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Abstract

The main result of this article establishes strong convergence rates on the whole probability space for explicit space-time discrete numerical approximations for a class of stochastic evolution equations with possibly non-globally monotone coefficients such as stochastic Burgers equations with additive trace-class noise. The key idea in the proof of our main result is (i) to bring the classical Alekseev-Gröbner formula from deterministic analysis into play and (ii) to employ uniform exponential moment estimates for the numerical approximations.

Contents

1 Introduction

In this article we study the problem of establishing strong convergence rates for explicit space-time discrete approximations of semilinear stochastic evolution equations (SEEs) with non-globally monotone coefficients (see, e.g., Liu & Röckner [55, (H2) in Chapter 4] for global monotonicity) such as stochastic Burgers equations. Proving strong convergence with rates for numerical approximations of SEEs with non-globally monotone coefficients is known to be challenging. In fact, there exist stochastic ordinary differential equations (SODEs) with smooth and globally bounded but non-globally monotone coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge strongly to their solutions faster than any given speed of convergence (see Jentzen et al. [48, Theorem 1.3], Hairer et al. [32], and also, e.g., [28, 34, 61, 69, 70]). In addition, the classical Euler-Maruyama method, the exponential Euler method, and the linear-implicit Euler method fail to converge strongly as well as weakly for some SEEs with superlinearly growing coefficients (see, e.g., Hutzenthaler et al. [39, Theorem 2.1] and Hutzenthaler et al. [41, Theorem 2.1] for SODEs and Beccari et al. [4] for stochastic partial differential equations (SPDEs)).

Recently, a series of appropriately modified versions of the explicit Euler method have been introduced and proven to converge strongly for some SEEs with superlinearly growing coefficients (see, e.g., [37, 38, 40, 63, 64, 66, 67] for SODEs and, e.g., [5, 7, 30, 42, 50, 51, 57] for SPDEs). These methods are easily implementable and tame the superlinearly growing terms in order to ensure strong convergence. Strong convergence rates for explicit time discrete and explicit space-time discrete numerical methods for SPDEs with a non-globally Lipschitz continuous but globally monotone nonlinearity have been derived in, e.g., Becker et al. [5, Theorems 1.1 and 5.5], Becker & Jentzen [7, Corollaries 6.15 and 6.17], Brehiér et al. [12, Theorem 3.1], and Jentzen & Pušnik [50, Theorem 1.1]. Moreover, suitable nonlinear-implicit approximation schemes are known to converge strongly in the case of several SEEs with superlinearly growing coefficients (see, e.g., [35, 36] for SODEs and, e.g., [13, 26, 27, 29, 53, 54, 56] for SPDEs). Strong convergence rates for temporal and spatio-temporal approximations of SEEs with non-globally monotone coefficients on suitable large subsets of the probability space (sometimes referred to as semi-strong convergence rates) have been established in, e.g., Bessaih et al. [8, Theorem 5.2], Carelli & Prohl [14, Theorems 3.1, 3.2, and 4.2], and Furihata et al. [27, Theorem 5.3]. These semi-strong convergence rates can imply convergence in probability, but they are not sufficient to prove strong convergence rates. For completeness, we also refer to, e.g., [1, 10, 11, 16, 52, 62, 71, 72, 73] for results concerning convergence in probability with and without rates, pathwise convergence with rates, and strong convergence without rates for numerical approximations of SEEs with superlinearly growing coefficients. Weak convergence with rates for splitting approximations of 2D stochastic Navier-Stokes equations has been established in [25]. In Bessaih & Millet [9, Theorem 4.6] strong convergence with rates is proven for fully drift-implicit Euler approximations in the case of 2D stochastic Navier-Stokes equations with additive trace-class noise by exploiting a rather specific property (see Bessaih & Millet [9, (2.4) in Section 2]) of the Navier-Stokes-nonlinearity (see also Bessaih & Millet [9, Theorems 3.6, 3.9, and 4.4 and Proposition 4.8] for further strong convergence results). These fully drift-implicit Euler approximations of 2D stochastic Navier-Stokes equations involve solutions of nonlinear equations that are not known to be unique and it is unknown how to approximate these solutions with positive convergence rates. Strong convergences rates for nonlinear-implicit numerical schemes for SEEs with non-globally monotone coefficients have also been analyzed in Cui & Hong [18, 19] and Cui et al. [21, 22] (cf. also, e.g., Cui et al. [20] and Yang & Zhang [68]).

To the best of our knowledge, there exist no results in the scientific literature establishing strong convergence with rates on the whole probability space for an explicit space-time discrete numerical method for an evolutionary SPDE with a non-globally monotone nonlinearity such as stochastic Burgers equations, stochastic Navier-Stokes equations, stochastic Kuramoto-Sivashinsky equations, Cahn-Hilliard-Cook equations, or stochastic nonlinear Schrödinger equations. It is the key contribution of this work to partially solve this problem and to establish strong convergence rates for an appropriately tamed-truncated exponential Euler-type method for SPDEs with a possibly non-globally monotone nonlinearity and additive trace-class noise (see Theorem 5.9 below). In particular, in Corollary 6.2 below we derive strong convergence rates for explicit space-time discrete approximations of stochastic Burgers equations. A slightly simplified version of Corollary 6.2 below is given in the following theorem.

Theorem 1.1. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue-Borel square-integrable functions from $(0,1)$ to $\mathbb{R}$, let $A\colon D(A) \subseteq H \to H$ be the Laplacian with zero Dirichlet boundary conditions on H, let $T \in (0,\infty)$, $c \in \mathbb{R}$, $\xi \in D(A)$, $\beta \in (0, 1/2]$, $B \in \mathrm{HS}(H, D((-A)^{\beta}))$, $(e_n)_{n\in\mathbb{N}} \subseteq H$ satisfy for every $n \in \mathbb{N}$ that $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot))$, let $(P_N)_{N\in\mathbb{N}} \subseteq L(H)$ satisfy for every $N \in \mathbb{N}$, $v \in H$ that $P_N(v) = \sum_{n=1}^{N} \langle e_n, v\rangle_H e_n$, let $F\colon D((-A)^{1/2}) \to H$ be the function which satisfies for every $v \in D((-A)^{1/2})$ that $F(v) = c\, v'v$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical Wiener process, let $W^N\colon [0,T]\times\Omega \to P_N(H)$, $N \in \mathbb{N}$, be stochastic processes which satisfy for every $N \in \mathbb{N}$, $t \in [0,T]$ that $\mathbb{P}(W_t^N = \int_0^t P_N B\, dW_s) = 1$, and let $\mathbf{X}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M, N \in \mathbb{N}$, be the stochastic processes which satisfy for every $M, N \in \mathbb{N}$, $m \in \{0, 1, \ldots, M-1\}$, $t \in (mT/M, (m+1)T/M]$ that $\mathbf{X}_0^{M,N} = P_N(\xi)$ and*

$$\begin{aligned}\mathbf{X}_t^{M,N} = e^{(t - mT/M)A}\Big(&\mathbf{X}_{mT/M}^{M,N} \\ &+ \mathbb{1}_{\{1+\|(-A)^{1/2}\mathbf{X}_{mT/M}^{M,N}\|_H^2 \le (M/T)^{1/19}\}}\Big[P_N F(\mathbf{X}_{mT/M}^{M,N})\,(t - (mT/M)) + \tfrac{W_t^N - W_{mT/M}^N}{1+\|W_t^N - W_{mT/M}^N\|_H^2}\Big]\Big).\end{aligned} \tag{1}$$

Then

(i) there exists an up to indistinguishability unique stochastic process $X\colon [0,T]\times\Omega \to D((-A)^{1/2})$ with continuous sample paths which satisfies that for every $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\, ds + \int_0^t e^{(t-s)A}B\, dW_s \tag{2}$$

and

(ii) for every $\varepsilon, p \in (0,\infty)$ there exists $C \in \mathbb{R}$ such that for every $M, N \in \mathbb{N}$ it holds that

$$\sup\nolimits_{t\in[0,T]} \big(\mathbb{E}[\|X_t - \mathbf{X}_t^{M,N}\|_H^p]\big)^{1/p} \le C\big(M^{(\varepsilon-\beta)} + N^{(\varepsilon-2\beta)}\big). \tag{3}$$

Theorem 1.1 is an immediate consequence of Corollary 6.2 in Section 6 below (with $T = T$, $\varepsilon = \varepsilon$, $c_0 = 1$, $c_1 = c$, $\varsigma = 1/19$, $p = \max\{p, 1\}$, $\beta = \beta$, $\gamma = 1/2$, $H = H$, $e_n = e_n$, $A = A$, $H_r = D((-A)^r)$, $B = B$, $\xi = \xi$, $F = F$, $P_N = P_N$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\mathbf{X}^{\{0,T/M,\ldots,T\},N} = \mathbf{X}^{M,N}$ for $M, N, n \in \mathbb{N}$, $\varepsilon, p \in (0,\infty)$, $r \in [0,\infty)$ in the notation of Corollary 6.2) and Hölder's inequality. Corollary 6.2, in turn, is a consequence of Theorem 5.9 in Subsection 5.2 below (the main result of this work). We note that if the diffusion coefficient B is a diagonal operator with respect to the orthonormal basis $(e_n)_{n\in\mathbb{N}} \subseteq H$, then the processes W^N, $N \in \mathbb{N}$, in Theorem 1.1

above are Wiener processes with computable covariance structure (cf. Corollary 5.3 below) and the approximation scheme (1) is directly implementable up to an additional approximation error resulting from the numerical evaluations of Galerkin projections P_N, $N \in \mathbb{N}$. We now briefly sketch the key ideas which we employ to prove Theorem 1.1. In the case of SPDEs with globally monotone nonlinearities one can, very roughly speaking, apply the Itô formula to the squared Hilbert space norm of the difference between the exact solution of the SPDE and its numerical approximation and, thereafter, employ the global monotonicity property together with Gronwall's lemma and suitable uniform moment bounds for the solution and the numerical approximations to establish strong convergence rates. This procedure, however, fails in the case of SPDEs with non-globally monotone coefficients. We overcome this issue by bringing the classical Alekseev-Gröbner formula from deterministic numerical analysis (see, e.g., Hairer et al. [31, Theorem 14.5]) into play and by employing the fact that the considered approximation processes $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$, (see (1) above) have uniformly bounded exponential moments. More specifically, we apply the extended version of the Alexeev-Gröbner formula in [46, Corollary 5.2] to a spatially semi-discrete version of the solution $(X_t)_{t\in[0,T]}$ of the considered SPDE (see (2) above) and its numerical approximations $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$, (see (1) above) in order to derive a suitable error representation (cf. Lemma 2.3 below). This allows us to estimate the strong approximation error by an appropriate integral expression involving two main terms (cf. (48) in Corollary 2.9 below) which we analyze independently. The first main term is, very roughly speaking, the derivative of the spatially semi-discrete version of $(X_t)_{t\in[0,T]}$ with respect to its initial value, evaluated in a function of the numerical approximations $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$, and the Wiener process $(W_t)_{t\in[0,T]}$. The second main term is a function of the numerical approximations $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$, and the Wiener process $(W_t)_{t\in[0,T]}$ but does not involve the spatially semi-discrete version of $(X_t)_{t\in[0,T]}$ (cf. Corollary 2.9 below). A key step in establishing strong convergence rates is, loosely speaking, to obtain a uniform moment bound for the derivative of the spatially semi-discrete version of $(X_t)_{t\in[0,T]}$ with respect to its initial value in terms of an appropriate functional of the spatially semi-discrete version of $(X_t)_{t\in[0,T]}$ and the numerical approximations $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$ (cf. Corollary 3.3 below). Applying a general result on exponential integrability from Cox et al. [17, Corollary 2.4], this moment bound is then further estimated by appropriate exponential moments of the numerical approximations $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$ (cf. Lemma 3.5 below). The exponential moments established in [45, 49] therefore yield a uniform upper bound for the first main term in the initial strong error estimate (cf. Proposition 4.5, Corollary 5.5, and the proof of Theorem 5.9 below). The fact that the numerical approximations $(\mathbf{X}_t^{M,N})_{t\in[0,T]}$, $M, N \in \mathbb{N}$, enjoy sufficient regularity properties (cf. Corollary 5.7 and the regularity results in [45, 47]) ensures that the second main term in the initial strong error estimate converges strongly with rates (cf. Proposition 4.5 and the proof of Theorem 5.9 below). Combining the estimates for both main terms in the initial strong error estimate finally establishes strong convergence rates for explicit space-time discrete approximations of the SPDE under consideration (cf. Theorem 5.9 and Corollaries 5.10, 6.1, and 6.2 below).

Let us comment on the optimality of the convergence rates obtained in Theorem 1.1. It is not clear to us whether the established strong convergence rates are essentially optimal or whether they can be substantially improved. In the simplified case $c = 0$, where the nonlinearity is omitted and the stochastic Burgers equation in (2) reduces to a stochastic heat equation, lower bounds for strong and weak approximation errors are well understood (see, e.g., Becker et al. [6], Conus et al. [15], Davie & Gaines [24], Jentzen & Kurniawan [44], Müller-Gronbach & Ritter [58], Müller-Gronbach et al. [59, 60], and the references mentioned therein). In particular, e.g, Becker et al.

[6, Theorem 1.1], Conus et al. [15, Lemma 7.2], Davie & Gaines [24, Section 2.1], and Müller-Gronbach et al. [60, Theorem 4.2] indicate that the convergence rates in Theorem 1.1 above might not be optimal in the case $c = 0$. In the case $c \neq 0$, where the nonlinearity does not vanish, lower bounds for strong and weak approximation errors remain on open problem for future research.

The remainder of this article is structured as follows. In Subsection 2.1 we apply the Alexeev-Gröbner formula from [46, Corollary 5.2] and establish in Lemma 2.5 below a general pathwise estimate. Combining this general pathwise estimate with suitable measurability results from the scientific literature allows us to establish in Corollary 2.9 in Subsection 2.2 below a strong $\mathcal{L}^p$ estimate for the difference between the spatially semi-discrete version of the solution of the considered SPDE and the considered numerical approximations. In Subsection 3.1 we employ Cox et al. [17, Corollary 2.4] to provide an appropriate a priori bound for the derivative of the spatially semi-discrete version of the solution of the considered SPDE with respect to its initial value (see (88) in Lemma 3.5 below). In Subsection 3.2 we combine the results from Section 2 and Subsection 3.1 to obtain in Proposition 3.6 a simplified upper bound for the strong error. In Subsection 4.1 we establish suitable uniform moment bounds for the spatially semi-discrete version of the considered SPDE which we then employ in Subsection 4.2 together with Proposition 3.6 to prove in Proposition 4.5 strong convergence with rates for space-time discrete numerical approximations with suitable integrability and regularity properties for a large class of SPDEs. In Subsection 5.1 we show that the considered tamed-truncated numerical scheme enjoys appropriate integrability and measurability properties. These properties are then used together with Proposition 4.5 to establish in Theorem 5.9 in Subsection 5.2 below (see also Corollary 5.10) strong convergence rates for the considered tamed-truncated numerical scheme. In Section 6 we combine in Corollaries 6.1 and 6.2 the results established in [47] with Corollary 5.10 in this article to establish strong convergence rates in the case of stochastic Burgers equations with additive trace-class noise.

1.1 General setting

Throughout this article the following setting is frequently used.

Setting 1.2. *For every measurable space $(\Omega_1, \mathcal{F}_1)$ and every measurable space $(\Omega_2, \mathcal{F}_2)$ let $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ be the set of all $\mathcal{F}_1/\mathcal{F}_2$-measurable functions, for every set X let $\mathcal{P}(X)$ be the power set of X, for every set X let $\mathcal{P}_0(X)$ be the set given by $\mathcal{P}_0(X) = \{\theta \in \mathcal{P}(X) \colon \theta \text{ is a finite set}\}$, for every $T \in (0,\infty)$ let ϖ_T be the set given by $\varpi_T = \{\theta \in \mathcal{P}_0([0,T]) \colon \{0,T\} \subseteq \theta\}$, for every $T \in (0,\infty)$ let $|\cdot|_T : \varpi_T \to [0,T]$ be the function which satisfies for every $\theta \in \varpi_T$ that*

$$|\theta|_T = \max\Big\{x \in (0,\infty) \colon \big(\exists\, a,b \in \theta \colon \big[x = b - a \text{ and } \theta \cap (a,\infty) \cap (-\infty,b) = \emptyset\big]\big)\Big\}, \tag{4}$$

for every $\theta \in (\cup_{T\in(0,\infty)} \varpi_T)$ let $\llcorner \cdot \lrcorner_\theta \colon [0,\infty) \to [0,\infty)$ be the function which satisfies for every $t \in (0,\infty)$ that $\llcorner t \lrcorner_\theta = \max([0,t) \cap \theta)$ and $\llcorner 0 \lrcorner_\theta = 0$, and for every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space $(S, \mathcal{S})$, every set R, and every function $f \colon \Omega \to R$ let $[f]_{\mu,\mathcal{S}}$ be the set given by $[f]_{\mu,\mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F}, \mathcal{S}) \colon (\exists\, A \in \mathcal{F} \colon \mu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A)\}$.

Setting 1.3. *Assume Setting 1.2, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be non-zero separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, let $\mathfrak{v} \colon \mathbb{H} \to \mathbb{R}$ be a function which satisfies $\sup_{h\in\mathbb{H}} \mathfrak{v}_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \{v \in H \colon \sum_{h\in\mathbb{H}} |\mathfrak{v}_h \langle h, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A) \colon Av = \sum_{h\in\mathbb{H}} \mathfrak{v}_h \langle h, v\rangle_H h$, and let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [65, Section 3.7]).*

Note that the assumption in Setting 1.3 above that $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, is a family of interpolation spaces associated to $-A$ ensures that for every $r \in [0,\infty)$ it holds that $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r}) = (D((-A)^r), \langle(-A)^r(\cdot),(-A)^r(\cdot)\rangle_H, \|(-A)^r(\cdot)\|_H)$.

2 Time discretization error estimates based on an Alexeev-Gröbner-type formula

Setting 2.1. *Assume Setting 1.3, assume that* $\dim(H) < \infty$*, let* $T \in (0,\infty)$*,* $\theta \in \varpi_T$*,* $\xi \in H$*,* $O \in \mathcal{C}([0,T],H)$*,* $\mathbf{O} \in \mathcal{M}(\mathcal{B}([0,T]),\mathcal{B}(H))$*,* $F \in \mathcal{C}^1(H,H)$*, let* $\mathbf{F}\colon H \to H$ *be a function, for every* $s \in [0,T]$*,* $x \in H$ *let* $X^x_{s,(\cdot)} = (X^x_{s,t})_{t\in[s,T]}\colon [s,T] \to H$ *be a continuous function which satisfies for every* $t \in [s,T]$ *that*

$$X^x_{s,t} = e^{(t-s)A}x + \int_s^t e^{(t-u)A}F(X^x_{s,u})\,du + O_t - e^{(t-s)A}O_s, \tag{5}$$

and let $\mathbf{X}\colon [0,T] \to H$ *be the function which satisfies for every* $t \in [0,T]$ *that*

$$\mathbf{X}_t = e^{tA}\xi + \int_0^t e^{(t-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\,du + \mathbf{O}_t. \tag{6}$$

Note that for every topological space (X,τ) it holds that $\mathcal{B}(X)$ is the smallest sigma-algebra on X which contains all elements of τ.

2.1 Pathwise temporal approximation error estimates

In this subsection we apply the extended Alekseev-Gröbner formula in [46, Corollary 5.2] to express the difference between the exact solution $(X^{\xi+O_0}_{0,t})_{t\in[0,T]}$ of the integral equation (5) above, started at time $s = 0$ in $x = \xi + O_0$, and the corresponding numerical approximation $(\mathbf{X}_t)_{t\in[0,T]}$ in (6) above in terms of an appropriate integral in Lemma 2.3 below. We then combine these auxiliary results with Lemma 2.3 and Lemma 2.4 to derive an upper bound for the approximation error in Lemma 2.5.

Lemma 2.2. *Assume Setting 1.3, assume that* $\dim(H) < \infty$*, let* $T \in (0,\infty)$*,* $s \in [0,T]$*,* $x \in H$*,* $Z \in \mathcal{M}(\mathcal{B}([s,T]),\mathcal{B}(H))$ *satisfy* $\int_s^T \|Z_u\|_H\,du < \infty$*, and let* $Y\colon [s,T] \to H$ *be the function which satisfies for every* $t \in [s,T]$ *that* $Y_t = e^{(t-s)A}x + \int_s^t e^{(t-u)A}Z_u\,du$*. Then*

(i) it holds that $Y \in \mathcal{C}([s,T],H)$ *and*

(ii) it holds for every $t \in [s,T]$ *that* $Y_t = x + \int_s^t [AY_u + Z_u]\,du$*.*

Proof of Lemma 2.2. Throughout this proof assume w.l.o.g. that $s \in [0,T)$. Note that the fact that $\dim(H) < \infty$ ensures that for every $t \in [s,T]$ it holds that $\int_s^T \|e^{(s-u)A}Z_u\|_H\,du < \infty$ and

$$Y_t = e^{(t-s)A}\bigg(x + \int_s^t e^{(s-u)A}Z_u\,du\bigg). \tag{7}$$

Moreover, observe that the dominated convergence theorem implies that

$$\bigg([s,T] \ni t \mapsto \int_s^t e^{(s-u)A}Z_u\,du\bigg) \in \mathcal{C}([s,T],H). \tag{8}$$

Combining (7) and the fact that $([s,T] \ni t \mapsto e^{(t-s)A} \in L(H)) \in \mathcal{C}([s,T], L(H))$ therefore establishes item (i). Next note that (7), the fact that $[s,T] \times H \ni (t,h) \mapsto e^{(t-s)A}h \in H$ is continuously differentiable, and, e.g., [46, Corollary 2.8] (with $(V, \|\cdot\|_V) = (H, \|\cdot\|_H)$, $(W, \|\cdot\|_W) = (H, \|\cdot\|_H)$, $a = s$, $b = T$, $F = ([s,T] \ni t \mapsto (x + \int_s^t e^{(s-u)A} Z_u \, du) \in H)$, $\phi = ([s,T] \times H \ni (t,h) \mapsto e^{(t-s)A}h \in H)$, $f = ([s,T] \ni u \mapsto (e^{(s-u)A} Z_u) \in H)$ in the notation of [46, Corollary 2.8]) show that for every $t \in [s,T]$ it holds that

$$\begin{aligned} Y_t - x &= \int_s^t \left[A e^{(u-s)A} \left(x + \int_s^u e^{(s-r)A} Z_r \, dr \right) + e^{(u-s)A} e^{(s-u)A} Z_u \right] du \\ &= \int_s^t [A Y_u + Z_u] \, du. \end{aligned} \tag{9}$$

This establishes item (ii). The proof of Lemma 2.2 is thus completed. □

Lemma 2.3. *Assume Setting 2.1. Then*

(i) it holds that $(\mathbf{X} - \mathbf{O}) \in \mathcal{C}([0,T], H)$,

(ii) it holds that $(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H \ni (s,t,x) \mapsto X_{s,t}^x \in H) \in \mathcal{C}^{0,0,1}(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H, H)$,

(iii) it holds for every $t \in [0,T]$ *that*

$$\begin{aligned} \big([0,t] \ni s \mapsto \big[\tfrac{\partial}{\partial x} X_{s,t}^{\mathbf{X}_s - \mathbf{O}_s + O_s} \big(e^{(s - \llcorner s \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_s - \mathbf{O}_s + O_s)\big)\big] \in H\big) \\ \in \mathcal{M}(\mathcal{B}([0,t]), \mathcal{B}(H)), \end{aligned} \tag{10}$$

(iv) it holds for every $t \in [0,T]$ *that*

$$\int_0^t \big\| \tfrac{\partial}{\partial x} X_{s,t}^{\mathbf{X}_s - \mathbf{O}_s + O_s} \big(e^{(s - \llcorner s \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_s - \mathbf{O}_s + O_s)\big) \big\|_H \, ds < \infty, \tag{11}$$

and

(v) it holds for every $t \in [0,T]$ *that*

$$\mathbf{X}_t - X_{0,t}^{\xi + O_0} = \mathbf{O}_t - O_t + \int_0^t \tfrac{\partial}{\partial x} X_{s,t}^{\mathbf{X}_s - \mathbf{O}_s + O_s} \big(e^{(s - \llcorner s \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_s - \mathbf{O}_s + O_s)\big) \, ds. \tag{12}$$

Proof of Lemma 2.3. Throughout this proof let $\lambda \colon \mathcal{B}([0,T]) \to [0,T]$ be the Lebesgue-Borel measure on $[0,T]$, let $\mathcal{Y} \colon [0,T] \to H$ be the function which satisfies for every $t \in [0,T]$ that $\mathcal{Y}_t = \mathbf{X}_t - \mathbf{O}_t$, and let $\mathcal{X}_{s,(\cdot)}^x = (\mathcal{X}_{s,t}^x)_{t \in [s,T]} \colon [s,T] \to H$, $s \in [0,T]$, $x \in H$, be the functions which satisfy for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$ that $\mathcal{X}_{s,t}^x = X_{s,t}^{x + O_s} - O_t$. Note that (5) implies that for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$ it holds that

$$\mathcal{X}_{s,t}^x = e^{(t-s)A} x + \int_s^t e^{(t-u)A} F(\mathcal{X}_{s,u}^x + O_u) \, du. \tag{13}$$

The fact that for every $s \in [0,T]$, $x \in H$ it holds that $([s,T] \ni t \mapsto F(\mathcal{X}_{s,t}^x + O_t) \in H) \in \mathcal{C}([s,T], H)$ and item (ii) of Lemma 2.2 (with $T = T$, $s = s$, $x = x$, $Z = ([s,T] \ni t \mapsto F(\mathcal{X}_{s,t}^x + O_t) \in H)$,

$Y = ([s,T] \ni t \mapsto \mathcal{X}^x_{s,t} \in H)$ for $s \in [0,T]$, $x \in H$ in the notation of item (ii) of Lemma 2.2) therefore ensure that for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$ it holds that

$$\mathcal{X}^x_{s,t} = x + \int_s^t [A\mathcal{X}^x_{s,u} + F(\mathcal{X}^x_{s,u} + O_u)]\, du. \tag{14}$$

Next note that (6) implies that for every $t \in [0,T]$ it holds that

$$\mathcal{Y}_t = e^{tA}\xi + \int_0^t e^{(t-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta})\, du. \tag{15}$$

In addition, observe that the fact that $[0,T] \ni u \mapsto e^{(u-\llcorner u \lrcorner_\theta)A} \in L(H)$ is bounded and left-continuous implies that

$$([0,T] \ni u \mapsto e^{(u-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta}) \in H) \in \mathcal{L}^1(\lambda; H). \tag{16}$$

Combining (15) and Lemma 2.2 (with $T = T$, $s = 0$, $x = \xi$, $Z = ([0,T] \ni u \mapsto e^{(u-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta}) \in H)$, $Y = \mathcal{Y}$ in the notation of Lemma 2.2) therefore proves that

(a) it holds that $\mathcal{Y} \in \mathcal{C}([0,T], H)$ and

(b) it holds for every $t \in [0,T]$ that

$$\mathcal{Y}_t = \xi + \int_0^t [A\mathcal{Y}_u + e^{(u-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta})]\, du. \tag{17}$$

Observe that item (a) and the fact that $\mathcal{Y} = \mathbf{X} - \mathbf{O}$ establish item (i). Furthermore, note that (16), the assumption that $O \in \mathcal{C}([0,T], H)$, the fact that $F \in \mathcal{C}(H, H)$, and item (a) ensure that

$$([0,T] \ni u \mapsto e^{(u-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta}) - F(\mathcal{Y}_u + O_u) \in H) \in \mathcal{L}^1(\lambda; H). \tag{18}$$

In addition, observe that the assumption that $\dim(H) < \infty$, the fact that $O \in \mathcal{C}([0,T], H)$, the fact that $F \in \mathcal{C}(H, H)$, and item (a) show that

$$([0,T] \ni u \mapsto A\mathcal{Y}_u + F(\mathcal{Y}_u + O_u) \in H) \in \mathcal{L}^1(\lambda; H). \tag{19}$$

This, (18), and item (b) imply that for every $t \in [0,T]$ it holds that

$$\mathcal{Y}_t = \xi + \int_0^t [A\mathcal{Y}_u + F(\mathcal{Y}_u + O_u)]\, du + \int_0^t [e^{(u-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta}) - F(\mathcal{Y}_u + O_u)]\, du. \tag{20}$$

Combining (14), (18), (19), the fact that $([0,T] \times H \ni (u,h) \mapsto Ah + F(h + O_u) \in H) \in \mathcal{C}^{0,1}([0,T] \times H, H)$, and [46, Corollary 5.2] (with $V = H$, $T = T$, $f = ([0,T] \times H \ni (u,h) \mapsto Ah + F(h + O_u) \in H)$, $Y = \mathcal{Y}$, $E = ([0,T] \ni u \mapsto e^{(u-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta} + \mathbf{O}_{\llcorner u \lrcorner_\theta}) - F(\mathcal{Y}_u + O_u) \in H)$, $X^x_{s,t} = \mathcal{X}^x_{s,t}$ for $x \in H$, $t \in [s,T]$, $s \in [0,T]$ in the notation of [46, Corollary 5.2]) hence proves that

(A) it holds that $(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H \ni (s,t,x) \mapsto \mathcal{X}^x_{s,t} \in H) \in \mathcal{C}^{0,0,1}(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H, H)$,

(B) it holds for every $t \in [0,T]$ that

$$\begin{aligned}\big([0,t] \ni s \mapsto \big[\tfrac{\partial}{\partial x}\mathcal{X}^{\mathcal{Y}_s}_{s,t}\big(e^{(s-\llcorner s \lrcorner_\theta)A}\mathbf{F}(\mathcal{Y}_{\llcorner s \lrcorner_\theta} + \mathbf{O}_{\llcorner s \lrcorner_\theta}) - F(\mathcal{Y}_s + O_s)\big)\big] \in H\big) \\ \in \mathcal{M}(\mathcal{B}([0,t]), \mathcal{B}(H)),\end{aligned} \tag{21}$$

(C) it holds for every $t \in [0,T]$ that

$$\int_0^t \left\| \tfrac{\partial}{\partial x} \mathcal{X}_{s,t}^{\mathcal{Y}_s} \big(e^{(s - \llcorner s \lrcorner_\theta)A} \mathbf{F}(\mathcal{Y}_{\llcorner s \lrcorner_\theta} + \mathbf{O}_{\llcorner s \lrcorner_\theta}) - F(\mathcal{Y}_s + O_s) \big) \right\|_H ds < \infty, \tag{22}$$

and

(D) it holds for every $t \in [0,T]$ that

$$\mathcal{Y}_t - \mathcal{X}_{0,t}^{\mathcal{Y}_0} = \int_0^t \tfrac{\partial}{\partial x} \mathcal{X}_{s,t}^{\mathcal{Y}_s} \big(e^{(s - \llcorner s \lrcorner_\theta)A} \mathbf{F}(\mathcal{Y}_{\llcorner s \lrcorner_\theta} + \mathbf{O}_{\llcorner s \lrcorner_\theta}) - F(\mathcal{Y}_s + O_s) \big)\, ds. \tag{23}$$

Observe that the fact that for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$ it holds that $X_{s,t}^x = \mathcal{X}_{s,t}^{x - O_s} + O_t$, the assumption that $O \in \mathcal{C}([0,T], H)$, and item (A) establish item (ii). Next note that item (B), the fact that for every $s \in [0,T]$, $t \in [s,T]$ it holds that $\frac{\partial}{\partial x} \mathcal{X}_{s,t}^{\mathcal{Y}_s} = \frac{\partial}{\partial x} X_{s,t}^{\mathcal{Y}_s + O_s}$, and the fact that for every $s \in [0,T]$ it holds that $\mathcal{Y}_s = \mathbf{X}_s - \mathbf{O}_s$ imply item (iii). In addition, observe that item (C), the fact that for every $s \in [0,T]$, $t \in [s,T]$ it holds that $\frac{\partial}{\partial x} \mathcal{X}_{s,t}^{\mathcal{Y}_s} = \frac{\partial}{\partial x} X_{s,t}^{\mathcal{Y}_s + O_s}$, and the fact that for every $s \in [0,T]$ it holds that $\mathcal{Y}_s = \mathbf{X}_s - \mathbf{O}_s$ show item (iv). Moreover, note that item (D), the fact that for every $t \in [0,T]$ it holds that $\mathcal{X}_{0,t}^{\xi} = X_{0,t}^{\xi + O_0} - O_t$, the fact that for every $s \in [0,T]$, $t \in [s,T]$ it holds that $\frac{\partial}{\partial x} \mathcal{X}_{s,t}^{\mathcal{Y}_s} = \frac{\partial}{\partial x} X_{s,t}^{\mathcal{Y}_s + O_s}$, and the fact that for every $s \in [0,T]$ it holds that $\mathcal{Y}_s = \mathbf{X}_s - \mathbf{O}_s$ establish item (v). The proof of Lemma 2.3 is thus completed. □

Lemma 2.4. *Assume Setting 2.1, let $C, c \in [1,\infty)$, $\gamma \in [0,1]$, $\delta \in [0,\gamma]$, $\iota \in [0, 1-\delta]$, $\kappa \in \mathbb{R}$, and assume for every $x, y \in H$ that $\|F(x) - F(y)\|_H \leq C\|x - y\|_{H_\delta}(1 + \|x\|_{H_\kappa}^c + \|y\|_{H_\kappa}^c)$. Then it holds for every $t \in [0,T]$ that*

$$\begin{aligned} &\|e^{(t - \llcorner t \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_t - \mathbf{O}_t + O_t)\|_H \\ &\leq [|\theta|_T]^{\gamma - \delta} \|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_{H_{\gamma-\delta}} + \|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H + C\Big([|\theta|_T]^{\gamma-\delta} \|\xi\|_{H_\gamma} \\ &\quad + [|\theta|_T]^{1-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H + [|\theta|_T]^{\iota} \int_0^{\llcorner t \lrcorner_\theta} (\llcorner t \lrcorner_\theta - \llcorner s \lrcorner_\theta)^{-\delta-\iota} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H \, ds \\ &\quad + \|\mathbf{O}_t - \mathbf{O}_{\llcorner t \lrcorner_\theta}\|_{H_\delta} + \|\mathbf{O}_t - O_t\|_{H_\delta} \Big) \big(1 + \|\mathbf{X}_{\llcorner t \lrcorner_\theta}\|_{H_\kappa}^c + (\|\mathbf{X}_t\|_{H_\kappa} + \|\mathbf{O}_t - O_t\|_{H_\kappa})^c\big). \end{aligned} \tag{24}$$

