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Abstract We elaborate so-called regular decompositions of vector fields on a three-dimensional Lipschitz domain where the field and its rotation/divergence belong to L^2 and where the tangential/normal component of the field vanishes on a sufficiently smooth "Dirichlet" part of the boundary. We impose no restrictions on the topology of the domain, its boundary, or the Dirichlet boundary parts.

The field is split into a regular vector field, whose Cartesian components lie in H^1 and vanish on the Dirichlet boundary, and a remainder contained in the kernel of the rotation/divergence operator. The decomposition is proved to be stable not only in the natural norms, but also with respect to the L^2 norm. Besides, for special cases of mixed boundary conditions, we show the existence of H^1-regular potentials that characterize the range of the rotation and divergence operator.

We conclude with results on discrete counterparts of regular decompositions for spaces of low-order discrete differential forms on simplicial meshes. Essentially, all results for function spaces carry over, though local correction terms may be necessary. These discrete regular decompositions have become an important tool in finite element exterior calculus (FEEC) and for the construction of preconditioners.

1 Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ recall the classical L^2-orthogonal *Helmholtz decompositions*

$$\mathbf{L}^2(\Omega) = \nabla H_0^1(\Omega) \oplus \mathbf{H}(\mathrm{div}\, 0, \Omega) = \nabla H^1(\Omega) \oplus \mathbf{H}_0(\mathrm{div}\, 0, \Omega)\,,$$

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see, e.g., [11, Ch. XI, Sect. I]. They can be used to derive decompositions of (subspaces of) $\mathbf{H}(\mathbf{curl}, \Omega)$:

$$\mathbf{H}_0(\mathbf{curl}, \Omega) = \nabla H_0^1(\Omega) \oplus \mathbf{X}_N(\Omega), \quad \mathbf{X}_N(\Omega) := \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}\, 0, \Omega),$$
$$\mathbf{H}(\mathbf{curl}, \Omega) = \nabla H^1(\Omega) \oplus \mathbf{X}_T(\Omega), \quad \mathbf{X}_T(\Omega) := \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}_0(\mathrm{div}\, 0, \Omega)\,.$$

If the domain Ω is convex then the respective complementary space, $\mathbf{X}_N(\Omega)$ or $\mathbf{X}_T(\Omega)$, is continuously embedded in the space $\mathbf{H}^1(\Omega)$ of vector fields with Cartesian components in $H^1(\Omega)$, cf. [1]. Then one can, for instance, write any $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ as

$$\mathbf{u} = \nabla p + \mathbf{z}, \tag{1}$$

with $p \in H^1(\Omega)$ and $\mathbf{z} \in \mathbf{H}^1(\Omega)$. Since $\|\nabla p\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$ one obtains (using the continuous embedding) the stability property[1]

$$\|\nabla p\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathbf{z}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,. \tag{2}$$

A similar decomposition can be found for $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$.

Generally, a decomposition of form (1) with the stability property (2) is called *regular decomposition*, even if $\mathbf{L}^2$-orthogonality does not hold. Actually, it turns out that (1)–(2) can be achieved even in cases where Ω is non-convex, in particular on non-smooth domains, or in cases where Ω or its boundary have non-trivial topology; only the $\mathbf{L}^2$-orthogonality has to be sacrificed, cf. [23].

Noting that $\nabla H^1(\Omega)$ is contained in the kernel of the $\mathbf{curl}$ operator and that—under mild smoothness assumptions on the domain—the whole kernel is spanned by $\nabla H^1(\Omega)$ plus a finite-dimensional *co-homology space* [17, Sect. 4] one can achieve a second decomposition,

$$\mathbf{u} = \mathbf{h} + \mathbf{z}\,, \tag{3}$$

with $\mathbf{h} \in \ker(\mathbf{curl}_{|\mathbf{H}(\mathbf{curl},\Omega)})$ and $\mathbf{z} \in \mathbf{H}^1(\Omega)$, where

$$\|\mathbf{h}\|_{\mathbf{L}^2(\Omega)} \leq C\,\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}, \qquad \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \leq C\,\|\,\mathbf{curl}\;\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\,. \tag{4}$$

The second stability estimate states that if $\mathbf{u}$ is already in the kernel of the $\mathbf{curl}$ operator, then $\mathbf{z}$ is zero. Hence, (i) the operator mapping $\mathbf{u}$ to $\mathbf{h}$ is a *projection* onto the kernel space and (ii) the complement operator projects $\mathbf{u}$ to the function $\mathbf{z}$ of higher regularity $\mathbf{H}^1(\Omega)$. For trivial topology of Ω and $\partial\Omega$, the two decompositions (1)–(2) and (3)–(4) coincide.

As a few among many more [20, Sect. 1.5], we would like to highlight two important applications of these regular decompositions.

(i) The second form (3)–(4), in the sequel called *rotation-bounded decomposition*, can be used to show that the operator underlying a certain boundary value prob-

[1] Here and below C stands for a positive “generic constant” that may depend only on Ω, unless specified otherwise.

lem for Maxwell's equations is a *Fredholm operator*. The key point is that the complement space of the kernel (from the view of the mentioned projections) is $\mathbf{H}^1(\Omega)$ which is *compactly* embedded in $\mathbf{L}^2(\Omega)$, see e.g., [16, 18] and references therein.

(ii) The first form (1)–(2), in the sequel called *gradient-based decomposition*, has been used to generate stable three-term splittings of a finite element subspace of $\mathbf{H}(\mathbf{curl}, \Omega)$, cf. [21, 22, 25, 23], which allows the construction of so-called *fictitious* or *auxiliary* space preconditioners for the ill-conditioned system matrix underlying the discretized Maxwell equations.

In both applications, it is desirable to obtain the decompositions for minimal smoothness of the domain, e.g., Lipschitz domains, which are not necessarily convex. Moreover, it is also desirable to go beyond decompositions of the entire space $\mathbf{H}(\mathbf{curl}, \Omega)$ and extend them to subspaces for which the appropriate trace vanishes on a "Dirichlet part" Γ_D of the boundary. In this case traces of the two summands should also vanish on Γ_D.

In the present paper, we provide regular decompositions of both types for subspaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ (in Section 3) and $\mathbf{H}(\mathrm{div}, \Omega)$ (in Section 4) comprising functions with vanishing trace on a part Γ_D of the boundary $\partial\Omega$ for Lipschitz domains Ω of *arbitrary topology*. In particular, Ω is allowed to have handles, and $\partial\Omega$ and Γ_D may have several connected components. The Dirichlet boundary Γ_D must satisfy a certain smoothness assumption that we shall introduce in Section 2.1. In addition to the stability estimtes (2), (4), we show that the decompositions are stable even in $\mathbf{L}^2(\Omega)$.

In the final part of the manuscript, in Section 5, we establish regular decompositions of spaces of Whitney forms, which are lowest-order conforming finite element subspaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$, respectively, built upon simplicial triangulations of Ω.

We point out that parts of this work are based on [20] and some results have already been obtained there and will be referenced precisely throughout the text.

2 Preliminaries

2.1 Geometric Setting

Since subtle geometric arguments will play a major role for parts of the theory, we start with a precise characterization of the geometric setting: Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected Lipschitz domain[2]. We write $\mathrm{d}(\Omega)$ for its diameter. Its boundary $\Gamma := \partial\Omega$ is partitioned according to $\Gamma = \Gamma_D \cup \Sigma \cup \Gamma_N$, with relatively open sets Γ_D and Γ_N. We assume that this provides a *piecewise C^1 dissection* of

[2] Strongly Lipschitz, in the sense that the boundary is locally the graph of a Lipschitz continuous function.

$\partial\Omega$ in the sense of [14, Definition 2.2]. Sloppily speaking, this means that Σ is the union of closed curves that are piecewise C^1.

Under the above assumptions on Ω and Γ_D, [14, Lemma 4.4] guarantees the existence of an open Lipschitz neighborhood Ω_Γ ("Lipschitz collar") of Γ and of a continuous vector field $\widetilde{\mathbf{n}}$ on Ω_Γ with $\|\widetilde{\mathbf{n}}(\mathbf{x})\| = 1$ that is *transversal* to Γ:

$$\exists \kappa > 0: \quad \widetilde{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \geq \kappa \quad \text{for almost all } \mathbf{x} \in \Gamma, \tag{5}$$

where $\mathbf{n}$ is the outward unit normal on Γ. Extrusion of Γ_D by the local flow induced by $\widetilde{\mathbf{n}}$ spawns the "bulge" $\Upsilon_D \subset \Omega_\Gamma \setminus \Omega$, see Section 2.1. We recall the properties of bulge domains from [14, Sect. 2, Thm. 2.3], also stated in [20, Thm. 2.2]:

Theorem 1 (Bulge-augmented domain)

There exists a Lipschitz domain $\Upsilon_D \subset \mathbb{R}^3 \setminus \overline{\Omega}$*, such that* $\overline{\Upsilon}_D \cap \overline{\Omega} = \Gamma_D$*,* $\Omega^e := \Upsilon_D \cup \Gamma_D \cup \Omega$ *is Lipschitz,* $\mathrm{d}(\Omega^e) \leq 2\,\mathrm{d}(\Omega)$*, and* $\overline{\Upsilon}_D \subset \Omega_\Gamma$*. Moreover, each connected component* $\Gamma_{D,k}$ *of* Γ_D *corresponds to a connected component* $\Upsilon_{D,k}$ *of* Υ_D*, and these have positive distance from each other.*

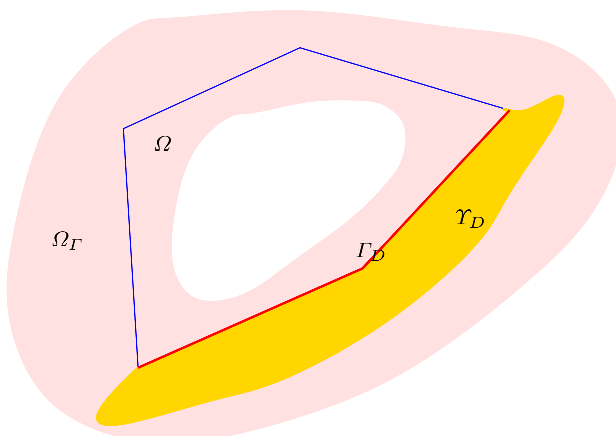

Fig. 1 Collar domain Ω_Γ (pink) and bulge domain Υ_D (gold)

2.2 De Rham Complex with Boundary Conditions

Let

$$\begin{aligned} H^1_{\Gamma_D}(\Omega) &:= \{u \in H^1(\Omega) \colon (\gamma u)_{|\Gamma_D} = 0\}, \\ \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) &:= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) \colon (\boldsymbol{\gamma}_\tau \mathbf{u})_{|\Gamma_D} = 0\}, \\ \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega) \colon (\gamma_n \mathbf{u})_{|\Gamma_D} = 0\}\,, \end{aligned}$$

denote the standard Sobolev spaces where the distributional gradient, curl, or divergence is in L^2 and where the pointwise trace γu, the tangential trace $\gamma_\tau \mathbf{u}$, or the normal trace $\gamma_n \mathbf{u}$, respectively, vanishes on the Dirichlet boundary Γ_D, see e.g. [7, 29, 3]. These space are linked via the *de Rham complex*,

$$\mathcal{K}_{\Gamma_D}(\Omega) \xrightarrow{\mathsf{id}} H^1_{\Gamma_D}(\Omega) \xrightarrow{\nabla} \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega), \tag{6}$$

where

$$\mathcal{K}_{\Gamma_D}(\Omega) := \{v \in H^1_{\Gamma_D}(\Omega)\colon v = \mathrm{const}\} = \begin{cases} \mathrm{span}\{1\}, & \text{if } \Gamma_D = \emptyset, \\ \{0\}, & \text{otherwise.} \end{cases}$$

The range of each operator in (6) lies in the kernel space of the succeeding one, cf. [3, Lemma 2.2]. We define

$$\begin{aligned} \mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega) &:= \{\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)\colon \ \mathbf{curl}\,\mathbf{v} = 0\}, \\ \mathbf{H}_{\Gamma_D}(\mathrm{div}\,0, \Omega) &:= \{\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)\colon \ \mathrm{div}\,\mathbf{v} = 0\}. \end{aligned} \tag{7}$$

Barring topological obstructions these kernels can be represented through potentials: Let $\beta_1(\Omega)$ denote the *first Betti number* of Ω (the number of "handles") and $\beta_2(\Omega)$ the *second Betti number* (the number of connected components of $\partial\Omega$ minus one). By the very definition of the Betti numbers as dimensions of co-homology spaces we have

$$\beta_1(\Omega) = 0 \quad \Longrightarrow \quad \mathbf{H}(\mathbf{curl}\,0, \Omega) = \nabla H^1(\Omega), \tag{8}$$

$$\beta_2(\Omega) = 0 \quad \Longrightarrow \quad \mathbf{H}(\mathrm{div}\,0, \Omega) = \mathbf{curl}\,\mathbf{H}(\mathbf{curl}, \Omega), \tag{9}$$

cf. [29]. We call Ω topologically trivial if $\beta_1(\Omega) = \beta_2(\Omega) = 0$.

3 Regular Decompositions and Potentials related to $\mathbf{H}(\mathbf{curl})$

Throughout we rely on the properties of Ω and Γ_D as introduced in Section 2.1 and use the notations from Theorem 1. We write C for positive "generic constants" and say that a constant "depends only on the shape of Ω and Γ_D", if it depends on the geometric setting alone, but is invariant with respect to similarity transformations. To achieve this the diameter of Ω will have to enter the estimates; we denote it by $\mathrm{d}(\Omega)$.

3.1 Gradient-Based Regular Decomposition of $\mathbf{H}(\mathbf{curl})$

The following theorem is essentially [20, Thm. 2.1].

