Quantitative estimates for stress concentration of the Stokes flow between adjacent circular cylinders

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Abstract

When two inclusions with high contrast material properties are located close to each other in a homogeneous medium, stress may become arbitrarily large in the narrow region between them. In this paper, we investigate such stress concentration in the two-dimensional Stokes flow when inclusions are the two-dimensional cross sections of circular cylinders of the same radii and the background velocity field is linear. We construct two vector-valued functions which completely capture the singular behavior of the stress and derive an asymptotic representation formula for the stress in terms of these functions as the distance between the two cylinders tends to zero. We then show, using the representation formula, that the stress always blows up by proving that either the pressure or the shear stress component of the stress tensor blows up. The blow-up rate is shown to be $\delta^{-1/2}$, where δ is the distance between the two cylinders.

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Key words. stress concentration, blow-up, Stokes flow, Stokes system, singular functions, bi-polar coordinates

1 Introduction and statements of the main results

When two close-to-touching inclusions with high contrast material properties are present, the stress may become arbitrarily large in the narrow region between them. Such stress blow-up occurs in electro-statics and elasto-statics, and quantitative understanding of such a phenomenon is important in relation with the field enhancement in the electro-static case, and with the stress in composites in the elasto-static case. Lately, significant progress has been made in understanding the field enhancement or the stress concentration. In the electro-static case, it is proved that the stress, which is the gradient of the solution to the conductivity equation, blows up in the narrow region between two perfect conductors (where the conductivity is infinite) at the rate of $\delta^{-1/2}$ [2, 27] in two dimensions and of $|\delta \log \delta|^{-1}$ in three dimensions [5], as the distance δ between the two inclusions tends to zero. The singular term of the stress concentration is also characterized in two dimensions

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[1]. This result has been extended to the elasticity in the context of the Lamé system of linear elasticity, showing that the blow-up rate of the stress in between two stiff inclusions (where the shear modulus is infinite) is $\delta^{-1/2}$ in two dimensions [6, 16]. References cited above are far from being complete. In fact, there is a long list of recent important achievements in this direction of research, for which we refer to the references in [16].

In this paper, we consider the stress concentration in the two-dimensional steady Stokes system when two adjacent circular cylinders are present. This problem is particularly interesting in comparison to the case of linear elasticity. In the linear elasticity case, the divergence of the displacement vector field blows up in general as the distance between two inclusions tends to zero, as was proved in [16]. However, the divergence of the velocity vector in Stokes flow is confined to be zero, namely, the flow is incompressible. Thus, it is not clear whether the stress blows up or not in the case of Stokes flow, and how large it is if it actually blows up. The stress in the Newtonian fluid including the Stokes flow consists of two components, the pressure and the shear gradient of the velocity field. We investigate the blow-up rate of each component when the distance between the two cylinders tends to zero.

More precisely, suppose that two circular cylinders, denoted by D_1 and D_2, of the same radius R are immersed in Stokes flow and they are separated by a distance $\delta > 0$. Since D_1 and D_2 are (rigid) cylinders, the boundary values of the steady flow on ∂D_1 and ∂D_2 are given as a linear combination of three vector fields representing rigid motions $\{\psi_j\}_{j=1}^3$, which are defined as

$$\psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} y \\ -x \end{bmatrix}. \tag{1.1}$$

Thus, we consider the following Stokes system in the exterior domain $D^e := \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$:

$$\begin{cases} \mu \Delta \mathbf{u} = \nabla p & \text{in } D^e, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } D^e, \\ \mathbf{u} = \sum_{j=1}^3 c_{ij} \psi_j & \text{on } \partial D_i, \ i = 1, 2, \\ (\mathbf{u} - \mathbf{U}, p - P) \in \mathcal{M}_0, \end{cases} \tag{1.2}$$

where μ represents the constant viscosity of the fluid, c_{ij} are constants to be determined from the equilibrium conditions (see (1.10) below), $(\mathbf{U}, P)$ is a given background solution to the homogeneous Stokes system in $\mathbb{R}^2$, namely,

$$\mu \Delta \mathbf{U} = \nabla P \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

and the class $\mathcal{M}_0$ is characterized by decay conditions at ∞. The precise definition of $\mathcal{M}_0$ is given later in Subsection 2.3. Here we just mention that the problem (1.2) admits a unique solution.

Throughout this paper, we assume that both the gradient $\nabla \mathbf{U}$ of the background velocity field and the pressure P are constant functions. Since the pressure is determined up to a constant, we assume that $P = 0$ and

$$\mathbf{U}(x, y) = \begin{bmatrix} a & c \\ d & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (a^2 + (c + d)^2 \neq 0) \tag{1.4}$$

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for some constants c and d. The fields in (1.4) are the only divergence-free fields in the case where $\nabla \mathbf{U}$ is constant. The condition $a^2 + (c+d)^2 \neq 0$ is imposed from the fact that otherwise

$$\mathbf{U}(x, y) = c \begin{bmatrix} y \\ -x \end{bmatrix},$$

and hence $\mathbf{U}$ with a constant p is the solution to the problem (1.2), and its gradient does not blow up. If we write $\mathbf{U}$ as

$$\mathbf{U} = a\mathbf{U}_{\mathrm{ex}} + \frac{c+d}{2}\mathbf{U}_{\mathrm{sh}} + \frac{c-d}{2} \begin{bmatrix} y \\ -x \end{bmatrix} := a \begin{bmatrix} x \\ -y \end{bmatrix} + \frac{c+d}{2} \begin{bmatrix} y \\ x \end{bmatrix} + \frac{c-d}{2} \begin{bmatrix} y \\ -x \end{bmatrix}, \qquad (1.5)$$

and denote respectively by $(\mathbf{u}_{\mathrm{ex}}, p_{\mathrm{ex}})$ and $(\mathbf{u}_{\mathrm{sh}}, p_{\mathrm{sh}})$ the solutions to (1.2) when $\mathbf{U} = \mathbf{U}_{\mathrm{ex}}$ and $\mathbf{U} = \mathbf{U}_{\mathrm{sh}}$, then the solution $(\mathbf{u}, p)$ is given by

$$\mathbf{u} = a\mathbf{u}_{\mathrm{ex}} + \frac{c+d}{2}\mathbf{u}_{\mathrm{sh}} + \frac{c-d}{2} \begin{bmatrix} y \\ -x \end{bmatrix} \qquad (1.6)$$

and

$$p = ap_{\mathrm{ex}} + \frac{c+d}{2}p_{\mathrm{sh}} + \frac{c-d}{2}\mathrm{const.} \qquad (1.7)$$

The singular behavior of the stress comes solely from those corresponding to $(\mathbf{u}_{\mathrm{ex}}, p_{\mathrm{ex}})$ and $(\mathbf{u}_{\mathrm{sh}}, p_{\mathrm{sh}})$. The flows $\mathbf{U}_{\mathrm{ex}}$ and $\mathbf{U}_{\mathrm{sh}}$ are called the extensional flow and the shear flow, respectively, which explains the subscripts ex and sh in our notation.

For the solution $(\mathbf{u}, p)$ to the Stokes system, the strain tensor, denoted by $\mathcal{E}[\mathbf{u}]$, is given by

$$\mathcal{E}[\mathbf{u}] = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \qquad (1.8)$$

where the superscript T denotes the transpose, and the corresponding stress tensor is given by

$$\sigma[\mathbf{u}, p] = -pI + 2\mu\mathcal{E}[\mathbf{u}], \qquad (1.9)$$

where I is the identity matrix. The constants c_{ij} appearing in (1.2) are determined by the boundary integral conditions

$$\int_{\partial D_i} \boldsymbol{\psi}_j \cdot \sigma[\mathbf{u}, p]\nu \, dl = 0, \quad i = 1, 2, \ j = 1, 2, 3. \qquad (1.10)$$

Here, ν denotes the unit normal on the boundary ∂D_i and dl is the line element. Physically, these integral conditions imply that each rigid inclusion is in equilibrium, namely, the net translational and rotational stress on each boundary is zero (see, e.g., [7]).

The following is the main result of this paper. It shows that the stress always blows up. There and in what follows, $A \lesssim B$ means that there is a constant C independent of δ such that $A \leq CB$, and $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. The supremum norm on D^e is denoted by $\|\cdot\|_\infty$.

Theorem 1.1. *Let D_1 and D_2 be disks of the same radii and let $(\mathbf{u}, p)$ be the the unique solution to (1.2) when U is of the form (1.4) and $P = 0$. Then,*

$$\|\sigma[\mathbf{u}, p]\|_\infty \approx \delta^{-1/2}. \qquad (1.11)$$

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In fact, we can separate our problem into the cases where the pressure or shear stress blows up as the following two theorems show, of which the main theorem is an immediate consequence. To present these results clearly, we assume for convenience that the centers of D_1 and D_2 are, respectively, given by

$$c_1 = (-R - \delta/2, 0) \quad \text{and} \quad c_2 = (R + \delta/2, 0) \tag{1.12}$$

after applying rotation and translation if necessary, where R is the common radius of the disks and δ is the distance between them. To describe the two-dimensional Stokes flow, we construct a pair of stream functions using the bipolar coordinates, and then use the stream function formulation to construct special solutions $(\mathbf{h}_j, p_j)$, $j = 1, 2$, to the Stokes system (see Section 3 for precise definitions of $(\mathbf{h}_j, p_j)$). It turns out that these special solutions, called singular functions, capture precisely the singular behavior of $\sigma[\mathbf{u}_{\text{ex}}, p_{\text{ex}}]$ and $\sigma[\mathbf{u}_{\text{sh}}, p_{\text{sh}}]$. As a result, we are able to characterize the blow-up of the pressure and the shear stress for the different configurations of the background velocity field $\mathbf{U}$: when $\mathbf{U} = \mathbf{U}_{\text{ex}}$, the pressure blows up at the rate of $\delta^{-1/2}$ while the shear stress is bounded; when $\mathbf{U} = \mathbf{U}_{\text{sh}}$, the other way around.

The precise statements of the results are presented in the following theorems. Here and afterwards, Π_δ denotes the narrow region between the two cylinders defined by

$$\Pi_\delta := ([-R - \delta/2, R + \delta/2] \times [-\sqrt{\delta}, \sqrt{\delta}]) \cap D^e. \tag{1.13}$$

Theorem 1.2. *Suppose that D_1 and D_2 are arranged so that (1.12) holds and that $\mathbf{U} = \mathbf{U}_{\text{ex}}$ and $P = 0$. It holds that*

$$\|\mathcal{E}[\mathbf{u}_{\text{ex}}]\|_\infty \lesssim 1 \quad \text{and} \quad \|p_{\text{ex}}\|_\infty \approx \delta^{-1/2} \tag{1.14}$$

as $\delta \to 0$. In the narrow region Π_δ,

$$\sigma[\mathbf{u}_{\text{ex}}, p_{\text{ex}}](x, y) = 2\mu\sqrt{R}\delta^{-1/2}\frac{(y^2 + 3R\delta)(y^2 - R\delta)}{(y^2 + R\delta)^2}I + O(1). \tag{1.15}$$

Theorem 1.3. *Suppose that D_1 and D_2 are arranged so that (1.12) holds and that $\mathbf{U} = \mathbf{U}_{\text{sh}}$ and $P = 0$. It holds that*

$$\|\mathcal{E}[\mathbf{u}_{\text{sh}}]\|_\infty \approx \delta^{-1/2} \quad \text{and} \quad \|p_{\text{sh}}\|_\infty \lesssim 1 \tag{1.16}$$

as $\delta \to 0$. In the narrow region Π_δ,

$$\sigma[\mathbf{u}_{\text{sh}}, p_{\text{sh}}](x, y) = 2\mu\sqrt{\frac{R}{\delta}}\frac{R\delta}{y^2 + R\delta}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + O(1). \tag{1.17}$$

Let $(\mathbf{u}, p)$ be the solution to (1.2). According to (1.6) and (1.7),

$$\sigma[\mathbf{u}, p] = a\sigma[\mathbf{u}_{\text{ex}}, p_{\text{ex}}] + \frac{c + d}{2}\sigma[\mathbf{u}_{\text{sh}}, p_{\text{sh}}] + O(1). \tag{1.18}$$

Thus, Theorem 1.1 is an immediate consequence of (1.15) and (1.17).

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What is actually shown in this paper is that if the background velocity field $\mathbf{U}$ is of the form (1.4) and $P = 0$, then the solution $(\mathbf{u}, p)$ is of the following form:

$$(\mathbf{u}, p) = a \frac{2}{\sqrt{R}} \delta^{3/2} (\mathbf{h}_1, p_1) + \frac{c+d}{2} \sqrt{R\delta} (\mathbf{h}_2, p_2) + (\mathbf{u}_0, p_0), \tag{1.19}$$

where $(\mathbf{u}_0, p_0)$ is a solution to the Stokes problem whose stress tensor is bounded. See the end of section 5 for a brief proof of this fact. Since the singular functions $(\mathbf{h}_j, p_j)$ $(j = 1, 2)$ are given explicitly, the decomposition formula (1.19) may cast light on the challenging problem of computing the Stokes flow in presence of closely located rigid cylinders.

Some historical remarks on the study of the Stokes flow in presence of two circular cylinders are in order. Jeffrey developed in [12] a separable solution method based on bipolar coordinates and then analyzed in [13] the flow generated by two rotating circular cylinders. Several other authors independently developed similar methods [23, 3]. Jeffrey's method has been applied to various problems of Stokes flow [20, 24, 25, 14, 22, 26, 8, 11]. In particular, Raasch derived the exact analytic solution for two circular cylinders under the equilibrium condition, which represents suspended particles in a viscous fluid [18] (see also [19, 9]). However, due to the high complexity of the solution, it is difficult to analyze the singular behavior of the solution when the cylinders are close-to-touching. In this work, this difficulty is successfully overcome by introducing the singular functions.

The organization of the paper is as follows. In the next section, we introduce the bipolar coordinates and review the stream function formulation for the Stokes system. Section 3 is to construct singular functions which are the building blocks in describing the singular behavior of the solution to the Stokes system (1.2) as the separating distance between D_1 and D_2 tends to zero. Sections 4 and 5 are to prove Theorems 1.2 and 1.3. Section 6 and 7 are to prove that stress does not blow up if the no-slip boundary condition is prescribed on the boundary of the circular inclusions. Appendices are to prove some auxiliary lemmas. The paper ends with a short discussion.

2 Preliminaries

2.1 Bipolar coordinates

Given a positive constant a, the bipolar coordinates (ζ, θ) are defined by

$$x + iy = a \frac{e^{\zeta - i\theta} + 1}{e^{\zeta - i\theta} - 1}, \tag{2.1}$$

so that

$$x = a \frac{\sinh \zeta}{\cosh \zeta - \cos \theta}, \quad y = a \frac{\sin \theta}{\cosh \zeta - \cos \theta}, \tag{2.2}$$

or equivalently,

$$\zeta = \log \frac{\sqrt{(x+a)^2 + y^2}}{\sqrt{(x-a)^2 + y^2}}, \quad \theta = \arg(x - a, y) - \arg(x + a, y). \tag{2.3}$$

The coordinate curve $\{\zeta = c\}$ represents a circle of radius $a/|\sinh c|$ centered at the point $(a/\tanh c, 0)$. Similarly, the curve $\{\theta = c\}$ represents a circle of radius $a/|\sin c|$ centered at

$(a/\tan c, 0)$. The point of infinity corresponds to $(\zeta, \theta) = (0,0)$. See, e.g., [12] for bipolar coordinates in relation with the Stokes system.

The geometry of two disks (the cross sections of the two circular cylinders) can be described efficiently in terms of bipolar coordinates. Let

$$a := \sqrt{\delta\left(R + \frac{\delta}{4}\right)}. \tag{2.4}$$

Then the boundary ∂D_i of the cylinder D_i can be parameterized by a ζ-coordinate curve as follows:

$$\partial D_1 = \{\zeta = -s\}, \quad \partial D_2 = \{\zeta = +s\}, \tag{2.5}$$

where

$$s = \sinh^{-1}(a/R). \tag{2.6}$$

We note that

$$s = \sqrt{\frac{\delta}{R}} + O(\delta^{3/2}) \quad \text{as } \delta \to 0. \tag{2.7}$$

The exterior domain D^e of $D_1 \cup D_2$ is characterized in bipolar coordinates (ζ, θ) by the rectangle

$$D^e = \{(\zeta, \theta) \in (-s, s) \times [0, 2\pi)\}. \tag{2.8}$$

In particular, $\{(\zeta, \pi), |\zeta| < s\}$ is the line segment connecting the two points $(-\delta/2, 0)$ and $(\delta/2, 0)$.

Let $\{\mathbf{e}_x, \mathbf{e}_y\}$ be the standard unit basis vectors in $\mathbb{R}^2$ and let $\{\mathbf{e}_\zeta, \mathbf{e}_\theta\}$ be the unit basis vectors in the bipolar coordinates, namely,

$$\mathbf{e}_\zeta = \frac{\nabla\zeta}{|\nabla\zeta|}, \quad \mathbf{e}_\theta = \frac{\nabla\theta}{|\nabla\theta|}.$$

Let $[\mathbf{e}_\zeta, \mathbf{e}_\theta]$ denote the 2×2 matrix whose columns are $\mathbf{e}_\zeta$ and $\mathbf{e}_\theta$. Then one can easily see from (2.3) that

$$\Xi := [\mathbf{e}_\zeta, \mathbf{e}_\theta] = \begin{bmatrix} \alpha(\zeta, \theta) & -\beta(\zeta, \theta) \\ -\beta(\zeta, \theta) & -\alpha(\zeta, \theta) \end{bmatrix}, \tag{2.9}$$

where

$$\alpha(\zeta, \theta) := \frac{1 - \cosh\zeta\cos\theta}{\cosh\zeta - \cos\theta}, \quad \beta(\zeta, \theta) := \frac{\sinh\zeta\sin\theta}{\cosh\zeta - \cos\theta}. \tag{2.10}$$

Since $\alpha^2 + \beta^2 = 1$, we have

$$\Xi^2 = I. \tag{2.11}$$

This means that Ξ is the transition transformation in the sense that

$$[\mathbf{e}_x, \mathbf{e}_y] = \Xi[\mathbf{e}_\zeta, \mathbf{e}_\theta]. \tag{2.12}$$

Define the scaling function

$$h(\zeta, \theta) := \frac{\cosh\zeta - \cos\theta}{a}. \tag{2.13}$$

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Then, for any scalar function f, its gradient ∇f can be expressed as

$$\nabla f = h(\zeta, \theta)[\partial_\zeta f\, \mathbf{e}_\zeta + \partial_\theta f\, \mathbf{e}_\theta] \tag{2.14}$$

(see, e.g., [21]). Here and throughout this paper, ∂_ζ and ∂_θ denote the partial derivatives with respect to the ζ and θ variables, respectively. Moreover, the line element, denoted by dl, on ∂D_2 is given by

$$dl = h(s, \theta)^{-1} d\theta. \tag{2.15}$$

One can easily check that, for $i = 1, 2$,

$$\partial_\nu f\big|_{\partial D_i} = (-1)^{i+1} h(\zeta, \theta) \partial_\zeta f\big|_{\zeta=(-1)^i s}, \tag{2.16}$$

and

$$\partial_T f\big|_{\partial D_i} = (-1)^i h(\zeta, \theta) \partial_\theta f\big|_{\zeta=(-1)^i s}, \tag{2.17}$$

where ∂_ν and ∂_T denote the normal and tangential derivatives, respectively.

Using (2.2) and (2.4), one can see that

$$\frac{\cos\theta}{\cosh\zeta - \cos\theta} = \frac{1}{2a^2}(x^2 + y^2) - \frac{1}{2} = \frac{1}{2R\delta}(x^2 + y^2) - \frac{1}{2} + O(\delta).$$

If (x, y) lies in the narrow region Π_δ defined in (1.13), then $|x| \lesssim \delta$, and hence

$$\frac{1}{2R\delta}(x^2 + y^2) - \frac{1}{2} = \frac{y^2}{2R\delta} - \frac{1}{2} + O(\delta).$$

Moreover, if (ζ, θ) lies in Π_δ, then there is a positive constant $C < \pi$ such that $|\theta - \pi| < C$. Since $|\zeta| < s \approx \sqrt{\delta}$, we have

$$\cosh\zeta - \cos\theta = 1 - \cos\theta + O(\zeta^2) = 1 - \cos\theta + O(\delta).$$

Thus we have

$$\frac{\cos\theta}{1 - \cos\theta} = \frac{y^2}{2R\delta} - \frac{1}{2} + O(\delta),$$

or equivalently,

$$\cos\theta = \frac{y^2 - R\delta}{y^2 + R\delta} + O(\delta) \tag{2.18}$$

in the region Π_δ. One can also easily see from (2.10) that in Π_δ

$$\alpha(\zeta, \theta) = 1 + O(\delta), \quad \beta(\zeta, \theta) = O(\sqrt{\delta}),$$

and hence

$$\Xi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + O(\sqrt{\delta}). \tag{2.19}$$

Using (2.2), one can see

$$|\mathbf{x}|^2 = x^2 + y^2 = \frac{\cosh\zeta + \cos\theta}{\cosh\zeta - \cos\theta}.$$

Since the following relation holds for large enough $|\mathbf{x}|$ (or small enough ζ and θ)

$$|\mathbf{x}|^{-2} = \frac{\cosh\zeta - \cos\theta}{\cosh\zeta + \cos\theta} = \frac{\frac{\zeta^2}{2} + \frac{\theta^2}{2}}{2 + O(\zeta^2 + \theta^2)},$$

we obtain

$$\frac{1}{8}(\zeta^2 + \theta^2) \leq |\mathbf{x}|^{-2} \leq \frac{1}{2}(\zeta^2 + \theta^2). \tag{2.20}$$

2.2 The stream function

Here we review the stream function formulation in the two-dimensional incompressible flow and collect some useful formulas.