Proof of Lemma 2.4. Note that the triangle inequality shows that for every $t \in [0,T]$ it holds that

$$\begin{aligned} &\|e^{(t - \llcorner t \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_t - \mathbf{O}_t + O_t)\|_H \\ &\leq \|(e^{(t - \llcorner t \lrcorner_\theta)A} - \mathrm{Id}_H) \mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H \\ &\quad + \|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H + \|F(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_t - \mathbf{O}_t + O_t)\|_H. \end{aligned} \tag{25}$$

In addition, observe that for every $t \in [0,T]$ it holds that

$$\begin{aligned} &\|(e^{(t - \llcorner t \lrcorner_\theta)A} - \mathrm{Id}_H) \mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H \leq \|(-A)^{\delta - \gamma}(e^{(t - \llcorner t \lrcorner_\theta)A} - \mathrm{Id}_H)\|_{L(H)} \|(-A)^{\gamma - \delta} \mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H \\ &\leq (t - \llcorner t \lrcorner_\theta)^{\gamma - \delta} \|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_{H_{\gamma-\delta}} \leq [|\theta|_T]^{\gamma-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_{H_{\gamma-\delta}}. \end{aligned} \tag{26}$$

Moreover, note that for every $t \in [0,T]$ it holds that

$$\begin{aligned} &\|F(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_t - \mathbf{O}_t + O_t)\|_H \\ &\leq C\|\mathbf{X}_{\llcorner t \lrcorner_\theta} - \mathbf{X}_t + \mathbf{O}_t - O_t\|_{H_\delta} \big(1 + \|\mathbf{X}_{\llcorner t \lrcorner_\theta}\|_{H_\kappa}^c + \|\mathbf{X}_t - \mathbf{O}_t + O_t\|_{H_\kappa}^c\big). \end{aligned} \tag{27}$$

The triangle inequality hence shows that for every $t \in [0,T]$ it holds that

$$\begin{aligned} &\|F(\mathbf{X}_{\llcorner t \lrcorner_\theta}) - F(\mathbf{X}_t - \mathbf{O}_t + O_t)\|_H \\ &\leq C\big(\|\mathbf{X}_{\llcorner t \lrcorner_\theta} - \mathbf{X}_t\|_{H_\delta} + \|\mathbf{O}_t - O_t\|_{H_\delta}\big)\big(1 + \|\mathbf{X}_{\llcorner t \lrcorner_\theta}\|_{H_\kappa}^c + (\|\mathbf{X}_t\|_{H_\kappa} + \|\mathbf{O}_t - O_t\|_{H_\kappa})^c\big). \end{aligned} \tag{28}$$

In the next step we observe that for every $t \in [0,T]$ it holds that

$$\begin{aligned} \|\mathbf{X}_t - \mathbf{X}_{\llcorner t \lrcorner_\theta}\|_{H_\delta} &\leq \|e^{\llcorner t \lrcorner_\theta A}(e^{(t-\llcorner t \lrcorner_\theta)A} - \mathrm{Id}_H)\xi\|_{H_\delta} + \int_{\llcorner t \lrcorner_\theta}^t \|e^{(t-\llcorner s \lrcorner_\theta)A}\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{H_\delta}\, ds \\ &\quad + \int_0^{\llcorner t \lrcorner_\theta} \|(e^{(t-\llcorner s \lrcorner_\theta)A} - e^{(\llcorner t \lrcorner_\theta - \llcorner s \lrcorner_\theta)A})\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{H_\delta}\, ds + \|\mathbf{O}_t - \mathbf{O}_{\llcorner t \lrcorner_\theta}\|_{H_\delta} \\ &\leq \|(-A)^{\delta-\gamma}(e^{(t-\llcorner t \lrcorner_\theta)A} - \mathrm{Id}_H)\|_{L(H)}\|\xi\|_{H_\gamma} + \int_{\llcorner t \lrcorner_\theta}^t \|(-A)^\delta e^{(t-\llcorner t \lrcorner_\theta)A}\|_{L(H)}\|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H\, ds \\ &\quad + \int_0^{\llcorner t \lrcorner_\theta} \|(-A)^{\delta+\iota} e^{(\llcorner t \lrcorner_\theta - \llcorner s \lrcorner_\theta)A}\|_{L(H)}\|(-A)^{-\iota}(e^{(t-\llcorner t \lrcorner_\theta)A} - \mathrm{Id}_H)\|_{L(H)}\|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H\, ds \\ &\quad + \|\mathbf{O}_t - \mathbf{O}_{\llcorner t \lrcorner_\theta}\|_{H_\delta} \\ &\leq (t - \llcorner t \lrcorner_\theta)^{\gamma-\delta}\|\xi\|_{H_\gamma} + (t - \llcorner t \lrcorner_\theta)^{1-\delta}\|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H \\ &\quad + \int_0^{\llcorner t \lrcorner_\theta} (\llcorner t \lrcorner_\theta - \llcorner s \lrcorner_\theta)^{-\delta-\iota}(t - \llcorner t \lrcorner_\theta)^\iota \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H\, ds + \|\mathbf{O}_t - \mathbf{O}_{\llcorner t \lrcorner_\theta}\|_{H_\delta} \\ &\leq [|\theta|_T]^{\gamma-\delta}\|\xi\|_{H_\gamma} + [|\theta|_T]^{1-\delta}\|\mathbf{F}(\mathbf{X}_{\llcorner t \lrcorner_\theta})\|_H \\ &\quad + [|\theta|_T]^\iota \int_0^{\llcorner t \lrcorner_\theta} (\llcorner t \lrcorner_\theta - \llcorner s \lrcorner_\theta)^{-\delta-\iota}\|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H\, ds + \|\mathbf{O}_t - \mathbf{O}_{\llcorner t \lrcorner_\theta}\|_{H_\delta}. \end{aligned} \tag{29}$$

Combining (25), (26), and (28) therefore establishes (24). The proof of Lemma 2.4 is thus completed. ☐

Lemma 2.5. *Assume Setting 2.1, let $C, c \in [1,\infty)$, $\gamma \in [0,1]$, $\delta \in [0,\gamma]$, $\iota \in [0, 1-\delta]$, $\kappa \in \mathbb{R}$, and assume for every $x, y \in H$ that $\|F(x) - F(y)\|_H \leq C\|x-y\|_{H_\delta}(1 + \|x\|_{H_\kappa}^c + \|y\|_{H_\kappa}^c)$. Then*

(i) it holds that $(\mathbf{X} - \mathbf{O}) \in \mathcal{C}([0,T], H)$,

(ii) it holds that $(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H \ni (s,t,x) \mapsto X_{s,t}^x \in H) \in \mathcal{C}^{0,0,1}(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H, H)$, and

(iii) it holds for every $t \in [0,T]$ that

$$\begin{aligned} \|\mathbf{X}_t - X_{0,t}^{\xi+O_0}\|_H &\leq \|\mathbf{O}_t - O_t\|_H + \int_0^t \big\|\tfrac{\partial}{\partial x} X_{s,t}^{\mathbf{X}_s - \mathbf{O}_s + O_s}\big\|_{L(H)} \Big\{[|\theta|_T]^{\gamma-\delta}\|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{H_{\gamma-\delta}} \\ &\quad + \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H + C\Big([|\theta|_T]^{\gamma-\delta}\|\xi\|_{H_\gamma} + [|\theta|_T]^{1-\delta}\|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H \\ &\quad + [|\theta|_T]^\iota \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - \llcorner u \lrcorner_\theta)^{-\delta-\iota}\|\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\|_H\, du + \|\mathbf{O}_s - \mathbf{O}_{\llcorner s \lrcorner_\theta}\|_{H_\delta} \\ &\quad + \|\mathbf{O}_s - O_s\|_{H_\delta}\Big)\big(1 + \|\mathbf{X}_{\llcorner s \lrcorner_\theta}\|_{H_\kappa} + \|\mathbf{X}_s\|_{H_\kappa} + \|\mathbf{O}_s - O_s\|_{H_\kappa}\big)^c\Big\}\, ds. \end{aligned} \tag{30}$$

Proof of Lemma 2.5. Observe that item (i) of Lemma 2.3 implies item (i). In addition, note that item (ii) of Lemma 2.3 establishes item (ii). Moreover, observe that items (iii) and (v) of

Lemma 2.3 and the triangle inequality show that for every $t \in [0, T]$ it holds that

$$\begin{aligned} \|\mathbf{X}_t - X_{0,t}^{\xi+O_0}\|_H &\leq \|\mathbf{O}_t - O_t\|_H \\ &\quad + \int_0^t \big\| \tfrac{\partial}{\partial x} X_{s,t}^{\mathbf{X}_s - \mathbf{O}_s + O_s} \big(e^{(s-\llcorner s \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_s - \mathbf{O}_s + O_s)\big) \big\|_H \, ds. \end{aligned} \tag{31}$$

Lemma 2.4 (with $C = C$, $c = c$, $\gamma = \gamma$, $\delta = \delta$, $\iota = \iota$, $\kappa = \kappa$ in the notation of Lemma 2.4) and the fact that $\forall\, a, b \in [0, \infty)$, $c \in [1, \infty)$: $1 + a^c + b^c \leq (1 + a + b)^c$ therefore establish item (iii). The proof of Lemma 2.5 is thus completed. □

2.2 Strong temporal approximation error estimates

In this subsection we recall in Lemma 2.6 (see, e.g., Aliprantis & Border [2, Lemma 4.51]) and Lemma 2.7 (see, e.g., Aliprantis & Border [2, Theorem 4.55]) some basic facts on measurability properties of functions. Thereafter, we combine Lemma 2.6 and Lemma 2.7 with Lemma 2.3 to establish in Lemma 2.8 suitable regularity properties for the solution of a stochastic version of the integral equation in (5) above (see (32) below). Combining Lemma 2.8 and Lemma 2.5 enables us to establish in Corollary 2.9 an upper moment bound for the difference between the solution of the considered SODE (cf. (46) below and (5) above) and its numerical approximation (cf. (47) below and (6) above).

Lemma 2.6. *Let $(\Omega, \mathcal{F})$ be a measurable space, let (X, d_X) be a separable metric space, let (Y, d_Y) be a metric space, let $f\colon X \times \Omega \to Y$ be a function, assume for every $x \in X$ that $\Omega \ni \omega \mapsto f(x, \omega) \in Y$ is $\mathcal{F}/\mathcal{B}(Y)$-measurable, and assume for every $\omega \in \Omega$ that $(X \ni x \mapsto f(x, \omega) \in Y) \in \mathcal{C}(X, Y)$. Then it holds that $f\colon X \times \Omega \to Y$ is $(\mathcal{B}(X) \otimes \mathcal{F})/\mathcal{B}(Y)$-measurable.*

Lemma 2.7. *Let $(\Omega, \mathcal{F})$ be a measurable space, let (X, d_X) be a compact metric space, let (Y, d_Y) be a separable metric space, let $\mathcal{C}(X, Y)$ be endowed with the topology of uniform convergence, let $f\colon X \times \Omega \to Y$ be a function, assume for every $x \in X$ that $\Omega \ni \omega \mapsto f(x, \omega) \in Y$ is $\mathcal{F}/\mathcal{B}(Y)$-measurable, and assume for every $\omega \in \Omega$ that $(X \ni x \mapsto f(x, \omega) \in Y) \in \mathcal{C}(X, Y)$. Then it holds that $\Omega \ni \omega \mapsto (X \ni x \mapsto f(x, \omega) \in Y) \in \mathcal{C}(X, Y)$ is $\mathcal{F}/\mathcal{B}(\mathcal{C}(X, Y))$-measurable.*

Lemma 2.8. *Assume Setting 1.3, assume that $\dim(H) < \infty$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T \in (0, \infty)$, $F \in \mathcal{C}^1(H, H)$, $Y, Z \in \mathcal{M}(\mathcal{B}([0, T]) \otimes \mathcal{F}, \mathcal{B}(H))$, let $O\colon [0, T] \times \Omega \to H$ be a stochastic process with continuous sample paths, and for every $s \in [0, T]$, $x \in H$ let $X_{s,(\cdot)}^x = (X_{s,t}^x)_{t \in [s,T]}\colon [s, T] \times \Omega \to H$ be a stochastic process with continuous sample paths which satisfies for every $t \in [s, T]$ that*

$$X_{s,t}^x = e^{(t-s)A} x + \int_s^t e^{(t-u)A} F(X_{s,u}^x)\, du + O_t - e^{(t-s)A} O_s. \tag{32}$$

Then

(i) it holds for every $\omega \in \Omega$ that $(\{(u, v) \in [0, T]^2 \colon u \leq v\} \times H \ni (s, t, x) \mapsto X_{s,t}^x(\omega) \in H) \in \mathcal{C}^{0,0,1}(\{(u, v) \in [0, T]^2 \colon u \leq v\} \times H, H)$,

(ii) it holds that $\big(\{(u, v) \in [0, T]^2 \colon u \leq v\} \times \Omega \ni (s, t, \omega) \mapsto X_{s,t}^{Y_s(\omega)}(\omega) \in H\big) \in \mathcal{M}(\mathcal{B}(\{(u, v) \in [0, T]^2 \colon u \leq v\}) \otimes \mathcal{F}, \mathcal{B}(H))$, and

(iii) it holds that $\big(\{(u,v)\in[0,T]^2\colon u\leq v\}\times\Omega\ni(s,t,\omega)\mapsto\frac{\partial}{\partial x}X_{s,t}^{Z_s(\omega)}(\omega)\in L(H)\big)\in\mathcal{M}(\mathcal{B}(\{(u,v)\in[0,T]^2\colon u\leq v\})\otimes\mathcal{F},\mathcal{B}(L(H)))$.

Proof of Lemma 2.8. Throughout this proof let $\angle_T=\{(u,v)\in[0,T]^2\colon u\leq v\}$, let $V=\mathcal{C}(\{w\in H\colon\|w\|_H\leq 1\},H)$, let $\|\cdot\|_V\colon V\to[0,\infty)$ be the function which satisfies for every $f\in V$ that

$$\|f\|_V=\sup\nolimits_{h\in\{w\in H\colon\|w\|_H\leq 1\}}\|f(h)\|_H,\tag{33}$$

and let $\iota\colon L(H)\to V$ be the function which satisfies for every $Q\in L(H)$ that

$$\iota(Q)=(\{w\in H\colon\|w\|_H\leq 1\}\ni h\mapsto Q(h)\in H).\tag{34}$$

Note that item (ii) of Lemma 2.3 (with $T=T$, $O_t=O_t(\omega)$, $F=F$, $X_{s,t}^x=X_{s,t}^x(\omega)$ for $(s,t)\in\angle_T$, $x\in H$, $\omega\in\Omega$ in the notation of item (ii) of Lemma 2.3) establishes item (i). This ensures that for every $\omega\in\Omega$ it holds that

$$(\angle_T\times H\ni(s,t,x)\mapsto X_{s,t}^x(\omega)\in H)\in\mathcal{C}(\angle_T\times H,H).\tag{35}$$

The fact that for every $(s,t)\in\angle_T$, $x\in H$ it holds that $(\Omega\ni\omega\mapsto X_{s,t}^x(\omega)\in H)\in\mathcal{M}(\mathcal{F},\mathcal{B}(H))$ and Lemma 2.6 (with $\Omega=\Omega$, $\mathcal{F}=\mathcal{F}$, $X=\angle_T\times H$, $d_X=([\angle_T\times H]^2\ni((s_1,t_1,x_1),(s_2,t_2,x_2))\mapsto[|s_1-s_2|^2+|t_1-t_2|^2+\|x_1-x_2\|_H^2]^{1/2}\in[0,\infty))$, $Y=H$, $d_Y=(H^2\ni(x_1,x_2)\mapsto\|x_1-x_2\|_H\in[0,\infty))$, $f=(\angle_T\times H\times\Omega\ni(s,t,x,\omega)\mapsto X_{s,t}^x(\omega)\in H)$ in the notation of Lemma 2.6) hence show that

$$(\angle_T\times H\times\Omega\ni(s,t,x,\omega)\mapsto X_{s,t}^x(\omega)\in H)\in\mathcal{M}(\mathcal{B}(\angle_T)\otimes\mathcal{B}(H)\otimes\mathcal{F},\mathcal{B}(H)).\tag{36}$$

The fact that $(\angle_T\times\Omega\ni(s,t,\omega)\mapsto(s,t,Y_s(\omega),\omega)\in\angle_T\times H\times\Omega)\in\mathcal{M}(\mathcal{B}(\angle_T)\otimes\mathcal{F},\mathcal{B}(\angle_T)\otimes\mathcal{B}(H)\otimes\mathcal{F})$ therefore establishes item (ii). Furthermore, observe that item (i) implies that for every $(s,t)\in\angle_T$, $x\in H$, $\omega\in\Omega$ it holds that

$$\begin{aligned}&\limsup_{r\searrow 0}\Big\|\Big(\{w\in H\colon\|w\|_H\leq 1\}\ni h\mapsto\tfrac{X_{s,t}^{x+rh}(\omega)-X_{s,t}^x(\omega)}{r}\in H\Big)-\iota\big(\tfrac{\partial}{\partial x}X_{s,t}^x(\omega)\big)\Big\|_V\\&=\limsup_{r\searrow 0}\Bigg[\sup_{h\in H,\|h\|_H\leq 1}\Big\|\tfrac{X_{s,t}^{x+rh}(\omega)-X_{s,t}^x(\omega)}{r}-\big(\tfrac{\partial}{\partial x}X_{s,t}^x(\omega)\big)h\Big\|_H\Bigg]=0.\end{aligned}\tag{37}$$

Moreover, note that Lemma 2.7 (with $\Omega=\Omega$, $\mathcal{F}=\mathcal{F}$, $X=\{w\in H\colon\|w\|_H\leq 1\}$, $d_X=(\{w\in H\colon\|w\|_H\leq 1\}\times\{w\in H\colon\|w\|_H\leq 1\}\ni(x,y)\mapsto\|x-y\|_H\in[0,\infty))$, $Y=H$, $d_Y=(H\times H\ni(x,y)\mapsto\|x-y\|_H\in[0,\infty))$, $f=(\{w\in H\colon\|w\|_H\leq 1\}\times\Omega\ni(h,\omega)\mapsto X_{s,t}^{x+rh}(\omega)\in H)$ for $(s,t)\in\angle_T$, $x\in H$, $r\in(0,\infty)$ in the notation of Lemma 2.7) assures that for every $(s,t)\in\angle_T$, $x\in H$, $r\in(0,\infty)$ it holds that

$$\big(\Omega\ni\omega\mapsto(\{w\in H\colon\|w\|_H\leq 1\}\ni h\mapsto X_{s,t}^{x+rh}(\omega)\in H)\in V\big)\in\mathcal{M}(\mathcal{F},\mathcal{B}(V)).\tag{38}$$

This and (37) prove that for every $(s,t)\in\angle_T$, $x\in H$ it holds that

$$\big(\Omega\ni\omega\mapsto\iota\big(\tfrac{\partial}{\partial x}X_{s,t}^x(\omega)\big)\in V\big)\in\mathcal{M}(\mathcal{F},\mathcal{B}(V)).\tag{39}$$

Hence, we obtain that for every $Q\in L(H)$, $\varepsilon\in(0,\infty)$, $(s,t)\in\angle_T$, $x\in H$ it holds that

$$\big\{\omega\in\Omega\colon\big\|\iota\big(\tfrac{\partial}{\partial x}X_{s,t}^x(\omega)\big)-\iota(Q)\big\|_V<\varepsilon\big\}\in\mathcal{F}.\tag{40}$$

In addition, observe that for every $Q_1, Q_2 \in L(H)$ it holds that

$$
\begin{aligned}
&\|Q_1 - Q_2\|_{L(H)} = \sup\nolimits_{h \in \{w \in H \colon \|w\|_H \leq 1\}} \|Q_1(h) - Q_2(h)\|_H \\
&= \sup\nolimits_{h \in \{w \in H \colon \|w\|_H \leq 1\}} \|\iota(Q_1)(h) - \iota(Q_2)(h)\|_H = \|\iota(Q_1) - \iota(Q_2)\|_V.
\end{aligned} \tag{41}
$$

Combining this and (40) ensures that for every $Q \in L(H)$, $\varepsilon \in (0, \infty)$, $(s,t) \in \angle_T$, $x \in H$ it holds that

$$
\big\{\omega \in \Omega \colon \big\|\tfrac{\partial}{\partial x} X^x_{s,t}(\omega) - Q\big\|_{L(H)} < \varepsilon\big\} = \big\{\omega \in \Omega \colon \big\|\iota\big(\tfrac{\partial}{\partial x} X^x_{s,t}(\omega)\big) - \iota(Q)\big\|_V < \varepsilon\big\} \in \mathcal{F}. \tag{42}
$$

The fact that $L(H)$ is a separable metric space and the fact that the Borel-sigma algebra on a separable metric space is generated by the set of open balls therefore prove that for every $(s,t) \in \angle_T$, $x \in H$ it holds that

$$
\big(\Omega \ni \omega \mapsto \tfrac{\partial}{\partial x} X^x_{s,t}(\omega) \in L(H)\big) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(L(H))). \tag{43}
$$

Moreover, note that item (i) ensures that for every $\omega \in \Omega$ it holds that

$$
\big(\angle_T \times H \ni (s,t,x) \mapsto \tfrac{\partial}{\partial x} X^x_{s,t}(\omega) \in L(H)\big) \in \mathcal{C}(\angle_T \times H, L(H)). \tag{44}
$$

Lemma 2.6 (with $\Omega = \Omega$, $\mathcal{F} = \mathcal{F}$, $X = \angle_T \times H$, $d_X = ([\angle_T \times H]^2 \ni ((s_1,t_1,x_1),(s_2,t_2,x_2)) \mapsto [|s_1 - s_2|^2 + |t_1 - t_2|^2 + \|x_1 - x_2\|_H^2]^{1/2} \in [0,\infty))$, $Y = L(H)$, $d_Y = ([L(H)]^2 \ni (A_1, A_2) \mapsto \|A_1 - A_2\|_{L(H)} \in [0,\infty))$, $f = (\angle_T \times H \times \Omega \ni (s,t,x,\omega) \mapsto \frac{\partial}{\partial x} X^x_{s,t}(\omega) \in L(H))$ in the notation of Lemma 2.6) and (43) therefore prove that

$$
\big(\angle_T \times H \times \Omega \ni (s,t,x,\omega) \mapsto \tfrac{\partial}{\partial x} X^x_{s,t}(\omega) \in L(H)\big) \in \mathcal{M}(\mathcal{B}(\angle_T) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(L(H))). \tag{45}
$$

The fact that $(\angle_T \times \Omega \ni (s,t,\omega) \mapsto (s,t,Z_s(\omega),\omega) \in \angle_T \times H \times \Omega) \in \mathcal{M}(\mathcal{B}(\angle_T) \otimes \mathcal{F}, \mathcal{B}(\angle_T) \otimes \mathcal{B}(H) \otimes \mathcal{F})$ hence establishes item (iii). The proof of Lemma 2.8 is thus completed. ☐

Corollary 2.9. *Assume Setting 1.3, assume that* $\dim(H) < \infty$*, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $T \in (0,\infty)$*,* $\theta \in \varpi_T$*,* $C, c, p \in [1,\infty)$*,* $\gamma \in [0,1)$*,* $\delta \in [0,\gamma]$*,* $\iota \in [0, 1-\delta)$*,* $\kappa \in \mathbb{R}$*,* $\xi \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$*,* $F \in \mathcal{C}^1(H,H)$*,* $\mathbf{F} \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(H))$*,* $\mathbf{O} \in \mathcal{M}(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B}(H))$*, let* $O \colon [0,T] \times \Omega \to H$ *be a stochastic process with continuous sample paths, assume for every* $x, y \in H$ *that* $\|F(x) - F(y)\|_H \leq C\|x - y\|_{H_\delta}(1 + \|x\|^c_{H_\kappa} + \|y\|^c_{H_\kappa})$*, for every* $s \in [0,T]$*,* $x \in H$ *let* $X^x_{s,(\cdot)} = (X^x_{s,t})_{t \in [s,T]} \colon [s,T] \times \Omega \to H$ *be a stochastic process with continuous sample paths which satisfies for every* $t \in [s,T]$ *that*

$$
X^x_{s,t} = e^{(t-s)A}x + \int_s^t e^{(t-u)A} F(X^x_{s,u})\, du + O_t - e^{(t-s)A} O_s, \tag{46}
$$

and let $\mathbf{X} \colon [0,T] \times \Omega \to H$ *be a function which satisfies for every* $t \in [0,T]$ *that*

$$
\mathbf{X}_t = e^{tA}\xi + \int_0^t e^{(t - \llcorner u \lrcorner_\theta)A} \mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\, du + \mathbf{O}_t. \tag{47}
$$

Then

(i) it holds that $\mathbf{X} \in \mathcal{M}(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B}(H))$*,*

(ii) *it holds for every* $\omega \in \Omega$ *that* $(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H \ni (s,t,x) \mapsto X^x_{s,t}(\omega) \in H) \in \mathcal{C}^{0,0,1}(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H, H)$,

(iii) *it holds for every* $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ *that* $\big(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times \Omega \ni (s,t,\omega) \mapsto X^{\zeta(\omega)+O_s(\omega)}_{s,t}(\omega) \in H\big) \in \mathcal{M}(\mathcal{B}(\{(u,v) \in [0,T]^2 \colon u \leq v\}) \otimes \mathcal{F}, \mathcal{B}(H))$,

(iv) *it holds for every* $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ *that* $\big(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times \Omega \ni (s,t,\omega) \mapsto \frac{\partial}{\partial x} X^{\mathbf{X}_s(\omega)-\mathbf{O}_s(\omega)+O_s(\omega)+e^{sA}(\zeta(\omega)-\xi(\omega))}_{s,t}(\omega) \in L(H)\big) \in \mathcal{M}(\mathcal{B}(\{(u,v) \in [0,T]^2 \colon u \leq v\}) \otimes \mathcal{F}, \mathcal{B}(L(H)))$, *and*

(v) *it holds for every* $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$, $t \in [0,T]$ *that*

$$\begin{aligned}
&\|\mathbf{X}_t - X^{\zeta+O_0}_{0,t}\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \|\mathbf{O}_t - O_t\|_{\mathcal{L}^p(\mathbb{P};H)} + \|\xi - \zeta\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&+ \tfrac{C \max\{T,1\}}{1-\delta-\iota} \int_0^t \big\| \tfrac{\partial}{\partial x} X^{\mathbf{X}_s - \mathbf{O}_s + O_s + e^{sA}(\zeta-\xi)}_{s,t} \big\|_{\mathcal{L}^{2p}(\mathbb{P};L(H))} \Big\{ [|\theta|_T]^{\gamma-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})} \\
&+ \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{\mathcal{L}^{2p}(\mathbb{P};H)} + \Big([|\theta|_T]^{1-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{\mathcal{L}^{4p}(\mathbb{P};H)} \\
&+ [|\theta|_T]^{\iota} \sup\nolimits_{u \in [0,T]} \|\mathbf{F}(\mathbf{X}_u)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} + \|\mathbf{O}_s - \mathbf{O}_{\llcorner s \lrcorner_\theta}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} + [|\theta|_T]^{\gamma-\delta} \|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)} \\
&+ \|\mathbf{O}_s - O_s\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} + \|\xi - \zeta\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \Big) \big[1 + \|\mathbf{X}_{\llcorner s \lrcorner_\theta}\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} + \|\mathbf{X}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} \\
&+ \|\mathbf{O}_s - O_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} + \|\xi - \zeta\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} \big]^c \Big\} \, ds.
\end{aligned} \tag{48}$$

Proof of Corollary 2.9. Observe that item (i) of Lemma 2.5 (with $T = T$, $\theta = \theta$, $\xi = \xi(\omega)$, $O_s = O_s(\omega)$, $\mathbf{O}_s = \mathbf{O}_s(\omega)$, $F = F$, $\mathbf{F} = \mathbf{F}$, $X^x_{s,t} = X^x_{s,t}(\omega)$, $\mathbf{X}_s = \mathbf{X}_s(\omega)$ for $\omega \in \Omega$, $t \in [s,T]$, $s \in [0,T]$, $x \in H$ in the notation of item (i) of Lemma 2.5) proves that for every $\omega \in \Omega$ it holds that

$$([0,T] \ni t \mapsto \mathbf{X}_t(\omega) - \mathbf{O}_t(\omega) \in H) \in \mathcal{C}([0,T], H). \tag{49}$$

Moreover, note that (47), the fact that for every $t \in [0,T]$ it holds that $(\Omega \ni \omega \mapsto \mathbf{O}_t(\omega) \in H) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$, and the assumption that $\xi \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ ensure that for every $t \in [0,T]$ it holds that

$$(\Omega \ni \omega \mapsto \mathbf{X}_t(\omega) - \mathbf{O}_t(\omega) \in H) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H)). \tag{50}$$

Combining this, (49), and Lemma 2.6 (with $\Omega = \Omega$, $\mathcal{F} = \mathcal{F}$, $X = [0,T]$, $d_X = ([0,T]^2 \ni (s,t) \mapsto |t-s| \in [0,\infty))$, $Y = H$, $d_Y = (H \times H \ni (x,y) \mapsto \|x-y\|_H \in [0,\infty))$, $f = \mathbf{X} - \mathbf{O}$ in the notation of Lemma 2.6) ensures that

$$([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbf{X}_t(\omega) - \mathbf{O}_t(\omega) \in H) \in \mathcal{M}(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B}(H)). \tag{51}$$

The assumption that $\mathbf{O} \in \mathcal{M}(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B}(H))$ therefore establishes item (i). Next note that Lemma 2.8 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $F = F$, $Y_s = \zeta + O_s$, $Z_s = \mathbf{X}_s - \mathbf{O}_s + O_s + e^{sA}(\zeta - \xi)$, $O_s = O_s$, $X^x_{s,t} = X^x_{s,t}$ for $x \in H$, $t \in [s,T]$, $s \in [0,T]$, $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ in the notation of Lemma 2.8) establishes items (ii)–(iv). In the next step we observe that for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$, $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ it holds that

$$X^x_{s,t} = e^{(t-s)A} x + \int_s^t e^{(t-u)A} F(X^x_{s,u}) \, du + (O_t + e^{tA}\zeta) - e^{(t-s)A}(O_s + e^{sA}\zeta) \tag{52}$$

and

$$\mathbf{X}_t = \int_0^t e^{(t-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\,du + \big[\mathbf{O}_t + e^{tA}\xi\big]. \tag{53}$$

Lemma 2.5 (with $T = T$, $\theta = \theta$, $\xi = 0$, $O_s = O_s(\omega) + e^{sA}\zeta(\omega)$, $\mathbf{O}_s = \mathbf{O}_s(\omega) + e^{sA}\xi(\omega)$, $F = F$, $\mathbf{F} = \mathbf{F}$, $X^x_{s,t} = X^x_{s,t}(\omega)$, $\mathbf{X}_s = \mathbf{X}_s(\omega)$, $C = C$, $c = c$, $\gamma = \gamma$, $\delta = \delta$, $\iota = \iota$, $\kappa = \kappa$ for $\omega \in \Omega$, $x \in H$, $t \in [s,T]$, $s \in [0,T]$, $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ in the notation of Lemma 2.5) therefore implies that for every $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|\mathbf{X}_t - X^{\zeta+O_0}_{0,t}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \|\mathbf{O}_t - O_t + e^{tA}(\xi-\zeta)\|_{\mathcal{L}^p(\mathbb{P};H)} + \int_0^t \bigg\| \big\|\tfrac{\partial}{\partial x} X^{\mathbf{X}_s - \mathbf{O}_s + O_s + e^{sA}(\zeta-\xi)}_{s,t}\big\|_{L(H)} \\
&\quad \cdot \Big([|\theta|_T]^{\gamma-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{H_{\gamma-\delta}} + \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H \\
&\quad + C\Big([|\theta|_T]^{1-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H + [|\theta|_T]^{\iota} \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - \llcorner u \lrcorner_\theta)^{-\delta-\iota} \|\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\|_H \,du \\
&\quad + \|\mathbf{O}_s - \mathbf{O}_{\llcorner s \lrcorner_\theta} + (e^{sA} - e^{\llcorner s \lrcorner_\theta A})\xi\|_{H_\delta} + \|\mathbf{O}_s - O_s + e^{sA}(\xi-\zeta)\|_{H_\delta}\Big) \\
&\quad \cdot \big[1 + \|\mathbf{X}_{\llcorner s \lrcorner_\theta}\|_{H_\kappa} + \|\mathbf{X}_s\|_{H_\kappa} + \|\mathbf{O}_s - O_s + e^{sA}(\xi-\zeta)\|_{H_\kappa}\big]^c\Big)\bigg\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} ds.
\end{aligned} \tag{54}$$

Hölder's inequality and the triangle inequality hence show that for every $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|\mathbf{X}_t - X^{\zeta+O_0}_{0,t}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \|\mathbf{O}_t - O_t\|_{\mathcal{L}^p(\mathbb{P};H)} + \|e^{tA}(\xi-\zeta)\|_{\mathcal{L}^p(\mathbb{P};H)} + \int_0^t \big\|\tfrac{\partial}{\partial x} X^{\mathbf{X}_s - \mathbf{O}_s + O_s + e^{sA}(\zeta-\xi)}_{s,t}\big\|_{\mathcal{L}^{2p}(\mathbb{P};L(H))} \\
&\quad \cdot \bigg\| [|\theta|_T]^{\gamma-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{H_{\gamma-\delta}} + \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H \\
&\quad + C\Big([|\theta|_T]^{1-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H + [|\theta|_T]^{\iota} \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - \llcorner u \lrcorner_\theta)^{-\delta-\iota} \|\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\|_H \,du \\
&\quad + \|\mathbf{O}_s - \mathbf{O}_{\llcorner s \lrcorner_\theta}\|_{H_\delta} + \|(e^{sA} - e^{\llcorner s \lrcorner_\theta A})\xi\|_{H_\delta} + \|\mathbf{O}_s - O_s\|_{H_\delta} + \|e^{sA}(\xi-\zeta)\|_{H_\delta}\Big) \\
&\quad \cdot \big[1 + \|\mathbf{X}_{\llcorner s \lrcorner_\theta}\|_{H_\kappa} + \|\mathbf{X}_s\|_{H_\kappa} + \|\mathbf{O}_s - O_s\|_{H_\kappa} + \|e^{sA}(\xi-\zeta)\|_{H_\kappa}\big]^c \bigg\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} ds.
\end{aligned} \tag{55}$$

Hölder's inequality and the triangle inequality therefore prove that for every $\zeta \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|\mathbf{X}_t - X^{\zeta+O_0}_{0,t}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \|\mathbf{O}_t - O_t\|_{\mathcal{L}^p(\mathbb{P};H)} + \|\xi-\zeta\|_{\mathcal{L}^p(\mathbb{P};H)} + \int_0^t \big\|\tfrac{\partial}{\partial x} X^{\mathbf{X}_s - \mathbf{O}_s + O_s + e^{sA}(\zeta-\xi)}_{s,t}\big\|_{\mathcal{L}^{2p}(\mathbb{P};L(H))} \\
&\quad \cdot \Big\{ [|\theta|_T]^{\gamma-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})} + \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta}) - F(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&\quad + C\bigg\| [|\theta|_T]^{1-\delta} \|\mathbf{F}(\mathbf{X}_{\llcorner s \lrcorner_\theta})\|_H + [|\theta|_T]^{\iota} \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - \llcorner u \lrcorner_\theta)^{-\delta-\iota} \|\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\|_H \,du \\
&\quad + \|\mathbf{O}_s - \mathbf{O}_{\llcorner s \lrcorner_\theta}\|_{H_\delta} + \|(e^{sA} - e^{\llcorner s \lrcorner_\theta A})\xi\|_{H_\delta} + \|\mathbf{O}_s - O_s\|_{H_\delta} + \|\xi-\zeta\|_{H_\delta}\bigg\|_{\mathcal{L}^{4p}(\mathbb{P};\mathbb{R})} \\
&\quad \cdot \big\|1 + \|\mathbf{X}_{\llcorner s \lrcorner_\theta}\|_{H_\kappa} + \|\mathbf{X}_s\|_{H_\kappa} + \|\mathbf{O}_s - O_s\|_{H_\kappa} + \|\xi-\zeta\|_{H_\kappa}\big\|^c_{\mathcal{L}^{4pc}(\mathbb{P};\mathbb{R})} \Big\} \,ds.
\end{aligned} \tag{56}$$

In addition, note that the fact that $\delta + \iota < 1$ assures that for every $s \in [0,T]$ it holds that

$$\begin{aligned}
&\left\| \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - \llcorner u \lrcorner_\theta)^{-\delta-\iota} \|\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\|_H \, du \right\|_{\mathcal{L}^{4p}(\mathbb{P};\mathbb{R})} \\
&\leq \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - \llcorner u \lrcorner_\theta)^{-\delta-\iota} \|\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\|_{\mathcal{L}^{4p}(\mathbb{P};H)} \, du \\
&\leq \sup\nolimits_{u \in [0,T]} \|\mathbf{F}(\mathbf{X}_u)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} \int_0^{\llcorner s \lrcorner_\theta} (\llcorner s \lrcorner_\theta - u)^{-\delta-\iota} \, du \\
&= \sup\nolimits_{u \in [0,T]} \|\mathbf{F}(\mathbf{X}_u)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} \tfrac{(\llcorner s \lrcorner_\theta)^{1-\delta-\iota}}{1-\delta-\iota} \\
&\leq \sup\nolimits_{u \in [0,T]} \|\mathbf{F}(\mathbf{X}_u)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} \tfrac{\max\{T,1\}}{1-\delta-\iota}.
\end{aligned} \tag{57}$$

Furthermore, observe that for every $s \in [0,T]$ it holds that

$$\begin{aligned}
&\|(e^{sA} - e^{\llcorner s \lrcorner_\theta A})\xi\|_{H_\delta} = \|e^{\llcorner s \lrcorner_\theta A}(e^{(s-\llcorner s \lrcorner_\theta)A} - \mathrm{Id}_H)\xi\|_{H_\delta} \leq \|(-A)^\delta (e^{(s-\llcorner s \lrcorner_\theta)A} - \mathrm{Id}_H)\xi\|_H \\
&\leq \|(-A)^{\delta-\gamma}(e^{(s-\llcorner s \lrcorner_\theta)A} - \mathrm{Id}_H)\|_{L(H)} \|\xi\|_{H_\gamma} \leq (s - \llcorner s \lrcorner_\theta)^{\gamma-\delta} \|\xi\|_{H_\gamma} \leq [|\theta|_T]^{\gamma-\delta} \|\xi\|_{H_\gamma}.
\end{aligned} \tag{58}$$

Combining this with (56) and (57) establishes item (v). The proof of Corollary 2.9 is thus completed. □

3 Moment bounds for the derivative process and resulting time discretization error estimates

3.1 A priori bounds for the derivative process

In this subsection we derive in Lemma 3.5 an appropriate moment bound for the pathwise derivatives of the solution processes $(X^x_{s,t})_{t \in [s,T]}$, $s \in [0,T]$, $x \in H$, with respect to their initial conditions appearing in item (v) of Corollary 2.9 above (see (88) in Lemma 3.5 below). We first demonstrate in Lemma 3.1 that the well known local monotonicity property (see (59) in Lemma 3.1 below and cf., e.g., Liu & Röckner [55, (H2′) in Chapter 5]) together with the continuous Fréchet differentiability of the nonlinearity F implies the property of F' that we are exploiting in this article (see (60) in Lemma 3.1 below). In addition, Proposition 3.2 (cf. Hairer & Mattingly [33, (4.8) in Section 4.4]) provides a suitable upper bound for the derivative process appearing in item (v) of Corollary 2.9 (see (64) in Proposition 3.2 below). Combining Lemma 2.8 and Proposition 3.2 implies Corollary 3.3 which we use together with Cox et al. [17, Corollary 2.4] as a tool to establish in Lemma 3.5 the desired moment bound.