Theorem 2 (Gradient-Based Regular Decomposition of $\mathbf{H}(\mathbf{curl})$)
Let (Ω, Γ_D) satisfy the assumptions of Section 2.1. Then for each $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ there exist $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and $p \in H^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{u}$ such that

(i) $$\mathbf{u} = \mathbf{z} + \nabla p,$$

(ii) $$\|\mathbf{z}\|_{0,\Omega} + \|\nabla p\|_{0,\Omega} \le C\,\|\mathbf{u}\|_{0,\Omega},$$

(iii) $$\|\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{z}\|_{0,\Omega} \le C\|\,\mathbf{curl}\;\mathbf{u}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{u}\|_{0,\Omega}\,,$$

with constants depending only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

For completeness, we give the proof, which makes use of three auxiliary results that we quote without proof:

Lemma 1 (Stein extension operator for H^k, [32])

Let $\mathcal{D}$ be a bounded Lipschitz domain with $\mathrm{d}(\mathcal{D}) = 1$. Then there exists a bounded linear extension operator $E^{\nabla,\mathit{Stein}}_{\mathcal{D}}\colon H^1(\mathcal{D}) \to H^1(\mathbb{R}^3)$ such that, for all integers $k \ge 1$, with constants depending only on $\mathcal{D}$ and k,

$$\|E^{\nabla,\mathit{Stein}}_{\mathcal{D}} u\|_{H^k(\mathbb{R}^3)} \le C\,\|u\|_{H^k(\mathcal{D})} \qquad \forall u \in H^k(\mathcal{D}). \tag{10}$$

Lemma 2 (Extension operator for $\mathbf{H}(\mathbf{curl})$, [20, Lemma 2.6])

Let $\mathcal{D}$ be a bounded Lipschitz domain with $\mathrm{d}(\mathcal{D}) = 1$. Then there exists a bounded linear extension operator $E^{\mathrm{curl}}_{\mathcal{D}}\colon \mathbf{L}^2(\mathcal{D}) \to \mathbf{L}^2(\mathbb{R}^3)$ such that, with constants depending only on $\mathcal{D}$,

$$\begin{aligned} \|E^{\mathrm{curl}}_{\mathcal{D}}\mathbf{v}\|_{0,\mathbb{R}^3} &\le C\|\mathbf{v}\|_{0,\mathcal{D}} && \forall \mathbf{v} \in \mathbf{L}^2(\mathcal{D}),\\ \|E^{\mathrm{curl}}_{\mathcal{D}}\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3)} &\le C\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\mathcal{D})} && \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl},\mathcal{D}), \end{aligned}$$

and $E^{\mathrm{curl}}_{\mathcal{D}} u$ has compact support.

Lemma 3 (Rotation-Preserving Projection onto $\mathbf{H}^1(\mathbb{R}^3)$, [20, Lemma 2.7])

There exists a bounded linear operator $\mathsf{L}^{\mathrm{curl}}\colon \mathbf{H}(\mathbf{curl},\mathbb{R}^3) \to \mathbf{H}^1(\mathbb{R}^3)$ such that for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl},\mathbb{R}^3)$

- (LC_1) $\mathbf{curl}\,\mathsf{L}^{\mathrm{curl}}\mathbf{v} = \mathbf{curl}\,\mathbf{v}$,
- (LC_2) $\mathrm{div}\,\mathsf{L}^{\mathrm{curl}}\mathbf{v} = 0$,
- (LC_3) $\|\mathsf{L}^{\mathrm{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \le \|\mathbf{v}\|_{0,\mathbb{R}^3}$ *and* $\|(\mathsf{id} - \mathsf{L}^{\mathrm{curl}})\mathbf{v}\|_{0,\mathbb{R}^3} \le \|\mathbf{v}\|_{0,\mathbb{R}^3}$,
- (LC_4) $\|\boldsymbol{\nabla}\,\mathsf{L}^{\mathrm{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \le \|\,\mathbf{curl}\,\mathbf{v}\|_{0,\mathbb{R}^3}$,
- (LC_5) $(\mathsf{L}^{\mathrm{curl}})^2\mathbf{v} = \mathsf{L}^{\mathrm{curl}}\mathbf{v}$, *i.e.,* $\mathsf{L}^{\mathrm{curl}}$ *is a projection.*

Proof (Theorem 2) Without loss of generality, we may assume that $\mathrm{d}(\Omega) = 1$. We follow the proof as in [23, Thm. 5.9], which is based on the ideas in [6, Prop. 5.1], and establish the L^2-stability using the ideas from [31, Lemma 2.2]. Let $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ be arbitrary but fixed.

Step 1: We extend $\mathbf{u}$ by zero to a function in $\mathbf{H}(\mathbf{curl}, \Omega^e)$, where Ω^e is the extended domain from Theorem 1, and then to $\widetilde{\mathbf{u}} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ using the operator $E^{\mathrm{curl}}_{\Omega^e}$ from Lemma 2. We observe $\widetilde{\mathbf{u}}_{|\Upsilon_D} = 0$ and that Lemma 2 implies

$$\|\widetilde{\mathbf{u}}\|_{0,\mathbb{R}^3} \;\leq\; C\,\|\mathbf{u}\|_{0,\Omega} \quad , \quad \|\,\mathbf{curl}\,\widetilde{\mathbf{u}}\|_{0,\mathbb{R}^3} \;\leq\; C\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,. \tag{11}$$

Step 2: Let $B \supseteq \Omega^e$ be a ball such that $1 \leq \mathrm{d}(B) \leq 2$ and define

$$\mathbf{w} := (\mathsf{L}^{\mathrm{curl}}\widetilde{\mathbf{u}})_{|B}.$$

Due to Lemma 3, (LC_1), $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\widetilde{\mathbf{u}}$ in B. Since B has trivial topology, there exists a scalar potential $\psi \in H^1(B)$ with zero average $\int_B \psi\, d\mathbf{x} = 0$ such that

$$\widetilde{\mathbf{u}} = \mathbf{w} + \nabla\,\psi.$$

Lemma 3 together with (11) implies

$$\begin{aligned}
\|\mathbf{w}\|_{0,B} = \|\mathsf{L}^{\mathrm{curl}}\widetilde{\mathbf{u}}\|_{0,B} &\overset{(\mathrm{LC}_3)}{\leq} \|\widetilde{\mathbf{u}}\|_{0,\mathbb{R}^3} \leq C\,\|\mathbf{u}\|_{0,\Omega}\,,\\
\|\,\nabla\,\psi\|_{0,B} = \|(I - \mathsf{L}^{\mathrm{curl}})\widetilde{\mathbf{u}}\|_{0,B} &\overset{(\mathrm{LC}_3)}{\leq} \|\widetilde{\mathbf{u}}\|_{0,\mathbb{R}^3} \leq C\,\|\mathbf{u}\|_{0,\Omega}\,,\\
\|\,\boldsymbol{\nabla}\,\mathbf{w}\|_{0,B} &\overset{(\mathrm{LC}_4)}{\leq} \|\,\mathbf{curl}\,\widetilde{\mathbf{u}}\|_{0,\mathbb{R}^3} \leq C\,\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,,\\
\|\psi\|_{0,B} &\leq C\,\|\,\nabla\,\psi\|_{0,B} \leq C\,\|\mathbf{u}\|_{0,\Omega}\,,
\end{aligned} \tag{12}$$

where the last estimate is due to Poincaré's inequality on the convex ball B [4].

Step 3: Since

$$0 = \mathbf{w} + \nabla\,\psi \quad \text{in } \Upsilon_D\,,$$

we conclude that $\psi_{|\Upsilon_D} \in H^2(\Upsilon_D)$. We define the extension $\widetilde{\psi} := (E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}\psi)_{|B} \in H^2(B)$. Note that if the bulge domain Υ_D has multiple connected components, we have to define $E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}$ by putting together individual extension operators using a partition of unity. From Lemma 1, we obtain

$$\begin{aligned}
\|\widetilde{\psi}\|_{0,B} &\leq C\,\|\psi\|_{0,\Upsilon_D} \leq C\,\|\mathbf{u}\|_{0,\Omega}\,,\\
\|\,\nabla\,\widetilde{\psi}\|_{0,B} &\leq C\,\|\psi\|_{1,\Upsilon_D} \leq C\,\|\,\nabla\,\psi\|_{0,B} \leq C\,\|\mathbf{u}\|_{0,\Omega}\,,\\
\|\,\boldsymbol{\nabla}\,\nabla\,\widetilde{\psi}\|_{0,B} &\leq C\,\big(\|\underbrace{\boldsymbol{\nabla}\,\nabla\,\psi}_{=-\,\boldsymbol{\nabla}\,\mathbf{w}}\|^2_{0,\Upsilon_D} + \|\psi\|^2_{1,\Upsilon_D}\big)^{1/2} \leq C\,\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,,
\end{aligned} \tag{13}$$

where $\boldsymbol{\nabla}\,\nabla$ designates the Hessian.

Step 4: In B, it holds that

$$\widetilde{\mathbf{u}} = \mathbf{w} + \nabla\,\psi = \underbrace{\mathbf{w} + \nabla\,\widetilde{\psi}}_{=:\mathbf{z}\in\mathbf{H}^1} + \nabla(\underbrace{\psi - \widetilde{\psi}}_{=:p\in H^1}).$$

It is easy to see that $p = 0$ in Υ_D and so $p \in H^1_{\Gamma_D}(\Omega)$. Correspondingly, $\nabla p = 0$ and $\widetilde{\mathbf{u}} = 0$ in Υ_D, and so $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. Combining (12) and (13) yields the desired estimates for $\mathbf{z}$ and p. □

Remark 1 An early decomposition of a subspace of $\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)$ into a regular part in $\mathbf{H}^1(\Omega)$ and a singular part in $\nabla H^1(\Omega)$ can be found in [5] and in [6, Proposition 5.1], see also [9, Sect. 3] and references therein. Theorem 2 was proved in [16, Lemma 2.4] for the case of $\Gamma_D = \partial\Omega$ and without the $\mathbf{L}^2$-stability estimate, following [6, Proposition 5.1]. Pasciak and Zhao [31, Lemma 2.2] provided a version for simply connected Ω and the case $\Gamma_D = \partial\Omega$ *with* pure $\mathbf{L}^2$-stability, but p is only constant on each connected component of $\partial\Omega$ (see also Theorem 5 and Remark 3). This result was refined in [26, Thm. 3.1]. For the case $\Gamma_D = \emptyset$, [16, Lemma 2.4] gives a similar decomposition but ∇p must be replaced by an element from $\mathbf{H}(\mathbf{curl}\, 0, \Omega)$ in general. Finally, Theorem 2 without the pure $\mathbf{L}^2$-stability was proved in [23, Thm. 5.2][3].

Remark 2 The constant C in Theorem 2 depends mainly on the stability constants of the extension operators $E^{\mathrm{curl}}_{\Omega^e}$ and $E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}$. If the bulge Υ_D has multiple components $\Upsilon_{D,k}$, the final estimate will depend on the relative distances between $\Upsilon_{D,k}$, $\Upsilon_{D,\ell}$, $k \neq \ell$ and the ratios $\mathrm{d}(\Upsilon_{D,k})/\,\mathrm{d}(\Omega)$ due to the construction of $E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}$ using the partition of unity.

Remark 3 If $\Gamma_D = \partial\Omega$, we can use the trivial extension by zero instead of $E^{\mathrm{curl}}_{\Omega^e}$. In that case, one may modify the construction of the scalar potentials ψ to $\psi_k \in H^2(\Upsilon_{D,k})$ with $\overline{\psi_k}^{\Upsilon_{D,k}} = 0$. Then, one obtains only $p \in H^1(\Omega)$ being constant on each connected component of Γ_D but the improved bound

$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1}\|\mathbf{z}\|_{0,\Omega} \leq C\,\|\,\mathbf{curl}\,\mathbf{u}\|_{0,\Omega}\,.$$

Results on regular decompositions in this special case can be found in [31, 26].

3.2 Regular Potentials for Some Divergence-Free Functions

Let the domain Ω and the Dirichlet boundary part Γ_D be as introduced in Section 2.1 and let Γ_i, $i = 0, \ldots, \beta_2(\Omega)$, denote the connected components of $\partial\Omega$, where $\beta_2(\Omega)$ is the second Betti number of Ω.

We define the space[4]

$$\mathbf{H}_{\Gamma_D}(\mathrm{div}\, 00, \Omega) := \Big\{\mathbf{q} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}\, 0, \Omega)\colon\, \langle \gamma_n \mathbf{q}, 1\rangle_{\Gamma_i} = 0,\; i = 0, \ldots, \beta_2(\Omega)\Big\}\,. \tag{14}$$

[3] This reference contains a typo which is easily identified when inspecting the proof: In general, $\mathbf{z}$ cannot be estimated in terms of $\|\,\mathbf{curl}\,\mathbf{u}\|_{0,\Omega}$ but one must use the full $\mathbf{H}(\mathbf{curl})$ norm.

[4] Alternatively we can define $\mathbf{H}_{\Gamma_D}(\mathrm{div}\, 00, \Omega)$ as the functions in $\mathbf{H}_{\Gamma_D}(\mathrm{div}\, 0, \Omega)$ orthogonal to the *harmonic Dirichlet fields* $\mathbf{H}(\mathrm{div}\, 0, \Omega) \cap \mathbf{H}_0(\mathbf{curl}\, 0, \Omega)$.

Above γ_n denotes the normal trace operator, and the duality pairing is that between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$. If $\Gamma_D = \emptyset$ we simply drop the subscript Γ_D. Obviously,

$$\mathbf{H}_{\Gamma_D}(\mathrm{div}\, 00, \Omega) \subset \mathbf{H}(\mathrm{div}\, 00, \Omega)\,.$$

The next result identifies the above space as the range of the curl operator.

