \gamma_j$. Then, we remark that there exists $C = C(\Omega, \sigma, \widetilde{\gamma}) > 0$, such that, for all $K \in \mathcal{T}_{\mathrm{vert}}$,

$$r_{|K}^{-\widetilde{\gamma}} h_K^{\gamma_K} \geqslant C, \qquad r_{|K}^{-\widetilde{\gamma}+1} h_K^{\gamma_K - 1} \geqslant C. \tag{6.24}$$

We start by considering elements abutting at a corner, i.e., the elements in $\mathcal{T}_{\mathrm{vert}}$, defined in (6.18). Let us split (6.20a) as

$$\begin{aligned} E_1^2 &= \sum_{K \in \mathcal{T}_{\mathrm{int}}} \left(|\boldsymbol{u} - \boldsymbol{v}|^2_{H^1(K)} + h_K^{-2} \|\boldsymbol{u} - \boldsymbol{v}\|^2_{L^2(K)} + \|p - q\|^2_{L^2(K)} \right) \\ &\quad + \sum_{K \in \mathcal{T}_{\mathrm{vert}}} \left(|\boldsymbol{u} - \boldsymbol{v}|^2_{H^1(K)} + h_K^{-2} \|\boldsymbol{u} - \boldsymbol{v}\|^2_{L^2(K)} + \|p - q\|^2_{L^2(K)} \right) \\ &= E^2_{1,\mathrm{int}} + E^2_{1,\mathrm{vert}}. \end{aligned} \tag{6.25}$$

Then, choose $\boldsymbol{v}_{|K} = \mathbf{0}$ and $q_{|K} = 0$ for all $K \in \mathcal{T}_{\text{vert}}$ and note that, thanks to (6.24),

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{\text{vert}}} \left(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)} + h_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(K)} + \|p-q\|^2_{L^2(K)} \right) \\
&\quad \leqslant C_1 \sum_{K\in\mathcal{T}_{\text{vert}}} \left(h_K^{2(\gamma_K-1)}\|r^{-\gamma_K+1}\nabla\boldsymbol{u}\|^2_{L^2(K)} + h_K^{2(\gamma_K-1)}\|r^{-\gamma_K}\boldsymbol{u}\|^2_{L^2(K)} + h_K^{2(\gamma_K-1)}\|r^{-\gamma_K+1}p\|^2_{L^2(K)} \right) \\
&\quad \leqslant C_2 \sum_{K\in\mathcal{T}_{\text{vert}}} \sigma^{2(\gamma_K-1)\ell} \left(\|\boldsymbol{u}\|^2_{\mathcal{K}^1_{\widetilde{\gamma}}(K)} + \|p\|^2_{\mathcal{K}^0_{\widetilde{\gamma}-1}(K)} \right),
\end{aligned}$$

where the last inequality follows from assumption (iii) of Section 6.3.1. Therefore, there exists $C_{\text{vert}}, b_{\text{vert}} > 0$ independent of ℓ such that

$$E^2_{1,\text{vert}} + E^2_3 \leqslant C_{\text{vert}} \exp(-b_{\text{vert}}\ell). \tag{6.26}$$

The terms from (6.20a)–(6.20c) left to estimate are thus given by

$$E^2_{1,\text{int}} + E^2_2 = \sum_{K\in\mathcal{T}_{\text{int}}} \left(\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} + \sum_{i=0,1} h_k^{2i} |p-q|^2_{H^i(K)} \right). \tag{6.27}$$

For a $j \in \mathbb{N}$, we consider an element $K \in \mathfrak{L}^\ell_j$. We start by considering the case where K is a quadrilateral. Let then $s \in \mathbb{N}$ such that $s \leqslant k_K$ and let $\beta \in \mathbb{N}^2_0$ be a multi index such that $|\beta| = s$. There exists a polynomial $\boldsymbol{v} \in \mathbb{Q}_{k_K}(K)^2$ such that

$$\begin{aligned}
\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} &\leqslant C h_K^{2s} \sum_{i=0,1,2} \left(\frac{(k_K-s)!}{(k_K+s+2-2i)!} \sum_{|\alpha|=1,2,3} h_K^{2|\alpha|-2} \|\partial^{\alpha+\beta}\boldsymbol{u}\|^2_{L^2(K)} \right) \\
&\leqslant C h_K^{2s} \sum_{i=0,1,2} \left(\frac{(k_K-s)!}{(k_K+s+2-2i)!} \sum_{|\alpha|=1,2,3} h_K^{2|\alpha|-2} \|\partial^{\alpha+\beta}\boldsymbol{u}\|^2_{L^2(K)} \right),
\end{aligned} \tag{6.28}$$

see Schwab (1998, Corollary 4.47). Then, by items (ii) and (iii) of Section 6.3.1, there holds $r_{|K} \simeq h_K \simeq \sigma^j$, uniformly in j and ℓ. Writing $\gamma_{\min} = \min \widetilde{\gamma}$, it holds furthermore