Lemma 3.1. *Assume Setting 1.3, let $\varepsilon, \mathbf{C}, \gamma \in [0,\infty)$, $F \in \mathcal{C}^1(H_\gamma, H)$, and assume for every $x, y \in H_{\max\{\gamma, 1/2\}}$ that*

$$\langle F(x) - F(y), x - y \rangle_H \leq (\varepsilon \|x\|^2_{H_{1/2}} + \mathbf{C}) \|x - y\|^2_H + \|x - y\|^2_{H_{1/2}}. \tag{59}$$

Then it holds for every $x, y \in H_{\max\{\gamma, 1/2\}}$ that

$$\langle F'(x)y, y \rangle_H \leq (\varepsilon \|x\|^2_{H_{1/2}} + \mathbf{C}) \|y\|^2_H + \|y\|^2_{H_{1/2}}. \tag{60}$$

Proof of Lemma 3.1. Observe that for every $x \in H_{\max\{\gamma,1/2\}}$, $y \in (H_{\max\{\gamma,1/2\}} \backslash \{0\})$ it holds that

$$\begin{aligned}
&\langle F'(x)y, y\rangle_H = \big\langle \lim_{r\searrow 0} \tfrac{F(x+ry)-F(x)}{r}, y\big\rangle_H = \lim_{r\searrow 0} \big\langle \tfrac{F(x+ry)-F(x)}{r}, y\big\rangle_H \\
&= \lim_{r\searrow 0} \big(\tfrac{1}{r^2}\langle F(x+ry)-F(x), ry\rangle_H\big) \\
&\leq \big((\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}\big) \limsup_{r\searrow 0} \left[\frac{\frac{1}{r^2}\langle F(x+ry)-F(x), ry\rangle_H}{(\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}}\right] \\
&= \big((\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}\big) \limsup_{r\searrow 0} \left[\frac{\langle F(x+ry)-F(x), ry\rangle_H}{(\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|ry\|^2_H + \|ry\|^2_{H_{1/2}}}\right] \\
&\leq \big((\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}\big) \sup_{r\in(0,1]} \left[\frac{\langle F(x+ry)-F(x), ry\rangle_H}{(\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|ry\|^2_H + \|ry\|^2_{H_{1/2}}}\right] \\
&\leq \big((\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}\big) \sup_{v\in H_{\max\{\gamma,1/2\}}\backslash\{0\}} \left[\frac{\langle F(x+v)-F(x), v\rangle_H}{(\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|v\|^2_H + \|v\|^2_{H_{1/2}}}\right].
\end{aligned} \tag{61}$$

Combining this and (59) establishes (61). The proof of Lemma 3.1 is thus completed. □

Proposition 3.2. *Assume Setting 1.3, assume that* $\dim(H) < \infty$, *let* $T \in (0,\infty)$, $\varepsilon, \mathbf{C} \in [0,\infty)$, $F \in \mathcal{C}^1(H,H)$, $O \in \mathcal{C}([0,T],H)$, *assume for every* $x, y \in H$ *that* $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}$, *and for every* $s \in [0,T]$, $x \in H$ *let* $\mathcal{X}^x_{s,(\cdot)} = (\mathcal{X}^x_{s,t})_{t\in[s,T]} \colon [s,T] \to H$ *be a continuous function which satisfies for every* $t \in [s,T]$ *that*

$$\mathcal{X}^x_{s,t} = x + \int_s^t \big(A\mathcal{X}^x_{s,u} + F(\mathcal{X}^x_{s,u} + O_u)\big)\,du. \tag{62}$$

Then

(i) it holds that $(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H \ni (s,t,x) \mapsto \mathcal{X}^x_{s,t} \in H) \in \mathcal{C}^{0,0,1}(\{(u,v) \in [0,T]^2 \colon u \leq v\} \times H, H)$,

(ii) it holds for every $s \in [0,T]$, $t \in [s,T]$, $x, y \in H$ *that*

$$\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,t}\big)y = y + \int_s^t \big[A\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y + F'(\mathcal{X}^x_{s,u} + O_u)\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big]\,du, \tag{63}$$

and

(iii) it holds for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$ *that*

$$\big\|\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,t}\big\|_{L(H)} \leq \exp\!\left(\int_s^t \big(\varepsilon\|\mathcal{X}^x_{s,u} + O_u\|^2_{H_{1/2}} + \mathbf{C}\big)\,du\right). \tag{64}$$

Proof of Proposition 3.2. Note that the fact that $([0,T] \times H \ni (u,h) \mapsto Ah + F(h + O_u) \in H) \in \mathcal{C}^{0,1}([0,T] \times H, H)$ and, e.g., [46, items (v) and (vi) of Lemma 4.8] (with $V = H$, $T = T$, $f = ([0,T] \times H \ni (u,h) \mapsto Ah + F(h + O_u) \in H)$, $X^x_{s,t} = \mathcal{X}^x_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$, $x \in H$ in the notation of [46, items (v) and (vi) of Lemma 4.8]) establish items (i) and (ii). Therefore, we obtain that for every $s \in [0,T]$, $t \in [s,T]$, $x,y \in H$ it holds that

$$
\begin{aligned}
&\big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,t}\big)y\big\|_H^2 - \|y\|_H^2 = 2\int_s^t \big\langle \big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y, A\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y + F'(\mathcal{X}^x_{s,u}+O_u)\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\rangle_H \, du \\
&= 2\int_s^t \Big[\big\langle F'(\mathcal{X}^x_{s,u}+O_u)\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y, \big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\rangle_H - \big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\|^2_{H_{1/2}}\Big]\, du \\
&\leq 2\int_s^t \Big[\big(\varepsilon\|\mathcal{X}^x_{s,u}+O_u\|^2_{H_{1/2}} + \mathbf{C}\big)\big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\|_H^2 + \big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\|^2_{H_{1/2}} - \big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\|^2_{H_{1/2}}\Big]\, du \\
&= 2\int_s^t \Big[\big(\varepsilon\|\mathcal{X}^x_{s,u}+O_u\|^2_{H_{1/2}} + \mathbf{C}\big)\big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,u}\big)y\big\|_H^2\Big]\, du.
\end{aligned}
\tag{65}
$$

Moreover, note that the assumption that $\dim(H) < \infty$ assures that for every $s \in [0,T]$, $x \in H$ it holds that

$$\big([s,T] \ni u \mapsto \|\mathcal{X}^x_{s,u}+O_u\|^2_{H_{1/2}} \in [0,\infty)\big) \in \mathcal{C}([s,T],[0,\infty)). \tag{66}$$

Combining this, item (i), and (65) with Gronwall's lemma demonstrates that for every $s \in [0,T]$, $t \in [s,T]$, $x,y \in H$ it holds that

$$\big\|\big(\tfrac{\partial}{\partial x}\mathcal{X}^x_{s,t}\big)y\big\|_H \leq \|y\|_H \exp\!\bigg(\int_s^t \big(\varepsilon\|\mathcal{X}^x_{s,u}+O_u\|^2_{H_{1/2}} + \mathbf{C}\big)\, du\bigg). \tag{67}$$

The proof of Proposition 3.2 is thus completed. ☐

Corollary 3.3. *Assume Setting 1.3, assume that* $\dim(H) < \infty$, *let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space, let* $T \in (0,\infty)$, $\varepsilon, \mathbf{C} \in [0,\infty)$, $p \in [1,\infty)$, $F \in \mathcal{C}^1(H,H)$, $Y \in \mathcal{M}(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B}(H))$, *let* $O\colon [0,T]\times\Omega \to H$ *be a stochastic process with continuous sample paths, assume for every* $x,y \in H$ *that* $\langle F'(x)y,y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|_H^2 + \|y\|^2_{H_{1/2}}$, *and for every* $s \in [0,T]$, $x \in H$ *let* $X^x_{s,(\cdot)} = (X^x_{s,t})_{t\in[s,T]}\colon [s,T]\times\Omega \to H$ *be a stochastic process with continuous sample paths which satisfies for every* $t \in [s,T]$ *that*

$$X^x_{s,t} = e^{(t-s)A}x + \int_s^t e^{(t-u)A}F(X^x_{s,u})\, du + O_t - e^{(t-s)A}O_s. \tag{68}$$

Then

(i) it holds for every $\omega \in \Omega$ *that* $(\{(u,v)\in[0,T]^2\colon u \leq v\}\times H \ni (s,t,x) \mapsto X^x_{s,t}(\omega) \in H) \in \mathcal{C}^{0,0,1}(\{(u,v)\in[0,T]^2\colon u\leq v\}\times H, H)$,

(ii) it holds that $\big(\{(u,v)\in[0,T]^2\colon u\leq v\}\times\Omega \ni (s,t,\omega) \mapsto \tfrac{\partial}{\partial x}X^{Y_s(\omega)}_{s,t}(\omega) \in L(H)\big) \in \mathcal{M}(\mathcal{B}(\{(u,v)\in[0,T]^2\colon u\leq v\})\otimes\mathcal{F}, \mathcal{B}(L(H))),$

(iii) it holds that $\big(\{(u,v)\in[0,T]^2\colon u\leq v\}\times\Omega \ni (s,t,\omega) \mapsto X^{Y_s(\omega)}_{s,t}(\omega) \in H_{1/2}\big) \in \mathcal{M}(\mathcal{B}(\{(u,v)\in[0,T]^2\colon u\leq v\})\otimes\mathcal{F}, \mathcal{B}(H_{1/2}))$, *and*

(iv) it holds for every $s \in [0,T]$, $t \in [s,T]$ *that*

$$\mathbb{E}\Big[\big\|\tfrac{\partial}{\partial x}X^{Y_s}_{s,t}\big\|^p_{L(H)}\Big] \leq \mathbb{E}\bigg[\exp\!\bigg(p\int_s^t \big(\varepsilon\|X^{Y_s}_{s,u}\|^2_{H_{1/2}} + \mathbf{C}\big)\, du\bigg)\bigg]. \tag{69}$$

Proof of Corollary 3.3. Throughout this proof let $\mathcal{X}^x_{s,(\cdot)} = (\mathcal{X}^x_{s,t})_{t\in[s,T]} \colon [s,T]\times\Omega \to H$, $s\in[0,T]$, $x\in H$, be the functions which satisfy for every $s\in[0,T]$, $t\in[s,T]$, $\omega\in\Omega$, $x\in H$ that

$$\mathcal{X}^x_{s,t}(\omega) = X^{x+O_s(\omega)}_{s,t}(\omega) - O_t(\omega). \tag{70}$$

Observe that items (i) and (iii) of Lemma 2.8 (with $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $T=T$, $F=F$, $Z_s = Y_s$, $O_s = O_s$, $X^x_{s,t} = X^x_{s,t}$ for $t\in[s,T]$, $s\in[0,T]$, $x\in H$ in the notation of items (i) and (iii) of Lemma 2.8) establish items (i) and (ii). Furthermore, note that item (ii) of Lemma 2.8 (with $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $T=T$, $F=F$, $Y_s = Y_s$, $O_s = O_s$, $X^x_{s,t} = X^x_{s,t}$ for $t\in[s,T]$, $s\in[0,T]$, $x\in H$ in the notation of item (ii) of Lemma 2.8) implies that

$$\begin{aligned}&\big(\{(u,v)\in[0,T]^2\colon u\le v\}\times\Omega \ni (s,t,\omega)\mapsto X^{Y_s(\omega)}_{s,t}(\omega)\in H\big)\\ &\qquad\qquad\in \mathcal{M}(\mathcal{B}(\{(u,v)\in[0,T]^2\colon u\le v\})\otimes\mathcal{F},\mathcal{B}(H)).\end{aligned} \tag{71}$$

The assumption that $\dim(H)<\infty$ hence establishes item (iii). Next observe that (70) and the fact that for every $s\in[0,T]$, $t\in[s,T]$, $\omega\in\Omega$, $x\in H$ it holds that

$$X^{x+O_s(\omega)}_{s,t}(\omega) = e^{(t-s)A}(x+O_s(\omega)) + \int_s^t e^{(t-u)A}F(X^{x+O_s(\omega)}_{s,u}(\omega))\,du + O_t(\omega) - e^{(t-s)A}O_s(\omega) \tag{72}$$

prove that for every $s\in[0,T]$, $t\in[s,T]$, $\omega\in\Omega$, $x\in H$ it holds that

$$\mathcal{X}^x_{s,t}(\omega) = e^{(t-s)A}x + \int_s^t e^{(t-u)A}F(\mathcal{X}^x_{s,u}(\omega)+O_u(\omega))\,du. \tag{73}$$

The fact that $F\in\mathcal{C}(H,H)$, the fact that $\forall\, s\in[0,T], \omega\in\Omega\colon ([s,T]\ni t\mapsto O_t(\omega)\in H)\in\mathcal{C}([s,T],H)$, the fact that $\forall\, s\in[0,T], \omega\in\Omega, x\in H\colon ([s,T]\ni t\mapsto \mathcal{X}^x_{s,t}(\omega)\in H)\in\mathcal{C}([s,T],H)$, and Lemma 2.2 (with $T=T$, $s=s$, $x=x$, $Z=([s,T]\ni t\mapsto F(\mathcal{X}^x_{s,t}(\omega)+O_t(\omega))\in H)$, $Y=([s,T]\ni t\mapsto\mathcal{X}^x_{s,t}(\omega)\in H)$ for $s\in[0,T]$, $\omega\in\Omega$, $x\in H$ in the notation of Lemma 2.2) therefore ensure that for every $s\in[0,T]$, $t\in[s,T]$, $\omega\in\Omega$, $x\in H$ it holds that

$$\mathcal{X}^x_{s,t}(\omega) = x + \int_s^t [A\mathcal{X}^x_{s,u}(\omega) + F(\mathcal{X}^x_{s,u}(\omega)+O_u(\omega))]\,du. \tag{74}$$

Item (i) of Proposition 3.2 (with $T=T$, $\varepsilon=\varepsilon$, $\mathbf{C}=\mathbf{C}$, $F=F$, $O_t = O_t(\omega)$, $\mathcal{X}^x_{s,t} = \mathcal{X}^x_{s,t}(\omega)$ for $t\in[s,T]$, $s\in[0,T]$, $\omega\in\Omega$, $x\in H$ in the notation of item (i) of Proposition 3.2) hence proves that for every $\omega\in\Omega$ it holds that

$$\begin{aligned}&(\{(u,v)\in[0,T]^2\colon u\le v\}\times H\ni(s,t,x)\mapsto\mathcal{X}^x_{s,t}(\omega)\in H)\\ &\qquad\qquad\in\mathcal{C}^{0,0,1}(\{(u,v)\in[0,T]^2\colon u\le v\}\times H, H).\end{aligned} \tag{75}$$

Moreover, observe that (74) and item (iii) of Proposition 3.2 (with $T=T$, $\varepsilon=\varepsilon$, $\mathbf{C}=\mathbf{C}$, $F=F$, $O_t=O_t(\omega)$, $\mathcal{X}^x_{s,t}=\mathcal{X}^x_{s,t}(\omega)$ for $t\in[s,T]$, $s\in[0,T]$, $\omega\in\Omega$, $x\in H$ in the notation of item (iii) of Proposition 3.2) ensure that for every $s\in[0,T]$, $t\in[s,T]$, $\omega\in\Omega$ it holds that

$$\big\|\tfrac{\partial}{\partial x}\mathcal{X}^{Y_s(\omega)-O_s(\omega)}_{s,t}(\omega)\big\|_{L(H)} \le \exp\!\bigg(\int_s^t\big(\varepsilon\|\mathcal{X}^{Y_s(\omega)-O_s(\omega)}_{s,u}(\omega)+O_u(\omega)\|^2_{H_{1/2}}+\mathbf{C}\big)\,du\bigg). \tag{76}$$

This and (70) show that for every $s \in [0,T]$, $t \in [s,T]$, $\omega \in \Omega$ it holds that

$$\big\| \tfrac{\partial}{\partial x} X_{s,t}^{Y_s(\omega)}(\omega) \big\|_{L(H)} \leq \exp\!\bigg(\int_s^t \big(\varepsilon \| X_{s,u}^{Y_s(\omega)}(\omega) \|_{H_{1/2}}^2 + \mathbf{C} \big) \, du \bigg). \tag{77}$$

Combining this and items (ii) and (iii) establishes item (iv). The proof of Corollary 3.3 is thus completed. ☐

Lemma 3.4. *Assume Setting 1.3, assume that* $\dim(H) < \infty$, *let* $T \in (0,\infty)$, $s \in [0,T]$, $B \in \mathrm{HS}(U,H)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical Wiener process, let* $\xi \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$, $Z \in \mathcal{M}(\mathcal{B}([s,T]) \otimes \mathcal{F}, \mathcal{B}(H))$ *satisfy for every* $\omega \in \Omega$ *that* $\int_s^T \|Z_u(\omega)\|_H \, du < \infty$, *and let* $Y\colon [s,T] \times \Omega \to H$ *and* $O\colon [0,T] \times \Omega \to H$ *be stochastic processes with continuous sample paths which satisfy for every* $t \in [s,T]$ *that* $[O_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-u)A} B \, dW_u$ *and*

$$\mathbb{P}\bigg(Y_t = e^{(t-s)A}\xi + \int_s^t e^{(t-u)A} Z_u \, du + O_t - e^{(t-s)A} O_s \bigg) = 1. \tag{78}$$

Then it holds for every $t \in [s,T]$ *that*

$$[Y_t]_{\mathbb{P},\mathcal{B}(H)} = \bigg[\xi + \int_s^t [AY_u + Z_u] \, du \bigg]_{\mathbb{P},\mathcal{B}(H)} + \int_s^t B \, dW_u. \tag{79}$$

Proof of Lemma 3.4. Throughout this proof let $\Sigma = \{\omega \in \Omega \colon (\forall\, t \in [s,T] \colon Y_t(\omega) = e^{(t-s)A}\xi(\omega) + \int_s^t e^{(t-u)A} Z_u(\omega) \, du + O_t(\omega) - e^{(t-s)A} O_s(\omega))\}$. Observe that item (i) of Lemma 2.2 (with $T = T$, $s = s$, $x = \xi(\omega)$, $Z_t = Z_t(\omega)$, $Y_t = e^{(t-s)A}\xi(\omega) + \int_s^t e^{(t-u)A} Z_u(\omega) \, du$ for $t \in [s,T]$, $\omega \in \Sigma$ in the notation of item (i) of Lemma 2.2) proves that for every $\omega \in \Omega$ it holds that

$$\bigg([s,T] \ni t \mapsto e^{(t-s)A}\xi(\omega) + \int_s^t e^{(t-u)A} Z_u(\omega) \, du \in H \bigg) \in \mathcal{C}([s,T], H). \tag{80}$$

The fact that O and Y have continuous sample paths and (78) therefore show that

$$\mathbb{P}(\Sigma) = 1. \tag{81}$$

Next note that the assumption that $\dim(H) < \infty$ ensures that for every $t \in [s,T]$ it holds that

$$\begin{aligned} [e^{-(t-s)A} O_t]_{\mathbb{P},\mathcal{B}(H)} &= \int_0^t e^{(s-u)A} B \, dW_u = \int_0^s e^{(s-u)A} B \, dW_u + \int_s^t e^{(s-u)A} B \, dW_u \\ &= [O_s]_{\mathbb{P},\mathcal{B}(H)} + \int_s^t e^{(s-u)A} B \, dW_u. \end{aligned} \tag{82}$$

This implies that for every $t \in [s,T]$ it holds that

$$\int_s^t e^{(s-u)A} B \, dW_u = [e^{-(t-s)A} O_t - O_s]_{\mathbb{P},\mathcal{B}(H)}. \tag{83}$$

Combining (82), the fact that $[s,T]\times H\ni(t,x)\mapsto e^{(t-s)A}x\in H$ is twice continuously differentiable, and Itô's formula hence shows that for every $t\in[s,T]$ it holds that

$$\begin{aligned}[O_t]_{\mathbb{P},\mathcal{B}(H)} &= [e^{(t-s)A}O_s]_{\mathbb{P},\mathcal{B}(H)} + e^{(t-s)A}\int_s^t e^{(s-u)A}B\,dW_u \\ &= \left[e^{(t-s)A}O_s + \int_s^t Ae^{(u-s)A}(e^{-(u-s)A}O_u - O_s)\,du\right]_{\mathbb{P},\mathcal{B}(H)} + \int_s^t e^{(u-s)A}e^{(s-u)A}B\,dW_u \\ &= \left[e^{(t-s)A}O_s + \int_s^t A(O_u - e^{(u-s)A}O_s)\,du\right]_{\mathbb{P},\mathcal{B}(H)} + \int_s^t B\,dW_u.\end{aligned} \tag{84}$$

This implies that for every $t\in[s,T]$ it holds that

$$[O_t - e^{(t-s)A}O_s]_{\mathbb{P},\mathcal{B}(H)} = \left[\int_s^t A(O_u - e^{(u-s)A}O_s)\,du\right]_{\mathbb{P},\mathcal{B}(H)} + \int_s^t B\,dW_u. \tag{85}$$

Moreover, observe that item (ii) of Lemma 2.2 (with $T=T$, $s=s$, $x=\xi(\omega)$, $Z_t=Z_t(\omega)$, $Y_t=Y_t(\omega)-(O_t(\omega)-e^{(t-s)A}O_s(\omega))$ for $t\in[s,T]$, $\omega\in\Sigma$ in the notation of item (ii) of Lemma 2.2) proves that for every $t\in[s,T]$, $\omega\in\Sigma$ it holds that

$$Y_t(\omega)-(O_t(\omega)-e^{(t-s)A}O_s(\omega)) = \xi(\omega)+\int_s^t \left[A\big(Y_u(\omega)-(O_u(\omega)-e^{(u-s)A}O_s(\omega))\big)+Z_u(\omega)\right]du. \tag{86}$$

Combining (81) and (85) therefore establishes (79). The proof of Lemma 3.4 is thus completed. □

Lemma 3.5. *Assume Setting 1.3, assume that* $\dim(H)<\infty$*, let* $T\in(0,\infty)$*,* $a,b,\mathbf{C},\rho\in[0,\infty)$*,* $p\in[1,\infty)$*,* $B\in\mathrm{HS}(U,H)$*,* $\varepsilon\in[0,(2\rho/p)\exp(-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})T)]$*,* $F\in\mathcal{C}^1(H,H)$*, assume for every* $x,y\in H$ *that* $\langle x,F(x)\rangle_H\le a+b\|x\|_H^2$ *and* $\langle F'(x)y,y\rangle_H\le(\varepsilon\|x\|^2_{H_{1/2}}+\mathbf{C})\|y\|_H^2+\|y\|^2_{H_{1/2}}$*, let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$*, let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $Y\colon[0,T]\times\Omega\to H$ *and* $O\colon[0,T]\times\Omega\to H$ *be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths, assume for every* $t\in[0,T]$ *that* $[O_t]_{\mathbb{P},\mathcal{B}(H)}=\int_0^t e^{(t-u)A}B\,dW_u$*, and for every* $s\in[0,T]$*,* $x\in H$ *let* $X^x_{s,(\cdot)}=(X^x_{s,t})_{t\in[s,T]}\colon[s,T]\times\Omega\to H$ *be an* $(\mathbb{F}_t)_{t\in[s,T]}$*-adapted stochastic process with continuous sample paths which satisfies for every* $t\in[s,T]$ *that*

$$X^x_{s,t}=e^{(t-s)A}x+\int_s^t e^{(t-u)A}F(X^x_{s,u})\,du+O_t-e^{(t-s)A}O_s. \tag{87}$$

Then

(i) it holds for every $\omega\in\Omega$ *that* $(\{(u,v)\in[0,T]^2\colon u\le v\}\times H\ni(s,t,x)\mapsto X^x_{s,t}(\omega)\in H)\in\mathcal{C}^{0,0,1}(\{(u,v)\in[0,T]^2\colon u\le v\}\times H,H)$*,*

(ii) it holds that $\big(\{(u,v)\in[0,T]^2\colon u\le v\}\times\Omega\ni(s,t,\omega)\mapsto\frac{\partial}{\partial x}X^{Y_s(\omega)}_{s,t}(\omega)\in L(H)\big)\in\mathcal{M}(\mathcal{B}(\{(u,v)\in[0,T]^2\colon u\le v\})\otimes\mathcal{F},\mathcal{B}(L(H)))$*, and*

(iii) it holds for every $s\in[0,T]$*,* $t\in[s,T]$ *that*

$$\mathbb{E}\Big[\big\|\tfrac{\partial}{\partial x}X^{Y_s}_{s,t}\big\|^p_{L(H)}\Big]\le\exp\big(\big(p\mathbf{C}+\rho(2a+\|B\|^2_{\mathrm{HS}(U,H)})\big)(t-s)\big)\mathbb{E}\big[e^{\rho\|Y_s\|^2_H}\big]. \tag{88}$$

Proof of Lemma 3.5. Throughout this proof let $\mathbb{B} \in L(H,U)$ satisfy for every $v \in H$, $u \in U$ that $\langle Bu, v\rangle_H = \langle u, \mathbb{B}v\rangle_U$, let $R\colon U \to [\ker(B)]^\perp$ be the orthogonal projection of U on $[\ker(B)]^\perp$, let $d = \dim(H)$, $m = \dim([\ker(B)]^\perp)$, and let $\iota\colon H \to \mathbb{R}^d$ and $\kappa\colon R(U) \to \mathbb{R}^m$ be isometric isomorphisms. Observe that the assumption that for every $x, y \in H$ it holds that $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}$ and items (i) and (ii) of Corollary 3.3 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $\varepsilon = \varepsilon$, $\mathbf{C} = \mathbf{C}$, $p = p$, $F = F$, $Y_s = Y_s$, $O_s = O_s$, $X^x_{s,t} = X^x_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$, $x \in H$ in the notation of items (i) and (ii) of Corollary 3.3) establish items (i) and (ii). Moreover, note that the assumption that for every $x, y \in H$ it holds that $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}$ and items (iii) and (iv) of Corollary 3.3 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $\varepsilon = \varepsilon$, $\mathbf{C} = \mathbf{C}$, $p = p$, $F = F$, $Y_s = Y_s$, $O_s = O_s$, $X^x_{s,t} = X^x_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$, $x \in H$ in the notation of items (iii) and (iv) of Corollary 3.3) prove that for every $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\mathbb{E}\Big[\big\|\tfrac{\partial}{\partial x} X^{Y_s}_{s,t}\big\|^p_{L(H)}\Big] \leq e^{p\mathbf{C}(t-s)}\mathbb{E}\bigg[\exp\bigg(p\varepsilon \int_s^t \|X^{Y_s}_{s,u}\|^2_{H_{1/2}}\,du\bigg)\bigg]. \tag{89}$$

In the next step we intend to apply Cox et al. [17, Corollary 2.4] in order to derive an a priori bound for the right-hand side of (89). For this note that the assumption that for every $x \in H$ it holds that $\langle x, F(x)\rangle_H \leq a + b\|x\|^2_H$ implies that for every $x \in H$ it holds that

$$\begin{aligned}
&2\rho\langle x, Ax + F(x)\rangle_H + \rho\|B\|^2_{\mathrm{HS}(U,H)} + 2\rho^2\|\mathbb{B}x\|^2_U \\
&\leq -2\rho\|x\|^2_{H_{1/2}} + 2\rho\langle x, F(x)\rangle_H + \rho\|B\|^2_{HS(U,H)} + 2\rho^2\|B\|^2_{\mathrm{HS}(U,H)}\|x\|^2_H \\
&\leq -2\rho\|x\|^2_{H_{1/2}} + 2\rho a + 2\rho b\|x\|^2_H + \rho\|B\|^2_{\mathrm{HS}(U,H)} + 2\rho^2\|B\|^2_{\mathrm{HS}(U,H)}\|x\|^2_H \\
&= -2\rho\|x\|^2_{H_{1/2}} + \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)}) + 2\rho(b + \rho\|B\|^2_{\mathrm{HS}(U,H)})\|x\|^2_H.
\end{aligned} \tag{90}$$

Next note that Lemma 3.4 (with $T = T$, $s = s$, $B = B$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = Y_s$, $Z_{s+t} = F(X^{Y_s}_{s,s+t})$, $Y_{s+t} = X^{Y_s}_{s,s+t}$, $O = O$ for $t \in [0, T-s]$, $s \in [0,T]$ in the notation of Lemma 3.4) ensures that for every $s \in [0,T]$, $t \in [0, T-s]$ it holds that

$$[X^{Y_s}_{s,s+t}]_{\mathbb{P},\mathcal{B}(H)} = [Y_s]_{\mathbb{P},\mathcal{B}(H)} + \bigg[\int_s^{s+t} \big[AX^{Y_s}_{s,u} + F(X^{Y_s}_{s,u})\big]\,du\bigg]_{\mathbb{P},\mathcal{B}(H)} + \int_s^{s+t} B\,dW_u. \tag{91}$$

Moreover, observe that the assumption that $\dim(H) < \infty$ ensures that $\dim([\ker(B)]^\perp) < \infty$ and $R \in \mathrm{HS}(U)$. This implies that there exists a stochastic process $\mathbb{W}\colon [0,T] \times \Omega \to R(U)$ with continuous sample paths which satisfies for every $t \in [0,T]$ that

$$[\mathbb{W}_t]_{\mathbb{P},\mathcal{B}(R(U))} = \int_0^t R\,dW_s. \tag{92}$$

Observe that (92) implies that for every $s \in [0,T]$, $t \in [0, T-s]$ it holds that

$$\int_s^{s+t} B\,dW_u = \int_s^{s+t} BR\,dW_u = \int_s^{s+t} (B|_{R(U)})\,d\mathbb{W}_u = [(B|_{R(U)})(\mathbb{W}_{s+t} - \mathbb{W}_s)]_{\mathbb{P},\mathcal{B}(H)}. \tag{93}$$

In addition, note that, e.g., [49, Lemma 3.2] (with $H = R(U)$, $U = U$, $T = T$, $Q = \mathrm{Id}_U$, $R = \mathrm{Id}_{R(U)}$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $(\mathcal{G}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(\tilde{W}_t)_{t\in[0,T]} = (\mathbb{W}_t)_{t\in[0,T]}$ in the notation of [49, Lemma 3.2]) proves that $(\mathbb{W}_t)_{t\in[0,T]}$ is an $\mathrm{Id}_{R(U)}$-standard $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process. Combining this, (90), and (93) with Cox et al. [17, Corollary 2.4] (with $d = \dim(H)$, $m = \dim([\ker(B)]^\perp)$, $T = T - s$, $O = \mathbb{R}^d$, $\mu = (\mathbb{R}^d \ni x \mapsto$

$(\iota \circ A \circ \iota^{-1})(x) + (\iota \circ F \circ \iota^{-1})(x) \in \mathbb{R}^d)$, $\sigma = (\mathbb{R}^d \ni x \mapsto \iota \circ (B|_{R(U)}) \circ \kappa^{-1} \in \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d))$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_u = \mathbb{F}_{s+u}$, $W_u = \kappa(\mathbb{W}_{s+u} - \mathbb{W}_s)$, $\alpha = 2b + 2\rho\|B\|^2_{\mathrm{HS}(U,H)}$, $U = (\mathbb{R}^d \ni x \mapsto \rho\|\iota^{-1}(x)\|^2_H \in \mathbb{R})$, $\bar{U} = ([0, T-s] \times \mathbb{R}^d \ni (r, x) \mapsto 2\rho\|\iota^{-1}(x)\|^2_{H_{1/2}} - \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)}) \in \mathbb{R})$, $\tau = (\Omega \ni \omega \mapsto t - s \in [0, T-s])$, $X_u = \iota \circ X^{Y_s}_{s,s+u}$ for $u \in [0, T-s]$, $t \in [s, T]$, $s \in [0, T)$ in the notation of Cox et al. [17, Corollary 2.4]) shows that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$\begin{aligned} \mathbb{E}\bigg[\exp\bigg(&\rho e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})(t-s)}\|X^{Y_s}_{s,t}\|^2_H \\ &+ \int_s^t e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})(u-s)}\big(2\rho\|X^{Y_s}_{s,u}\|^2_{H_{1/2}} - \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})\big)\,du\bigg)\bigg] \leq \mathbb{E}\big[e^{\rho\|Y_s\|^2_H}\big]. \end{aligned} \tag{94}$$

This implies that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$\begin{aligned} &\mathbb{E}\bigg[\exp\bigg(\rho e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})(t-s)}\|X^{Y_s}_{s,t}\|^2_H + 2\rho\int_s^t e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})(u-s)}\|X^{Y_s}_{s,u}\|^2_{H_{1/2}}\,du\bigg)\bigg] \\ &\leq \exp\bigg(\rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})\int_s^t e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})(u-s)}\,du\bigg)\mathbb{E}\big[e^{\rho\|Y_s\|^2_H}\big]. \end{aligned} \tag{95}$$

Therefore, we obtain that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$\begin{aligned} &\mathbb{E}\bigg[\exp\bigg(2\rho e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})T}\int_s^t \|X^{Y_s}_{s,u}\|^2_{H_{1/2}}\,du\bigg)\bigg] \\ &\leq \exp\big(\rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})(t-s)\big)\mathbb{E}\big[e^{\rho\|Y_s\|^2_H}\big]. \end{aligned} \tag{96}$$

The assumption that $p\varepsilon \leq 2\rho\exp(-2(b + \rho\|B\|^2_{\mathrm{HS}(U,H)})T)$ and (89) hence demonstrate that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$\begin{aligned} \mathbb{E}\Big[\big\|\tfrac{\partial}{\partial x}X^{Y_s}_{s,t}\big\|^p_{L(H)}\Big] &\leq e^{p\mathbf{C}(t-s)}\mathbb{E}\bigg[\exp\bigg(2\rho e^{-2(b+\rho\|B\|^2_{\mathrm{HS}(U,H)})T}\int_s^t \|X^{Y_s}_{s,u}\|^2_{H_{1/2}}\,du\bigg)\bigg] \\ &\leq \exp\big(p\mathbf{C}(t-s) + \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})(t-s)\big)\mathbb{E}\big[e^{\rho\|Y_s\|^2_H}\big]. \end{aligned} \tag{97}$$

The proof of Lemma 3.5 is thus completed. □

3.2 Strong error estimates for exponential Euler-type approximations

In this subsection we combine the results from Subsections 2.2 and 3.1 to establish in Proposition 3.6 an upper bound for the strong error between the exact solution of an SODE with additive noise and given initial value (see (99) below) and its numerical approximation (see (98) below).