Theorem 3 (Regular potential of range($\mathbf{curl}$)**)**

Let (Ω, Γ_D) be as in Section 2.1 and assume in addition that each connected component $\Upsilon_{D,k}$ of the bulge has vanishing first Betti number, $\beta_1(\Upsilon_{D,k}) = 0$. Then

$$\mathbf{H}_{\Gamma_D}(\mathrm{div}\ 00, \Omega) = \mathbf{curl}\ \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) = \mathbf{curl}\ \mathbf{H}^1_{\Gamma_D}(\Omega)\,,$$

and for each $\mathbf{q} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}\ 00, \Omega)$ there exists $\boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{q}$ such that

$$\mathbf{curl}\ \boldsymbol{\psi} = \mathbf{q} \quad \textit{and} \quad \|\nabla \boldsymbol{\psi}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\, \|\boldsymbol{\psi}\|_{0,\Omega} \le C\, \|\mathbf{q}\|_{0,\Omega}\,,$$

where C depends only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$

The proof is along the lines of [13, Lemma 3.4] and [16, Lemma 2.4] (which follows [6, Proposition 5.1]) and splits into two parts.

Proof (Theorem 3, part 1) Let $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ be arbitrary but fixed, and observe that $\mathbf{curl}\, \mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$. For fixed $i = 0, \ldots, \beta_2(\Omega)$, let $\mu_i \in C^\infty(\overline{\Omega})$ be a smooth cut-off function that is one in a neighborhood of Γ_i and zero on the other components, cf. [13, p. 45]. Then $\mu_i \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and by Gauss' theorem

$$\langle \gamma_n\, \mathbf{curl}\ \mathbf{u}, 1\rangle_{\Gamma_i} = \langle \gamma_n\, \mathbf{curl}(\mu_i \mathbf{u}), 1\rangle_{\partial\Omega} = \int_\Omega \mathrm{div}\, \mathbf{curl}(\mu_i \mathbf{u})\, d\mathbf{x} = 0.$$

This proves that

$$\mathbf{curl}\, \mathbf{H}^1_{\Gamma_D}(\Omega) \subset \mathbf{curl}\, \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \subset \mathbf{H}_{\Gamma_D}(\mathrm{div}\ 00, \Omega).$$

In the second part, we show that $\mathbf{H}_{\Gamma_D}(\mathrm{div}\ 00, \Omega) \subset \mathbf{curl}\, \mathbf{H}^1_{\Gamma_D}(\Omega)$, for which we need two auxiliary results about the divergence-free extension of vector fields and $\mathbf{H}^1$-regular vector potentials.

Lemma 4 (Divergence-free extension)

Let $\mathcal{D}$ be a bounded Lipschitz domain. There exists a bounded extension operator $E^{\mathrm{div},0}_{\mathcal{D}}\colon \mathbf{H}(\mathrm{div}\ 00, \mathcal{D}) \to \mathbf{H}(\mathrm{div}\, 0, \mathbb{R}^3)$ such that, with a constant C depending only on the shape of $\mathcal{D}$, but not on $\mathrm{d}(\Omega)$,

$$\|E^{\mathrm{div},0}_{\mathcal{D}} \mathbf{q}\|_{0,\mathbb{R}^3} \le C \|\mathbf{q}\|_{0,\mathcal{D}} \qquad \forall \mathbf{q} \in \mathbf{H}(\mathrm{div}\ 00, \mathcal{D}),$$

and $E^{\mathrm{div},0}_{\mathcal{D}} \mathbf{q}$ has compact support.

Proof This result is a by-product of the proof of [13, Thm. 3.1]. For each connected component of $\partial\mathcal{D}$, one solves a Neumann problem for the Laplace equation with Neumann data $\gamma_n\mathbf{q}$ and uses the gradient of the solution as the extension. $\square$

Lemma 5 ($\mathbf{H}^1$-regular vector potential on $\mathbb{R}^3$)

For every $\widetilde{\mathbf{q}} \in \mathbf{H}(\operatorname{div} 0, \mathbb{R}^3)$ *with compact support, there is* $\mathbf{w} \in \mathbf{H}^1(\mathbb{R}^3)$*, linearly depending on* $\widetilde{\mathbf{q}}$*, such that* $\mathbf{curl}\,\mathbf{w} = \widetilde{\mathbf{q}}$ *and* $\operatorname{div}\mathbf{w} = 0$*, and* $\|\boldsymbol{\nabla}\mathbf{w}\|_{0,\mathbb{R}^3} \leq \|\widetilde{\mathbf{q}}\|_{0,\mathbb{R}^3}$.

Proof On $\mathbb{R}^3$ it is natural to use Fourier techniques, cf. the proof of [20, Lemma 2.7].

Let $\widehat{\mathbf{q}} = \mathcal{F}\mathbf{q}$ be the Fourier transform of $\widetilde{\mathbf{q}}$ and set

$$\widehat{\mathbf{w}}(\boldsymbol{\xi}) := -(2\pi\mathrm{i})^{-1}|\boldsymbol{\xi}|^{-2}(\boldsymbol{\xi}\times\widehat{\mathbf{q}}(\boldsymbol{\xi})).$$

It turns out that $\widehat{\mathbf{w}}$ is the Fourier transform of a function $\mathbf{w} \in \mathbf{H}^1(\mathbb{R}^3)$, cf. [13, p. 46], and it is easily seen that $\mathbf{curl}\,\mathbf{w} = \widetilde{\mathbf{q}}$ and $\operatorname{div}\mathbf{w} = 0$. By construction, $\mathbf{w}$ depends linearly on $\widetilde{\mathbf{q}}$. Plancherel's theorem allows for the estimate

$$\|\boldsymbol{\nabla}\mathbf{w}\|^2_{0,\mathbb{R}^3} = \sum_{k=1}^{3}\|2\pi\mathrm{i}\xi_k\widehat{\mathbf{w}}(\boldsymbol{\xi})\|^2_{0,\mathbb{R}^3} \leq \|\widehat{\mathbf{q}}\|^2_{0,\mathbb{R}^3} = \|\widetilde{\mathbf{q}}\|^2_{0,\mathbb{R}^3}\,,$$

which finishes the proof. $\square$

Proof (Theorem 3, part 2) Let $\mathbf{q} \in \mathbf{H}_{\Gamma_D}(\operatorname{div}\,00, \Omega)$ be arbitrary but fixed.

Step 1: Using Lemma 4 we extend $\mathbf{q}$ by zero from Ω to Ω^e, resulting in $\mathbf{q}^e \in \mathbf{H}(\operatorname{div}\,00, \Omega^e)$. We define $\widetilde{\mathbf{q}} := E^{\operatorname{div},0}_{\Omega^e}\mathbf{q}^e$ and the estimate of Lemma 4 shows that $\widetilde{\mathbf{q}} \in \mathbf{H}(\operatorname{div} 0, \mathbb{R}^3)$ with $\widetilde{\mathbf{q}}_{|\Omega} = \mathbf{q}$ and $\|\widetilde{\mathbf{q}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{q}\|_{0,\Omega}$.

Step 2: Appealing to Lemma 5 we find $\mathbf{w} \in \mathbf{H}^1(\mathbb{R}^3)$, depending linearly on $\widetilde{\mathbf{q}}$, such that $\mathbf{curl}\,\mathbf{w} = \widetilde{\mathbf{q}}$ and $\|\boldsymbol{\nabla}\mathbf{w}\|_{0,\mathbb{R}^3} \leq C\|\widetilde{\mathbf{q}}\|_{0,\Omega}$.

Step 3: Let B be the smallest ball containing Ω^e such that $\mathrm{d}(B) = 2\,\mathrm{d}(\Omega)$, see Sect. 2.1. We define $\mathbf{w}_1 \in \mathbf{H}^1(B)$ by

$$\mathbf{w}_1 := \mathbf{w}_{|B} - \overline{\mathbf{w}}^B\,,$$

where $\overline{\mathbf{w}}^B$ is the constant vector field that agrees with the average of $\mathbf{w}$ over B. Thus, $\overline{\mathbf{w}_1}^B = 0$ and still $\mathbf{curl}\,\mathbf{w}_1 = \widetilde{\mathbf{q}}$ in B.

Step 4: Recall that $\widetilde{\mathbf{q}}_{|\Upsilon_D} = 0$. Hence,

$$\mathbf{curl}\,\mathbf{w}_1 = 0 \qquad \text{in } \Upsilon_D.$$

Since we have assumed that each connected component $\Upsilon_{D,k}$ of Υ_D has vanishing first Betti number, due to (8), there exist scalar potentials $\varphi_k \in H^1(\Upsilon_{D,k})$ with $\overline{\varphi_k}^{\Upsilon_{D,k}} = 0$ such that

$$\mathbf{w}_1 = \nabla\varphi_k \qquad \text{in } \Upsilon_{D,k} \qquad \forall k = 1,\ldots,N\,,$$

where N is the number of connected components of Υ_D. This shows that $\varphi_k \in H^2(\Upsilon_{D,k})$. Using an adaptation $E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}$ of the Stein extension operator from

Lemma 1, we obtain a function $\widetilde{\varphi} \in H^2(\mathbb{R}^3)$ such that $\widetilde{\varphi}_{|\Upsilon_{D,k}} = \varphi_k$. Moreover, thanks to the stability of the extension,

$$\| \nabla \widetilde{\varphi} \|^2_{0,\mathbb{R}^3} \leq C \sum_{k=1}^{N} \| \underbrace{\nabla \varphi_k}_{=\mathbf{w}_1} \|^2_{0,\Upsilon_{D,k}} , \tag{15}$$

$$\| \boldsymbol{\nabla} \nabla \widetilde{\varphi} \|^2_{0,\mathbb{R}^3} \leq C \sum_{k=1}^{N} \mathrm{d}(\Upsilon_{D,k})^2 \| \underbrace{\nabla \varphi_k}_{=\mathbf{w}_1} \|^2_{0,\Upsilon_{D,k}} + \| \underbrace{\boldsymbol{\nabla} \nabla \varphi_k}_{=\boldsymbol{\nabla}\, \mathbf{w}_1} \|^2_{0,\Upsilon_{D,k}} . \tag{16}$$

Note that the L^2 norm of φ_k on $\Upsilon_{D,k}$ has been estimated in terms of its gradient by Poincaré's inequality, making use of $\overline{\varphi_k}^{\Upsilon_{D,k}} = 0$.

Step 5: We obtain

$$\mathbf{curl}(\mathbf{w}_1 - \nabla \widetilde{\varphi}) = \widetilde{\mathbf{q}} \qquad \text{in } B$$

and we define $\boldsymbol{\psi} := (\mathbf{w}_1 - \nabla \widetilde{\varphi})_{|\Omega} \in \mathbf{H}^1(\Omega)$. Since $\mathbf{w}_1 - \nabla \widetilde{\varphi} = 0$ on Υ_D, it follows that $\boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. Moreover, (15) yields

$$\begin{aligned} \|\boldsymbol{\psi}\|_{0,\Omega} &\leq \|\mathbf{w}_1\|_{0,\Omega} + \| \nabla \widetilde{\varphi} \|_{0,\Omega} \leq C \, \|\mathbf{w}_1\|_{0,B} \\ &\leq C \ \mathrm{d}(B) \, \| \boldsymbol{\nabla}\, \mathbf{w}_1 \|_{0,B} \leq C \ \mathrm{d}(\Omega) \|\mathbf{q}\|_{0,\Omega} , \end{aligned}$$

where we have used Poincaré's inequality and the fact that $\overline{\mathbf{w}_1}^B = 0$. Finally, (16) and the same arguments yield

$$\begin{aligned} \| \boldsymbol{\nabla} \boldsymbol{\psi} \|_{0,\Omega} \leq \| \boldsymbol{\nabla}\, \mathbf{w}_1 \|_{0,\Omega} + \| \boldsymbol{\nabla} \nabla \widetilde{\varphi} \|_{0,\Omega} &\leq C \big[\mathrm{d}(\Omega) \, \|\mathbf{w}_1\|_{0,B} + \| \boldsymbol{\nabla}\, \mathbf{w}_1 \|_{0,B} \big] \\ &\leq C \, \| \boldsymbol{\nabla}\, \mathbf{w}_1 \|_{0,B} \leq C \ \mathrm{d}(\Omega) \|\mathbf{q}\|_{0,\Omega} . \end{aligned}$$

This concludes the proof of Theorem 3. □

Remark 4 For the case that $\Gamma_D = \emptyset$, we reproduce the classical result

$$\mathbf{H}(\mathrm{div}\, 00, \Omega) = \mathbf{curl}\ \mathbf{H}(\mathbf{curl}, \Omega) = \mathbf{curl}\ \mathbf{H}^1(\Omega),$$

see [13, Thm. 3.4]. In that case, Step 4 of the proof can be left out and $\boldsymbol{\psi} = \mathbf{w}_1$ which is why $\mathrm{div}\, \boldsymbol{\psi} = 0$ in Ω. This property, however, is lost in the general case.

Remark 5 The constant C in Theorem 4 depends essentially on the stability constants of the divergence-free extension operator $E^{\mathrm{div},0}_{\Omega^e}$ and the (adapted) Stein extension operator $E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}$. For $\Gamma_D = \partial\Omega$, one can replace $E^{\mathrm{div},0}_{\Omega^e}$ by the trivial extension by zero.

3.3 Rotation-Bounded Regular Decomposition of $\mathbf{H}(\mathrm{curl})$

We can now formulate another *new* variety of regular decompositions, for which the $\mathbf{H}^1$-component will vanish for curl-free fields.