It is well known that any solution $(\mathbf{u}, p)$ to the Stokes system, $\mu\Delta\mathbf{u} = \nabla p$ and $\nabla\cdot\mathbf{u} = 0$, can be written using a scalar function Ψ satisfying the biharmonic equation $\Delta^2\Psi = 0$. The function Ψ is called the stream function. Once the function Ψ is known, the velocity field $\mathbf{u} = (u_x, u_y)^T$ can be determined from the relations

$$u_x = \partial_y\Psi, \quad u_y = -\partial_x\Psi, \tag{2.21}$$

and the pressure p is a harmonic conjugate of $\mu\Delta\Psi$ (see, e.g., [4]).

Let us write the stream function formulation in terms of bipolar coordinates. It is also known (see, e.g, [24, 25]) that the Laplacian in Cartesian coordinates is related to bipolar coordinates via

$$\Delta_{x,y}\Psi = \frac{1}{a}\left((\cosh\zeta - \cos\theta)\Delta_{\zeta,\theta} + (\cosh\zeta + \cos\theta) - 2(\sinh\zeta\partial_\zeta + \sin\theta\partial_\theta)\right)(h\Psi), \tag{2.22}$$

where h is the function defined in (2.13). Using this formula, the biharmonic equation $\Delta^2\Psi = 0$ can be rewritten as

$$\left(\partial_\zeta^4 + 2\partial_\zeta^2\partial_\theta^2 + \partial_\theta^4 - 2\partial_\zeta^2 + 2\partial_\theta^2 + 1\right)(h\Psi) = 0, \tag{2.23}$$

and the general solution to the above equation takes the following form:

$$
\begin{aligned}
(h\Psi)&(\zeta, \theta) \\
&= K(\cosh\zeta - \cos\theta)\ln(2\cosh\zeta - 2\cos\theta) + a_0\cosh\zeta + b_0\zeta\cosh\zeta + c_0\sinh\zeta + d_0\zeta\sinh\zeta \\
&+ (a_1\cosh 2\zeta + b_1 + c_1\sinh 2\zeta + d_1\zeta)\cos\theta + (\widetilde{a}_1\cosh 2\zeta + \widetilde{b}_1 + \widetilde{c}_1\sinh 2\zeta + \widetilde{d}_1\zeta)\sin\theta \\
&+ \sum_{n=2}^{\infty}\left(a_n\cosh(n+1)\zeta + b_n\cosh(n-1)\zeta + c_n\sinh(n+1)\zeta + d_n\sinh(n-1)\zeta\right)\cos n\theta \\
&+ \sum_{n=2}^{\infty}\left(\widetilde{a}_n\cosh(n+1)\zeta + \widetilde{b}_n\cosh(n-1)\zeta + \widetilde{c}_n\sinh(n+1)\zeta + \widetilde{d}_n\sinh(n-1)\zeta\right)\sin n\theta.
\end{aligned}
\tag{2.24}
$$

Using (2.14) and (2.21), one can see that the components of the velocity $\mathbf{u} = u_\zeta\mathbf{e}_\zeta + u_\theta\mathbf{e}_\theta$ are given as follows:

$$u_\zeta = -h\partial_\theta\Psi = \left(-\partial_\theta + \frac{\sin\theta}{\cosh\zeta - \cos\theta}\right)(h\Psi), \tag{2.25}$$

$$u_\theta = +h\partial_\zeta\Psi = \left(\partial_\zeta - \frac{\sinh\zeta}{\cosh\zeta - \cos\theta}\right)(h\Psi), \tag{2.26}$$

and the pressure p satisfies the relations

$$\partial_\zeta p = -\mu\partial_\theta\Delta\Psi, \quad \partial_\theta p = \mu\partial_\zeta\Delta\Psi. \tag{2.27}$$

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The entries of the strain tensor $\mathcal{E}[\mathbf{u}]$ when represented in terms of the basis $\{\mathbf{e}_\zeta, \mathbf{e}_\theta\}$ are given by

$$\mathcal{E}_{\zeta\zeta} = -h\partial_\zeta (h\partial_\theta \Psi) - h\partial_\zeta \Psi \partial_\theta h, \tag{2.28}$$

$$\mathcal{E}_{\theta\theta} = +h\partial_\theta (h\partial_\zeta \Psi) + h\partial_\theta \Psi \partial_\zeta h, \tag{2.29}$$

$$\mathcal{E}_{\zeta\theta} = \frac{1}{2} \left(\partial_\zeta \left(h^2 \partial_\zeta \Psi \right) - \partial_\theta \left(h^2 \partial_\theta \Psi \right) \right). \tag{2.30}$$

Therefore, the following relation holds:

$$\mathcal{E}[\mathbf{u}] = \Xi \begin{bmatrix} \mathcal{E}_{\zeta\zeta} & \mathcal{E}_{\zeta\theta} \\ \mathcal{E}_{\zeta\theta} & \mathcal{E}_{\theta\theta} \end{bmatrix} \Xi, \tag{2.31}$$

where Ξ is the matrix given in (2.9). The entries of the stress tensor in bipolar coordinates are given by

$$\sigma_{\zeta\zeta} = -p + 2\mu\mathcal{E}_{\zeta\zeta}, \quad \sigma_{\theta\theta} = -p + 2\mu\mathcal{E}_{\zeta\theta}, \quad \sigma_{\zeta\theta} = 2\mu\mathcal{E}_{\zeta\theta}. \tag{2.32}$$

Similarly, we have the following relation for the stress tensor:

$$\sigma[\mathbf{u}, p] = \Xi \begin{bmatrix} \sigma_{\zeta\zeta} & \sigma_{\zeta\theta} \\ \sigma_{\zeta\theta} & \sigma_{\theta\theta} \end{bmatrix} \Xi. \tag{2.33}$$

Since each component of Ξ is bounded, it follows from (2.11), (2.31) and (2.33) that

$$\|\mathcal{E}[\mathbf{u}]\|_{L^\infty(K)} \approx \|\mathcal{E}_{\zeta\zeta}\|_{L^\infty(K)} + \|\mathcal{E}_{\theta\theta}\|_{L^\infty(K)} + \|\mathcal{E}_{\zeta\theta}\|_{L^\infty(K)}, \tag{2.34}$$

and

$$\|\sigma[\mathbf{u}, p]\|_{L^\infty(K)} \approx \|\sigma_{\zeta\zeta}\|_{L^\infty(K)} + \|\sigma_{\theta\theta}\|_{L^\infty(K)} + \|\sigma_{\zeta\theta}\|_{L^\infty(K)} \tag{2.35}$$

for any subset K of D^e.

Using integrations by parts on the exterior domain D^e, we have for any solutions $(\mathbf{u}, p), (\mathbf{v}, q)$ to the Stokes system such that $\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ that

$$\int_{\partial D^e} \mathbf{u} \cdot \sigma[\mathbf{v}, q]\nu = -\int_{D^e} \mathcal{E}[\mathbf{u}] : \sigma[\mathbf{v}, q]$$

$$= \int_{D^e} (\nabla \cdot \mathbf{u})q - 2\mu \int_{D^e} \mathcal{E}[\mathbf{u}] : \mathcal{E}[\mathbf{v}]$$

$$= -2\mu \int_{D^e} \mathcal{E}[\mathbf{u}] : \mathcal{E}[\mathbf{v}]. \tag{2.36}$$

This implies in particular that the following Green's theorem holds:

$$\int_{\partial D^e} \mathbf{u} \cdot \sigma[\mathbf{v}, q]\nu = \int_{\partial D^e} \mathbf{v} \cdot \sigma[\mathbf{u}, p]\nu. \tag{2.37}$$

2.3   An exterior Dirichlet problem

Let $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{ij}(\mathbf{x}))_{i,j=1,2}$ be

$$\Gamma_{ij}(\mathbf{x}) = -\frac{1}{4\pi\mu}(\delta_{ij} \log |\mathbf{x}| + \frac{x_i x_j}{|\mathbf{x}|^2}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus \{(0,0)\},$$

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and define $\mathbf{p} = (p_j)_{j=1,2}$ by

$$\mathbf{p} = -\frac{1}{2\pi} \frac{\mathbf{x}}{|\mathbf{x}|^2}, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Then, $(\mathbf{\Gamma}, \mathbf{p})$ is the fundamental solution to the Stokes system, namely,

$$\mu \Delta \mathbf{\Gamma} - \nabla \mathbf{p} = \delta(\mathbf{x})\mathbf{I}.$$

Let Γ_Δ be the fundamental solution to the Laplacian given by

$$\Gamma_\Delta(\mathbf{x}) = \frac{1}{2\pi} \log|\mathbf{x}|.$$

The existence and uniqueness of the exterior Dirichlet problem, proved in [17, Theorem 9.15], is as follows.

Theorem 2.1. *Assume that Ω is a bounded Lipschitz domain. Then the exterior Dirichlet problem*

$$\begin{cases} \mu \Delta \mathbf{u} = \nabla p & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \quad (2.38)$$

with the decaying conditions

$$\begin{cases} \mathbf{u}(\mathbf{x}) = \mathbf{\Gamma}(\mathbf{x})\mathbf{A} + C + O(|\mathbf{x}|^{-1}), \\ \partial_j \mathbf{u}(\mathbf{x}) = \partial_j \mathbf{\Gamma}(\mathbf{x})\mathbf{A} + O(|\mathbf{x}|^{-2}), \\ p(\mathbf{x}) = \nabla\Gamma_\Delta \cdot \mathbf{A} + O(|\mathbf{x}|^{-2}) \end{cases}$$

as $|\mathbf{x}| \to \infty$ for some constant C, has a solution, which is unique modulo adding functions to the pressure term which are locally constant in $\mathbb{R}^2$. Here, $\mathbf{A} \in \mathbb{R}^2$ is a priori given constant.

We shall consider the exterior Dirichlet problem with $\mathbf{A} = 0$. Let $\mathcal{M}$ be the set of all pairs of functions $(\mathbf{u}, p)$ satisfying

$$\begin{cases} \mathbf{u}(\mathbf{x}) = C + O(|\mathbf{x}|^{-1}), \\ \nabla\mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \\ p(\mathbf{x}) = O(|\mathbf{x}|^{-2}) \end{cases} \quad (2.39)$$

as $|\mathbf{x}| \to \infty$ for some constant C. We denote by $\mathcal{M}_0$ the set of all pairs of functions $(\mathbf{u}, p)$ satisfying the decay conditions (2.39) with $C = 0$.

3   The singular functions for the Stokes system

In what follows, we construct the singular functions $(\mathbf{h}_j, p_j)$, $j = 1, 2$, which is the unique solution to the following problem:

$$\begin{cases} \mu \Delta \mathbf{h}_j = \nabla p_j & \text{in } D^e, \\ \nabla \cdot \mathbf{h}_j = 0 & \text{in } D^e, \\ \mathbf{h}_j = \frac{(-1)^i}{2}\psi_j & \partial D_i, \ i = 1, 2, \\ (\mathbf{h}_j, p_j) \in \mathcal{M}. \end{cases} \quad (3.1)$$

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We then provide quantitative estimates of the blow-up of these functions in the subsequent propositions. We call the solutions $(\mathbf{h}_j, p_j)$ the singular functions since they are the building blocks in describing the singular behavior, i.e., the stress tensor blow-up, of the solution to (1.2). In fact, we will see that the solution to (1.2) can be expressed as a linear combination of singular functions (modulo a regular function) and the nature of the stress tensor blow-up is characterized by $(\mathbf{h}_1, p_1)$ and $(\mathbf{h}_2, p_2)$.

Proposition 3.1. *Define two constants A_1 and B_1 by*

$$A_1 := \frac{1}{2s - \tanh 2s}, \quad B_1 := -\frac{1}{2\cosh 2s} A_1. \tag{3.2}$$

(i) *The stream function Ψ_1 associated with the singular functions $(\mathbf{h}_1, p_1)$ is given by*

$$\Psi_1(\zeta, \theta) = \frac{1}{h(\zeta, \theta)}(A_1\zeta + B_1 \sinh 2\zeta) \sin \theta. \tag{3.3}$$

(ii) *The components of the velocity $\mathbf{h}_1 = h_{1\zeta}\mathbf{e}_\zeta + h_{1\theta}\mathbf{e}_\theta$ are given by*

$$h_{1\zeta} = (A_1\zeta + B_1 \sinh 2\zeta)\frac{1 - \cosh \zeta \cos \theta}{\cosh \zeta - \cos \theta}, \tag{3.4}$$

$$h_{1\theta} = \sin \theta \left(A_1 + 2B_1 \cosh 2\zeta - \frac{\sinh \zeta (A_1\zeta + B_1 \sinh 2\zeta)}{\cosh \zeta - \cos \theta} \right). \tag{3.5}$$

(iii) *The pressure p_1 is given by*

$$p_1 = \frac{2\mu}{a}((A_1 - 2B_1)\cosh \zeta \cos \theta + B_1 \cosh 2\zeta \cos 2\theta) - \frac{2\mu}{a}(A_1 - B_1). \tag{3.6}$$

Proof. The formulas (3.3)-(3.5) are derived in the following way. We use the expansion (2.24) for the general solution to the Stokes system, and then determine its unknown constant coefficients by matching the boundary conditions on ∂D^e, given by $\{\zeta = \pm s\}$, and using formulas (2.12), (2.25), and (2.26). Let us show that the boundary conditions are fulfilled. If $\zeta = \pm s$, we have

$$h_{1\zeta}|_{\zeta = \pm s} = \pm \frac{1 - \cosh s \cos \theta}{2(\cosh s - \cos \theta)} = \pm \frac{1}{2}\alpha(s, \theta),$$

$$h_{1\theta}|_{\zeta = \pm s} = \mp \frac{\sinh s \sin \theta}{2(\cosh s - \cos \theta)} = \mp \frac{1}{2}\beta(s, \theta).$$

One can see from the relation (2.12) that the boundary conditions on $\partial D_1 \cup \partial D_2$ are satisfied.

The formula (3.6) follows from (2.27) and (3.3). In fact, applying (2.22) to Ψ_1 given in (3.3), we see that

$$\mu \Delta \Psi_1 = \frac{-2\mu}{a}((A_1 - 2B_1)\sinh \zeta \sin \theta + B_1 \sinh 2\zeta \sin 2\theta).$$

The harmonic conjugate of this function vanishing at $(\zeta, \theta) = (0, 0)$ is nothing but the one given in (3.6).

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We now prove that $(\mathbf{h}_1, p_1)$ belongs to $\mathcal{M}$. We first prove that $\mathbf{h}_1(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$, which amounts to proving

$$\mathbf{h}(\zeta, \theta) = O(|\zeta| + |\theta|), \quad (\zeta, \theta) \to (0, 0), \tag{3.7}$$

thanks to (2.20). We have from (3.4) and (3.5) that

$$|h_{1\zeta}| \le C(|\zeta| + |\zeta| \frac{|\zeta|^2 + |\theta|^2}{|\zeta|^2 + |\theta|^2}) \le C|\zeta|,$$

$$|h_{1\theta}| \le C|\theta|\Big(1 + |\zeta| \frac{|\zeta|}{|\zeta|^2 + |\theta|^2}\Big) \le C|\theta|. \tag{3.8}$$

Here and throughout this proof, the constant C may depend on s, but is independent of (ζ, θ). This proves (3.7).

Similarly, one can show that

$$|\partial_\zeta h_{1\zeta}| \le C, \quad |\partial_\zeta h_{1\zeta}| \le C, \quad |\partial_\zeta h_{1\zeta}| \le C, \quad |\partial_\zeta h_{1\zeta}| \le C. \tag{3.9}$$

Since $\mathbf{h}_1 = h_{1\zeta}\mathbf{e}_\zeta + h_{1\theta}\mathbf{e}_\theta$, we have

$$|\nabla \mathbf{h}_1| \le C(|\nabla h_{1\zeta}| + |h_{1\zeta}\nabla \mathbf{e}_\zeta| + |\nabla h_{1\theta}| + |h_{1\theta}\nabla \mathbf{e}_\theta|). \tag{3.10}$$

It then follows from (3.8) and the following lemma, whose proof will be given in Appendix A, that

$$|\nabla \mathbf{h}_1| \le C(|\nabla h_{1\zeta}| + |\nabla h_{1\theta}| + |\zeta|^2 + |\theta|^2). \tag{3.11}$$

Lemma 3.2. *It holds that*
$$|\nabla \mathbf{e}_\zeta| + |\nabla \mathbf{e}_\theta| \lesssim |\zeta| + |\theta|. \tag{3.12}$$

We then have from (3.9) that

$$|\nabla \mathbf{h}_1| \le C(|h\partial_\zeta h_{1\zeta}| + |h\partial_\theta h_{1\zeta}| + |h\partial_\zeta h_{1\theta}| + |h\partial_\theta h_{1\theta}| + |\zeta|^2 + |\theta|^2) \le C(|h| + |\zeta|^2 + |\theta|^2).$$

One can see from the definition of the function h that

$$|h(\zeta, \theta)| \le C(|\zeta|^2 + |\theta|^2),$$

and hence

$$|\nabla \mathbf{h}_1| \le C(|\zeta|^2 + |\theta|^2),$$

or equivalently, $\nabla \mathbf{h}_1(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$.

Note that $p(\zeta, \theta) = O(|\zeta|^2 + |\theta|^2)$ as $(\zeta, \theta) \to (0, 0)$. Thus, $p(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, and hence $(\mathbf{h}_1, p_1) \in \mathcal{M}$. This completes the proof. $\qquad \square$

It is helpful to write $\mathbf{h}_1$ in terms of Cartesian coordinates. By (2.2), we have

$$\Psi_1 = A_1 y\zeta + B_1 y \sinh \zeta,$$

and hence

$$\nabla \Psi_1 = A_1 \zeta \mathbf{e}_y + A_1 y \nabla \zeta + B_1 \sinh \zeta \mathbf{e}_y + B_1 y \cosh \zeta \nabla \zeta.$$

Then, since $\mathbf{h}_1 = (\nabla \Psi_1)^\perp$, we have

$$\mathbf{h}_1 = (A_1 \zeta + B_1 \sinh \zeta)\mathbf{e}_x + (A_1 + B_1 \cosh \zeta)y(\nabla \zeta)^\perp. \tag{3.13}$$

Here, $(x, y)^\perp = (y, -x)$.

Proposition 3.3. *We have*

$$\|\mathcal{E}[\mathbf{h}_1]\| \lesssim \delta^{-3/2} \quad and \quad \|p_1\|_\infty \approx \delta^{-2} \tag{3.14}$$

as $\delta \to 0$. *In the narrow region* Π_δ,

$$\sigma[\mathbf{h}_1, p_1](x, y) = -\frac{3}{4}\mu R \delta^{-2} \frac{(y^2 + 3R\delta)(y^2 - R\delta)}{(y^2 + R\delta)^2} I + O(\delta^{-3/2}). \tag{3.15}$$

Proof. One can see from the explicit forms of the constants A_1 and B_1 in (3.2) that

$$A_1 = \frac{3}{8}s^{-3} + O(s^{-1}), \quad B_1 = -\frac{3}{16}s^{-3} + O(s^{-1}). \tag{3.16}$$

Using (2.28)-(2.30) and Proposition 3.1 (i), we have

$$\mathcal{E}_{\zeta\zeta} = -h(\zeta, \theta)(A_1 + 2B_1 \cosh 2\xi) \cos\theta, \tag{3.17}$$
$$\mathcal{E}_{\theta\theta} = h(\zeta, \theta)(A_1 + 2B_1 \cosh 2\xi) \cos\theta, \tag{3.18}$$
$$\mathcal{E}_{\zeta\theta} = h(\zeta, \theta)2B_1 \sinh 2\zeta \sin\theta. \tag{3.19}$$

We first estimate $\mathcal{E}_{\zeta\zeta}$. It follows from the Taylor expansions of $\cosh 2\zeta$ and $\sinh 2\zeta$, and from (3.16) that

$$\mathcal{E}_{\zeta\zeta} = -\frac{1 + \cos\theta + O(\zeta^2)}{a}(A_1 + 2B_1 + O(\zeta^2)) \cos\theta.$$

Observe from (3.16) that $A_1 + 2B_1 = O(s^{-1})$. Since $|\zeta| \le s$ and $a, s \approx \sqrt{\delta}$, we have

$$|\mathcal{E}_{\zeta\zeta}| \lesssim \delta^{-1}.$$

Estimates for $\mathcal{E}_{\theta\theta}$ and $\mathcal{E}_{\zeta\theta}$ are simpler. In fact, one can see immediately from (3.18) and (3.19) that

$$|\mathcal{E}_{\theta\theta}| = |\mathcal{E}_{\zeta\zeta}| \lesssim \delta^{-1}$$

and

$$|\mathcal{E}_{\zeta\theta}| \lesssim a^{-1}|B_1\zeta| \lesssim \delta^{-3/2}.$$

Then (2.34) yields the first estimate in (3.14).