Proposition 3.6. *Assume Setting 1.3, assume that* $\dim(H) < \infty$, *let* $T \in (0, \infty)$, $\theta \in \varpi_T$, $a, b, \mathbf{C}, \rho \in [0, \infty)$, $C, c, p \in [1, \infty)$, $\gamma \in [0, 1)$, $\delta \in [0, \gamma]$, $\kappa \in \mathbb{R}$, $B \in \mathrm{HS}(U, H)$, $\varepsilon \in [0, (\rho/p)\exp(-2(b + \rho\|B\|^2_{\mathrm{HS}(U,H)})T)]$, $F \in \mathcal{C}^1(H, H)$, $\mathbf{F} \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(H))$, $\Phi \in \mathcal{C}(H, [0, \infty))$, *assume for every* $x, y \in H$ *that* $\langle x, F(x)\rangle_H \leq a + b\|x\|^2_H$, $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}$, $\|F(x) - F(y)\|_H \leq C\|x - y\|_{H_\delta}(1 + \|x\|^c_{H_\kappa} + \|y\|^c_{H_\kappa})$, *and* $\langle x, Ax + F(x + y)\rangle_H \leq \Phi(y)(1 + \|x\|^2_H)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$, *let* $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(H))$, *let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $O\colon [0, T] \times \Omega \to H$ *be a stochastic process with continuous sample paths which satisfies for every* $t \in [0, T]$ *that* $[O_t]_{\mathbb{P},\mathcal{B}(H)} =$

$\int_0^t e^{(t-u)A} B\, dW_u$, *and let* $\mathbf{X}\colon [0,T]\times\Omega\to H$ *and* $\mathbf{O}\colon [0,T]\times\Omega\to H$ *be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths which satisfy for every* $t\in[0,T]$ *that*

$$\mathbb{P}\bigg(\mathbf{X}_t = e^{tA}\xi + \int_0^t e^{(t-\llcorner u \lrcorner_\theta)A}\mathbf{F}(\mathbf{X}_{\llcorner u \lrcorner_\theta})\, du + \mathbf{O}_t\bigg) = 1. \tag{98}$$

Then

(i) there exists a unique stochastic process $X\colon [0,T]\times\Omega\to H$ *with continuous sample paths which satisfies for every* $t\in[0,T]$ *that*

$$X_t = e^{tA}\xi + \int_0^t e^{(t-u)A}F(X_u)\, du + O_t, \tag{99}$$

(ii) it holds that X *is* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted, and*

(iii) it holds for every $t\in[0,T]$ *that*

$$\begin{aligned}
&\|\mathbf{X}_t - X_t\|_{\mathcal{L}^p(\mathbb{P};H)} \le \sup\nolimits_{s\in[0,T]} \|\mathbf{O}_s - O_s\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\quad + \tfrac{C[\max\{T,1\}]^2}{1-\gamma}\exp\!\big(\big(\mathbf{C}+\rho(2a+\|B\|^2_{\mathrm{HS}(U,H)})\big)t\big)\bigg[\int_0^t \mathbb{E}\big[e^{\rho\|\mathbf{X}_s-\mathbf{O}_s+O_s\|_H^2}\big]\, ds\bigg] \\
&\quad \cdot\Big\{[|\theta|_T]^{\gamma-\delta}\sup\nolimits_{s\in[0,T]}\|\mathbf{F}(\mathbf{X}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})} + \sup\nolimits_{s\in[0,T]}\|\mathbf{F}(\mathbf{X}_s)-F(\mathbf{X}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&\quad + \Big(2[|\theta|_T]^{\gamma-\delta}\sup\nolimits_{s\in[0,T]}\|\mathbf{F}(\mathbf{X}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} + \sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s-\mathbf{O}_{\llcorner s\lrcorner_\theta}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \\
&\quad + [|\theta|_T]^{\gamma-\delta}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)} + \sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s-O_s\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}\Big) \\
&\quad \cdot\big[1+2\sup\nolimits_{s\in[0,T]}\|\mathbf{X}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} + \sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s-O_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}\big]^c\Big\}.
\end{aligned} \tag{100}$$

Proof of Proposition 3.6. Throughout this proof let $\Sigma\subseteq\Omega$ be the set which satisfies that

$$\Sigma = \bigg\{\omega\in\Omega\colon \bigg(\forall\, t\in[0,T]\colon \mathbf{X}_t(\omega) = e^{tA}\xi(\omega) + \int_0^t e^{(t-\llcorner u\lrcorner_\theta)A}\mathbf{F}(\mathbf{X}_{\llcorner u\lrcorner_\theta}(\omega))\, du + \mathbf{O}_t(\omega)\bigg)\bigg\}, \tag{101}$$

let $\mathcal{Y}\colon [0,T]\times\Omega\to H$ be the function which satisfies for every $t\in[0,T]$, $\omega\in\Omega$ that

$$\mathcal{Y}_t(\omega) = \begin{cases}\mathbf{X}_t(\omega) & : \omega\in\Sigma\\ 0 & : \omega\in(\Omega\backslash\Sigma),\end{cases} \tag{102}$$

and let $\mathcal{O}\colon [0,T]\times\Omega\to H$ be the function which satisfies for every $t\in[0,T]$, $\omega\in\Omega$ that

$$\mathcal{O}_t(\omega) = \begin{cases}\mathbf{O}_t(\omega) & : \omega\in\Sigma\\ -e^{tA}\xi(\omega) - \int_0^t e^{(t-\llcorner u\lrcorner_\theta)A}\mathbf{F}(0)\, du & : \omega\in(\Omega\backslash\Sigma).\end{cases} \tag{103}$$

Note that the assumption that for every $x,y\in H$ it holds that

$$\|F(x)-F(y)\|_H \le C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\kappa}+\|y\|^c_{H_\kappa}), \tag{104}$$

the assumption that for every $x, y \in H$ it holds that

$$\langle x, Ax + F(x+y)\rangle_H \leq \Phi(y)(1 + \|x\|_H^2), \tag{105}$$

and, e.g., [47, Corollary 2.4] (with $H = H$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $T = T$, $s = 0$, $C = C$, $c = c$, $\delta = \delta$, $\kappa = \kappa$, $F = F$, $\Phi = \Phi$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $\xi = \xi + O_0$, $O = O$ in the notation of [47, Corollary 2.4]) establish items (i) and (ii). In the next step we are going to use Corollary 2.9 and Lemma 3.5 to prove (100). For this observe that (104), (105), and, e.g., [47, Corollary 2.4] (with $H = H$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $T = T$, $s = s$, $C = C$, $c = c$, $\delta = \delta$, $\kappa = \kappa$, $F = F$, $\Phi = \Phi$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $\xi = (\Omega \ni \omega \mapsto x \in H)$, $O = O$ for $s \in [0,T]$, $x \in H$ in the notation of [47, Corollary 2.4]) demonstrate that there exist stochastic processes $\mathcal{X}^x_{s,(\cdot)} = (\mathcal{X}^x_{s,t})_{t\in[s,T]} \colon [s,T] \times \Omega \to H$, $s \in [0,T]$, $x \in H$, with continuous sample paths which satisfy for every $s \in [0,T]$, $t \in [s,T]$, $x \in H$ that $\mathcal{X}^x_{s,(\cdot)}$ is $(\mathbb{F}_u)_{u\in[s,T]}$-adapted and

$$\mathcal{X}^x_{s,t} = e^{(t-s)A}x + \int_s^t e^{(t-u)A} F(\mathcal{X}^x_{s,u})\, du + O_t - e^{(t-s)A} O_s. \tag{106}$$

Moreover, note that (98) and the fact that $\mathbf{X}$ and $\mathbf{O}$ are stochastic processes with continuous sample paths ensure that

$$\Sigma \in \mathcal{F} \qquad \text{and} \qquad \mathbb{P}(\Sigma) = 1. \tag{107}$$

The fact that $(\mathbb{F}_t)_{t\in[0,T]}$ is a normal filtration and the fact that $\mathbf{X}$ and $\mathbf{O}$ are $(\mathbb{F}_t)_{t\in[0,T]}$-adapted therefore implies that

(a) it holds that $\mathcal{Y}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted,

(b) it holds that $\mathcal{O}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted,

(c) it holds for every $t \in [0,T]$ that $\mathbb{P}(\mathcal{Y}_t = \mathbf{X}_t) = 1$, and

(d) it holds for every $t \in [0,T]$ that $\mathbb{P}(\mathcal{O}_t = \mathbf{O}_t) = 1$.

In addition, note that (106) implies that for every $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\mathcal{X}^{\xi(\omega)+O_0(\omega)}_{0,t}(\omega) = e^{tA}\xi(\omega) + \int_0^t e^{(t-u)A} F\big(\mathcal{X}^{\xi(\omega)+O_0(\omega)}_{0,u}(\omega)\big)\, du + O_t(\omega). \tag{108}$$

Furthermore, observe that item (i) ensures that for every $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X_t(\omega) = e^{tA}\xi(\omega) + \int_0^t e^{(t-u)A} F(X_u(\omega))\, du + O_t(\omega). \tag{109}$$

Combining this, (108), and, e.g., [47, item (i) of Corollary 2.4] (with $H = H$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $T = T$, $s = 0$, $C = C$, $c = c$, $\delta = \delta$, $\kappa = \kappa$, $F = F$, $\Phi = \Phi$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $\xi = \xi + O_0$, $O = O$ in the notation of [47, item (i) of Corollary 2.4]) shows that for every $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X_t(\omega) = \mathcal{X}^{\xi(\omega)+O_0(\omega)}_{0,t}(\omega). \tag{110}$$

Moreover, observe that (101)–(103) prove that for every $t \in [0,T]$ it holds that

$$\mathcal{Y}_t = e^{tA}\xi + \int_0^t e^{(t-\llcorner u \lrcorner_\theta)A} \mathbf{F}(\mathcal{Y}_{\llcorner u \lrcorner_\theta})\, du + \mathcal{O}_t. \tag{111}$$

Combining item (c), (104), (106), (110), and Corollary 2.9 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $\theta = \theta$, $C = C$, $c = c$, $p = p$, $\gamma = \gamma$, $\delta = \delta$, $\iota = \gamma - \delta$, $\kappa = \kappa$, $\xi = \xi$, $F = F$, $\mathbf{F} = \mathbf{F}$, $\mathbf{O}_s = \mathcal{O}_s$, $O_s = O_s$, $X^x_{s,t} = \mathcal{X}^x_{s,t}$, $\mathbf{X}_s = \mathcal{Y}_s$, $\zeta = \xi$ for $t \in [s,T]$, $s \in [0,T]$, $x \in H$ in the notation of Corollary 2.9) therefore establishes that

(A) it holds for every $s \in [0,T]$, $t \in [s,T]$, $\omega \in \Omega$ that $H \ni x \mapsto \mathcal{X}^x_{s,t}(\omega) \in H$ is differentiable,

(B) it holds for every $t \in [0,T]$ that $\big(\Omega \ni \omega \mapsto \mathcal{X}^{\xi(\omega)+O_0(\omega)}_{0,t}(\omega) \in H\big) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$,

(C) it holds for every $t \in [0,T]$ that $\big([0,t] \times \Omega \ni (s,\omega) \mapsto \frac{\partial}{\partial x}\mathcal{X}^{\mathcal{Y}_s(\omega)-\mathcal{O}_s(\omega)+O_s(\omega)}_{s,t}(\omega) \in L(H)\big) \in \mathcal{M}(\mathcal{B}([0,t]) \otimes \mathcal{F}, \mathcal{B}(L(H)))$, and

(D) it holds for every $t \in [0,T]$ that

$$\begin{aligned}
&\|\mathbf{X}_t - X_t\|_{\mathcal{L}^p(\mathbb{P};H)} = \|\mathcal{Y}_t - \mathcal{X}^{\xi+O_0}_{0,t}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - O_s\|_{\mathcal{L}^p(\mathbb{P};H)} + \tfrac{C\max\{T,1\}}{1-\gamma}\left[\int_0^t \big\|\tfrac{\partial}{\partial x}\mathcal{X}^{\mathcal{Y}_s-\mathcal{O}_s+O_s}_{s,t}\big\|_{\mathcal{L}^{2p}(\mathbb{P};L(H))}\,ds\right] \\
&\quad\cdot \Big\{[|\theta|_T]^{\gamma-\delta} \sup\nolimits_{s\in[0,T]} \|\mathbf{F}(\mathcal{Y}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})} + \sup\nolimits_{s\in[0,T]} \|\mathbf{F}(\mathcal{Y}_s) - F(\mathcal{Y}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&\quad+ \Big(([|\theta|_T]^{1-\delta} + [|\theta|_T]^{\gamma-\delta}) \sup\nolimits_{s\in[0,T]} \|\mathbf{F}(\mathcal{Y}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} + \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - \mathcal{O}_{\llcorner s \lrcorner_\theta}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \\
&\quad+ [|\theta|_T]^{\gamma-\delta}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)} + \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - O_s\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}\Big) \\
&\quad\cdot \big[1 + 2\sup\nolimits_{s\in[0,T]} \|\mathcal{Y}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} + \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - O_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}\big]^c\Big\}.
\end{aligned} \tag{112}$$

Moreover, note that (106), the fact that $\mathcal{Y}$, $\mathcal{O}$, and O are $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths, the assumption that for every $x, y \in H$ it holds that $\langle x, F(x)\rangle_H \leq a + b\|x\|^2_H$ and $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + \mathbf{C})\|y\|^2_H + \|y\|^2_{H_{1/2}}$, and Lemma 3.5 (with $T = T$, $a = a$, $b = b$, $\mathbf{C} = \mathbf{C}$, $\rho = \rho$, $p = 2p$, $B = B$, $\varepsilon = \varepsilon$, $F = F$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $Y_s = \mathcal{Y}_s - \mathcal{O}_s + O_s$, $O_s = O_s$, $X^x_{s,u} = \mathcal{X}^x_{s,u}$ for $u \in [s,T]$, $s \in [0,T]$, $x \in H$ in the notation of Lemma 3.5) prove that for every $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\mathbb{E}\Big[\big\|\tfrac{\partial}{\partial x}\mathcal{X}^{\mathcal{Y}_s-\mathcal{O}_s+O_s}_{s,t}\big\|^{2p}_{L(H)}\Big] \leq \exp\!\big(\big(2p\mathbf{C} + \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})\big)t\big)\,\mathbb{E}\big[e^{\rho\|\mathcal{Y}_s-\mathcal{O}_s+O_s\|^2_H}\big]. \tag{113}$$

This and (112) show that for every $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|\mathbf{X}_t - X_t\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - O_s\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\quad+ \tfrac{C[\max\{T,1\}]^2}{1-\gamma} \exp\!\big(\big(\mathbf{C} + \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})\big)t\big)\left[\int_0^t \big(\mathbb{E}\big[e^{\rho\|\mathcal{Y}_s-\mathcal{O}_s+O_s\|^2_H}\big]\big)^{1/(2p)}\,ds\right] \\
&\quad\cdot \Big\{[|\theta|_T]^{\gamma-\delta} \sup\nolimits_{s\in[0,T]} \|\mathbf{F}(\mathcal{Y}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})} + \sup\nolimits_{s\in[0,T]} \|\mathbf{F}(\mathcal{Y}_s) - F(\mathcal{Y}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&\quad+ \Big(2[|\theta|_T]^{\gamma-\delta} \sup\nolimits_{s\in[0,T]} \|\mathbf{F}(\mathcal{Y}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} + \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - \mathcal{O}_{\llcorner s \lrcorner_\theta}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \\
&\quad+ [|\theta|_T]^{\gamma-\delta}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)} + \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - O_s\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}\Big) \\
&\quad\cdot \big[1 + 2\sup\nolimits_{s\in[0,T]} \|\mathcal{Y}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} + \sup\nolimits_{s\in[0,T]} \|\mathcal{O}_s - O_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}\big]^c\Big\}.
\end{aligned} \tag{114}$$

Combining this and items (c) and (d) establishes item (iii). The proof of Proposition 3.6 is thus completed. ∎

4 Strong convergence rates for space-time discrete exponential Euler-type approximations with assuming finite exponential moments

4.1 Moment bounds for spatial spectral Galerkin approximations

In this subsection we prove in Lemma 4.1 suitable a priori moment bounds for exact solutions of SODEs. Corollary 4.2 then establishes uniform a priori moment bounds for spectral Galerkin approximations of exact solutions of semilinear SPDEs with additive noise.

Lemma 4.1. *Assume Setting 1.3, assume that* $\dim(H) < \infty$*, let* $T \in (0,\infty)$*,* $a, b \in [0,\infty)$*,* $p \in [2,\infty)$*,* $s \in [0,T]$*,* $B \in \mathrm{HS}(U,H)$*,* $F \in \mathcal{C}(H,H)$*, assume for every* $x \in H$ *that* $\langle x, F(x)\rangle_H \leq a + b\|x\|_H^2$*, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$*, let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $\xi \in \mathcal{M}(\mathbb{F}_s, \mathcal{B}(H))$*, let* $O\colon [0,T]\times\Omega \to H$ *be a stochastic process with continuous sample paths which satisfies for every* $t \in [0,T]$ *that* $[O_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-u)A} B\, dW_u$*, and let* $X\colon [s,T]\times\Omega \to H$ *be an* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic process with continuous sample paths which satisfies for every* $t \in [s,T]$ *that*

$$\mathbb{P}\bigg(X_t = e^{(t-s)A}\xi + \int_s^t e^{(t-u)A} F(X_u)\, du + O_t - e^{(t-s)A} O_s\bigg) = 1. \tag{115}$$

Then

$$\sup\nolimits_{t\in[s,T]} \mathbb{E}[\|X_t\|_H^p] \leq \big(\mathbb{E}[\|\xi\|_H^p] + 2T\big[a + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big]^{p/2}\big) \exp((pb + p - 2)T). \tag{116}$$

Proof of Lemma 4.1. Throughout this proof let $\mathbb{U} \subseteq U$ be an orthonormal basis of U. Note that Lemma 3.4 (with $T = T$, $s = s$, $B = B$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $Z_t = F(X_t)$, $Y_t = X_t$, $O = O$ for $t \in [s,T]$ in the notation of Lemma 3.4) shows that for every $t \in [s,T]$ it holds that

$$[X_t]_{\mathbb{P},\mathcal{B}(H)} = \bigg[\xi + \int_s^t [AX_u + F(X_u)]\, du\bigg]_{\mathbb{P},\mathcal{B}(H)} + \int_s^t B\, dW_u. \tag{117}$$

Furthermore, observe that the fact that X has continuous sample paths ensures that there exist $(\mathbb{F}_t)_{t\in[s,T]}$-stopping times $\tau_r\colon \Omega \to [s,T]$, $r \in (0,\infty)$, which satisfy for every $r \in (0,\infty)$ that

$$\tau_r = \inf(\{T\} \cup \{t \in [s,T]\colon \|X_t\|_H \geq r\}). \tag{118}$$

Note that Itô's formula, (117), and (118) demonstrate that for every $r \in (0,\infty)$, $t \in [s,T]$ it holds that

$$
\begin{aligned}
&[\|X_{\min\{\tau_r,t\}}\|_H^p]_{\mathbb{P},\mathcal{B}(\mathbb{R})} = \left[\|\xi\|_H^p + \int_s^{\min\{\tau_r,t\}} p\|X_u\|_H^{p-2}\langle X_u, AX_u + F(X_u)\rangle_H \, du\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})} \\
&\quad + \int_s^t p\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^{p-2}\langle X_u, B\, dW_u\rangle_H \\
&\quad + \left[\tfrac{1}{2}\int_s^{\min\{\tau_r,t\}} \sum_{\mathbf{u}\in\mathbb{U}} \left[p\|X_u\|_H^{p-2}\|B\mathbf{u}\|_H^2 + p(p-2)\mathbb{1}_{\{X_u\neq 0\}}\|X_u\|_H^{p-4}|\langle X_u, B\mathbf{u}\rangle_H|^2\right] du\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})} \\
&\leq \left[\|\xi\|_H^p + \int_s^{\min\{\tau_r,t\}} p\|X_u\|_H^{p-2}\langle X_u, AX_u + F(X_u)\rangle_H \, du\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})} \\
&\quad + \int_s^t p\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^{p-2}\langle X_u, B\, dW_u\rangle_H + \left[\tfrac{p(p-1)}{2}\|B\|_{\mathrm{HS}(U,H)}^2 \int_s^{\min\{\tau_r,t\}} \|X_u\|_H^{p-2}\, du\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})}.
\end{aligned}
\tag{119}
$$

Moreover, observe that for every $r \in (0,\infty)$, $t \in [s,T]$ it holds that

$$
\begin{aligned}
&\int_s^t \mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^{2(p-2)}\|(U \ni v \mapsto \langle X_u, B(v)\rangle_H \in \mathbb{R})\|_{\mathrm{HS}(U,\mathbb{R})}^2\, du \\
&\leq \int_s^t \mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^{2(p-1)}\|B\|_{\mathrm{HS}(U,H)}^2\, du \\
&\leq \int_s^t r^{2(p-1)}\|B\|_{\mathrm{HS}(U,H)}^2\, du \leq \int_0^T r^{2(p-1)}\|B\|_{\mathrm{HS}(U,H)}^2\, du < \infty.
\end{aligned}
\tag{120}
$$

Combining this, the assumption that for every $x \in H$ it holds that $\langle x, F(x)\rangle_H \leq a + b\|x\|_H^2$, (119), Tonelli's theorem, and Young's inequality proves that for every $r \in (0,\infty)$, $t \in [s,T]$ it holds that

$$
\begin{aligned}
&\mathbb{E}[\|\mathbb{1}_{\{\tau_r \geq t\}}X_t\|_H^p] \leq \mathbb{E}[(\|\mathbb{1}_{\{\tau_r \geq t\}}X_{\min\{\tau_r,t\}}\|_H + \|\mathbb{1}_{\{\tau_r < t\}}X_{\min\{\tau_r,t\}}\|_H)^p] = \mathbb{E}[\|X_{\min\{\tau_r,t\}}\|_H^p] \\
&\leq \mathbb{E}[\|\xi\|_H^p] + p\mathbb{E}\left[\int_s^{\min\{\tau_r,t\}} \|X_u\|_H^{p-2}\big(\langle X_u, AX_u + F(X_u)\rangle_H + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big)\, du\right] \\
&\leq \mathbb{E}[\|\xi\|_H^p] + p\mathbb{E}\left[\int_s^{\min\{\tau_r,t\}} \|X_u\|_H^{p-2}\big(a + b\|X_u\|_H^2 + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big)\, du\right] \\
&= \mathbb{E}[\|\xi\|_H^p] + p\mathbb{E}\left[\int_s^t \mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^{p-2}\big(a + b\|X_u\|_H^2 + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big)\, du\right] \\
&= \mathbb{E}[\|\xi\|_H^p] + p\int_s^t \mathbb{E}\big[\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^{p-2}\big(a + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big) + b\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^p\big]\, du \\
&\leq \mathbb{E}[\|\xi\|_H^p] + p\int_s^t \mathbb{E}\big[\tfrac{p-2}{p}\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^p + \tfrac{2}{p}\big(a + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big)^{p/2} + b\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^p\big]\, du \\
&= \mathbb{E}[\|\xi\|_H^p] + (pb + p - 2)\int_s^t \mathbb{E}[\mathbb{1}_{\{\tau_r \geq u\}}\|X_u\|_H^p]\, du + 2(t-s)\big(a + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big)^{p/2} \\
&\leq \mathbb{E}[\|\xi\|_H^p] + (pb + p - 2)(t-s)r^p + 2(t-s)\big(a + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big)^{p/2}.
\end{aligned}
\tag{121}
$$

Gronwall's lemma therefore shows that for every $r \in (0,\infty)$, $t \in [s,T]$ it holds that

$$
\mathbb{E}[\mathbb{1}_{\{\tau_r \geq t\}}\|X_t\|_H^p] \leq \big(\mathbb{E}[\|\xi\|_H^p] + 2(t-s)\big[a + \tfrac{p-1}{2}\|B\|_{\mathrm{HS}(U,H)}^2\big]^{p/2}\big)\exp((pb + p - 2)(t-s)). \tag{122}
$$

The fact that for every $n \in \mathbb{N}$, $t \in [0,T]$ it holds that $\mathbb{1}_{\{\tau_n \geq t\}} \leq \mathbb{1}_{\{\tau_{n+1} \geq t\}}$ and the monotone convergence theorem hence establish (116). The proof of Lemma 4.1 is thus completed. ☐

Corollary 4.2. *Assume Setting 1.3, let $T \in (0,\infty)$, $a,b \in [0,\infty)$, $p \in [1,\infty)$, $\beta \in [0, 1/2)$, $\gamma, \eta_1 \in [0, 1/2+\beta)$, $\eta_2 \in [\eta_1, 1/2+\beta)$, $\iota \in [\eta_2, 1/2+\beta)$, $\alpha_1 \in [0, 1-\eta_1)$, $\alpha_2 \in [0, 1-\eta_2)$, $B \in \mathrm{HS}(U, H_\beta)$, $F \in \mathcal{C}(H_\gamma, H)$, $(P_I)_{I \in \mathcal{P}(\mathbb{H})} \subseteq L(H)$ satisfy for every $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ that $P_I(x) = \sum_{h \in I} \langle h, x \rangle_H h$, assume for every $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ that $\langle x, F(x) \rangle_H \leq a + b\|x\|_H^2$ and*

$$\left[\sup_{v \in H_{\max\{\gamma,\eta_2\}}} \tfrac{\|F(v)\|_H}{1+\|v\|^2_{H_{\eta_2}}}\right] + \left[\sup_{v \in H_{\max\{\gamma,\eta_1\}}} \tfrac{\|F(v)\|_{H_{-\alpha_2}}}{1+\|v\|^2_{H_{\eta_1}}}\right] + \left[\sup_{v \in H_\gamma} \tfrac{\|F(v)\|_{H_{-\alpha_1}}}{1+\|v\|^2_H}\right] < \infty, \tag{123}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, let $\xi \in \mathcal{L}^{4p}(\mathbb{P}|_{\mathbb{F}_0}; H_\iota)$ satisfy $\mathbb{E}[\|\xi\|_H^{8p}] < \infty$, and let $X^I\colon [0,T] \times \Omega \to P_I(H)$, $I \in \mathcal{P}_0(\mathbb{H})$, and $O^I\colon [0,T] \times \Omega \to P_I(H)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for every $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ that $[O^I_t]_{\mathbb{P},\mathcal{B}(P_I(H))} = \int_0^t e^{(t-s)A} P_I B \, dW_s$ and

$$X^I_t = e^{tA} P_I \xi + \int_0^t e^{(t-s)A} P_I F(X^I_s)\, ds + O^I_t. \tag{124}$$

Then

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \|X^I_t\|_{\mathcal{L}^p(\mathbb{P}; H_\iota)} < \infty. \tag{125}$$

Proof of Corollary 4.2. Throughout this proof let $\mathcal{A}_I\colon P_I(H) \to P_I(H)$, $I \in \mathcal{P}_0(\mathbb{H})$, be the functions which satisfy for every $I \in \mathcal{P}_0(\mathbb{H})$, $v \in P_I(H)$ that $\mathcal{A}_I v = Av$ and for every $I \in \mathcal{P}_0(\mathbb{H})$ let $(\mathcal{H}_{I,r}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{I,r}}, \|\cdot\|_{\mathcal{H}_{I,r}})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-\mathcal{A}_I$. Note that the Burkholder-Davis-Gundy-type inequality in Da Prato & Zabczyk [23, Lemma 7.7] proves that for every $t \in [0,T]$, $q \in [2,\infty)$ it holds that

$$\begin{aligned}
&\sup_{I \in \mathcal{P}_0(\mathbb{H})} \|O^I_t\|^2_{\mathcal{L}^q(\mathbb{P}; H_\iota)} \leq \tfrac{q(q-1)}{2} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \int_0^t \|e^{(t-s)A} P_I B\|^2_{\mathrm{HS}(U, H_\iota)}\, ds \\
&\leq \tfrac{q(q-1)}{2} \int_0^t \|(-A)^{\iota-\beta} e^{(t-s)A}\|^2_{L(H)} \|B\|^2_{\mathrm{HS}(U,H_\beta)}\, ds \leq \tfrac{q(q-1)}{2} \int_0^t (t-s)^{2\beta-2\iota} \|B\|^2_{\mathrm{HS}(U,H_\beta)}\, ds \\
&\leq \tfrac{q(q-1)}{2} \tfrac{t^{1+2\beta-2\iota}}{1+2\beta-2\iota} \|B\|^2_{\mathrm{HS}(U,H_\beta)} < \infty.
\end{aligned} \tag{126}$$

Next observe that the fact that $\xi \in \mathcal{L}^{8p}(\mathbb{P}|_{\mathbb{F}_0}; H)$, the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that $\langle x, F(x) \rangle_H \leq a + b\|x\|^2_H$, and Lemma 4.1 (with $H = P_I(H)$, $\mathbb{H} = P_I(\mathbb{H})$, $\mathfrak{v} = (I \ni h \mapsto \mathfrak{v}_h \in \mathbb{R})$, $A = \mathcal{A}_I$, $(H_s)_{s\in\mathbb{R}} = (\mathcal{H}_{I,s})_{s\in\mathbb{R}}$, $T = T$, $a = a$, $b = b$, $p = 8p$, $s = 0$, $B = (U \ni u \mapsto P_I B(u) \in P_I(H))$, $F = (P_I(H) \ni x \mapsto P_I F(x) \in P_I(H))$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = (\Omega \ni \omega \mapsto P_I \xi(\omega) \in P_I(H))$, $O = O^I$, $X = X^I$ for $I \in (\mathcal{P}_0(\mathbb{H}) \backslash \{\emptyset\})$ in the notation of Lemma 4.1) imply that

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \|X^I_t\|_{\mathcal{L}^{8p}(\mathbb{P}; H)} < \infty. \tag{127}$$

Combining the assumption that $\xi \in \mathcal{L}^{4p}(\mathbb{P}|_{\mathbb{F}_0}; H_\iota)$, (123), and (126) with, e.g., [47, Lemma 3.4] (with $H = H$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $\beta = 1/2 + \beta$, $\gamma = \gamma$, $\xi = (\Omega \ni \omega \mapsto P_I(\xi(\omega)) \in H_{1/2+\beta})$, $F = (H_\gamma \ni x \mapsto P_I F(x) \in H)$, $\kappa = ([0,T] \ni t \mapsto t \in [0,T])$, $Z = ([0,T] \times \Omega \ni (t,\omega) \mapsto X^I_t(\omega) \in H_\gamma)$, $O = ([0,T] \times \Omega \ni (t,\omega) \mapsto O^I_t(\omega) \in H_{1/2+\beta})$, $Y = ([0,T] \times \Omega \ni (t,\omega) \mapsto X^I_t(\omega) \in H)$, $p = p$, $\rho = \eta_1$, $\eta = \eta_2$, $\iota = \iota$, $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$ for $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of [47, Lemma 3.4]) therefore establishes (125). The proof of Corollary 4.2 is thus completed. ☐

4.2 Strong error estimates for space-time discrete truncated exponential Euler-type approximations

In this subsection we study numerical approximations for a class of semilinear SPDEs with additive noise and establish in Proposition 4.5 below strong convergence rates for truncated exponential Euler-type approximation processes $(\mathbf{X}_t^{\theta,I})_{t\in[0,T]}$, $I \in \mathcal{P}_0(\mathbb{H})$, $\theta \in \varpi_T$, (see (143) in Proposition 4.5 below) under (i) the assumption that the truncated exponential Euler-type approximations satisfy suitable exponential moment bounds and (ii) suitable approximatibility assumptions on the stochastic convolution process. Our proof of Proposition 4.5 employs Proposition 3.6 and Corollary 4.2 above as well as the elementary truncation error estimate in Lemma 4.3 below.