Theorem 4 (Rotation-Bounded Regular Decomposition of $\mathbf{H}(\mathbf{curl})$ (I))
Let (Ω, Γ_D) be as in Section 2.1 and assume, in addition, that each connected component $\Upsilon_{D,k}$ of the bulge has vanishing first Betti number, $\beta_1(\Upsilon_{D,k}) = 0$. Then, for each $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ there exist $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and a curl-free vector field $\mathbf{h} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}\, 0, \Omega)$, depending linearly on $\mathbf{u}$ such that

$$\begin{aligned} \mathbf{u} &= \mathbf{z} + \mathbf{h}, \\ \|\mathbf{h}\|_{0,\Omega} &\leq \|\mathbf{u}\|_{0,\Omega} + C\ \mathrm{d}(\Omega)\, \|\ \mathbf{curl}\ \mathbf{u}\|_{0,\Omega}, \\ \|\ \boldsymbol{\nabla}\, \mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{z}\|_{0,\Omega} &\leq C\, \|\ \mathbf{curl}\ \mathbf{u}\|_{0,\Omega}\,, \end{aligned}$$

where C depends only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

Proof Let $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ be arbitrary but fixed. Due to Theorem 3, $\mathbf{curl}\ \mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}\ 00, \Omega)$ and there exists a regular potential $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{curl}\ \mathbf{u}$ such that

$$\begin{aligned} \mathbf{curl}\, \mathbf{z} &= \mathbf{curl}\, \mathbf{u}, \\ \|\ \boldsymbol{\nabla}\, \mathbf{z}\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1}\|\mathbf{z}\|_{0,\Omega} &\leq C\, \|\ \mathbf{curl}\, \mathbf{u}\|_{0,\Omega}\,. \end{aligned}$$

Since $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega) \subset \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ we conclude that $\mathbf{h} := \mathbf{u} - \mathbf{z} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}\, 0, \Omega)$ and that $\mathbf{u} = \mathbf{z} + \mathbf{h}$ with

$$\|\mathbf{h}\|_{0,\Omega} \leq \|\mathbf{u} - \mathbf{z}\|_{0,\Omega} \leq \|\mathbf{u}\|_{0,\Omega} + C\, \mathrm{d}(\Omega)\, \|\ \mathbf{curl}\, \mathbf{u}\|_{0,\Omega}\,.$$

Remark 6 The constant C in Theorem 4 depends essentially on the stability constants of the divergence-free extension operator $E^{\mathrm{div},0}_{\Omega^e}$ and the (adapted) Stein extension operator $E^{\nabla,\mathrm{Stein}}_{\Upsilon_D}$.

Another *stronger* version of the rotation-bounded regular decomposition of $\mathbf{H}(\mathbf{curl})$ gets rid of the assumptions on the topology of the Dirichlet boundary and has improved stability properties (though with less explicit constants).

Theorem 5 (Rotation-Bounded Regular Decomposition of $\mathbf{H}(\mathbf{curl})$ (II))
Let (Ω, Γ_D) be as in Sect. 2.1. Then for each $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ there exist $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and a curl-free $\mathbf{h} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}\, 0, \Omega)$ depending linearly on $\mathbf{u}$ such that

$$\begin{aligned} \mathbf{u} &= \mathbf{z} + \mathbf{h}\,, \\ \|\mathbf{z}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Omega} &\leq C\, \|\mathbf{u}\|_{0,\Omega}\,, \\ \|\ \boldsymbol{\nabla}\, \mathbf{z}\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1}\|\mathbf{z}\|_{0,\Omega} &\leq C\, \|\ \mathbf{curl}\ \mathbf{u}\|_{0,\Omega}\,, \end{aligned}$$

where C depends only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

Remark 7 For the case $\Gamma_D = \partial\Omega$ the result of the theorem is already proved by Remark 3 since we obtain $\mathbf{u} = \mathbf{z} + \nabla p$ with $\nabla p \in \nabla H^1_{0,\mathrm{const}}(\Omega) = \mathbf{H}_0(\mathbf{curl}, \Omega)$.

For the proof of Theorem 5 we need a Friedrichs type inequality.

Lemma 6 (Friedrichs-Type inequality in $\mathbf{H}(\mathbf{curl})$)
Let (Ω, Γ_D) be as introduced in Section 2.1 and write $\Pi_{\mathbf{curl}\,0,\Gamma_D}$ for the $\mathbf{L}^2$-orthogonal projector from $\mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ onto $\mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega)$. Then

$$\|\mathbf{u} - \Pi_{\mathbf{curl}\,0,\Gamma_D}\mathbf{u}\|_{0,\Omega} \le C\ \mathrm{d}(\Omega)\,\|\,\mathbf{curl}\,\mathbf{u}\|_{0,\Omega} \qquad \forall \mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega),$$

with a constant C depending only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

Note that if each connected component of the bulge Υ_D has vanishing first Betti number, then Lemma 6 can be derived directly from Theorem 4 using that $\Pi_{\mathbf{curl}\,0,\Gamma_D}$ is $\mathbf{L}^2$-orthogonal:

$$\|\mathbf{u} - \Pi_{\mathbf{curl}\,0,\Gamma_D}\mathbf{u}\|_{0,\Omega} \le \|\mathbf{u} - \mathbf{h}\|_{0,\Omega} = \|\mathbf{z}\|_{0,\Omega} \le C\ \mathrm{d}(\Omega)\,\|\,\mathbf{curl}\,\mathbf{u}\|_{0,\Omega}\,.$$

In general, one obtains Lemma 6 from the more general Gaffney inequality

$$\|\mathbf{u}\|_{0,\Omega} \le C\ \mathrm{d}(\Omega)\Big(\int_\Omega |\,\mathbf{curl}\,\mathbf{u}|^2 + |\,\mathrm{div}\,\mathbf{u}|^2\,dx\Big)^{1/2}, \tag{17}$$

for all $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \cap \mathbf{H}_{\Gamma_N}(\mathrm{div}, \Omega)$ that are $\mathbf{L}^2$-orthogonal to the relative co-homology space $\mathcal{H}_{\Gamma_D}(\Omega) := \mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega) \cap \mathbf{H}_{\Gamma_N}(\mathrm{div}\,0, \Omega)$. The Gaffney inequality, in turn, is a consequence of the compact embedding of $\mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \cap \mathbf{H}_{\Gamma_N}(\mathrm{div}, \Omega)$ in $\mathbf{L}^2(\Omega)$, which holds for our assumptions on (Ω, Γ_D) from Section 2.1, see [3] and references therein.

Proof (Theorem 5) Let $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ be arbitrary but fixed and set

$$\mathbf{u}_0 := \Pi_{\mathbf{curl}\,0,\Gamma_D}\mathbf{u} \quad , \quad \mathbf{u}_1 := \mathbf{u} - \mathbf{u}_0\,,$$

such that

$$\|\mathbf{u}_0\|_{0,\Omega} \le \|\mathbf{u}\|_{0,\Omega}\,, \qquad \|\mathbf{u}_1\|_{0,\Omega} \le \|\mathbf{u}\|_{0,\Omega}\,, \tag{18}$$

and by Lemma 6,

$$\|\mathbf{u}_1\|_{0,\Omega} \le C\ \mathrm{d}(\Omega)\,\|\,\mathbf{curl}\,\mathbf{u}\|_{0,\Omega}\,. \tag{19}$$

As a next step, we apply Theorem 2 to $\mathbf{u}_1$ to obtain

$$\mathbf{u}_1 = \mathbf{z} + \nabla p \quad , \quad \mathbf{u} = \mathbf{z} + \underbrace{\nabla p + \mathbf{u}_0}_{=:\mathbf{h}}, \tag{20}$$

with $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and $p \in H^1_{\Gamma_D}(\Omega)$ and

$$\|\mathbf{z}\|_{0,\Omega} + \|\nabla p\|_{0,\Omega} \le C\,\|\mathbf{u}_1\|_{0,\Omega}\,, \tag{21}$$

$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} \le C\big(\|\,\mathbf{curl}\,\mathbf{u}_1\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1}\|\mathbf{u}_1\|_{0,\Omega}\big)\,. \tag{22}$$

Using that $\mathbf{curl}\,\mathbf{u}_1 = \mathbf{curl}\,\mathbf{u}$ and combining (21)–(22) with (18)–(19) yields the desired result. □

Remark 8 We would like to emphasize that both in Theorem 2 and Theorem 5, the domain Ω may be non-convex, non-smooth, and may have non-trivial topology: It may have handles and its boundary may have multiple components. Also the Dirichlet boundary Γ_D may have multiple components, each of which with non-trivial topology. Moreover, we have the pure $\mathbf{L}^2(\Omega)$-stability in both theorems. In this sense, the results of Theorem 2 and Theorem 5 are superior to those found, e.g., in [9, Thm 3.4], [22] or the more recent ones in [10, Thm. 2.3], [24].

Remark 9 If either $\Gamma_D = \partial\Omega$ or if each component of Υ_D has vanishing first Betti number, the constant C in Theorem 5 can be tracked back to extension constants. In the general case, we are not aware of any explicit estimates of the Friedrichs constant.

Remark 10 If Ω has vanishing first Betti number, $\beta_1(\Omega) = 0$, then $\mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega) = \nabla H^1_{\Gamma_D,\mathrm{const}}(\Omega)$. Hence, we can split each $\mathbf{u} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ into $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and ∇p with $p \in H^1(\Omega)$ being constant on each connected component of Γ_D. If Γ_D is connected, then $p \in H^1_{\Gamma_D}(\Omega)$. Summarizing, if Ω has no handles and if Γ_D is connected, then we have the combined features of Theorem 2 and Theorem 5.

Finally, we mention that the regular decomposition theorems spawn projection operators that play a fundamental role in the analysis of weak formulations of Maxwell's equations in frequency domain [16, Sect. 5].

Corollary 1 *Let (Ω, Γ_D) be as in Sect. 2.1. Then there exist continuous projection operators* $\mathsf{R}\colon \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \to \mathbf{H}^1_{\Gamma_D}(\Omega)$ *and* $\mathsf{N}\colon \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega)$ *such that* $\mathsf{R} + \mathsf{N} = \mathsf{id}$ *and*

$$\|\mathsf{R}\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|\mathsf{N}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \le C\,\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \qquad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega),$$

where C is a constant independent of $\mathbf{v}$. *Moreover,* $\mathsf{F}\colon \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ *defined by* $\mathsf{F}\mathbf{v} := \mathsf{R}\mathbf{v} - \mathsf{N}\mathbf{v}$ *is an isomorphism.*

Proof We define $\mathsf{R}\mathbf{u} := \mathbf{z}$ and $\mathsf{N}\mathbf{u} := \mathbf{h}$ with $\mathbf{z}$ and $\mathbf{h}$ from Theorem 5. Thanks to the estimates R and N are bounded linear operators. As a consequence of the last estimate from Theorem 5,

$$\mathbf{curl}\ \mathbf{u} = 0 \implies \mathbf{z} = 0,$$

which is why

$$\mathsf{N}\mathbf{v} = \mathbf{v} \qquad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega).$$

This in turn implies that $\mathrm{range}(\mathsf{N}) = \mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega)$, which is a *closed* subspace of $\mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$. It also implies that $\mathsf{N}^2 = \mathsf{N}$ and so N is a *projection* (but different from the $\mathbf{L}^2$-orthogonal projection onto $\mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega)$). Hence, the operator $\mathsf{R} = \mathsf{id} - \mathsf{N}$ is a projection too and $\mathrm{range}(\mathsf{R}) = \ker(\mathsf{N})$ is closed in $\mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$. Finally, $\mathsf{F} = \mathsf{R} - \mathsf{N} = \mathsf{id} - 2\mathsf{N}$. Since N is a projection, one can easily see that $\mathsf{F}^2 = \mathsf{id}$, so F is an isomorphism. □

Remark 11 The $\mathbf{L}^2$-estimates from Theorem 4 then show that the corresponding operator R can be extended to a continuous operator mapping from $\mathbf{L}^2(\Omega)$ to $\mathbf{L}^2(\Omega)$.

4 Regular Decompositions and Potentials Related to $\mathrm{H}(\mathrm{div})$

The developments of this section are largely parallel to those of Section 3 with some new aspects concerning extensions and topological considerations.

4.1 Rotation-Based Regular Decomposition of $\mathrm{H}(\mathrm{div})$

The following theorem is the $\mathbf{H}(\mathrm{div})$-counterpart of Theorem 2.

Theorem 6 (Rotation-Based Regular Decomposition of $\mathbf{H}(\mathrm{div})$)
Let (Ω, Γ_D) satisfy the assumptions made in Section 2.1. Then for each $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$ there exist $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and $\mathbf{q} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{v}$ such that

$$\mathbf{v} = \mathbf{z} + \mathbf{curl}\,\mathbf{q},$$

$$\|\mathbf{z}\|_{0,\Omega} + \|\,\mathbf{curl}\,\mathbf{q}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{q}\|_{0,\Omega} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,,$$

$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\,\boldsymbol{\nabla}\,\mathbf{q}\|_{0,\Omega} \leq C\big(\|\,\mathbf{curl}\;\mathbf{v}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{v}\|_{0,\Omega}\big)\,,$$

with constant C depending only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

For the proof we need a counterpart of the Stein extension Lemma 1 in $\mathbf{H}(\mathbf{curl})$, which was shown in [19]: Let $\mathbf{H}^k(\mathbf{curl}, \mathcal{D}) := \{\mathbf{v} \in \mathbf{H}^k(\mathcal{D})\colon\ \mathbf{curl}\,\mathbf{v} \in \mathbf{H}^k(\mathcal{D})\}$, be equipped with norm $\|\mathbf{v}\|_{\mathbf{H}^k(\mathbf{curl},\mathcal{D})} := (\|\mathbf{v}\|^2_{\mathbf{H}^k(\mathcal{D})} + \|\,\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{H}^k(\mathcal{D})})^{1/2}$.

Lemma 7 *Let $\mathcal{D}$ be a bounded Lipschitz domain with $\mathrm{d}(\mathcal{D}) = 1$. Then there exists a bounded linear operator extension operator $E^{\mathbf{curl},Stein}_{\mathcal{D}}\colon \mathbf{H}(\mathbf{curl}, \mathcal{D}) \to \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ such that, for all integers $k \geq 1$, with constants depending only on $\mathcal{D}$ and k,*

$$\|E^{\mathbf{curl},Stein}_{\mathcal{D}}\mathbf{v}\|_{\mathbf{H}^k(\mathbf{curl},\mathbb{R})} \leq C\|\mathbf{v}\|_{\mathbf{H}^k(\mathbf{curl},\mathcal{D})} \qquad \forall \mathbf{v} \in \mathbf{H}^k(\mathbf{curl}, \mathcal{D}). \tag{23}$$

We also need a counterpart to Lemma 2: an extension operator in $\mathbf{H}(\mathrm{div})$ that is continuous in $\mathbf{L}^2$ and $\mathbf{H}(\mathrm{div})$. The proof essentially follows that of [20, Lemma 2.6] and is left to the reader.

