Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

Localized sensitivity analysis at high-curvature boundary points of reconstructing inclusions in transmission problems

H. Ammari and Y.T. Chow and H. Liu

Research Report No. 2020-01

January 2020
Latest revision: January 2020

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 200021--172483

LOCALIZED SENSITIVITY ANALYSIS AT HIGH-CURVATURE BOUNDARY POINTS OF RECONSTRUCTING INCLUSIONS IN TRANSMISSION PROBLEMS

HABIB AMMARI, YAT TIN CHOW, AND HONGYU LIU

Abstract. In this paper, we are concerned with the recovery of the geometric shapes of inhomogeneous inclusions from the associated far field data in electrostatics and acoustic scattering. We present a local resolution analysis and show that the local shape around a boundary point with a high magnitude of mean curvature can be reconstructed more easily and stably. In proving this, we develop a novel mathematical scheme by analyzing the generalized polarisation tensors (GPTs) and the scattering coefficients (SCs) coming from the associated scattered fields, which in turn boils down to the analysis of the layer potential operators that sit inside the GPTs and SCs via microlocal analysis. In a delicate and subtle manner, we decompose the reconstruction process into several steps, where all but one steps depend on the global geometry, and one particular step depends on the mean curvature at a given boundary point. Then by a sensitivity analysis with respect to local perturbations of the curvature of the boundary surface, we establish the local resolution effects. Our study opens up a new field of mathematical analysis on wave super-resolution imaging.

Keywords: electrostatics and wave scattering, inverse inclusion problems, mean curvature, localized sensitivity, super-resolution, layer potential operators, microlocal analysis
2010 Mathematics Subject Classification: 35R30, 35J25, 86A20, 35A27, 31B10, 58J60

1. Introduction

We are concerned with the recovery of inhomogeneous inclusions from measurement of the scattered fields in electrostatics and acoustic scattering. We are particularly interested in studying and analysing how the mean curvature of a shape of an inhomogeneity would affect propagation of information of the shape via the scattered field. Let D signify the shape of an inhomogeneity. We show that information from points $x \in \partial D$ with high magnitude of mean curvature $|H(x)|$ propagates with a significantly larger magnitude. This is reflected by the sensitivity analysis of the scattered field with respect to the change of the shape. Indeed, we can localize our analysis at those boundary points with high mean curvature. Such larger sensitivity of information allows one more easily to locate these points of high mean curvature and also in a stable manner to reconstruct the local shape around these points.

The study of the correspondence between the geometry of an inhomogeneous inclusion and its scattered field has attracted significant attention in the literature. It can also find important applications in practice including medical imaging and geophysical exploration [8, 22]. From a physical intuition, "pathological" geometries should help to improve the transmission of scattering information and hence enhance the imaging effect. One of the "pathological" geometries that has been studied extensively in the literature is the corner/edge singularity, where the surface tangential vectors are discontinuous. In [19], it is shown that the corner singularity of an inhomogeneity always scatters a probing field nontrivially and in [16], the authors further quantified the result by establishing a positive lower bound of the scattering energy. That means, a corner singularity of an inhomogeneous inclusion can always generate significant scattering, and this is consistent with the aforementioned physical intuition. From a geometric perspective, a corner singularity indicates that the "extrinsic" curvature is infinity. Hence, it is natural to consider the scattering due to curvatures. In [17], the scattering by curvatures was considered and it is shown that if there is a boundary point on a generic inclusion with a sufficiently high curvature, then it scatters every probing field nontrivially. It is remarked that in [17], the "pathological" boundary point possesses both high mean and Gaussian curvatures. We would also like to mention in passing the related study in the literature on the scattering from impenetrable obstacles, or the so-called cavities, with "pathological" geometries; see [21, 28–30] and the references therein. In particular, in [30] the recovery of the boundary curvature of a convex acoustic obstacle from the associated high-frequency scattered fields was established and in [21] characterisations of the generalized polarisation tensors (GPTs) [8, 9] of the scattered field due to corner singularities of an insulating cavity in electrostatics were derived.

In this paper, we rigorously reinforce the aforementioned physical intuition from a reconstruction perspective by performing sensitivity analysis of the reconstruction around the boundary point with high mean curvature. That means, we include the corner/edge singularity as an extreme case. We consider our study for the electrostatics and the wave scattering in the quasi-static regime, where the reconstructions are severely more ill-conditioned than the corresponding high-frequency reconstruction. Indeed, we know that the corresponding reconstructions are exponentially ill-posed [14, 15, 23]. One of the major findings in our study can be roughly described as follows by taking the reconstruction in electrostatics for the discussion. The generalized polarised tensors (GPTs) of the scattered field are a natural and powerful shape descriptor of the underlying inclusion [7, 8]. It is a fact that the high-frequency information of the shape of the inclusion, namely the fine details of shape, enters into the higher order GPTs. Thus the boundary information around the high-curvature point enters into the high-order GPTs. However, GPTs decay exponentially; and hence, as the scattered field propagates away from the inclusion, the fine-detail information of the inclusion becomes less visible and will be contaminated by the noise. However, if we have very large magnitude of high curvature information, these higher order GPTs, albeit exponentially decay, will be pushed up to relatively high magnitudes, making them more apparent. Therefore after a further perturbation around such a point of high curvature, the fine details near it will be more apparent in the far field and stably reconstructable. Hence, it is unobjectionable to claim that one can produce super-resolution reconstruction of the inclusion around the high curvature point. On the other hand, it is emphasized that in this work we are not suggesting a new reconstruction method. In fact stable ways of reconstructing inclusions using the GPTs via the optimisation approach can be found in [1, 5, 6, 10]. In our study, Newton's method is considered, but the aim of mentioning that is rather to analyse the local sensitivity of the scattered field measurement at the high-curvature point.

Although it is physically intuitive to expect that the local geometry of the shape of an inclusion should have an effect on the local resolution, it turns out that the corresponding derivation is highly technical. In fact, decoding the local geometric information of an inclusion from the corresponding scattered field is highly challenging. In this paper, we develop a novel mathematical scheme to understand this correspondence through analyzing the GPTs and scattering coefficients (SCs) coming from the associated scattered fields, which in turn boils down to the analysis of the operators that sit inside the GPTs and SCs via microlocal analysis. By doing so, we are able to decompose the reconstruction process into several steps in a delicate and subtle manner, where all but one steps depend only on the global geometry, and one particular step that depends on the mean curvature of the surface at that point (c.f. Corollary 2.11). Finally, we are able to see clearly how sensitivity of local perturbations relates to local curvature information of the surface. Our study has important applications in super-resolution wave imaging.

The rest of the paper is organised as follows. In Section 2, we consider the electrostatic transmission problem. We compute the semi-classical symbols of several related operators on the boundary of the inclusion including the Neumann-Poincaré operator and its variants, and perform a sensitivity analysis on the generalized polarization tensors and hence the scattering coefficients. We establish an increased sensitivity at points with mean curvature of high magnitude. Then we move on to the inverse wave scattering in the low-frequency regime governed by the Helmholtz system with a small wavenumber and observe a similar property in Section 3.

2. Localized sensitivity analysis for reconstructions in electrostatics

In this section, we consider the reconstruction of an inhomogeneous inclusion in electrostatics. We first introduce the electrostatic transmission problem as well as the associated layer potential operators that are crucial in our subsequent analysis. We compute the semi-classical symbols of those operators when viewing them as pseudo-differential operators. Then we conduct the localised sensitivity analysis at the high-curvature point on the boundary of the inclusion.

2.1. **Electrostatic transmission problem and layer-potential operators.** We introduce the electrostatic transmission problem and the associated layer-potential operators. Consider an open connected domain D with a $\mathcal{C}^{2,\alpha}$, $0<\alpha<1$, boundary ∂D and a connected complement $\mathbb{R}^d\backslash\overline{D}$, $d\geq 2$. Physically, D is the support of an inhomogeneous dielectric inclusion. Let ε_c and ε_m be two positive constants, signifying the electric permittivities. Consider a medium configuration as follows,

$$\varepsilon_D=\varepsilon_c\chi(D)+\varepsilon_m\chi(\mathbb{R}^d\backslash\overline{D}), \tag{2.1}$$

where and also in what follows, χ stands for the characteristic function of a domain. Let u_0 be a given harmonic function that signifies a probing field of the inclusion D. The electrostatic transmission problem is given for a potential field $u\in H^1_{loc}(\mathbb{R}^d)$ as follows,

$$\begin{cases}\nabla\cdot(\varepsilon_D\nabla u)=0 & \text{in } \ \mathbb{R}^d,\\ u-u_0=o(|x|^{1-d}) & \text{as } \ |x|\to\infty.\end{cases} \tag{2.2}$$

We proceed to introduce the single-layer potential operator and the Neumann-Poincaré operator associated with (2.2). They are crucial in solving (2.2) via the layer-potential theory, and moreover they provide critical ingredients in solving the inverse problem of reconstructing the inclusion D from the associated scattered field $u-u_0$.

Given a density function $\phi\in L^2(\partial D)$, the single-layer and double-layer potentials, $\mathcal{S}_{\partial D}[\phi]$ and $\mathcal{D}_{\partial D}[\phi]$, are respectively defined as follows,

$$\mathcal{S}_{\partial D}[\phi](x) := \int_{\partial D} G(x-y)\phi(y)d\sigma(y), \tag{2.3}$$

$$\mathcal{D}_{\partial D}[\phi](x) := \int_{\partial D} \frac{\partial}{\partial\nu_y}G(x-y)\phi(y)d\sigma(y), \tag{2.4}$$

for $x\in\mathbb{R}^d$, where G is the fundamental solution of the Laplacian in $\mathbb{R}^d$:

$$G(x-y)=\begin{cases}-\frac{1}{2\pi}\log|x-y| & \text{if } \ d=2\,,\\ \frac{1}{(2-d)\varpi_d}|x-y|^{2-d} & \text{if } \ d>2\,,\end{cases} \tag{2.5}$$

with ϖ_d denoting the surface area of the unit sphere in $\mathbb{R}^d$. The single-layer potential satisfies the following jump relation across ∂D:

$$\frac{\partial}{\partial \nu}\left(\mathcal{S}_{\partial D}[\phi]\right)^{\pm} = (\pm\frac{1}{2}I + \mathcal{K}^*_{\partial D})[\phi]\,, \tag{2.6}$$

where the superscripts $\pm$ indicate the limits from outside and inside D respectively, and $\mathcal{K}^*_{\partial D} : L^2(\partial D) \to L^2(\partial D)$ is the Neumann-Poincaré operator defined by

$$\mathcal{K}^*_{\partial D}[\phi](x) := \frac{1}{\varpi_d}\int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^d}\phi(y)d\sigma(y)\,, \tag{2.7}$$

with ν_x being the outward normal at $x \in \partial D$. It is noted that $\mathcal{K}^*_{\partial D}$ maps $L^2_0(\partial D)$ onto itself, where

$$L^2_0(\partial D) := \{\phi \in L^2(\partial D); \int_{\partial D} \phi\ d\sigma = 0\}.$$

The transmission problem (2.2) can be rewritten as

$$\begin{cases} \Delta u = 0 & \text{in } \ D\bigcup(\mathbb{R}^d\backslash\overline{D})\,, \\ u^+ = u^- & \text{on } \ \partial D\,, \\ \varepsilon_c\frac{\partial u^+}{\partial \nu} = \varepsilon_m\frac{\partial u^-}{\partial \nu} & \text{on } \ \partial D\,, \\ u - u_0 = O(|x|^{1-d}) & \text{as } \ |x| \to \infty\,. \end{cases} \tag{2.8}$$

With the help of the single-layer potential, one can rewrite the perturbation $u - u_0$, which is due to the inclusion D, as

$$u - u_0 = \mathcal{S}_{\partial D}[\phi]\,, \tag{2.9}$$

where $\phi \in L^2(\partial D)$ is an unknown density, and $\mathcal{S}_{\partial D}[\phi]$ signifies the refraction part of the potential in the presence of the inclusion. By virtue of the jump relation (2.6), solving the above system (2.8) is equivalent to solving the density function $\phi \in L^2(\partial D)$ of the following integral equation

$$\frac{\partial u_0}{\partial \nu} = \left(\frac{\varepsilon_c + \varepsilon_m}{2(\varepsilon_c - \varepsilon_m)}I - \mathcal{K}^*_{\partial D}\right)[\phi]\,. \tag{2.10}$$

This gives

$$u - u_0 = \mathcal{S}_{\partial D} \circ (\lambda I - \mathcal{K}^*_{\partial D})^{-1}\left[\frac{\partial u_0}{\partial \nu}\right]\,, \tag{2.11}$$

where

$$\lambda := \frac{\varepsilon_c + \varepsilon_m}{2(\varepsilon_c - \varepsilon_m)}.$$

The invertibility of the operator $(\frac{\varepsilon_c+\varepsilon_m}{2(\varepsilon_c-\varepsilon_m)}I - \mathcal{K}^*_{\partial D})$ from $L^2(\partial D)$ onto $L^2(\partial D)$ and from $L^2_0(\partial D)$ onto $L^2_0(\partial D)$ is proved (cf. [8,26]), provided that $|\frac{\varepsilon_c+\varepsilon_m}{2(\varepsilon_c-\varepsilon_m)}| > 1/2$ via the Fredholm alternative, which holds when the constants ε_c and ε_m are positive.

From (2.11), we see that in order to understand the quantitative behaviour of the scattered field, one needs to investigate the mapping properties of the Neumann-Poincaré operator. Since ∂D is $\mathcal{C}^{2,\alpha}$, the operator $\mathcal{K}^*_{\partial D} : L^2(\partial D) \to L^2(\partial D)$ is compact (cf. [4]), and its spectrum is discrete and accumulates at zero. All the eigenvalues are real and bounded by $1/2$. Moreover, $1/2$ is always an eigenvalue and its associated eigenspace is of dimension one, which is nothing else but the kernel of the single-layer potential $\mathcal{S}_{\partial D}$. In two dimensions, it is proved that if $\lambda_i \neq 1/2$ is an eigenvalue of $\mathcal{K}^*_{\partial D}$, then $-\lambda_i$ is an eigenvalue as well. This property is known as the twin spectrum property; see [31]. The Fredholm eigenvalues are the eigenvalues of $\mathcal{K}^*_{\partial D}$. It is easy to see, from the properties of $\mathcal{K}^*_{\partial D}$, that they are invariant with respect to rigid motions and scaling. They can be explicitly computed for ellipses and spheres. In fact, if a and b denote the semi-axis lengths of an ellipse then it can be shown that $\pm(\frac{a-b}{a+b})^i$ are the Fredholm eigenvalues [27]. For the sphere, they are given by $1/(2(2i+1))$; see [25]. Some other computations of Neumann-Poincaré eigenvalues for different shapes can be found in [18, 20, 24]. In three dimensions, in [32, 33], it is derived that

$$\lambda_j(\mathcal{K}^*_{\partial D}) \sim \left\{ \frac{3\int\limits_{\partial D} H^2(x)d\sigma_x - \int\limits_{\partial D} K(x)d\sigma_x}{128\pi} \right\}^{\frac{1}{2}} j^{-\frac{1}{2}}, \tag{2.12}$$

where λ_j denotes the j-th Neumann-Poincaré eigenvalue, and $H(x)$ and $K(x)$ are respectively the mean and Gaussian curvatures at the point $x \in \partial D$. Therefore, one sees that the magnitude of λ_j not only has a decay order, but also depends on a constant related to the curvature of the inclusion.

From (2.12), it is natural to expect that the curvature of the boundary of the inclusion should also enter into the scattered field in an explicit way. In fact, in what follows, we shall establish such an explicit dependence locally at a boundary point with a high magnitude of mean curvature. It turns out that the corresponding derivation is technical and tricky. For that purpose, we need to introduce the so-called generalized polarisation tensor (GPT) in arbitrary dimensions in the next subsection.