Lemma 4.3. *Assume Setting 1.3, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space, let $\varsigma \in [0,\infty)$, $p \in [1,\infty)$, $\alpha, c, h \in (0,\infty)$, $Y \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V))$, $r \in \mathcal{M}(\mathcal{B}(V), \mathcal{B}([0,\infty)))$, $P \in L(H)$, $F \in \mathcal{M}(\mathcal{B}(V), \mathcal{B}(H))$, $D \in \mathcal{B}(V)$ satisfy $\{v \in V : r(v) \leq ch^{-\varsigma}\} \subseteq D$. Then it holds that*

$$\begin{aligned}\|\mathbb{1}_D(Y)\, PF(Y) - F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)} &\leq c^{-\alpha} h^{\alpha\varsigma} \|r(Y)\|^{\alpha}_{\mathcal{L}^{2p\alpha}(\mathbb{P};\mathbb{R})} \|PF(Y)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\ &\quad + \|(P - \mathrm{Id}_H)F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)}.\end{aligned} \tag{128}$$

Proof of Lemma 4.3. Observe that the triangle inequality and Hölder's inequality prove that

$$\begin{aligned}&\|\mathbb{1}_D(Y)\, PF(Y) - F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)} \\ &\leq \|(\mathbb{1}_D(Y) - 1)PF(Y)\|_{\mathcal{L}^p(\mathbb{P};H)} + \|PF(Y) - F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)} \\ &\leq \|\mathbb{1}_D(Y) - 1\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \|PF(Y)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} + \|(P - \mathrm{Id}_H)F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)}.\end{aligned} \tag{129}$$

Moreover, note that Markov's inequality shows that

$$\begin{aligned}\|\mathbb{1}_D(Y) - 1\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} &= \|\mathbb{1}_{V\setminus D}(Y)\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \leq \|\mathbb{1}_{\{r(Y) > ch^{-\varsigma}\}}\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \\ &= [\mathbb{P}(|r(Y)|^{2p\alpha} > (ch^{-\varsigma})^{2p\alpha})]^{1/(2p)} \leq (ch^{-\varsigma})^{-\alpha} (\mathbb{E}[|r(Y)|^{2p\alpha}])^{1/(2p)}.\end{aligned} \tag{130}$$

This and (129) imply that

$$\begin{aligned}\|\mathbb{1}_D(Y)\, PF(Y) - F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)} &\leq c^{-\alpha} h^{\alpha\varsigma} (\mathbb{E}[|r(Y)|^{2p\alpha}])^{1/(2p)} \|PF(Y)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\ &\quad + \|(P - \mathrm{Id}_H)F(Y)\|_{\mathcal{L}^p(\mathbb{P};H)}.\end{aligned} \tag{131}$$

The proof of Lemma 4.3 is thus completed. ∎

Lemma 4.4. *Assume Setting 1.3, let $C, c, \gamma \in [0,\infty)$, $\delta, \kappa \in [0,\gamma]$, $F \in \mathcal{C}(H_\gamma, H)$, let $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ satisfy for every $I \in \mathcal{P}(\mathbb{H})$, $v \in H$ that $P_I(v) = \sum_{h\in I} \langle h, v\rangle_H h$, and assume for every $I \in \mathcal{P}_0(\mathbb{H})$, $u, v \in P_I(H)$ that $\|P_I F(u) - P_I F(v)\|_H \leq C\|u - v\|_{H_\delta}(1 + \|u\|^c_{H_\kappa} + \|v\|^c_{H_\kappa})$. Then it holds for every $u, v \in H_\gamma$ that*

$$\|F(u) - F(v)\|_H \leq C\|u - v\|_{H_\delta}(1 + \|u\|^c_{H_\kappa} + \|v\|^c_{H_\kappa}). \tag{132}$$

Proof of Lemma 4.4. Throughout this proof let $I_n \subseteq \mathbb{H}$, $n \in \mathbb{N}$, be sets which satisfy for every $n \in \mathbb{N}$ that $I_n \subseteq I_{n+1}$ and $\cup_{m\in\mathbb{N}} I_m = \mathbb{H}$. Note that the triangle inequality implies that for every $m, n \in \mathbb{N}$, $u, v \in H_\gamma$ it holds that

$$\begin{aligned}&\|F(u) - F(v)\|_V \leq \|F(u) - P_{I_m}F(u)\|_H + \|P_{I_m}F(u) - P_{I_m}F(P_{I_n}u)\|_H \\ &+ \|P_{I_m}F(P_{I_n}u) - P_{I_m}F(P_{I_n}v)\|_H + \|P_{I_m}F(P_{I_n}v) - P_{I_m}F(v)\|_H + \|P_{I_m}F(v) - F(v)\|_H.\end{aligned} \tag{133}$$

Next observe that for every $v \in H$ it holds that

$$\limsup_{n\to\infty} \|v - P_{I_n} v\|_H = 0. \tag{134}$$

This ensures that for every $u, v \in H_\gamma$ it holds that

$$\limsup_{m\to\infty} \big(\|F(u) - P_{I_m} F(u)\|_H + \|P_{I_m} F(v) - F(v)\|_H\big) = 0. \tag{135}$$

In addition, observe that for every $u \in H_\gamma$ it holds that

$$\limsup_{n\to\infty} \|u - P_{I_n} u\|_{H_\gamma} = 0. \tag{136}$$

The assumption that $F \in \mathcal{C}(H_\gamma, H)$ hence implies that for every $m \in \mathbb{N}$, $u, v \in H_\gamma$ it holds that

$$\begin{aligned} &\limsup_{n\to\infty} \big(\|P_{I_m} F(u) - P_{I_m} F(P_{I_n} u)\|_H + \|P_{I_m} F(P_{I_n} v) - P_{I_m} F(v)\|_H\big) \\ &\leq \limsup_{n\to\infty} \big(\|F(u) - F(P_{I_n} u)\|_H + \|F(P_{I_n} v) - F(v)\|_H\big) = 0. \end{aligned} \tag{137}$$

Moreover, note that the fact that $\forall\, n \in \mathbb{N},\ u, v \in P_{I_n}(H)\colon \|P_{I_n} F(u) - P_{I_n} F(v)\|_H \leq C\|u - v\|_{H_\delta}(1 + \|u\|_{H_\kappa}^c + \|v\|_{H_\kappa}^c)$ and the fact that $\forall\, m \in \mathbb{N},\ n \in ([m,\infty) \cap \mathbb{N}),\ u \in H\colon \|P_{I_m} u\|_H = \|P_{I_m} P_{I_n} u\|_H \leq \|P_{I_n} u\|_H$ show that for every $m \in \mathbb{N}$, $n \in ([m,\infty) \cap \mathbb{N})$, $u, v \in H_\gamma$ it holds that

$$\begin{aligned} \|P_{I_m} F(P_{I_n} u) - P_{I_m} F(P_{I_n} v)\|_H &\leq \|P_{I_n} F(P_{I_n} u) - P_{I_n} F(P_{I_n} v)\|_H \\ &\leq C\|P_{I_n} u - P_{I_n} v\|_{H_\delta}(1 + \|P_{I_n} u\|_{H_\kappa}^c + \|P_{I_n} v\|_{H_\kappa}^c). \end{aligned} \tag{138}$$

The fact that $\delta, \kappa \in [0, \gamma]$ and (136) therefore prove that for every $m \in \mathbb{N}$, $u, v \in H_\gamma$ it holds that

$$\limsup_{n\to\infty} \|P_{I_m} F(P_{I_n} u) - P_{I_m} F(P_{I_n} v)\|_H \leq C\|u - v\|_{H_\delta}(1 + \|u\|_{H_\kappa}^c + \|v\|_{H_\kappa}^c). \tag{139}$$

Combining (133) and (137) hence implies that for every $m \in \mathbb{N}$, $u, v \in H_\gamma$ it holds that

$$\begin{aligned} \|F(u) - F(v)\|_H \leq{}& \|F(u) - P_{I_m} F(u)\|_H + C\|u - v\|_{H_\delta}(1 + \|u\|_{H_\kappa}^c + \|v\|_{H_\kappa}^c) \\ &+ \|P_{I_m} F(v) - F(v)\|_H. \end{aligned} \tag{140}$$

This and (135) establish (132). The proof of Lemma 4.4 is thus completed. ☐

Proposition 4.5. *Assume Setting 1.3, let* $T, \mathsf{v}, \varsigma, \alpha \in (0,\infty)$, $a, \iota, \rho \in [0,\infty)$, $C, c, p \in [1,\infty)$, $\beta \in [0, 1/2)$, $\gamma \in [2\beta, 1/2 + \beta)$, $\delta, \kappa \in [0,\gamma]$, $\eta_1 \in [0, 1/2 + \beta)$, $\eta_2 \in [\eta_1, 1/2 + \beta)$, $\alpha_1 \in [0, 1 - \eta_1)$, $\alpha_2 \in [0, 1 - \eta_2)$, $B \in \mathrm{HS}(U, H_\beta)$, $\varepsilon \in [0, (\rho/p)\exp(-2(a + \rho\|B\|_{\mathrm{HS}(U,H)}^2)T)]$, $F \in \mathcal{C}^1(H_\gamma, H)$, $r \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}([0,\infty)))$, $(D_h^I)_{h\in(0,T], I\in\mathcal{P}_0(\mathbb{H})} \subseteq \mathcal{B}(H_\gamma)$, *let* $\Phi\colon H \to [0,\infty)$ *be a function, let* $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ *satisfy for every* $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ *that* $P_I(x) = \sum_{h\in I}\langle h, x\rangle_H h$, *assume for every* $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ *that* $\{v \in P_I(H)\colon r(v) \leq \mathsf{v} h^{-\varsigma}\} \subseteq D_h^I$ *and* $(P_I(H) \ni v \mapsto \Phi(v) \in [0,\infty)) \in \mathcal{C}(P_I(H), [0,\infty))$, *assume for every* $I \in \mathcal{P}_0(\mathbb{H})$, $x, y \in P_I(H)$ *that* $\langle x, F(x)\rangle_H \leq a(1 + \|x\|_H^2)$, $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|_{H_{1/2}}^2 + C)\|y\|_H^2 + \|y\|_{H_{1/2}}^2$, $\|P_I(F(x) - F(y))\|_H \leq C\|x - y\|_{H_\delta}(1 + \|x\|_{H_\kappa}^c + \|y\|_{H_\kappa}^c)$, $\langle x, Ax + F(x+y)\rangle_H \leq \Phi(y)(1 + \|x\|_H^2)$, *and*

$$\left[\sup_{v\in H_{\max\{\gamma,\eta_2\}}} \tfrac{\|F(v)\|_H}{1+\|v\|_{H_{\eta_2}}^2}\right] + \left[\sup_{v\in H_{\max\{\gamma,\eta_1\}}} \tfrac{\|F(v)\|_{H_{-\alpha_2}}}{1+\|v\|_{H_{\eta_1}}^2}\right] + \left[\sup_{v\in H_\gamma} \tfrac{\|F(v)\|_{H_{-\alpha_1}}}{1+\|v\|_H^2}\right] < \infty, \tag{141}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$, *let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $\xi \in \mathcal{L}^{4p\max\{c,2\}}(\mathbb{P}|_{\mathbb{F}_0}; H_{\max\{\gamma,\eta_2\}})$ *satisfy* $\mathbb{E}[\|\xi\|_H^{16p}] < \infty$, *let* $X\colon [0,T]\times\Omega\to H_\gamma$ *and* $O\colon[0,T]\times\Omega\to H_\gamma$ *be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths which satisfy for every* $t\in[0,T]$ *that* $[O_t]_{\mathbb{P},\mathcal{B}(H_\gamma)}=\int_0^t e^{(t-s)A}B\,dW_s$ *and*

$$\mathbb{P}\bigg(X_t=e^{tA}\xi+\int_0^t e^{(t-s)A}F(X_s)\,ds+O_t\bigg)=1, \tag{142}$$

let $\mathbf{X}^{\theta,I}\colon[0,T]\times\Omega\to P_I(H)$*,* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$*, and* $\mathbf{O}^{\theta,I}\colon[0,T]\times\Omega\to P_I(H)$*,* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$*, be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths which satisfy for every* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$*,* $t\in[0,T]$ *that*

$$\mathbb{P}\Bigg(\mathbf{X}_t^{\theta,I}=e^{tA}P_I\xi+\int_0^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_{\llcorner s\lrcorner_\theta})\,e^{(t-\llcorner s\lrcorner_\theta)A}P_IF(\mathbf{X}^{\theta,I}_{\llcorner s\lrcorner_\theta})\,ds+\mathbf{O}_t^{\theta,I}\Bigg)=1, \tag{143}$$

and assume for every $\theta\in\varpi_T$*,* $I,\mathcal{I}\in\mathcal{P}_0(\mathbb{H})$ *with* $I\subseteq\mathcal{I}$ *that*

$$\sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s^{\theta,I}-\mathbf{O}^{\theta,I}_{\llcorner s\lrcorner_\theta}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}\leq C[|\theta|_T]^\alpha, \tag{144}$$

$$\sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s^{\theta,I}-P_{\mathcal{I}}O_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_{\max\{\kappa,\delta\}})}\leq C(\|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)}+[|\theta|_T]^\alpha), \tag{145}$$

$$\sup\nolimits_{J,K\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{\vartheta\in\varpi_T}\int_0^T\mathbb{E}\big[\exp\big(\rho\|\mathbf{X}_s^{\vartheta,K}-\mathbf{O}_s^{\vartheta,K}+P_JO_s+e^{sA}P_{J\setminus K}\xi\|_H^2\big)\big]\,ds<\infty, \tag{146}$$

$$\sup\nolimits_{J\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{\vartheta\in\varpi_T}\sup\nolimits_{s\in[0,T]}\big[\|P_JF(\mathbf{X}_s^{\vartheta,J})\|_{\mathcal{L}^{4p}(\mathbb{P};H_{\gamma-\delta})}+\|P_JF(\mathbf{X}_s^{\vartheta,J})\|_{\mathcal{L}^{2p}(\mathbb{P};H_\iota)}\big]<\infty, \tag{147}$$

$$\textit{and}\quad \sup\nolimits_{J\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{\vartheta\in\varpi_T}\sup\nolimits_{s\in[0,T]}\big[\|\mathbf{X}_s^{\vartheta,J}\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}+\|r(\mathbf{X}_s^{\vartheta,J})\|_{\mathcal{L}^{4p\alpha/\varsigma}(\mathbb{P};\mathbb{R})}\big]<\infty. \tag{148}$$

Then there exists $\mathfrak{c}\in\mathbb{R}$ *such that for every* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$ *it holds that*

$$\sup\nolimits_{t\in[0,T]}\|X_t-\mathbf{X}_t^{\theta,I}\|_{\mathcal{L}^p(\mathbb{P};H)}\leq\mathfrak{c}\big(\|P_{\mathbb{H}\setminus I}(-A)^{-\min\{\gamma-\delta,\iota\}}\|_{L(H)}+[|\theta|_T]^{\min\{\gamma-\delta,\alpha\}}\big). \tag{149}$$

Proof of Proposition 4.5. Throughout this proof let $O^I\colon[0,T]\times\Omega\to P_I(H)$, $I\in\mathcal{P}_0(\mathbb{H})$, be the $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes which satisfy for every $I\in\mathcal{P}_0(\mathbb{H})$, $t\in[0,T]$ that $O_t^I=P_IO_t$, let $\mathcal{A}_I\colon P_I(H)\to P_I(H)$, $I\in\mathcal{P}_0(\mathbb{H})$, be the functions which satisfy for every $I\in\mathcal{P}_0(\mathbb{H})$, $v\in P_I(H)$ that $\mathcal{A}_Iv=Av$, for every $I\in\mathcal{P}_0(\mathbb{H})$ let $(\mathcal{H}_{I,s},\langle\cdot,\cdot\rangle_{\mathcal{H}_{I,s}},\|\cdot\|_{\mathcal{H}_{I,s}})$, $s\in\mathbb{R}$, be a family of interpolation spaces associated to $-\mathcal{A}_I$, and let $I_m\in(\mathcal{P}_0(\mathbb{H})\backslash\{\emptyset\})$, $m\in\mathbb{N}$, be sets which satisfy $\cup_{n\in\mathbb{N}}(\cap_{m\in\{n+1,n+2,\ldots\}}I_m)=\mathbb{H}$. Note that the fact that for every $I\in\mathcal{P}_0(\mathbb{H})$, $x\in P_I(H)$ it holds that

$$\langle x,P_IF(x)\rangle_H\leq a(1+\|x\|_H^2), \tag{150}$$

the fact that for every $I\in\mathcal{P}_0(\mathbb{H})$, $x,y\in P_I(H)$ it holds that $\langle (P_IF)'(x)y,y\rangle_H\leq(\varepsilon\|x\|^2_{H_{1/2}}+C)\|y\|_H^2+\|y\|^2_{H_{1/2}}$, $\langle x,Ax+P_IF(x+y)\rangle_H\leq\Phi(y)(1+\|x\|_H^2)$, and

$$\|P_I(F(x)-F(y))\|_H\leq C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\kappa}+\|y\|^c_{H_\kappa}), \tag{151}$$

Proposition 3.6 (with $H=P_{I_n}(H)$, $\mathbb{H}=P_{I_n}(\mathbb{H})$, $\mathfrak{v}=(I_n\ni h\mapsto\mathfrak{v}_h\in\mathbb{R})$, $A=\mathcal{A}_{I_n}$, $(H_s)_{s\in\mathbb{R}}=(\mathcal{H}_{I_n,s})_{s\in\mathbb{R}}$, $T=T$, $\theta=\theta$, $a=a$, $b=a$, $\mathbf{C}=C$, $\rho=\rho$, $C=C$, $c=c$, $p=p$, $\gamma=\gamma$, $\delta=\delta$, $\kappa=\kappa$, $B=(U\ni u\mapsto P_{I_n}B(u)\in P_{I_n}(H))$, $\varepsilon=\varepsilon$, $F=(P_{I_n}(H)\ni x\mapsto P_{I_n}F(x)\in P_{I_n}(H))$, $\mathbf{F}=(P_{I_n}(H)\ni x\mapsto\mathbb{1}_{D^I_{|\theta|_T}}(x)P_IF(x)\in P_{I_n}(H))$, $\Phi=(P_{I_n}(H)\ni x\mapsto\Phi(x)\in[0,\infty))$, $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]}=(\mathbb{F}_t)_{t\in[0,T]}$, $\xi=(\Omega\ni\omega\mapsto P_{I_n}\xi(\omega)\in P_{I_n}(H))$, $(W_t)_{t\in[0,T]}=(W_t)_{t\in[0,T]}$, $O=O^{I_n}$, $\mathbf{X}=([0,T]\times\Omega\ni(t,\omega)\mapsto\mathbf{X}_t^{\theta,I}(\omega)\in P_{I_n}(H))$, $\mathbf{O}=([0,T]\times\Omega\ni(t,\omega)\mapsto\mathbf{O}_t^{\theta,I}(\omega)-e^{tA}P_{I_n\setminus I}\xi(\omega)\in P_{I_n}(H))$, $X=\mathcal{X}^n$ for $\theta\in\varpi_T$, $I\in\mathcal{P}(I_n)$, $n\in\mathbb{N}$ in the notation of Proposition 3.6), and the triangle inequality prove that

(a) it holds that there exist $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes $\mathcal{X}^n\colon [0,T]\times\Omega \to P_{I_n}(H)$, $n\in\mathbb{N}$, with continuous sample paths which satisfy for every $n\in\mathbb{N}$, $t\in[0,T]$ that

$$\mathcal{X}_t^n = e^{tA}P_{I_n}\xi + \int_0^t e^{(t-u)A}P_{I_n}F(\mathcal{X}_u^n)\,du + O_t^{I_n} \tag{152}$$

and

(b) it holds for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $n\in\mathbb{N}$, $t\in[0,T]$ with $I\subseteq I_n$ that

$$\begin{aligned}
&\|\mathbf{X}_t^{\theta,I}-\mathcal{X}_t^n\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s^{\theta,I}-O_s^{I_n}\|_{\mathcal{L}^p(\mathbb{P};H)} + \|P_{I_n\setminus I}\xi\|_{\mathcal{L}^p(\mathbb{P};H)}\\
&+\tfrac{C[\max\{T,1\}]^2}{1-\gamma}\exp\!\big(\big(C+\rho(2a+\|B\|^2_{\mathrm{HS}(U,H)})\big)T\big)\\
&\cdot\left[\int_0^T \mathbb{E}\big[e^{\rho\|\mathbf{X}_s^{\theta,I}-\mathbf{O}_s^{\theta,I}+O_s^{I_n}+e^{sA}P_{I_n\setminus I}\xi\|_H^2}\big]\,ds\right]\Big\{[|\theta|_T]^{\gamma-\delta}\sup\nolimits_{s\in[0,T]}\|P_IF(\mathbf{X}_s^{\theta,I})\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})}\\
&+\sup\nolimits_{s\in[0,T]}\|\mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}_s^{\theta,I})P_IF(\mathbf{X}_s^{\theta,I})-P_{I_n}F(\mathbf{X}_s^{\theta,I})\|_{\mathcal{L}^{2p}(\mathbb{P};H)}\\
&+\Big(2[|\theta|_T]^{\gamma-\delta}\sup\nolimits_{s\in[0,T]}\|P_IF(\mathbf{X}_s^{\theta,I})\|_{\mathcal{L}^{4p}(\mathbb{P};H)}+\sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s^{\theta,I}-\mathbf{O}_{\llcorner s\lrcorner_\theta}^{\theta,I}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}\\
&+[|\theta|_T]^{\gamma-\delta}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)}+\sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s^{\theta,I}-O_s^{I_n}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}+\|P_{I_n\setminus I}\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)}\Big)\\
&\cdot\big[1+2\sup\nolimits_{s\in[0,T]}\|\mathbf{X}_s^{\theta,I}\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}+\sup\nolimits_{s\in[0,T]}\|\mathbf{O}_s^{\theta,I}-O_s^{I_n}\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}+\|P_{I_n\setminus I}\xi\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}\big]^c\Big\}.
\end{aligned} \tag{153}$$

Moreover, observe that the triangle inequality implies that for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $n\in\mathbb{N}$, $t\in[0,T]$ it holds that

$$\|\mathbf{X}_t^{\theta,I}-X_t\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \|\mathbf{X}_t^{\theta,I}-\mathcal{X}_t^n\|_{\mathcal{L}^p(\mathbb{P};H)} + \|\mathcal{X}_t^n-X_t\|_{\mathcal{L}^p(\mathbb{P};H)}. \tag{154}$$

Next note that (141), (150), the fact that $\xi\in\mathcal{L}^{8p}(\mathbb{P}|_{\mathbb{F}_0};H_{\max\{\gamma,\eta_2\}})$, the fact that $\mathbb{E}[\|\xi\|_H^{16p}]<\infty$, and Corollary 4.2 (with $T=T$, $a=a$, $b=a$, $p=2p$, $\beta=\beta$, $\gamma=\gamma$, $\eta_1=\eta_1$, $\eta_2=\eta_2$, $\iota=\max\{\gamma,\eta_2\}$, $\alpha_1=\alpha_1$, $\alpha_2=\alpha_2$, $B=B$, $F=F$, $P_I=P_I$, $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_s)_{s\in[0,T]}=(\mathbb{F}_s)_{s\in[0,T]}$, $(W_s)_{s\in[0,T]}=(W_s)_{s\in[0,T]}$, $\xi=\xi$, $X_t^{I_n}=\mathcal{X}_t^n$, $O_t^I=O_t^I$ for $t\in[0,T]$, $n\in\mathbb{N}$, $I\in\mathcal{P}_0(\mathbb{H})$ in the notation of Corollary 4.2) demonstrate that

$$\sup\nolimits_{n\in\mathbb{N}}\sup\nolimits_{t\in[0,T]}\|\mathcal{X}_t^n\|_{\mathcal{L}^{2p}(\mathbb{P};H_\gamma)}<\infty. \tag{155}$$

In addition, observe that (151), Lemma 4.4 (with $C=C$, $c=c$, $\gamma=\gamma$, $\delta=\delta$, $\kappa=\kappa$, $F=F$, $P_I=P_I$ for $I\in\mathcal{P}(\mathbb{H})$ in the notation of Lemma 4.4), and the fact that $\gamma\geq\max\{2\beta,\kappa,\delta\}$ show that for every $R\in(0,\infty)$, $x,y\in H_\gamma$ with $\max\{\|x\|_{H_\gamma},\|y\|_{H_\gamma}\}\leq R$ it holds that

$$\begin{aligned}
&\|F(x)-F(y)\|_{H_{2\beta-\gamma}} \leq \|(-A)^{2\beta-\gamma}\|_{L(H)}\|F(x)-F(y)\|_H\\
&\leq C\|(-A)^{2\beta-\gamma}\|_{L(H)}\|x-y\|_{H_\delta}(1+2(\|(-A)^{\kappa-\gamma}\|_{L(H)}R)^c)\\
&\leq C\|(-A)^{2\beta-\gamma}\|_{L(H)}\|(-A)^{\delta-\gamma}\|_{L(H)}\|x-y\|_{H_\gamma}(1+2(\|(-A)^{\kappa-\gamma}\|_{L(H)}R)^c)<\infty.
\end{aligned} \tag{156}$$

Combining this, (152), (155), and the fact that $2\beta-\gamma\leq 0$ with, e.g., [50, Corollary 6.5] (with $H=H$, $U=U$, $\mathbb{H}=\mathbb{H}$, $\lambda=\mathfrak{v}$, $A=A$, $\gamma=\gamma$, $T=T$, $p=2p$, $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $(\mathcal{F}_t)_{t\in[0,T]}=(\mathbb{F}_t)_{t\in[0,T]}$ $\xi=\xi$, $(W_t)_{t\in[0,T]}=(W_t)_{t\in[0,T]}$, $\eta=2(\gamma-\beta)$, $F=(H_\gamma\ni x\mapsto F(x)\in$

$H_{2\beta-\gamma})$, $B = (H_\gamma \ni x \mapsto B \in \mathrm{HS}(U, H_\beta))$, $I_n = I_n$, $X^n = \mathcal{X}^n$, $X^0 = X$, $q = p$, $K = C\|(-A)^{2\beta-\gamma}\|_{L(H)}\|(-A)^{\delta-\gamma}\|_{L(H)}(1 + 2(\|(-A)^{\kappa-\gamma}\|_{L(H)}R)^c)$ for $n \in \mathbb{N}$, $R \in (0,\infty)$ in the notation of [50, Corollary 6.5]) ensures that

$$\limsup_{n\to\infty} \big(\sup_{t\in[0,T]} \|\mathcal{X}^n_t - X_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \big) = 0. \tag{157}$$

In addition, note that the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that $\{v \in P_I(H) : r(v) \leq \mathbf{v}h^{-\varsigma}\} \subseteq D^I_h$ and Lemma 4.3 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $V = P_I(H)$, $\varsigma = \varsigma$, $p = 2p$, $\alpha = \frac{\alpha}{\varsigma}$, $c = \mathbf{v}$, $h = |\theta|_T$, $Y = (\Omega \ni \omega \mapsto \mathbf{X}^{\theta,I}_t(\omega) \in P_I(H))$, $r = (P_I(H) \ni x \mapsto r(x) \in [0,\infty))$, $P = P_I$, $F = (P_I(H) \ni x \mapsto P_{I_n}F(x) \in H)$, $D = D^I_{|\theta|_T}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$, $t \in [0,T]$ in the notation of Lemma 4.3) prove that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$, $t \in [0,T]$ with $I \subseteq I_n$ it holds that

$$\begin{aligned}
&\|\mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_t)\, P_I F(\mathbf{X}^{\theta,I}_t) - P_{I_n}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&= \|\mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_t)\, P_I(P_{I_n}F(\mathbf{X}^{\theta,I}_t)) - P_{I_n}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&\leq |\mathbf{v}|^{-\alpha/\varsigma}[|\theta|_T]^\alpha \|r(\mathbf{X}^{\theta,I}_t)\|^{\alpha/\varsigma}_{\mathcal{L}^{4p\alpha/\varsigma}(\mathbb{P};\mathbb{R})}\|P_I P_{I_n}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} \\
&\quad + \|(P_I - \mathrm{Id}_H)P_{I_n}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \\
&= |\mathbf{v}|^{-\alpha/\varsigma}[|\theta|_T]^\alpha \|r(\mathbf{X}^{\theta,I}_t)\|^{\alpha/\varsigma}_{\mathcal{L}^{4p\alpha/\varsigma}(\mathbb{P};\mathbb{R})}\|P_I F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} + \|P_{I_n\setminus I}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H)}.
\end{aligned} \tag{158}$$

Moreover, note that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$ with $I \subseteq I_n$ it holds that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|P_{I_n\setminus I}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \leq \|P_{I_n\setminus I}(-A)^{-\iota}\|_{L(H)} \sup_{t\in[0,T]} \|P_{I_n\setminus I}F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H_\iota)} \\
&\leq \|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)} \sup_{J\in\mathcal{P}_0(\mathbb{H})} \sup_{t\in[0,T]} \|P_J F(\mathbf{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H_\iota)}.
\end{aligned} \tag{159}$$

In addition, observe that for every $I \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$ with $I \subseteq I_n$ it holds that

$$\|P_{I_n\setminus I}\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \leq \|P_{\mathbb{H}\setminus I}(-A)^{\delta-\gamma}\|_{L(H)}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)}. \tag{160}$$

Next note that (145) ensures that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$ with $I \subseteq I_n$ it holds that

$$\begin{aligned}
&\sup_{s\in[0,T]} \|\mathbf{O}^{\theta,I}_s - O^{I_n}_s\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq C \max\{\|(-A)^{-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}(\|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)} + [|\theta|_T]^\alpha),
\end{aligned} \tag{161}$$

$$\begin{aligned}
&\sup_{s\in[0,T]} \|\mathbf{O}^{\theta,I}_s - O^{I_n}_s\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \\
&\leq C \max\{\|(-A)^{\delta-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}\big(\|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)} + [|\theta|_T]^\alpha\big),
\end{aligned} \tag{162}$$

and

$$\begin{aligned}
&\sup_{s\in[0,T]} \|\mathbf{O}^{\theta,I}_s - O^{I_n}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} \\
&\leq C \max\{\|(-A)^{\kappa-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}(\|(-A)^{-\iota}\|_{L(H)} + [\max\{T,1\}]^\alpha).
\end{aligned} \tag{163}$$

Combining (144), item (b), and (158)–(160) hence implies that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\begin{aligned}
&\limsup_{n\to\infty} \big(\sup_{t\in[0,T]} \|\mathbf{X}^{\theta,I}_t - \mathcal{X}^n_t\|_{\mathcal{L}^p(\mathbb{P};H)} \big) \\
&\leq C \max\{\|(-A)^{-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}\big(\|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)} + [|\theta|_T]^\alpha\big) + \|P_{\mathbb{H}\setminus I}\xi\|_{\mathcal{L}^p(\mathbb{P};H)}
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{C[\max\{T,1\}]^2}{1-\gamma} \exp\!\big(\big(C + \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})\big)T\big) \\
&\cdot \bigg[\sup\nolimits_{n\in\mathbb{N}} \int_0^T \mathbb{E}\big[e^{\rho\|\mathbf{X}^{\theta,I}_s - \mathbf{O}^{\theta,I}_s + O^{I_n}_s + e^{sA}P_{I_n\setminus I}\xi\|^2_H}\big]\, ds\bigg]\Big\{[|\theta|_T]^{\gamma-\delta} \sup\nolimits_{s\in[0,T]} \|P_I F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})} \\
&+ |\boldsymbol{\nu}|^{-\alpha/\varsigma} \sup\nolimits_{t\in[0,T]} \|r(\mathbf{X}^{\theta,I}_t)\|^{\alpha/\varsigma}_{\mathcal{L}^{4p\alpha/\varsigma}(\mathbb{P};\mathbb{R})} \sup\nolimits_{s\in[0,T]} \|P_I F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} [|\theta|_T]^{\alpha} \\
&+ \|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)} \sup\nolimits_{J\in\mathcal{P}_0(\mathbb{H})} \sup\nolimits_{s\in[0,T]} \|P_J F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_\iota)} \\
&+ \Big(2[|\theta|_T]^{\gamma-\delta} \sup\nolimits_{s\in[0,T]} \|P_I F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)} + C[|\theta|_T]^{\alpha} + [|\theta|_T]^{\gamma-\delta}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)} \\
&+ C\max\{\|(-A)^{\delta-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}(\|P_{\mathbb{H}\setminus I}(-A)^{-\iota}\|_{L(H)} + [|\theta|_T]^{\alpha}) \\
&+ \|P_{\mathbb{H}\setminus I}(-A)^{\delta-\gamma}\|_{L(H)}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)}\Big)\big[1 + 2\sup\nolimits_{s\in[0,T]} \|\mathbf{X}^{\theta,I}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} \\
&+ C\max\{\|(-A)^{\kappa-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}\big(\|(-A)^{-\iota}\|_{L(H)} + [\max\{T,1\}]^{\alpha}\big) + \|\xi\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}\big]^{c}\Big\}.
\end{aligned} \tag{164}
$$