We now consider the pressure p_1. Since $a \approx \sqrt{\delta}$, we have

$$|p_1(\zeta, \theta)| \lesssim \delta^{-2}(\cosh\zeta \,|\cos\theta| + 1).$$

Since $|\zeta| \le s \approx \sqrt{\delta}$ by (2.7) if $(\zeta, \theta) \in D^e$, we have

$$|p_1(\zeta, \theta)| \lesssim \delta^{-2}.$$

Using the Taylor expansion of $\cosh\zeta$, we see

$$p_1 = \frac{3}{2}\mu R \delta^{-2}\left(\cos\theta - \frac{1}{2}\cos^2\theta\right) + O(\delta^{-1}).$$

In particular, we have $\|p_1\|_\infty \gtrsim \delta^{-2}$, and the second statement in (3.14) follows. Now the expansion (3.15) in the narrow region follows from (1.9) and (2.18). $\square$

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The expressions for the solution $(\mathbf{h}_2, p_2)$ are quite involved even though it is possible to express it explicitly. However, its singular part, which is to be used in the rest of the paper, can be expressed in a rather simple way. To express the singular part, which is denoted by $(\widetilde{\mathbf{h}}_2, \widetilde{p}_2)$, let

$$A_2 = -\frac{1}{2s + \sinh 2s}. \tag{3.20}$$

Then, the components of the velocity field $\widetilde{\mathbf{h}}_2 = \widetilde{h}_{2\zeta}\mathbf{e}_\zeta + \widetilde{h}_{2\theta}\mathbf{e}_\theta$ are given by

$$\widetilde{h}_{2\zeta} = A_2\zeta\beta(\zeta, \theta), \tag{3.21}$$

$$\widetilde{h}_{2\theta} = A_2\zeta\alpha(\zeta, \theta) + A_2\sinh\zeta, \tag{3.22}$$

and the pressure $\widetilde{p}_2$ is given by

$$\widetilde{p}_2 = -\frac{2\mu}{a}A_2\sinh\zeta\sin\theta. \tag{3.23}$$

Then one can see easily that $(\widetilde{\mathbf{h}}_2, \widetilde{p}_2)$ belongs to $\mathcal{M}$ and is a solution to the Stokes system. Moreover, $\widetilde{\mathbf{h}}_2$ satisfies

$$\widetilde{\mathbf{h}}_2|_{\partial D_i} = \frac{(-1)^i}{2}\boldsymbol{\psi}_2 - C_2\boldsymbol{\psi}_3, \quad i = 1, 2, \tag{3.24}$$

where $\boldsymbol{\psi}_3$ is the one given in (1.1) and C_2 is the constant given by

$$C_2 = \frac{\sinh^2 s}{a}A_2. \tag{3.25}$$

In fact, one can easily check using (2.12) that

$$\mathbf{e}_\theta|_{\partial D_2} = -\cosh s\,\boldsymbol{\psi}_2 - \frac{\sinh s}{a}\boldsymbol{\psi}_3. \tag{3.26}$$

It then follows from (3.21) and (3.22) that

$$\widetilde{\mathbf{h}}_2|_{\partial D_2} = A_2 s(\beta\mathbf{e}_\zeta + \alpha\mathbf{e}_\theta) + A_2\sinh s\,\mathbf{e}_\theta$$

$$= A_2 s(-\boldsymbol{\psi}_2) + A_2(-\sinh s\cosh s)\boldsymbol{\psi}_2 - \frac{\sinh^2 s}{a}A_2\boldsymbol{\psi}_3.$$

This proves (3.24) on ∂D_2. (3.24) on ∂D_1 can be proved in the same way. In Cartesian coordinates, $\widetilde{\mathbf{h}}_2$ is represented in a simple form as

$$\widetilde{\mathbf{h}}_2 = -A_2\zeta\mathbf{e}_y + A_2 x(\nabla\zeta)^\perp. \tag{3.27}$$

Some words about how to derive $(\widetilde{\mathbf{h}}_2, \widetilde{p}_2)$ may be helpful. As in Proposition 3.1, we first derive the relevant stream function $\widetilde{\Psi}_2$ using the expansion (2.24) for the general solution, which turns out to be

$$\widetilde{\Psi}_2(\zeta, \theta) = \frac{1}{h(\zeta, \theta)}A_2\zeta\sinh\zeta. \tag{3.28}$$

We then let $(\widetilde{\mathbf{h}}_2, \widetilde{p}_2)$ be its associated solution to the Stokes system.

Thanks to (3.24), how to find the solution $(\mathbf{h}_2, p_2)$ is clear. Let $(\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}})$ be the solution to

$$
\begin{cases}
\mu \Delta \mathbf{h}_{\mathrm{rot}} = \nabla p_{\mathrm{rot}} & \text{in } D^e, \\
\nabla \cdot \mathbf{h}_{\mathrm{rot}} = 0 & \text{in } D^e, \\
\mathbf{h}_{\mathrm{rot}} = \boldsymbol{\psi}_3 & \partial D_1 \cup \partial D_2, \\
(\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}) \in \mathcal{M}.
\end{cases}
\tag{3.29}
$$

The existence and uniqueness of the solution are guaranteed by Theorem 2.1. We will prove the following proposition later in section 7.

Theorem 3.4. *We have*

$$
\|\mathcal{E}[\mathbf{h}_{\mathrm{rot}}]\|_\infty \lesssim 1, \quad \|p_{\mathrm{rot}}\|_\infty \lesssim 1,
\tag{3.30}
$$

and

$$
\|\sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\|_\infty \lesssim 1.
\tag{3.31}
$$

We immediately have the following proposition.

Proposition 3.5. *Let $(\widetilde{\mathbf{h}}_2, \widetilde{p}_2)$ be as given in (3.21)-(3.23) and C_2 the constant given in (3.25). The solution $(\mathbf{h}_2, p_2)$ to (3.1) is given by*

$$
(\mathbf{h}_2, p_2) = (\widetilde{\mathbf{h}}_2, \widetilde{p}_2) + C_2 (\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}).
\tag{3.32}
$$

Proposition 3.6. *It holds that*

$$
\|\mathcal{E}[\mathbf{h}_2]\|_\infty \approx \delta^{-1} \quad \text{and} \quad \|p_2\|_\infty \approx \delta^{-1/2},
\tag{3.33}
$$

as $\delta \to 0$. In the narrow region Π_δ,

$$
\sigma[\mathbf{h}_2, p_2](x, y) = \mu \delta^{-1} \frac{R\delta}{y^2 + R\delta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + O(\delta^{-1/2}).
\tag{3.34}
$$

Proof. We first note that

$$
A_2 = -\frac{1}{4s} + O(s).
\tag{3.35}
$$

Since $|\zeta| \le s \approx \sqrt{\delta}$ and $a \approx \sqrt{\delta}$, the second estimate in (3.33) immediately follows from (3.23).

Since $a \approx s$ as one can see from (2.6), it follows from (3.35) and the definition of C_2 in (3.25) that C_2 is bounded regardless of δ. In view of (3.30), we only need to derive estimates related to $(\widetilde{\mathbf{h}}_2, \widetilde{p}_2)$. Using (2.28)-(2.30) and (3.28), we have

$$
\mathcal{E}_{\zeta\zeta}[\widetilde{\mathbf{h}}_2] = 0,
\tag{3.36}
$$

$$
\mathcal{E}_{\theta\theta}[\widetilde{\mathbf{h}}_2] = 0,
\tag{3.37}
$$

$$
\mathcal{E}_{\zeta\theta}[\widetilde{\mathbf{h}}_2] = \frac{\cosh \zeta - \cos \theta}{a} A_2 \cosh \zeta.
\tag{3.38}
$$

We then have from (3.38) that

$$
|\mathcal{E}_{\zeta\theta}[\widetilde{\mathbf{h}}_2]| \lesssim \delta^{-1},
$$

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and hence
$$\|\mathcal{E}[\widetilde{\mathbf{h}}_2]\|_\infty \lesssim \delta^{-1}.$$

We see from (2.4), (2.7), (3.35) and (3.38) that

$$\mathcal{E}_{\zeta\theta}[\widetilde{\mathbf{h}}_2] = -\frac{1}{4\delta}(\cosh\zeta - \cos\theta) + O(1).$$

In the narrow region Π_δ, we have

$$\mathcal{E}_{\zeta\theta}[\widetilde{\mathbf{h}}_2] = -\frac{1}{4\delta}(1 - \cos\theta) + O(1).$$

In particular, $|\mathcal{E}_{\zeta\theta}| \gtrsim \delta^{-1}$, and the first estimate in (3.33) follows. The asymptotic formula (3.34) follows from (1.9), (2.18), (2.19) and (2.33). □

4 Proof of Theorem 1.2

Thanks to the symmetry of the problem (1.2) with $\mathbf{U} = \mathbf{U}_{\text{ex}} = (x, -y)^T$ and $P = 0$, the velocity $\mathbf{u}$ enjoys the following symmetry:

$$u_x(x, y) = u_x(x, -y) = -u_x(-x, y),$$
$$u_y(x, y) = -u_y(x, -y) = u_y(-x, y),$$

and the pressure p does:

$$p(x, y) = p(-x, y), \quad p(x, y) = p(x, -y).$$

Thus, we infer

$$c_{11} = -c_{21} \quad \text{and} \quad c_{i2} = c_{i3} = 0 \ \text{ for } i = 1, 2.$$

In other words, we have

$$\mathbf{u} = -c_{21}\boldsymbol{\psi}_1 \quad \text{on } \partial D_1, \quad \mathbf{u} = c_{21}\boldsymbol{\psi}_1 \quad \text{on } \partial D_2. \tag{4.1}$$

Therefore, the solution $(\mathbf{u}, p) := (\mathbf{u}_{\text{ex}}, p_{\text{ex}})$ admits the decomposition in terms of the singular function

$$\mathbf{u} = \mathbf{v}_1 + 2c_{21}\mathbf{h}_1, \quad p = q_1 + 2c_{21}p_1 \quad \text{in } D^e, \tag{4.2}$$

where $(\mathbf{v}_1, q_1)$ is the solution with the no-slip boundary condition, namely,

$$\begin{cases} \mu\Delta\mathbf{v}_1 = \nabla q_1 & \text{in } D^e, \\ \nabla \cdot \mathbf{v}_1 = 0 & \text{in } D^e, \\ \mathbf{v}_1 = 0 & \text{on } \partial D_1 \cup \partial D_2, \\ (\mathbf{v}_1 - \mathbf{U}_{\text{ex}}, q_1) \in \mathcal{M}. \end{cases} \tag{4.3}$$

The following theorem will be proved in section 7.

Theorem 4.1. *Let $(\mathbf{v}_1, q_1)$ be the solution to (4.3). Then, the following estimates hold:*

$$\|\mathcal{E}[\mathbf{v}_1]\|_\infty \lesssim 1, \quad \|q_1\|_\infty \lesssim 1, \tag{4.4}$$

and

$$\|\sigma[\mathbf{v}_1, q_1]\|_\infty \lesssim 1. \tag{4.5}$$

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It then follows from (4.2) that

$$\mathcal{E}[\mathbf{u}] = 2c_{21}\mathcal{E}[\mathbf{h}_1] + O(1),$$
$$p = 2c_{21}p_1 + O(1), \tag{4.6}$$
$$\sigma[\mathbf{u}, p] = 2c_{21}\sigma[\mathbf{h}_1, p_1] + O(1),$$

as $\delta \to 0$. Here, $O(1)$ means that the supremum norms of the remainder terms are bounded on D^e regardless of δ. Because of (3.14), it is now sufficient to estimate the constant c_{21}.

We first express c_{21} in terms of boundary integrals. To do so, we let

$$\mathcal{I}_1 := \int_{\partial D_2} \mathbf{e}_x \cdot \sigma[\mathbf{h}_1, p_1]\nu \, dl \quad \text{and} \quad \mathcal{J}_1 := \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_1, p_1]\nu \, dl, \tag{4.7}$$

with $\mathbf{U} = \mathbf{U}_{\text{ex}} = (x, -y)^T$.

Lemma 4.2. *We have*

$$c_{21} = \frac{\mathcal{J}_1}{\mathcal{I}_1}. \tag{4.8}$$

Proof. By Green's identity for the Stokes system on D^e, we obtain that

$$\int_{\partial D^e} (\mathbf{u} - \mathbf{U}) \cdot \sigma[\mathbf{h}_1, p_1]\big|_+\nu - \sigma[\mathbf{u} - \mathbf{U}, p]\big|_+\nu \cdot \mathbf{h}_1 = 0. \tag{4.9}$$

Since $\mathbf{h}_1|_{\partial D_i} = (-1)^i \frac{1}{2}\psi_1$, it follows from the boundary integral conditions (1.10) that

$$\int_{\partial D_i} \sigma[\mathbf{u}, p]\big|_+\nu \cdot \mathbf{h}_1 = 0, \quad i = 1, 2.$$

Applying Green's identity on D_i, we have

$$\int_{\partial D_i} \sigma[\mathbf{U}, 0]\big|_+\nu \cdot \mathbf{h}_1 = \int_{\partial D_i} \sigma[\mathbf{U}, p_0]\big|_-\nu \cdot \mathbf{h}_1 = 0, \quad i = 1, 2.$$

It then follows from (4.9) that

$$\int_{\partial D^e} (\mathbf{u} - \mathbf{U}) \cdot \sigma[\mathbf{h}_1, p_1]\big|_+\nu = 0,$$

or equivalently

$$\int_{\partial D^e} \mathbf{u} \cdot \sigma[\mathbf{h}_1, p_1]\big|_+\nu = \int_{\partial D^e} \mathbf{U} \cdot \sigma[\mathbf{h}_1, p_1]\big|_+\nu.$$

By symmetry, we have

$$\int_{\partial D_2} \mathbf{u} \cdot \sigma[\mathbf{h}_1, p_1]\big|_+\nu = \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_1, p_1]\big|_+\nu.$$

Then the conclusion follows from (4.1). $\qquad\square$

We have the following lemma whose proof is given in Appendix B.

Lemma 4.3. *As* $\delta \to 0$, *we have*

$$\mathcal{I}_1 = -\frac{3\pi\mu}{2}\left(\frac{R}{\delta}\right)^{3/2} + O(\delta^{-1/2}), \tag{4.10}$$

and

$$\mathcal{J}_1 = -3\pi\mu R + O(\delta). \tag{4.11}$$

As an immediate consequence of Lemmas 4.2 and 4.3, we have the following corollary:

Corollary 4.4. *As* $\delta \to 0$, *we have*

$$c_{21} = \frac{2}{\sqrt{R}}\delta^{3/2} + O(\delta^{5/2}). \tag{4.12}$$

Now Theorem 1.2 follows from Proposition 3.3, (4.6), and Corollary 4.4.

5 Proof of Theorem 1.3

Assume that $\mathbf{U}(x, y) = \mathbf{U}_{\mathrm{sh}} = (y, x)^T$. We write $(\mathbf{u}, p)$ for $(\mathbf{u}_{\mathrm{sh}}, p_{\mathrm{sh}})$ for ease of notation. In this case the velocity $\mathbf{u}$ satisfies

$$u_x(x, y) = -u_x(x, -y) = u_x(-x, y),$$
$$u_y(x, y) = u_y(x, -y) = -u_y(-x, y), \tag{5.1}$$

and the pressure p satisfies:

$$p(x, y) = -p(-x, y), \quad p(x, y) = -p(x, -y).$$

Then, we see that $c_{22} = -c_{12}$, $c_{23} = c_{13}$ and $c_{i1} = c_{i1} = 0$ for $i = 1, 2$. As a result, we have
,

$$\mathbf{u} = -c_{22}\boldsymbol{\psi}_2 + c_{23}\boldsymbol{\psi}_3 \quad \text{on } \partial D_1, \quad \mathbf{u} = c_{22}\boldsymbol{\psi}_2 + c_{23}\boldsymbol{\psi}_3 \quad \text{on } \partial D_2. \tag{5.2}$$

Let us decompose the solution $(\mathbf{u}, p)$ in D^e as

$$(\mathbf{u}, p) = (\mathbf{v}_2, q_2) + 2c_{22}(\mathbf{h}_2, p_2) + c_{23}(\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}), \tag{5.3}$$

where $(\mathbf{v}_2, q_2)$ is the solution to

$$\begin{cases} \mu\Delta\mathbf{v}_2 = \nabla q_2 & \text{in } D^e, \\ \nabla \cdot \mathbf{v}_2 = 0 & \text{in } D^e, \\ \mathbf{v}_2 = 0 & \partial D_1 \cup \partial D_2, \\ (\mathbf{v}_2 - \mathbf{U}_{\mathrm{sh}}, q_2) \in \mathcal{M}, \end{cases} \tag{5.4}$$

and $(\mathbf{h}_{\mathrm{rot}}, q_{\mathrm{rot}})$ is the solution to (3.29). Note that $\mathbf{v}_2$ also satisfies the no-slip boundary condition like $\mathbf{v}_1$, and the following theorem analogous to the one for $\mathbf{v}_1$ holds. We shall give its proof in section 7.

Theorem 5.1. *We have*
$$\|\mathcal{E}[\mathbf{v}_2]\|_\infty \lesssim 1, \quad \|q_2\|_\infty \lesssim 1, \tag{5.5}$$

and
$$\|\sigma[\mathbf{v}_2, q_2]\|_\infty \lesssim 1. \tag{5.6}$$

It follows from (5.3) that
$$\begin{aligned}
\mathcal{E}[\mathbf{u}] &= 2c_{22}\mathcal{E}[\mathbf{h}_2] + c_{23}\mathcal{E}[\mathbf{h}_{\mathrm{rot}}] + O(1), \\
p &= 2c_{22}p_2 + c_{23}p_{\mathrm{rot}} + O(1), \\
\sigma[\mathbf{u}, p] &= 2c_{22}\sigma[\mathbf{h}_2, p_2] + c_{23}\sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}] + O(1),
\end{aligned} \tag{5.7}$$

as $\delta \to 0$.

As before, we represent the constant c_{22} using the integrals
$$\mathcal{I}_{2j} := \int_{\partial D_2} \boldsymbol{\psi}_j \cdot \sigma[\mathbf{h}_2, p_2]\nu \, dl, \quad j = 2, 3, \tag{5.8}$$

$$\mathcal{I}_{\mathrm{rot}} := \int_{\partial D_2} \boldsymbol{\psi}_3 \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\nu \, dl, \tag{5.9}$$

$$\mathcal{J}_2 := \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_2, p_2]\nu \, dl, \tag{5.10}$$

$$\mathcal{J}_{\mathrm{rot}} := \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\nu \, dl, \tag{5.11}$$

where $\mathbf{U} = \mathbf{U}_{\mathrm{sh}} = (y, x)^T$. We have the following lemma whose proof is similar to the one of Lemma 4.2.

Lemma 5.2. *We have*
$$\begin{bmatrix} \mathcal{I}_{22} & \mathcal{I}_{23} \\ \mathcal{I}_{23} & \mathcal{I}_{\mathrm{rot}} \end{bmatrix} \begin{bmatrix} c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_2 \\ \mathcal{J}_{\mathrm{rot}} \end{bmatrix}.$$

Proof. As in the proof of Lemma 4.2, we have
$$\int_{\partial D_2} \mathbf{u} \cdot \sigma[\mathbf{h}_2, p_2]\big|_+\nu = \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_2, p_2]\big|_+\nu,$$

and
$$\int_{\partial D_2} \mathbf{u} \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\big|_+\nu = \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\big|_+\nu.$$

Then, by (5.2), we see
$$\mathcal{I}_{22}c_{22} + \mathcal{I}_{23}c_{23} = \mathcal{J}_2,$$

and
$$\int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\big|_+\nu \cdot c_{22} + \mathcal{I}_{\mathrm{rot}}c_{23} = \mathcal{J}_{\mathrm{rot}}.$$

Then, Green's identity yields
$$\int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\big|_+\nu = \int_{\partial D^e} \mathbf{h}_2 \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\big|_+\nu = \int_{\partial D^e} \mathbf{h}_{\mathrm{rot}} \cdot \sigma[\mathbf{h}_2, p_2]\big|_+\nu$$
$$= \int_{\partial D_2} \boldsymbol{\psi}_3 \cdot \sigma[\mathbf{h}_2, p_2]\big|_+\nu = \mathcal{I}_{23}, \tag{5.12}$$

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and hence the conclusion follows. □

We have the following lemma whose proof is given in Appendix C. Let

$$f_0(x) := \frac{4e^{-x}\sinh^2 x(\cosh x + \sinh x) - 4x^2}{x^3(\sinh 2x + 2x)}, \tag{5.13}$$

$$g_0(x) := \frac{4x}{\sinh 2x + 2x}, \tag{5.14}$$

and let

$$F_0 := \int_0^\infty f_0(x)dx, \quad G_0 := \int_0^\infty g_0(x)dx. \tag{5.15}$$

Lemma 5.3. *As* $\delta \to 0$, *we have*

$$\mathcal{I}_{22} = -\pi\mu\sqrt{\frac{R}{\delta}} + O(1), \tag{5.16}$$

$$\mathcal{I}_{23} = \frac{\pi\mu R}{F_0} + O(\sqrt{\delta}), \tag{5.17}$$

$$\mathcal{I}_{\mathrm{rot}} = -\frac{4\pi\mu R^2}{F_0} + O(\sqrt{\delta}). \tag{5.18}$$

$$\mathcal{J}_2 = -\pi\mu R\left(1 - \frac{1 - G_0}{F_0}\right) + O(\sqrt{\delta}), \tag{5.19}$$

$$\mathcal{J}_{\mathrm{rot}} = -4\pi\mu R^2 \frac{1 - G_0}{F_0} + O(\sqrt{\delta}). \tag{5.20}$$

As an immediate consequence, the following corollary holds:

Corollary 5.4. *As* $\delta \to 0$, *we have*

$$c_{22} = \sqrt{R\delta} + O(\delta), \quad c_{23} = O(1). \tag{5.21}$$

Now, Theorem 1.3 follows from Theorem 3.4, Proposition 3.6, (5.7) and Corollary 5.4. One can also see that the decomposition formula (1.19) for the solution $(\mathbf{u}, p)$ is an immediate consequence of (4.6), (4.12), (5.7) and (5.21).