Lemma 8 (Extension Operator for $\mathbf{H}(\mathrm{div})$)

Let $\mathcal{D}$ be a bounded Lipschitz domain with $\mathrm{d}(\mathcal{D}) = 1$. *Then there exists a bounded linear extension operator* $E_{\mathcal{D}}^{\mathrm{div}} \colon \mathbf{L}^2(\mathcal{D}) \to \mathbf{L}^2(\mathbb{R}^3)$ *such that, with constants depending only on* $\mathcal{D}$,

$$\begin{aligned} \|E_{\mathcal{D}}^{\mathrm{div}}\mathbf{v}\|_{0,\mathbb{R}^3} &\leq C\|\mathbf{v}\|_{0,\mathcal{D}} && \forall \mathbf{v} \in \mathbf{L}^2(\mathcal{D})\,, \\ \|E_{\mathcal{D}}^{\mathrm{div}}\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\mathbb{R}^3)} &\leq C\|\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\mathcal{D})} && \forall \mathbf{v} \in \mathbf{H}(\mathrm{div},\mathcal{D})\,, \end{aligned}$$

and $E_{\mathcal{D}}^{\mathrm{div}}\mathbf{v}$ *has compact support.*

Finally, we need a counterpart to the projection introduced in Lemma 3. The proof of the following lemma is very close to that of [20, Lemma 2.7] and we skip it.

Lemma 9 (Fourier-based Projection for $\mathbf{H}(\mathrm{div})$)

There exists a bounded linear operator $\mathsf{L}^{\mathrm{div}} \colon \mathbf{H}(\mathrm{div},\mathbb{R}^3) \to \mathbf{H}^1(\mathbb{R}^3)$ *such that for all* $\mathbf{v} \in \mathbf{H}(\mathrm{div},\mathbb{R}^3)$

(LD$_1$) $\quad \mathrm{div}\,\mathsf{L}^{\mathrm{div}}\mathbf{v} = \mathrm{div}\,\mathbf{v}$,

(LD$_2$) $\quad \mathbf{curl}\,\mathsf{L}^{\mathrm{div}}\mathbf{v} = 0$,

(LD$_3$) $\quad \|\mathsf{L}^{\mathrm{div}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{v}\|_{0,\mathbb{R}^3}$ *and* $\|(\mathsf{id} - \mathsf{L}^{\mathrm{div}})\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{v}\|_{0,\mathbb{R}^3}$,

(LD$_4$) $\quad \|\boldsymbol{\nabla}\,\mathsf{L}^{\mathrm{div}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\,\mathrm{div}\,\mathbf{v}\|_{0,\mathbb{R}^3}$,

(LD$_5$) $\quad (\mathsf{L}^{\mathrm{div}})^2\mathbf{v} = \mathsf{L}^{\mathrm{div}}\mathbf{v}$, *i.e.,* $\mathsf{L}^{\mathrm{div}}$ *is a projection.*

Proof (Theorem 6) Without loss of generality, we may assume that $\mathrm{d}(\Omega) = 1$. Let $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div},\Omega)$ be arbitrary but fixed.

Step 1: We extend $\mathbf{v}$ by zero to a function in $\mathbf{H}(\mathrm{div},\Omega^e)$, where Ω^e is the extended domain as defined in Section 2.1, and then to $\widetilde{\mathbf{v}} \in \mathbf{H}(\mathrm{div},\mathbb{R}^3)$ using $E_{\Omega^e}^{\mathrm{div}}$. We observe that $\widetilde{\mathbf{v}}_{|\Upsilon_D} = 0$ and that Lemma 8 implies

$$\|\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{v}\|_{0,\Omega} \quad , \quad \|\,\mathrm{div}\,\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega)}\,. \tag{24}$$

Step 2: Let $B \supset \Omega^e$ be a ball such that $1 \leq \mathrm{d}(B) \leq 2$ and define

$$\mathbf{w} := (\mathsf{L}^{\mathrm{div}}\widetilde{\mathbf{v}})_{|B}\,.$$

Due to Lemma 9, $\mathrm{div}\,\mathbf{w} = \mathrm{div}\,\widetilde{\mathbf{v}}$ in B. Since B has trivial topology, we find from (9) that $\widetilde{\mathbf{v}}_{|B} - \mathbf{w} \in \mathbf{H}(\mathrm{div}\,0,B) = \mathbf{H}(\mathrm{div}\,00,B)$, and conclude from Theorem 3 that there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(B)$ such that

$$\widetilde{\mathbf{v}} = \mathbf{w} + \mathbf{curl}\,\boldsymbol{\psi} \qquad \text{in } B, \tag{25}$$

and (using Lemma 9)

$$\begin{aligned} \|\mathbf{w}\|_{0,B} &\leq C\|\widetilde{\mathbf{v}}\|_{0,B} \leq C\|\mathbf{v}\|_{0,\Omega}\,, \\ \|\boldsymbol{\nabla}\,\mathbf{w}\|_{0,B} &\leq C\|\,\mathrm{div}\,\widetilde{\mathbf{v}}\|_{0,B} \leq C\|\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega)}\,, \\ \|\boldsymbol{\psi}\|_{\mathbf{H}^1(B)} &\leq C\|\widetilde{\mathbf{v}} - \mathbf{w}\|_{0,B} \leq C\|\widetilde{\mathbf{v}}\|_{0,B} \leq C\|\mathbf{v}\|_{0,\Omega}\,. \end{aligned} \tag{26}$$

Step 3: Since

$$0 = \mathbf{w} + \mathbf{curl}\,\boldsymbol{\psi} \qquad \text{in } \Upsilon_D\,,$$

we conclude that $\boldsymbol{\psi} \in \mathbf{H}^1(\mathbf{curl}, \Upsilon_D)$. We define $\widetilde{\boldsymbol{\psi}} := (E^{\mathbf{curl},\text{Stein}}_{\Upsilon_D}\boldsymbol{\psi})_{|B} \in \mathbf{H}^1(\mathbf{curl}, B)$. Note that if the bulge domain Υ_D has multiple components, we have to define $E^{\mathbf{curl},\text{Stein}}_{\Upsilon_D}$ by blending individual extension operators using a partition of unity. From Lemma 7, we obtain

$$\begin{aligned}
\|\widetilde{\boldsymbol{\psi}}\|_{\mathbf{H}(\mathbf{curl},B)} &\le C\|\boldsymbol{\psi}\|_{\mathbf{H}(\mathbf{curl},\Upsilon_D)} \le C\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Upsilon_D)} \le C\|\mathbf{v}\|_{0,\Omega}\,,\\
\|\widetilde{\boldsymbol{\psi}}\|_{\mathbf{H}^1(\mathbf{curl},B)} &\le C\|\boldsymbol{\psi}\|_{\mathbf{H}^1(\mathbf{curl},\Upsilon_D)}\\
&\le C(\|\underbrace{\boldsymbol{\nabla}\,\mathbf{curl}\,\boldsymbol{\psi}}_{=-\boldsymbol{\nabla}\,\mathbf{w}}\|_{0,\Upsilon_D} + \|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Upsilon_D)}) \le C\|\mathbf{v}\|_{\mathbf{H}(\text{div},\Omega)}\,.
\end{aligned} \tag{27}$$

Step 4: In the ball B,

$$\widetilde{\mathbf{v}} = \mathbf{w} + \mathbf{curl}\,\boldsymbol{\psi} = \underbrace{\mathbf{w} + \mathbf{curl}\,\widetilde{\boldsymbol{\psi}}}_{=:\mathbf{z}\in\mathbf{H}^1(B)} + \mathbf{curl}(\underbrace{\boldsymbol{\psi} - \widetilde{\boldsymbol{\psi}}}_{=:\mathbf{q}\in\mathbf{H}^1(B)})\,.$$

It is easy to see that $\mathbf{q} = 0$ in Υ_D and so $\mathbf{q}_{|\Omega} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. By the same argument, $\mathbf{curl}\,\mathbf{q} = 0$ and $\widetilde{\mathbf{v}} = 0$ in Υ_D and so $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. Combining (26) and (27) yields the desired estimates for $\mathbf{z}$ and $\mathbf{q}$. □

4.2 Regular Potential with Prescribed Divergence

The next result carries Theorem 3 over to $\mathbf{H}(\text{div})$.

Theorem 7 (Regular Potentials for the Image Space of div**)**
Let (Ω, Γ_D) be as in Section 2.1 and, in addition, assume that each connected component $\Upsilon_{D,k}$ of the bulge has a connected boundary, i.e., $\beta_2(\Upsilon_{D,k}) = 0$. Then

$$L^2(\Omega) = \text{div}\,\mathbf{H}_{\Gamma_D}(\text{div}, \Omega) = \text{div}\,\mathbf{H}^1_{\Gamma_D}(\Omega).$$

Moreover, for each $v \in L^2(\Omega)$ there exists $\mathbf{q} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ depending linearly on v such that, with a constant C depending on Ω and Γ_D but not on $\text{d}(\Omega)$,

$$\text{div}\,\mathbf{q} = v \quad \textit{and} \quad \|\boldsymbol{\nabla}\,\mathbf{q}\|_{0,\Omega} + \frac{1}{\text{d}(\Omega)}\|\mathbf{q}\|_{0,\Omega} \le C\,\|v\|_{0,\Omega}\,.$$

Note that the assumption on Υ_D rules out the case $\Gamma_D = \partial\Omega$ even for a domain with trivial topology, since at least one component of the bulge would have a boundary with multiple components. Necessarily so, because $\text{div}\,\mathbf{H}_{\partial\Omega}(\text{div}, \Omega)$ contains only functions with vanishing mean.

Proof From Gauss' theorem, one can easily deduce

$$\operatorname{div} \mathbf{H}^1_{\Gamma_D}(\Omega) \subseteq \operatorname{div} \mathbf{H}_{\Gamma_D}(\operatorname{div}, \Omega) \subset L^2(\Omega), \tag{28}$$

such that it remains to show $L^2(\Omega) \subseteq \operatorname{div} \mathbf{H}^1_{\Gamma_D}(\Omega)$ and the stability estimate for the potential. Let $v \in L^2(\Omega)$ be arbitrary but fixed and assume without loss of generality that $\mathrm{d}(\Omega) = 1$.

Step 1: We extend v by zero to a function $\widetilde{v} \in L^2(\mathbb{R}^3)$.

Step 2: We denote by $\widehat{v} := \mathcal{F}\widetilde{v}$ the Fourier transform of $\widetilde{v}$ and define

$$\widehat{w}(\boldsymbol{\xi}) := (2\pi \mathrm{i})^{-1} |\boldsymbol{\xi}|^{-2} \boldsymbol{\xi} \widehat{v}(\boldsymbol{\xi}). \tag{29}$$

One can show that $\mathbf{w} := \mathcal{F}^{-1}\widehat{\mathbf{w}} \in \mathbf{H}^1(\mathbb{R}^3)$ and that $\mathbf{curl}\, \mathbf{w} = 0$ and $\operatorname{div} \mathbf{w} = \widetilde{v}$.[5] By construction $\mathbf{w}$ depends linearly on $\widetilde{v}$. Due to Plancherel's theorem,

$$\| \boldsymbol{\nabla} \mathbf{w} \|^2_{0,\mathbb{R}^3} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} |2\pi \mathrm{i} \xi_k \widehat{\mathbf{w}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le \|\widehat{v}\|^2_{0,\mathbb{R}^3} = \|v\|^2_{0,\Omega} \,. \tag{30}$$

Step 3: Let $B \supset \Omega^e$ a ball with $1 \le \mathrm{d}(B) \le 2$. We define $\mathbf{w}_1 \in \mathbf{H}^1(B)$ by

$$\mathbf{w}_1 := \mathbf{w}_{|B} - \overline{\mathbf{w}}^B, \tag{31}$$

where $\overline{\mathbf{w}}^B$ is the component-wise mean value over B, such that

$$\operatorname{div} \mathbf{w}_1 = \widetilde{v} \qquad \text{in } B\,. \tag{32}$$

Step 4: Recall that $\widetilde{v}$ is zero outside of Ω. In particular,

$$\operatorname{div} \mathbf{w}_1 = 0 \qquad \text{in } \Upsilon_{D,k} \qquad \forall k = 1, \ldots, N. \tag{33}$$

Since each connected component $\Upsilon_{D,k}$ of Υ_D is assumed to have a connected boundary, we obtain $\mathbf{H}(\operatorname{div} 0, \Upsilon_{D,k}) = \mathbf{H}(\operatorname{div} 00, \Upsilon_{D,k})$ and so Theorem 3 guarantees the existence of a vector potential $\boldsymbol{\psi}_k \in \mathbf{H}^1(\Upsilon_{D,k})$ depending linearly on $\mathbf{w}_1$ such that

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{\psi}_k &= \mathbf{w}_1 \qquad \text{in } \Upsilon_{D,k}, \\ (\| \boldsymbol{\nabla} \boldsymbol{\psi}_k \|^2_{0,\Upsilon_{D,k}} + \|\boldsymbol{\psi}_k\|^2_{0,\Upsilon_{D,k}})^{1/2} &\le C \, \|\mathbf{w}_1\|_{0,\Upsilon_{D,k}} \,. \end{aligned} \tag{34}$$

This shows that $\boldsymbol{\psi}_k \in \mathbf{H}^1(\mathbf{curl}, \Upsilon_{D,k})$. We extend $\{\boldsymbol{\psi}_k\}_{k=1}^N$ to a function $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^1(\mathbf{curl}, \mathbb{R}^3)$ by patching together the extensions $E^{\mathbf{curl},\text{Stein}}_{\Upsilon_{D,k}} \boldsymbol{\psi}_k$ by a partition of unity such that (using Lemma 7, (34), (31), and the Poincaré inequality)