2.2. **Generalized polarization tensor in arbitrary dimensions.** For $|x| > |y|$, one has

$$\begin{aligned} \Gamma(x-y) =& c_d \sum_{k=0}^{\infty} \frac{|y|^k}{|x|^{k+d-2}} C_{d,k}^{(\frac{d-2}{2})}(\langle \omega_x \,,\, \omega_y \rangle) \\ =& \sum_{k=0}^{\infty} c_{d,k} \frac{|y|^k}{|x|^{k+d-2}} \sum_{|l_1|\le l_2 \le \ldots \le l_{d-1}=k} Y_{l_1,\ldots,l_{d-1}}(\omega_x)\,\overline{Y_{l_1,\ldots,l_{d-1}}(\omega_y)}, \end{aligned} \tag{2.13}$$

where $\omega_x := x/|x|, \omega_y = y/|y| \in \mathbb{S}^{d-1}$, $c_d, c_{k,d}$ are some dimensional constants, $C_{d,k}^{(\frac{d-2}{2})}$ are the Gegenbauer polynomials (which are generalization of Legandre polynomials when $d = 3$) and $Y_{l_1,\ldots,l_{d-1}}(\omega)$ with $|l_1| \le l_2 \le \ldots \le l_{d-1}$ are the spherical harmonics. Similar to the expansions given by generalized polarisation tensors in two and three dimensions [5, 8], by virtue of (2.11) and (2.13), one can expand the scattered potential for all $|x| > \sup\{|x| : x \in D\}$ as

$$\begin{aligned} (u-u_0)(x) = & \sum_{k=0}^{\infty} \sum_{|l_1|\le l_2 \le \ldots \le l_{d-1}=k} c_{d,k}\, |x|^{-k-d+2}\, Y_{l_1,\ldots,l_{d-1}}(\omega_x) \\ & \times \int_{\partial D} |y|^k \overline{Y_{l_1,\ldots,l_{d-1}}(\omega_y)} \left\{ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \left[\frac{\partial u_0}{\partial \nu} \right] \right\} (y) d\sigma(y). \end{aligned} \tag{2.14}$$

The generalized polarisation tensor (GPT) is obtained by choosing incident harmonic function $u_0(x) = |x|^k Y_{l_1,\ldots,l_{d-1}}(\omega_y)$ and then taking the coefficient with respect to the function $Y_{l_1,\ldots,l_{d-1}}(\omega_x)$ in (2.14). That is,

Definition 2.1. *The generalized polarisation tensors (GPTs) of dimension d with a given λ and a domain $D \subset \mathbb{R}^d$ with a $\mathcal{C}^{2,\alpha}$ boundary are defined as*

$$\begin{aligned} & \mathcal{M}_{(l_1,\ldots,l_{d-1}),(m_1,\ldots,m_{d-1})}(\lambda, D) \\ := & \int_{\partial D} |y|^{l_{d-1}} \overline{Y_{l_1,\ldots,l_{d-1}}(\omega_y)} \left\{ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \left[\partial_\nu \left(r^{m_{d-1}} Y_{m_1,\ldots,m_{d-1}}(\omega) \right) \right] \right\} (y) d\sigma(y), \end{aligned} \tag{2.15}$$

where $|l_1| \le l_2 \le \ldots \le l_{d-1}$ and $|m_1| \le l_2 \le \ldots \le m_{d-1}$.

By writing $L = (l_1, \ldots, l_{d-1})$ and $I_k = \{L : l_1| \le l_2 \le \ldots \le l_{d-1} = k\}$, we handily obtain the following lemma.

Lemma 2.2. *Consider a domain $D \subset \mathbb{R}$ of $\mathcal{C}^{2,\alpha}$ class. The solution to (2.2) with*

$$u_0(x) = \sum_{k=0}^{\infty} \sum_{M \in I_k} a_M \, r_x^k \, Y_M(\omega_x)$$

and $|x| > \sup\{|x| : x \in D\}$ is given by

$$(u - u_0)(x) = \sum_{k=0}^{\infty} \sum_{L \in I_k} \sum_{n=0}^{\infty} \sum_{M \in I_n} c_{d,k} \, a_M \, |x|^{-k-d+2} \, Y_L(\omega_x) \, \mathcal{M}_{L,M}(\lambda, D) \,.$$

Hence, for $x \in R \cdot \mathbb{S}^{d-1}$ with $R > \sup\{|x| : x \in D\}$, one has

$$\mathcal{M}_{L,M}(\lambda, D) = \frac{1}{c_{d,k}} |R|^{2k+d-2} \int_{\mathbb{S}^{d-1}} \overline{Y_L(\omega_x)} \Big(u - r^n Y_M(\omega) \Big)(R\,\omega_x) d\omega_x \,. \tag{2.16}$$

Notice that this definition extends the definition of genearlized polarization tensors to an arbitrary dimension d. Moreover, it is easy to show that the transformation rules and decaying properties in high dimensions are similar to those in [5,8]. The above lemma indicates that the scattering information is fully encoded in the GPTs, $\mathcal{M}_{(l_1,\dots,l_{d-1}),(m_1,\dots,m_{d-1})}(\lambda, D) = \mathcal{M}_{L,M}(\lambda, D)$.

2.3. **Sensitivity analysis of the Neumann-Poincaré operator.** In this section, we present the shape derivative of the Neumann-Poincaré operator (2.7) associated with a shape D sitting inside a general space $\mathbb{R}^d$ for any $d \geq 2$. The special two-dimensional case was first treated in [12], and the general case was considered in [1]. Since this result is of fundamental importance for our future analysis, we shall briefly derive it here for the sake of completeness.

Given a shape D sitting inside $\mathbb{R}^d$, we consider a regular parametrization of the surface ∂D as

$$\begin{aligned} \mathbb{X} : U \subset \mathbb{R}^{d-1} \quad &\to \quad \partial D \subset \mathbb{R}^d, \\ u = (u_1, u_2, ..., u_{d-1}) \quad &\mapsto \quad \mathbb{X}(u) \,. \end{aligned}$$

For notational sake, we often write the vector $\mathbb{X}_i := \frac{\partial \mathbb{X}}{\partial u_i}$. For a given $d-1$ vector $\{v_i\}_{i=1}^{d-1}$, we denote the $d-1$ cross product $\times_{i=1}^{d-1} v_i = v_1 \times v_2 ... \times v_{d-1}$ as the dual vector of the functional $\det(\,\cdot\,, v_1, v_2, ..., v_{d-1})$, i.e., $\langle w, \times_{i=1}^{d-1} v_i \rangle = \det(w, v_1, v_2, ..., v_{d-1})$ for any w, which is guaranteed to exist by the Reisz representation theorem. Then, from the fact that $\mathbb{X}$ is regular, we know $\times_{i=1}^{d-1} \mathbb{X}_i$ is non-zero, and the normal vector $\nu := \times_{i=1}^{d-1} \mathbb{X}_i / | \times_{i=1}^{d-1} \mathbb{X}_i|$ is well-defined.

Now we consider an ε-perturbation of D, namely ∂D_ε given by

$$\partial D_\varepsilon := \{ \widetilde{x} \,\big|\, \widetilde{x} = x + \varepsilon h(u) \nu(x) \,,\, x \in \partial D \} \,, \tag{2.17}$$

with $h \in \mathcal{C}^{2,\alpha}(\partial D)$. Let $\Psi_\varepsilon(x) := x + \varepsilon h(u)\nu(x)$ be the diffeomorphism from ∂D to ∂D_ε. It is directly verified that

$$\begin{aligned} \mathbb{X}^\varepsilon : U \subset \mathbb{R}^{d-1} \quad &\to \quad \partial D_\varepsilon \subset \mathbb{R}^d, \\ u = (u_1, u_2, ..., u_{d-1}) \quad &\mapsto \quad \Psi_\varepsilon[u] = \mathbb{X}(u) + \varepsilon h(u) \nu(\mathbb{X}(u)) \,, \end{aligned}$$

is a regular parametrization over ∂D_ε for sufficiently small $\varepsilon \in \mathbb{R}_+$. Writing g to be the induced metric on ∂D from $\mathbb{R}^d$, directly from the definition, we have $\mathbb{X}_i^\varepsilon = \mathbb{X}_i + \varepsilon \frac{\partial h}{\partial u_i} \nu + \varepsilon h \sum_{j=1}^{d-1} \sum_{k=1}^{d-1} g^{ik} A_{kj} \mathbb{X}_j$, where the matrix A_{ij} is defined as

$$A := (A_{ij}) = \langle \mathbf{II}(\mathbb{X}_i, \mathbb{X}_j), \nu \rangle \,,$$

and $\mathbf{II}$ is the second fundamental form given by

$$\begin{aligned} \mathbf{II} : T(\partial D) \times T(\partial D) &\to T^{\perp}(\partial D), \\ \mathbf{II}(v,w) &= -\langle \bar{\nabla}_v \nu, w\rangle \nu = \langle \nu, \bar{\nabla}_v w\rangle \nu, \end{aligned}$$

where $\bar{\nabla}$ is the standard covariant derivative on the ambient space $\mathbb{R}^d$. From the multi-linearity and alternating property of the $d-1$ cross product, we can readily calculate at any point $b \in U$ that

$$\begin{aligned} \times_{i=1}^{d-1} \mathbb{X}_i^{\varepsilon}(b) &= \times_{i=1}^{d-1} \left(\mathbb{X}_i + \varepsilon \frac{\partial h}{\partial u_i} \nu + \varepsilon h \sum_{j,k=1}^{d-1} g^{ik} A_{kj} \mathbb{X}_j \right) \\ &= (1 + \varepsilon h(b) \mathrm{tr}_g(A)(b)) \left(\times_{i=1}^{d-1} \mathbb{X}_i(b) \right) + O(\varepsilon^2), \end{aligned}$$

where the constant in large O is bounded by $|A(\mathbb{X}(b))|$ at the point $\mathbb{X}(b)$ and $||h||_{\mathcal{C}^1}$ and

$$\mathrm{tr}_g(A)(b) := \mathrm{tr}_g(A)(\mathbb{X}(b)) = \sum_{j,k=1}^{d-1} g^{jk} A_{kj}(\mathbb{X}(b)) := (d-1) H(\mathbb{X}(b)),$$

with $(g^{ij}) = g^{-1}$ and $H(\mathbb{X}(b))$ being the mean curvature at the point $\mathbb{X}(b)$. Hence it yields that

$$\begin{aligned} \langle \cdot , \nu^{\varepsilon}(b) \rangle \, d\sigma^{\varepsilon}(b) &= \langle \cdot , \times_{i=1}^{d-1} \mathbb{X}_i^{\varepsilon}(b) \rangle \, db \\ &= \left\langle \cdot , (1 + \varepsilon h(b) \mathrm{tr}_g(A)(b)) \left(\times_{i=1}^{d-1} \mathbb{X}_i(b) \right) \right\rangle db + O(\varepsilon^2) \\ &= (1 + \varepsilon h(b) \mathrm{tr}_g(A)(b)) \langle \cdot , \nu(b) \rangle \, d\sigma^{\varepsilon}(b) + O(\varepsilon^2), \end{aligned} \tag{2.18}$$

where $\nu^{\varepsilon}(b)$ denotes the normal vector at $\mathbb{X}^{\varepsilon}(b)$. Moreover, for two arbitrary points $x, y \in \partial D$ given by $x = \mathbb{X}(a), y = \mathbb{X}(b)$ for some $a, b \in U$, we have

$$\mathbb{X}^{\varepsilon}(a) - \mathbb{X}^{\varepsilon}(b) = \mathbb{X}(a) - \mathbb{X}(b) + \varepsilon K(a,b)[h], \tag{2.19}$$

where $K(a,b)[h] := h(a)\nu(a) - h(b)\nu(b)$. Hence, by the Taylor expansion of $|\mathbb{X}^{\varepsilon}(a) - \mathbb{X}^{\varepsilon}(b)|^{-d}$ in ε, it follows that

$$\begin{aligned} &|\mathbb{X}^{\varepsilon}(a) - \mathbb{X}^{\varepsilon}(b)|^{-d} \\ =&|\mathbb{X}(a) - \mathbb{X}(b)|^{-d} - d\,\varepsilon |\mathbb{X}(a) - \mathbb{X}(b)|^{-d-2} \langle \mathbb{X}(a) - \mathbb{X}(b), K(a,b)[h] \rangle + O(\varepsilon^2). \end{aligned} \tag{2.20}$$

Combining (2.18) and (2.20), we obtain the following series expression

$$\frac{\langle \mathbb{X}^{\varepsilon}(a) - \mathbb{X}^{\varepsilon}(b) \, , \nu^{\varepsilon}(b) \rangle}{|\mathbb{X}^{\varepsilon}(a) - \mathbb{X}^{\varepsilon}(b)|^d} \, d\sigma^{\varepsilon}(b) := \sum_{n=0}^{\infty} \varepsilon^n \mathbb{K}_{h,n}(a,b) \, d\sigma(b),$$

where

$$\begin{aligned} \mathbb{K}_{h,0}(a,b) &:= \frac{\langle \mathbb{X}(a) - \mathbb{X}(b) \, , \nu(b) \rangle}{|\mathbb{X}(a) - \mathbb{X}(b)|^d} \\ \mathbb{K}_{h,1}(a,b) &:= \frac{\langle \mathbb{X}(a) - \mathbb{X}(b) \, , h(b) \mathrm{tr}(A)(b) \nu(b) \rangle + \langle K(a,b)[h], \nu(b) \rangle}{|\mathbb{X}(a) - \mathbb{X}(b)|^d} \\ &\quad -d \, \frac{\langle \mathbb{X}(a) - \mathbb{X}(b) \, , K(a,b)[h] \rangle \langle \mathbb{X}(a) - \mathbb{X}(b), \nu(b) \rangle}{|\mathbb{X}(a) - \mathbb{X}(b)|^{d+2}}, \end{aligned}$$

and the higher order terms $\mathbb{K}_{h,i}$ can be explicitly calculated from (2.20) in a similar fashion. Therefore we see that the kernel of the Neumann-Poincaré operator varies analytically with respect to ε along any direction $h \in \mathcal{C}^{2,\alpha}(\partial D)$.

Starting from now on, whenever the context is clear, by an abuse of notation we shall not distinguish between $F(x)$ and $F(a)$ for any function F over ∂D if $x = \mathbb{X}(a) \in \partial D$. Now we define a sequence of integral operators $\mathcal{K}^{(n)}_{D,h}$: $L^2(\partial D) \to L^2(\partial D)$ by

$$\mathcal{K}^{(n)}_{D,h}[\phi](x) := \int_{\partial D} \mathbb{K}_{h,n}(x,y)\phi(y)d\sigma(y),\ \forall\, \phi \in L^2(\partial D)\,, \tag{2.21}$$

for $n \geq 0$. Notice here that the notion $\mathbb{K}_{h,0}(x,y)$ is nothing but the kernal of $\mathcal{K}^*_{\partial D}$ itself. Then we can directly obtain the following result from (2.21).

Theorem 2.3. *For $N \in \mathbb{N}$, there exists a constant C depending only on N, $||\mathbb{X}||_{\mathcal{C}^2}$ and $||h||_{\mathcal{C}^2}$ such that the following estimate holds for any $\widetilde{\phi} \in L^2(\partial D_\varepsilon)$ and $\phi := \tilde{\phi} \circ \Psi_\varepsilon$:*

$$\left|\left|\mathcal{K}^*_{\partial D_\varepsilon}[\tilde{\phi} \circ \Psi_\varepsilon] - \mathcal{K}^*_{\partial D}[\phi] - \sum_{n=1}^{N} \varepsilon^n \mathcal{K}^{(n)}_{D,h}[\phi]\right|\right|_{L^2(\partial D)} \leq C\varepsilon^{N+1}||\phi||_{L^2(\partial D)}\,. \tag{2.22}$$

In particular, the kernel of $\mathcal{K}^{(1)}_{D,h}$ can be explicitly expressed by

$$\begin{aligned}\mathbb{K}_{h,1}(x,y) =& \frac{\langle x-y\,,\nu(y)\rangle h(y)tr_g(A)(y) + \langle K(x,y)[h], \nu(y)\rangle}{|x-y|^d}\\ &- d\,\frac{\langle x-y\,, K(x,y)[h]\rangle\langle x-y, \nu(y)\rangle}{|x-y|^{d+2}}\,,\end{aligned}$$

where $K(x,y)[h] := h(x)\nu(x) - h(y)\nu(y)$ and $tr_g(A)(y) = (d-1)H(y)$ with $H(y)$ denoting the mean curvature of the surface at y.

2.4. **$\mathcal{K}^*_{\partial D}$ and $\mathcal{K}^{(1)}_{D,h}$ as pseudo-differential operators.** In this subsection, we derive some crucial properties of $\mathcal{K}^*_{\partial D}$ and $\mathcal{K}^{(1)}_{D,h}$, verifying that they are pseudo-differential operators with particular orders and obtain their principal symbols.

First, we note that for a fixed $x \in \partial\Omega$, if we take the geodesic normal coordinate $v \in T_x(\partial\Omega) \cong \mathbb{R}^{d-1} \mapsto \mathbb{X}(v) := \exp_x(v) \in \partial\Omega$, then we have $g_{ij}(x) = \delta_{ij}$ and $\Gamma^k_{ij}(x) = 0$, where Γ^k_{ij} are the Christoffel symbols. For $y = \exp_x(\delta\,\omega)$ with $|\,\omega\,| = 1$, we have

$$\begin{aligned}\nu(y) &= \nu(x) + \delta A(x)\omega + \delta^2\frac{1}{2}[\partial_\omega A(x)\omega + A(x)A(x)\omega + |A(x)\omega|^2\nu(x)] + O(\delta^3)\,,\\ \sqrt{\det(g(y))} &= 1 + \delta^2\,\mathrm{Ric}(\omega,\omega) + O(\delta^3)\,.\end{aligned}$$

Therefore we obtain the following expansion for the kernel of $\mathcal{K}^*_{\partial D}$,

$$\mathbb{K}_{h,0}(x,y)d\sigma(y) \;=\; \delta^{-d+2}\,\langle A(x)\,\omega,\,\omega\rangle\,dy + O(\delta^{-d+3}).$$

Following [32, 33], via a Fourier transform of the kernel of $\mathbb{K}_{h,0}(x,y)$ with respect to $v := \delta\omega$, we obtain the symbol around $x = y$ in the geodesic normal coordinate (noting that $g_{ij}(x) = \delta_{ij}$) as

$$\begin{aligned}&p_{\mathcal{K}^*_{\partial D}}(x,\xi) := \mathcal{F}_v\left[\langle A(x)\,\omega,\,\omega\rangle\,|v|^{-d}\right](\xi) + O(|\xi|^{-2})\\ &=\sum_{i,j=1}^{d-1} A_{ij}(x)\,\mathcal{F}_v\left[v_iv_j\,|v|^{-d}\right](\xi) + O(|\xi|^{-2}) = \sum_{i,j=1}^{d-1} A_{ij}(x)\,\partial_i\partial_j\,|\,\xi\,| + O(|\xi|^{-2})\\ &=\sum_{i,j=1}^{d-1} A_{ij}(x)\,\left(\frac{\delta_{ij}}{|\xi|} - \frac{\xi_i\xi_j}{|\xi|^3}\right) + O(|\xi|^{-2})\\ &=(d-1)H(x)\,|\xi|^{-1} - \langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-3} + O(|\xi|^{-2})\,.\end{aligned}$$

Therefore $\mathcal{K}^*_{\partial D}$ is a pseudodifferential operator of order -1 on ∂D and hence in the Schatten p class S_p for $p > d-1$ for $d > 2$ via the Weyl asymptotics. We summerize the above discussion in the following theorem, which generalizes the three-dimensional result in [32,33].