6   No blow-up with no-slip boundary conditions I

In this and next sections, we show that the stress tensor does not blow up under the no-slip boundary condition, that is, we prove Theorems 3.4, 4.1 and 5.1. Theorem 3.4 is for the problem with the boundary condition given by $\boldsymbol{\psi}_3$, and Theorems 4.1 and 5.1 for those with the no-slip boundary conditions. For doing so we first construct solutions $(\mathbf{v}_j, q_j)$, $j = 1, 2$, and $(\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}})$ by using the stream function formulation and bipolar coordinates. To avoid notational confusion, we denote the stream functions by Φ in this section instead of Ψ which was used in previous sections.

6.1   Construction of stream functions

In the following three lemmas we present stream functions for $(\mathbf{v}_1 - \mathbf{U}_{\mathrm{ex}}, q_1)$, $(\mathbf{v}_2 - \mathbf{U}_{\mathrm{sh}}, q_2)$, and $(\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}})$. Each stream function is found using the general form (2.24) and matching the boundary conditions using the formula (2.25) and (2.26) for $\mathbf{u}$ components of the solution.

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6.1.1 Stream function for $(\mathbf{v}_1 - \mathbf{U}_{\text{ex}}, q_1)$

Lemma 6.1. *Let Φ_1 be the stream function associated with the solution $(\mathbf{v}_1 - \mathbf{U}_{\text{ex}}, q_1)$. We have*

$$(h\Phi_1)(\zeta, \theta) = a_1 \sinh 2\zeta \sin \theta + b_1 \zeta \sin \theta$$

$$+ \sum_{n=2}^{\infty} \Big(a_n \sinh(n+1)\zeta + b_n \sinh(n-1)\zeta \Big) \sin n\theta, \qquad (6.1)$$

where

$$a_1 = -\frac{2ae^{-s}(\sinh s - se^{-s})}{\sinh 2s - 2s \cosh 2s},$$

$$b_1 = \frac{4a \sinh^2 s}{\sinh 2s - 2s \cosh 2s},$$

$$a_n = -\frac{2a(e^{-ns} \sinh ns - e^{-s} n \sinh s)}{\sinh 2ns - n \sinh 2s}, \quad n \geq 2,$$

$$b_n = \frac{2a(e^{-ns} \sinh ns - e^s n \sinh s)}{\sinh 2ns - n \sinh 2s}, \quad n \geq 2.$$

Proof. We need to show that the solution $(\mathbf{v}_1, q_1)$ constructed from Φ_1 satisfies the no-slip condition $\mathbf{v}_1 = 0$ on ∂D_1 and ∂D_2, and $(\mathbf{v}_1 - \mathbf{U}_{\text{ex}}, q_1) \in \mathcal{M}$.

We first observe that $\Phi_1^0 := xy$ is the stream function associated to the background solution $(\mathbf{U}, 0)$. Here and throughout this proof $\mathbf{U} = \mathbf{U}_{\text{ex}}$. In fact, $\mathbf{U} = (x, -y) = (\nabla \Phi_1^0)^{\perp}$ and a harmonic conjugate of $\mu \Delta \Phi_1^0 = 0$ is constant. We see from (2.2) that

$$\Phi_1^0 = \frac{a^2 \sinh \zeta \sin \theta}{(\cosh \zeta - \cos \theta)^2}$$

in bipolar coordinates.

Notice that Φ_1^0 has the odd symmetry in both ζ and θ. So we look for Φ_1 with the same symmetry property. We assume that Φ_1 of the form (6.1) which is the part with such symmetry of the general solution (2.24), and determine the coefficients a_n and b_n from the no-slip boundary condition.

For that, define Φ_1^{tot} by

$$\Phi_1^{\text{tot}} := \Phi_1^0 + \Phi_1,$$

so that Φ_1^{tot} is the stream function associated with $(\mathbf{v}_1, q_1)$. If we write $\mathbf{v}_1 = v_{1\zeta} \mathbf{e}_\zeta + v_{1\theta} \mathbf{e}_\theta$, then the no-slip boundary condition becomes

$$v_{1\zeta} = 0 \quad \text{on } \zeta = \pm s, \qquad (6.2)$$

$$v_{1\theta} = 0 \quad \text{on } \zeta = \pm s. \qquad (6.3)$$

Then, from the formula (2.17) for the tangential derivative and (2.25) for the stream function in bipolar coordinates, we have

$$0 = v_{1\zeta}|_{\zeta = \pm s} = -(h\partial_\theta \Phi_1^{\text{tot}})|_{\zeta = \pm s} = \mp \partial_T \Phi_1^{\text{tot}} \Big|_{\zeta = \pm s}.$$

This amounts to Φ_1^{tot} being constant on $\{\zeta = s\}$ and $\{\zeta = -s\}$. Since Φ_1^{tot} is odd in ζ, we further require that

$$\Phi_1^{\text{tot}} = 0 \quad \text{on } \zeta = \pm s.$$

We also have from (2.26) and (6.3) that on $\{\zeta = \pm s\}$

$$0 = v_{1\theta} = h\partial_\zeta \Phi_1^{\text{tot}} = \left(\partial_\zeta - \frac{\sinh \zeta}{\cosh \zeta - \cos \theta} \right) (h\Phi_1^{\text{tot}}) = \partial_\zeta (h\Phi_1^{\text{tot}}).$$

Thus the no-slip boundary condition is fulfilled if

$$\begin{cases} h\Phi_1^{\text{tot}} = 0 & \text{on } \zeta = \pm s, \\ \partial_\zeta (h\Phi_1^{\text{tot}}) = 0 & \text{on } \zeta = \pm s, \end{cases}$$

or equivalently

$$\begin{cases} h\Phi_1 = -h\Phi_1^0 & \text{on } \zeta = \pm s, \\ \partial_\zeta (h\Phi_1) = -\partial_\zeta (h\Phi_1^0) & \text{on } \zeta = \pm s. \end{cases} \tag{6.4}$$

Note that

$$(h\Phi_1^0)(\zeta, \theta) = \frac{a \sinh \zeta \sin \theta}{\cosh \zeta - \cos \theta} = 2a \sinh \zeta \sum_{n=1}^{\infty} e^{-n|\zeta|} \sin n\theta, \quad \zeta \neq 0. \tag{6.5}$$

We then see from (6.1) that (6.4) is equivalent to the following linear systems for a_n and b_n:

$$\begin{bmatrix} \sinh 2s & s \\ 2\cosh 2s & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} -2a \sinh s e^{-s} \\ -2a \cosh s e^{-s} + 2a \sinh s e^{-s} \end{bmatrix}, \tag{6.6}$$

and

$$\begin{bmatrix} \sinh(n+1)s & \sinh(n-1)s \\ (n+1)\cosh(n+1)s & (n-1)\cosh(n-1)s \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$
$$= \begin{bmatrix} -2a \sinh s e^{-ns} \\ -2a \cosh s e^{-ns} + 2a \sinh s n e^{-ns} \end{bmatrix}, \quad n \geq 2. \tag{6.7}$$

Solving these linear systems yield the expressions for a_n and b_n.

We now show

$$(\mathbf{v}_1 - \mathbf{U}, q_1) \in \mathcal{M}. \tag{6.8}$$

We first prove

$$(\mathbf{v}_1 - \mathbf{U})(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| \to \infty. \tag{6.9}$$

Since $|\mathbf{x}| \to \infty$ is equivalent to $(\zeta, \theta) \to (0, 0)$, it is equivalent to proving

$$(\mathbf{v}_1 - \mathbf{U})(\zeta, \theta) = O(|\zeta| + |\theta|), \quad (\zeta, \theta) \to (0, 0). \tag{6.10}$$

One can see from the explicit forms of a_n and b_n that there is a constant C independent of n (C may dependent on s) such that

$$|a_n| + |b_n| \leq Cne^{-2ns}$$

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for all n. Thus for any positive number k there is a constant C such that

$$\sum_{n=1}^{\infty} n^k e^{ns}(|a_n| + |b_n|) \leq C. \tag{6.11}$$

The constant C may differ at each appearance.

If we write $\mathbf{v}_1 - \mathbf{U} = f_\zeta \mathbf{e}_\zeta + f_\theta \mathbf{e}_\theta$, then it follows from (2.25) and (2.26) that

$$f_\zeta = -h\partial_\theta \Phi_1 = (-\partial_\theta + F)(h\Phi_1), \tag{6.12}$$
$$f_\theta = +h\partial_\zeta \Phi_1 = (\partial_\zeta - G)(h\Phi_1), \tag{6.13}$$

where

$$F := \frac{\sin\theta}{\cosh\zeta - \cos\theta}, \quad G := \frac{\sinh\zeta}{\cosh\zeta - \cos\theta}. \tag{6.14}$$

According to (6.1), $h\Phi_1$ can be written as

$$(h\Phi_1)(\zeta, \theta) = a_1 \sinh 2\zeta \sin\theta + b_1 \zeta \sin\theta + \sum_{n=2}^{\infty} \left(a_n w_n^+(\zeta, \theta) + b_n w_n^-(\zeta, \theta) \right), \tag{6.15}$$

where

$$w_n^\pm(\zeta, \theta) := \sinh(n \pm 1)\zeta \sin n\theta. \tag{6.16}$$

One can see that

$$|w_n^\pm(\zeta, \theta)| \lesssim n^2 e^{ns}|\zeta\theta|. \tag{6.17}$$

It thus follows from (6.11) that

$$|(h\Phi_1)(\zeta, \theta)| \lesssim |\zeta\theta| + |\zeta\theta| \sum_{n=2}^{\infty} n^2 e^{ns}(|a_n| + |b_n|) \lesssim |\zeta\theta|. \tag{6.18}$$

Similarly, one can show that there is C independent of (ζ, θ) such that

$$|\partial_\zeta(h\Phi_1)| \leq C|\theta|, \quad |\partial_\theta(h\Phi_1)| \leq C|\zeta|,$$
$$|\partial_\zeta^2(h\Phi_1)| \leq C|\zeta\theta|, \quad |\partial_\theta^2(h\Phi_1)| \leq C|\zeta\theta|, \quad |\partial_\zeta\partial_\theta(h\Phi_1)| \leq C. \tag{6.19}$$

Since

$$|F| \approx \frac{|\theta|}{\zeta^2 + \theta^2}, \quad |G| \approx \frac{|\zeta|}{\zeta^2 + \theta^2} \tag{6.20}$$

as $(\zeta, \theta) \to 0$, we have from (6.12), (6.13), (6.18) and (6.19) that

$$|f_\zeta| + |f_\theta| \leq C(|\zeta| + |\theta|) \tag{6.21}$$

for some constant C (depending on s, and hence on δ), which implies (6.9).

Next, we prove

$$\nabla(\mathbf{v}_1 - \mathbf{U})(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}| \to \infty, \tag{6.22}$$

or equivalently

$$\nabla(\mathbf{v}_1 - \mathbf{U})(\zeta, \theta) = O(\zeta^2 + \theta^2), \quad (\zeta, \theta) \to (0, 0). \tag{6.23}$$

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Since $\mathbf{v}_1 - \mathbf{U} = f_\zeta \mathbf{e}_\zeta + f_\theta \mathbf{e}_\theta$, we have

$$|\nabla(\mathbf{v}_1 - \mathbf{U})| \leq C(|\nabla f_\zeta| + |f_\zeta \nabla \mathbf{e}_\zeta| + |\nabla f_\theta| + |f_\theta \nabla \mathbf{e}_\theta|).$$

Lemma 3.2 and (6.21) yield

$$|\nabla \mathbf{v}_1 - \mathbf{U}| \lesssim |\nabla f_\zeta| + |\nabla f_\theta| + (\zeta^2 + \theta^2).$$

We see from (6.12)

$$\partial_\zeta f_\zeta = (-\partial_\zeta \partial_\theta + F \partial_\zeta + \partial_\zeta F)(h\Phi_1).$$

One can see easily that $|\partial_\zeta F| \lesssim (\zeta^2 + \theta^2)^{-1}$. Thus we obtain from (6.18), (6.19) and (6.20)

$$\partial_\zeta f_\zeta = O(1).$$

Similarly, one can show

$$\partial_\theta f_\zeta = O(1), \quad \partial_\zeta f_\theta = O(1), \quad \partial_\theta f_\theta = O(1).$$

Therefore, we have

$$\nabla f_\zeta = O(|h\partial_\zeta f_\zeta| + |h\partial_\theta f_\zeta|) = O(|h|) = O(\zeta^2 + \theta^2), \tag{6.24}$$

$$\nabla f_\theta = O(|h\partial_\zeta f_\theta| + |h\partial_\theta f_\theta|) = O(|h|) = O(\zeta^2 + \theta^2). \tag{6.25}$$

This proves (6.22).

We now prove the estimate of the pressure:

$$q_1(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}| \to \infty, \tag{6.26}$$

or equivalently,

$$q_1 = O(\zeta^2 + \theta^2), \quad (\zeta, \theta) \to (0, 0). \tag{6.27}$$

Let

$$w_n(\zeta, \theta) := \sinh n\zeta \sin n\theta, \quad \tilde{w}_n(\zeta, \theta) = \cosh n\zeta \cos n\theta. \tag{6.28}$$

The pressure q_1 is given by

$$q_1 = C - a_1 \frac{2\mu}{a}(2\tilde{w}_1 - \tilde{w}_2) + b_1 \frac{2\mu}{a}\tilde{w}_1 - \frac{2\mu}{a}\sum_{n=2}^{\infty}((n+1)a_n - (n-1)b_n)\tilde{w}_n$$

$$+ \frac{2\mu}{a}\sum_{n=2}^{\infty} na_n \tilde{w}_{n+1} - \frac{2\mu}{a}\sum_{n=2}^{\infty} nb_n \tilde{w}_{n-1}, \tag{6.29}$$

for some constant C. In fact, one can see from (2.22) that

$$\Delta\Phi_1 = a_1 \frac{2}{a}(2w_1 - w_2) - b_1 \frac{2}{a}w_1 + \frac{2}{a}\sum_{n=2}^{\infty}((n+1)a_n - (n-1)b_n)w_n$$

$$- \frac{2}{a}\sum_{n=2}^{\infty} na_n w_{n+1} + \frac{2}{a}\sum_{n=2}^{\infty} nb_n w_{n-1}.$$

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Since q_1 is a harmonic conjugate of $\mu\Delta\Phi_1$ and $-\tilde{w}_n$ is a harmonic conjugate of w_n, (6.29) follows.

We choose the constant C to be

$$C = \frac{2\mu}{a}a_1 - \frac{2\mu}{a}b_1 + \frac{2\mu}{a}\sum_{n=2}^{\infty}(a_n + b_n). \tag{6.30}$$

Then, q_1 take the form

$$q_1 = C - a_1\frac{2\mu}{a}(2(\tilde{w}_1 - 1) - (\tilde{w}_2 - 1)) + b_1\frac{2\mu}{a}(\tilde{w}_1 - 1)$$

$$- \frac{2\mu}{a}\sum_{n=2}^{\infty}((n+1)a_n - (n-1)b_n)(\tilde{w}_n - 1)$$

$$+ \frac{2\mu}{a}\sum_{n=2}^{\infty}na_n(\tilde{w}_{n+1} - 1) - \frac{2\mu}{a}\sum_{n=2}^{\infty}nb_n(\tilde{w}_{n-1} - 1). \tag{6.31}$$

Note that

$$|\tilde{w}_n(\zeta, \theta) - 1| \lesssim n^2 e^{ns}(\zeta^2 + \theta^2). \tag{6.32}$$

This together with (6.11) yields

$$|q_1| \lesssim \left(1 + \sum_{n=2}^{\infty}n^3 e^{ns}(|a_n| + |b_n|)\right)|\zeta^2 + \theta^2| = O(\zeta^2 + \theta^2).$$

This proves (6.26) and hence (6.8). The proof is completed. $\qquad\square$

6.1.2 Stream function for $(\mathbf{v}_2 - \mathbf{U}_{\mathrm{sh}}, q_2)$

Lemma 6.2. *The stream function Φ_2 associated with the solution $(\mathbf{v}_2 - \mathbf{U}_{\mathrm{sh}}, q_2)$ is given by*

$$(h\Phi_2)(\zeta, \theta) = K_v(\cosh\zeta - \cos\theta)\ln(2\cosh\zeta - 2\cos\theta) + c_0\cosh\zeta + d_0\zeta\sinh\zeta$$

$$+ \sum_{n=1}^{\infty}\left(c_n\cosh(n+1)\zeta + d_n\cosh(n-1)\zeta\right)\cos n\theta, \tag{6.33}$$

where

$$K_v = \frac{a\left(1 - \tanh s - \frac{2\sinh^2 s}{2s + \sinh 2s} - M'\right)}{\frac{1}{2} + \frac{s(\sinh 2s - 2\tanh s)}{2s + \sinh 2s} + M}$$

with

$$M = \sum_{n=2}^{\infty}\frac{4n\sinh s\cosh s + e^{-ns}\sinh ns - 4n^2\sinh^2 s}{n(n^2 - 1)(\sinh 2ns + n\sinh 2s)}, \tag{6.34}$$

and

$$M' = \sum_{n=2}^{\infty}\frac{4n\sinh^2 s}{\sinh 2ns + n\sinh 2s}, \tag{6.35}$$

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$$c_0 = -\frac{a}{2} + \frac{a\sinh^2 s}{\sinh s\cosh s + s} + K_v\frac{-1 + e^{-2s} - 2s(1+s)}{2s + \sinh 2s},$$

$$d_0 = \frac{a}{\sinh s\cosh s + s} - K_v\frac{\sinh^2 s}{s + \cosh s\sinh s},$$

$$c_1 = a(-1 + \coth 2s) + K_v\frac{1}{1 + e^{2s}},$$

$$d_1 = \frac{a}{2} - \frac{a}{\sinh 2s} + K_v(1 + s - \frac{\tanh s}{2}),$$

$$c_n = \frac{2a(e^{-ns}\cosh ns - e^{-s}n\sinh s)}{\sinh 2ns + n\sinh 2s} + 2K_v\frac{e^{-ns}\sinh ns + e^{-s}n\sinh s}{n(n+1)(\sinh 2ns + n\sinh 2s)},$$

$$d_n = -\frac{2a(e^{-ns}\cosh ns - e^{s}n\sinh s)}{\sinh 2ns + n\sinh 2s} - 2K_v\frac{e^{-ns}\sinh ns + e^{s}n\sinh s}{n(n-1)(\sinh 2ns + n\sinh 2s)}.$$

Proof. Like the proof of Lemma 6.1, one can see that the stream function associated with the background solution $(\mathbf{U}, 0)$ is given by

$$\Phi_2^0 = \frac{1}{2}(-x^2 + y^2). \tag{6.36}$$

Here and throughout this proof, $\mathbf{U} = \mathbf{U}_{\mathrm{sh}}$. One can see from (2.2) that

$$\Phi_2^0 = \frac{1}{2}\frac{a^2(-\sinh^2\zeta + \sin^2\theta)}{(\cosh\zeta - \cos\theta)^2}. \tag{6.37}$$

Since Φ_2^0 has the even symmetry in both ζ and θ, we seek Φ_2 in the form (6.33) which has the same symmetric property. Let

$$(h\Phi_2^K)(\zeta, \theta) := K_v(\cosh\zeta - \cos\theta)\ln(2\cosh\zeta - 2\cos\theta), \tag{6.38}$$

$$(h\Phi_2^F)(\zeta, \theta) := c_0\cosh\zeta + d_0\zeta\sinh\zeta$$
$$+ \sum_{n=1}^{\infty}\big(c_n\cosh(n+1)\zeta + d_n\cosh(n-1)\zeta\big)\cos n\theta, \tag{6.39}$$

so that

$$\Phi_2 = \Phi_2^K + \Phi_2^F. \tag{6.40}$$

Let

$$\Phi_2^{\mathrm{tot}} := \Phi_2^0 + \Phi_2.$$

Then, Φ_2^{tot} is the stream function associated with $(\mathbf{v}_2, q_2)$. We determine the coefficients c_n and d_n from the no-slip boundary condition $\mathbf{v}_2 = 0$ on ∂D_1 and ∂D_2. One can show as in the proof of Lemma 6.1 that this condition is fulfilled if

$$\begin{cases} h\Phi_2^{\mathrm{tot}} = 0 & \text{on } \zeta = \pm s, \\ \partial_\zeta(h\Phi_2^{\mathrm{tot}}) = 0 & \text{on } \zeta = \pm s. \end{cases} \tag{6.41}$$