This proves that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$
\begin{aligned}
&\limsup\nolimits_{n\to\infty} \big(\sup\nolimits_{t\in[0,T]} \|\mathbf{X}^{\theta,I}_t - \mathcal{X}^n_t\|_{\mathcal{L}^p(\mathbb{P};H)}\big) \\
&\leq C\max\{\|(-A)^{-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}\|(-A)^{-\iota+\min\{\gamma-\delta,\iota\}}\|_{L(H)}\|P_{\mathbb{H}\setminus I}(-A)^{-\min\{\gamma-\delta,\iota\}}\|_{L(H)} \\
&\quad + C\max\{\|(-A)^{-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}[|\theta|_T]^{\alpha} \\
&\quad + \|P_{\mathbb{H}\setminus I}(-A)^{-\min\{\gamma-\delta,\iota\}}\|_{L(H)}\|(-A)^{\min\{\gamma-\delta,\iota\}+\delta-\gamma}\|_{L(H)}\|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{\gamma-\delta})} \\
&\quad + \tfrac{C[\max\{T,1\}]^2}{1-\gamma} \exp\!\big(\big(C + \rho(2a + \|B\|^2_{\mathrm{HS}(U,H)})\big)T\big)\big[1 + 2\sup\nolimits_{s\in[0,T]} \|\mathbf{X}^{\theta,I}_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)} \\
&\quad + C\max\{\|(-A)^{\kappa-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}\big(\|(-A)^{-\iota}\|_{L(H)} + [\max\{T,1\}]^{\alpha}\big) + \|\xi\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}\big]^{c} \\
&\quad \cdot \bigg[\sup\nolimits_{n\in\mathbb{N}} \int_0^T \mathbb{E}\big[e^{\rho\|\mathbf{X}^{\theta,I}_s - \mathbf{O}^{\theta,I}_s + O^{I_n}_s + e^{sA}P_{I_n\setminus I}\xi\|^2_H}\big]\, ds\bigg]\Big\{\sup\nolimits_{s\in[0,T]} \|P_I F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})}[|\theta|_T]^{\gamma-\delta} \\
&\quad + |\boldsymbol{\nu}|^{-\alpha/\varsigma} \sup\nolimits_{t\in[0,T]} \|r(\mathbf{X}^{\theta,I}_t)\|^{\alpha/\varsigma}_{\mathcal{L}^{4p\alpha/\varsigma}(\mathbb{P};\mathbb{R})} \sup\nolimits_{s\in[0,T]} \|P_I F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)}[|\theta|_T]^{\alpha} \\
&\quad + \sup\nolimits_{J\in\mathcal{P}_0(\mathbb{H})} \sup\nolimits_{s\in[0,T]} \|P_J F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{2p}(\mathbb{P};H_\iota)} \\
&\quad \cdot \|(-A)^{-\iota+\min\{\gamma-\delta,\iota\}}\|_{L(H)}\|P_{\mathbb{H}\setminus I}(-A)^{-\min\{\gamma-\delta,\iota\}}\|_{L(H)} \\
&\quad + 2\sup\nolimits_{s\in[0,T]} \|P_I F(\mathbf{X}^{\theta,I}_s)\|_{\mathcal{L}^{4p}(\mathbb{P};H)}[|\theta|_T]^{\gamma-\delta} + C[|\theta|_T]^{\alpha} + \|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)}[|\theta|_T]^{\gamma-\delta} \\
&\quad + C\max\{\|(-A)^{\delta-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}\|(-A)^{-\iota+\min\{\gamma-\delta,\iota\}}\|_{L(H)}\|P_{\mathbb{H}\setminus I}(-A)^{-\min\{\gamma-\delta,\iota\}}\|_{L(H)} \\
&\quad + C\max\{\|(-A)^{\delta-\max\{\kappa,\delta\}}\|_{L(H)}, 1\}[|\theta|_T]^{\alpha} \\
&\quad + \|(-A)^{\delta-\gamma+\min\{\gamma-\delta,\iota\}}\|_{L(H)}\|P_{\mathbb{H}\setminus I}(-A)^{-\min\{\gamma-\delta,\iota\}}\|_{L(H)}\|\xi\|_{\mathcal{L}^{4p}(\mathbb{P};H_\gamma)}\Big\}.
\end{aligned} \tag{165}
$$

Moreover, note that (154) and (157) ensure that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$
\|\mathbf{X}^{\theta,I}_t - X_t\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \limsup\nolimits_{n\to\infty} \big(\sup\nolimits_{t\in[0,T]} \|\mathbf{X}^{\theta,I}_t - \mathcal{X}^n_t\|_{\mathcal{L}^p(\mathbb{P};H)}\big). \tag{166}
$$

Combining the fact that $\xi \in \mathcal{L}^{4pc}(\mathbb{P}; H_\gamma)$, (146)–(148), and (165) therefore establishes (149). The proof of Proposition 4.5 is thus completed. ☐

5 Strong convergence rates for space-time discrete tamed-truncated exponential Euler-type approximations without assuming finite exponential moments

Setting 5.1. *Assume Setting 1.3, let* $T \in (0,\infty)$, $a, b, \nu \in [0,\infty)$, $\varsigma \in (0, 1/18)$, $\epsilon \in (0,1]$, $\beta \in [0, 1/2)$, $\gamma \in [0, 1/2 + \beta)$, $B \in \mathrm{HS}(U, H_\beta)$, $F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H))$, $(D_h^I)_{h\in(0,T], I\in\mathcal{P}_0(\mathbb{H})} \subseteq \mathcal{B}(H_\gamma)$, *let* $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ *satisfy for every* $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ *that* $P_I(x) = \sum_{h\in I}\langle h, x\rangle_H h$, *assume for every* $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$, $x \in D_h^I$ *that* $D_h^I \subseteq \{v \in P_I(H) \colon \|B\|_{\mathrm{HS}(U,H)} + \epsilon\|v\|_H^2 \leq \nu h^{-\varsigma}\}$, $\max\{\|P_I F(x)\|_H, \|B\|_{\mathrm{HS}(U,H)}\} \leq \nu h^{-\varsigma}$, *and* $\langle x, F(x)\rangle_H \leq a + b\|x\|_H^2$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$, *let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(H_\gamma))$ *satisfy* $\mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$, *and let* $\mathbf{X}^{\theta,I} \colon [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, *be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths which satisfy for every* $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ *that* $\mathbf{X}_0^{\theta,I} = P_I\xi$ *and*

$$\begin{aligned}[\mathbf{X}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} &= \left[e^{(t-\llcorner t\lrcorner_\theta)A}\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta} + \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I F(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})(t-\llcorner t\lrcorner_\theta)\right]_{\mathbb{P},\mathcal{B}(P_I(H))} \\ &\quad + \frac{\int_{\llcorner t\lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I B\, dW_s}{1 + \|\int_{\llcorner t\lrcorner_\theta}^t P_I B\, dW_s\|_H^2}. \end{aligned} \tag{167}$$

5.1 Finite exponential moments for tamed-truncated Euler-type approximations

In this subsection we establish in Corollary 5.5 below uniformly bounded exponential moments for the space-time discrete tamed-truncated exponential Euler-type approximation processes $(\mathbf{X}_t^{\theta,I})_{t\in[0,T]}$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, (see (167) above). Our proof of Corollary 5.5 uses the exponential moment estimate in [49, Corollary 3.4]. We then employ Corollary 5.5 to establish in Corollary 5.6 below for every $p \in (0,\infty)$ uniformly bounded $\mathcal{L}^p$-moments for the considered approximation processes. Moreover, combining Corollary 5.6 with [45, Corollary 3.1] and [47, Lemma 3.4] allows us to establish in Corollary 5.7 below for every $p \in (0,\infty)$ strengthened uniformly bounded $\mathcal{L}^p$-moments for the considered approximation processes.

Lemma 5.2. *Assume Setting 1.2, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be a non-zero separable* $\mathbb{R}$*-Hilbert space, let* $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ *be a separable* $\mathbb{R}$*-Hilbert space, let* $\mathfrak{N} = [1, \dim(H)] \cap \mathbb{N}$, *let* $(h_n)_{n\in\mathfrak{N}} \subseteq H$ *be an orthonormal basis of* H, *let* $\mathbb{H} = \{h_n \colon n \in \mathfrak{N}\}$, *let* $B \colon U \to H$ *be a linear function, let* $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ *satisfy for every* $I \in \mathcal{P}(\mathbb{H})$, $v \in H$ *that* $P_I(v) = \sum_{h\in I}\langle h, v\rangle_H h$, *for every* $n \in \mathfrak{N}$ *let* $\mathbb{U}_n \subseteq [\ker(P_{\{h_1,h_2,\ldots,h_n\}}B)]^\perp$ *be an orthonormal basis of* $[\ker(P_{\{h_1,h_2,\ldots,h_n\}}B)]^\perp$, *assume for every* $n \in (\mathfrak{N}\backslash\{\sup(\mathfrak{N})\})$ *that* $\mathbb{U}_n \subseteq \mathbb{U}_{n+1}$, *and let* $(\mathfrak{P}_I)_{I\in\mathcal{P}(\cup_{n\in\mathfrak{N}}\mathbb{U}_n)} \subseteq L(U)$ *satisfy for every* $I \in \mathcal{P}(\cup_{n\in\mathfrak{N}}\mathbb{U}_n)$, $u \in U$ *that* $\mathfrak{P}_I u = \sum_{\mathfrak{u}\in I}\langle \mathfrak{u}, u\rangle_U \mathfrak{u}$. *Then there exists a function* $\Gamma \colon \mathcal{P}_0(\mathbb{H}) \to \mathfrak{N}$ *which satisfies that*

(i) it holds for every $I \in \mathcal{P}_0(\mathbb{H})$ *that* $[\ker(P_I B)]^\perp \subseteq \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}(U)$,

(ii) it holds for every $n \in \mathfrak{N}$ *that* $\Gamma(\{h_1, h_2, \ldots, h_n\}) \leq n$, *and*

(iii) it holds for every $I \in \mathcal{P}_0(\mathbb{H})$ *that* $P_I B = P_I B \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}$.

Proof of Lemma 5.2. Throughout this proof let $\Gamma\colon \mathcal{P}_0(\mathbb{H}) \to \mathbb{N} \cup \{\infty\}$ be the function which satisfies for every $I \in \mathcal{P}_0(\mathbb{H})$ that

$$\Gamma(I) = \inf(\{n \in \mathfrak{N}\colon [\ker(P_I B)]^\perp \subseteq \mathfrak{P}_{\mathbb{U}_n}(U)\} \cup \{\infty\}). \tag{168}$$

Observe that for every $n \in \mathfrak{N}$ it holds that

$$[\ker(P_{\{h_1,h_2,\dots,h_n\}}B)]^\perp = \mathfrak{P}_{\mathbb{U}_n}(U). \tag{169}$$

Moreover, note that for every $I \in \mathcal{P}_0(\mathbb{H})$ there exists $n \in \mathfrak{N}$ such that $I \subseteq \{h_1, h_2, \dots, h_n\}$. This ensures that for every $I \in \mathcal{P}_0(\mathbb{H})$ there exists $n \in \mathfrak{N}$ such that

$$\ker(P_{\{h_1,h_2,\dots,h_n\}}B) \subseteq \ker(P_I B). \tag{170}$$

This and (169) imply that for every $I \in \mathcal{P}_0(\mathbb{H})$ there exists $n \in \mathfrak{N}$ such that

$$[\ker(P_I B)]^\perp \subseteq \mathfrak{P}_{\mathbb{U}_n}(U). \tag{171}$$

Therefore, we obtain that for every $I \in \mathcal{P}_0(\mathbb{H})$ it holds that $\Gamma(I) \in \mathfrak{N}$. Combining this, (168), and (169) establishes items (i) and (ii). Moreover, note that item (i) implies that for every $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$P_I B = P_I B \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}. \tag{172}$$

This implies item (iii). The proof of Lemma 5.2 is thus completed. ☐

Corollary 5.3. *Assume Setting 1.2, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a non-zero separable $\mathbb{R}$-Hilbert space, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathfrak{N} = [1, \dim(H)] \cap \mathbb{N}$, let $(h_N)_{N\in\mathfrak{N}} \subseteq H$ be an orthonormal basis of H, let $T \in (0,\infty)$, $B \in \mathrm{HS}(U,H)$, let $\mathbb{B} \in L(H,U)$ satisfy for every $v \in H$, $u \in U$ that $\langle Bu, v\rangle_H = \langle u, \mathbb{B}v\rangle_U$, let $(P_N)_{N\in\mathfrak{N}} \subseteq L(H)$ satisfy for every $N \in \mathfrak{N}$, $v \in H$ that $P_N(v) = \sum_{n=1}^N \langle h_n, v\rangle_H h_n$, for every $N \in \mathfrak{N}$ let $\mathbb{U}_N \subseteq [\ker(P_N B)]^\perp$ be an orthonormal basis of $[\ker(P_N B)]^\perp$, assume for every $N \in (\mathfrak{N}\backslash\{\sup(\mathfrak{N})\})$ that $\mathbb{U}_N \subseteq \mathbb{U}_{N+1}$, let $(\mathfrak{P}_N)_{N\in\mathfrak{N}} \subseteq L(U)$ satisfy for every $N \in \mathfrak{N}$, $u \in U$ that $\mathfrak{P}_N u = \sum_{\mathfrak{u}\in\mathbb{U}_N} \langle \mathfrak{u}, u\rangle_U \mathfrak{u}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical Wiener process, and for every $N \in \mathfrak{N}$ let $W^N\colon [0,T] \times \Omega \to P_N(H)$ be a stochastic process with continuous sample paths which satisfies for every $t \in [0,T]$ that $[W_t^N]_{\mathbb{P},\mathcal{B}(P_N(H))} = \int_0^t P_N B \, dW_s$. Then*

(i) it holds for every $N \in \mathfrak{N}$ that $P_N B \mathfrak{P}_N = P_N B$,

(ii) it holds for every $N \in \mathfrak{N}$, $t \in [0,T]$ that $[W_t^N]_{\mathbb{P},\mathcal{B}(P_N(H))} = \int_0^t P_N B \mathfrak{P}_N \, dW_s$, and

(iii) it holds for every $N \in \mathfrak{N}$ that $(W_t^N)_{t\in[0,T]}$ is a $(P_N B \mathbb{B}|_{P_N(H)})$-Wiener process.

Proof of Corollary 5.3. Throughout this proof let $(\mathbb{F}_t)_{t\in[0,T]}$ be the normal filtration generated by $(W_t)_{t\in[0,T]}$. Observe that Lemma 5.2 (with $H = H$, $U = U$, $\mathfrak{N} = \mathfrak{N}$, $h_n = h_n$, $B = B$, $P_{\{h_1,h_2,\dots,h_n\}} = P_n$, $\mathbb{U}_n = \mathbb{U}_n$, $\mathfrak{P}_{\mathbb{U}_n} = \mathfrak{P}_n$ for $n \in \mathfrak{N}$ in the notation of Lemma 5.2) ensures that for every $N \in \mathfrak{N}$, $t \in [0,T]$ it holds that

$$P_N B = P_N B \mathfrak{P}_N. \tag{173}$$

This establishes items (i) and (ii). Combining (173) and, e.g, [49, Lemma 3.2] (with $H = P_N(H)$, $U = U$, $T = T$, $Q = \mathrm{Id}_U$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $R = (U \ni u \mapsto P_N B(u) \in P_N(H))$, $(\tilde{W}_t)_{t\in[0,T]} = (W_t^N)_{t\in[0,T]}$ for $N \in \mathfrak{N}$ in the notation of [49, Lemma 3.2]) establishes item (iii). The proof of Corollary 5.3 is thus completed. ☐

Lemma 5.4. *Assume Setting 1.3, let $T \in (0,\infty)$, $\theta \in \varpi_T$, $\beta \in [0, 1/2)$, $\gamma \in [0, 1/2 + \beta)$, $B \in \mathrm{HS}(U, H_\beta)$, $F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H))$, $D \in \mathcal{B}(H_\gamma)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, let $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(H_\gamma))$, $I \in \mathcal{P}_0(\mathbb{H})$, $P \in L(H)$ satisfy for every $x \in H$ that $P(x) = \sum_{h\in I} \langle h, x\rangle_H h$, let $\mathcal{W}\colon [0,T] \times \Omega \to P(H)$ be a stochastic process with continuous sample paths which satisfies for every $t \in [0,T]$ that $[\mathcal{W}_t]_{\mathbb{P},\mathcal{B}(P(H))} = \int_0^t PB\, dW_s$, and let $\mathbf{X}\colon [0,T]\times\Omega \to P(H)$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process which satisfies for every $t \in [0,T]$ that $\mathbf{X}_0 = P\xi$ and*

$$[\mathbf{X}_t]_{\mathbb{P},\mathcal{B}(P(H))} = \big[e^{(t-\llcorner t \lrcorner_\theta)A}\mathbf{X}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_D(\mathbf{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PF(\mathbf{X}_{\llcorner t \lrcorner_\theta})(t - \llcorner t \lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_D(\mathbf{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PB\, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t PB\, dW_s\|_H^2}. \tag{174}$$

Then there exists an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process $\mathcal{X}\colon [0,T]\times\Omega \to P(H)$ with continuous sample paths which satisfies that

(i) it holds that $\mathcal{X}_0 = P\xi$,

(ii) it holds for every $t \in [0,T]$ that

$$\mathcal{X}_t = e^{(t-\llcorner t \lrcorner_\theta)A}\mathcal{X}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A}\left[PF(\mathcal{X}_{\llcorner t \lrcorner_\theta})(t - \llcorner t \lrcorner_\theta) + \frac{(\mathcal{W}_t - \mathcal{W}_{\llcorner t \lrcorner_\theta})}{1 + \|\mathcal{W}_t - \mathcal{W}_{\llcorner t \lrcorner_\theta}\|_H^2}\right], \tag{175}$$

(iii) it holds for every $t \in [0,T]$ that

$$[\mathcal{X}_t]_{\mathbb{P},\mathcal{B}(P(H))} = \big[e^{(t-\llcorner t \lrcorner_\theta)A}\mathcal{X}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PF(\mathcal{X}_{\llcorner t \lrcorner_\theta})(t - \llcorner t \lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PB\, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t PB\, dW_s\|_H^2}, \tag{176}$$

and

(iv) it holds for every $t \in [0,T]$ that $\mathbb{P}(\mathcal{X}_t = \mathbf{X}_t) = 1$.

Proof of Lemma 5.4. Throughout this proof let $\mathcal{X}\colon [0,T]\times\Omega \to P(H)$ be the stochastic process which satisfies for every $t \in [0,T]$ that $\mathcal{X}_0 = P\xi$ and

$$\mathcal{X}_t = e^{(t-\llcorner t \lrcorner_\theta)A}\mathcal{X}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A}\left[PF(\mathcal{X}_{\llcorner t \lrcorner_\theta})(t - \llcorner t \lrcorner_\theta) + \frac{(\mathcal{W}_t - \mathcal{W}_{\llcorner t \lrcorner_\theta})}{1 + \|\mathcal{W}_t - \mathcal{W}_{\llcorner t \lrcorner_\theta}\|_H^2}\right]. \tag{177}$$

Note that the fact that for every $s \in [0,T]$ it holds that $[s,T]\times H \ni (t,x) \mapsto e^{(t-s)A}x \in P(H)$ is continuous, the fact that $\mathcal{W}$ has continuous sample paths, and (177) ensure that $\mathcal{X}$ has continuous sample paths. Moreover, observe that the assumption that $(\mathbb{F}_t)_{t\in[0,T]}$ is a normal filtration and the assumption that for every $t \in [0,T]$ it holds that $[\mathcal{W}_t]_{\mathbb{P},\mathcal{B}(P(H))} = \int_0^t PB\, dW_s$ show that $\mathcal{W}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted. Combining this, (177), the fact that $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(P(H)))$, and the assumption that $(\mathbb{F}_t)_{t\in[0,T]}$ is a normal filtration therefore shows that $\mathcal{X}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted. This, (177), and the fact that $\mathcal{X}$ has continuous sample paths establish items (i) and (ii). Next note that the fact that $\mathcal{X}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted ensures that for every $t \in [0,T]$ it holds that

$$\left[\mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A}\frac{(\mathcal{W}_t - \mathcal{W}_{\llcorner t \lrcorner_\theta})}{1 + \|\mathcal{W}_t - \mathcal{W}_{\llcorner t \lrcorner_\theta}\|_H^2}\right]_{\mathbb{P},\mathcal{B}(P(H))} = \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PB\, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t PB\, dW_s\|_H^2}. \tag{178}$$

Combining this and (177) demonstrates that for every $t \in [0,T]$ it holds that

$$[\mathcal{X}_t]_{\mathbb{P},\mathcal{B}(P(H))} = \big[e^{(t-\llcorner t \lrcorner_\theta)A}\mathcal{X}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PF(\mathcal{X}_{\llcorner t \lrcorner_\theta})(t-\llcorner t \lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_D(\mathcal{X}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} PB\, dW_s}{1 + \| \int_{\llcorner t \lrcorner_\theta}^t PB\, dW_s\|_H^2}. \tag{179}$$

This establishes item (iii). Moreover, observe that (174), (179), and item (i) assure that for every $t \in [0,T]$ it holds that

$$\mathbb{P}(\mathcal{X}_t = X_t) = 1. \tag{180}$$

This establishes item (iv). The proof of Lemma 5.4 is thus completed. □

Corollary 5.5. *Assume Setting 5.1. Then*

$$\sup\nolimits_{\theta\in\varpi_T} \sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})} \sup\nolimits_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\frac{\epsilon}{e^{2(b+\|B\|^2_{\mathrm{HS}(U,H)})T}}\|\mathbf{X}_t^{\theta,I}\|_H^2\bigg)\bigg] < \infty. \tag{181}$$

Proof of Corollary 5.5. Throughout this proof let $c = 2\max\{\epsilon a, \epsilon\|B\|_{\mathrm{HS}(U,H)}, \epsilon, \mathrm{v}, 1\}$, let $\mathfrak{N} = [1,\dim(H)] \cap \mathbb{N}$, let $h_n \in H$, $n \in \mathfrak{N}$, satisfy for every $m,n \in \mathbb{N}$ that $h_m \neq h_n$ and $\mathbb{H} = \{h_N : N \in \mathfrak{N}\}$, let $\mathbb{U}_1 \subseteq [\ker(P_{\{h_1\}}B)]^\perp$ be an orthonormal basis of $[\ker(P_{\{h_1\}}B)]^\perp$, for every $n \in ([2,\infty)\cap\mathfrak{N})$ let $\mathbb{U}_n \subseteq [\ker(P_{\{h_1,h_2,\dots,h_n\}}B)]^\perp$ be an orthonormal basis of $[\ker(P_{\{h_1,h_2,\dots,h_n\}}B)]^\perp$ with $\mathbb{U}_{n-1} \subseteq \mathbb{U}_n$, let $\mathcal{U} \subseteq U$ be an orthonormal basis of U with $\mathcal{U} \supseteq \cup_{n\in\mathbb{N}}\mathbb{U}_n$, let $\mathfrak{P}_I \in L(U)$, $I \in \mathcal{P}(\mathcal{U})$, satisfy for every $I \in \mathcal{P}(\mathcal{U})$, $u \in U$ that $\mathfrak{P}_I u = \sum_{\mathfrak{u}\in I}\langle \mathfrak{u}, u\rangle_U \mathfrak{u}$, and let $\mathfrak{X}^{\theta,I,J} : [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathcal{U})$, be $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes which satisfy for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathcal{U})$, $t \in [0,T]$ that $\mathfrak{X}_0^{\theta,I,J} = P_I\xi$ and

$$[\mathfrak{X}_t^{\theta,I,J}]_{\mathbb{P},\mathcal{B}(P_I(H))} = \big[e^{(t-\llcorner t \lrcorner_\theta)A}\mathfrak{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_{D^I_{|\theta|_T}}(\mathfrak{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} P_I F(\mathfrak{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta})(t-\llcorner t \lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P_I(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathfrak{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} P_I B \mathfrak{P}_J\, dW_s}{1 + \| \int_{\llcorner t \lrcorner_\theta}^t P_I B\mathfrak{P}_J\, dW_s\|_H^2}. \tag{182}$$

Observe that Lemma 5.4 (with $T = T$, $\theta = \theta$, $\beta = \beta$, $\gamma = \gamma$, $B = B\mathfrak{P}_J$, $F = F$, $D = D^I_{|\theta|_T}$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $I = I$, $P = P_I$, $\mathbf{X}^{\theta,I} = \mathfrak{X}^{\theta,I,J}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathcal{U})$ in the notation of Lemma 5.4) ensures that there exist $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes $\mathcal{X}^{\theta,I,J} : [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathcal{U})$, with continuous sample paths which satisfy for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathcal{U})$, $t \in [0,T]$ that $\mathcal{X}_0^{\theta,I,J} = P_I\xi$ and

$$[\mathcal{X}_t^{\theta,I,J}]_{\mathbb{P},\mathcal{B}(P_I(H))} = \big[e^{(t-\llcorner t \lrcorner_\theta)A}\mathcal{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} P_I F(\mathcal{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta})(t-\llcorner t \lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P_I(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I,J}_{\llcorner t \lrcorner_\theta})\, e^{(t-\llcorner t \lrcorner_\theta)A} P_I B \mathfrak{P}_J\, dW_s}{1 + \| \int_{\llcorner t \lrcorner_\theta}^t P_I B\mathfrak{P}_J\, dW_s\|_H^2}. \tag{183}$$

Next note that Lemma 5.2 (with $H = H$, $U = U$, $\mathfrak{N} = \mathfrak{N}$, $h_n = h_n$, $\mathbb{H} = \mathbb{H}$, $B = (U \ni u \mapsto B(u) \in H)$, $P_I = P_I$, $\mathbb{U}_n = \mathbb{U}_n$, $\mathfrak{P}_J = \mathfrak{P}_J$ for $I \in \mathcal{P}(\mathbb{H})$, $n \in \mathfrak{N}$, $J \in \mathcal{P}(\cup_{n\in\mathfrak{N}}\mathbb{U}_n)$ in the notation of Lemma 5.2) assures that there exists a function $\Gamma : \mathcal{P}_0(\mathbb{H}) \to \mathfrak{N}$ which satisfies for every $I \in \mathcal{P}_0(\mathbb{H})$ that

$$P_I B = P_I B \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}. \tag{184}$$

Combining (167) and (184) demonstrates that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$[\mathbf{X}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} = \big[e^{(t-\llcorner t \lrcorner_\theta)A}\mathbf{X}_{\llcorner t \lrcorner_\theta}^{\theta,I} + \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}_{\llcorner t \lrcorner_\theta}^{\theta,I})\, e^{(t-\llcorner t \lrcorner_\theta)A} P_I F(\mathbf{X}_{\llcorner t \lrcorner_\theta}^{\theta,I})(t-\llcorner t \lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P_I(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}_{\llcorner t \lrcorner_\theta}^{\theta,I})\, e^{(t-\llcorner t \lrcorner_\theta)A} P_I B \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}\, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t P_I B \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}\, dW_s\|_H^2}. \tag{185}$$

This and (183) ensure that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$\mathbf{X}^{\theta,I} = \mathcal{X}^{\theta,I,\mathbb{U}_{\Gamma(I)}}. \tag{186}$$

In addition, note that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$D_h^I \subseteq \{v \in P_I(H) \colon \|B\|_{\mathrm{HS}(U,H)} + \epsilon\|v\|_H^2 \leq \mathfrak{v} h^{-\varsigma}\} \subseteq \{v \in H \colon \|B\|_{\mathrm{HS}(U,H)} + \epsilon\|v\|_H^2 \leq c h^{-\varsigma}\}. \tag{187}$$

Furthermore, observe that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$, $x \in D_h^I$ it holds that

$$\max\{\|P_I F(x)\|_H, \|P_I B \mathfrak{P}_{\mathbb{U}_{\Gamma(I)}}\|_{\mathrm{HS}(U,H)}\} \leq \max\{\|P_I F(x)\|_H, \|B\|_{\mathrm{HS}(U,H)}\} \leq \mathfrak{v} h^{-\varsigma} \leq c h^{-\varsigma}. \tag{188}$$

Moreover, note that the fact that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that $D_h^I \subseteq P_I(H)$ demonstrates that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$, $x \in D_h^I$ it holds that

$$\langle x, P_I F(x)\rangle_H = \langle x, F(x)\rangle_H \leq a + b\|x\|_H^2. \tag{189}$$

Combining this and (185)–(188) with [49, Corollary 3.4] (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathbb{U} = \mathcal{U}$, $\lambda = \mathfrak{v}$, $A = A$, $T = T$, $\gamma = \gamma$, $\delta = \varsigma$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $F = F$, $B = (H_\gamma \ni x \mapsto B \in \mathrm{HS}(U,H))$, $D_h^I = D_h^I$, $P_I = P_I$, $\hat{P}_J = \mathfrak{P}_J$, $\vartheta = \|B\|^2_{\mathrm{HS}(U,H)}$, $b_1 = a$, $b_2 = b$, $\varepsilon = \epsilon$, $\varsigma = \varsigma$, $c = c$, $Y^{\theta,I,J} = \mathcal{X}^{\theta,I,J}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathcal{U})$, $h \in (0,T]$ in the notation of [49, Corollary 3.4]) shows that

$$\sup\nolimits_{\theta\in\varpi_T} \sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})} \sup\nolimits_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{\epsilon\|\mathbf{X}_t^{\theta,I}\|_H^2}{e^{2(b+\epsilon\|B\|^2_{\mathrm{HS}(U,H)})t}}\right)\right] < \infty. \tag{190}$$

In addition, note that the fact that $\epsilon \leq 1$ assures that for every $t \in [0,T]$ it holds that

$$\frac{\epsilon}{e^{2(b+\epsilon\|B\|^2_{\mathrm{HS}(U,H)})t}} \geq \frac{\epsilon}{e^{2(b+\epsilon\|B\|^2_{\mathrm{HS}(U,H)})T}} \geq \frac{\epsilon}{e^{2(b+\|B\|^2_{\mathrm{HS}(U,H)})T}}. \tag{191}$$

This and (190) establish (181). The proof of Corollary 5.5 is thus completed. □

Corollary 5.6. *Assume Setting 5.1 and let $p \in (0,\infty)$. Then it holds that*

$$\sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})} \sup\nolimits_{\theta\in\varpi_T} \sup\nolimits_{t\in[0,T]} \|\mathbf{X}_t^{\theta,I}\|_{\mathcal{L}^p(\mathbb{P};H)} < \infty. \tag{192}$$

Proof of Corollary 5.6. Throughout this proof let $N \in ([\frac{p}{2}, \frac{p}{2}+1) \cap \mathbb{N})$. Observe that Corollary 5.5 shows that there exists $M \in [0,\infty)$ such that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$, $\varepsilon \in (0, \epsilon\exp(-2(b + \|B\|^2_{\mathrm{HS}(U,H)})T)]$ it holds that

$$\mathbb{E}\big[\exp\big(\varepsilon\|\mathbf{X}_t^{\theta,I}\|_H^2\big)\big] \leq M. \tag{193}$$

In addition, note that Young's inequality ensures that for every $x \in (0,\infty)$ it holds that

$$\begin{aligned} x^{p/2} &= x^{(N-1)(N-(p/2))} x^{N((p/2)-N+1)} \leq (N - \tfrac{p}{2}) x^{N-1} + (\tfrac{p}{2} - N + 1) x^N \\ &\leq N x^{N-1} + x^N = (N!) \Big(\tfrac{x^{N-1}}{(N-1)!} + \tfrac{x^N}{N!} \Big) \leq (N!) e^x. \end{aligned} \tag{194}$$

Therefore, we obtain that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$\mathbb{E}\Big[\big| \varepsilon \| \mathbf{X}_t^{\theta,I} \|_H^2 \big|^{p/2} \Big] \leq (N!) \mathbb{E}\big[\exp\big(\varepsilon \| \mathbf{X}_t^{\theta,I} \|_H^2 \big) \big]. \tag{195}$$

This and (193) imply that there exists $M \in [0,\infty)$ such that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$, $\varepsilon \in (0, \epsilon \exp(-2(b + \|B\|_{\mathrm{HS}(U,H)}^2)T)]$ it holds that

$$\Big(\mathbb{E}\Big[\big| \varepsilon \| \mathbf{X}_t^{\theta,I} \|_H^2 \big|^{p/2} \Big] \Big)^{2/p} \leq ((N!)M)^{2/p}. \tag{196}$$

This completes the proof of Corollary 5.6. ◻

Corollary 5.7. *Assume Setting 5.1, let $p \in (0,\infty)$, $\eta_1 \in [0, 1/2 + \beta)$, $\eta_2 \in [\eta_1, 1/2 + \beta)$, $\iota \in [\eta_2, 1/2 + \beta)$, $\alpha_1 \in [0, 1 - \eta_1)$, $\alpha_2 \in [0, 1 - \eta_2)$, and assume that $\mathbb{E}[\|\xi\|_{H_\iota}^{4\max\{p,1\}}] < \infty$ and*

$$\Big[\sup_{v \in H_{\max\{\gamma,\eta_2\}}} \tfrac{\|F(v)\|_H}{1 + \|v\|_{H_{\eta_2}}^2} \Big] + \Big[\sup_{v \in H_{\max\{\gamma,\eta_1\}}} \tfrac{\|F(v)\|_{H_{-\alpha_2}}}{1 + \|v\|_{H_{\eta_1}}^2} \Big] + \Big[\sup_{v \in H_\gamma} \tfrac{\|F(v)\|_{H_{-\alpha_1}}}{1 + \|v\|_H^2} \Big] < \infty. \tag{197}$$

Then it holds that

$$\sup_{\theta \in \varpi_T} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \| \mathbf{X}_t^{\theta,I} \|_{\mathcal{L}^p(\mathbb{P}; H_\iota)} < \infty. \tag{198}$$

Proof of Corollary 5.7. Throughout this proof let $(\mathbb{G}_t)_{t \in [0,T]}$ be the normal filtration generated by $(W_t)_{t \in [0,T]}$, let $\mathbb{U}$ be an orthonormal basis of U, and let $\mathbf{O}^{\theta,I} \colon [0,T] \times \Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, be stochastic processes which satisfy for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ that

$$\mathbf{O}_t^{\theta,I} = \mathbf{X}_t^{\theta,I} - \bigg(e^{tA} P_I \xi + \int_0^t \mathbb{1}_{D_{|\theta|_T}^I}(\mathbf{X}_{\llcorner s \lrcorner_\theta}^{\theta,I}) \, e^{(t - \llcorner s \lrcorner_\theta)A} P_I F(\mathbf{X}_{\llcorner s \lrcorner_\theta}^{\theta,I}) \, ds \bigg). \tag{199}$$

Observe that [45, Corollary 3.1] (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $\beta = \beta$, $T = T$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{G}_t)_{t \in [0,T]})$, $(W_t)_{t \in [0,T]} = (W_t)_{t \in [0,T]}$, $B = B$, $\mathbb{U} = \mathbb{U}$, $P_I = P_I$, $\hat{P}_\mathbb{U} = \mathrm{Id}_U$, $\chi^{\theta,I,\mathbb{U}} = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbb{1}_{D_{|\theta|_T}^I}(\mathbf{X}_t^{\theta,I}(\omega)) \in [0,1])$, $\mathbf{O}^{\theta,I,\mathbb{U}} = \mathbf{O}^{\theta,I}$, $p = \max\{p,1\}$, $\gamma = \iota$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of [45, Corollary 3.1]) shows that

$$\sup_{\theta \in \varpi_T} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \| \mathbf{O}_t^{\theta,I} \|_{\mathcal{L}^{4\max\{p,1\}}(\mathbb{P}; H_\iota)} < \infty. \tag{200}$$

Next note that Corollary 5.6 (with $p = 8\max\{p,1\}$ in the notation of Corollary 5.6) proves that

$$\sup_{\theta \in \varpi_T} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \| \mathbf{X}_t^{\theta,I} \|_{\mathcal{L}^{8\max\{p,1\}}(\mathbb{P}; H)} < \infty. \tag{201}$$

Combining this, (197), and (200) with, e.g., [47, Lemma 3.4] (with $H = H$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $\beta = 1/2 + \beta$, $\gamma = \gamma$, $\xi = (\Omega \ni \omega \mapsto P_I(\xi(\omega)) \in H_{1/2+\beta})$, $F = (H_\gamma \ni x \mapsto \mathbb{1}_{D_{|\theta|_T}^I}(x) P_I F(x) \in H)$, $\kappa = ([0,T] \ni t \mapsto \llcorner t \lrcorner_\theta \in [0,T])$, $Z = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbf{X}_{\llcorner t \lrcorner_\theta}^{\theta,I}(\omega) \in H_\gamma)$, $O = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbf{O}_t^{\theta,I}(\omega) \in H_{1/2+\beta})$, $Y = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbf{X}_t^{\theta,I}(\omega) \in H)$, $p = \max\{p,1\}$, $\rho = \eta_1$, $\eta = \eta_2$, $\iota = \iota$, $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of [47, Lemma 3.4]) shows that

$$\sup_{\theta \in \varpi_T} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \| \mathbf{X}_t^{\theta,I} \|_{\mathcal{L}^{\max\{p,1\}}(\mathbb{P}; H_\iota)} < \infty. \tag{202}$$

Hölder's inequality therefore establishes (198). The proof of Corollary 5.7 is thus completed. ◻

5.2 Strong error estimates for tamed-truncated Euler-type approximations

In this subsection we establish the main result of this article in Theorem 5.9 below. To do so, we first prove an elementary exponential moment estimate in Lemma 5.8. Combining Corollaries 5.5–5.7, Lemma 5.8, and [45, Corollaries 3.2–3.4] allows us to apply Proposition 4.5 to derive in Theorem 5.9 strong convergence rates for the numerical approximations $(\mathbf{X}_t^{\theta,I})_{t\in[0,T]}$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, (see (212) below) for a general class of semilinear SPDEs with additive noise and a possibly non-globally monotone nonlinearity. Moreover, in Corollary 5.10 we briefly present and prove a simplified version of Theorem 5.9.