Theorem 2.4. *The operator $\mathcal{K}^*_{\partial D}$ is a pseudodifferential operator of order -1 on ∂D if $\partial D \in \mathcal{C}^{2,\alpha}$ with its symbol given as follows in the geodesic normal coodinate around each point x:*

$$p_{\mathcal{K}^*_{\partial D}}(x,\xi) = (d-1)H(x)\,|\xi|^{-1} - \langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-3} + O(|\xi|^{-2})\,,$$

where the large O depends on $\|\mathbb{X}\|_{\mathcal{C}^2}$. Hence $\mathcal{K}^*_{\partial D}$ is a compact operator of Schatten p class S_p for $p > d-1$ for $d > 2$.

A remark is that the above result holds also for $\mathcal{K}_{\partial D}$ instead of $\mathcal{K}^*_{\partial D}$ when we only look at the first order term. We would also like to remark that if the geodesic normal coordinate is not chosen, and for a general coordinate, tracing back the above steps, we have instead

$$p_{\mathcal{K}^*_{\partial D}}(x,\xi) \;=\; (d-1)H(x)\,|\xi|^{-1}_{g(x)} - \langle A(x)\,g^{-1}(x)\,\xi,\,g^{-1}(x)\,\xi\rangle\,|\xi|^{-3}_{g(x)} + O(|\xi|^{-2}_{g(x)})\,.$$

The above remark is in force to indicate that our choice of geodesic normal coordinate is just for simplification of the resulting computation and is not a necessary move. In a similar manner, we can obtain:

Theorem 2.5. *Let $\partial D \in \mathcal{C}^{2,\alpha}$. The operator $\mathcal{K}^{(1)}_{D,h}$ defined in (2.21) can be decomposed as*

$$\mathcal{K}^{(1)}_{D,h} = \mathcal{K}^{(1)}_{D,h,0} + \mathcal{K}^{(1)}_{D,h,-1} + \mathcal{K}^{(1)}_{D,h,-2}\,, \tag{2.23}$$

where $\mathcal{K}^{(1)}_{D,h,0}$, $\mathcal{K}^{(1)}_{D,h,-1}$ and $\mathcal{K}^{(1)}_{D,h,-2}$ are pseudodifferential operators of order $0,-1,-2$ respectively on ∂D with their symbols given as follows in the geodesic normal coordinate around each point x:

$$\begin{aligned}
p_{\mathcal{K}^{(1)}_{D,h,0}}(x,\xi) \;&=\; -\partial_\xi h(x)|\,\xi\,|^{-1} = O(1)\,,\\
p_{\mathcal{K}^{(1)}_{D,h,-1}}(x,\xi) \;&=\; h(x)\Big\{(4d-1)|H(x)|^2\,|\,\xi\,|^{-1} + (5d+1)H(x)\langle A(x)\xi\,,\,\xi\rangle\,|\,\xi\,|^{-3}\\
&\quad +\frac{1}{2}|A(x)|^2_F\,|\,\xi\,|^{-1} + \frac{24\,d-1}{2}|A(x)\xi|^2\,|\,\xi\,|^{-3} - 9d\,|\,\langle A(x)\xi\,,\,\xi\rangle\,|^2\,|\,\xi\,|^{-5}\Big\}\\
&\quad -\frac{12d+1}{2}\,\Delta h(x)\,|\,\xi\,|^{-1} - \frac{1}{2}\mathrm{Hess}_{\xi,\xi}h(x)\,|\,\xi\,|^{-3}\\
&=\; O(|\xi|^{-1})\,,\\
p_{\mathcal{K}^{(1)}_{D,h,-2}}(x,\xi) \;&=\; O(|\xi|^{-2})\,,
\end{aligned}$$

where $|\cdot|_F$ is the Frobenius norm of the matrix and the constant of the large O depends on $\|\mathbb{X}\|_{\mathcal{C}^2}$ and $\|h\|_{\mathcal{C}^2}$. Moreover, $\mathcal{K}^{(1)}_{D,h,-1}$ is a compact operator of Schatten p class S_p for $p > d-1$ and $d > 2$.

Proof. By straightforward calculations, we can obtain the kernel of $\mathcal{K}^{(1)}_{D,h}$ in the geodesic normal coordinate as follows,

$$\begin{aligned}
&\mathbb{K}_{h,1}(x,y)d\sigma(y)\\
=\;& -\delta^{-d+1}\,\partial_\omega h(x)\,dy\\
&+\delta^{-d+2}\bigg(h(x)\,\Big[(d-1)H(x)\,\langle A(x)\,\omega,\,\omega\rangle - d|\,\langle A(x)\,\omega,\,\omega\rangle\,|^2 + \frac{1}{2}\,|\,A(x)\,\omega\,|^2\Big]\\
&-\frac{1}{2}\mathrm{Hess}_{\omega,\omega}h(x)\bigg)\,dy + O(\delta^{-d+3}),
\end{aligned}$$

where the constant of large O now depends on $||h||_{\mathcal{C}^3}$. Using the Fourier transform of the kernel $\mathbb{K}_{h,1}(x,y)$ with respect to $v := \delta\omega$, $\omega \in \mathbb{S}^{d-1}$, we obtain the symbol around $x = y$ in the geodesic normal coordinate (and noticing $\nabla_v v = 0$) that

$$
\begin{aligned}
& p_{\mathcal{K}^{(1)}_{D,h}}(x,\xi) \\
:= \; & -\mathcal{F}_v\left[\,|v|^{-d}\,\partial_v h(x)\right](\xi) \\
& +\mathcal{F}_v\Bigg[\,|v|^{-d-2}\Bigg(h(x)\left[(d-1)H(x)\,\langle A(x)\,v,\,v\rangle\,|v|^2 - d|\,\langle A(x)\,v,\,v\rangle\,|^2 + \frac{1}{2}\,|\,A(x)\,v\,|^2|v|^2\right] \\
& -\frac{1}{2}\mathrm{Hess}_{v,v}h(x)|v|^2\Bigg)\Bigg](\xi) + O(|\xi|^{-2}) \\
= \; & -\sum_{i=1}^{d-1}\partial_i h(x)\,\mathcal{F}_v\left[v_i\,|v|^{-d}\right](\xi) \\
& +\sum_{i,j=1}^{d-1}\left((d-1)h(x)H(x)\,A_{ij}(x) + h(x)\frac{1}{2}\sum_{k=1}^{d-1}A_{ik}(x)A_{kj} - \frac{1}{2}\partial_i\partial_j h(x)\right)\mathcal{F}_v\left[v_iv_j|v|^{-d}\right](\xi) \\
& -d\,h(x)\sum_{i,j,k,l=1}^{d-1}A_{ij}(x)A_{kl}(x)\,\mathcal{F}_v\left[v_iv_jv_kv_l\,|v|^{-d-2}\right](\xi) + O(|\xi|^{-2}) \\
= \; & -\sum_{i=1}^{d-1}\partial_i h(x)\,\partial_i\,|\,\xi\,| + \sum_{i,j=1}^{d-1}\Bigg((d-1)h(x)H(x)\,A_{ij}(x) + h(x)\frac{1}{2}\sum_{k=1}^{d-1}A_{ik}(x)A_{kj} \\
& -\frac{1}{2}\partial_i\partial_j h(x)\Bigg)\,\partial_i\partial_j\,|\,\xi\,| - d\,h(x)\sum_{i,j,k,l=1}^{d-1}A_{ij}(x)A_{kl}(x)\,\partial_i\partial_j\partial_k\partial_l\,|\,\xi\,|^3 + O(|\xi|^{-2})\,.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
p_{\mathcal{K}^{(1)}_{D,h}}(x,\xi) \; = \; & -\partial_\xi h(x)|\,\xi\,|^{-1} \\
& +h(x)\Bigg\{(4d-1)|H(x)|^2\,|\,\xi\,|^{-1} + (5d+1)H(x)\langle A(x)\xi\,,\,\xi\rangle\,|\,\xi\,|^{-3} \\
& +\frac{1}{2}|A(x)|_F^2\,|\,\xi\,|^{-1} + \frac{24\,d-1}{2}|A(x)\xi|^2\,|\,\xi\,|^{-3} - 9d\,|\,\langle A(x)\xi\,,\,\xi\rangle\,|^2\,|\,\xi\,|^{-5}\Bigg\} \\
& -\frac{12d+1}{2}\,\Delta h(x)\,|\,\xi\,|^{-1} - \frac{1}{2}\mathrm{Hess}_{\xi,\xi}h(x)\,|\,\xi\,|^{-3} \\
& +O(|\xi|^{-2}),
\end{aligned}
$$

which readily yields (2.23).

The proof is complete. □

2.5. **A property of the generalized polarisation tensors.** By (2.11), in order to analyse the quantitative behaviour of the scattered field, one needs to analyse the operator $(\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \partial_\nu$. In this subsection, we analyse the symbol of the operator $(\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \partial_\nu$ in terms of the GPT $\mathcal{M}_{L,M}(\lambda, D)$. We have the following lemma for a subsequent use.

Lemma 2.6. *The GPT $\mathcal{M}_{L,M}(\lambda, \partial D)$ in (2.16) has the following representation*

$$
\mathcal{M}_{L,M}(\lambda,\partial D) = \Big\langle |r|^k Y_L(\omega)\,,\,(P_{D,1} + P_{D,0} + P_{D,-1})\,(|r|^n Y_M(\omega))\Big\rangle_{L^2(\partial D, d\sigma)}, \qquad (2.24)
$$

where $P_{D,m}$ are pseudo-differential operators of order m for $m = 1, 0, -1$, and in the geodesic normal coordinate around each point x, there holds

$$\begin{aligned} p_{P_{D,1}}(x,\xi) =& \lambda^{-1}|\xi|, \\ p_{P_{D,1}}(x,\xi) =& \lambda^{-1}\left(\lambda^{-1}-\frac{1}{2}\right)\left((d-1)H(x) - \langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-2}\right), \\ p_{P_{D,-1}}(x,\xi) =& O\left(\lambda^{-1}|\xi|^{-1}\right)\,, \end{aligned} \tag{2.25}$$

where the constant of the large O depends on $\|\mathbb{X}\|_{\mathcal{C}^2}$ and $\|h\|_{\mathcal{C}^2}$.

Proof. First, by straightforward calculations, we can render the symbol of the operator $(\lambda I - \mathcal{K}^*_{\partial D})^{-1}$ as follows

$$p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi) = \lambda^{-1}+\lambda^{-2}(d-1)H(x)\,|\xi|^{-1}-\lambda^{-2}\langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-3}+O\left(\lambda^{-2}|\xi|^{-2}\right)\,, \tag{2.26}$$

where it is noted that $|\lambda| > \frac{1}{2}$.

Next, it is noticed that the function $u_0(x) = r^k\,Y_L(\omega)$ satisfies $\Delta u_0 = 0$, where $(r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ is the spherical coordinate of $x \in \mathbb{R}^d$. Hence, the map $\Lambda_0 : r^k\,Y_L(\omega) \mapsto \partial_\nu r^k\,Y_L(\omega)$ is in fact a Dirichlet-to-Neumann (DtN) map associated with the Laplacian. In [35], it is shown that the Laplacian can be factorised into a product of two operators modulo a smoothing operator, and it is a pseudo-differential operator of order 1. In the sequel, we choose a coordinate on a neighbourhood of $x \in \partial D$ as (a,s) with $\tilde{\mathbb{X}}(a,s) = \exp_x(a) + s\,\nu(\exp_x(a))$.

We proceed to compute the symbol of the composition operator, $(\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \Lambda_0$. Following [35] and considering the fact that

$$\Delta = \partial_\nu^2 + (d-1)H(x)\partial_\nu + \Delta_{\partial D}\,, \tag{2.27}$$

and under our choice of coordinates, one has

$$\partial_\varepsilon\,|_{\varepsilon=0}\,g_{ij}(x+\varepsilon\nu(x)) = \partial_\varepsilon\,|_{\varepsilon=0}\,\langle\tilde{\mathbb{X}}(e_i,\varepsilon),\tilde{\mathbb{X}}(e_j,\varepsilon)\rangle = -A_{ij}(x)\,.$$

Therefore using the recursive formula (3.11)-(3.14) in [35] and keeping in mind that the second-order derivatives of g do not vanish, we obtain that the symbol of Λ_0 and its derivative with respect to x are given by,

$$\begin{aligned} p_{\Lambda_0}(x,\xi) =& |\xi| + \frac{1}{2}\langle A(x)\xi,\xi\rangle\,|\xi|^{-2} - \frac{d-1}{2}H(x) \\ &- \frac{1}{4}\langle\partial_\nu A(x)\xi,\xi\rangle\,|\xi|^{-3} - \frac{1}{2}|A(x)\xi|^2\,|\xi|^{-3} + \frac{1}{4}|\langle A(x)\xi,\xi\rangle|^2\,|\xi|^{-5} \\ &+ \frac{d-1}{4}\partial_\nu H(x)\,|\xi|^{-1} - \frac{1}{4}(d-1)H(x)\langle A(x)\xi,\xi\rangle\,|\xi|^{-3} \\ &+ \frac{(d-1)^2}{4}|H(x)|^2\,|\xi|^{-1} - \frac{1}{4}\langle\partial_\xi A(x)\xi,\xi\rangle\,|\xi|^{-4} - \frac{d-1}{4}\partial_\xi H(x)\,|\xi|^{-1} \\ &- \frac{1}{4}\langle\Delta g(x)\xi,\xi\rangle|\xi|^{-3} - \frac{1}{4}\langle\mathrm{Hess}_{\xi,\xi}\,g(x)\xi,\xi\rangle|\xi|^{-5} + O\left(|\xi|^{-2}\right)\,, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_l}p_{\Lambda_0}(x,\xi) =& \frac{1}{2}\langle\partial_l A(x)\xi,\xi\rangle\,|\xi|^{-2} - \frac{d-1}{2}\partial_l H(x) - \frac{1}{2}\sum_{i,j,k=1}^{d-1}\partial_l\partial_k g^{ij}(x)\xi_i\xi_j\xi_k|\xi|^{-3} \\ &+ \frac{1}{2}\sum_{i,k=1}^{d-1}\partial_l\Gamma^k_{ii}(x)\xi_k|\xi|^{-1} + O\left(|\xi|^{-1}\right)\,. \end{aligned} \tag{2.29}$$

It is worth noticing that in (2.28), the symbol $p_{\Lambda_0}(x,\xi)$ is computed with one more term for our later use in the next section. By combining (2.28) and (2.29), we have

$$\begin{aligned}
&p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\circ\Lambda_0}(x,\xi)\\
=&p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi)p_{\Lambda_0}(x,\xi)+\frac{\partial}{\partial\xi}p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi)\frac{\partial}{\partial x}p_{\Lambda_0}(x,\xi)+O\left(\lambda^{-1}|\xi|^{-1}\right)\\
=&\lambda^{-1}|\xi|+\lambda^{-1}\left(\lambda^{-1}-\frac{1}{2}\right)\left((d-1)H(x)-\langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-2}\right)+O\left(\lambda^{-1}|\xi|^{-1}\right),
\end{aligned}$$

which together with (2.16) readily gives (2.24)–(2.25).

The proof is complete.

□

2.6. Sensitivity analysis of the generalized polarisation tensor and the scattered potential field.

In this subsection, we compute the shape derivative of the generalized polarisation tensor $\mathcal{M}_{L,M}(\lambda,D)$, $L\in I_k, M\in I_n$, which fully accounts for the shape derivative of the scattered field $(u-u_0)(x)$ associated with D. From that, we can analyse the semi-classical symbols of the operators involved in the sensitivity of $\mathcal{M}_{L,M}(\lambda,D)$. This is of crucial importance to understand how the (local) sensitivity of the scattered field behaves under the influence of the mean curvature in the next subsection.

The main result of this subsection is contained in the following theorem.