In other words,

$$\begin{cases} h\Phi_2^F = -h\Phi_2^0 - h\Phi_2^K & \text{on } \zeta = \pm s, \\ \partial_\zeta(h\Phi_2^F) = -\partial_\zeta(h\Phi_2^0) - \partial_\zeta(h\Phi_2^K) & \text{on } \zeta = \pm s. \end{cases} \tag{6.42}$$

Let

$$(h\Phi_2^0)(\zeta,\theta) = \frac{1}{2}\frac{a^2(-\sinh^2\zeta + \sin^2\theta)}{\cosh\zeta - \cos\theta}$$

$$= \frac{a}{2}e^{-|\zeta|} + \frac{a}{2}e^{-2|\zeta|}\cos\theta - a\sinh|\zeta|\sum_{n=2}^{\infty}e^{-n|\zeta|}\cos n\theta$$

$$=: \sum_{n=0}^{\infty}\phi_n^0(\zeta)\cos n\theta, \tag{6.43}$$

and

$$(h\Phi_2^K)(\zeta,\theta) = K_v(|\zeta|\cosh\zeta + e^{-|\zeta|}) - K_v\left(1 + \frac{e^{-2|\zeta|}}{2} + |\zeta|\right)\cos\theta$$

$$+ K_v\sum_{n=2}^{\infty}\left(\frac{e^{-(n-1)|\zeta|}}{n-1} - 2\cosh\zeta\frac{e^{-n|\zeta|}}{n} + \frac{e^{-(n+1)|\zeta|}}{n+1}\right)\cos n\theta$$

$$=: \sum_{n=0}^{\infty}\phi_n^K(\zeta)\cos n\theta. \tag{6.44}$$

Then one can infer from (6.42) that the following system of equations for c_n, d_n hold:

$$\begin{bmatrix}\cosh s & s\sinh s \\ \sinh s & \sinh s + s\cosh s\end{bmatrix}\begin{bmatrix}c_0 \\ d_0\end{bmatrix} = \begin{bmatrix}-\phi_0^K(s) - \phi_0^K(s) \\ -(\phi_0^K)'(s) - (\phi_0^K)'(s)\end{bmatrix},$$

$$\begin{bmatrix}\cosh 2s & 1 \\ 2\sinh 2s & 0\end{bmatrix}\begin{bmatrix}c_1 \\ d_1\end{bmatrix} = \begin{bmatrix}-\phi_1^K(s) - \phi_1^K(s) \\ -(\phi_1^K)'(s) - (\phi_1^K)'(s)\end{bmatrix},$$

and

$$\begin{bmatrix}\cosh(n+1)s & \cosh(n-1)s \\ (n+1)\sinh(n+1)s & (n-1)\sinh(n-1)s\end{bmatrix}\begin{bmatrix}c_n \\ d_n\end{bmatrix} = \begin{bmatrix}-\phi_n^K(s) - \phi_n^K(s) \\ -(\phi_n^K)'(s) - (\phi_n^K)'(s)\end{bmatrix},$$

for $n \geq 2$. Solving these linear systems yields the expressions given in the lemma for c_n and d_n in terms of K_v. We then determine the constant K_v by imposing the condition

$$c_0 + \sum_{n=1}^{\infty}(c_n + d_n) = 0. \tag{6.45}$$

This condition is required to prove

$$(\mathbf{v}_2 - \mathbf{U}, q_2) \in \mathcal{M}. \tag{6.46}$$

We will be brief in presenting the proof of (6.46) since it is parallel to (6.8). We only mention why the condition (6.45) is required, and write down the formula for the pressure term q_2 since it will be used in latter part of this section.

Similarly to (6.11), one can show that for any positive number k there is a constant C such that

$$\sum_{n=1}^{\infty}n^k e^{ns}(|c_n| + |d_n|) \leq C. \tag{6.47}$$

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Note that

$$(h\Phi^{e,F})(\zeta,\theta) = c_0\cosh\zeta + d_0\zeta\sinh\zeta + \sum_{n=1}^{\infty}\left(c_n\tilde{w}_n^+(\zeta,\theta) + d_n\tilde{w}_n^-(\zeta,\theta)\right),$$

where

$$\tilde{w}_n^{\pm}(\zeta,\theta) = \cosh(n\pm 1)\zeta\cos n\theta.$$

Thanks to (6.45), we have

$$(h\Phi^{e,F})(\zeta,\theta) = c_0(\cosh\zeta - 1) + d_0\zeta\sinh\zeta + \sum_{n=1}^{\infty}c_n(\tilde{w}_n^+(\zeta,\theta) - 1) + d_n(\tilde{w}_n^-(\zeta,\theta) - 1).$$

We then use (6.32) to obtain

$$(h\Phi^{e,F})(\zeta,\theta) = O(\zeta^2 + \theta^2).$$

We use (2.22) to see that $\Delta\Phi_2^K = 0$ and

$$\Delta\Phi_2 = \Delta\Phi_2^F = \frac{2}{a}c_0 + d_0\frac{2}{a}(1 - \tilde{w}_1) + \frac{2}{a}\sum_{n=1}^{\infty}((n+1)c_n - (n-1)d_n)\tilde{w}_n$$

$$- \frac{2}{a}\sum_{n=1}^{\infty}nc_n\tilde{w}_{n+1} + \frac{2}{a}\sum_{n=1}^{\infty}nd_n\tilde{w}_{n-1}.$$

Since the pressure q_2 is a harmonic conjugate of $\mu\Delta\Phi^e$, we have

$$q_2 = \frac{2\mu}{a}d_0 w_1 + \frac{2\mu}{a}\sum_{n=1}^{\infty}((n+1)c_n - (n-1)d_n)w_n$$

$$- \frac{2\mu}{a}\sum_{n=1}^{\infty}nc_n w_{n+1} + \frac{2\mu}{a}\sum_{n=1}^{\infty}nd_n w_{n-1} + C \qquad (6.48)$$

for some constant C. We choose $C = 0$. Then, since

$$|w_n(\zeta,\theta)| \lesssim n^2 e^{ns}(\zeta^2 + \theta^2), \qquad (6.49)$$

we have $q_2 = O(\zeta^2 + \theta^2)$ as $(\zeta,\theta) \to 0$, namely, (6.46) holds. $\qquad\square$

6.1.3 Stream function for $(\mathbf{h}_{\rm rot}, p_{\rm rot})$

Lemma 6.3. *The stream function $\Phi_{\rm rot}$ associated with the solution $(\mathbf{h}_{\rm rot}, p_{\rm rot})$ is given by*

$$(h\Phi_{\rm rot})(\zeta,\theta) = K_{\rm rot}(\cosh\zeta - \cos\theta)\ln(2\cosh\zeta - 2\cos\theta) + a_0'\cosh\zeta + d_0'\zeta\sinh\zeta$$

$$+ \sum_{n=1}^{\infty}\left(a_n'\cosh(n+1)\zeta + b_n'\cosh(n-1)\zeta\right)\cos n\theta, \qquad (6.50)$$

where

$$K_{\rm rot} = -a\left(\frac{s\sinh^2 s\tanh s}{\sinh s\cosh s + s} + \frac{1}{2} + M\right)^{-1}$$

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with M given in (6.34), and

$$a_0' = a - \frac{K_{\rm rot}(s^2 + s + e^{-s}\sinh s)}{\sinh s \cosh s + s}, \quad d_0' = -\frac{K_{\rm rot}\sinh^2 s}{\sinh s \cosh s + s},$$

$$a_1' = \frac{1}{2}K_{\rm rot}e^{-s}{\rm sech}\, s, \quad b_1' = K_{\rm rot}(s + 1 - \frac{1}{2}\tanh s),$$

$$a_n' = \frac{2K_{\rm rot}(ne^{-s}\sinh s + e^{-ns}\sinh ns)}{n(n+1)(\sinh 2ns + n\sinh 2s)},$$

$$b_n' = -\frac{2K_{\rm rot}(ne^s\sinh s + e^{-ns}\sinh ns)}{n(n-1)(\sinh 2ns + n\sinh 2s)}.$$

Proof. Let

$$(\tilde{\mathbf{h}}_{\rm rot}, \tilde{p}_{\rm rot}) := (\mathbf{h}_{\rm rot}, p_{\rm rot}) - (\boldsymbol{\psi}_3, 0).$$

Then $(\tilde{\mathbf{h}}_{\rm rot}, \tilde{p}_{\rm rot})$ is the solution to

$$\begin{cases} \mu\Delta\tilde{\mathbf{h}}_{\rm rot} = \nabla\tilde{p}_{\rm rot} & \text{in } D^e, \\ \nabla \cdot \tilde{\mathbf{h}}_{\rm rot} = 0 & \text{in } D^e, \\ (\tilde{\mathbf{h}}_{\rm rot}, \tilde{p}_{\rm rot}) - (-\boldsymbol{\psi}_3, 0) \in \mathcal{M}, \end{cases} \tag{6.51}$$

with the no-slip boundary condition, namely,

$$\tilde{\mathbf{h}}_{\rm rot}|_{\partial D_1} = 0, \quad \tilde{\mathbf{h}}_{\rm rot}|_{\partial D_2} = 0. \tag{6.52}$$

Observe that the above equation is similar to the equation (5.4) for $(\mathbf{v}_2, q_2)$ with the only difference being that the background solution $\mathbf{U}_{\rm sh}$ is replaced with $-\boldsymbol{\psi}_3$.

It is easy to see that the function $\Phi^0_{\rm rot}$ defined by

$$\Phi^0_{\rm rot} = -\frac{1}{2}(x^2 + y^2)$$

is a stream function associated with the solution $(-\boldsymbol{\psi}_3, 0)$. In bipolar coordinates,

$$\Phi^0_{\rm rot} = \frac{1}{2}\frac{a^2(-\sinh^2\zeta + \sin^2\theta)}{(\cosh\zeta - \cos\theta)^2}. \tag{6.53}$$

Note that $\Phi^0_{\rm rot}$ has the even symmetry in both ζ and θ. In exactly the same way as in the proof of Lemma 6.2, we can find the stream function associated with $(\tilde{\mathbf{h}}_{\rm rot}, \tilde{p}_{\rm rot})$, which immediately yields Lemma 6.3. $\square$

6.2 Asymptotics of K_v and $K_{\rm rot}$

Lemma 6.4. *As $\delta \to 0$, we have*

$$K_v = R\frac{1 - G_0}{F_0}\sqrt{\frac{R}{\delta}} + O(1), \tag{6.54}$$

$$K_{\rm rot} = -\frac{R}{F_0}\sqrt{\frac{R}{\delta}} + O(1), \tag{6.55}$$

where F_0 and G_0 are the numbers given in (5.15).

Proof. The proof is based on a special case of the Euler-Maclaurin summation formula: if $f \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, then, for a small parameter $s > 0$, we have

$$s \sum_{n=0}^{\infty} f(x_0 + ns) = \int_{x_0}^{\infty} f(x)dx + R_1, \tag{6.56}$$

where the remainder term R_1 satisfies

$$|R_1| \lesssim s \left(|f(x_0)| + \int_{x_0}^{\infty} |f'(x)|dx \right).$$

We first consider the asymptotics of the series M defined by (6.34). One can easily see that

$$M + 2\sum_{n=2}^{\infty} \frac{1}{n(n^2-1)} = \sum_{n=2}^{\infty} \frac{4e^{-ns}\sinh^2(ns)(\cosh ns + \sinh ns) - 4n^2\sinh^2 s}{n(n^2-1)(\sinh 2ns + n\sinh 2s)},$$

and

$$2\sum_{n=2}^{\infty} \frac{1}{n(n^2-1)} = \frac{1}{2}.$$

Thus, the Euler-Maclaurin summation formula yields

$$M + \frac{1}{2} = s^3 \sum_{n=2}^{\infty} \frac{4e^{-ns}\sinh^2(ns)(\cosh ns + \sinh ns) - 4(ns)^2(\sinh s/s)^2}{(ns)((ns)^2 - s^2)(\sinh 2ns + ns(\sinh 2s/s))}$$

$$= s^2 \int_{2s}^{\infty} f_s(x)dx + s^2 R_1, \tag{6.57}$$

where

$$f_s(x) := \frac{4e^{-x}\sinh^2 x(\cosh x + \sinh x) - 4x^2(\sinh s/s)^2}{x(x^2-s^2)(\sinh 2x + (\sinh 2s/s)x)},$$

and

$$|R_1| \lesssim s \left(|f_s(2s)| + \int_{2s}^{\infty} |f_s'(x)|dx \right).$$

By straightforward but tedious computations, one can see that

$$|f_s(2s)| \leq C, \quad \int_{2s}^{\infty} |f_s'(x)|dx \leq C',$$

where C and C' are constants independent of $s > 0$. Therefore, as $s \to 0$, we obtain

$$M + \frac{1}{2} = s^2 \int_0^{\infty} f_0(x)dx + O(s^3) = s^2 F_0 + O(s^3).$$

So, for small s, we have

$$K_{\text{rot}} = -a \left(\frac{1}{2} + M + O(s^3) \right)^{-1} = -\frac{a}{s^2 F_0} + O(1).$$

The other quantity K_v can be estimated similarly, and the proof is completed. $\quad\square$

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7 No blow-up with no-slip boundary conditions II

7.1 Proof of Theorem 4.1

We first estimate the strain tensor $\mathcal{E}[\mathbf{v}_1]$. Since $\mathcal{E}[\mathbf{U}] = O(1)$, we estimate $\mathcal{E}[\mathbf{v}_1 - \mathbf{U}]$.

The following formulae are derived using the relations (2.28)–(2.30) between the strain tensor and the stream function Φ_1 and (6.1):

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1 - \mathbf{U}] = -h(\zeta,\theta)2a_1\cosh 2\zeta\cos\theta - h(\zeta,\theta)b_1\cos\theta$$
$$- h(\zeta,\theta)\sum_{n=2}^{\infty}\left(\tilde{a}_n\cosh(n+1)\zeta + \tilde{b}_n\cosh(n-1)\zeta\right)\cos n\theta,$$

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_1 - \mathbf{U}] = h(\zeta,\theta)2a_1\sinh 2\zeta\sin\theta$$
$$+ h(\zeta,\theta)\sum_{n=2}^{\infty}\left(\tilde{a}_n\sinh(n+1)\zeta + \tilde{b}_n\sinh(n-1)\zeta\right)\sin n\theta,$$

$$\mathcal{E}_{\theta\theta}[\mathbf{v}_1 - \mathbf{U}] = -\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1 - \mathbf{U}],$$

where

$$\tilde{a}_n = n(n+1)a_n, \quad \tilde{b}_n = n(n-1)b_n. \tag{7.1}$$

Here, a_n and b_n are given in Lemma 6.1.

Using the hyperbolic identities

$$\cosh(n+1)\zeta + \cosh(n-1)\zeta = 2\cosh n\zeta\cosh\zeta,$$
$$\cosh(n+1)\zeta - \cosh(n-1)\zeta = 2\sinh n\zeta\sinh\zeta,$$

we can rewrite $\mathcal{E}_{\zeta\zeta}$ and $\mathcal{E}_{\zeta\theta}$ as

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1 - \mathbf{U}] = -h(\zeta,\theta)2a_1\cosh 2\zeta\cos\theta - h(\zeta,\theta)b_1\cos\theta$$
$$- h(\zeta,\theta)\sum_{n=2}^{\infty}\left((\tilde{a}_n + \tilde{b}_n)\cosh n\zeta\cosh\zeta + (\tilde{a}_n - \tilde{b}_n)\sinh n\zeta\sinh\zeta\right)\cos n\theta, \tag{7.2}$$

and

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_1 - \mathbf{U}] = h(\zeta,\theta)2a_1\sinh 2\zeta\sin\theta$$
$$+ h(\zeta,\theta)\sum_{n=2}^{\infty}\left((\tilde{a}_n + \tilde{b}_n)\sinh n\zeta\cosh\zeta + (\tilde{a}_n - \tilde{b}_n)\cosh n\zeta\sinh\zeta\right)\sin n\theta. \tag{7.3}$$

From the expressions of a_n and b_n given in Lemma 6.1, we have, for $n \geq 2$,

$$\tilde{a}_n + \tilde{b}_n = -\frac{4a}{s}\frac{nse^{-ns}\sinh ns - (ns)^2\eta_2 + (ns)^3\eta_1}{\sinh 2ns - 2ns\eta_2},$$

$$\tilde{a}_n - \tilde{b}_n = -\frac{4a}{s^2}\frac{(ns)^2 e^{-ns}\sinh ns - (ns)^3\eta_2 + s^2(ns)^2\eta_1}{\sinh 2ns - 2ns\eta_2},$$

where

$$\eta_1 = \eta_1(s) := \frac{\sinh^2 s}{s^2}, \quad \eta_2 = \eta_2(s) := \frac{\sinh 2s}{2s}. \tag{7.4}$$

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If we define

$$f_1(x) := \frac{xe^{-x}\sinh x - x^2\eta_2 + x^3\eta_1}{\sinh 2x - 2x\eta_2}, \quad f_2(x) := \frac{x^2 e^{-x}\sinh x - x^3\eta_2 + s^2 x^2\eta_1}{\sinh 2x - 2x\eta_2}, \quad (7.5)$$

then $\tilde{a}_n + \tilde{b}_n$ and $\tilde{a}_n - \tilde{b}_n$ can be rewritten as

$$\tilde{a}_n + \tilde{b}_n = -\frac{4a}{s}f_1(ns), \quad \tilde{a}_n - \tilde{b}_n = -\frac{4a}{s^2}f_2(ns). \quad (7.6)$$

It follows from $\mathcal{E}_{\zeta\zeta}$ and $\mathcal{E}_{\zeta\theta}$ that from (7.2) and (7.6) that

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1 - \mathbf{U}] = -h(\zeta,\theta)2c_1\cosh 2\zeta \cos\theta - h(\zeta,\theta)d_1\cos\theta$$

$$+ \frac{4a}{s}h(\zeta,\theta)\sum_{n=2}^{\infty}\left(f_1(ns)\cosh n\zeta \cosh\zeta\right)\cos n\theta$$

$$+ \frac{4a}{s^2}h(\zeta,\theta)\sum_{n=2}^{\infty}\left(f_2(ns)\sinh n\zeta \sinh\zeta\right)\cos n\theta. \quad (7.7)$$

Let, for $j = 1, 2$,

$$A_{j,n}^+(\zeta) = f_j(ns)\cosh n\zeta, \quad A_{j,n}^-(\zeta) = f_j(ns)\sinh n\zeta, \quad (7.8)$$

and then define S_j^{++}, S_j^{-+}, etc, by

$$S_j^{\pm+} = \sum_{n=2}^{\infty}A_{j,n}^{\pm}(\zeta)ah(\zeta,\theta)\cos n\theta, \quad S_j^{\pm-} = \sum_{n=2}^{\infty}A_{j,n}^{\pm}(\zeta)ah(\zeta,\theta)\sin n\theta. \quad (7.9)$$

Then, (7.7) reads

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1 - \mathbf{U}] = S_0 + \frac{4}{s}\cosh\zeta S_1^{++} + \frac{4}{s^2}\sinh\zeta S_2^{-+}, \quad (7.10)$$

where
$$S_0 = -h(\zeta,\theta)2a_1\cosh 2\zeta \cos\theta - h(\zeta,\theta)b_1\cos\theta. \quad (7.11)$$

Similarly, one can see that $\mathcal{E}_{\zeta\theta}$ is written as

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_1 - \mathbf{U}] = \widetilde{S}_0 - \frac{4}{s}\cosh\zeta S_1^{--} - \frac{4}{s^2}\sinh\zeta S_2^{+-}, \quad (7.12)$$

where
$$\widetilde{S}_0 = h(\zeta,\theta)2b_1\sinh 2\zeta \sin\theta. \quad (7.13)$$

We use the following lemma here and present its proof in Appendix D.

Lemma 7.1. *If $2s \leq x$, then*

$$|f_j(x)| + |f_j'(x)| + |f_j''(x)| \lesssim (1 + x^3)e^{-2x}, \quad j = 1, 2. \quad (7.14)$$

If $2s \leq x \leq 1$, then

$$\left|f_1'(x) - \frac{1}{2}\right| \lesssim x, \quad |f_2''(x) - 1| \lesssim x. \quad (7.15)$$

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Lemma 7.2. *The following asymptotic formulas hold for $j = 1, 2$:*

$$S_j^{++} = -\frac{1}{2} f_j(2s) \cosh 2\zeta \cos \theta + f_j(2s) \cosh 2\zeta \cos 2\theta$$

$$-\frac{1}{2} f_j(3s) \cosh 3\zeta \cos 2\theta + O(s), \tag{7.16}$$

$$S_j^{+-} = -\frac{1}{2} f_j(2s) \cosh 2\zeta \sin \theta + f_j(2s) \cosh 2\zeta \sin 2\theta$$

$$-\frac{1}{2} f_j(3s) \cosh 3\zeta \sin 2\theta + O(s), \tag{7.17}$$

$$S_j^{-\pm} = O(s), \tag{7.18}$$

as $s \to 0$.