Lemma 5.8. *Assume Setting 1.3, let $T \in (0,\infty)$, $B \in \mathrm{HS}(U,H)$, let $(P_I)_{I\in\mathcal{P}_0(\mathbb{H})} \subseteq L(H)$ satisfy for every $I \in \mathcal{P}_0(\mathbb{H})$, $v \in H$ that $P_I(v) = \sum_{h\in I}\langle h, v\rangle_H h$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical Wiener process. Then it holds for every $t \in [0,T]$ with $2t\|B\|^2_{\mathrm{HS}(U,H)} < 1$ that*

$$\sup_{I\in\mathcal{P}_0(\mathbb{H})} \mathbb{E}\big[e^{\|\int_0^t e^{(t-s)A}P_I B\,dW_s\|_H^2}\big] \leq \tfrac{2}{1-4t^2\|B\|^4_{\mathrm{HS}(U,H)}}. \tag{203}$$

Proof of Lemma 5.8. Throughout this proof let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $(\mathbb{F}_t)_{t\in[0,T]}$ be the normal filtration generated by $(W_t)_{t\in[0,T]}$, and let $O^I\colon [0,T]\times\Omega \to P_I(H)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for every $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ that $[O_t^I]_{\mathbb{P},\mathcal{B}(P_I(H))} = \int_0^t P_I e^{(t-s)A}B\,dW_s$. Observe that for every $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$[O_t^I]_{\mathbb{P},\mathcal{B}(P_I(H))} = \left[\int_0^t AO_s^I\,ds\right]_{\mathbb{P},\mathcal{B}(P_I(H))} + \int_0^t P_I B\,dW_s. \tag{204}$$

Itô's formula therefore shows that for every $p \in [2,\infty)$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$\begin{aligned}
[\|O_t^I\|_H^p]_{\mathbb{P},\mathcal{B}(\mathbb{R})} &= \left[\int_0^t p\|O_s^I\|_H^{p-2}\langle O_s^I, AO_s^I\rangle_H\,ds\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})} + \int_0^t p\|O_s^I\|_H^{p-2}\langle O_s^I, B\,dW_s\rangle_H \\
&+ \left[\tfrac{1}{2}\int_0^t \sum_{\mathbf{u}\in\mathbb{U}} \big[p\|O_s^I\|_H^{p-2}\|B\mathbf{u}\|_H^2 + p(p-2)\mathbb{1}_{\{O_s^I\neq 0\}}\|O_s^I\|_H^{p-4}|\langle O_s^I, B\mathbf{u}\rangle_H|^2\big]\,ds\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})}.
\end{aligned} \tag{205}$$

Moreover, note that the Burkholder-Davis-Gundy-type inequality in Da Prato & Zabczyk [23, Lemma 7.7] proves that for every $p \in [2,\infty)$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\int_0^t \mathbb{E}\big[\|O_s^I\|_H^{2(p-2)}\|(U \ni u \mapsto \langle O_s^I, B(u)\rangle_H \in \mathbb{R})\|^2_{\mathrm{HS}(U,\mathbb{R})}\big]\,ds \\
&\leq \int_0^t \mathbb{E}\big[\|O_s^I\|_H^{2(p-1)}\|B\|^2_{\mathrm{HS}(U,H)}\big]\,ds = \|B\|^2_{\mathrm{HS}(U,H)}\int_0^t \|O_s^I\|^{2(p-1)}_{\mathcal{L}^{2(p-1)}(\mathbb{P};H)}\,ds \\
&\leq \|B\|^2_{\mathrm{HS}(U,H)}\int_0^t [(p-1)(2p-3)]^{(p-1)}\left[\int_0^s \|P_I e^{(s-u)A}B\|^2_{\mathrm{HS}(U,H)}\,du\right]^{(p-1)}ds \\
&\leq \|B\|^2_{\mathrm{HS}(U,H)}\int_0^t [(p-1)(2p-3)]^{(p-1)}\left[\int_0^s \|e^{(s-u)A}\|^2_{L(H)}\|B\|^2_{\mathrm{HS}(U,H)}\,du\right]^{(p-1)}ds \\
&\leq \|B\|^{2p}_{\mathrm{HS}(U,H)}[(p-1)(2p-3)]^{(p-1)}\int_0^t\left[\int_0^s du\right]^{(p-1)}ds < \infty.
\end{aligned} \tag{206}$$

Combining (205), the fact that for every $x \in H_1$ it holds that $\langle x, Ax\rangle_H = -\|x\|^2_{H_{1/2}} \leq 0$, Cauchy-Schwarz's inequality, and Tonelli's theorem therefore implies that for every $p \in [2,\infty)$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\mathbb{E}[\|O^I_t\|^p_H] \leq \tfrac{1}{2}\mathbb{E}\bigg[\int_0^t \sum_{\mathbf{u}\in\mathbb{U}} \big[p\|O^I_s\|^{p-2}_H\|B\mathbf{u}\|^2_H + p(p-2)\mathbb{1}_{\{O^I_s\neq 0\}}\|O^I_s\|^{p-2}_H\|B\mathbf{u}\|^2_H\big]\,ds\bigg]\\
&= \tfrac{1}{2}\|B\|^2_{\mathrm{HS}(U,H)}\mathbb{E}\bigg[\int_0^t \big[p\|O^I_s\|^{p-2}_H + p(p-2)\mathbb{1}_{\{O^I_s\neq 0\}}\|O^I_s\|^{p-2}_H\big]\,ds\bigg]\\
&= \tfrac{1}{2}\|B\|^2_{\mathrm{HS}(U,H)}\int_0^t \mathbb{E}\big[p\|O^I_s\|^{p-2}_H + p(p-2)\|O^I_s\|^{p-2}_H\big]\,ds = \tfrac{p(p-1)\|B\|^2_{\mathrm{HS}(U,H)}}{2}\int_0^t \mathbb{E}\big[\|O^I_s\|^{p-2}_H\big]\,ds.
\end{aligned} \tag{207}$$

This ensures that for every $I \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$, $t_0 \in [0,T]$ it holds that

$$\begin{aligned}
\mathbb{E}[\|O^I_{t_0}\|^{2n}_H] &\leq \tfrac{2n(2n-1)\|B\|^2_{\mathrm{HS}(U,H)}}{2}\int_0^{t_0}\mathbb{E}\big[\|O^I_s\|^{2(n-1)}_H\big]\,ds\\
&\leq \tfrac{(2n)!\|B\|^{2n}_{\mathrm{HS}(U,H)}}{2^n}\int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_n\cdots dt_2\,dt_1 = \tfrac{(2n)!\|B\|^{2n}_{\mathrm{HS}(U,H)}}{2^n n!}t_0^n.
\end{aligned} \tag{208}$$

Moreover, note that for every $x \in [0,\infty)$ it holds that $e^x \leq 2\sum_{n=0}^\infty \frac{x^{2n}}{(2n)!}$ (see, e.g., Hutzenthaler et al. [43, Lemma 2.4]). Combining this, (208), Tonelli's theorem, and the fact that for every $n \in \mathbb{N}$ it holds that $(4n)! \leq 2^{4n}[(2n)!]^2$ implies that for every $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,\infty)$ with $2t\|B\|^2_{\mathrm{HS}(U,H)} < 1$ it holds that

$$\begin{aligned}
\mathbb{E}\Big[e^{\|\int_0^t e^{(t-s)A}P_I B\,dW_s\|^2_H}\Big] &= \mathbb{E}\big[e^{\|O^I_t\|^2_H}\big] \leq 2\,\mathbb{E}\Big[\sum\nolimits_{n=0}^\infty \tfrac{\|O^I_t\|^{4n}_H}{(2n)!}\Big] = 2\sum\nolimits_{n=0}^\infty \tfrac{\mathbb{E}[\|O^I_t\|^{4n}_H]}{(2n)!}\\
&\leq 2\sum\nolimits_{n=0}^\infty \tfrac{(4n)!\|B\|^{4n}_{\mathrm{HS}(U,H)}t^{2n}}{[(2n)!]^2 2^{2n}} \leq 2\sum\nolimits_{n=0}^\infty 2^{2n}\|B\|^{4n}_{\mathrm{HS}(U,H)}t^{2n}\\
&= 2\sum\nolimits_{n=0}^\infty \big(4\|B\|^4_{\mathrm{HS}(U,H)}t^2\big)^n = \tfrac{2}{1-4t^2\|B\|^4_{\mathrm{HS}(U,H)}}.
\end{aligned} \tag{209}$$

This completes the proof of Lemma 5.8. □

Theorem 5.9. *Assume Setting 1.3, let* $T, \mathsf{v} \in (0,\infty)$, $\varsigma \in (0, 1/18)$, $a \in [0,\infty)$, $C, c, p \in [1,\infty)$, $\beta \in [0, 1/2)$, $\gamma \in [2\beta, 1/2+\beta) \cap (0,\infty)$, $\delta \in (\gamma - 1/2, \gamma) \cap [0,\infty)$, $\kappa \in [0,\gamma] \cap [0, 1/2 + \beta - \gamma + \delta)$, $\eta_0 = 0$, $\sigma, \nu, \eta_1 \in [0, 1/2+\beta)$, $\eta_2 \in [\eta_1, 1/2+\beta)$, $\alpha_1 \in [0, 1-\eta_1)$, $\alpha_2 \in [0, 1-\eta_2)$, $\alpha_3 = 0$, $B \in \mathrm{HS}(U, H_\beta)$, $\epsilon \in (0, \exp(-2(a + \|B\|^2_{\mathrm{HS}(U,H)})T)]$, $\varepsilon \in [0, \frac{1}{16p}\exp(-2(a+\|B\|^2_{\mathrm{HS}(U,H)})T)\min\{\epsilon\exp(-2(a+\|B\|^2_{\mathrm{HS}(U,H)})T), 1/(8\max\{\|B\|^2_{\mathrm{HS}(U,H)},1\}\max\{T,1\})^2\})$, $F \in \mathcal{C}^1(H_\gamma, H)$, $r \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}([0,\infty)))$, $(D^I_h)_{h\in(0,T], I\in\mathcal{P}_0(\mathbb{H})} \subseteq \mathcal{B}(H_\gamma)$, *let* $\Phi\colon H \to [0,\infty)$ *be a function, let* $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ *satisfy for every* $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ *that* $P_I(x) = \sum_{h\in I}\langle h, x\rangle_H h$, *assume for every* $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ *that* $D^I_h = \{v \in P_I(H)\colon r(v) \leq \mathsf{v}h^{-\varsigma}\}$ *and* $(P_I(H) \ni v \mapsto \Phi(v) \in [0,\infty)) \in \mathcal{C}(P_I(H), [0,\infty))$, *assume for every* $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$, $x \in D^I_h$ *that* $\max\{\|P_I F(x)\|_H, \|B\|_{\mathrm{HS}(U,H)}\} \leq \mathsf{v}h^{-\varsigma}$, *assume for every* $I \in \mathcal{P}_0(\mathbb{H})$, $x, y \in P_I(H)$ *that* $\|B\|_{\mathrm{HS}(U,H)} + \epsilon\|x\|^2_H \leq r(x) \leq C(1+\|x\|^2_{H_\nu})$, $\langle x, F(x)\rangle_H \leq a(1+\|x\|^2_H)$, $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + C)\|y\|^2_H + \|y\|^2_{H_{1/2}}$, $\|P_I(F(x)-F(y))\|_H \leq C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\kappa}+\|y\|^c_{H_\kappa})$, $\langle x, Ax + F(x+y)\rangle_H \leq \Phi(y)(1+\|x\|^2_H)$, *and*

$$\Big[\sup\nolimits_{J\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{v\in P_J(H)}\tfrac{\|P_J F(v)\|_{H_{\gamma-\delta}}}{1+\|v\|^2_{H_\sigma}}\Big] + \sum_{i=0}^2\Big[\sup\nolimits_{v\in H_{\max\{\gamma,\eta_i\}}}\tfrac{\|F(v)\|_{H_{-\alpha_{i+1}}}}{1+\|v\|^2_{H_{\eta_i}}}\Big] < \infty, \tag{210}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$*, let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $\xi \in \mathcal{L}^{32pc\max\{(\gamma-\delta)/\varsigma,1\}}(\mathbb{P}|_{\mathbb{F}_0}; H_{\max\{\eta_2,\sigma,\nu,\gamma\}})$ *satisfy* $\mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$*, let* $X\colon [0,T]\times\Omega \to H_\gamma$ *be an* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic process with continuous sample paths which satisfies for every* $t\in[0,T]$ *that*

$$[X_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} = \left[e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,ds\right]_{\mathbb{P},\mathcal{B}(H_\gamma)} + \int_0^t e^{(t-s)A}B\,dW_s, \tag{211}$$

and let $\mathbf{X}^{\theta,I}\colon [0,T]\times\Omega \to P_I(H)$*,* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$*, be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes which satisfy for every* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$*,* $t\in[0,T]$ *that* $\mathbf{X}_0^{\theta,I} = P_I\xi$ *and*

$$\begin{aligned}[\mathbf{X}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} &= \big[e^{(t-\llcorner t\lrcorner_\theta)A}\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta} + \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I F(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})(t-\llcorner t\lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P_I(H))} \\ &\quad + \frac{\int_{\llcorner t\lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I B\,dW_s}{1+\|\int_{\llcorner t\lrcorner_\theta}^t P_I B\,dW_s\|_H^2}.\end{aligned} \tag{212}$$

Then there exists $\mathfrak{c}\in\mathbb{R}$ *such that for every* $\theta\in\varpi_T$*,* $I\in\mathcal{P}_0(\mathbb{H})$ *it holds that*

$$\sup\nolimits_{t\in[0,T]}\|X_t - \mathbf{X}_t^{\theta,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \le \mathfrak{c}\big(\|P_{\mathbb{H}\setminus I}(-A)^{\delta-\gamma}\|_{L(H)} + [|\theta|_T]^{\gamma-\delta}\big). \tag{213}$$

Proof of Theorem 5.9. Throughout this proof let $\rho\in(0,\infty)$ satisfy that

$$\varepsilon p e^{2(a+\|B\|^2_{\mathrm{HS}(U,H)})T} \le \rho < \tfrac{1}{16}\min\Big\{\epsilon e^{-2(a+\|B\|^2_{\mathrm{HS}(U,H)})T}, \tfrac{1}{(8\max\{\|B\|^2_{\mathrm{HS}(U,H)},1\}\max\{T,1\})^2}\Big\}. \tag{214}$$

Note that Lemma 5.4 (with $T=T$, $\theta=\theta$, $\beta=\beta$, $\gamma=\gamma$, $B=B$, $F=F$, $D=D^I_{|\theta|_T}$, $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]}=(\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]}=(W_t)_{t\in[0,T]}$, $\xi=\xi$, $I=I$, $P=P_I$, $\mathbf{X}=\mathbf{X}^{\theta,I}$ for $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$ in the notation of Lemma 5.4) proves that there exist $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes $\mathcal{X}^{\theta,I}\colon[0,T]\times\Omega\to P_I(H)$, $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, with continuous sample paths which satisfy for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $t\in[0,T]$ that $\mathbb{P}(\mathcal{X}_t^{\theta,I}=\mathbf{X}_t^{\theta,I})=1$ and

$$\begin{aligned}[\mathcal{X}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} &= \big[e^{(t-\llcorner t\lrcorner_\theta)A}\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta} + \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I F(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})(t-\llcorner t\lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P_I(H))} \\ &\quad + \frac{\int_{\llcorner t\lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I B\,dW_s}{1+\|\int_{\llcorner t\lrcorner_\theta}^t P_I B\,dW_s\|_H^2}.\end{aligned} \tag{215}$$

Next let $\mathbf{O}^{\theta,I}\colon[0,T]\times\Omega\to P_I(H)$, $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, be stochastic processes which satisfy for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $t\in[0,T]$ that

$$\mathbf{O}_t^{\theta,I} = \mathcal{X}_t^{\theta,I} - \left(e^{tA}P_I\xi + \int_0^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_{\llcorner s\lrcorner_\theta})\, e^{(t-\llcorner s\lrcorner_\theta)A}P_I F(\mathcal{X}^{\theta,I}_{\llcorner s\lrcorner_\theta})\,ds\right). \tag{216}$$

We intend to prove Theorem 5.9 through an application of Proposition 4.5 (with $\alpha=\gamma-\delta$, $\iota=\gamma-\delta$, $\mathbf{X}^{\theta,I}=\mathcal{X}^{\theta,I}$, $\mathbf{O}^{\theta,I}=\mathbf{O}^{\theta,I}$ for $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$ in the notation of Proposition 4.5). For this we now verify the hypotheses (144)–(148) in Proposition 4.5. Observe that (215) and (216) imply that for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $t\in[0,T]$ it holds that

$$[\mathbf{O}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} = [e^{(t-\llcorner t\lrcorner_\theta)A}\mathbf{O}^{\theta,I}_{\llcorner t\lrcorner_\theta}]_{\mathbb{P},\mathcal{B}(P_I(H))} + \frac{\int_{\llcorner t\lrcorner_\theta}^t \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\, e^{(t-\llcorner t\lrcorner_\theta)A}P_I B\,dW_s}{1+\|\int_{\llcorner t\lrcorner_\theta}^t P_I B\,dW_s\|_H^2}. \tag{217}$$

This and [45, Corollary 3.2] (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $\beta = \beta$, $T = T$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $B = B$, $P_I = P_I$, $\hat{P}_{\mathbb{U}} = \mathrm{Id}_U$, $\chi^{\theta,I,\mathbb{U}} = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_t(\omega)) \in [0,1])$, $\mathbf{O}^{\theta,I,\mathbb{U}} = \mathbf{O}^{\theta,I}$, $p = 4p$, $\gamma = \delta$, $\rho = \gamma - \delta$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of [45, Corollary 3.2]) show that there exists $\mathfrak{C} \in \mathbb{R}$ which satisfies that for every $\theta \in \varpi_T$ it holds that

$$\sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{s\in[0,T]} \|\mathbf{O}^{\theta,I}_s - \mathbf{O}^{\theta,I}_{\llcorner s \lrcorner_\theta}\|_{\mathcal{L}^{4p}(\mathbb{P};H_\delta)} \leq \mathfrak{C}\, [|\theta|_T]^{\gamma-\delta}. \tag{218}$$

Moreover, note that the fact that $\gamma < 1/2 + \beta$ ensures that there exists an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process $O\colon [0,T] \times \Omega \to H_\gamma$ with continuous sample paths which satisfies for every $t \in [0,T]$ that

$$[O_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} = \int_0^t e^{(t-s)A} B \, dW_s \tag{219}$$

(cf., e.g., [47, Lemma 5.5]). Next let $O^I\colon [0,T] \times \Omega \to P_I(H)$, $I \in \mathcal{P}_0(\mathbb{H})$, be stochastic processes which satisfy for every $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ that

$$O^I_t = P_I O_t. \tag{220}$$

Observe that (220) and Hölder's inequality imply that for every $\theta \in \varpi_T$, $I, J \in \mathcal{P}_0(H)$, $s \in [0,T]$ it holds that

$$\begin{aligned} &\mathbb{E}\big[\exp\big(\rho\|\mathcal{X}^{\theta,J}_s - \mathbf{O}^{\theta,J}_s + O^I_s + e^{sA} P_{I\setminus J}\xi\|_H^2\big)\big] \\ &\leq \mathbb{E}\big[\exp\big(4\rho(\|\mathcal{X}^{\theta,J}_s\|_H^2 + \|\mathbf{O}^{\theta,J}_s\|_H^2 + \|O^I_s\|_H^2 + \|\xi\|_H^2)\big)\big] \\ &\leq \big[\mathbb{E}[\exp(16\rho\|\mathcal{X}^{\theta,J}_s\|_H^2)]\, \mathbb{E}[\exp(16\rho\|\mathbf{O}^{\theta,J}_s\|_H^2)]\, \mathbb{E}[\exp(16\rho\|O^I_s\|_H^2)]\, \mathbb{E}[\exp(16\rho\|\xi\|_H^2)]\big]^{1/4}. \end{aligned} \tag{221}$$

Moreover, note that the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that

$$\|B\|_{\mathrm{HS}(U,H)} + \epsilon\|x\|_H^2 \leq r(x) \leq C(1 + \|x\|_{H_\nu}^2) \tag{222}$$

and the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$D^I_h = \{v \in P_I(H)\colon r(v) \leq \mathbf{v} h^{-\varsigma}\} \tag{223}$$

ensure that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$\{v \in P_I(H)\colon C(1 + \|v\|_{H_\nu}^2) \leq \mathbf{v} h^{-\varsigma}\} \subseteq D^I_h \subseteq \{v \in P_I(H)\colon \|B\|_{\mathrm{HS}(U,H)} + \epsilon\|v\|_H^2 \leq \mathbf{v} h^{-\varsigma}\}. \tag{224}$$

Combining this, the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$, $x \in D^I_h$ it holds that

$$\max\{\|P_I F(x)\|_H, \|B\|_{\mathrm{HS}(U,H)}\} \leq \mathbf{v} h^{-\varsigma}, \tag{225}$$

the assumption that $\mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$, the fact that $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(H_\gamma))$, the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that

$$\langle x, F(x)\rangle_H \leq a(1 + \|x\|_H^2), \tag{226}$$

the fact that $16\rho \leq \epsilon \exp(-2(a + \|B\|_{\mathrm{HS}(U,H)}^2)T)$, (215), and Corollary 5.5 (with $T = T$, $a = a$, $b = a$, $\mathbf{v} = \mathbf{v}$, $\varsigma = \varsigma$, $\epsilon = \epsilon$, $\beta = \beta$, $\gamma = \gamma$, $B = B$, $F = F$, $D^I_h = D^I_h$, $P_I = P_I$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$,

$(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $\mathbf{X}^{\theta,I} = \mathcal{X}^{\theta,I}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ in the notation of Corollary 5.5) proves that

$$\sup_{\theta\in\varpi_T} \sup_{J\in\mathcal{P}_0(H)} \sup_{s\in[0,T]} \mathbb{E}[\exp(16\rho\|\mathcal{X}_s^{\theta,J}\|_H^2)] < \infty. \tag{227}$$

In addition, note that the fact that $16\rho < {}^1\!/\!_{(8\max\{\|B\|^2_{\mathrm{HS}(U,H)},1\}\max\{T,1\})^2}$, (217), and [45, Corollary 3.4] (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $\beta = \beta$, $T = T$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $B = B$, $P_I = P_I$, $\hat{P}_{\mathbb{U}} = \mathrm{Id}_U$, $\chi^{\theta,I,\mathbb{U}} = ([0,T]\times\Omega \ni (t,\omega) \mapsto \mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}_t^{\theta,I}(\omega)) \in [0,1])$, $\mathbf{O}^{\theta,I,\mathbb{U}} = \mathbf{O}^{\theta,I}$, $\varepsilon = 16\rho$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of [45, Corollary 3.4]) assure that

$$\sup_{\theta\in\varpi_T} \sup_{J\in\mathcal{P}_0(H)} \sup_{s\in[0,T]} \mathbb{E}[\exp(16\rho\|\mathbf{O}_s^{\theta,J}\|_H^2)] < \infty. \tag{228}$$

Furthermore, note that Lemma 5.8 (with $T = T$, $B = (U \ni u \mapsto 4\sqrt{\rho}Bu \in H)$, $P_I = P_I$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$ for $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of Lemma 5.8) shows that for every $I \in \mathcal{P}_0(\mathbb{H})$, $s \in [0,T]$ with $32\rho s\|B\|^2_{\mathrm{HS}(U,H)} < 1$ it holds that

$$\mathbb{E}[\exp(16\rho\|O_s^I\|_H^2)] \leq \tfrac{2}{1-1024\rho^2 s^2\|B\|^4_{\mathrm{HS}(U,H)}}. \tag{229}$$

Next observe that the fact that for every $x \in [0,\infty)$ it holds that $x < 2e^x$ implies that $4T\|B\|^2_{\mathrm{HS}(U,H)} < 2e^{4T\|B\|^2_{\mathrm{HS}(U,H)}}$. This shows that $2T\|B\|^2_{\mathrm{HS}(U,H)}e^{-4T\|B\|^2_{\mathrm{HS}(U,H)}} < 1$. Therefore, we obtain that

$$\begin{aligned} 32\rho T\|B\|^2_{\mathrm{HS}(U,H)} &\leq \tfrac{32\epsilon T\|B\|^2_{\mathrm{HS}(U,H)}}{16} e^{-2T(a+\|B\|^2_{\mathrm{HS}(U,H)})} \leq 2T\|B\|^2_{\mathrm{HS}(U,H)} e^{-4T(a+\|B\|^2_{\mathrm{HS}(U,H)})} \\ &\leq 2T\|B\|^2_{\mathrm{HS}(U,H)} e^{-4T\|B\|^2_{\mathrm{HS}(U,H)}} < 1. \end{aligned} \tag{230}$$

This and (229) imply that

$$\sup_{I\in\mathcal{P}_0(H)} \sup_{s\in[0,T]} \mathbb{E}[\exp(16\rho\|O_s^I\|_H^2)] < \infty. \tag{231}$$

Combining this, (221), (227), (228), the assumption that $\mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$, and the fact that $16\rho < \epsilon$ demonstrates that

$$\sup_{I,J\in\mathcal{P}_0(\mathbb{H})} \sup_{\theta\in\varpi_T} \sup_{s\in[0,T]} \mathbb{E}\big[\exp\big(\rho\|\mathcal{X}_s^{\theta,J} - \mathbf{O}_s^{\theta,J} + O_s^I + e^{sA}P_{I\setminus J}\xi\|_H^2\big)\big] < \infty. \tag{232}$$

Next observe that the fact that $\xi \in \mathcal{L}^{32pc\max\{(\gamma-\delta)/\varsigma,1\}}(\mathbb{P}; H_{\max\{\eta_2,\sigma,\nu,\gamma\}})$, the fact that

$$\Big[\sup_{v\in H_{\max\{\gamma,\eta_2\}}} \tfrac{\|F(v)\|_H}{1+\|v\|^2_{H_{\eta_2}}}\Big] + \Big[\sup_{v\in H_{\max\{\gamma,\eta_1\}}} \tfrac{\|F(v)\|_{H_{-\alpha_2}}}{1+\|v\|^2_{H_{\eta_1}}}\Big] + \Big[\sup_{v\in H_\gamma} \tfrac{\|F(v)\|_{H_{-\alpha_1}}}{1+\|v\|^2_H}\Big] < \infty, \tag{233}$$

the assumption that $\mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$, (215), (224)–(226), the fact that $\epsilon \leq \exp(-2(a + \|B\|^2_{\mathrm{HS}(U,H)})T)$, and Corollary 5.7 (with $T = T$, $a = a$, $b = a$, $\boldsymbol{\nu} = \boldsymbol{\nu}$, $\varsigma = \varsigma$, $\epsilon = \epsilon$, $\beta = \beta$, $\gamma = \gamma$, $B = B$, $F = F$, $D_h^I = D_h^I$, $P_I = P_I$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $\mathbf{X}^{\theta,I} = \mathcal{X}^{\theta,I}$, $p = 8pc\max\{{}^{(\gamma-\delta)}\!/\!_{\varsigma}, 1\}$, $\eta_1 = \eta_1$, $\eta_2 = \eta_2$, $\iota = \max\{\eta_2, \sigma, \nu, \gamma\}$, $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$ for $h \in (0,T]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of Corollary 5.7) prove that

$$\sup_{\theta\in\varpi_T} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{t\in[0,T]} \|\mathcal{X}_t^{\theta,I}\|_{\mathcal{L}^{8pc\max\{(\gamma-\delta)/\varsigma,1\}}(\mathbb{P}; H_{\max\{\eta_2,\sigma,\nu,\gamma\}})} < \infty. \tag{234}$$

Combining this and the fact that $\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{v\in P_I(H)}\big({(\|P_IF(v)\|_{H_{\gamma-\delta}})}/{(1+\|v\|^2_{H_\sigma})}\big)<\infty$ shows that

$$\begin{aligned}
&\sup\nolimits_{\theta\in\varpi_T}\sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{t\in[0,T]}\|P_IF(\mathcal{X}^{\theta,I}_t)\|_{\mathcal{L}^{4pc\max\{(\gamma-\delta)/\varsigma,1\}}(\mathbb{P};H_{\gamma-\delta})}\\
&\leq\left[\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{v\in P_I(H)}\tfrac{\|P_IF(v)\|_{H_{\gamma-\delta}}}{1+\|v\|^2_{H_\sigma}}\right]\left[1+\sup_{\theta\in\varpi_T}\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{t\in[0,T]}\|\mathcal{X}^{\theta,I}_t\|^2_{\mathcal{L}^{8pc\max\{(\gamma-\delta)/\varsigma,1\}}(\mathbb{P};H_\sigma)}\right]<\infty.
\end{aligned}\tag{235}$$

This and (234) assure that

$$\sup\nolimits_{\theta\in\varpi_T}\sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{t\in[0,T]}\big[\|P_IF(\mathcal{X}^{\theta,I}_t)\|_{\mathcal{L}^{4p}(\mathbb{P};H_{\gamma-\delta})}+\|P_IF(\mathcal{X}^{\theta,I}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{\gamma-\delta})}\big]<\infty\tag{236}$$

and

$$\sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{\theta\in\varpi_T}\sup\nolimits_{t\in[0,T]}\big[\|\mathcal{X}^{\theta,I}_t\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}+\|\mathcal{X}^{\theta,I}_t\|_{\mathcal{L}^{2\max\{4p(\gamma-\delta)/\varsigma,1\}}(\mathbb{P};H_\nu)}\big]<\infty.\tag{237}$$

In addition, note that (222) and (237) prove that

$$\sup\nolimits_{I\in\mathcal{P}_0(\mathbb{H})}\sup\nolimits_{\theta\in\varpi_T}\sup\nolimits_{t\in[0,T]}\big[\|\mathcal{X}^{\theta,I}_t\|_{\mathcal{L}^{4pc}(\mathbb{P};H_\kappa)}+\|r(\mathcal{X}^{\theta,I}_t)\|_{\mathcal{L}^{4p(\gamma-\delta)/\varsigma}(\mathbb{P};\mathbb{R})}\big]<\infty.\tag{238}$$

Moreover, observe that (224) and Markov's inequality ensure that for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $h\in(0,T]$, $t\in[0,T]$ it holds that

$$\begin{aligned}
&\big\|1-\mathbb{1}_{D^I_h}(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\big\|_{\mathcal{L}^{4pc}(\mathbb{P};\mathbb{R})}=\big\|\mathbb{1}_{P_I(H)\backslash D^I_h}(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\big\|_{\mathcal{L}^{4pc}(\mathbb{P};\mathbb{R})}\leq\big\|\mathbb{1}_{\{C(1+\|\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_\nu})>\mathbf{v}h^{-\varsigma}\}}\big\|_{\mathcal{L}^{4pc}(\mathbb{P};\mathbb{R})}\\
&\leq\big[\mathbb{P}\big(|C(1+\|\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_\nu})|^{4pc(\gamma-\delta)/\varsigma}>(\mathbf{v}h^{-\varsigma})^{4pc(\gamma-\delta)/\varsigma}\big)\big]^{1/(4pc)}\\
&\leq(\mathbf{v}h^{-\varsigma})^{-(\gamma-\delta)/\varsigma}\big(\mathbb{E}\big[|C(1+\|\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_\nu})|^{4pc(\gamma-\delta)/\varsigma}\big]\big)^{1/(4pc)}\\
&=|\mathbf{v}|^{-(\gamma-\delta)/\varsigma}h^{\gamma-\delta}C^{(\gamma-\delta)/\varsigma}\big\|1+\|\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_\nu}\big\|^{(\gamma-\delta)/\varsigma}_{\mathcal{L}^{4pc(\gamma-\delta)/\varsigma}(\mathbb{P},\mathbb{R})}\\
&\leq|\mathbf{v}|^{-(\gamma-\delta)/\varsigma}h^{\gamma-\delta}C^{(\gamma-\delta)/\varsigma}\big(1+\|\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{\mathcal{L}^{2\max\{4pc(\gamma-\delta)/\varsigma,1\}}(\mathbb{P},H_\nu)}\big)^{(\gamma-\delta)/\varsigma}.
\end{aligned}\tag{239}$$

Combining this and (237) demonstrates that there exists $\mathbf{C}\in[1,\infty)$ which satisfies that for every $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $t\in[0,T]$ it holds that

$$\big\|1-\mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\big\|_{\mathcal{L}^{4pc}(\mathbb{P};\mathbb{R})}\leq\mathbf{C}\,[|\theta|_T]^{\gamma-\delta}.\tag{240}$$