[5] Alternatively, one can set $\mathbf{w} := \nabla \Delta^{-1} \widetilde{v}$, where Δ^{-1} corresponds to solving the variational Laplace problem with right-hand side in $L^2(\mathbb{R}^3)$ and the energy space $L^2_{\text{loc}}(\Omega)$ with gradient in $\mathbf{L}^2(\mathbb{R}^3)$. By a standard regularity argument [28], the gradient of the solution is even in $\mathbf{H}^1(\mathbb{R}^3)$.

$$
\begin{aligned}
\|\widetilde{\boldsymbol{\psi}}\|^2_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3)} &\leq C \sum_{k=1}^{N} \left(\| \mathbf{curl}\, \boldsymbol{\psi}_k \|^2_{0,\Upsilon_{D,k}} + \| \boldsymbol{\psi}_k \|^2_{0,\Upsilon_{D,k}} \right) \\
&\leq C \sum_{k=1}^{N} \| \mathbf{w}_1 \|^2_{0,\Upsilon_{D,k}} \leq C \, \| \mathbf{w}_1 \|^2_{0,B} \leq C \, \| \boldsymbol{\nabla} \mathbf{w}_1 \|^2_{0,B} \,, \\
\| \boldsymbol{\nabla} \mathbf{curl}\, \widetilde{\boldsymbol{\psi}} \|^2_{0,\mathbb{R}^2} &\leq C \sum_{k=1}^{N} \Big(\| \boldsymbol{\nabla} \underbrace{\mathbf{curl}\, \boldsymbol{\psi}_k}_{=\mathbf{w}_1} \|^2_{0,\Upsilon_{D,k}} + \| \boldsymbol{\nabla} \boldsymbol{\psi}_k \|^2_{0,\Upsilon_{D,k}} + \| \boldsymbol{\psi}_k \|^2_{0,\Upsilon_{D,k}} \Big) \\
&\leq C \, \| \boldsymbol{\nabla} \mathbf{w}_1 \|^2_{0,B} + \| \mathbf{w}_1 \|_{0,B} \leq C \, \| \boldsymbol{\nabla} \mathbf{w}_1 \|^2_{0,B} \,.
\end{aligned}
\tag{35}
$$

Step 5: We obtain

$$
\operatorname{div}(\mathbf{w}_1 - \mathbf{curl}\, \widetilde{\boldsymbol{\psi}}) = \widetilde{v} \qquad \text{in} \quad B \,, \tag{36}
$$

and define $\mathbf{q} := (\mathbf{w}_1 - \mathbf{curl}\, \widetilde{\boldsymbol{\psi}})_{|\Omega} \in \mathbf{H}^1(\Omega)$. Since $\mathbf{w}_1 - \mathbf{curl}\, \widetilde{\boldsymbol{\psi}} = 0$ in Υ_D, it follows that $\mathbf{q} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. Moreover, due to (35) and (30),

$$
\begin{aligned}
\| \mathbf{q} \|_{0,\Omega} &\leq \| \mathbf{w}_1 \|_{0,\Omega} + \| \mathbf{curl}\, \widetilde{\boldsymbol{\psi}} \|_{0,\Omega} \leq C \, \| \mathbf{w}_1 \|_{0,B} \leq C \, \| \boldsymbol{\nabla} \mathbf{w}_1 \|_{0,B} \leq C \, \| v \|_{0,\Omega} \,, \\
\| \boldsymbol{\nabla} \mathbf{q} \|_{0,\Omega} &\leq \| \boldsymbol{\nabla} \mathbf{w}_1 \|_{0,\Omega} + \| \boldsymbol{\nabla} \mathbf{curl}\, \widetilde{\boldsymbol{\psi}} \|_{0,\Omega} \leq C \, \| \boldsymbol{\nabla} \mathbf{w}_1 \|_{0,B} \leq C \, \| v \|_{0,\Omega} \,,
\end{aligned}
$$

which concludes the proof. □

Remark 12 The constant C in Theorem 7 depends essentially on the stability constants of the $\mathbf{H}^k(\mathbf{curl})$-extension operator from Lemma 7 and the partition of unity. It therefore involves the relative diameters $\mathrm{d}(\Upsilon_{D,k}) / \, \mathrm{d}(\Omega)$ and the relative distances between $\Upsilon_{D,k}$ and $\Upsilon_{D,\ell}$ for $k \neq \ell$.

4.3 Divergence-Bounded Regular Decompositions of $\mathbf{H}(\mathrm{div})$

We can now formulate other variants of regular decompositions of $\mathbf{H}(\mathrm{div})$ in analogy to what we did in Section 3.3.

Theorem 8 (Divergence-Bounded Regular Decomposition of $\mathbf{H}(\mathrm{div})$ (I))
Let (Ω, Γ_D) be as in Section 2.1. In addition, assume that each connected component $\Upsilon_{D,k}$ of the bulge has a connected boundary, i.e., $\beta_2(\Upsilon_{D,k}) = 0$. Then, for each $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$ there exists $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and a divergence-free vector field $\mathbf{h} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}\, 0, \Omega)$ depending linearly on $\mathbf{v}$ such that

$$\mathbf{v} = \mathbf{z} + \mathbf{h}, \tag{37}$$

$$\|\mathbf{h}\|_{0,\Omega} \leq \|\mathbf{v}\|_{0,\Omega} + C\,\mathrm{d}(\Omega)\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,, \tag{38}$$

$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{z}\|_{0,\Omega} \leq C\,\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,, \tag{39}$$

where C depends only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

Proof Let $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$ be arbitrary but fixed. Then $\mathrm{div}\,\mathbf{v} \in L^2_{\Gamma_D}(\Omega)$ and, due to Theorem 7, there exists a regular potential $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathrm{div}\,\mathbf{v}$ such that

$$\mathrm{div}\,\mathbf{z} = \mathrm{div}\,\mathbf{v},$$
$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1}\|\mathbf{z}\|_{0,\Omega} \leq C\,\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,.$$

Since $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ we conclude that $\mathbf{h} := \mathbf{v} - \mathbf{z} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}\, 0, \Omega)$ and that $\mathbf{v} = \mathbf{z} + \mathbf{h}$ with

$$\|\mathbf{h}\|_{0,\Omega} \leq \|\mathbf{v} - \mathbf{z}\|_{0,\Omega} \leq \|\mathbf{v}\|_{0,\Omega} + C\,\mathrm{d}(\Omega)\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,.$$

Remark 13 The constant C is the same as in Theorem 7.

The last variant of $\mathbf{H}(\mathrm{div})$ regular decomposition of $\mathbf{H}(\mathrm{div})$ dispenses with the assumptions on the topology of the Dirichlet boundary and has better stability properties than the splitting from Theorem 8 (though with less explicit constants).

Theorem 9 (Divergence-Bounded Regular Decomposition of $\mathbf{H}(\mathrm{div})$ (II))
Let (Ω, Γ_D) be as in Section 2.1. Then, for each $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$ there exists $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and a divergence-free vector field $\mathbf{h} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}\, 0, \Omega)$ depending linearly on $\mathbf{v}$ such that

$$\mathbf{v} = \mathbf{z} + \mathbf{h}, \tag{40}$$

$$\|\mathbf{z}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Omega} \leq \|\mathbf{v}\|_{0,\Omega}\,, \tag{41}$$

$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\|\mathbf{z}\|_{0,\Omega} \leq C\,\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,, \tag{42}$$

where C depends only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

For the proof of Theorem 9 we need a Friedrichs type inequality. Let $\Pi_{\mathrm{div}\,0,\Gamma_D}$ be the $\mathbf{L}^2$-orthogonal projector from $\mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$ to $\mathbf{H}_{\Gamma_D}(\mathrm{div}\, 0, \Omega)$.

Lemma 10 (Friedrichs-Type Inequality for $\mathbf{H}(\mathrm{div})$)
Let (Ω, Γ_D) be as introduced in Section 2.1. Then

$$\|\mathbf{v} - \Pi_{\mathrm{div}\,0,\Gamma_D}\mathbf{v}\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\,\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega} \qquad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega),$$

with a constant C depending only on the shape of Ω and Γ_D, but not on $\mathrm{d}(\Omega)$.

Note that, if each connected component of the bulge Υ_D has connected boundary, then Lemma 10 can be derived directly from Theorem 7 using that $\Pi_{\mathrm{div}\,0,\Gamma_D}$ is $\mathbf{L}^2$-orthogonal:

$$\|\mathbf{v} - \Pi_{\mathrm{div}\,0,\Gamma_D}\mathbf{v}\|_{0,\Omega} \leq \|\mathbf{v} - \mathbf{h}\|_{0,\Omega} = \|\mathbf{z}\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\,\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,.$$

In general, one obtains Lemma 10 from the Gaffney inequality

$$\|\mathbf{v}\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\Big(\int_{\Omega} |\,\mathbf{curl}\,\mathbf{v}|^2 + |\,\mathrm{div}\,\mathbf{v}|^2\,dx\Big)^{1/2}, \tag{43}$$

for all $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega) \cap \mathbf{H}_{\Gamma_N}(\mathbf{curl}, \Omega)$ that are $\mathbf{L}^2$-orthogonal to the relative co-homology space $\mathcal{H}_{\Gamma_N}(\Omega) := \mathbf{H}_{\Gamma_N}(\mathbf{curl}\,0, \Omega) \cap \mathbf{H}_{\Gamma_D}(\mathrm{div}\,0, \Omega)$, see (17). As mentioned earlier, the inequality (43), is an immediate consequence of the compact embedding of $\mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega) \cap \mathbf{H}_{\Gamma_N}(\mathbf{curl}, \Omega)$ in $\mathbf{L}^2(\Omega)$, which holds under the assumptions on (Ω, Γ_D) made in Section 2.1, see [3, Thm. 5.1].

Proof (Theorem 9) Let $\mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$ be arbitrary but fixed and set

$$\mathbf{v}_0 := \Pi_{\mathrm{div}\,0,\Gamma_D}\mathbf{v}, \qquad \mathbf{v}_1 := \mathbf{v} - \mathbf{v}_0\,,$$

such that

$$\|\mathbf{v}_0\|_{0,\Omega} \leq \|\mathbf{v}\|_{0,\Omega}\,, \qquad \|\mathbf{v}_1\|_{0,\Omega} \leq \|\mathbf{v}\|_{0,\Omega}\,, \tag{44}$$

and by Lemma 10,

$$\|\mathbf{v}_1\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\,\|\,\mathrm{div}\,\mathbf{v}\|_{0,\Omega}\,. \tag{45}$$

As a next step, we apply Theorem 6 to $\mathbf{v}_1$ to obtain

$$\mathbf{v}_1 = \mathbf{z} + \mathbf{curl}\,\mathbf{q}, \qquad \mathbf{v} = \mathbf{z} + \underbrace{\mathbf{curl}\,\mathbf{q} + \mathbf{v}_0}_{=:\mathbf{h}}, \tag{46}$$

with $\mathbf{z} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and $\mathbf{q} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and

$$\|\mathbf{z}\|_{0,\Omega} + \|\mathbf{h}\|_{0,\Omega} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,, \tag{47}$$

$$\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega} \leq C\|\,\mathrm{div}\,\mathbf{v}_1\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1}\|\mathbf{v}_1\|_{0,\Omega}\,. \tag{48}$$

Using that $\mathrm{div}\,\mathbf{v}_1 = \mathrm{div}\,\mathbf{v}$ and combining all the stability estimates obtained so far yields the desired result. □

5 Discrete Counterparts of the Regular Decompositions

The discrete setting to which we want to extend the concept of regular decompositions is provided by finite element exterior calculus (FEEC, [2]) which introduces finite element subspaces of $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$ as special instances of spaces of discrete differential forms. In this section we confine ourselves to the lowest-order case of piecewise linear finite element functions.

Throughout, we assume that (Ω, Γ_D) is as in Section 2.1, and, additionally, that Ω is a polyhedron and that $\partial\Gamma_D$ consists of straight line segments. All considerations take for granted a shape-regular family of meshes $\{\mathcal{T}^h\}_h$ of Ω, consisting of tetrahedral elements, and resolving Γ_D in the sense that Γ_D is a union of faces of some of the tetrahedra.

The following finite element spaces will be relevant:

- the space $\mathcal{W}^0_{h,\Gamma_D}(\Omega)$ of $H^1_{\Gamma_D}(\Omega)$-conforming piecewise linear Lagrangian finite element functions,
- the space $\boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\Omega)$ of $\mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$-conforming lowest order Nédélec elements, also known as *edge elements*,
- the space $\boldsymbol{\mathcal{W}}^2_{h,\Gamma_D}(\Omega)$ of $\mathbf{H}_{\Gamma_D}(\mathrm{div}, \Omega)$-conforming lowest order tetrahedral Raviart-Thomas finite elements, aka, *face elements*,
- the space $\boldsymbol{\mathcal{W}}^0_{h,\Gamma_D}(\Omega) := [\mathcal{W}^0_{h,\Gamma_D}(\Omega)]^3$ of piecewise linear globally continuous vector fields vanishing on Γ_D.

Functions in $\boldsymbol{\mathcal{W}}^\ell_{h,\Gamma_D}(\Omega)$, $\ell = 1, 2, 3$, are so-called Whitney forms, lowest-order discrete differential forms of the first family as introduced in [15] and [2, Sect. 5].