Proof. We first have the following identity:

$$ah(\zeta, \theta) \cos n\theta = \cosh \zeta \cos n\theta - \cos \theta \cos n\theta$$

$$= -\frac{1}{2} \big(\cos(n+1)\theta - 2 \cosh \zeta \cos n\theta + \cos(n-1)\theta \big)$$

$$= -\frac{1}{2} \big(\cos(n+1)\theta - 2 \cos n\theta + \cos(n-1)\theta \big) + (\cosh \zeta - 1) \cos n\theta$$

$$= -\frac{1}{2} \big(\cos(n+1)\theta - 2 \cos n\theta + \cos(n-1)\theta \big) + \sinh^2(\zeta/2) \cos n\theta.$$

By substituting this identity into (7.9) and then rearranging indices, we arrive at

$$S_j^{\pm+} = -\frac{1}{2} (A_{j,2}^{\pm} \cos \theta - 2A_{j,2}^{\pm} \cos 2\theta + A_{j,3}^{\pm} \sin 2\theta)$$

$$-\frac{1}{2} \sum_{n=3}^{\infty} \left(A_{j,n+1}^{\pm} - 2A_{j,n}^{\pm} + A_{j,n-1}^{\pm} \right) \cos n\theta$$

$$+ \sinh^2(\zeta/2) \sum_{n=2}^{\infty} A_{j,n}^{\pm} \cos n\theta =: S_{j,0}^{\pm+} + S_{j,1}^{\pm+} + S_{j,2}^{\pm+}. \tag{7.19}$$

Note that $A_{j,n}^{\pm}$ is of the form

$$A_{j,n}^{\pm} = F_{j,n} G_n^{\pm},$$

where

$$F_{j,n} = f_j(ns), \quad G_n^{+} = \cosh n\zeta, \quad G_n^{-} = \sinh n\zeta.$$

We then have

$$A_{j,n+1}^{\pm} - 2A_{j,n}^{\pm} + A_{j,n-1}^{\pm} = F_{j,n+1} G_{n+1}^{\pm} - 2F_{j,n} G_n^{\pm} + F_{j,n-1} G_{n-1}^{\pm}$$

$$= (F_{j,n+1} - 2F_{j,n} + F_{j,n-1}) G_n^{\pm} + F_{j,n} (G_{n+1}^{\pm} - 2G_n^{\pm} + G_{n-1}^{\pm})$$

$$+ (F_{j,n+1} - F_{j,n})(G_{n+1}^{\pm} - G_n^{\pm}) + (F_{j,n} - F_{j,n-1})(G_n^{\pm} - G_{n-1}^{\pm}).$$

One can easily see that

$$G_{n+1}^{\pm} - G_n^{\pm} = \sinh(\zeta/2)(e^{(n+\frac{1}{2})\zeta} \mp e^{-(n+\frac{1}{2})\zeta}),$$

$$G_{n+1}^{\pm} - 2G_n^{\pm} + G_{n-1}^{\pm} = 2\sinh^2(\zeta/2)(e^{n\zeta} \pm e^{-n\zeta}).$$

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Since $|\zeta| \leq s$, we have

$$|G_n^{\pm}| \lesssim e^{ns}, \quad |G_{n+1}^{\pm} - G_n^{\pm}| \lesssim se^{ns}, \quad |G_{n+1}^{\pm} - 2G_n^{\pm} + G_{n-1}^{\pm}| \lesssim s^2 e^{ns}. \qquad (7.20)$$

Next, we estimate $F_{j,n}$ and its finite differences. By the mean value theorem, there exist $x_n^* \in (ns, (n+1)s), x_n^{**} \in ((n-1)s, (n+1)s)$ such that

$$\frac{F_{j,n+1} - F_{j,n}}{s} = f_j'(x_n^*), \quad \frac{F_{j,n+1} - 2F_{j,n} + F_{j,n-1}}{s^2} = f_j''(x_n^{**}).$$

Then, by (7.14), we infer

$$|F_{j,n}| \lesssim (1 + (ns)^3)e^{-2ns}, \qquad (7.21)$$

$$|F_{j,n+1} - F_{j,n}| \lesssim s(1 + (ns)^3)e^{-2ns}, \qquad (7.22)$$

$$|F_{j,n+1} - 2F_{j,n} + F_{j,n-1}| \lesssim s^2(1 + (ns)^3)e^{-2ns}. \qquad (7.23)$$

These estimates together with (7.20) lead us to

$$|A_n^{\pm}| \lesssim e^{ns}|F_{j,n}| \lesssim (1 + (ns)^3)e^{-ns},$$

and

$$|A_{n+1}^{\pm} - 2A_n^{\pm} + A_{n-1}^{\pm}| \lesssim \left|F_{n+1} - 2F_n + F_{n-1}\right|e^{ns} + F_n s^2 e^{ns}$$
$$+ |F_{n+1} - F_n|se^{ns} + |F_n - F_{n-1}|se^{ns}$$
$$\lesssim s^2(1 + (ns)^3)e^{-ns}.$$

Using (6.56), we have

$$|S_{j,1}^{\pm+}| \lesssim \sum_{n=2} |A_{n+1}^{\pm} - 2A_n^{\pm} + A_{n-1}^{\pm}| \lesssim \sum_{n=2} s^2(1 + (ns)^3)e^{-ns}$$
$$\lesssim s \int_0^{\infty} (1 + x^3)e^{-x}dx \lesssim s,$$

and

$$|S_{j,2}^{\pm+}| \lesssim s^2 \sum_{n=2} |A_n^{\pm}| \lesssim s^2 \sum_{n=2} (1 + (ns)^3)e^{-ns}$$
$$\lesssim s \int_0^{\infty} (1 + x^3)e^{-x}dx \lesssim s.$$

Therefore, from (7.19) and (7.24), we see that

$$S_j^{++} = S_{j,0}^{++} + O(s),$$

which is the formula (7.16). Similarly,

$$S_j^{-+} = S_{j,0}^{-+} + O(s)$$
$$= -\frac{1}{2}(A_{j,2}^- \cos\theta - 2A_{j,2}^- \cos 2\theta + A_{j,3}^- \cos 2\theta) + O(s)$$
$$= -\frac{1}{2}(f_j(2s)\sinh 2\zeta \cos\theta - 2f_j(2s)\sinh 2\zeta \cos 2\theta + f_j(3s)\sinh 3\zeta \cos 2\theta) + O(s).$$

Since $\sinh\zeta = O(s)$ and $|f_j(2s)| + |f_j(3s)|$ is bounded thanks to (7.14), the estimate (7.18) for S_j^{-+} follows.

Using the identity

$$ah(\zeta,\theta)\sin n\theta = -\frac{1}{2}\big(\sin(n+1)\theta - 2\sin n\theta + \sin(n-1)\theta\big) + \sinh^2(\zeta/2)\sin n\theta,$$

one can see that

$$
\begin{aligned}
S_j^{\pm-} &= -\frac{1}{2}(A_{j,2}^{\pm}\sin\theta - 2A_{j,2}^{\pm}\sin 2\theta + A_{j,3}^{\pm}\sin 2\theta)\\
&\quad -\frac{1}{2}\sum_{n=3}^{\infty}\Big(A_{j,n+1}^{\pm} - 2A_{j,n}^{\pm} + A_{j,n-1}^{\pm}\Big)\sin n\theta + \sinh^2(\zeta/2)\sum_{n=2}^{\infty}A_{j,n}^{\pm}\sin n\theta. \quad (7.24)
\end{aligned}
$$

The other formulas, namely, (7.17) and (7.18) for S_j^{--}, can be proved in the same way. The proof is completed. $\qquad\square$

We are now prepared for estimating $\mathcal{E}_{\zeta\zeta}$ and $\mathcal{E}_{\zeta\theta}$. By applying Lemma 7.2 to (7.10) and (7.12), we have

$$
\begin{aligned}
\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1 - \mathbf{U}] &= S_0 + \frac{4}{s}\cosh\zeta S_1^{++} + O(1)\\
&= S_0 - \frac{2}{s}\cosh\zeta\Big[f_1(2s)\cosh 2\zeta\cos\theta\\
&\qquad - 2f_1(2s)\cosh 2\zeta\cos 2\theta + f_1(3s)\cosh 3\zeta\cos 2\theta\Big] + O(1),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{E}_{\zeta\theta}[\mathbf{v}_1 - \mathbf{U}] &= \widetilde{S}_0 - \frac{4}{s^2}\sinh\zeta S_2^{+-} + O(1)\\
&= \widetilde{S}_0 + \frac{2}{s^2}\sinh\zeta\Big[f_j(2s)\cosh 2\zeta\sin\theta\\
&\qquad - 2f_j(2s)\cosh 2\zeta\sin 2\theta + f_j(3s)\cosh 3\zeta\sin 2\theta\Big] + O(1).
\end{aligned}
$$

By applying Taylor expansions to f_j given in (7.5), we see that

$$
\begin{aligned}
f_1(2s) &= s + O(s^2), & f_1(3s) &= \frac{3}{2}s + O(s^2),\\
f_2(2s) &= -\frac{3}{2}s + O(s^2), & f_2(3s) &= -\frac{9}{4}s + O(s^2).
\end{aligned}
$$

So we have

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1] = S_0 + O(1), \qquad \mathcal{E}_{\zeta\theta}[\mathbf{v}_1] = \widetilde{S}_0 + O(1).$$

It remains to estimate S_0 and $\widetilde{S}_0$. Applying Taylor expansions to a_1 and b_1 given in Lemma 6.1, we have

$$a_1 = \frac{3a}{4s} + O(s), \quad b_1 = -\frac{3a}{2s} + O(s). \tag{7.25}$$

Then, from (7.11) and (7.13), we have

$$S_0 = -h(\zeta, \theta)(2a_1 \cosh 2\zeta + b_1) \cos \theta = -h(\zeta, \theta)(2a_1 + b_1) + O(1) = O(1),$$
$$\widetilde{S}_0 = h(\zeta, \theta) 2a_1 \sinh 2\zeta \sin \theta = O(1).$$

Therefore, we obtain that

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_1] = O(1), \qquad \mathcal{E}_{\zeta\theta}[\mathbf{v}_1] = O(1). \tag{7.26}$$

We now prove that the pressure q_1 is bounded regardless of δ. Recall from (6.31) that q_1 is given by

$$q_1 = -a_1 \frac{2\mu}{a}(2(\tilde{w}_1 - 1) - (\tilde{w}_2 - 1)) + b_1 \frac{2\mu}{a}(\tilde{w}_1 - 1)$$
$$- \frac{2\mu}{a} \sum_{n=2}^{\infty} ((n+1)a_n - (n-1)b_n)(\tilde{w}_n - 1)$$
$$+ \frac{2\mu}{a} \sum_{n=2}^{\infty} na_n(\tilde{w}_{n+1} - 1) - \frac{2\mu}{a} \sum_{n=2}^{\infty} nb_n(\tilde{w}_{n-1} - 1),$$

where $\tilde{w}_n(\zeta, \theta) = \cosh n\zeta \cos n\theta$. Using notation in (7.1), we have

$$q_1 = \frac{2\mu}{a}(-2a_1 + b_1 - 2b_2)\tilde{w}_1 + \frac{2\mu}{a}(a_1 - 3a_2 + b_2 - 3b_3)\tilde{w}_2$$
$$- \frac{2\mu}{a} \sum_{n=3}^{\infty} \frac{1}{n}(\tilde{a}_n - \tilde{b}_n - \tilde{a}_{n-1} + \tilde{b}_{n+1})\tilde{w}_n. \tag{7.27}$$

Note that

$$\tilde{a}_n - \tilde{b}_n - \tilde{a}_{n-1} + \tilde{b}_{n+1} = \frac{1}{2}\Big((\tilde{a}_{n+1} + \tilde{b}_{n+1}) - (\tilde{a}_{n-1} + \tilde{b}_{n-1})\Big)$$
$$- \frac{1}{2}\Big((\tilde{a}_{n+1} - \tilde{b}_{n+1}) - 2(\tilde{a}_n - \tilde{b}_n) + (\tilde{a}_{n-1} - \tilde{b}_{n-1})\Big).$$

Then we have from (7.6) that

$$\tilde{a}_n - \tilde{b}_n - \tilde{a}_{n-1} + \tilde{b}_{n+1} = -\frac{2a}{s}\big(f_1((n+1)s) - f_1((n-1)s)\big)$$
$$+ \frac{2a}{s^2}\big(f_2((n+1)s) - 2f_2(ns) + f_2((n-1)s)\big).$$

Therefore, by (7.27), we have

$$|q_1| \lesssim \frac{1}{s}|-2a_1 + b_1 - 2b_2| + \frac{1}{s}|a_1 - 3a_2 + b_2 - 3b_3|$$
$$+ \sum_{n=3}^{\infty} \frac{s \cosh ns}{ns}\Big|2\frac{f_1((n+1)s) - f_1((n-1)s)}{2s}$$
$$- \frac{1}{s^2}\big(f_2((n+1)s) - 2f_2(ns) + f_2((n-1)s)\big)\Big|. \tag{7.28}$$

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By applying the mean value theorem, we have

$$|q_1| \lesssim \frac{1}{s}|-2a_1 + b_1 - 2b_2| + \frac{1}{s}|a_1 - 3a_2 + b_2 - 3b_3|$$
$$+ \sum_{n=3}^{\infty} \frac{s \cosh ns}{ns}\left|2f_1'(x_n^*) - f_2''(x_n^{**})\right| \tag{7.29}$$

for some $x_n^* \in ((n-1)s, (n+1)s)$ and $x_n^{**} \in ((n-1)s, (n+1)s)$.

By regarding the infinite series in (7.29) as a Riemann sum, we infer

$$I := \sum_{n=3}^{\infty} \frac{s \cosh ns}{ns}\left|2f_1'(x_n^*) - f_2''(x_n^{**})\right| \lesssim \int_0^{\infty} \frac{\cosh x}{x}|2f_1'(x) - f_2''(x)|dx.$$

It then follows that

$$I \lesssim \int_0^1 + \int_1^{\infty} \frac{\cosh x}{x}|2f_1'(x) - f_2''(x)|dx$$
$$= \int_0^1 \frac{\cosh x}{x}|2(f_1'(x) - \frac{1}{2}) - (f_2''(x) - 1)|dx + \int_1^{\infty} \frac{\cosh x}{x}|2f_1'(x) - f_2''(x)|dx$$
$$\lesssim \int_0^1 \frac{\cosh x}{x}(x + x)dx + \int_1^{\infty} \frac{\cosh x}{x}x^3 e^{-2x}dx$$
$$\lesssim 1 + \int_1^{\infty} x^2 e^{-x}dx \lesssim 1,$$

where the third inequality follows from (7.15).

We next estimate the first two terms in the right-hand side of (7.28). By Taylor expansions we obtain

$$a_2 = \frac{1}{2}\frac{a}{s} + O(s), \quad b_2 = -\frac{3}{2}\frac{a}{s} + O(s), \quad b_3 = -\frac{3}{4}\frac{a}{s} + O(s). \tag{7.30}$$

These together with (7.25) yield

$$-2a_1 + b_1 - 2b_2 = O(s), \quad a_1 - 3a_2 + b_2 - 3b_3 = O(s).$$

Therefore, from (7.29), we have

$$|q_1| \lesssim 1.$$

This completes the proof. □

7.2   Proofs of Theorems 3.4 and 5.1

We only prove Theorem 5.1. Thanks to the similarity between the stream functions Φ_2 and Φ_{rot}, Theorem 3.4 can be proved in exactly the same way.

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We first estimate the strain tensor $\mathcal{E}[\mathbf{v}_2 - \mathbf{U}]$. In this proof, $\mathbf{U} = \mathbf{U}_{\text{sh}}$. Using (2.28)-(2.30) and (6.33), one can see that

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}] = -K_v \frac{\sinh\zeta}{a}\sin\theta + h(\zeta,\theta)2c_1\sinh 2\zeta\sin\theta$$
$$+ h(\zeta,\theta)\sum_{n=2}^{\infty}\left(\tilde{c}_n\sinh(n+1)\zeta + \tilde{d}_n\sinh(n-1)\zeta\right)\sin n\theta, \qquad (7.31)$$

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_2 - \mathbf{U}] = K_v \frac{\cosh 2\zeta - 2\cosh\zeta\cos\theta + \cos 2\theta}{2a}$$
$$+ h(\zeta,\theta)d_0\cosh\zeta + h(\zeta,\theta)2c_1\cosh 2\zeta\cos\theta$$
$$+ h(\zeta,\theta)\sum_{n=2}^{\infty}\left(\tilde{c}_n\cosh(n+1)\zeta + \tilde{d}_n\cosh(n-1)\zeta\right)\cos n\theta, \qquad (7.32)$$

$$\mathcal{E}_{\theta\theta}[\mathbf{v}_2 - \mathbf{U}] = -\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}], \qquad (7.33)$$

where

$$\tilde{c}_n = n(n+1)c_n, \quad \tilde{d}_n = n(n-1)b_n.$$

Using the hyperbolic identities

$$\sinh(n+1)\zeta + \sinh(n-1)\zeta = 2\sinh n\zeta\cosh\zeta,$$
$$\sinh(n+1)\zeta - \sinh(n-1)\zeta = 2\cosh n\zeta\sinh\zeta,$$

we can rewrite $\mathcal{E}_{\zeta\zeta}$ and $\mathcal{E}_{\zeta\theta}$ as

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}] = -K_v \frac{\sinh\zeta}{a}\sin\theta + h(\zeta,\theta)2c_1\sinh 2\zeta\sin\theta$$
$$+ h(\zeta,\theta)\sum_{n=2}^{\infty}\left((\tilde{c}_n + \tilde{d}_n)\sinh n\zeta\cosh\zeta + (\tilde{c}_n - \tilde{d}_n)\cosh n\zeta\sinh\zeta\right)\sin n\theta, \qquad (7.34)$$

and

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_2 - \mathbf{U}] = K_v \frac{\cosh 2\zeta - 2\cosh\zeta\cos\theta + \cos 2\theta}{2a}$$
$$+ h(\zeta,\theta)d_0\cosh\zeta + h(\zeta,\theta)2c_1\cosh 2\zeta\cos\theta$$
$$+ h(\zeta,\theta)\sum_{n=2}^{\infty}\left((\tilde{c}_n + \tilde{d}_n)\cosh n\zeta\cosh\zeta + (\tilde{c}_n - \tilde{d}_n)\sinh n\zeta\sinh\zeta\right)\cos n\theta. \qquad (7.35)$$

From the expressions of c_n and d_n given in Lemma 6.2, one can see that, for $n \geq 2$,

$$\tilde{c}_n + \tilde{d}_n = \frac{4a}{s}\frac{nse^{-ns}\cosh ns - (ns)^2\eta_2 + (ns)^3\eta_1}{\sinh 2ns + 2ns\eta_2} - 4K_v s\eta_1\frac{ns}{\sinh 2ns + 2ns\eta_2},$$

$$\tilde{c}_n - \tilde{d}_n = \frac{4a}{s^2}\frac{(ns)^2e^{-ns}\cosh ns - (ns)^3\eta_2 + (ns)^2 s^2\eta_1}{\sinh 2ns + 2ns\eta_2} + 4K_v\frac{e^{-ns}\sinh ns + ns\eta_2}{\sinh 2ns + 2ns\eta_2},$$

where η_1 and η_2 are the quantities given in (7.4). Define, for $0 < x < \infty$,

$$g_1(x) := \frac{xe^{-x}\cosh x - x^2\eta_2 + x^3\eta_1}{\sinh 2x + 2x\eta_2},$$

$$g_2(x) := \frac{x}{\sinh 2x + 2x\eta_2},$$

$$g_3(x) := \frac{x^2 e^{-x}\cosh x - x^3\eta_2 + x^2 s^2\eta_1}{\sinh 2x + 2x\eta_2},$$

$$g_4(x) := \frac{e^{-x}\sinh x + x\eta_2}{\sinh 2x + 2x\eta_2}. \tag{7.36}$$

Then, we have

$$\tilde{c}_n + \tilde{d}_n = \frac{4a}{s}g_1(ns) - 4K_v s\eta_1 g_2(ns),$$

$$\tilde{c}_n - \tilde{d}_n = \frac{4a}{s^2}g_3(ns) + 4K_v g_4(ns). \tag{7.37}$$

It follows from (7.34) that

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}] = -K_v\frac{\sinh \zeta}{a}\sin\theta + h(\zeta,\theta)2c_1\sinh 2\zeta\sin\theta$$

$$+ \frac{4a}{s}h(\zeta,\theta)\sum_{n=2}^{\infty}\big(g_1(ns)\sinh n\zeta\cosh\zeta\big)\sin n\theta$$

$$- 4K_v h(\zeta,\theta)\sum_{n=2}^{\infty}\big(g_2(ns)\sinh n\zeta\cosh\zeta\big)\sin n\theta$$

$$+ \frac{4a}{s^2}h(\zeta,\theta)\sum_{n=2}^{\infty}\big(g_3(ns)\cosh n\zeta\sinh\zeta\big)\sin n\theta$$

$$+ 4K_v h(\zeta,\theta)\sum_{n=2}^{\infty}\big(g_4(ns)\cosh n\zeta\sinh\zeta\big)\sin n\theta. \tag{7.38}$$

If we define

$$T_j^{++}(\zeta,\theta) := ah\sum_{n=2}^{\infty}g_j(ns)\cosh n\zeta\cos n\theta,$$

$$T_j^{+-}(\zeta,\theta) := ah\sum_{n=2}^{\infty}g_j(ns)\cosh n\zeta\sin n\theta,$$

$$T_j^{-+}(\zeta,\theta) := ah\sum_{n=2}^{\infty}g_j(ns)\sinh n\zeta\cos n\theta,$$

$$T_j^{--}(\zeta,\theta) := ah\sum_{n=2}^{\infty}g_j(ns)\sinh n\zeta\sin n\theta,$$

then the component $\mathcal{E}_{\zeta\zeta}$ can be rewritten as

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}] = T_0 + \frac{4}{s}\cosh\zeta T_1^{--} - \frac{4K_v s\eta_1}{a}\cosh\zeta T_2^{--}$$

$$+ \frac{4}{s^2}\sinh\zeta T_3^{+-} + \frac{4K_v}{a}\sinh\zeta T_4^{+-}, \tag{7.39}$$

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where

$$T_0 = -K_v \frac{\sinh \zeta}{a} \sin \theta + h(\zeta, \theta) 2c_1 \sinh 2\zeta \sin \theta. \tag{7.40}$$

Similarly, $\mathcal{E}_{\zeta\theta}$ can be written as

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_2 - \mathbf{U}] = \widetilde{T}_0 + \frac{4}{s} \cosh \zeta T_1^{++} - \frac{4K_v s\eta_1}{a} \cosh \zeta T_2^{++}$$
$$+ \frac{4}{s^2} \sinh \zeta T_3^{-+} + \frac{4K_v}{a} \sinh \zeta T_4^{-+}, \tag{7.41}$$

where

$$\widetilde{T}_0 = K_v \frac{\cosh 2\zeta - 2\cosh \zeta \cos \theta + \cos 2\theta}{2a}$$
$$+ h(\zeta, \theta) d_0 \cosh \zeta + h(\zeta, \theta) 2c_1 \cosh 2\zeta \cos \theta. \tag{7.42}$$

The proof the following lemma is given in appendix E.