Theorem 2.7. *For $N\in\mathbb{N}$, there exists a positive constant C depending only on N, $L\in I_k, M\in I_n$, $||\mathbb{X}||_{\mathcal{C}^2}$ and $||h||_{\mathcal{C}^2}$ such that*

$$\left|\mathcal{M}_{L,M}(\lambda,D_\varepsilon)-\mathcal{M}_{L,M}(\lambda,D)-\sum_{n=1}^{N}\varepsilon^n\mathcal{M}^{(n)}_{L,M}(\lambda,D,h)\right|\le C\varepsilon^{N+1},\tag{2.30}$$

for some $\mathcal{M}^{(n)}_{L,M}(\lambda,D,h)$, with $\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)$ given as

$$\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)=\left\langle|r|^kY_L(\omega)\,,\,Q_{D,h}\left(|r|^nY_M(\omega)\right)\right\rangle_{L^2(\partial D,d\sigma)},$$

where

$$Q_{D,h}=Q_{D,h,1,I}+Q_{D,h,1,II}+Q_{D,h,0},\tag{2.31}$$

with $Q_{D,h,1,I},Q_{D,h,1,II}$ being pseudo-differential operators of order 1 and $Q_{D,h,0}$ being of order 0, and that in normal coordinate around each point x,

$$\begin{aligned}
p_{Q_{D,h,1,I}}(x,\xi)&=\lambda^{-2}\partial_\xi h(x)=O(\lambda^{-2}|\xi|),\\
p_{Q_{D,h,1,II}}(x,\xi)&=-\lambda^{-1}\Big((d-1)h(x)H(x)|\xi|-h(x)\langle A(x)\xi,\xi\rangle\,|\xi|^{-1}\Big)=O(\lambda^{-1}|\xi|),\\
p_{Q_{D,h,0}}(x,\xi)&=O\left(\lambda^{-1}\right),
\end{aligned}$$

where the constant of the large O depends on $\|\mathbb{X}\|_{\mathcal{C}^2}$ and $\|h\|_{\mathcal{C}^2}$.

Proof. Consider a point $y\in\partial D$ given by $y=\mathbb{X}(b)$ for some $b\in U$. We have $\mathbb{X}^\varepsilon(b)=y+h(y)\nu(y)$. By using (2.18), the decomposition (2.27) and the understanding that ∂_ν acting on $r^kY_L(\omega)$ is in fact a DtN map Λ_0 which is self-adjoint on $\langle\cdot\,,\,\cdot\rangle_{\frac{1}{2},-\frac{1}{2}}$ coupling, we can deduce that

$$
\begin{aligned}
& \mathcal{M}_{L,M}(\lambda, D_\varepsilon) - \mathcal{M}_{L,M}(\lambda, D) \\
= \; & \frac{\varepsilon}{c_{d,k}} |R|^{d-2+k} \int_{\mathbb{S}^{d-1}} \overline{Y_L(\omega_x)} \, \frac{\delta}{\delta h} \left[\Big(u - r^n Y_M(\omega) \Big) (R\,\omega_x) \right] (h) d\omega_x + O(\varepsilon^2) \\
= \; & \varepsilon \int_{\partial D} |y|^k \, \overline{Y_L(\omega_y)} \, \left\{ h \circ \Lambda_0 \circ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \Lambda_0 \left[r^n \, Y_M(\omega) \right] \right\} (y) \, d\sigma(y) \\
& - \varepsilon \int_{\partial D} |y|^k \, \overline{Y_L(\omega_y)} \, \left(\left\{ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \mathcal{K}^{(1)}_{D,h} \circ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \Lambda_0 \right\} \left[r^n \, Y_M(\omega) \right] \right) (y) \, d\sigma(y) \\
& + \varepsilon (d-1) \int_{\partial D} |y|^k \, \overline{Y_L(\omega_y)} \, \left(\left\{ \left[h\, H \,, (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \right] \circ \Lambda_0 \right\} (r^n \, Y_M(\omega)) \right) (y) \; d\sigma(y) \\
& - \varepsilon \int_{\partial D} |y|^k \, \overline{Y_L(\omega_y)} \, \left\{ \left\{ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ h \circ \Delta_{\partial D} \right\} \left[r^n \, Y_M(\omega) \right] \right\} (y) \, d\sigma(y) \\
& + O(\varepsilon^2) \,,
\end{aligned} \tag{2.32}
$$

where $[A, B]$ is the commutator of A and B. Here and also in what follows, when a function is written as an operator, it signfies the multiplicative operator as multiplication by the function.

To compute derivatives of the symbols, we need to be more careful. After keeping in mind that the second derivatives of $g(x)$ do not vanish, we obtain the followings:

$$
\frac{\partial}{\partial x_l} p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) = \lambda^{-2} (d-1) \partial_l H(x) \, |\xi|^{-1} - \lambda^{-2} \langle \partial_l A(x) \, \xi, \, \xi \rangle \, |\xi|^{-3} + O\left(\lambda^{-2} |\xi|^{-2} \right) \,,
$$

$$
p_{h\Delta_{\partial D}}(x, \xi) = h(x) |\xi|^2 \,, \quad \frac{\partial}{\partial x_l} p_{h\Delta_{\partial D}}(x, \xi) = \partial_l h(x) \, |\xi^2| + h(x) \sum_{i,k=1}^{d-1} \partial_l \Gamma^k_{ii}(x) \, \xi_k \,.
$$

Together with the symbol of Λ_0 and its derivatives, we could render the symbols of the following 4 operators in concern in the geodesic normal coordinate (where $|\lambda| > \frac{1}{2}$) as follows:

$$
\begin{aligned}
& p_{h \circ \Lambda_0 \circ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \Lambda_0}(x, \xi) \\
= \; & h(x) p_{\Lambda_0}(x, \xi) p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) p_{\Lambda_0}(x, \xi) + h(x) \frac{\partial}{\partial \xi} p_{\Lambda_0}(x, \xi) \frac{\partial}{\partial x} p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) p_{\Lambda_0}(x, \xi) \\
& + h(x) \frac{\partial}{\partial \xi} p_{\Lambda_0}(x, \xi) p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) \frac{\partial}{\partial x} p_{\Lambda_0}(x, \xi) \\
& + h(x) p_{\Lambda_0}(x, \xi) \frac{\partial}{\partial \xi} p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) \frac{\partial}{\partial x} p_{\Lambda_0}(x, \xi) + O(\lambda^{-1}) \\
= \; & \lambda^{-1} h(x) |\xi|^2 + \left(\lambda^{-2} - \lambda^{-1} \right) \Big((d-1) h(x) H(x) |\xi| - h(x) \langle A(x) \xi, \xi \rangle \, |\xi|^{-1} \Big) + O(\lambda^{-1}) \,,
\end{aligned}
$$

and

$$
\begin{aligned}
& p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \mathcal{K}^{(1)}_{D,h} \circ (\lambda I - \mathcal{K}^*_{\partial D})^{-1} \circ \Lambda_0}(x, \xi) \\
= \; & p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) p_{\mathcal{K}^{(1)}_{D,h}}(x, \xi) p_{(\lambda I - \mathcal{K}^*_{\partial D})^{-1}}(x, \xi) p_{\Lambda_0}(x, \xi) + O(\lambda^{-1} 1) \\
= \; & -\lambda^{-2} \partial_\xi h(x) + O(\lambda^{-1}) \,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
& p_{\left[h\,H\,,(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\right]\circ\Lambda_0}(x,\xi) \\
= & \left(-\frac{\partial}{\partial x}(h(x)H(x))\,\frac{\partial}{\partial \xi}\,p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi)+\frac{\partial}{\partial \xi}(h(x)H(x))\,\frac{\partial}{\partial x}\,p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi)\right)p_{\Lambda_0}(x,\xi) \\
& +O\left(\lambda^{-2}|\xi|^{-2}\right) \\
= & \lambda^{-2}\langle[(d-1)H(x)I+2A(x)]\xi\,,\,\partial(h(x)H(x))\rangle\,|\xi|^{-2}-3\lambda^{-2}\partial_\xi(h(x)H(x))\langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-4} \\
& +O\left(\lambda^{-2}|\xi|^{-2}\right) \\
= & O\left(\lambda^{-2}|\xi|^{-1}\right)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
& p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\circ h\circ\Delta_{\partial D}}(x,\xi) \\
= & p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi)p_{h\Delta_{\partial D}}(x,\xi)+\frac{\partial}{\partial\xi}p_{(\lambda I-\mathcal{K}^*_{\partial D})^{-1}}(x,\xi)\frac{\partial}{\partial x}p_{h\Delta_{\partial D}}(x,\xi)+O\left(\lambda^{-2}1\right) \\
= & \lambda^{-1}h(x)|\xi|^2+\lambda^{-2}\left((d-1)h(x)H(x)\,|\xi|-h(x)\langle A(x)\,\xi,\,\xi\rangle\,|\xi|^{-1}\right)+O\left(\lambda^{-2}|\xi|^{0}\right)\,.
\end{aligned}
$$

Therefore we can combine the above results to obtain

$$
\begin{aligned}
& p_{\mathcal{Q}}(x,\xi) \\
= & \lambda^{-2}\partial_\xi h(x)-\lambda^{-1}\left((d-1)h(x)H(x)|\xi|-h(x)\langle A(x)\xi,\xi\rangle\,|\xi|^{-1}\right)+O\left(\lambda^{-1}|\xi|^{0}\right),
\end{aligned}
$$

where the operator $\mathcal{Q}$ is given as

$$
\begin{aligned}
\mathcal{Q}:=& h\circ\Lambda_0\circ(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\circ\Lambda_0-(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\circ\mathcal{K}^{(1)}_{D,h}\circ(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\circ\Lambda_0 \\
&+(d-1)\left[h\,H\,,(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\right]\circ\Lambda_0-(\lambda I-\mathcal{K}^*_{\partial D})^{-1}\circ h\circ\Delta_{\partial D}.
\end{aligned}
$$

This completes the proof.

□

2.7. **Localization of sensitivity of generalized polarization tensors at points of high mean curvature.** Consider the space

$$
\mathrm{tr}_{\partial D}\mathrm{Ker}(\Delta):=\{u\,|_{\partial D}\colon\,\Delta u=0\text{ in }\mathbb{R}^d\}.
$$

Notice that $\overline{\mathrm{tr}_{\partial D}\mathrm{Ker}(\Delta)}^{H^s(\partial D,d\sigma)}=H^s(\partial D,d\sigma)$ for all $s\in\mathbb{R}$. Considering the fact that $Q_{D,h}$ is a pseudo-differential operator of order 1, and the closure of operators under the weak operator topology, we can have that

$$
\begin{aligned}
& \left\{\,\langle\psi\,,\,Q_{D,h}\,\phi\rangle_{L^2(\partial D,d\sigma)}\,:\,\psi\in H^s(\partial D,d\sigma),\phi\in H^t(\partial D,d\sigma),s,t\in\mathbb{R},s+t-1=0\right\} \\
= & \left\{\sum_{k,m}\sum_{L\in I_k,M\in I_m}a_Lb_M\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\,:\,a_L,b_M\in\mathbb{C}\text{ such that the sum converges}\right\},
\end{aligned}
$$

where from now on we abuse the notation of $\langle\,\psi\,,\;\phi\,\rangle_{L^2(\partial D,d\sigma)}$ as an L^2-pivoting as soon as the resulting $\overline{\psi}\,\phi\in L^1(\partial D,d\sigma)$. Therefore, the map $h\mapsto\left(\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\right)_{L\in I_k,M\in I_n,k,n\in\mathbb{N}}$ can fully reconstruct the opeartor-valued map $h\mapsto Q_{D,h}=Q_{D,h,1,I}+Q_{D,h,1,II}+Q_{D,h,0}$. Now with suitable choices of $\psi\in H^s(\partial D,d\sigma),\phi\in H^t(\partial D,d\sigma)$ such that $s,t\in\mathbb{R},s+t-1=0$, we can obtain the principal symbol in the geodesic normal coodinate at each point x as follows

$$\lim_{t\to\infty} t^{-1}e^{-it\varphi_{x,\xi}}Q_{D,h}e^{it\varphi_{x,\xi}}\chi_x = p_{Q_{D,h,1,I}}(x,\xi) + p_{Q_{D,h,1,II}}(x,\xi)\,,$$

where $\xi \in \mathbb{S}^{d-1}$ and $\varphi_{x,\xi}(\,\cdot\,) = \langle \xi, \log_x(\,\cdot\,)\rangle$ in half of the injective radius of the convex neighborhood and is zero outside 3/4 of the injective radius, and $\chi_x(\,\cdot\,)$ is a cut off function such that $\chi_x(\,x\,) = 1$ and its value is zero outside 3/4 of injective radius. Then $p_{Q_{D,h,1,I}}(x,\xi) + p_{Q_{D,h,1,II}}(x,\xi)$ can be reconstructed in full by the property of being homogenous of degree one. On the other hand, one can recover $h(x)H(x)$ from $p_{Q_{D,h,1,I}}(x,\xi) + p_{Q_{D,h,1,II}}(x,\xi)$ as follows via Theorem (2.7),

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \left(p_{Q_{D,h,1,I}}(x,\xi) + p_{Q_{D,h,1,II}}(x,\xi)\right)|\xi|^{-1}d\sigma(\xi)\\
=\ &\int_{\mathbb{S}^{d-1}} \left(\lambda^{-2}\,\partial_\xi h(x)|\xi|^{-1} - \lambda^{-1}(d-1)h(x)H(x) + \lambda^{-1}h(x)\langle A(x)\xi,\xi\rangle\,|\xi|^{-2}\right)d\sigma_\xi\\
=\ &[(1-d)\omega_d + 1]\,\lambda^{-1}h(x)H(x).
\end{aligned}$$

Hence the inverse composition map

$$\begin{aligned}
&\left(\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\right)_{L\in I_k, M\in I_n, k,n\in\mathbb{N}}\\
&\mapsto^{\mathrm{inv}_1} Q_{D,h} \mapsto^{\mathrm{inv}_2} p_{Q_{D,h,1,I}}(x,\xi) + p_{Q_{D,h,1,II}}(x,\xi) \mapsto^{\mathrm{inv}_3} h(x)H(x)
\end{aligned}$$

is well-defined. To make the above description more precise, let us consider the following complete orthornormal bases on $L^2(\partial D, d\sigma)$:

$$\{\eta_{p,\partial D}\}_{k\in\mathbb{N}} \quad \text{where} \quad -\Delta_{\partial D}\eta_{p,\partial D} = \lambda_p^2\eta_{p,\partial D}\,,$$

and write $\lambda(\Delta_{\partial D})$ to be the eigenvalues λ satisfying the above. By Weyl's asymptotics, we have at least that $\lambda_p^{-1} \sim p^{-\frac{1}{d-1}}$. Therefore for any smooth function ϕ on ∂D, we have $\langle \eta_{p,\partial D}\,,\,\phi\rangle_{L^2(\partial D,d\sigma)} = O(p^{-l})$ for any l. By the density of its subspace $\mathrm{tr}_{\partial D}\mathrm{Ker}(\Delta)$ in $L^2(\partial D, d\sigma)$, $\{r^n Y_M(\omega)|_{\partial D}\}_{M\in I_n, n\in\mathbb{N}}$ is also a complete frame in $L^2(\partial D, d\sigma)$. Therefore there is a change of basis map $(U_{p,L,\partial D})$ which is the matrix for the change of the basis to the corresponding orthonormal one. We write $\left(U^{-1}_{L,p,\partial D}\right)$ as its inverse. Moreover, since $\eta_{p,\partial D}$ is orthornomal,

$$U^{-1}_{L,p,\partial D} = \langle r^k Y_L(\omega)\,,\,\eta_{p,\partial D}\rangle_{L^2(\partial D,d\sigma)}\,.$$

Combining the above discussions, we have the following theorem in force.

Theorem 2.8. *We have the following inversion formula for $\partial D \in \mathcal{C}^{2,\alpha}$ and $h \in \mathbb{C}^{2,\alpha}$*

$$\begin{aligned}
&[(1-d)\omega_d + 1]\,h(x)H(x)\\
=\ &\mathrm{inv}_3\circ\mathrm{inv}_2\circ\mathrm{inv}_1\left[\left(\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\right)_{L\in I_k, M\in I_n, k,n\in\mathbb{N}}\right]\\
:=\ &\int_{\mathbb{S}^{d-1}} \lim_{t\to\infty}\mathcal{G}(\xi,t,x)d\sigma(\xi),
\end{aligned} \tag{2.33}$$

where

$$\mathcal{G}(\xi,t,x) := \sum_{\substack{L\in I_k,\ M\in I_n\ \tilde{M}\in I_{\tilde{n}},\\ k,n,r,s\in\mathbb{N}}} |\xi|^{-1}t^{-1}e^{-it\varphi_{x,\xi}}\,\eta_{s,\partial D}\,U_{s,L,\partial D}\,\mathcal{M}^{(1)}_{L,M}(\lambda,D,h) \times U^{-1}_{M,r,\partial D}\,\langle \eta_{r,\partial D}\,,\,\chi_x\, e^{it\varphi_{x,\xi}}\rangle_{L^2(\partial D,d\sigma)}.$$

Now let ∂D_ϵ be given as in (2.32), and $u_{\partial D_\epsilon}$ be u satisfying (2.8) with $u_0 = r^nY_M(\omega)$ and the support of the inhomogeneity D_ϵ. We further define

$$\frac{\partial}{\partial\epsilon}\left(\left\langle Y_L(\omega_x)\,,\,\Big(u_{\partial D_\epsilon} - r^nY_M(\omega)\Big)(R\,\omega_x)\right\rangle_{L^2(R\mathbb{S}^{d-1},d\omega_x)}\right)_{L\in I_k,M\in I_n,k,n\in\mathbb{N}}$$
$$\mapsto^{\mathrm{inv}_0}\ \Big(\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\Big)_{L\in I_k,M\in I_n,k,n\in\mathbb{N}}.$$

Hence we have the following corollary:

Corollary 2.9. *For $\partial D\in\mathcal{C}^{2,\alpha}$ and $h\in\mathcal{C}^{2,\alpha}$, we have*

$$\begin{aligned}&[(1-d)\omega_d+1]\,h(x)H(x)\\ &=\mathrm{inv}_3\circ\mathrm{inv}_2\circ\mathrm{inv}_1\circ\mathrm{inv}_0\left[\frac{\partial}{\partial\epsilon}\left(\left\langle Y_L(\omega_x)\,,\,\Big(u_{\partial D_\epsilon} - r^nY_M(\omega)\Big)(R\,\omega_x)\right\rangle_{L^2(R\mathbb{S}^{d-1},d\omega_x)}\right)_{L\in I_k,M\in I_n,k,n\in\mathbb{N}}\right].\end{aligned} \tag{2.34}$$

We would like to remark that the change of coordinate maps are indeed unbounded maps from l_2 to l_2. In general, it is well known that the inverse problem is exponentially ill-posed. However we would like to dissect the composition map in Corollary 2.9 and understand more on how the ill-posedness are given by different properties of the domain. Now let us gaze at the composition of $\mathrm{inv}_3\circ\mathrm{inv}_2\circ\mathrm{inv}_1$ in (2.34) and establish the properties of the composition of maps under a specific assumption on ∂D.