5.1 Discrete Regular Decompositions for Edge Elements

Commuting projectors, also known as co-chain projectors, are the linchpin of FEEC theory [2, Sect. 7], and it is not different with our developments. Thus, let

$$\begin{aligned} & R^0_{h,\Gamma_D} \colon H^1_{\Gamma_D}(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\Omega) \\ \text{and} \quad & \mathbf{R}^1_{h,\Gamma_D} \colon \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \to \mathcal{W}^1_{h,\Gamma_D}(\Omega) \end{aligned}$$

denote the *continous*, *boundary-aware* cochain projectors from [20, Sect. 3.2.6], which extend the pioneering work [12] by Falk and Winther. These two linear operators are projectors onto their ranges, they fulfill the commuting property

$$\nabla(R^0_{h,\Gamma_D}\varphi) = \mathbf{R}^1_{h,\Gamma_D}(\nabla\varphi) \qquad \forall\varphi \in H^1_{\Gamma_D}(\Omega)\,, \tag{49}$$

and the local stability estimates

$$\|R^0_{h,\Gamma_D} u\|_{0,T} \le C\big(\|u\|_{0,\omega_T} + h_T \|\nabla u\|_{0,\omega_T}\big) \qquad \forall u \in H^1_{\Gamma_D}(\Omega), \tag{50}$$

$$\|\mathbf{R}^0_{h,\Gamma_D} \mathbf{v}\|_{0,T} \le C\big(\|\mathbf{v}\|_{0,\omega_T} + h_T \|\mathbf{curl}\,\mathbf{v}\|_{0,\omega_T}\big) \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega), \tag{51}$$

for all mesh elements T, where ω_T is the element patch of T, the union of neighboring elements, and h_T the element diameter. The constant C is uniform in T and depends only on the shape regularity of the mesh $\mathcal{T}^h(\Omega)$.

Theorem 10 ([20, Thm. 1.2])
For each $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\Omega)$ *there exists a continuous and piecewise linear vector field* $\mathbf{z}_h \in \boldsymbol{\mathcal{W}}^0_{h,\Gamma_D}(\Omega)$*, a continuous and piecewise linear scalar function* $p_h \in \mathcal{W}^0_{h,\Gamma_D}(\Omega)$*, and a remainder* $\widetilde{\mathbf{v}}_h \in \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\Omega)$*, all depending linearly on* $\mathbf{v}_h$*, providing the discrete regular decomposition*

$$\mathbf{v}_h = \mathbf{R}^1_{h,\Gamma_D}\mathbf{z}_h + \widetilde{\mathbf{v}}_h + \nabla p_h$$

and satisfying the stability estimates

$$\|\mathbf{z}_h\|_{0,\Omega} + \|\nabla p_h\|_{0,\Omega} + \|\widetilde{\mathbf{v}}_h\|_{0,\Omega} \leq C\,\|\mathbf{v}_h\|_{0,\Omega}\,, \tag{52}$$

$$\|\boldsymbol{\nabla}\,\mathbf{z}_h\|_{0,\Omega} + \|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega} \leq C\big(\|\,\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega} + \tfrac{1}{\mathrm{d}(\Omega)}\|\mathbf{v}_h\|_{0,\Omega}\big)\,, \tag{53}$$

where C *is a generic constant that depends only on the shape of* (Ω,Γ_D)*, but not on* $\mathrm{d}(\Omega)$*, and on the shape regularity constant of* $\mathcal{T}^h(\Omega)$*. Above,* h^{-1} *is the piecewise constant function that is equal to* h_T^{-1} *on every element* T*.*

Obviously, this is a discrete counterpart of the regular decomposition of $\mathbf{H}(\mathbf{curl})$ from Theorem 2. The following theorem appears to be new and it corresponds to the rotation-bounded regular decomposition of Theorem 5. For the sake of brevity define the discrete nullspace of the curl operator

$$\mathcal{N}^1_h := \{\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\Omega)\colon\ \mathbf{curl}\,\mathbf{v}_h = 0\}\,. \tag{54}$$

If Ω and Γ_D have simple topology, $\mathcal{X}_h = \nabla\,\mathcal{W}^0_{h,\Gamma_D}(\Omega)$, but if the first Betti number of Ω is non-zero, or if Γ_D has multiple components, then a finite-dimensional cohomology space has to be added [2, Sect. 5.6].

Theorem 11 (Rotation-bounded discrete regular decomposition for edge elements)
For each $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\Omega)$ *there exists a continuous and piecewise linear vector field* $\mathbf{z}_h \in \boldsymbol{\mathcal{W}}^0_{h,\Gamma_D}(\Omega)$*, an* curl-free *edge element function* $\mathbf{h}_h \in \mathcal{N}^1_h$*, and a remainder* $\widetilde{\mathbf{v}}_h \in \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\Omega)$*, all depending linearly on* $\mathbf{v}_h$*, providing the discrete regular decomposition*

$$\mathbf{v}_h = \mathbf{R}^1_{h,\Gamma_D}\mathbf{z}_h + \widetilde{\mathbf{v}}_h + \mathbf{h}_h$$

and satisfying the stability bounds

$$\left.\begin{array}{l}\|\mathbf{z}_h\|_{0,\Omega}\\ \|\mathbf{h}_h\|_{0,\Omega}\\ \|\widetilde{\mathbf{v}}_h\|_{0,\Omega}\end{array}\right\} \leq C\,\|\mathbf{v}_h\|_{0,\Omega}\,, \qquad \left.\begin{array}{l}\|\,\boldsymbol{\nabla}\,\mathbf{z}_h\|_{0,\Omega}\\ \|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega}\end{array}\right\} \leq C\,\|\,\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\,,$$

where C *is a uniform constant that depends only on the shape of* (Ω, Γ_D)*, but not on* $\mathrm{d}(\Omega)$*, and on the shape regularity constant of* $\mathcal{T}^h(\Omega)$*.*

For the proof of Theorem 11 we need a discrete Friedrichs inequality, otherwise it runs parallel to the proof of Theorem 5. The discrete Friedrichs inequality can elegantly be derived using the co-chain projector $\mathbf{R}^1_{h,\Gamma_D}$.

Lemma 11 (discrete Friedrichs inequality for $\boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}$)
Let $\Pi^1_{h,\Gamma_D} : \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\Omega) \to \mathcal{N}^1_h$ *denote the* $\mathbf{L}^2$*-orthogonal projection onto the discrete curl-free subspace* $\mathcal{N}^1_h \subset \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\Omega)$*. Then*

$$\|\mathbf{v}_h - \Pi^1_{h,\Gamma_D}\mathbf{v}_h\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\|\,\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega} \qquad \forall \mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\Omega),$$

with a uniform constant C *that depends on the constant in the continuous Friedrichs inequality (Lemma 6) and on the shape-regularity constant of* $\mathcal{T}^h(\Omega)$*.*

Proof Recall the $\mathbf{L}^2$-orthogonal projector $\Pi_{\mathbf{curl}\,0,\Gamma_D}$ mapping from $\mathbf{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ to $\mathbf{H}_{\Gamma_D}(\mathbf{curl}\,0, \Omega)$. Due to the $\mathbf{L}^2$-minimization property of Π^1_{h,Γ_D},

$$\begin{aligned}\|\mathbf{v}_h - \Pi^1_{h,\Gamma_D}\mathbf{v}_h\|_{0,\Omega} &= \min_{\mathbf{w}_h \in \mathcal{N}^1_h} \|\mathbf{v}_h - \mathbf{w}_h\|_{0,\Omega}\\ &\leq \|\mathbf{v}_h - \mathbf{R}^1_{h,\Gamma_D}\Pi_{\mathbf{curl},\Gamma_D}\mathbf{v}_h\|_{0,\Omega}\\ &= \|\mathbf{R}^1_{h,\Gamma_D}\mathbf{v}_h - \mathbf{R}^1_{h,\Gamma_D}\Pi_{\mathbf{curl},\Gamma_D}\mathbf{v}_h\|_{0,\Omega}\\ &\leq C\big(\|\mathbf{v}_h - \Pi_{\mathbf{curl},\Gamma_D}\mathbf{v}_h\|_{0,\Omega} + h_{\max}\|\,\mathbf{curl}(\mathbf{v}_h - \Pi_{\mathbf{curl},\Gamma_D}\mathbf{v}_h)\|_{0,\Omega}\big),\end{aligned}$$

where we have used the projection property $\mathbf{R}^1_{h,\Gamma_D}$ and the stability estimate (51), and $h_{\max}$ is the maximal element diameter. The proof is concluded by observing that $\mathbf{curl}\,\Pi_{\mathbf{curl},\Gamma_D} = 0$ and $h_{\max} \leq \mathrm{d}(\Omega)$, and by the continuous Friedrichs inequality Lemma 6. □

Proof (Theorem 11) Let $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\Omega)$ be arbitrary but fixed and set

$$\mathbf{v}_{h,0} := \Pi^1_{h,\Gamma_D}\mathbf{v}_h\,, \qquad \mathbf{v}_{h,1} := \mathbf{v}_h - \mathbf{v}_{h,0}$$

such that

$$\|\mathbf{v}_{h,0}\|_{0,\Omega} \leq \|\mathbf{v}_h\|_{0,\Omega}\,, \qquad \|\mathbf{v}_{h,1}\|_{0,\Omega} \leq \|\mathbf{v}_h\|_{0,\Omega}\,, \tag{55}$$

and by Lemma 11,

$$\|\mathbf{v}_{h,1}\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\, \|\,\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\,. \tag{56}$$

We now apply Theorem 10 to $\mathbf{v}_{h,1}$ and obtain

$$\mathbf{v}_{h,1} = \mathbf{R}^1_{h,\Gamma_D}\mathbf{z}_h + \widetilde{\mathbf{v}} + \nabla p_h$$

Setting $\mathbf{h}_h := \nabla p_h + \mathbf{v}_{h,0}$ and combining the stability estimates from Theorem 10 with (55)–(56) concludes the proof. □

We stress that the statements of Theorem 10 and Theorem 11 do not hinge on *any* assumptions on the topological properties of Ω and Γ_D.

5.2 Discrete Regular Decompositions for Face Elements

For face elements, the construction of a boundary-aware co-chain projection operator

$$\mathbf{R}^2_{h,\Gamma_D}\colon \mathbf{H}_{\Gamma_D}(\mathrm{div},\Omega) \to \boldsymbol{\mathcal{W}}^2_{h,\Gamma_D}(\Omega)$$

that commutes with $\mathbf{R}^1_{h,\Gamma_D}$ and the $\mathbf{curl}$-operator has not yet been accomplished. Fortunately, in the case $\Gamma_D = \emptyset$, this operator is available from [12]. Thus, in the following, we treat only the case $\Gamma_D = \emptyset$ and just omit the subscript Γ_D. Then, from [12] we can borrow a linear operator $\mathbf{R}^2_h\colon \mathbf{H}(\mathrm{div},\Omega) \to \boldsymbol{\mathcal{W}}^2_h(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{R}^1_h\mathbf{u} = \mathbf{R}^2_h\,\mathbf{curl}\,\mathbf{u} \qquad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl},\Omega)\,, \tag{57}$$

and

$$\|\mathbf{R}^2_h\mathbf{v}\|_{0,T} \leq C(\|\mathbf{v}\|_{0,T} + h_T\|\,\mathrm{div}\,\mathbf{v}\|_{0,T}) \qquad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div},\Omega)\,. \tag{58}$$

The next result takes Theorem 6 to the discrete setting.

Theorem 12 (Discrete Regular Decomposition of $\mathcal{W}_h^2(\Omega)$)
For each vector field $\mathbf{v}_h$ in the lowest-order Raviart-Thomas space $\mathcal{W}_h^2(\Omega)$, there exists a continuous and piecewise linear vector field $\mathbf{z}_h \in \mathcal{W}_h^0(\Omega)$, a vector field $\mathbf{q}_h$ in the lowest-order Nédélec space $\mathcal{W}_h^1(\Omega)$, and a remainder $\widetilde{\mathbf{v}}_h \in \mathcal{W}_h^2(\Omega)$, all depending linearly on $\mathbf{v}_h$, providing the discrete regular decomposition

$$\mathbf{v}_h = \mathbf{R}_h^2 \mathbf{z}_h + \widetilde{\mathbf{v}}_h + \mathbf{curl}\, \mathbf{q}_h \,,$$

and the stability estimates

$$\|\mathbf{curl}\, \mathbf{q}_h\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)}\left.\begin{matrix} \|\mathbf{z}_h\|_{0,\Omega} \\ \|\mathbf{q}_h\|_{0,\Omega} \\ \|\widetilde{\mathbf{v}}_h\|_{0,\Omega} \end{matrix}\right\} \leq C\, \|\mathbf{v}_h\|_{0,\Omega} \,,$$

$$\left.\begin{matrix} \|\nabla \mathbf{z}_h\|_{0,\Omega} \\ \|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega} \end{matrix}\right\} \leq C \|\operatorname{div} \mathbf{v}_h\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)} \|\mathbf{v}_h\|_{0,\Omega} \,.$$

The constant C depends only on the shape of Ω, but not on $\mathrm{d}(\Omega)$, and the shape-regularity of $\mathcal{T}^h(\Omega)$.