Lemma 7.3. *For $j = 1, 2, 3, 4$, we have*

$$|g_j(x)| + |g_j'(x)| + |g_j''(x)| \lesssim (1 + x^3)e^{-2x}, \quad 0 < x < \infty. \tag{7.43}$$

We omit the proof of the following lemma since it can be proved using Lemma 7.3 in the same way as Lemma 7.2:

Lemma 7.4. *For $j = 1, 2, 3, 4$,*

$$T_j^{++} = -\frac{1}{2} g_j(2s) \cosh 2\zeta \cos \theta + g_j(2s) \cosh 2\zeta \cos 2\theta$$
$$- \frac{1}{2} g_j(3s) \cosh 3\zeta \cos 2\theta + O(s), \tag{7.44}$$
$$T_j^{+-} = -\frac{1}{2} g_j(2s) \cosh 2\zeta \sin \theta + g_j(2s) \cosh 2\zeta \sin 2\theta$$
$$- \frac{1}{2} g_j(3s) \cosh 3\zeta \sin 2\theta + O(s), \tag{7.45}$$
$$T_j^{-\pm} = O(s). \tag{7.46}$$

We infer using the definition (7.36) of g_j that for $k = 2, 3$,

$$g_1(ks) = \frac{1}{4} + O(s), \quad g_2(ks) = \frac{1}{4} + O(s),$$
$$g_3(ks) = \frac{1}{4}s + O(s^2), \quad g_4(ks) = \frac{1}{2} + O(s). \tag{7.47}$$

Thus, we have

$$T_3^{+-} = O(s)$$

and

$$T_4^{+-} = -\frac{1}{4} \cosh 2\zeta \sin \theta + \frac{1}{2} \cosh 2\zeta \sin 2\theta - \frac{1}{4} \cosh 3\zeta \sin 2\theta + O(s)$$
$$= -\frac{1}{4}(\sin \theta - \sin 2\theta) + O(s),$$

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as $s \to 0$. It then follows from (7.39) that

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}] = T_0 + \frac{K_v \zeta}{a}(-\sin\theta + \sin 2\theta) + O(1). \tag{7.48}$$

Since

$$c_1 = a(-1 + \coth 2s) + K_v \frac{1}{1 + e^{2s}} = \frac{1-s}{2}K_v + \frac{a}{2s} + O(s), \tag{7.49}$$

it follows from (7.40) that

$$
\begin{aligned}
T_0 &= -K_v \frac{\sinh\zeta}{a}\sin\theta + \frac{\cosh\zeta}{a}2c_1 \sinh 2\zeta \sin\theta - \frac{\cos\theta}{a}2c_1 \sinh 2\zeta \sin\theta \\
&= -K_v \frac{\zeta}{a}\sin\theta + 2K_v\frac{\zeta}{a}\sin\theta - K_v\frac{\zeta}{a}\sin 2\theta + O(1) \\
&= K_v\frac{\zeta}{a}\sin\theta - K_v\frac{\zeta}{a}\sin 2\theta + O(1).
\end{aligned} \tag{7.50}
$$

This together with (7.48) yields the desired estimate

$$\mathcal{E}_{\zeta\zeta}[\mathbf{v}_2 - \mathbf{U}] = O(1). \tag{7.51}$$

Likewise, we use (7.41), Lemma 7.4 and (7.47) to ensure

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_2 - \mathbf{U}] = \widetilde{T_0} + (-\frac{1}{2s} + \frac{K_v s}{2a})\cos\theta + (+\frac{1}{2s} - \frac{K_v s}{2a})\cos 2\theta + O(1).$$

Then, using the estimate

$$d_0 = \frac{a}{\sinh s \cosh s + s} - K_v \frac{\sinh^2 s}{s + \cosh s \sinh s} = \frac{a}{2s} - K_v\frac{s}{2} + O(s) \tag{7.52}$$

in addition to (7.49), we obtain

$$
\begin{aligned}
T_0' &= \frac{K_v}{2a} - \frac{K_v}{a}\cos\theta + \frac{1}{2}\frac{K_v}{a}\cos 2\theta + \frac{1-\cos\theta}{a}(\frac{a}{2s} + \frac{K_v s}{2}) \\
&\quad + \frac{1-\cos\theta}{a}(K_v - sK_v + \frac{a}{s})\cos\theta + O(1) \\
&= \frac{K_v}{a}(\frac{1}{2}) + \frac{1}{2s} - \frac{K_v s}{2a} - \frac{1}{2}(\frac{K_v}{a} - \frac{K_v s}{a} + \frac{1}{s}) + (-\frac{1}{2s} + \frac{K_v s}{2a} - \frac{K_v s}{a} + \frac{1}{s})\cos\theta \\
&\quad + (\frac{1}{2}\frac{K_v}{a} - \frac{1}{2}(\frac{K_v}{a} - \frac{K_v s}{a} + \frac{1}{s}))\cos 2\theta + O(1) \\
&= (\frac{1}{2s} - \frac{K_v s}{2a})\cos\theta + (\frac{K_v s}{2a} - \frac{1}{2s})\cos 2\theta + O(1),
\end{aligned} \tag{7.53}
$$

where we have used the following identity for the second equality:

$$(1 - \cos\theta)\cos\theta = -\frac{1}{2} + \cos\theta - \frac{1}{2}\cos 2\theta.$$

Thus, we have

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}_2 - \mathbf{U}] = O(1).$$

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So far we proved that

$$\|\mathcal{E}[\mathbf{v}_2]\|_\infty \lesssim 1.$$

We now prove that the pressure q_2 is bounded independently of δ. It was shown in (6.48)

$$q_2 = \frac{2\mu}{a}d_0 w_1 + \frac{2\mu}{a}\sum_{n=1}^\infty ((n+1)c_n - (n-1)d_n)w_n$$
$$- \frac{2\mu}{a}\sum_{n=1}^\infty nc_n w_{n+1} + \frac{2\mu}{a}\sum_{n=1}^\infty nd_n w_{n-1},$$

where $w_n(\zeta, \theta) := \sinh n\zeta \sin n\theta$. It can be rewritten as

$$q_2 = \frac{2\mu}{a}(-d_0 + 2(c_1 + d_2))w_1 + \frac{2\mu}{a}\sum_{n=2}^\infty \frac{1}{n}(\tilde{c}_n - \tilde{d}_n - \tilde{c}_{n-1} + \tilde{d}_{n+1})w_n, \qquad (7.54)$$

where $\tilde{c}_n = n(n+1)c_n$ and $\tilde{d}_n = n(n-1)d_n$ for $n \geq 2$.

Note that

$$\tilde{c}_n - \tilde{d}_n - \tilde{c}_{n-1} + \tilde{d}_{n+1} = \frac{1}{2}(\tilde{c}_{n+1} + \tilde{d}_{n+1}) - \frac{1}{2}(\tilde{c}_{n-1} + \tilde{d}_{n-1})$$
$$+ \frac{1}{2}\Big(2(\tilde{c}_n - \tilde{d}_n) - (\tilde{c}_{n-1} - \tilde{d}_{n-1}) - (\tilde{c}_{n+1} - \tilde{d}_{n+1})\Big).$$

It then follows from (7.37) that

$$\tilde{c}_n - \tilde{d}_n - \tilde{c}_{n-1} + \tilde{d}_{n+1} = \frac{2a}{s}(g_1((n+1)s) - g_1((n-1)s))$$
$$- 2K_v s^2 \eta_1(g_2((n+1)s) - g_2((n-1)s))$$
$$- 2a(g_3((n+1)s) - 2g_3(ns) + g_3((n-1)s))$$
$$- 2K_v s^2(g_4((n+1)s) - 2g_4(ns) + g_4((n-1)s)). \qquad (7.55)$$

By the mean value theorem, there are $x_{j,n} \in ((n-1)s, (n+1)s)$ such that

$$|g_j((n+1)s) - g_j((n-1)s)| \lesssim s|g_j'(x_{j,n})|, \quad j = 1, 2,$$

and

$$|g_j((n+1)s) - 2g_j(ns) + g_j((n-1)s))| \lesssim s^2|g_j''(x_{j,n})|, \quad j = 3, 4.$$

We then infer from (7.43) that

$$|g_j((n+1)s) - g_j((n-1)s)| \lesssim s(1 + (ns)^3)e^{-2ns}, \quad j = 1, 2,$$

and

$$|g_j((n+1)s) - 2g_j(ns) + g_j((n-1)s))| \lesssim s^2(1 + (ns)^3)e^{-2ns}, \quad j = 3, 4.$$

Since $a \approx s$ and $K_v = O(s^{-1})$, it then follows from (7.55) that

$$|\tilde{c}_n - \tilde{d}_n - \tilde{c}_{n-1} + \tilde{d}_{n+1}| \lesssim s(1 + (ns)^3)e^{-2ns},$$

and from (7.54) that

$$|q_2| \lesssim (|d_0| + 2|c_1 + d_2|) + \sum_{n=2}^{\infty} \frac{1}{n}(1 + (ns)^3)e^{-2ns} \sinh n\zeta. \qquad (7.56)$$

One can see from (7.52) that

$$d_0 = \frac{a}{2s} - \frac{K_v s}{2} + O(s) = O(1).$$

One can also see from expressions of c_1 and d_2 in Lemma 6.2 that

$$c_1 = \frac{1}{2}K_v + O(1), \quad d_2 = -\frac{1}{2}K_v + O(1),$$

and hence $c_1 + d_2 = O(1)$. Thus we have from (7.56)

$$|q_2| \lesssim 1 + \sum_{n=2}^{\infty} \frac{s \sinh ns}{ns}(1 + (ns)^3)e^{-2ns} \lesssim 1 + \int_0^{\infty} \frac{\sinh t}{t}(1 + t^3)e^{-2t}dt \lesssim 1.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Concluding remarks

In this paper, we have investigated the problem of quantifying the stress concentration in the narrow region between two rigid cylinders and derived precise estimates for the stress blow-up in the Stokes system when inclusions are circular cylinders of the same radii. We have shown that, even though the divergence of the velocity is confined to be zero, either the pressure component or the shear stress component of the stress tensor always blows up, and that the blow-up rate is $\delta^{-1/2}$, where δ is the distance between the cylinders. This blow-up rate coincides with the ones for elasto-statics and elasto-statics. In the course of deriving the results, it is proved that the blow-up of the stress tensor does not occur when the no-slip boundary is prescribed. We also derived an asymptotic decomposition formula which explicitly characterizes the singular behaviour of the solution. This formula may play an important role in computing the Stokes flow in presence of closely located rigid cylinders.

Since the method of bipolar coordinates is employed, extension of this paper's results to the case of circular cylinders with different radii is not a big issue. However, it is quite challenging to extend them to the more general case when the cross sections of the cylinders are strictly convex. In particular, proving no blow-up for the problem with the no-slip boundary condition on the convex boundaries seems already quite challenging.

A   Proof of Lemma 3.2

We have from (2.12) that

$$\mathbf{e}_\zeta = \alpha\mathbf{e}_x - \beta\mathbf{e}_y, \quad \mathbf{e}_\theta = -\beta\mathbf{e}_x - \alpha\mathbf{e}_y.$$

So we have

$$|\nabla \mathbf{e}_\zeta| + |\nabla \mathbf{e}_\theta| = 2(|\nabla \alpha| + |\nabla \beta|) \lesssim |h\partial_\zeta \alpha| + |h\partial_\theta \alpha| + |h\partial_\zeta \beta| + |h\partial_\theta \beta|.$$

Since

$$h\partial_\zeta \alpha = -\frac{\sinh \zeta \sin^2 \theta}{a(\cosh \zeta - \cos \theta)},$$

one can see that

$$|h\partial_\zeta \alpha| \lesssim |\theta| \leq |\zeta| + |\theta|.$$

Similarly one can show that

$$|h\partial_\theta \alpha|, |h\partial_\zeta \beta|, |h\partial_\theta \beta| \lesssim |\zeta| + |\theta|.$$

This completes the proof. $\qquad\qquad\square$

B Proof of Lemma 4.3

According to the transition relation (2.33), the stress tensor $\sigma[\mathbf{h}_1, p_1]$ is given by

$$\sigma[\mathbf{h}_1, p_1] = \Xi \begin{bmatrix} \sigma_{1,\zeta\zeta} & \sigma_{1,\zeta\theta} \\ \sigma_{1,\zeta\theta} & \sigma_{1,\theta\theta} \end{bmatrix} \Xi.$$

In particular, we have

$$\mathbf{e}_\zeta \cdot \sigma[\mathbf{h}_1, p_1]\mathbf{e}_\zeta = \sigma_{1,\zeta\zeta}, \quad \mathbf{e}_\zeta \cdot \sigma[\mathbf{h}_1, p_1]\mathbf{e}_\theta = \sigma_{1,\zeta\theta}. \tag{B.1}$$

On ∂D_2 which is parametrized by $\{\zeta = s\}$, the outward unit normal ν is given by

$$\nu|_{\partial D_2} = -\mathbf{e}_\zeta|_{\zeta=s},$$

and, according to (2.12), $\mathbf{e}_x$ is expressed as $\mathbf{e}_x = \alpha(\zeta, \theta)\mathbf{e}_\zeta - \beta(\zeta, \theta)\mathbf{e}_\theta$, where α and β are defined by (2.10). So, we have

$$\mathbf{e}_x \cdot \sigma[\mathbf{h}_1, p_1]\nu = -\big(\alpha(s, \theta)\sigma_{1,\zeta\zeta} - \beta(s, \theta)\sigma_{1,\zeta\theta}\big).$$

Due to (2.15), we have

$$\mathcal{I}_1 = -\int_{-\pi}^{\pi} \big(\alpha(s, \theta)\sigma_{1,\zeta\zeta} - \beta(s, \theta)\sigma_{1,\zeta\theta}\big) h(s, \theta)^{-1} d\theta. \tag{B.2}$$

We now compute $(\alpha(s, \theta)\sigma_{1,\zeta\zeta} - \beta(s, \theta)\sigma_{1,\zeta\theta}) h(s, \theta)^{-1}$. It follows from the formulas (2.28) and (2.30) of the strain in bipolar coordinates, the strain-stress relation (2.32), and the formula (3.3) for the stream function that

$$\sigma_{1,\zeta\zeta}|_{\zeta=s} = \frac{A_2\mu}{a}(2 + \operatorname{sech} 2s - 4\cosh^3 s \operatorname{sech} 2s \cos \theta + \cos 2\theta), \tag{B.3}$$

$$\sigma_{1,\zeta\theta}|_{\zeta=s} = -\frac{A_2\mu}{a}(\cosh s - \cos \theta)2\tanh 2s \sin \theta. \tag{B.4}$$

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Using the definitions (2.10) of α and β, (B.3) and (B.4), we arrive at

$$\big(\alpha(s,\theta)\sigma_{1,\zeta\zeta} - \beta(s,\theta)\sigma_{1,\zeta\theta}\big)h(s,\theta)^{-1} = 2\mu A_1(-1 + \cosh s \operatorname{sech} 2s \cos\theta). \tag{B.5}$$

Then by integrating both sides of (B.5) over $[-\pi, \pi]$, we obtain

$$\mathcal{I}_1 = -4\pi\mu A_1 = -\frac{4\pi\mu}{2s - \tanh 2s} = -\frac{3\pi\mu}{2}\frac{1}{s^3} + O(s^{-1}). \tag{B.6}$$

Thanks to the asymptotic formula (2.7) of s as δ tends to 0, we get the asymptotic formula (4.10) for $\mathcal{I}_1$.

Next we consider $\mathcal{J}_1$. Similarly to the case of $\mathcal{I}_1$, we have

$$\int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_1, p_1]\nu\, dl = \int_{\partial D_2} \mathbf{U} \cdot \sigma[\mathbf{h}_1, p_1](-\mathbf{e}_\zeta)\, dl$$

$$= -\int_{-\pi}^{\pi} (U_\zeta\mathbf{e}_\zeta + U_\theta\mathbf{e}_\theta) \cdot (\sigma_{1,\zeta\zeta}\mathbf{e}_\zeta + \sigma_{1,\zeta\theta}\mathbf{e}_\theta)h(s,\theta)^{-1}d\theta$$

$$= -\int_{-\pi}^{\pi} (U_\zeta\sigma_{1,\zeta\zeta} + U_\theta\sigma_{1,\zeta\theta})|_{\zeta=s}h(s,\theta)^{-1}d\theta.$$

Since $U_\zeta = U \cdot \mathbf{e}_\zeta$ and $U_\theta = U \cdot \mathbf{e}_\theta$, it follows from (2.2) and (2.10) that

$$U_\zeta|_{\zeta=s} = \frac{a \sinh s \left(1 - \cosh s \cos\theta + \sin^2\theta\right)}{(\cosh s - \cos\theta)^2},$$

$$U_\theta|_{\zeta=s} = \frac{a \sin\theta \left(1 - \cosh s \cos\theta - \sinh^2 s\right)}{(\cosh s - \cos\theta)^2}. \tag{B.7}$$

It is convenient to use the following functions:

$$q_n = q_n(s,\theta) := \frac{\cos n\theta}{\cosh s - \cos\theta}, \quad n = 0, 1, 2, \ldots. \tag{B.8}$$

We obtain, by using (B.3), (B.4) and (B.7), that

$$(U_\zeta\sigma_{1,\zeta\zeta} + U_\theta\sigma_{1,\zeta\theta})h(s,\theta)^{-1}$$
$$= -a\mu A_1 \operatorname{sech} 2s \sinh s((-1 + 2\cosh 2s)q_0 - 2\cosh s q_1 + q_2). \tag{B.9}$$

As before, by integrating both sides of the equality in (B.9) over $[-\pi, \pi]$, we arrive at

$$\mathcal{J}_1 = -a\mu A_1 \operatorname{sech} 2s \sinh s((-1 + 2\cosh 2s)\mathcal{Q}_1^0 - 2\cosh s\mathcal{Q}_1^1 + \mathcal{Q}_1^2), \tag{B.10}$$

where

$$\mathcal{Q}_n = \mathcal{Q}_n(s) := \int_{-\pi}^{\pi} q_n(s,\theta)d\theta. \tag{B.11}$$

We also obtain the following asymptotic expansion of $\mathcal{Q}_n$, whose proof will be given after the current proof is completed.

Lemma B.1. *It holds that*

$$\mathcal{Q}_n(s) = \frac{2\pi}{s} - 2n\pi + (n^2 - 1/3)\pi s + O(s^2), \quad as\ s \to 0. \tag{B.12}$$

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Then, together with the asymptotic formula (2.7) of s as δ tends to 0, applying Lemma B.1 to (B.10) yields (4.11). $\qquad\square$

Proof of Lemma B.1. One can easily see that $Q_n(s)$ is the real part of the following contour integral:

$$-2\int_C \frac{z^n}{z^2 - 2z\cosh s + 1}dz,$$

where C is the unit circle. Then the residue theorem yields

$$Q_n(s) = \frac{2\pi e^{-ns}}{\sinh s}, \tag{B.13}$$

from which (B.12) follows. $\qquad\square$

C The asymptotics of the boundary integrals

Here we compute the asymptotics of the boundary integrals $\mathcal{I}_{22}, \mathcal{I}_{23}, \mathcal{J}_2, \mathcal{I}_{\mathrm{rot}},$ and $\mathcal{J}_{\mathrm{rot}},$ and prove Lemma 5.3.

C.1 A lemma

The following result will be used to prove Lemma 5.3.