Before we continue to understand the inverse problem, let us understand the perturbation of the change of the above coordinate maps and how they affect the condition number of a restriction of U to a particular finite dimensional subspace. For this purpose, let us consider the following restriction and extension:

$$\begin{aligned}U^{-1}_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D},W_{s,\partial D})} &:= \mathrm{Proj}^*_{W_{s,\partial D}}\circ U^{-1}_{L,M,\partial D}\circ\mathrm{Proj}_{V_{s,\partial D}},\\ U_{q,L,\partial D}|_{\mathcal{L}(W_{s,\partial D},V_{s,\partial D})} &:= \mathrm{Proj}^*_{V_{s,\partial D}}\circ U_{M,L,\partial D}\circ\mathrm{Proj}_{W_{s,\partial D}},\end{aligned}$$

where $\mathrm{Proj}_{V_{s,\partial D}} : L^2(\partial D,d\sigma)\to V_s := \mathrm{Span}\{\eta_{p,\partial D}\}_{p<|\{M:m_{d-1}\le s\}|}$ and $\mathrm{Proj}_{W_{s,\partial D}} : L^2(\partial D,d\sigma)\to W_s := \mathrm{Span}\{r^nY_M(\omega)|_{\partial D}\}_{m_{d-1}\le s}$ with $s\in\mathbb{N}$.

Lemma 2.10. *Given a general $\partial D\in\mathcal{C}^{1,\alpha}$, let ∂D^ϵ be an ε-perturbation under a direction $h\in\mathcal{C}^{1,\alpha}$ and let $S=|\{T:t_{d-1}\le s\}|$, for $\varepsilon\in\mathbb{R}_+$ small enough, we have*

$$\begin{aligned}&\max\Bigg\{\Big|\|U^{-1}_{L,p,\partial D^\varepsilon}|_{\mathcal{L}(V_{s,\partial D^\varepsilon},W_{s,\partial D^\varepsilon})}\|_{l^2\to l^2} - \|U^{-1}_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D},W_{s,\partial D})}\|_{l^2\to l^2}\Big|\,,\\ &\qquad\Big|\|U_{L,p,\partial D^\varepsilon}|_{\mathcal{L}(V_{s,\partial D^\varepsilon},W_{s,\partial D^\varepsilon})}\|^{-1}_{l^2\to l^2} - \|U_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D},W_{s,\partial D})}\|^{-1}_{l^2\to l^2}\Big|\Bigg\}\\ <\ &2\varepsilon\max_{1\le P\le S}\Bigg\{\max\Bigg\{1,\max_{z\neq\lambda_P,z\in\lambda(\Delta_{\partial D})}\frac{\|g\|^2_{C^1}}{|z^2-\lambda_P^2|}\Bigg\}\|h\|_{C^0}\|A\|_{C^1}\lambda_P^2\|r^s|_{\partial D}\|_{L^2(\partial D,d\sigma)}\\ &\qquad+\|h\|_{C^1}\|\partial_\nu(r^sY_T(\omega))|_{\partial D}\|_{L^2(\partial D,d\sigma)} + \varepsilon(d-1)\|h\|_{C^0}\|H\|_{C^0}\|r^sY_T(\omega)|_{\partial D}\|_{L^2(\partial D,d\sigma)}\Bigg\}.\end{aligned}$$

Proof. For a general $\partial D \in \mathcal{C}^{1,\alpha}$, considering ∂D^ϵ under a perturbation h, and comparing the surface Laplacian on ∂D with the one on ∂D^ε, we have by direct computations that

$$\Delta_{\partial D^\epsilon} - \Delta_{\partial D} = \varepsilon h(x)\left(\partial_\nu g^{ij}(x)\partial_i\partial_j + \partial_\nu g^{ij}(x)\Gamma^k_{ij}(x)\partial_k - g^{ij}(x)\partial_\nu\Gamma^k_{ij}(x)\partial_k\right) + O(\varepsilon^2)$$

where by definition and by the Gauss-Codazzi formula, we have

$$\partial_\nu g^{ij}(x) = g^{il}(x)A_{lm}(x)g^{mj}(x) \quad \text{and} \quad \partial_\nu\Gamma^k_{ij}(x) = \frac{1}{2}g^{kl}\left(\nabla_j A_{il}(x) + \Gamma^r_{ij}(x)A_{rl}(x)\right), \tag{2.35}$$

after absorbing the notations. Since $\Delta^{-1}_{\partial D^\epsilon}$ is collectively compact with respect to ε, by Osborn's Theorem [34] and that for repeated eigenvalues, we have, after applying $\Delta^{-1}_{\partial D^\epsilon} = -\varepsilon\Delta^{-1}_{\partial D^\epsilon}\partial_\varepsilon\Delta_{\partial D^\epsilon}\,|_{\varepsilon=0}\,\Delta^{-1}_{\partial D^\epsilon} + O(\varepsilon^2)$, that if we consider $-\lambda^2 \neq 0$ an eigenvalue of $\Delta_{\partial D}$ with multiplicity m and E_λ be its eigenspace, then there exists $\{\eta_{\lambda,s,\partial D}\}_{s=1}^m$ a basis of E_λ such that

$$\begin{aligned} &\lambda^2_{s,\varepsilon} - \lambda^2 \\ = \; & -\varepsilon\left\langle \eta_{\lambda,s,\partial D}, h(x)\left(\partial_\nu g^{ij}(x)\partial_i\partial_j + \partial_\nu g^{ij}(x)\Gamma^k_{ij}(x)\partial_k - g^{ij}(x)\partial_\nu\Gamma^k_{ij}(x)\partial_k\right)\eta_{\lambda,s,\partial D}\right\rangle_{L^2(\partial D,d\sigma)} \\ & + O(\varepsilon^2), \end{aligned}$$

and

$$\begin{aligned} &[\eta_{\lambda,s,\partial D^\varepsilon}\circ\Phi_\varepsilon - \eta_{\lambda,s,\partial D}]\big|_{E_\lambda^\perp} \\ = \; & \varepsilon\sum_{z\neq\lambda}\frac{\left\langle \eta_{z,\partial D}, h(x)\left(\partial_\nu g^{ij}(x)\partial_i\partial_j + \partial_\nu g^{ij}(x)\Gamma^k_{ij}(x)\partial_k - g^{ij}(x)\partial_\nu\Gamma^k_{ij}(x)\partial_k\right)\eta_{\lambda,s,\partial D}\right\rangle_{L^2(\partial D,d\sigma)}}{z^2-\lambda^2}\,\eta_{\lambda,\partial D} \\ & + O(\varepsilon^2), \end{aligned}$$

where $\left(-\lambda^2_{s,\varepsilon}, \eta_{\lambda,s,,\partial D^\varepsilon}\right)$ is an eigenpair of $\Delta_{\partial D^\epsilon}$, and Φ_ε brings ∂D to ∂D^ε. By Poincaré inequalities and Cauchy inequality,

$$\begin{aligned} & \|[\eta_{\lambda,s,\partial D^\varepsilon}\circ\Phi_\varepsilon - \eta_{\lambda,s,\partial D}]|_{E_\lambda^\perp}\|^2_{L^2(\partial D,d\sigma)} \\ \leq \; & \varepsilon^2 \max_{z\neq\lambda, z\in\lambda(\Delta_{\partial D})}\frac{1}{|z^2-\lambda^2|^2}\times \\ & \left\| h(x)\left(\partial_\nu g^{ij}(x)\partial_i\partial_j + \partial_\nu g^{ij}(x)\Gamma^k_{ij}(x)\partial_k - g^{ij}(x)\partial_\nu\Gamma^k_{ij}(x)\partial_k\right)\eta_{\lambda,s,\partial D}\right\|^2_{L^2(\partial D,d\sigma)} + O(\varepsilon^3) \\ \leq \; & \varepsilon^2 \max_{z\neq\lambda, z\in\lambda(\Delta_{\partial D})}\frac{1}{|z^2-\lambda^2|^2}\|h\|^2_{C^0}\|g\|^4_{C^1}\|A\|^2_{C^0}\lambda^4 + O(\varepsilon^3). \end{aligned}$$

From the fact that $\|\eta_{\lambda,s,\partial D^\varepsilon}\|^2_{L^2(\partial D^\varepsilon,d\sigma)} = 1$, $d\sigma_{\partial D^\varepsilon} = (1+\varepsilon(d-1)h(x)H(x))d\sigma_{\partial D} + O(\varepsilon^2)$ and $L^2(\partial D, d\sigma) = E_\lambda \oplus E_\lambda^\perp$, one can show that for $s = 1, ..., m$,

$$\|\eta_{\lambda,s,\partial D^\varepsilon}\circ\Phi_\varepsilon - \eta_{\lambda,s,\partial D}\|^2_{L^2(\partial D,d\sigma)} \leq \varepsilon^2 \max\left\{1, \max_{z\neq\lambda}\frac{\|g\|^4_{C^1}}{|z^2-\lambda^2|^2}\right\}\|h\|^2_{C^0}\|A\|^2_{C^1}\lambda^4 + O(\varepsilon^3).$$

Therefore we have that if $\varepsilon \in \mathbb{R}_+$ is sufficiently small,

$$\begin{aligned} & \left|\langle r^k Y_L(\omega)\,,\,\eta_{p,\partial D^\varepsilon}\rangle_{L^2(\partial D^\varepsilon,d\sigma)} - \langle r^k Y_L(\omega)\,,\,\eta_{p,\partial D}\rangle_{L^2(\partial D,d\sigma)}\right| \\ \leq \; & 2\varepsilon\left\{\max\left\{1, \max_{z\neq\lambda_p, z\in\lambda(\Delta_{\partial D})}\frac{\|g\|^2_{C^1}}{|z^2-\lambda_p^2|}\right\}\|h\|_{C^0}\|A\|_{C^1}\lambda_p^2\|r^k|_{\partial D}\|_{L^2(\partial D,d\sigma)}\right. \\ & \left. + \|h\|_{C^1}\|\partial_\nu(r^k Y_L(\omega))|_{\partial D}\|_{L^2(\partial D,d\sigma)} + \varepsilon(d-1)\|h\|_{C^0}\|H\|_{C^0}\|r^k Y_L(\omega)|_{\partial D}\|_{L^2(\partial D,d\sigma)}\right\}. \end{aligned}$$

Next we recall that

$$\|U^{-1}_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D},W_{s,\partial D})}\|_{l^2\to l^2} = \sigma_{\max}(U^{-1}_{L,M,\partial D}|_{\mathcal{L}(V_{s,\partial D},W_{s,\partial D})}),$$
$$\|U_{q,L,\partial D}|_{\mathcal{L}(W_{s,\partial D},V_{s,\partial D})}\|_{l^2\to l^2} = 1/\sigma_{\min}(U_{M,L,\partial D}|_{\mathcal{L}(W_{s,\partial D},V_{s,\partial D})}),$$

where $\sigma_{\max}(T)$ and $\sigma_{\min}(T)$ are the respective maximum and minimum singular values of an operator T. Finally, by Osborn's theorem once again, one can show the lemma. □

To consolidate our study, we next compute $\kappa(U_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D},W_{s,\partial D})})$ for several concrete examples of ∂D.

Example I.1. Let us first consider $\partial D = R_0\mathbb{S}^{d-1}$ where $R_0 < 1$. In this case, instead of indexing $\eta_{k,\partial D}$ via $k \in \mathbb{N}$, we may instead order them with $M \in I_n, n \in \mathbb{N}$, since

$$\eta_{M,R_0\mathbb{S}^{d-1}} = \omega_{d-1}^{-1} R_0^{-\frac{d-1}{2}} Y_L(\omega),$$

where ω_{d-1} is the volumn of $\mathbb{S}^{d-1}$. We have

$$U^{-1}_{L,M,R_0\mathbb{S}^{d-1}} = \omega_{d-1}^{-1} R_0^{k-\frac{d-1}{2}} \langle Y_L(\omega)\,,\, Y_M(\omega)\rangle_{L^2(\partial D, d\sigma)} = \omega_{d-1}^{-1} R_0^{k-\frac{d-1}{2}} \delta_{LM}\,.$$

Hence, if R_0 is small, we have

$$\begin{aligned}
&\|U^{-1}_{L,M,R_0\mathbb{S}^{d-1}}|_{\mathcal{L}(V_{s,R_0\mathbb{S}^{d-1}},W_{s,R_0\mathbb{S}^{d-1}})}\|_{l^2\to l^2}\\
=\ &\sigma_{\max}(U^{-1}_{L,M,R_0\mathbb{S}^{d-1}}|_{\mathcal{L}(V_{s,R_0\mathbb{S}^{d-1}},W_{s,R_0\mathbb{S}^{d-1}})}) = \omega_{d-1}^{-1}\, R_0^{-\frac{d-1}{2}} \max\{R_0^s, 1\},\\
&\|U_{M,L,R_0\mathbb{S}^{d-1}}|_{\mathcal{L}(W_{s,R_0\mathbb{S}^{d-1}},V_{s,R_0\mathbb{S}^{d-1}})}\|_{l^2\to l^2}\\
=\ &1/\sigma_{\min}(U_{M,L,R_0\mathbb{S}^{d-1}}|_{\mathcal{L}(W_{s,R_0\mathbb{S}^{d-1}},V_{s,R_0\mathbb{S}^{d-1}})}) = \omega_{d-1}\, R_0^{\frac{d-1}{2}} \max\{R_0^{-s}, 1\}.
\end{aligned}$$

Therefore we have the following estimate of the condition number when $\partial D = R_0\mathbb{S}^{d-1}$:

$$\kappa(U_{M,L,R_0\mathbb{S}^{d-1}}|_{\mathcal{L}(W_{s,R_0\mathbb{S}^{d-1}},V_{s,R_0\mathbb{S}^{d-1}})}) = \max\{R_0^{-s}, R_0^s\}\,.$$

Example I.2. Consider ∂D^δ as follows: $\partial D = R_0\mathbb{S}^{d-1}$ with perturbation $k \in \mathcal{C}^1$, and $\|k\|_{\mathcal{C}^1} < 1$ and small magnitude δ. Now, applying Lemma 2.10, together with $\lambda_{T,\mathbb{S}^d} = s(s+d-2)$ for $t_{d-1} = s$, we have for $\delta \in \mathbb{R}_+$ sufficiently small and $s \in \mathbb{R}_+$ sufficiently large that

$$\begin{aligned}
&\max\Bigg\{ \left| \|U^{-1}_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},W_{s,\partial D^\delta})}\|_{l^2\to l^2} - \omega_{d-1}^{-1}\, R_0^{-\frac{d-1}{2}} \max\{R_0^s, 1\} \right|,\\
&\qquad \left| \|U_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},W_{s,\partial D^\delta})}\|^{-1}_{l^2\to l^2} - \omega_{d-1}^{-1}\, R_0^{-\frac{d-1}{2}} \min\{R_0^s, 1\} \right| \Bigg\}\\
<\ &2\delta\omega_d \max\left\{ s(s+d-2)R_0^{s+1} + sR_0^{s-2} + R_0^{s-1}\,,\, (d-1)R_0^2 + R_0^{-1} + 1 \right\}\\
<\ &2\delta\omega_d \max\left\{ (s+d-2)^2 R_0^{s+1}\,,\, R_0^{-1} + d \right\}.
\end{aligned}$$

Hence for small enough $\delta \in \mathbb{R}_+$, we obtain

$$
\begin{aligned}
&\kappa(U_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},W_{s,\partial D^\delta})})\\
\leq\;& \frac{\max\{R_0^s,1\}+2\delta\omega_d^2 R_0^{\frac{d-1}{2}}\max\left\{(s+d-2)^2R_0^{s+1}\,,\,R_0^{-1}+d\right\}}{\min\{R_0^s,1\}-2\delta\omega_d^2 R_0^{\frac{d-1}{2}}\max\left\{(s+d-2)^2R_0^{s+1}\,,\,R_0^{-1}+d\right\}}\\
\leq\;& \begin{cases} \frac{R_0^s+2\delta\omega_d^2R_0^{\frac{d-1}{2}}(s+d-2)^2R_0^{s+1}}{1-2\delta\omega_d^2R_0^{\frac{d-1}{2}}(s+d-2)^2R_0^{s+1}} & \text{if } R_0\geq 1\,,\\ \frac{1+2\delta\omega_d^2R_0^{\frac{d-1}{2}}(R_0^{-1}+d)}{R_0^s-2\delta\omega_d^2R_0^{\frac{d-1}{2}}(R_0^{-1}+d)} & \text{if } R_0\leq 1\,. \end{cases}
\end{aligned}
$$

From the above example, we have the following corollary.