This, (217), and [45, Corollary 3.3] (with $H=H$, $U=U$, $\mathbb{H}=\mathbb{H}$, $\mathfrak{v}=\mathfrak{v}$, $A=A$, $\beta=\beta$, $T=T$, $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]})=(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]})$, $(W_t)_{t\in[0,T]}=(W_t)_{t\in[0,T]}$, $B=B$, $P_I=P_I$, $\hat{P}_{\mathbb{U}}=\mathrm{Id}_U$, $\chi^{\theta,I,\mathbb{U}}=([0,T]\times\Omega\ni(t,\omega)\mapsto\mathbb{1}_{D^I_{|\theta|_T}}(\mathcal{X}^{\theta,I}_t(\omega))\in[0,1])$, $\mathbf{O}^{\theta,I,\mathbb{U}}=\mathbf{O}^{\theta,I}$, $p=4pc$, $C=\mathbf{C}$, $\gamma=\max\{\delta,\kappa\}$, $\eta=\gamma-\delta$, $\rho=\gamma-\delta$, $O=O$ for $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$ in the notation of [45, Corollary 3.3]) demonstrate that there exists $\mathscr{C}\in\mathbb{R}$ which satisfies that for every $I,J\in\mathcal{P}_0(\mathbb{H})$ with $I\subseteq J$ it holds that

$$\sup\nolimits_{s\in[0,T]}\|\mathbf{O}^{\theta,I}_s-O^J_s\|_{\mathcal{L}^{4pc}(\mathbb{P};H_{\max\{\delta,\kappa\}})}\leq\mathscr{C}\big(\|P_{\mathbb{H}\backslash I}(-A)^{\delta-\gamma}\|_{L(H)}+[|\theta|_T]^{\gamma-\delta}\big).\tag{241}$$

Moreover, observe that (216) and the fact that $(\mathcal{X}^{\theta,I}_t)_{t\in[0,T]}$, $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, are $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths ensure that $(\mathbf{O}^{\theta,I}_t)_{t\in[0,T]}$, $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, are $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths. This, the assumption that for every $I\in\mathcal{P}_0(\mathbb{H})$, $x,y\in P_I(H)$ it holds that $\langle F'(x)y,y\rangle_H\leq(\varepsilon\|x\|^2_{H_{1/2}}+C)\|y\|^2_H+\|y\|^2_{H_{1/2}}$, $\|P_I(F(x)-F(y))\|_H\leq C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\kappa}+\|y\|^c_{H_\kappa})$, and

$\langle x, Ax + F(x+y)\rangle_H \leq \Phi(y)(1 + \|x\|_H^2)$, the fact that $\varepsilon \leq \frac{\rho}{p}\exp(-2(a + \|B\|^2_{\mathrm{HS}(U,H)})T)$, the fact that $\xi \in \mathcal{L}^{4p\max\{c,2\}}(\mathbb{P}|_{\mathbb{F}_0}; H_{\max\{\gamma,\eta_2\}})$, the fact that $\mathbb{E}\big[\|\xi\|_H^{16p}\big] < \infty$, (215), (216), (218), (219), (223), (226), (232), (233), (236), (238), (241), and Proposition 4.5 (with $T = T$, $\mathbf{v} = \mathbf{v}$, $\varsigma = \varsigma$, $\alpha = \gamma - \delta$, $a = a$, $\iota = \gamma - \delta$, $\rho = \rho$, $C = \max\{C, \mathfrak{C}, \mathscr{C}\}$, $c = c$, $p = p$, $\beta = \beta$, $\gamma = \gamma$, $\delta = \delta$, $\kappa = \kappa$, $\eta_1 = \eta_1$, $\eta_2 = \eta_2$, $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$, $B = B$, $\varepsilon = \varepsilon$, $F = F$, $r = r$, $D_h^I = D_h^I$, $\Phi = \Phi$, $P_I = P_I$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $X = X$, $O = O$, $\mathbf{X}^{\theta,I} = \mathcal{X}^{\theta,I}$, $\mathbf{O}^{\theta,I} = \mathbf{O}^{\theta,I}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ in the notation of Proposition 4.5) therefore establish (213). The proof of Theorem 5.9 is thus completed. □

Corollary 5.10. *Assume Setting 1.3, let $T \in (0,\infty)$, $\varsigma \in (0, 1/18)$, $a \in [0,\infty)$, $C, c, p \in [1,\infty)$, $(C_\varepsilon)_{\varepsilon\in(0,\infty)} \subseteq [0,\infty)$, $\beta \in [0, 1/2)$, $\gamma \in [2\beta, 1/2 + \beta) \cap (0,\infty)$, $\delta \in (\gamma - 1/2, \gamma) \cap [0,\infty)$, $\kappa \in [0,\gamma] \cap [0, 1/2 + \beta - \gamma + \delta)$, $\eta_0 = 0$, $\sigma, \nu, \eta_1 \in [0, 1/2 + \beta)$, $\eta_2 \in [\eta_1, 1/2 + \beta)$, $\alpha_1 \in [0, 1 - \eta_1)$, $\alpha_2 \in [0, 1 - \eta_2)$, $\alpha_3 = 0$, $B \in \mathrm{HS}(U, H_\beta)$, $F \in \mathcal{C}^1(H_\gamma, H)$, let $\Phi\colon H \to [0,\infty)$ be a function, let $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ satisfy for every $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ that $P_I(x) = \sum_{h\in I}\langle h, x\rangle_H h$, assume for every $I \in \mathcal{P}(\mathbb{H})$ that $(P_I(H) \ni v \mapsto \Phi(v) \in [0,\infty)) \in \mathcal{C}(P_I(H), [0,\infty))$, assume for every $I \in \mathcal{P}_0(\mathbb{H})$, $x, y \in P_I(H)$, $\varepsilon \in (0,\infty)$ that $\langle x, F(x)\rangle_H \leq a(1 + \|x\|_H^2)$, $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + C_\varepsilon)\|y\|_H^2 + \|y\|^2_{H_{1/2}}$, $\|F(x) - F(y)\|_H \leq C\|x - y\|_{H_\delta}(1 + \|x\|^c_{H_\kappa} + \|y\|^c_{H_\kappa})$, $\langle x, Ax + F(x+y)\rangle_H \leq \Phi(y)(1 + \|x\|_H^2)$, and*

$$\left[\sup_{J\in\mathcal{P}_0(\mathbb{H})}\sup_{v\in P_J(H)}\left\{\tfrac{\|P_J F(v)\|_H}{1+\|v\|^2_{H_\nu}} + \tfrac{\|P_J F(v)\|_{H_{\gamma-\delta}}}{1+\|v\|^2_{H_\sigma}}\right\}\right] + \sum_{i=0}^{2}\left[\sup_{v\in H_{\max\{\gamma,\eta_i\}}}\tfrac{\|F(v)\|_{H_{-\alpha_{i+1}}}}{1+\|v\|^2_{H_{\eta_i}}}\right] < \infty, \tag{242}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, let $\xi \in \mathcal{L}^{32pc\max\{(\gamma-\delta)/\varsigma, 1\}}(\mathbb{P}|_{\mathbb{F}_0}; H_{\max\{\eta_2,\sigma,\nu,\gamma\}})$ satisfy $\inf_{\epsilon\in(0,\infty)}\mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$, let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [0,T]$ that

$$[X_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} = \left[e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,ds\right]_{\mathbb{P},\mathcal{B}(H_\gamma)} + \int_0^t e^{(t-s)A}B\,dW_s, \tag{243}$$

and let $\mathbf{X}^{\theta,I}\colon [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes which satisfy for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in [0,T]$ that $\mathbf{X}_0^{\theta,I} = P_I\xi$ and

$$\begin{aligned}[\mathbf{X}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} &= \big[e^{(t-\llcorner t\lrcorner_\theta)A}\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta} + \mathbb{1}_{\{1+\|\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_\nu}\leq[|\theta|_T]^{-\varsigma}\}}e^{(t-\llcorner t\lrcorner_\theta)A}P_I F(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})(t - \llcorner t\lrcorner_\theta)\big]_{\mathbb{P},\mathcal{B}(P_I(H))} \\ &\quad + \frac{\int_{\llcorner t\lrcorner_\theta}^t \mathbb{1}_{\{1+\|\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_\nu}\leq[|\theta|_T]^{-\varsigma}\}}e^{(t-\llcorner t\lrcorner_\theta)A}P_I B\,dW_s}{1 + \|\int_{\llcorner t\lrcorner_\theta}^t P_I B\,dW_s\|_H^2}.\end{aligned} \tag{244}$$

Then there exists $\mathfrak{c} \in \mathbb{R}$ such that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\sup_{t\in[0,T]}\|X_t - \mathbf{X}_t^{\theta,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \mathfrak{c}\big(\|P_{\mathbb{H}\setminus I}(-A)^{\delta-\gamma}\|_{L(H)} + [|\theta|_T]^{\gamma-\delta}\big). \tag{245}$$

Proof of Corollary 5.10. Throughout this proof let $D_h^I \in \mathcal{P}(H)$, $h \in (0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$, be the sets which satisfy for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ that

$$D_h^I = \{v \in P_I(H)\colon 1 + \|v\|^2_{H_\nu} \leq h^{-\varsigma}\}, \tag{246}$$

let $\epsilon \in (0, \exp(-2(a + \|B\|^2_{\mathrm{HS}(U,H)})T)]$, $\varepsilon, \mathbf{C} \in (0,\infty)$ satisfy that

$$\begin{aligned}\mathbf{C} &= \max\{C_\varepsilon, 1\}\max\{\|B\|_{\mathrm{HS}(U,H)}, 1\}\max\{\|(-A)^{-\nu}\|^2_{L(H)}, 1\} \\ &\quad + \max\Big\{\sup_{J\in\mathcal{P}_0(\mathbb{H})}\sup_{v\in P_J(H)}\tfrac{\|P_J F(v)\|_H}{1+\|v\|^2_{H_\nu}}, \|B\|_{\mathrm{HS}(U,H)}\Big\},\end{aligned} \tag{247}$$

$$\varepsilon < \tfrac{\exp(-2(a+\|B\|^2_{\mathrm{HS}(U,H)})T)}{16p} \min\Big\{\epsilon \exp(-2(a + \|B\|^2_{\mathrm{HS}(U,H)})T), \tfrac{1}{(8\max\{\|B\|^2_{\mathrm{HS}(U,H)},1\}\max\{T,1\})^2}\Big\}, \tag{248}$$

and $\mathbb{E}[\exp(\epsilon\|\xi\|^2_H)] < \infty$, and let $r\colon H_\gamma \to [0,\infty)$ be the function which satisfies for every $v \in H_\gamma$ that

$$r(v) = \begin{cases} \mathbf{C}(1 + \|v\|^2_{H_\nu}) & : v \in H_{\max\{\nu,\gamma\}} \\ 0 & : v \in (H_\gamma \backslash H_{\max\{\nu,\gamma\}}). \end{cases} \tag{249}$$

Observe that, e.g., Becker et al. [5, Lemma 5.3] (with $V = H_{\max\{\nu,\gamma\}}$, $W = H_\gamma$, $(S,\mathcal{S}) = ([0,\infty), \mathcal{B}([0,\infty)))$, $\Psi = r$ in the notation of Becker et al. [5, Lemma 5.3]) ensures that

$$r \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}([0,\infty))). \tag{250}$$

Next note that for every $x \in H_{\max\{\nu,\gamma\}}$ it holds that

$$\begin{aligned} &\|B\|_{\mathrm{HS}(U,H)} + \epsilon\|x\|^2_H \leq \max\{\|B\|_{\mathrm{HS}(U,H)}, \epsilon\}(1 + \|x\|^2_H) \\ &\leq \max\{\|B\|_{\mathrm{HS}(U,H)}, \epsilon\} \max\{\|(-A)^{-\nu}\|^2_{L(H)}, 1\}(1 + \|x\|^2_{H_\nu}) \leq \mathbf{C}(1 + \|x\|^2_{H_\nu}) = r(x). \end{aligned} \tag{251}$$

Moreover, observe that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$D^I_h = \{v \in P_I(H)\colon r(v) \leq \mathbf{C}h^{-\varsigma}\}. \tag{252}$$

This, (250), and, e.g., Andersson et al. [3, Lemma 2.2] (with $V_0 = H_\gamma$, $V_1 = P_I(H)$ for $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of Andersson et al. [3, Lemma 2.2]) assure that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$D^I_h \in \mathcal{B}(H_\gamma). \tag{253}$$

Furthermore, note that (246) and (247) imply that for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$, $x \in D^I_h$ it holds that

$$\begin{aligned} &\max\{\|P_I F(x)\|_H, \|B\|_{\mathrm{HS}(U,H)}\} \\ &\leq \max\Big\{\Big(\sup_{J\in\mathcal{P}_0(\mathbb{H})}\sup_{v\in P_J(H)} \tfrac{\|P_J F(v)\|_H}{1+\|v\|^2_{H_\nu}}\Big)(1 + \|x\|^2_{H_\nu}), \|B\|_{\mathrm{HS}(U,H)}\Big\} \\ &\leq \max\Big\{\sup_{J\in\mathcal{P}_0(\mathbb{H})}\sup_{v\in P_J(H)} \tfrac{\|P_J F(v)\|_H}{1+\|v\|^2_{H_\nu}}, \|B\|_{\mathrm{HS}(U,H)}\Big\}(1 + \|x\|^2_{H_\nu}) \leq \mathbf{C}h^{-\varsigma}. \end{aligned} \tag{254}$$

Combining this, (242), (248), (250)–(253), the fact that $\mathbb{E}[\exp(\epsilon\|\xi\|^2_H)] < \infty$, the assumption that for every $I \in \mathcal{P}_0(\mathbb{H})$, $x, y \in P_I(H)$ it holds that $\langle F'(x)y, y\rangle_H \leq (\varepsilon\|x\|^2_{H_{1/2}} + C_\varepsilon)\|y\|^2_H + \|y\|^2_{H_{1/2}}$, $\langle x, F(x)\rangle_H \leq a(1+\|x\|^2_H)$, $\|F(x)-F(y)\|_H \leq C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\kappa} + \|y\|^c_{H_\kappa})$, and $\langle x, Ax+F(x+y)\rangle_H \leq \Phi(y)(1 + \|x\|^2_H)$, and Theorem 5.9 (with $T = T$, $\mathbf{v} = \mathbf{C}$, $\varsigma = \varsigma$, $a = a$, $C = \mathbf{C}$, $c = c$, $p = p$, $\beta = \beta$, $\gamma = \gamma$, $\delta = \delta$, $\kappa = \kappa$, $\eta_0 = \eta_0$, $\sigma = \sigma$, $\nu = \nu$, $\eta_1 = \eta_1$, $\eta_2 = \eta_2$, $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$, $\alpha_3 = \alpha_3$, $B = B$, $\epsilon = \epsilon$, $\varepsilon = \varepsilon$, $F = F$, $r = r$, $D^I_h = D^I_h$, $\Phi = \Phi$, $P_I = P_I$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $X = X$, $\mathbf{X}^{\theta,I} = \mathbf{X}^{\theta,I}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ in the notation of Theorem 5.9) establishes (245). The proof of Corollary 5.10 is thus completed. $\square$

6 Strong convergence rates for space-time discrete approximations of stochastic Burgers equations

In this section we illustrate Corollary 5.10 in the case of stochastic Burgers equations. For this we combine some of the regularity results in [47] with Corollary 5.10 to prove in Corollary 6.1 strong convergence for the numerical approximations $(\mathbf{X}_t^{\theta,I})_{t\in[0,T]}$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, (see (256) below) of the mild solution of a stochastic Burgers equation with additive trace-class noise (see (255) below). Finally, Corollary 6.2 presents the findings from Corollary 6.1 in a further simplified setting.

Corollary 6.1. *Assume Setting 1.2, let* $T, c_0 \in (0,\infty)$, $c_1 \in \mathbb{R}$, $\varsigma \in (0, 1/18)$, $p \in [1,\infty)$, $\beta \in (0, 1/2)$, $\gamma \in ([\max\{1/2, 2\beta\}, 1/2+\beta)\backslash\{1/2, 3/4\})$, *let* $\lambda\colon \mathcal{B}((0,1)) \to [0,1]$ *be the Lebesgue-Borel measure on* $(0,1)$, *let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(\lambda;\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda;\mathbb{R})}, \|\cdot\|_{L^2(\lambda;\mathbb{R})})$, *let* $(e_n)_{n\in\mathbb{N}} \subseteq H$ *satisfy for every* $n \in \mathbb{N}$ *that* $e_n = [(\sqrt{2}\sin(n\pi x))_{x\in(0,1)}]_{\lambda,\mathcal{B}(\mathbb{R})}$, *let* $\mathbb{H} \subseteq H$ *satisfy that* $\mathbb{H} = \{e_n : n \in \mathbb{N}\}$, *let* $A\colon D(A) \subseteq H \to H$ *be the linear operator which satisfies* $D(A) = \{v \in H : \sum_{n=1}^\infty |n^2\langle e_n, v\rangle_H|^2 < \infty\}$ *and* $\forall\, v \in D(A)\colon Av = -c_0 \sum_{n=1}^\infty \pi^2 n^2 \langle e_n, v\rangle_H e_n$, *let* $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$, *for every* $v \in W^{1,2}((0,1),\mathbb{R})$ *let* $\partial v \in H$ *satisfy for every* $\varphi \in \mathcal{C}^\infty_{cpt}((0,1),\mathbb{R})$ *that* $\langle \partial v, [\varphi]_{\lambda,\mathcal{B}(\mathbb{R})}\rangle_H = -\langle v, [\varphi']_{\lambda,\mathcal{B}(\mathbb{R})}\rangle_H$, *let* $B \in \mathrm{HS}(H, H_\beta)$, *let* $F\colon H_{1/2} \to H$ *be the function which satisfies for every* $v \in H_{1/2}$ *that* $F(v) = c_1 v\partial v$, *let* $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ *satisfy for every* $I \in \mathcal{P}(\mathbb{H})$, $v \in H$ *that* $P_I(v) = \sum_{h\in I} \langle h, v\rangle_H h$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$, *let* $(W_t)_{t\in[0,T]}$ *be an* Id_H*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $\xi \in \mathcal{L}^{32p\max\{(2\gamma-1)/(2\varsigma),1\}}(\mathbb{P}|_{\mathbb{F}_0}; H_\gamma)$ *satisfy* $\inf_{\epsilon\in(0,\infty)} \mathbb{E}[\exp(\epsilon\|\xi\|_H^2)] < \infty$, *let* $X\colon [0,T]\times\Omega \to H_\gamma$ *be an* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic process with continuous sample paths which satisfies for every* $t \in [0,T]$ *that*

$$[X_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} = \left[e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,ds\right]_{\mathbb{P},\mathcal{B}(H_\gamma)} + \int_0^t e^{(t-s)A}B\,dW_s, \tag{255}$$

and let $\mathbf{X}^{\theta,I}\colon [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, *be* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic processes which satisfy for every* $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $t \in (0,T]$ *that* $\mathbf{X}_0^{\theta,I} = P_I(\xi)$ *and*

$$\begin{split}[\mathbf{X}_t^{\theta,I}]_{\mathbb{P},\mathcal{B}(P_I(H))} = \Big[e^{(t-\llcorner t\lrcorner_\theta)A}\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta} + \mathbb{1}_{\{1+\|\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_{1/2}} \le [|\theta|_T]^{-\varsigma}\}} e^{(t-\llcorner t\lrcorner_\theta)A} P_I F(\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta})\,(t - \llcorner t\lrcorner_\theta)\Big]_{\mathbb{P},\mathcal{B}(P_I(H))} \\ + \frac{\int_{\llcorner t\lrcorner_\theta}^t \mathbb{1}_{\{1+\|\mathbf{X}^{\theta,I}_{\llcorner t\lrcorner_\theta}\|^2_{H_{1/2}} \le [|\theta|_T]^{-\varsigma}\}} e^{(t-\llcorner t\lrcorner_\theta)A} P_I B\,dW_s}{1 + \|\int_{\llcorner t\lrcorner_\theta}^t P_I B\,dW_s\|_H^2}.\end{split} \tag{256}$$

Then there exists $C \in \mathbb{R}$ *such that for every* $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ *it holds that*

$$\sup\nolimits_{t\in[0,T]} \|X_t - \mathbf{X}_t^{\theta,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \le C\big(\|P_{\mathbb{H}\backslash I}(-A)^{(1/2)-\gamma}\|_{L(H)} + [|\theta|_T]^{\gamma-(1/2)}\big). \tag{257}$$

Proof of Corollary 6.1. Throughout this proof let $\Phi\colon H \to [0,\infty)$ be the function which satisfies for every $w \in H$ that

$$\Phi(w) = \begin{cases} \frac{3|c_1|^2}{8|c_0|}\left[\sup_{u\in H_{1/2}\backslash\{0\}} \frac{\|u\|_{L^\infty(\lambda;\mathbb{R})}}{\|u\|_{H_{1/2}}} + \sup_{u\in H_{1/2}\backslash\{0\}} \frac{\|u\|^2_{L^4(\lambda;\mathbb{R})}}{\|u\|^2_{H_{1/2}}}\right]^2 (1+\|w\|^2_{H_{1/2}})^2 & : w \in H_{1/2} \\ 0 & : w \in (H\backslash H_{1/2}). \end{cases} \tag{258}$$

We intend to prove Corollary 6.1 through an application of Corollary 5.10. For this note that, e.g., [47, item (ii) of Lemma 4.13] shows that for every $v, w \in H_\gamma \subseteq H_{1/2}$ it holds that

$$\|F(v) - F(w)\|_H \leq \tfrac{|c_1|}{\sqrt{3}\,c_0}(\|v\|_{H_{1/2}} + \|w\|_{H_{1/2}})\|v - w\|_{H_{1/2}}. \tag{259}$$

In addition, observe that, e.g., [47, Lemma 4.19] and the fact that $H_\gamma \subseteq H_{1/2}$ continuously imply that

(a) it holds that $F \in \mathcal{C}^1(H_\gamma, H)$ and

(b) it holds that there exists $C \in (0, \infty)$ which satisfies for every $\varepsilon \in (0, \infty)$, $v, w \in H_\gamma \subseteq H_{1/2}$ that

$$\langle F'(w)v, v\rangle_H \leq \varepsilon \|w\|^2_{H_{1/2}} \|v\|^2_H + \tfrac{C}{\varepsilon^2}\|v\|^2_H + \|v\|^2_{H_{1/2}}. \tag{260}$$

Furthermore, note that the fact that $0 \leq \gamma - \frac{1}{2} < \frac{1}{2}$, the fact that $\gamma \neq \frac{3}{4}$, and, e.g., [47, Lemma 4.20] (with $\alpha = \gamma - \frac{1}{2}$ in the notation of [47, Lemma 4.20]) ensure that

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{v \in H_\gamma \setminus \{0\}} \left(\frac{\|P_I F(v)\|_{H_{\gamma-(1/2)}}}{\|v\|^2_{H_\gamma}} \right) < \infty. \tag{261}$$

Moreover, observe that, e.g., [47, Lemma 4.20] (with $\alpha = 0$ in the notation of [47, Lemma 4.20]) proves that

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{v \in H_{1/2} \setminus \{0\}} \left(\frac{\|P_I F(v)\|_H}{\|v\|^2_{H_{1/2}}} \right) < \infty. \tag{262}$$

In addition, note that, e.g., [47, Lemma 4.23] proves that for every $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that

$$\langle x, F(x)\rangle_H = 0. \tag{263}$$

Furthermore, observe that, e.g., [47, Corollary 4.22] (with $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$ for $\alpha_1 \in (3/4, \infty)$, $\alpha_2 \in (1/4, 1/2]$ in the notation of [47, Corollary 4.22]) shows that for every $\alpha_1 \in (3/4, \infty)$, $\alpha_2 \in (1/4, 1/2]$ it holds that

$$\left[\sup_{v \in H_{1/2} \setminus \{0\}} \frac{\|F(v)\|_H}{\|v\|^2_{H_{1/2}}}\right] + \left[\sup_{v \in H_{1/2} \setminus \{0\}} \frac{\|F(v)\|_{H_{-\alpha_2}}}{\|v\|^2_{H_{(1-\alpha_2)/3}}}\right] + \left[\sup_{v \in H_{1/2} \setminus \{0\}} \frac{\|F(v)\|_{H_{-\alpha_1}}}{\|v\|^2_H}\right] < \infty. \tag{264}$$

Moreover, note that, e.g., [47, Corollary 4.24] (with $\iota = 1/2$, $v = v$, $w = w$ for $v, w \in H_{1/2}$ in the notation of [47, Corollary 4.24]) assures that for every $v, w \in H_{1/2}$ it holds that

$$\langle v, F(v + w)\rangle_H \leq \Phi(w)(1 + \|v\|^2_H) - \langle v, Av\rangle_H. \tag{265}$$

Combining this, the assumption that $\inf_{\epsilon \in (0,\infty)} \mathbb{E}[\exp(\epsilon \|\xi\|^2_H)] < \infty$, items (a) and (b), (219), (259), and (261)–(264) with Corollary 5.10 (with $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U) = (H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, $\mathbb{H} = \mathbb{H}$, $\mathfrak{v}_{e_n} = -c_0\pi^2 n^2$, $A = A$, $H_r = H_r$, $T = T$, $\varsigma = \varsigma$, $a = 0$, $C = \max\{1, |c_1|/c_0\}$, $c = 1$, $p = p$, $C_\varepsilon = C/\varepsilon^2$, $\beta = \beta$, $\gamma = \gamma$, $\delta = 1/2$, $\kappa = 1/2$, $\sigma = \gamma$, $\nu = 1/2$, $\eta_1 = (1-\alpha_2)/3$, $\eta_2 = 1/2$, $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$, $B = B$, $F = (H_\gamma \ni x \mapsto F(x) \in H)$, $\Phi = \Phi$, $P_I = P_I$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = \xi$, $X = X$, $\mathbf{X}^{\theta,I} = \mathbf{X}^{\theta,I}$ for $n \in \mathbb{N}$, $r \in \mathbb{R}$, $\varepsilon \in (0, \infty)$, $\alpha_2 \in (1/4, 1/2)$, $\alpha_1 \in (3/4, (2+\alpha_2)/3)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$ in the notation of Corollary 5.10) therefore establishes (257). The proof of Corollary 6.1 is thus completed. ☐

Corollary 6.2. *Assume Setting 1.2, let $T, \varepsilon, c_0 \in (0,\infty)$, $c_1 \in \mathbb{R}$, $\varsigma \in (0, 1/18)$, $p \in [1,\infty)$, $\beta \in (0, 1/2]$, $\gamma \in [1/2, 1/2+\beta)$, let $\lambda \colon \mathcal{B}((0,1)) \to [0,1]$ be the Lebesgue-Borel measure on $(0,1)$, let $(H, \langle \cdot,\cdot \rangle_H, \|\cdot\|_H) = (L^2(\lambda;\mathbb{R}), \langle \cdot,\cdot \rangle_{L^2(\lambda;\mathbb{R})}, \|\cdot\|_{L^2(\lambda;\mathbb{R})})$, let $(e_n)_{n\in\mathbb{N}} \subseteq H$ satisfy for every $n \in \mathbb{N}$ that $e_n = [(\sqrt{2}\sin(n\pi x))_{x\in(0,1)}]_{\lambda,\mathcal{B}(\mathbb{R})}$, let $A \colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \{v \in H \colon \sum_{n=1}^\infty |n^2 \langle e_n, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A) \colon Av = -c_0 \sum_{n=1}^\infty \pi^2 n^2 \langle e_n, v \rangle_H e_n$, let $(H_r, \langle \cdot,\cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, for every $v \in W^{1,2}((0,1),\mathbb{R})$ let $\partial v \in H$ satisfy for every $\varphi \in \mathcal{C}^\infty_{cpt}((0,1),\mathbb{R})$ that $\langle \partial v, [\varphi]_{\lambda,\mathcal{B}(\mathbb{R})}\rangle_H = -\langle v, [\varphi']_{\lambda,\mathcal{B}(\mathbb{R})}\rangle_H$, let $B \in \mathrm{HS}(H, H_\beta)$, $\xi \in H_{1/2+\beta}$, let $F \colon H_{1/2} \to H$ be the function which satisfies for every $v \in H_{1/2}$ that $F(v) = c_1 v \partial v$, let $(P_N)_{N\in\mathbb{N}} \subseteq L(H)$ satisfy for every $N \in \mathbb{N}$, $v \in H$ that $P_N(v) = \sum_{n=1}^N \langle e_n, v\rangle_H e_n$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, and let $\mathbf{X}^{\theta,N} \colon [0,T]\times\Omega \to P_N(H)$, $\theta \in \varpi_T$, $N \in \mathbb{N}$, be $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes which satisfy for every $\theta \in \varpi_T$, $N \in \mathbb{N}$, $t \in (0,T]$ that $\mathbf{X}_0^{\theta,N} = P_N(\xi)$ and*

$$
\begin{split}
[\mathbf{X}_t^{\theta,N}]_{\mathbb{P},\mathcal{B}(P_N(H))} &= \frac{\int_{\llcorner t \lrcorner_\theta}^t \mathbb{1}_{\{1+\|\mathbf{X}^{\theta,N}_{\llcorner t \lrcorner_\theta}\|^2_{H_{1/2}} \le [|\theta|_T]^{-\varsigma}\}} e^{(t-\llcorner t \lrcorner_\theta)A} P_N B \, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t P_N B\, dW_s\|_H^2} \\
&+ \Big[e^{(t-\llcorner t \lrcorner_\theta)A} \mathbf{X}^{\theta,N}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_{\{1+\|\mathbf{X}^{\theta,N}_{\llcorner t \lrcorner_\theta}\|^2_{H_{1/2}} \le [|\theta|_T]^{-\varsigma}\}} e^{(t-\llcorner t \lrcorner_\theta)A} P_N F(\mathbf{X}^{\theta,N}_{\llcorner t \lrcorner_\theta})\,(t - \llcorner t \lrcorner_\theta) \Big]_{\mathbb{P},\mathcal{B}(P_N(H))}.
\end{split}
\tag{266}
$$

Then

(i) there exists an up to indistinguishability unique $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process $X \colon [0,T]\times\Omega \to H_\gamma$ with continuous sample paths which satisfies for every $t \in [0,T]$ that

$$
[X_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} = \left[e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\, ds \right]_{\mathbb{P},\mathcal{B}(H_\gamma)} + \int_0^t e^{(t-s)A} B\, dW_s \tag{267}
$$

and

(ii) there exists $C \in \mathbb{R}$ such that for every $\theta \in \varpi_T$, $N \in \mathbb{N}$ it holds that

$$
\sup\nolimits_{t\in[0,T]} \|X_t - \mathbf{X}_t^{\theta,N}\|_{\mathcal{L}^p(\mathbb{P};H)} \le C\big(N^{(\varepsilon-2\beta)} + [|\theta|_T]^{(\beta-\varepsilon)}\big). \tag{268}
$$

Proof of Corollary 6.2. Observe that [47, Theorem 5.10] (with $T = T$, $\varepsilon = 1/2+\beta-\boldsymbol{\gamma}$, $c_0 = c_0$, $c_1 = c_1$, $\beta = \beta$, $\gamma = \boldsymbol{\gamma}$, $H = H$, $e_n = e_n$, $A = A$, $H_r = H_r$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $B = B$, $\xi = (\Omega \ni \omega \mapsto \xi \in H_{1/2+\beta})$ for $r \in \mathbb{R}$, $n \in \mathbb{N}$, $\boldsymbol{\gamma} \in [1/2, 1/2+\beta)$ in the notation of [47, Theorem 5.10]) shows that there exist up to modification unique $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes $X^{\boldsymbol{\gamma}} \colon [0,T]\times\Omega \to H_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in [1/2, 1/2+\beta)$, with continuous sample paths which satisfy for every $\boldsymbol{\gamma} \in [1/2, 1/2+\beta)$, $t \in [0,T]$ that

$$
[X_t^{\boldsymbol{\gamma}}]_{\mathbb{P},\mathcal{B}(H_{\boldsymbol{\gamma}})} = \left[e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s^{\boldsymbol{\gamma}})\, ds \right]_{\mathbb{P},\mathcal{B}(H_{\boldsymbol{\gamma}})} + \int_0^t e^{(t-s)A} B\, dW_s. \tag{269}
$$

This establishes item (i). In the next step we note that for every $\iota \in (0,\infty)$, $N \in \mathbb{N}$, $v \in H$ it holds that

$$
\begin{split}
\|(\mathrm{Id}_H - P_N)(-A)^{-\iota} v\|_H^2 &= |c_0|^{-2\iota} \sum_{n=N+1}^\infty (\pi^2 n^2)^{-2\iota} |\langle v, e_n\rangle_H|^2 \\
&\le |c_0|^{-2\iota} (\pi^2 N^2)^{-2\iota} \sum_{n=N+1}^\infty |\langle v, e_n\rangle_H|^2 \le |c_0|^{-2\iota} (\pi^2 N^2)^{-2\iota} \|v\|_H^2.
\end{split}
\tag{270}
$$

This shows that for every $\iota \in (0, \infty)$, $N \in \mathbb{N}$ it holds that

$$\|(\mathrm{Id}_H - P_N)(-A)^{-\iota}\|_{L(H)} \leq |c_0|^{-\iota} \pi^{-2\iota} N^{-2\iota} \leq |c_0|^{-\iota} N^{-2\iota}. \tag{271}$$

The fact that for every $\theta \in \varpi_T$, $\epsilon \in (0, \infty)$ it holds that $[|\theta|_T]^{\beta-(\epsilon/2)} \leq T^{\epsilon/2}[|\theta|_T]^{(\beta-\epsilon)}$, (269), and Corollary 6.1 (with $T = T$, $c_0 = c_0$, $c_1 = c_1$, $\varsigma = \varsigma$, $p = p$, $\beta = \beta - \frac{\epsilon}{4}$, $\gamma = \frac{1}{2} + \beta - \frac{\epsilon}{2}$, $H = H$, $e_n = e_n$, $A = A$, $H_r = H_r$, $B = B$, $F = F$, $P_{\{e_1,e_2,\ldots,e_n\}} = P_n$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\xi = (\Omega \ni \omega \mapsto \xi \in H_{(1/2)+\beta-(\epsilon/2)})$, $X = ([0, T] \times \Omega \ni (t, \omega) \mapsto X_t^{(1/2)+\beta-(\epsilon/2)}(\omega) \in H_{(1/2)+\beta-(\epsilon/2)})$, $\mathbf{X}^{\theta,\{e_1,e_2,\ldots,e_n\}} = \mathbf{X}^{\theta,n}$ for $r \in \mathbb{R}$, $\theta \in \varpi_T$, $n \in \mathbb{N}$, $\epsilon \in ((0, 2\beta)\backslash\{2\beta - 1/2\})$ in the notation of Corollary 6.1) therefore establish item (ii). The proof of Corollary 6.2 is thus completed. □

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