Lemma C.1. *Suppose that a solution $(\mathbf{v}, q)$ to the Stokes system on the exterior region D^e satisfies $(\mathbf{v}, q) \in \mathcal{M}$, and that its corresponding stream function Ψ is given by*

$$(h\Psi)(\zeta, \theta) = K(\cosh\zeta - \cos\theta)\ln(2\cosh\zeta - 2\cos\theta) + a_0\cosh\zeta + d_0\zeta\sinh\zeta$$

$$+ \sum_{n=1}^{\infty} \big(a_n\cosh(n+1)\zeta + b_n\cosh(n-1)\zeta\big)\cos n\theta. \tag{C.1}$$

Then we have the following formulas for the boundary integrals:

$$\int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{v}, q]\nu = d_0 4\pi\mu, \tag{C.2}$$

$$\int_{\partial D_2} \boldsymbol{\psi}_3 \cdot \sigma[\mathbf{v}, q]\nu = K4\pi\mu a. \tag{C.3}$$

where a is the number defined in (2.4).

Proof. Let us write in terms of bipolar coordinates the stress tensor $\sigma[\mathbf{v}, q]$ and the strain tensor $\mathcal{E}[\mathbf{v}, q]$ as

$$\sigma[\mathbf{v}, q] = \Xi \begin{bmatrix} \sigma_{\zeta\zeta} & \sigma_{\zeta\theta} \\ \sigma_{\zeta\theta} & \sigma_{\theta\theta} \end{bmatrix} \Xi, \quad \mathcal{E}[\mathbf{v}, q] = \Xi \begin{bmatrix} \mathcal{E}_{\zeta\zeta} & \mathcal{E}_{\zeta\theta} \\ \mathcal{E}_{\zeta\theta} & \mathcal{E}_{\theta\theta} \end{bmatrix} \Xi.$$

One can show in the same way as of deriving (B.2) that

$$\mathcal{K}_2 := \int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{v}, q]\nu = \int_{-\pi}^{\pi} \big(\beta(s, \theta)\sigma_{\zeta\zeta} + \alpha(s, \theta)\sigma_{\zeta\theta}\big)h(s, \theta)^{-1}d\theta. \tag{C.4}$$

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Using (3.26), one can also see that

$$\mathcal{K}_3 := \int_{\partial D_2} \boldsymbol{\psi}_3 \cdot \sigma[\mathbf{v}, q]\nu = -\frac{a}{\tanh s} \int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{v}, q]\nu - \frac{a}{\sinh s} \int_{\partial D_2} \mathbf{e}_\theta \cdot \sigma[\mathbf{v}, q]\nu$$

$$= -\frac{a}{\tanh s}\mathcal{K}_2 + \frac{a}{\sinh s}\int_{-\pi}^{\pi} \sigma_{\zeta\theta}|_{\zeta=s} h(s,\theta)^{-1} d\theta. \qquad (C.5)$$

We assume for a moment that the stream function Ψ is given by

$$(h\Psi)(\zeta,\theta) = K(\cosh\zeta - \cos\theta)\ln(2\cosh\zeta - 2\cos\theta). \qquad (C.6)$$

Applying the formula (2.22) for the Laplacian in bipolar coordinates, we see that $\mu\Delta\Psi = 0$. Together with the relation (2.27) between the pressure and the stream function and the condition $q \to 0$ as $|\mathbf{x}| \to \infty$, this implies that the corresponding pressure $q = 0$. Then, by (2.28)-(2.30) (the strain-stream function relation) and (2.32) (the stress-strain relation), we obtain

$$\sigma_{\zeta\zeta}|_{\zeta=s} = -K\frac{2\mu}{a}\sinh s \sin\theta, \qquad (C.7)$$

$$\sigma_{\zeta\theta}|_{\zeta=s} = \frac{K\mu}{a}(\sinh^2 s - \sin^2\theta + (\cosh\zeta - \cos\theta)^2). \qquad (C.8)$$

We also have

$$\big(\beta(s,\theta)\sigma_{\zeta\zeta} + \alpha(s,\theta)\sigma_{\zeta\theta}\big)|_{\zeta=s} h(s,\theta)^{-1} = -2\mu\cosh s \cos\theta.$$

Then, by integrating over $[-\pi, \pi]$, we arrive at

$$\mathcal{K}_2 = 0.$$

We now consider $\mathcal{K}_3$. We see from (C.5) and (C.8) that

$$\mathcal{K}_3 = 0 + \frac{a}{\sinh s}K\mu\int_{-\pi}^{\pi}\left(\frac{\sinh^2 s - \sin^2\theta}{\cosh s - \cos\theta} + \cosh s - \cos\theta\right)d\theta$$

$$= \frac{a}{\sinh s}K\mu(2\pi\sinh s - 2\pi e^{-s} + 2\pi\cosh s)$$

$$= K4\pi\mu a,$$

where we have used (B.13) for the second equality. So far, we have computed $\mathcal{K}_2$ and $\mathcal{K}_3$ when the stream function Ψ is given by (C.6).

Next we assume that Ψ is given by

$$(h\Psi)(\zeta,\theta) = d_0\zeta\sinh\zeta + a_0\cosh\zeta + \sum_{n=1}^{\infty}\big(a_n\cosh(n+1)\zeta + b_n\cosh(n-1)\zeta\big)\cos n\theta. \quad (C.9)$$

By symmetry and from the fact that $\mathbf{h}_2|_{\partial D_i} = (-1)^i\frac{1}{2}\Psi_2$, we have

$$\mathcal{K}_2 = \int_{\partial D_1}\frac{-1}{2}\Psi_2 \cdot \sigma[\mathbf{v}, q]\nu + \int_{\partial D_2}\frac{1}{2}\Psi_2 \cdot \sigma[\mathbf{v}, q]\nu$$

$$= \int_{\partial D^e}\mathbf{h}_2 \cdot \sigma[\mathbf{v}, q]\nu.$$

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Then, by (2.36) (the divergence theorem), we have

$$\mathcal{K}_2 = -2\mu \int_{D^e} \mathcal{E}[\mathbf{h}_2] : \mathcal{E}[\mathbf{v}].$$

Recall from (3.36)-(3.38) that

$$\mathcal{E}_{\zeta\zeta}[\mathbf{h}_2] = 0, \quad \mathcal{E}_{\theta\theta}[\mathbf{h}_2] = 0, \quad \mathcal{E}_{\zeta\theta}[\mathbf{h}_2] = h(\zeta, \theta) A_2 \cosh\zeta. \tag{C.10}$$

By (2.30), one can easily check that

$$\mathcal{E}_{\zeta\theta}[\mathbf{v}] = h(\zeta, \theta) d_0 \cosh\zeta$$
$$+ h(\zeta, \theta) \sum_{n=1}^{\infty} (n(n+1)a_n \cosh(n+1)\zeta + n(n-1)b_n \cosh(n-1)\zeta) \cos n\theta. \tag{C.11}$$

So we obtain

$$\mathcal{K}_2 = -2\mu \int_{D^e} \mathcal{E}_{\zeta\zeta}[\mathbf{h}_2]\mathcal{E}_{\zeta\zeta}[\mathbf{v}] + 2\mathcal{E}_{\zeta\theta}[\mathbf{h}_2]\mathcal{E}_{\zeta\theta}[\mathbf{v}] + \mathcal{E}_{\theta\theta}[\mathbf{h}_2]\mathcal{E}_{\theta\theta}[\mathbf{v}]$$
$$= -2\mu \int_{-s}^{s} \int_{-\pi}^{\pi} 2\mathcal{E}_{\zeta\theta}[\mathbf{h}_2]\mathcal{E}_{\zeta\theta}[\mathbf{v}] \frac{1}{h(\zeta, \theta)^2} d\theta d\zeta$$
$$= -2\mu(2\pi) \int_{-s}^{s} 2A_2 d_0 \cosh^2\zeta \, d\zeta = -d_0 4\pi\mu A_2(2s + \sinh 2s) = d_0 4\pi\mu.$$

We now compute $\mathcal{K}_3$. We have from (C.5) and (C.11) that

$$\mathcal{K}_3 = -\frac{a}{\tanh s}\mathcal{K}_2 + \frac{2\mu a}{\sinh s} \int_{-\pi}^{\pi} \mathcal{E}_{\zeta\theta}|_{\zeta=s} h(s, \theta)^{-1} d\theta \tag{C.12}$$
$$= -\frac{a}{\tanh s}(d_0 4\pi\mu) + \frac{2\mu a}{\sinh s} \int_{-\pi}^{\pi} d_0 \cosh s d\theta$$
$$= 0. \tag{C.13}$$

The proof is completed. □

C.2 Proof of Lemma 5.3

Now we are ready to compute the asymptotics of integrals $\mathcal{I}_{22}, \mathcal{I}_{23}, \mathcal{J}_2, \mathcal{I}_{\mathrm{rot}}$, and $\mathcal{J}_{\mathrm{rot}}$.

We first consider $\mathcal{I}_{22}$. We see from Theorem 3.4 and Proposition 3.5 that

$$\|\sigma[\mathbf{h}_2 - \widetilde{\mathbf{h}}_2, p_2 - \widetilde{p}_2]\|_\infty \lesssim 1. \tag{C.14}$$

So we get

$$\mathcal{I}_{22} = \int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\widetilde{\mathbf{h}}_2, \widetilde{p}_2]\nu + O(1).$$

Therefore, Lemma C.1 with $\Psi = \widetilde{\Psi}_2$ yields

$$\mathcal{I}_{22} = A_2 4\pi\mu + O(1).$$

Hence, since $s \approx \sqrt{\delta}$, the asymptotic formula (3.35) for A_2 yields (5.16).

We now consider $\mathcal{I}_{23}$. Recall from (5.12) that

$$\mathcal{I}_{23} = \int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{h}_{\mathrm{rot}}, p_{\mathrm{rot}}]\big|_+ \nu.$$

Then, Lemma C.1 with $\Psi = \Psi_{\mathrm{rot}}$ yields

$$\mathcal{I}_{23} = -\frac{K_{\mathrm{rot}} \sinh^2 s}{\sinh s \cosh s + s} 4\pi\mu.$$

Similarly, we have

$$\mathcal{I}_{\mathrm{rot}} = K_{\mathrm{rot}} 4\pi\mu a.$$

Since $s \approx \sqrt{\delta}$, the asymptotic formula (6.55) for K_{rot} yields (5.17) and (5.18).

Next we consider $\mathcal{J}_2$ and $\mathcal{J}_{\mathrm{rot}}$. Using the symmetry and the fact that $(\mathbf{v}_2 - \mathbf{U})|_{\partial D_e} = -\mathbf{U}$, we have

$$\mathcal{J}_2 = \frac{1}{2} \int_{\partial D^e} (-1)(\mathbf{v}_2 - \mathbf{U}) \cdot \sigma[\mathbf{h}_2, p_2]\nu.$$

Here $\mathbf{U}(x, y) = \mathbf{U}_{\mathrm{sh}} = (y, x)^T$. Thanks to Green's formula (2.37), the following holds:

$$\mathcal{J}_2 = -\frac{1}{2} \int_{\partial D^e} \mathbf{h}_2 \cdot \sigma[\mathbf{v}_2 - \mathbf{U}, q_2]\nu = -\frac{1}{2} \int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{v}_2 - \mathbf{U}, q_2]\nu.$$

It can be proved in the same way as the proof of Lemma 4.2 that

$$\int_{\partial D^e} \mathbf{h}_2 \cdot \sigma[\mathbf{v}_2 - \mathbf{U}, q_2]\nu = 0.$$

Thus,

$$\mathcal{J}_2 = -\frac{1}{2} \int_{\partial D_2} \boldsymbol{\psi}_2 \cdot \sigma[\mathbf{v}_2 - \mathbf{U}, q_2]\nu.$$

Similarly, we have

$$\mathcal{J}_{\mathrm{rot}} = -\int_{\partial D_2} \boldsymbol{\psi}_3 \cdot \sigma[\mathbf{v}_2 - \mathbf{U}, q_2]\nu.$$

Since the stream function $\Psi_{v,2}$ associated with $(\mathbf{v}_2 - \mathbf{U}, q_2)$ is given in (6.33), we may apply Lemma C.1 to have

$$\mathcal{J}_2 = -\frac{4\pi\mu}{2} \left(\frac{a}{\sinh s \cosh s + s} - K_v \frac{\sinh^2 s}{s + \cosh s \sinh s} \right),$$
$$\mathcal{J}_{\mathrm{rot}} = -K_v 4\pi\mu a.$$

Since $s \approx a \approx \sqrt{\delta}$, the asymptotic formula (6.54) for K_v yields (5.19) and (5.20). The proof is then completed. $\qquad\square$

D   Proof of Lemma 7.1

If $1 < x < \infty$, then one can easily see that

$$|f_j(x)| + |f_j'(x)| + |f_j''(x)| \lesssim x^3 e^{-2x}$$

for $j = 1, 2$. So we consider the case when $2s \leq x \leq 1$, and prove

$$|f_j(x)| + |f_j'(x)| + |f_j''(x)| \lesssim 1, \quad j = 1, 2, \tag{D.1}$$

and (7.15).
Let

$$\alpha_1(x) = xe^{-x} \sinh x - x^2 \eta_2 + x^3 \eta_1,$$
$$\alpha_2(x) = x^2 e^{-x} \sinh x - x^3 \eta_2 + s^2 x^2 \eta_1,$$
$$\beta(x) = \frac{1}{\sinh 2x - 2x\eta_2},$$

so that the following relations hold:

$$f_1(x) = \alpha_1(x)\beta(x), \quad f_2(x) = \alpha_2(x)\beta(x).$$

One can see from the definition (7.4) of η_j that

$$\eta_1 = 1 + O(s^2), \quad \eta_2 = 1 + \frac{2}{3}s^2 + R_1(s), \tag{D.2}$$

where the remainder term $R_1(s)$ satisfies

$$|R_1(s)| \leq \frac{4}{15}s^4, \tag{D.3}$$

provided that s is sufficiently small.
Suppose $2s < x \leq 1$. Since

$$\alpha_1(x) = (1 - \eta_2)x^2 + (-1 + \eta_1)x^3 + \frac{2}{3}x^4 + O(x^5),$$

we have

$$\alpha_1 = \frac{2}{3}a + O(x^5), \quad \alpha_1'(x) = \frac{2}{3}a' + O(x^4), \quad \alpha_1''(x) = \frac{2}{3}a'' + O(x^3), \tag{D.4}$$

where

$$a(x) := x^4 - s^2 x^2. \tag{D.5}$$

Likewise, since

$$\alpha_2(x) = (1 - \eta_2)x^3 + \frac{2}{3}x^5 - (x^4 - s^2 x^2 \eta_1) + O(x^6),$$

we have

$$\alpha_2 = \tilde{a} + O(x^6), \quad \alpha_2' = \tilde{a}' + O(x^5), \quad \alpha_2'' = \tilde{a}'' + O(x^4), \tag{D.6}$$

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where
$$\tilde{a}(x) := \left(-1 + \frac{2}{3}x\right)a(x). \tag{D.7}$$

Let
$$w(x) := \sinh 2x - 2x\eta_2.$$

so that $\beta(x) = w(x)^{-1}$. Note that
$$\sinh 2x = 2x + \frac{4}{3}x^3 + R_2(x),$$

where the remainder term R_2 satisfies $R_2(x) = O(x^5)$ and
$$R_2(x) \geq \frac{4}{15}x^5. \tag{D.8}$$

Then
$$w(x) = \frac{4}{3}x^3 - \frac{4}{3}s^2x + R,$$

where $R := R_2(x) - R_1(s)x$. Since $x > 2s$, it follows from (D.3) and (D.8) that
$$R \geq \frac{4}{15}x(x^4 - s^s) \geq Cx^5$$

for some positive constant C. Therefore, we have
$$w(x) = \frac{4}{3}b(x)(1 + O(x^2)),$$

where the remainder term $O(x^2)$ is larger than Cx^2 for some positive constant C and
$$b(x) = x^3 - s^2x = \frac{a(x)}{x}. \tag{D.9}$$

Thus we have
$$\beta(x) = w(x)^{-1} = \frac{3}{4}\frac{1}{b(x)} + O(x^{-1}). \tag{D.10}$$

Since $\beta' = -\beta^2 w'$ and $\beta'' = 2\beta^3(w')^2 - \beta^2 w''$, we have
$$\beta'(x) = -\frac{3}{4}\frac{b'}{b^2} + O(x^{-2}), \quad \beta''(x) = \frac{3}{4}\frac{2(b')^2 - bb''}{b^3} + O(x^{-3}). \tag{D.11}$$

Now it is easy to see that $f_1(x) = O(x)$ and $f_1'(x) = O(1)$. To prove the first part of (7.15), we invoke (D.4), (D.10) and (D.11) to derive
$$f_1' = \frac{1}{2}\frac{a'b - ab'}{b^2} + O(x).$$

Since $a = xb$, we have
$$\frac{a'b - ab'}{b^2} = 1, \tag{D.12}$$

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which yields the first part of (7.15). To prove that f_1'' is bounded, we again use (D.4), (D.10) and (D.11) to derive

$$f_1'' = \frac{1}{2}\frac{a''b^2 - 2a'b'b + 2a(b')^2 - abb''}{b^3} + O(1).$$

One can easily see that

$$\frac{a''b^2 - 2a'b'b + 2a(b')^2 - abb''}{b^3} = \left(\frac{a'b - ab'}{b^2}\right)'.$$

Thus, thanks to (D.12), we infer

$$\frac{a''b^2 - 2a'b'b + 2a(b')^2 - abb''}{b^3} = 0. \tag{D.13}$$

This proves (D.1) for $j = 1$.

It is easy to see that $f_2(x) = O(x)$ and $f_2'(x) = O(1)$. On the other hand, we have

$$f_2'' = \frac{3}{4}\frac{\tilde{a}''b^2 - 2\tilde{a}'b'b + 2\tilde{a}(b')^2 - \tilde{a}bb''}{b^3} + O(x).$$

Because of (D.7), (D.12) and (D.13), we have

$$\frac{\tilde{a}''b^2 - 2\tilde{a}'b'b + 2\tilde{a}(b')^2 - \tilde{a}bb''}{b^3} = \frac{\frac{4}{3}a'b^2 - \frac{4}{3}ab'b}{b^3} = \frac{4}{3}.$$

Thus $f_2'' = 1 + O(x)$, which proves the second part of (7.15) as well as (D.1) for $j = 2$. This completes the proof. $\qquad\square$

E   Proof of Lemma 7.3

The functions g_j can be rewritten as

$$g_1(x) = \left(e^{-x}\cosh x - \eta_2 x + \eta_1 x^2\right)v(x),$$
$$g_2(x) = v(x),$$
$$g_3(x) = \left(xe^{-x}\cosh x - x^2\eta_2 + s^2\eta_1 x\right)v(x),$$
$$g_4(x) = \left(\frac{e^{-x}\sinh x}{x} + \eta_2\right)v(x),$$

where

$$v(x) = \frac{x}{\sinh 2x + 2x\eta_2}, \quad x > 0. \tag{E.1}$$

We estimate v first. Since $\eta_2 = \sinh(2s)/(2s) \geq 1$, we have

$$|v(x)| \leq \frac{x}{\sinh 2x + 2x},$$

and hence

$$|v(x)| \lesssim (1+x)e^{-2x}. \tag{E.2}$$

By straight-forward computations, one can see that

$$v'(x) = \gamma_1(x)(v(x))^2, \quad v''(x) = \gamma_2(x)(v(x))^3,$$

where

$$\gamma_1(x) := \frac{\sinh 2x - 2x \cosh 2x}{x^2},$$

$$\gamma_2(x) := \frac{2(3x + x \cosh 4x - \sinh 4x) + 4\eta_2(2x \cosh 2x - (1 + 2x^2) \sinh 2x)}{x^3}.$$

By Taylor expansions, it is easy to see that γ_1 and γ_2 are bounded if $0 < x \leq 1$. It is also easy to see that $|\gamma_1(x)| \lesssim x^{-1} e^{2x}$ and $|\gamma_2(x)| \lesssim x^{-2} e^{4x}$ if $1 < x < \infty$. Putting these estimates together, we have

$$|\gamma_1(x)| \lesssim \frac{e^{2x}}{1+x}, \quad |\gamma_2(x)| \lesssim \frac{e^{4x}}{1+x^2}, \quad 0 < x < \infty.$$

Then, from (E.2), we obtain

$$|v'(x)| \lesssim \frac{e^{2x}}{1+x}|v(x)|^2 \lesssim (1+x)e^{-2x}, \tag{E.3}$$

$$|v''(x)| \lesssim \frac{e^{4x}}{1+x^2}|v(x)|^3 \lesssim (1+x)e^{-2x}. \tag{E.4}$$

Since $g_2 = v$, the estimate (7.43) for $j = 2$ is already proved. Let us prove it for $j = 1$. We are ready to estimate g_j and their derivatives. We consider g_1 only for simplicity. We write

$$g_1(x) = \gamma(x)v(x), \quad \text{where} \quad \gamma(x) = e^{-x} \cosh x - \eta_2 x + \eta_1 x^2.$$

It is easy to show that

$$|\gamma(x)| \lesssim 1 + x^2, \quad |\gamma'(x)| \lesssim 1 + x, \quad |\gamma''(x)| \lesssim 1, \tag{E.5}$$

and the estimate (7.43) for $j = 1$ is an easy consequence of (E.2)-(E.5). (7.43) for $j = 3, 4$ can be proved in the same way. $\square$

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