$$\begin{aligned}
&\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} \\
&\quad \leqslant C^{s+1} h_K^{2(\gamma_{\min}-1)} \sum_{i=0,1,2} \left(\frac{(k_K-s)!}{(k_K+s+2-2i)!} \sum_{|\alpha|=1,2,3} \|r^{|\alpha|+|\beta|-\widetilde{\gamma}}\partial^{\alpha+\beta}\boldsymbol{u}\|^2_{L^2(K)} \right) \\
&\quad \leqslant C^{s+1} \sigma^{2(\gamma_{\min}-1)j} \sum_{i=0,1,2} \left(\frac{(k_K-s)!}{(k_K+s+2-2i)!} \sum_{|\alpha|=1,2,3} \|r^{|\alpha|+|\beta|-\widetilde{\gamma}}\partial^{\alpha+\beta}\boldsymbol{u}\|^2_{L^2(K)} \right).
\end{aligned} \tag{6.29}$$

We now consider the case where K is a triangle and denote it by $T = K$. We then consider the quadrilateral Q_T defined in (6.22) and remark that for all integer k, $\widehat{\mathbb{Q}}_{\lfloor k/2 \rfloor} \subset \widehat{\mathbb{P}}_k$; hence, a function in $\mathbb{Q}_{\lfloor k_T/2 \rfloor}(Q_T)$ can be restricted to a function in $\mathbb{P}_{k_T}(T)$. Then, the function $\boldsymbol{u}$ can be approximated in the quadrilateral Q_T and subsequently be restricted to T, i.e., for $i = 0, 1, 2$ there exists a polynomial $\boldsymbol{v} \in \mathbb{Q}_{\lfloor k_T/2 \rfloor}(Q_T)^2$ such that $\boldsymbol{v}_{|T} \in \mathbb{P}_{k_T}(T)^2$ and

$$|\boldsymbol{u}-\boldsymbol{v}|_{H^i(T)} \leqslant C |\boldsymbol{u}-\boldsymbol{v}|_{H^i(Q_T)}.$$

where the constant $C > 0$ depends only on the shape regularity of T, but is independent of h_T and of k_T. Since by items (iii) and (vi) of Section 6.3.1, $r_{|Q_T} \simeq h_{Q_T} \simeq h_T \simeq \sigma^j$ we obtain, by the argument above, the existence of a polynomial $\boldsymbol{v} \in \mathbb{P}_{k_T}(K)^2$ such that for $s \leqslant \lfloor k_T/2 \rfloor$,

$$\sum_{i=0,1,2} h_T^{2(i-1)} |\boldsymbol{u} - \boldsymbol{v}|^2_{H^i(T)} \leqslant C^{s+1} \sigma^{2(\gamma_{\min}-1)j} \sum_{i=0,1,2} \left(\frac{(\lfloor k_T/2 \rfloor - s)!}{(\lfloor k_T/2 \rfloor + s + 2 - 2i)!} \sum_{|\alpha|=1,2,3} \|r^{|\alpha|+|\beta|-\widetilde{\gamma}} \partial^{\alpha+\beta} \boldsymbol{u}\|^2_{L^2(T)} \right). \tag{6.30}$$

As before, we denote $\mathbb{S}_{k_K}(K) = \mathbb{Q}_{k_K}(K)$ if K is a quadrilateral, and $\mathbb{S}_{k_K}(K) = \mathbb{P}_{k_K}(K)$ if it is a triangle. We also remark that estimate (6.30) is weaker than (6.29), hence it also holds for quadrilateral elements and will be used in the sequel.

Now, since $\boldsymbol{u} \in \mathcal{K}^{\varpi}_{\widetilde{\gamma}}(\Omega)$, there exists $C_u, A_u > 1$ such that, for all multi indices $\alpha \in \mathbb{N}_0^2$ there holds $\|r^{|\alpha|-\widetilde{\gamma}} \partial^\alpha \boldsymbol{u}\|_{L^2(\Omega)} \leqslant C_u A_u^{|\alpha|} |\alpha|!$. Therefore, choosing $\widetilde{C}_1, \widetilde{A}_1$ independent of s such that $9C^{s+1} C_u^2 A_u^{2(s+3)} (s+3)^3 \leqslant \widetilde{C}_1 \widetilde{A}_1^s$ in (6.29) or (6.30), we obtain that for all $K \in \mathcal{T}$ there exists $\boldsymbol{v} \in \mathbb{S}_{k_K}(K)$ such that

$$\begin{aligned}\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u} - \boldsymbol{v}|^2_{H^i(K)} &\leqslant C^{s+1} C_u^2 \sigma^{2(\gamma_{\min}-1)j} \sum_{i=0,1,2} \left(\frac{(\lfloor k_K/2 \rfloor - s)!}{(\lfloor k_K/2 \rfloor + s + 2 - 2i)!} \sum_{n=1,2,3} A_u^{2(s+n)} (s+n)!^2 \right) \\ &\leqslant \widetilde{C}_1 \sigma^{2(\gamma_{\min}-1)j} \frac{(\lfloor k_K/2 \rfloor - s)!}{(\lfloor k_K/2 \rfloor + s)!} \widetilde{A}_1^s (s!)^2.\end{aligned} \tag{6.31}$$