Proof Let $\mathbf{v}_h \in \mathcal{W}_h^2(\Omega) \subset \mathbf{H}(\mathrm{div}, \Omega)$ be arbitrary but fixed. Due to Theorem 6, there exist $\mathbf{z}$ and $\mathbf{q} \in \mathbf{H}^1(\Omega)$ (depending linearly on $\mathbf{v}_h$) such that

$$\mathbf{v}_h = \mathbf{z} + \mathbf{curl}\, \mathbf{q}, \tag{59}$$

$$\|\mathbf{z}\|_{0,\Omega} + \|\mathbf{curl}\, \mathbf{q}\|_{0,\Omega} + \mathrm{d}(\Omega)^{-1} \|\mathbf{q}\|_{0,\Omega} \leq C\, \|\mathbf{v}_h\|_{0,\Omega} \,, \tag{60}$$

$$\|\nabla \mathbf{z}\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)} \|\nabla \mathbf{q}\|_{0,\Omega} \leq C\, \|\mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega} + \frac{1}{\mathrm{d}(\Omega)} \|\mathbf{v}_h\|_{0,\Omega}^2 \,. \tag{61}$$

Let $\mathbf{M}_h \colon \mathbf{L}^2(\Omega) \to \mathcal{W}^0(\Omega)$ denote the (vectorized) Clément quasi-interpolation operator. The projection property of $\mathbf{R}_h^2$ implies $\mathbf{R}_h^2 \mathbf{v}_h = \mathbf{v}_h$, and so

$$\mathbf{v}_h = \mathbf{R}_h^2 \mathbf{z} + \underbrace{\mathbf{R}_h^2\, \mathbf{curl}\, \mathbf{q}}_{=\mathbf{curl}\, \mathbf{R}_h^1 \mathbf{q}} = \mathbf{R}_h^2 \underbrace{\mathbf{M}_h \mathbf{z}}_{=:\mathbf{z}_h} + \underbrace{\mathbf{R}_h^2 (I - \mathbf{M}_h) \mathbf{z}}_{=:\widetilde{\mathbf{v}}_h} + \mathbf{curl}\, \underbrace{\mathbf{R}_h^1 \mathbf{q}}_{=:\mathbf{q}_h}, \tag{62}$$

where in the last step, we have used the commuting property (57). Due to the properties of the Clément operator (see [8] and [20, Sect. 3.2.3]),

$$\|\mathbf{M}_h \mathbf{z}\|_{0,T} \leq C\, \|\mathbf{z}\|_{0,\omega_T} \,, \tag{63}$$

$$\|\nabla \mathbf{M}_h \mathbf{z}\|_{0,T} \leq C\, \|\nabla \mathbf{z}\|_{0,\omega_T} \,, \tag{64}$$

$$\|\mathbf{z} - \mathbf{M}_h \mathbf{z}\|_{0,T} \leq C\, h_T\, \|\nabla \mathbf{z}\|_{0,\omega_T} \,. \tag{65}$$

Using these estimates, the properties of $\mathbf{R}_h^2$, and a standard inverse inequality, one obtains

$$
\begin{aligned}
\|\mathbf{z}_h\|_{0,\Omega} &\le C\,\|\mathbf{v}_h\|_{0,\Omega}\,,\\
\|\,\boldsymbol{\nabla}\,\mathbf{z}_h\|_{0,\Omega} &\le C\|\,\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega} + \mathrm{d}(\Omega)\|\mathbf{v}_h\|_{0,\Omega}\,,\\
\|\mathbf{R}_h^2\mathbf{z}_h\|_{0,\Omega} &\le C\|\mathbf{M}_h\mathbf{z}\|_{0,\Omega} + \Big(\sum_{T\in\mathcal{T}^h(\Omega)} \underbrace{h_T^2\|\,\mathrm{div}\,\mathbf{M}_h\mathbf{z}\|_{0,\omega_T}^2}_{\le C\|\mathbf{M}_h\mathbf{z}\|_{0,\omega_T}^2}\Big)^{1/2} \le \|\mathbf{v}_h\|_{0,\Omega}\,.
\end{aligned}
$$

From the properties of $\mathbf{R}_h^1$ and $\mathbf{R}_h^2$ we derive

$$
\begin{aligned}
\|\mathbf{q}_h\|_{0,\Omega} &\le C\big(\|\mathbf{q}\|_{0,\Omega} + \underbrace{h_{\text{max}}}_{\le\mathrm{d}(\Omega)}\,\|\,\mathbf{curl}\,\mathbf{q}\|_{0,\Omega}\big) \le C\;\mathrm{d}(\Omega)\,\|\mathbf{v}_h\|_{0,\Omega}\,,\\
\|\,\mathbf{curl}\,\mathbf{q}_h\|_{0,\Omega} &\le C\|\,\mathbf{curl}\,\mathbf{q}\|_{0,\Omega} \le C\|\mathbf{v}_h\|_{0,\Omega}\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|\widetilde{\mathbf{v}}_h\|_{0,\Omega} &= \|\mathbf{v}_h - \mathbf{R}_h^2\mathbf{z}_h - \mathbf{curl}\,\mathbf{q}_h\|_{0,\Omega} \le C\|\mathbf{v}_h\|_{0,\Omega}\,,\\
\|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega}^2 &= \sum_{T\in\mathcal{T}^h(\Omega)} h_T^{-2}\|\mathbf{R}_h^2(I-\mathbf{M}_h)\mathbf{z}\|_{0,T}^2\\
&\le C\Big(\sum_{T\in\mathcal{T}^h(\Omega)} h_T^{-2}\|(I-\mathbf{M}_h)\mathbf{z}\|_{0,\omega_T}^2 + \|\,\mathrm{div}(I-\mathbf{M}_h)\mathbf{z}\|_{0,\omega_T}^2\Big)\\
&\le C\,\|\,\boldsymbol{\nabla}\,\mathbf{z}\|_{0,\Omega}^2 \le C\|\,\mathrm{div}\,\mathbf{v}_h\|_{0,\Omega}^2 + \mathrm{d}(\Omega)^{-1}\|\mathbf{v}_h\|_{0,\Omega}^2\,,
\end{aligned}
$$

which concludes the proof. □

Finally, we present a counterpart to the divergence-bounded regular decomposition of Theorem 9. For convenience we introduce the space of divergence-free face element functions

$$
\mathcal{N}_h^2 := \{\mathbf{q}_h \in \boldsymbol{\mathcal{W}}_h^2(\Omega)\colon\ \mathrm{div}\,\mathbf{q}_h = 0\}\,. \tag{66}
$$

Theorem 13 (Divergence-Bounded Discrete Regular Decomposition of $\boldsymbol{\mathcal{W}}_h^2(\Omega)$)
For each vector field $\mathbf{v}_h$ in the lowest-oreder Raviart-Thomas space $\boldsymbol{\mathcal{W}}_h^2(\Omega)$, there exists a continuous and piecewise linear vector field $\mathbf{z}_h \in \boldsymbol{\mathcal{W}}_h^0(\Omega)$, an element $\mathbf{h}_h$ in the discrete divergence-free subspace $\mathcal{N}_h^2$, and a remainder $\widetilde{\mathbf{v}}_h \in \boldsymbol{\mathcal{W}}_h^2(\Omega)$, all depending linearly on $\mathbf{v}_h$, providing the discrete regular decomposition

$$\mathbf{v}_h = \mathbf{R}_h^2 \mathbf{z}_h + \widetilde{\mathbf{v}}_h + \mathbf{h}_h$$

and the stability estimates

$$\left.\begin{array}{r} \|\mathbf{z}_h\|_{0,\Omega} \\ \|\widetilde{\mathbf{v}}_h\|_{0,\Omega} \\ \|\mathbf{h}_h\|_{0,\Omega} \end{array}\right\} \leq C\, \|\mathbf{v}_h\|_{0,\Omega}\,, \qquad \left.\begin{array}{r} \|\boldsymbol{\nabla}\, \mathbf{z}_h\|_{0,\Omega} \\ \|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega} \end{array}\right\} \leq C\, \|\operatorname{div} \mathbf{v}_h\|_{0,\Omega}\,.$$

The constants C depend only on the shape of Ω, but not on $\mathrm{d}(\Omega)$, and the shape regularity of $\mathcal{T}^h(\Omega)$.

For the proof, we need a discrete Friedrichs type inequality:

Lemma 12 (discrete Friedrichs inequality for $\boldsymbol{\mathcal{W}}_h^2(\Omega)$)
Let $\Pi_h^2 \colon \boldsymbol{\mathcal{W}}_h^2(\Omega) \to \mathcal{N}_h^2$ denote the $\mathbf{L}^2$-orthogonal projection onto the discrete divergence-free subspace $\mathcal{N}_h^2$. Then

$$\|\mathbf{v}_h - \Pi_h^2 \mathbf{v}_h\|_{0,\Omega} \leq C\ \mathrm{d}(\Omega)\, \|\operatorname{div} \mathbf{v}_h\|_{0,\Omega} \qquad \forall \mathbf{v}_h \in \boldsymbol{W}_h^2(\Omega),$$

with a uniform constant C that depends on the constant in the continuous Friedrichs inequality (Lemma 10) and on the shape-regularity constant of $\mathcal{T}^h(\Omega)$.

Proof Recall the $\mathbf{L}^2$-orthogonal projector $\Pi_{\operatorname{div} 0}$ mapping $\mathbf{H}(\operatorname{div}, \Omega)$ to $\mathbf{H}(\operatorname{div} 0, \Omega)$. Due to the $\mathbf{L}^2$-minimization property of Π_h^2,

$$\begin{aligned}
\|\mathbf{v}_h - \Pi_h^2 \mathbf{v}_h\|_{0,\Omega} &= \min_{\mathbf{h} \in \mathcal{N}_h^2} \|\mathbf{v}_h - \mathbf{h}_h\|_{0,\Omega} \\
&\leq \|\mathbf{v}_h - \mathbf{R}_h^2 \Pi_{\operatorname{div} 0} \mathbf{v}_h\|_{0,\Omega} \\
&= \|\mathbf{R}_h^2 \mathbf{v}_h - \mathbf{R}_h^2 \Pi_{\operatorname{div} 0} \mathbf{v}_h\|_{0,\Omega} \\
&\leq C\big(\|\mathbf{v}_h - \Pi_{\operatorname{div} 0} \mathbf{v}_h\|_{0,\Omega} + h_{\text{max}} \|\operatorname{div}(\mathbf{v}_h - \Pi_{\operatorname{div} 0} \mathbf{v}_h\|_{0,\Omega}\big),
\end{aligned}$$

where we have used the projection property of $\mathbf{R}_h^2$ and the stability estimate (58). The proof is concluded by observing that $\operatorname{div} \Pi_{\operatorname{div} 0} = 0$ and $h_{\text{max}} \leq \mathrm{d}(\Omega)$, and using the continuous Friedrichs inequality Lemma 10. □

Proof (Theorem 13) Let $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}_h^2(\Omega)$ be arbitrary but fixed and let

$$\mathbf{v}_{h,0} := \Pi_h^2 \mathbf{v}_h\,, \qquad \mathbf{v}_{h,1} := \mathbf{v}_h - \mathbf{v}_{h,0}\,,$$

such that with Lemma 12,

$$\|\mathbf{v}_{h,0}\|_{0,\Omega} \leq \|\mathbf{v}_h\|, \qquad \|\mathbf{v}_{h,1}\|_{0,\Omega} \leq \|\mathbf{v}_h\|_{0,\Omega}, \tag{67}$$

$$\|\mathbf{v}_{h,1}\|_{0,\Omega} \leq C \ \mathrm{d}(\Omega) \|\operatorname{div}\mathbf{v}_h\|_{0,\Omega}. \tag{68}$$

We apply Theorem 12 to $\mathbf{v}_{h,1}$ such that

$$\mathbf{v}_h = \mathbf{R}_h^2\mathbf{z}_h + \widetilde{\mathbf{v}}_h + \underbrace{\mathbf{curl}\,\mathbf{q}_h + \mathbf{v}_{h,0}}_{=:\mathbf{h}_h} \tag{69}$$

with

$$\left.\begin{array}{r}\|\mathbf{z}_h\|_{0,\Omega}\\ \|\mathbf{curl}\,\mathbf{q}_h\|_{0,\Omega}\\ \|\widetilde{\mathbf{v}}_h\|_{0,\Omega}\end{array}\right\} \leq C\,\|\mathbf{v}_{h,1}\|_{0,\Omega} \leq C\|\mathbf{v}_h\|_{0,\Omega},$$

$$\left.\begin{array}{r}\|\boldsymbol{\nabla}\,\mathbf{z}_h\|_{0,\Omega}\\ \|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega}\end{array}\right\} \leq C\big(\|\underbrace{\operatorname{div}\mathbf{v}_{h,1}}_{=\operatorname{div}\mathbf{v}_h}\|_{0,\Omega} + \underbrace{\mathrm{d}(\Omega)^{-1}\|\mathbf{v}_{h,1}\|_{0,\Omega}}_{\leq C\|\operatorname{div}\mathbf{v}_h\|_{0,\Omega}^2}\big),$$

which concludes the proof. □

Remark 14 The result of Theorem 13 can be viewed as an improvement of the decompositions in [27] which are elaborated for the case of essential boundary conditions on $\partial\Omega$.

Corollary 2 *If the second Betti number of Ω vanishes, that is, if $\partial\Omega$ is connected, then $\mathbf{h}_h$ in Theorem 13 can be chosen as $\mathbf{h}_h = \mathbf{curl}\,\mathbf{q}_h$ with $\mathbf{q}_h \in \boldsymbol{\mathcal{W}}_h^1(\Omega)$ such that*

$$\mathbf{v}_h = \mathbf{R}_h^2\mathbf{z} + \widetilde{\mathbf{v}}_h + \mathbf{curl}\,\mathbf{q}_h,$$

with the bounds

$$\left.\begin{array}{r}\|\mathbf{z}_h\|_{0,\Omega}\\ \|\widetilde{\mathbf{v}}_h\|_{0,\Omega}\\ \|\mathbf{curl}\,\mathbf{q}_h\|_{0,\Omega}\\ \mathrm{d}(\Omega)^{-1}\|\mathbf{q}_h\|_{0,\Omega}\end{array}\right\} \leq C\,\|\mathbf{v}_h\|_{0,\Omega}, \qquad \left.\begin{array}{r}\|\boldsymbol{\nabla}\,\mathbf{z}_h\|_{0,\Omega}\\ \|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\Omega}\end{array}\right\} \leq C\,\|\operatorname{div}\mathbf{v}_h\|_{0,\Omega}.$$

Proof If the second Betti number of Ω vanishes, then $\mathcal{N}_h^2 = \mathbf{curl}\,\boldsymbol{\mathcal{W}}_h^1(\Omega)$. So $\mathbf{h}_h = \mathbf{curl}\,\widetilde{\mathbf{q}}_h$ for some $\widetilde{\mathbf{q}}_h \in \boldsymbol{\mathcal{W}}_h^1(\Omega)$. Setting $\mathbf{q}_h := \widetilde{\mathbf{q}}_h - \Pi_h^1\widetilde{\mathbf{q}}_h$ and using Lemma 11 concludes the proof. □

Remark 15 The result of Corollary 2 is an improvement of [22, Lemma 5.2] which assumes a domain Ω that is smooth enough to allow H^2-regularity of the Laplace problem (2-regular case, for details see [22, Sect. 3]). This lemma is used in [30] in a domain decomposition framework, where convex subdomains are assumed. With our improved version, this assumption can be weakened considerably.

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