Corollary 2.11. *Let us consider ∂D^δ as a δ-perturbation of $\partial D = R_0\mathbb{S}^{d-1}$ along the direction $k \in C^2(\partial D)$ with $\|k\|_{\mathcal{C}^2} < 1$ for sufficiently small $\delta \in \mathbb{R}_+$. Then for $h \in \mathcal{C}^2(\partial D^\delta)$ with $\|h\|_{\mathcal{C}^2} < 1$, considering an ε-perturbation of ∂D^δ along the direction h, $(\partial D^\delta)_\varepsilon$, we have*

$$
\begin{aligned}
&|[\mathrm{Proj}_{V_{s,\partial D^\delta}}(hH)](x)|\\
\leq\;& \begin{cases} C_d\frac{R_0^s+2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(s+d-2)^2R_0^{s+1}}{1-2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(s+d-2)^2R_0^{s+1}}\,\|\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})} & \textit{if } R_0\geq 1,\\ C_d\frac{1+2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(R_0^{-1}+d)}{R_0^s-2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(R_0^{-1}+d)}\,\|\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})} & \textit{if } R_0\leq 1, \end{cases}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|[\mathrm{Proj}_{V_{s,\partial D^\delta}}(hH)](x)|\\
\leq\;& \begin{cases} C_d\frac{R^{2s+d-2}}{c_{d,s}}\,\frac{R_0^s+2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(s+d-2)^2R_0^{s+1}}{1-2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(s+d-2)^2R_0^{s+1}}\times & \\ \left\|\frac{\partial}{\partial\epsilon}\left(\left\langle Y_L(\omega_x)\,,\,\left(u_{(\partial D^\delta)_\epsilon}-r^nY_M(\omega)\right)(R\,\omega_x)\right\rangle_{L^2(R\mathbb{S}^{d-1},d\omega_x)}\right)\right\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})}, & \textit{if } R_0\geq 1,\\ C_d\frac{R^{2s+d-2}}{c_{d,s}}\,\frac{1+2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(R_0^{-1}+d)}{R_0^s-2\varepsilon\omega_d^2R_0^{\frac{d-1}{2}}(R_0^{-1}+d)}\times & \\ \left\|\frac{\partial}{\partial\epsilon}\left(\left\langle Y_L(\omega_x)\,,\,\left(u_{(\partial D^\delta)_\epsilon}-r^nY_M(\omega)\right)(R\,\omega_x)\right\rangle_{L^2(R\mathbb{S}^{d-1},d\omega_x)}\right)\right\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})}, & \textit{if } R_0\leq 1. \end{cases}
\end{aligned}
$$

Proof. For a given resolution $s \in \mathbb{N}$, we have

$$
\begin{aligned}
&|[\mathrm{Proj}_{V_{s,\partial D^\delta}}(hH)](x)|\\
=\;& \left|\mathrm{inv}_3\circ\mathrm{inv}_2\circ\mathrm{inv}_1\left[\left(\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\right)_{L\in I_k,M\in I_n,k,n\leq s}\right]\right|\\
\leq\;& C_d\,\kappa(U_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},W_{s,\partial D^\delta})})\limsup_{t\to\infty}\|\chi_x\,e^{it\varphi_{x,\xi}}\|_{L^2(\partial S,d\sigma)}\,\|\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})},
\end{aligned}
$$

for some constant C_d. The results follow from the computations in Example 2 and (2.16). $\square$

It is now clear that the reconstruction of $h(x)$ from $\mathcal{M}^{(1)}_{L,M}(\lambda,D,h)$ is more sensitive with points of high mean curvature $|H(x)|^2$ if D is not too far from $R_0\mathbb{S}^{d-1}$. This can be more explicitly explained as follows. If we have $\left((u - r^n Y_M(\omega))^{\text{meas}} |_\Gamma \right)_{M \in I_n, n \in \mathbb{N}}$, we obtain $\left((\mathcal{M}_{L,M})^{\text{meas}}(\lambda, D_{\text{exact}}) \right)_{L \in I_k, M \in I_n, k,n \in \mathbb{N}}$. Then one may reconstruct D via the following Newton type iteration of (D^n, h^n), $n = 1, 2, \ldots$, where they are recursively defined as follows:

$$\begin{aligned} \mathcal{M}^{(1)}_{L,M}(\lambda, D^n, h^n) =& (\mathcal{M}_{L,M})^{\text{meas}}(\lambda, D_{\text{exact}}) - \mathcal{M}_{L,M}(\lambda, D^n)\,, \\ D^{n+1} =& \{x + h(x)\nu(x) \,:\, x \in D^n\}\,, \end{aligned} \tag{2.36}$$

and the reconstruction step for h^n is again more sensitive with points of high mean curvature $|H(x)|^2$.

Remark 2.12. *It is remarked that the recovery of h^n in (2.36) shall be numerically performed via (2.22) and (2.32) instead of $inv_3 \circ inv_2 \circ inv_1$ in (2.33) and division by $H(x)$. The composition of the three operators are considered only for theorectial analysis of sensitivity, and it is not ideal to perform that numerically.*

Remark 2.13. *It is emphasized that we did not claim that*

$$h \mapsto \left(\left[\left(u - r^n Y_M(\omega) \right) (R\,\omega_x) \right] (h) \right)_{M \in I_n, n \in \mathbb{N}}$$

has a bounded inverse, but only that

$$[\mathrm{Proj}_{V_{s,\partial D}}(hH)](x) \mapsto \left(\mathcal{M}^{(1)}_{L,M}(\lambda, D, h) \right)_{L \in I_k, M \in I_n, k,n \leq s}$$

has a bounded inverse under the weighted norm, considering the fact that the inverse problem to reconstruct D from the scattered fields is exponentially ill-posed as is shown in [14, 15, 23] and indicated by the decay order of $\mathcal{M}_{L,M}(\lambda, D)$.

Remark 2.14. *We remark that the mechanism of detecting geometric singularities in [30] is of high frequency nature, while our analysis is in the low frequency regime. High resolution boundary information of the inclusion only enters scattered fields as high order GPTs or SCs as will be shown in the next section, which decay exponentially. This is consistent with the well-known exponential ill-posedness [14, 15, 23], that high resolution information is more prone to the noise contamination. However, if we have large curvature points, higher order GPTs and SCs, albeit still exponentially decaying, will be pushed up to a high magnitude. If a perturbation is further applied, fine details of the perturbation near a high curvature point will be amplified in the far field and easily reconstructable (c.f. Theorem 2.8 and Corollary 2.11.)*

We can describe the above understanding using an example in two dimensions. For instance, we may take a shape D_1 coming as an ε_0-perturbation of a circle D_0 along a direction

$$\tilde{h} = h/\|h\|,$$

where

$$h := \sum_{k=-N}^{N} C^{|k|} e^{ik\theta} = 1 + 2\frac{1 - C\cos(\theta) + C^{N+1}\cos((N-1)\theta) - C^N \cos(N\theta)}{1 + C^2},$$

for a fixed small ε_0 and a fixed large N where $C > 1$, which is highly irragular around the point zero if N is very large. As we expect, the resulting object has the property that the corresponding GPT with order (m,n) has a decaying order of $\delta_{mn}(R/C)^{2n} + \varepsilon_0 (R/C)^m (R/C)^n + O(\varepsilon_0^2)$. We see that if C is comparable with R, the decay is less rapid (up till order N), and hence the high frequency information of the boundary inclusion enters the scattered field more stably. If we further perturb D_1 to D_2 as a δ-perturbation of D_1, with the fact *that the geometry of D_1 has already made GPT with order (m,n) where $|m|,|n|<N$ have a relatively larger magnitude, the scattered measurements projected onto these components are no longer hindered by noise and propagate to the far field. Therefore, the perturbations from D_1 to D_2 can be better detected from the far-field measurements.*

3. Transmission Helmholtz problem in the low frequency regiem and its sensitivity analysis

In this section, we consider the transmission Helmholtz problem. For given positive constants $\varepsilon_0,\mu_0,\varepsilon_1,\mu_1$, we let $k_0=\omega\sqrt{\varepsilon_0\mu_0}$ and $k_1=\omega\sqrt{\varepsilon_1\mu_1}$, and consider u satisfying

$$\begin{cases}\nabla\cdot(\frac{1}{\mu_D}\nabla u)+\omega^2\varepsilon_D u=0 & \text{in } \mathbb{R}^d,\\ (\frac{\partial}{\partial|x|}-ik_0)(u-u_0)=o(|x|^{-\frac{d-1}{2}}) & \text{as } |x|\to\infty,\end{cases}\tag{3.1}$$

where $\mu_D=\mu_1\chi(D)+\mu_0\chi(\mathbb{R}^d\backslash\overline{D})$, $\varepsilon_D=\varepsilon_1\chi(D)+\varepsilon_0\chi(\mathbb{R}^d\backslash\overline{D})$ and u_0 satisfies $(\Delta+k_0^2)=0$. In (3.1), $\omega\in\mathbb{R}_+$ denotes the operating frequency. Throughout the rest of our study, we consider the case that $\omega\ll 1$, or equivalently $k_0\ll 1$. This is referred to as the quasi-static regime. It is in fact an important regime regarding the fact that when ω is small, the resolution of the corresponding inverse problem is considerably poor.

For a given $k\in\mathbb{R}_+$, we introduce

$$\mathcal{S}_{\partial D}^k[\phi](x) := \int_{\partial D} G_k(x-y)\phi(y)d\sigma(y),\tag{3.2}$$

$$\mathcal{D}_{\partial D}^k[\phi](x) := \int_{\partial D}\frac{\partial}{\partial\nu_y}G_k(x-y)\phi(y)d\sigma(y),\tag{3.3}$$

for $x\in\mathbb{R}^d$ with $d\geq 2$, where G_k is the fundamental solution of the Helmholtz equation with outgoing radiation condition in $\mathbb{R}^d$ as follows:

$$\Gamma_k(x-y)=C_{k,d}(k|x-y|)^{-\frac{d-2}{2}}H^{(1)}_{\frac{d-2}{2}}(k|x-y|),\tag{3.4}$$

with C_d some constant depending only on d, and $H^{(1)}_{\frac{d-2}{2}}$ is the Hankel function of the first kind and order $(d-2)/2$. It is known that the single-layer potential $\mathcal{S}_{\partial D}^k$ satisfies the following jump condition on ∂D:

$$\frac{\partial}{\partial\nu}\left(\mathcal{S}_{\partial D}^k[\phi]\right)^{\pm}=(\pm\frac{1}{2}I+{\mathcal{K}_{\partial D}^k}^*)[\phi]\,,\tag{3.5}$$

where the superscripts $\pm$ indicate the limits from outside and inside D respectively, and ${\mathcal{K}_{\partial D}^k}^*:L^2(\partial D)\to L^2(\partial D)$ is the Neumann-Poincaré operator defined by

$${\mathcal{K}_{\partial D}^k}^*[\phi](x):=\int_{\partial D}\partial_{\nu_x}\Gamma_k(x-y)\phi(y)d\sigma(y)\,.\tag{3.6}$$

With the above preparations, $u\in H^1_{loc}(\mathbb{R}^d)$ in (3.1) can be given by

$$u=\begin{cases}u_0+\mathcal{S}_{\partial D}^{k_0}[\psi] & \text{on } \mathbb{R}^d\backslash\overline{D},\\ \mathcal{S}_{\partial D}^{k_1}[\phi] & \text{on } D,\end{cases}$$

where $(\phi,\psi)\in L^2(\partial D)\times L^2(\partial D)$ is the unique solution to (provided that k_1^2 is not a Dirichlet eigenvalue of the Laplacian in D)

$$\begin{cases}\mathcal{S}_{\partial D}^{k_1}[\phi]-\mathcal{S}_{\partial D}^{k_0}[\psi]=u_0\\ \frac{1}{\mu_1}(-\frac{1}{2}I+{\mathcal{K}_{\partial D}^{k_1}}^*)[\phi]-\frac{1}{\mu_0}(\frac{1}{2}I+{\mathcal{K}_{\partial D}^{k0}}^*)[\psi]=\frac{1}{\mu_0}\frac{\partial u_0}{\partial\nu}\,,\end{cases}$$

or that

$$\begin{aligned}
&\left\{\frac{1}{2}\left(\frac{1}{\mu_0}I+\frac{1}{\mu_1}\left(\mathcal{S}_{\partial D}^{k_1}\right)^{-1}\mathcal{S}_{\partial D}^{k_0}\right)+\frac{1}{\mu_0}\mathcal{K}_{\partial D}^{k0\,*}-\frac{1}{\mu_1}\mathcal{K}_{\partial D}^{k_1\,*}\left(\mathcal{S}_{\partial D}^{k_1}\right)^{-1}\mathcal{S}_{\partial D}^{k_0}\right\}[\psi]\\
=\;&\frac{1}{\mu_1}(-\frac{1}{2}I+\mathcal{K}_{\partial D}^{k_1\,*})\circ\left(\mathcal{S}_{\partial D}^{k_1}\right)^{-1}[u_0]-\frac{1}{\mu_0}\frac{\partial u_0}{\partial\nu}\\
=\;&\left(\frac{1}{\mu_1}-\frac{1}{\mu_0}\right)\frac{\partial u_0}{\partial\nu}.
\end{aligned}$$

As in [13], we can now write

$$\begin{aligned}
u-u_0=\left(\frac{1}{\mu_1}-\frac{1}{\mu_0}\right)\mathcal{S}_{\partial D}^{k_0}\circ&\left\{\frac{1}{2}\left(\frac{1}{\mu_0}I+\frac{1}{\mu_1}\left(\mathcal{S}_{\partial D}^{k_1}\right)^{-1}\mathcal{S}_{\partial D}^{k_0}\right)\right.\\
&\left.+\frac{1}{\mu_0}\mathcal{K}_{\partial D}^{k0\,*}-\frac{1}{\mu_1}\mathcal{K}_{\partial D}^{k_1\,*}\left(\mathcal{S}_{\partial D}^{k_1}\right)^{-1}\mathcal{S}_{\partial D}^{k_0}\right\}^{-1}\left[\frac{\partial u_0}{\partial\nu}\right]
\end{aligned}\tag{3.7}$$

where the inverse in the equation exists by the Fredholm alternative theorem.

From the following asymptotics as $z\to+0$,

$$J_\alpha(z)=\frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha+O(z^{\alpha+2})\quad\text{and}\quad Y_\alpha(z)=\begin{cases}\frac{2}{\pi}\left(\log\left(\frac{z}{2}\right)+\gamma\right)+O(z)\\ \frac{\Gamma(\alpha+1)}{\pi}\left(\frac{z}{2}\right)^{-\alpha}+O(z^{-\alpha+2})\end{cases},\tag{3.8}$$

we have

$$\Gamma_k(x-y)=\begin{cases}C_k\left[C_1\log\left(k|x-y|\right)+C_2+O\left(k|x-y|\right)\right] & \text{if } d=2,\\ C_{k,d}\left[k^{2-d}|x-y|^{2-d}+O\left(k^{4-d}|x-y|^{4-d}\right)\right] & \text{if } d>2\,,\end{cases}$$

where C_d only depends on d and C_2 is another constant. Since $\mathcal{S}_{\partial D}$ and $\mathcal{K}_{\partial D}{}^*$ are both of order-1, we have the following lemma as in [13].

Lemma 3.1. *We have the following decompositions for the boundary potential operators,*

$$\mathcal{S}_{\partial D}^k=\mathcal{S}_{\partial D}+\omega^2\,\mathcal{S}_{\partial D,-3}^k\quad\textit{and}\quad\mathcal{K}_{\partial D}^{k\;*}=\mathcal{K}_{\partial D}{}^*+\omega^2\,\mathcal{K}_{\partial D,-3}^k\,,$$

where $\mathcal{K}_{\partial D,-3}^k,\mathcal{S}_{\partial D,-3}^k$ are uniformly bounded w.r.t. ω and are of order -3.

Next, by following the same arguments as those for establishing Theorem 2.3, one can perturb ∂D along the normal direction ν, which in turn gives the perturbations of the boundary potential operators, being pseudo-differential operators with one order higher than the respective original operators. We actually have the following result.

Corollary 3.2. *For all k, $N\in\mathbb{N}$, there exists a constant C depending only on N, $||\mathbb{X}||_{\mathcal{C}^2}$ and $||h||_{\mathcal{C}^2}$ such that the following estimate holds for any $\widetilde{\phi}\in L^2(\partial D_\varepsilon)$ and $\phi:=\widetilde{\phi}\circ\Psi_\varepsilon$:*

$$\left|\left|\mathcal{S}_{\partial D_\epsilon}^k[\tilde{\phi}\circ\Psi_\varepsilon]-\mathcal{S}_{\partial D}^k[\phi]-\sum_{n=1}^N\varepsilon^n\left(\mathcal{S}^k\right)_{D,h}^{(n)}[\phi]\right|\right|_{L^2(\partial D)}\le C\varepsilon^{N+1}||\phi||_{L^2(\partial D)}\,,$$

$$\left|\left|\mathcal{K}_{\partial D_\epsilon}^{k\;*}[\tilde{\phi}\circ\Psi_\varepsilon]-\mathcal{K}_{\partial D}^{k\;*}[\phi]-\sum_{n=1}^N\varepsilon^n\left(\mathcal{K}^k\right)_{D,h}^{(n)}[\phi]\right|\right|_{L^2(\partial D)}\le C\varepsilon^{N+1}||\phi||_{L^2(\partial D)}\,,$$

with

$$\left(\mathcal{S}^k\right)_{D,h}^{(1)}=(\mathcal{S}^0)_{D,h}^{(1)}+\omega^2\left(\mathcal{S}^k\right)_{D,h,-2}^{(1)}\quad\textit{and}\quad\left(\mathcal{K}^k\right)_{D,h}^{(1)}=(\mathcal{K})_{D,h}^{(1)}+\omega^2\left(\mathcal{K}^k\right)_{D,h,-2}^{(1)}\,,$$

where $\left(\mathcal{K}^k\right)_{D,h,-2}^{(1)},\left(\mathcal{S}^k\right)_{D,h,-3}^{(1)}$ are uniformly bounded w.r.t. ω and are of order -2.