This gives an estimate on the velocity term at the right hand side of (6.27). An equivalent estimate for the second term at the right hand side of (6.27) can be obtained following the same steps. Specifically, we remark that p has the same regularity as the elements of $\nabla \boldsymbol{u}$ (i.e., both p and the components of $\nabla \boldsymbol{u}$ are in $\mathcal{K}^{\varpi}_{\widetilde{\gamma}-1}(\Omega)$). Hence, by the same arguments as in (6.28), (6.29), and by (6.31), we obtain that there exists a polynomial $q \in \mathbb{Q}_{k_K}(K)$ and a constant $\widetilde{C}_2 > 0$ such that

$$\sum_{i=0,1} h_K^{2(i-1)} |p - q|^2_{H^i(K)} \leqslant \widetilde{C}_2 \sigma^{2(\gamma_{\min}-1)j} \frac{(\lfloor k_K/2 \rfloor - s)!}{(\lfloor k_K/2 \rfloor + s)!} \widetilde{A}_2^s (s!)^2.$$

Then, denoting $k_j = k_{\min} + \mathfrak{s}\lfloor \ell - j \rfloor$, and $\widetilde{A} = \max(\widetilde{A}_1, \widetilde{A}_2)$, there exist constants $C_{\text{int}}, b_{\text{int}} > 0$ such that for all $\ell \geqslant 1$

$$E^2_{1,\text{int}} + E^2_2 \leqslant C \sum_{j=0}^{\ell-1} \sigma^{2(\gamma_{\min}-1)j} \min_{s=1,\dots,\lfloor k_j/2 \rfloor} \frac{(\lfloor k_j/2 \rfloor - s)!}{(\lfloor k_j/2 \rfloor + s)!} \widetilde{A}^s (s!)^2 \leqslant C_{\text{int}} e^{-b_{\text{int}} \ell}, \tag{6.32}$$

where the second inequality is a consequence of, e.g., Schötzau *et al.* (2013, Lemma 5.12). The proof is then concluded by combining (6.26), (6.32) and noting that $N \simeq \ell^3$.

□

7. Conclusions and Generalizations

We proved exponential convergence of an *hp*-version DG finite element discretization of the saddle point formulation of stationary, viscous, incompressible Navier-Stokes equations in a plane polygon, provided the problem data are analytic in a setting of countably-normed, weighted spaces. The analysis required a small data hypothesis to ensure uniqueness of the Leray-Hopf solution, and uniqueness and quasioptimality of the DG projection. The present result is the first exponential convergence proof for a nonlinear PDE in a nonsmooth domain, and we sketch possible generalizations.

7.1 *Other Boundary Conditions*

The present analysis and the analytic regularity result in our paper (Marcati & Schwab, 2019) were formulated for homogeneous essential boundary conditions for the velocity (so-called "no-slip" conditions). The analytic regularity results and the present hp-error analysis do allow extension to the "usual", variational boundary conditions for the Navier-Stokes equations, as described e.g. in Maz'ya & Rossmann (2010, Chap. 11.2). Naturally, the extension of the analytic regularity to the three-dimensional setting remains to be done. Here, for small data, Leray-Hopf solutions are once more unique, and admit, in bounded polyhedral domains, regularity shifts for the velocity and pressure in scales of corner and edge-weighted Sobolev spaces, see Maz'ya & Rossmann (2010, Chap. 11).

More general nonlinearities, in particular in the context of non-Newtonian rheological models, could also be considered in the context of so-called "three-field formulations" (see, e.g., Schwab & Suri (1999) and the references there) of the Stokes, resp. Navier-Stokes equations.

7.2 *Other hp-DG FEMs*

The hp-DG FEM considered in the present paper is nonconforming as far as the velocity field is concerned. In recent years, a number of alternative, so-called hybridized mixed discretizations of the NSE have been proposed and also analyzed, mostly in an h-version setting (see also the hp analysis in Waluga (2012)), and under strong (unrealistic) regularity hypothesis in standard, non-weighted Sobolev scales $H^k(\Omega)^2 \times H^{k-1}(\Omega)$ with $k \geqslant 2$. Using the analytic regularity results from Marcati & Schwab (2019) and corresponding hp-versions of these hybridized DG-FEMs will allow to prove also for these methods exponential convergence rates. The details are, however, yet to be worked out.

7.3 *Standard Mixed hp-FEM*

Although the present results were formulated for a mixed DG FEM discretization of the NSE, corresponding exponential convergence rate bounds can reasonably be expected also for several classes of "standard", $H^1(\Omega)$ conforming mixed hp-FEM, i.e., with continuous piecewise polynomial velocity approximation: based on the regularity results in Marcati & Schwab (2019) and on the proof arguments in Schötzau & Wihler (2003), a version of our main result (i.e., Theorem 6.3) can be established also for these FEMs. The detailed development of these lines of research will be presented elsewhere.

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