Moreover similar to the argument in Section 2.5, since u_0 satisfies $(\Delta + k_0^2)u_0 = 0$, the mapping $u_0 \mapsto \frac{\partial u_0}{\partial \nu}$ can be viewed as the following Dirichlet to Neumann map with respect to the operator $\Delta + k_0^2$, which we denote Λ_{k_0}. Now again using the factorization method as in [35], we have that Λ_{k_0} is a pseudo-differnetial operator of order 1 with its symbol in geodesic normal coordinate for each x being given by,

$$p_{\Lambda_{k_0}}(x,\xi) = p_{\Lambda_0}(x,\xi) - \frac{1}{2}k_0^2\,|\xi|^{-1} + O\left(|\xi|^{-2}\right) .$$

Comparing this symbol with $p_{\Lambda_0}(x,\xi)$, we can obtain the following lemma.

Lemma 3.3. *The following decomposition on ∂D holds:*

$$\Lambda_{k_0} = \Lambda_0 + \omega^2\,\Lambda_{k_0,-1}\,,$$

where $\Lambda_{k_0,-1}$ is uniformly bounded w.r.t. ω and is a pseudo-differntial operator of order -1 with its symbol given by

$$p_{\Lambda_{k_0,-1}}(x,\xi) = -\frac{1}{2}\varepsilon_0\mu_0\,|\xi|^{-1} + O\left(|\xi|^{-2}\right) .$$

Similarly, when we consider the perturbation of Λ_{k_0} with respect to h along the normal direction of ∂D, we may consider the decomposition (2.27) and follow Subsection 2.6 to obtain the perturbation of Λ_{k_0} in terms of $(d-1)H(x)h(x)\Lambda_{k_0}$ and $h(x)\Delta_{\partial D}$, but now with an additional term $h(x)k_0^2$ on the boundary. Therefore, we readily obtain the following:

Corollary 3.4. *For all $k,\ N \in \mathbb{N}$, there exists a constant C depending only on N, $||\mathbb{X}||_{\mathcal{C}^2}$ and $||h||_{\mathcal{C}^2}$ such that the following estimate holds for any $\widetilde{\phi} \in H^{\frac{1}{2}}(\partial D_\varepsilon)$ and $\phi := \tilde{\phi} \circ \Psi_\varepsilon$:*

$$\left|\left|\Lambda_{k_0,\sigma D_\epsilon}[\tilde{\phi}\circ\Psi_\varepsilon] - \Lambda_{k_0,\sigma D}[\phi] - \sum_{n=1}^{N}\varepsilon^n\Lambda^{(n)}_{k_0,D,h}[\phi]\right|\right|_{H^{\frac{1}{2}}(\partial D)} \leq C\varepsilon^{N+1}||\phi||_{L^2(\partial D)}\,,$$

where

$$\Lambda^{(1)}_{k_0,D,h} = \Lambda^{(1)}_{0,D,h} + \omega^2\Lambda^{(1)}_{k_0,D,h,0}\,.$$

Here $\Lambda^{(1)}_{k_0,D,h,0}$ is uniformly bounded w.r.t. ω and is a pseudo-differntial operator of order 0 with

$$p_{\Lambda^{(1)}_{k_0,D,h,0}}(x,\xi) = -\varepsilon_0\mu_0\,h(x) + O\left(|\xi|^{-1}\right) .$$

We would like to remark that the variational derivative of $\Lambda^{(1)}_{k_0,D,h}$ contains the term $\Lambda^{(1)}_{k_0,D,h,0}$ of order 0 instead of -1, in contrast to what one may have expected.

3.1. **Scattering coefficients in arbitrary dimensions.** With the above preparations, we have all the tools to analyse the scattered field in (3.1) in terms of the scattering coefficients as defined in [3,11], which we shall first extend to arbitrary dimensions in what follows.

From an analogous form to Graf's addition formula, we have

$$\begin{aligned}
&(k_0|x-y|)^{-\frac{d-2}{2}}H^{(1)}_{\frac{d-2}{2}}(k_0|x-y|)\\
=\ &\sum_{k=0}^{\infty}(k_0|x|)^{-\frac{d-2}{2}+k}H^{(1)}_{\frac{d-2}{2}+k}(k_0|x|)(k_0|y|)^{-\frac{d-2}{2}+k}J_{\frac{d-2}{2}+k}(k_0|y|)\sum_{L\in I_k}Y_L(\omega_x)\,\overline{Y_L(\omega_y)}
\end{aligned}$$

for $|x| > |y|$. From that, the scattering coefficients can then be defined by putting $u_0 = (kr)^{-\frac{d-2}{2}+k}J_{\frac{d-2}{2}+k}(kr)Y_L(\omega)$ which satisfies $(\Delta + k_0^2)u_0 = 0$, and then taking the coefficient with respect to the function $Y_L(\omega_x)$, i.e. as follows:

Definition 3.5. *The scattering coefficients associated with the scattered field $u - u_0$ given by (3.1) with given $\mu_0, \mu_1, \varepsilon_0, \varepsilon_1, \omega$ and a domain $D \subset \mathbb{R}$ with a $\mathcal{C}^{2,\alpha}$ boundary are defined by*

$$
\begin{aligned}
&\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D)\\
:=\ &\int_{\partial D} (k_0|y|)^{-\frac{d-2}{2}+k} J_{\frac{d-2}{2}+k}(k_0|y|)\,\overline{Y_L(\omega_y)}\Big\{\left(\frac{1}{\mu_1}-\frac{1}{\mu_0}\right)\left\{\frac{1}{2}\left(\frac{1}{\mu_0}I+\frac{1}{\mu_1}\left(\mathcal{S}^{k_1}_{\partial D}\right)^{-1}\mathcal{S}^{k_0}_{\partial D}\right)\right.\\
&\left.+\frac{1}{\mu_0}\mathcal{K}^{k0\ *}_{\partial D}-\frac{1}{\mu_1}\mathcal{K}^{k_1\ *}_{\partial D}\left(\mathcal{S}^{k_1}_{\partial D}\right)^{-1}\mathcal{S}^{k_0}_{\partial D}\right\}^{-1}\circ\Lambda_{k_0}\Big\}\\
&\qquad\times\left[(k_0 r)^{-\frac{d-2}{2}+n}J_{\frac{d-2}{2}+n}(k_0 r)Y_M(\omega)\right](y)\,d\sigma(y)\,,
\end{aligned}
$$

where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ and $k_1 = \omega\sqrt{\mu_1\varepsilon_1}$.

By direct calculations, one can establish the following lemma.

Lemma 3.6. *Let $D \subset \mathbb{R}^d$ be a bounded domain with a $\mathcal{C}^{2,\alpha}$ boundary. Consider the solution to (2.2) with*

$$
u_0(x) = \sum_{k=0}^{\infty}\sum_{L\in I_k} a_L\,(k_0 r_x)^{-\frac{d-2}{2}+k}J_{\frac{d-2}{2}+k}(k_0 r)Y_L(\omega_x)\,.
$$

Then the scattered field for $|x| > \sup\{|x| : x \in D\}$ is given by

$$
\begin{aligned}
&(u-u_0)(x)\\
=&C_{k_0,d}\sum_{k=0}^{\infty}\sum_{L\in I_k}\sum_{n=0}^{\infty}\sum_{M\in I_n} a_M\,(k_0|x|)^{-\frac{d-2}{2}+k}H^{(1)}_{\frac{d-2}{2}+k}(k_0|x|)Y_L(\omega_x)\,\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D)\,.
\end{aligned}
\tag{3.9}
$$

Hence, for $x \in R\,\mathbb{S}^{d-1}$ where $R > \sup\{|x| : x \in D\}$, we have

$$
\begin{aligned}
&\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D)\\
=&\frac{1}{C_{k_0,d}}(k_0|x|)^{\frac{d-2}{2}-k}\frac{1}{H^{(1)}_{\frac{d-2}{2}+k}(k_0R)}\int_{\mathbb{S}^{d-1}}\overline{Y_L(\omega_x)}\Big(u-(k_0r)^{-\frac{d-2}{2}+n}J_{\frac{d-2}{2}+n}(k_0r)Y_M(\omega)\Big)(R\,\omega_x)d\omega_x\,.
\end{aligned}
\tag{3.10}
$$

We note one important property in force:

$$
\begin{aligned}
&\left(\frac{1}{\mu_1}-\frac{1}{\mu_0}\right)\left\{\frac{1}{2}\left(\frac{1}{\mu_0}I+\frac{1}{\mu_1}\left(\mathcal{S}^{k_1}_{\partial D}\right)^{-1}\mathcal{S}^{k_0}_{\partial D}\right)+\frac{1}{\mu_0}\mathcal{K}^{k0\ *}_{\partial D}-\frac{1}{\mu_1}\mathcal{K}^{k_1\ *}_{\partial D}\left(\mathcal{S}^{k_1}_{\partial D}\right)^{-1}\mathcal{S}^{k_0}_{\partial D}\right\}^{-1}\circ\Lambda_{k_0}(u_0)\\
&=\{\lambda I-\mathcal{K}_{\partial D}{}^*\}^{-1}\circ\Lambda_{k_0}(u_0)+\omega^2\,R_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,\partial D,-1}(u_0),
\end{aligned}
$$

where $\lambda = \frac{\mu_0+\mu_1}{2(\mu_0-\mu_1)}$ which is the same equation as that in Section 2, modulus $\omega^2\,R_{\partial D,-1}(u_0)$ for a certain operator $R_{\partial D,-1}$ that is uniformly bounded with respect to ω and is of order -1. Therefore we obtain:

Theorem 3.7. *We have*

$$
u-u_0=\mathcal{S}^{k_0}_{\partial D}\circ\Big(\{\lambda I-\mathcal{K}_{\partial D}{}^*\}^{-1}\circ\Lambda_{k_0}(u_0)+\omega^2\,R_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,\partial D,-1}(u_0)\Big)\,,
$$

where $R_{\partial D,-1}$ is uniformly bounded with respect to ω and is of order -1.

Using the asymptotic properties of J_α and Y_α in (3.8), we can further obtain the following corollary by straightforward calculations.

Lemma 3.8. *We have*

$$\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D)=\mathcal{M}_{L,M}(\lambda,D)+O(\omega^2)\,,$$

where $\lambda=\frac{\mu_0+\mu_1}{2(\mu_0-\mu_1)}$.

3.2. **Sensitivity analysis of the scattering coefficients in the low frequency regime.** Combining Lemmas 3.7 and 2.6, we readily obtain:

Lemma 3.9. *The scattering coefficient* $M_{L,M}(\lambda,\partial D)$ *has the following representation*

$$\begin{aligned}&\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D)\\ =&\Big\langle (k_0r)^{-\frac{d-2}{2}+k}J_{\frac{d-2}{2}+k}(k_0r)Y_L(\omega)\,,\Big(P_{D,1}+P_{D,0}+P_{D,-1}\\ &\qquad+\omega^2\,R_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,\partial D,-1}\Big)\Big((k_0r)^{-\frac{d-2}{2}+n}J_{\frac{d-2}{2}+n}(k_0r)Y_M(\omega)\Big)\Big\rangle_{L^2(\partial D,d\sigma)},\end{aligned}\tag{3.11}$$

where $P_{D,m}$ *are pseudo-differential operators of order* m *for* $m=1,0,-1$ *as given in Lemma 2.6, and* $R_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,\partial D,-1}$ *is of order* -1 *and is uniformly bounded with respect to* ω.

Following the proof of Theorem 2.7 and utilizing Lemmas 3.1-3.3 and 3.7, we can also obtain that

Theorem 3.10. *For* $N\in\mathbb{N}$, *there exists a constant* C *depending only on* N, $L\in I_k, M\in I_n$, $||\mathbb{X}||_{\mathcal{C}^2}$ *and* $||h||_{\mathcal{C}^2}$ *such that*

$$\begin{aligned}&\left|\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D_\varepsilon)-\mathcal{W}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D)-\sum_{n=1}^{N}\varepsilon^n\mathcal{W}^{(n)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D,h)\right|\\ &\le C\varepsilon^{N+1},\end{aligned}\tag{3.12}$$

for some $\mathcal{W}^{(n)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D,h)$ *with* $\mathcal{W}^{(1)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D,h)$ *given by*

$$\begin{aligned}&\mathcal{W}^{(1)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D,h)\\ =&\Big\langle (k_0r)^{-\frac{d-2}{2}+k}J_{\frac{d-2}{2}+k}(k_0r)Y_L(\omega)\,,\,Q_{\mu_0,\widetilde{\mu_1,\varepsilon_0,\varepsilon_1},\omega,D,h}\Big((k_0r)^{-\frac{d-2}{2}+n}J_{\frac{d-2}{2}+n}(k_0r)Y_M(\omega)\Big)\Big\rangle_{L^2(\partial D,d\sigma)},\end{aligned}$$

where

$$Q_{\mu_0,\widetilde{\mu_1,\varepsilon_0,\varepsilon_1},\omega,D,h}=Q_{D,h,1,I}+Q_{D,h,1,II}+Q_{D,h,0}+\omega^2R_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,\partial D,0}$$

with $Q_{D,h,1,I}, Q_{D,h,1,II}, Q_{D,h,0}$ *being the same as those in Theorem 2.7, and* $R_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,\partial D,0}$ *a pseudo-differential operator of order* 0 *and uniformly bounded with respect to* ω.

3.3. **Localization of sensitivity of scattering coefficients at points of high mean curvature.** Similar to Section 2.7, let us consider

$$\mathrm{tr}_{\partial D}\mathrm{Ker}(\Delta+k_0^2):=\{u\,|_{\partial D}\colon (\Delta+k_0^2)u=0\text{ in }\mathbb{R}^d\},$$

where we notice that $\overline{\mathrm{tr}_{\partial D}\mathrm{Ker}(\Delta+k_0)^2)}^{H^s(\partial D,d\sigma)}=H^s(\partial D,d\sigma)$ for all $s\in\mathbb{R}$. Similar to the situation for the generalized polarization tensors, one has

$$\begin{aligned}&\left\{\Big\langle\psi\,,\,Q_{\mu_0,\widetilde{\mu_1,\varepsilon_0,\varepsilon_1},\omega,D,h}\,\phi\Big\rangle_{L^2(\partial D,d\sigma)}\;:\;\psi\in H^s(\partial D,d\sigma),\phi\in H^t(\partial D,d\sigma),s,t\in\mathbb{R},s+t-1=0\right\}\\ =&\left\{\sum_{k,m}\sum_{L\in I_k,M\in I_m}a_Lb_M\mathcal{W}^{(1)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D,h)\;:\;a_L,b_M\in\mathbb{C}\text{ such that the sum converges}\right\}.\end{aligned}$$

It is worth emphasizing in what follows that with suitable choices of $\psi \in H^s(\partial D, d\sigma), \phi \in H^t(\partial D, d\sigma)$ such that $s, t \in \mathbb{R}, s + t - 1 = 0$, one can obtain the principal symbol in the geodesic normal coordinate at each point x as follows

$$\lim_{t\to\infty} t^{-1} e^{-it\varphi_{x,\xi}} \widetilde{Q_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,D,h}} e^{it\varphi_{x,\xi}} \chi_x = p_{Q_{D,h,1,I}}(x,\xi) + p_{Q_{D,h,1,II}}(x,\xi)\,,$$

where $\xi \in \mathbb{S}^{d-1}$ and $\varphi_{x,\xi}(\,\cdot\,) = \langle \xi, \log_x(\,\cdot\,)\rangle$ in half of the injective radius. Let us consider the same complete orthornormal bases on $L^2(\partial D, d\sigma)$ as in Section 2.7, namely $\{\eta_{k,\partial D}\}_{k\in\mathbb{N}}$, and $\{(k_0 r)^{-\frac{d-2}{2}+n} J_{\frac{d-2}{2}+n}(k_0 r) Y_M(\omega)\}_{M\in I_n, n\in\mathbb{N}}$ are also a complete frame by density of $\mathrm{tr}_{\partial D}\mathrm{Ker}(\Delta + k_0^2)$. For r_0 such that $J_{\frac{d-2}{2}+n}(k_0 r_0) \neq 0$ for all $n \in \mathbb{N}$ (otherwise some basis needs to be dropped), let us denote by $\left(\tilde{U}_{k,L,\partial D}\right)$ the map that changes the basis to the orthonormal one and $\left(\tilde{U}^{-1}_{L,k,\partial D}\right)$ as its inverse. Then we render the following:

Theorem 3.11. *For r_0 such that $J_{\frac{d-2}{2}+n}(k_0 r_0) \neq 0$ for all $n \in \mathbb{N}$, we have the following inversion formula for $\partial D \in \mathcal{C}^{2,\alpha}$ and $h \in \mathcal{C}^{2,\alpha}$,*

$$[(1-d)\omega_d + 1]\, h(x) H(x) = \int_{\mathbb{S}^{d-1}} \lim_{t\to\infty} \widetilde{\mathcal{G}}(\xi, t, x) d\sigma(\xi), \tag{3.13}$$

where

$$\widetilde{\mathcal{G}}(\xi,t,x) := \sum_{\substack{L \in I_k,\ M \in I_n\ \tilde{M} \in I_{\tilde{n}},\\ k,n,r,s \in \mathbb{N}}} |\xi|^{-1} t^{-1} e^{-it\varphi_{x,\xi}}\, \eta_{s,\partial D}\, \tilde{U}_{s,L,\partial D}\, \mathcal{W}^{(1)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D,h)$$
$$\times\, \tilde{U}^{-1}_{M,r,\partial D}\, \left\langle \eta_{r,\partial D}\,,\, \chi_x\, e^{it\varphi_{x,\xi}} \right\rangle_{L^2(\partial D, d\sigma)}\,.$$

In the following, we define

$$\mathrm{Proj}_{\tilde{W}_{s,\partial D}} : L^2(\partial D, d\sigma) \to \tilde{W}_s := \mathrm{Span}\{(k_0 r)^{-\frac{d-2}{2}+n} J_{\frac{d-2}{2}+n}(k_0 r) Y_M(\omega)|_{\partial D}\}_{m_{d-1}\le s},$$

for $s \in \mathbb{N}$. Then, via a perturbation analysis, we have

Lemma 3.12. *Given a general $\partial D \in \mathcal{C}^{2,\alpha}$, let ∂D^ϵ be an ε-perturbation along $h \in \mathcal{C}^{2,\alpha}$ and let $S = |\{T : t_{d-1} \le s\}|$. Then for $\varepsilon \in \mathbb{R}_+$ sufficiently small, we have*

$$\begin{aligned}
&\max\Bigg\{ \Big| \|\tilde{U}^{-1}_{L,p,\partial D^\varepsilon}|_{\mathcal{L}(V_{s,\partial D^\varepsilon}, \tilde{W}_{s,\partial D^\varepsilon})}\|_{l^2\to l^2} - \|\tilde{U}^{-1}_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D}, \tilde{W}_{s,\partial D})}\|_{l^2\to l^2} \Big|\,,\\
&\qquad \Big| \|U_{L,p,\partial D^\varepsilon}|_{\mathcal{L}(V_{s,\partial D^\varepsilon}, W_{s,\partial D^\varepsilon})}\|^{-1}_{l^2\to l^2} - \|U_{L,p,\partial D}|_{\mathcal{L}(V_{s,\partial D}, W_{s,\partial D})}\|^{-1}_{l^2\to l^2} \Big| \Bigg\}\\
<\ & 2\varepsilon \max_{1\le P\le S} \Bigg\{ \max\Bigg\{1, \max_{z\neq\lambda_P, z\in\lambda(\Delta_{\partial D})} \frac{\|g\|^2_{C^1}}{|z^2 - \lambda_P^2|}\Bigg\} \|h\|_{C^0} \|A\|_{C^1} \lambda_P^2 \|(k_0 r)^{-\frac{d-2}{2}+s}\\
&\quad \times J_{\frac{d-2}{2}+s}(k_0 r)|_{\partial D}\|_{L^2(\partial D, d\sigma)} + \|h\|_{C^1} \|\partial_\nu((k_0 r)^{-\frac{d-2}{2}+s} J_{\frac{d-2}{2}+s}(k_0 r) Y_T(\omega))|_{\partial D}\|_{L^2(\partial D, d\sigma)}\\
&\quad + \varepsilon(d-1)\|h\|_{C^0}\|H\|_{C^0} \|(k_0 r)^{-\frac{d-2}{2}+s} J_{\frac{d-2}{2}+s}(k_0 r) Y_T(\omega)|_{\partial D}\|_{L^2(\partial D, d\sigma)} \Bigg\}.
\end{aligned}$$

Similar to the specific examples in Section 2, we can have the following results.

Example II.1. Let us consider $\partial D = R_0 \mathbb{S}^{d-1}$ with $R_0 < 1$. Using the previous notations, we have

$$\tilde{U}^{-1}_{L,M,R_0\mathbb{S}^{d-1}} = \omega^{-1}_{d-1} (k_0 R_0)^{k-\frac{d-2}{2}} [J_{\frac{d-2}{2}+k}(k_0 R_0)]^{-1} \delta_{LM}\,.$$

For $k_0R_0/2 \ll 1$, we have

$$(k_0R_0)^{k-\frac{d-2}{2}}[J_{\frac{d-2}{2}+k}(k_0R_0)]^{-1} \sim \Gamma\left(\frac{d-2}{2}+k\right)\left(\frac{k_0^2R_0^2}{2}\right)^{\frac{d-2}{2}-k}.$$

Hence we have the following estimate of the condition number when $\partial D = R_0\mathbb{S}^{d-1}$,

$$\kappa(\tilde{U}_{M,L,R_0\mathbb{S}^{d-1}}|_{\mathcal{L}(\tilde{W}_{s,R_0\mathbb{S}^{d-1}},V_{s,R_0\mathbb{S}^{d-1}})}) \leq C\sqrt{\frac{s+d-2}{2}}\left(\frac{k_0^2R_0^2\,(s+d-2)}{2}\right)^{\frac{d-2}{2}+s}.$$

Example II.2. Consider ∂D^δ as follows: $\partial D = R_0\mathbb{S}^{d-1}$ with perturbation $k \in C^2$, and $\|k\|_{C^2} < 1$ and small magnitude δ. Now, applying Lemma 3.12, together with $\lambda_{T,\mathbb{S}^d} = s(s+d-2)$ for $t_{d-1} = s$, we have for δ small enough and large enough s,

$$\begin{aligned}
&\max\Bigg\{\left|\|\tilde{U}^{-1}_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},\tilde{W}_{s,\partial D^\delta})}\|_{l^2\to l^2} - \omega_{d-1}^{-1}\,R_0^{-\frac{d-1}{2}}\right|,\\
&\qquad\left|\|\tilde{U}_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},\tilde{W}_{s,\partial D^\delta})}\|^{-1}_{l^2\to l^2} - \omega_{d-1}^{-1}\,R_0^{-\frac{d-1}{2}}\Gamma\left(\frac{d-2}{2}+k\right)\left(\frac{k_0^2R_0^2}{2}\right)^{\frac{d-2}{2}-s}\right|\Bigg\}\\
<\;& 2\delta(2R_0+2s-d+2)(2s-d+2)\left(\left(\frac{d-2}{2}+s\right)k_0^2\right)^{\frac{d-2}{2}-s}R_0^{-2s-1},
\end{aligned}$$

and therefore, for small enough δ, we obtain

$$\begin{aligned}
&\kappa(\tilde{U}_{L,p,\partial D^\delta}|_{\mathcal{L}(V_{s,\partial D^\delta},W_{s,\partial D^\delta})})\\
\leq\;& \frac{k_0^{d-2-2s}R_0^{\frac{d-2}{2}-2s} + 2\delta\,\omega_{d-1}(2R_0+2s-d+2)k_0^{d-2-2s}R_0^{-2s-1}}{(2s-d+2)^{-\frac{d-2}{2}-s-1}\,R_0^{-\frac{d-1}{2}} - 2\delta\,\omega_{d-1}(2R_0+2s-d+2)k_0^{d-2-2s}R_0^{-2s-1}}.
\end{aligned}$$

From the above example, we have similar results to those in Corollary 2.11.

Corollary 3.13. *Let ∂D^δ be a δ-perturbation of $\partial D = R_0\mathbb{S}^{d-1}$ along the direction $k \in \mathcal{C}^2(\partial D)$ with $\|k\|_{\mathcal{C}^2} < 1$ for sufficiently small $\delta \in \mathbb{R}_+$. Then for $h \in \mathcal{C}^2(\partial D^\delta)$ with $\|h\|_{\mathcal{C}^2} < 1$, considering an ε-perturbation of ∂D^δ along the direction h, $(\partial D^\delta)_\varepsilon$, we have*

$$\begin{aligned}
&|[\mathrm{Proj}_{V_{s,\partial D^\delta}}(hH)](x)|\\
\leq\;& C_d\frac{k_0^{d-2-2s}R_0^{\frac{d-2}{2}-2s} + 2\delta\,\omega_{d-1}(2R_0+2s-d+2)k_0^{d-2-2s}R_0^{-2s-1}}{(2s-d+2)^{-\frac{d-2}{2}-s-1}\,R_0^{-\frac{d-1}{2}} - 2\delta\,\omega_{d-1}(2R_0+2s-d+2)k_0^{d-2-2s}R_0^{-2s-1}}\\
&\times\|\mathcal{W}^{(1)}_{L,M}(\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,D^\delta,h)\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|[\mathrm{Proj}_{V_{s,\partial D^\delta}}(hH)](x)|\\
\leq\;& C_dC_{k_0,d}(k_0|x|)^{\frac{d-2}{2}-k}\frac{1}{|H^{(1)}_{\frac{d-2}{2}+k}(k_0R)|}\\
\times\;& \frac{k_0^{d-2-2s}R_0^{\frac{d-2}{2}-2s} + 2\delta\,\omega_{d-1}(2R_0+2s-d+2)k_0^{d-2-2s}R_0^{-2s-1}}{(2s-d+2)^{-\frac{d-2}{2}-s-1}\,R_0^{-\frac{d-1}{2}} - 2\delta\,\omega_{d-1}(2R_0+2s-d+2)k_0^{d-2-2s}R_0^{-2s-1}}\\
\times\;& \left\|\frac{\partial}{\partial\epsilon}\left(\left\langle Y_L(\omega_x)\,,\,\left(u_{(\partial D^\delta)_\epsilon} - (k_0r)^{-\frac{d-2}{2}+n}J_{\frac{d-2}{2}+n}(k_0r)Y_M(\omega)\right)(R\,\omega_x)\right\rangle_{L^2(R\mathbb{S}^{d-1},d\omega_x)}\right)\right\|_{\mathcal{L}(V_{s,\mathbb{S}^{d-1}},V_{s,\mathbb{S}^{d-1}})}.
\end{aligned}$$

The proof of Corollary 3.13 is similar to that of Corollary 2.11, and therefore we skip it. By Corollary 3.13, we readily see that the reconstruction of $h(x)$ from $\mathcal{W}^{(1)}_{L,M}(\mu_0, \mu_1, \varepsilon_0, \varepsilon_1, D, h)$ is more sensitive at points with high mean curvature $|H(x)|^2$ when D is not too far from $R_0\mathbb{S}^{d-1}$.

Acknowledgment

The work of H. Ammari was supported by the SNF grant 200021–172483. The work of H. Liu was supported by the Hong Kong UGC General Research Funds, 12302017, 12301218 and 12302919.

References

[1] H. Ammari, Y.T. Chow, K. Liu, and J. Zou, *Optimal Shape Design by Partial Spectral Data*, SIAM J. Sci. Comput., **37** (2015), B855-B883.

[2] H. Ammari, Y.T. Chow, and J. Zou, *Super-resolution in imaging high contrast targets from the perspective of scattering coefficients*, J. Math. Pures Appl. **111** 191-226, 2018.

[3] H. Ammari, Y.T. Chow and J. Zou, *The concept of heterogeneous scattering coefficients and its application in inverse medium scattering*, SIAM J. Math. Anal., **46** (2014), pp. 2905-2935.

[4] H. Ammari, G. Ciraolo, H. Kang, H. Lee, and G. Milton, *Spectral analysis of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance*, Archive for Rational Mechanics and Analysis, **208** (2013), 667–692.

[5] H. Ammari, Y. Deng, H. Kang and H. Lee *Reconstruction of Inhomogeneous Conductivities via the Concept of Generalized Polarization Tensors,* Annales de l'Institut Henri Poincare (C) Non Linear Analysis, **31** (2014), 877-897.

[6] H. Ammari, B. Fitzpatrick, H. Kang, M. Ruiz, S. Yu, and H. Zhang, Mathematical and Computational Methods in Photonics and Phononics. Mathematical Surveys and Monographs, Volume 235, American Mathematical Society, Providence, 2018.

[7] H. Ammari, J. Garnier, H. Kang, M. Lim, and S. Yu, *Generalized polarization tensors for shape description*, Numer. Math., DOI 10.1007/s00211-013-0561-5.

[8] H. Ammari and H. Kang, *Polarization and Moment Tensors: With Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences 162, Springer-Verlag, New York, 2007.

[9] H. Ammari and H. Kang, *Reconstruction of small inhomogeneities from boundary measurements*, Springer, 2004.

[10] H. Ammari, H. Kang and H. Lee, Layer Potential Techniques in Spectral Analysis. Mathematical Surveys and Monographs, Volume 153, American Mathematical Society, Providence, 2009.

[11] H. Ammari, H. Kang, H. Lee, and M. Lim, *Enhancement of near-cloaking. Part II: the Helmholtz equation*, Comm. Math. Phys., **317** (2013), pp. 485-502.

[12] H. Ammari, H. Kang, M. Lim, and H. Zribi, *The generalized polarization tensors for resolved imaging. Part I: Shape reconstruction of a conductivity inclusion*, Math. Comp., **81** (2012), 367–386.

[13] H. Ammari, P. Millien, M. Ruiz, H. Zhang, Mathematical Analysis of Plasmonic Nanoparticles: The Scalar Case, Archive for Rational Mechanics and Analysis, **224** (2017), pp. 597-658.

[14] G. Bao, J. Lin F. Triki, A multi-frequency inverse source problem, J. Differential Equations, **249** (2010), pp. 3443-3465.

[15] G. Bao, P. Li, J. Lin F. Triki, Inverse scattering problems with multifrequencies, Inverse Problems, **31** (2015), 093001 (21pp).

[16] E. Blåsten and H. Liu, *On corners scattering stably and stable shape determination by a single far-field pattern*, Indiana Univ. Math. J, in press, 2019.

[17] E. Blåsten and H. Liu, *Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems,* arXiv: 1808.01425, 2018.

[18] E. Blåsten, H. Li, H. Liu and Y. Wang, *Localization and geometrization in plasmon resonances and geometric structures of Neumann-Poincaré eigenfunctions*, preprint, arXiv:1809.08533, 2018.

[19] E. Blåsten, L. Päivärinta and J. Sylvester, *Corners always scatter*, Comm. Math. Phys., **331** (2014), 725–753.

[20] E. Bonnetier and H. Zhang, *Characterization of the essential spectrum of the Neumann-Poincaré operator in 2D domains with corner via Weyl sequences*, Rev. Mat. Iber., in press, 2018.

[21] D. Choi, J. Helsing and M. Lim, *Corner effects on the perturbation of an electric potential* , SIAM J. Appl. Math., **78** (2018), 1577-1601.

[22] D. Colton and R. Kress, *Looking back on inverse scattering theory*, SIAM Rev., **60** (2018), 779–807.

[23] M. D. Cristo and L. Rondi, *Exponential instability for inverse elliptic problems with unknown boundaries*, J. Phys.: Conf. Ser., **73** (2007), 012005 (18 pp).
[24] H. Kang, M. Lim and S. Yu, *Spectral resolution of the Neumann-Poincaré operator on intersecting disks and analysis of plasmon resonance*, Arch. Rati. Mech. Anal., **226** (2017), 83–115.
[25] H. Kang and J.K. Seo, *Inverse conductivity problem with one measurement: uniqueness of balls in* $\mathbb{R}^3$, SIAM J. Appl. Math., **59** (1999), 851–867.
[26] O.D. Kellogg, Foundations of Potential Theory, Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31 Springer-Verlag, Berlin-New York, 1967.
[27] D. Khavinson, M. Putinar, and H.S. Shapiro, Poincaré's variational problem in potential theory, Arch. Rational Mech. Anal., **185** (2007) 143–184.
[28] H. Liu, M. Petrini, L. Rondi and J. Xiao, *Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements*, J. Differential Equations, **262** (2017), no. 3, 1631–1670.
[29] H. Liu, L. Rondi and J. Xiao, *Mosco convergence for* H(curl) *spaces, higher integrability for Maxwell's equations, and stability in direct and inverse EM scattering problems*, Journal of the European Mathematical Society (JEMS), in press, 2018.
[30] A. Majda, *High frequency asymptotics for the scattering matrix and the inverse problem of acoustical scattering*, Comm. Pure Appl. Math., **29** (1976), 261–291.
[31] I.D. Mayergoyz, D.R. Fredkin, and Z. Zhang, Electrostatic (plasmon) resonances in nanoparticles, Phys. Rev. B, **72** (2005), 155412.
[32] Y. Miyanishi, *Weyl's law for the eigenvalues of the Neumann-Poincaré operators in three dimensions: Willmore energy and surface geometry,* Preprint, arXiv:1806.03657
[33] Y. Miyanishi and G. Rozenblum, *Eigenvalues of the Neumann-Poincaré operators in dimension 3: Weyl's Law and geometry,* Preprint, arXiv:1812.00582
[34] J.E. Osborn, Spectral approximation for compact operators, Math. Comp., 29 (1975), 712–725.
[35] G. Uhlmann, *The Dirichlet to Neumann Map and Inverse Problems,* preprint.

Department of Mathematics, ETH Zürich, Rämistrasse 101, CH-8092, Switzerland
E-mail address: habib.ammari@math.ethz.ch

Department of Mathematics, University of California, Riverside, USA
E-mail address: yattinc@ucr.edu

Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong SAR, China
E-mail address: hongyu.liuip@gmail.com, hongyuliu@hkbu.edu.hk