Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Time-Dependent Polarization Tensors: Derivation of Asymptotic Expansions for the Transient Wave Equation

L. Baldassari

Research Report No. 2019-16
March 2019

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 200021-172483

TIME-DEPENDENT POLARIZATION TENSORS: DERIVATION OF ASYMPTOTIC EXPANSIONS FOR THE TRANSIENT WAVE EQUATION

LORENZO BALDASSARI

Abstract. This paper aims at introducing the concept of time-dependent polarization tensors for the Helmholtz equation. One considers solutions to the frequency-domain Helmholtz equation in two and three dimensions. Based on layer potential techniques one provides for such solutions a rigorous systematic derivation of complete asymptotic expansions of perturbations resulting from the presence of diametrically small targets with constitutive parameters different from those of the background and size less than the operating wavelength. Such asymptotic expansions are based on careful and precise estimates of the dependence with respect to the frequency of the remainders. By truncating the high frequencies of the Fourier transform of these asymptotic expansions, one recovers the time-domain formulas. The threshold of the truncation is determined by the size of the target. The time-dependent asymptotic expansions are written in terms of the new concept of time-dependent polarization tensors. It is expected that our results will find important applications for developing time-domain algorithms for target classification.

1. Introduction

Echolocating bats can detect changes as small as 500 nanoseconds in the arrival time of echoes when these changes appear as jitter or alternations in arrival time from one echo to the next. The bat perceives the phase of the sounds, which cover the 25- to 100-kilohertz frequency range, as these are represented in the auditory system after peripheral transformation. The acoustic image of a sonar target is apparently derived from time-domain or periodicity information processing by the nervous system [20, 21].

The aim of this paper is to introduce the concept of time-dependent polarization tensors for the Helmholtz equation. Experimental data suggest that bats use temporal information for most, if not all, perceptual tasks [20]. Therefore, it is expected that our results will find important application for developing time-domain methods for target classification in echolocation by extending the correspondent frequency-domain methods [2, 11] and the elecro-sensing case [9]. Since a signal in time contains

This work was supported by the SNF grant 200021-172483

2010 *Mathematics Subject Classification.* 35R30, 35B30, 35C20.

Key words and phrases. Helmholtz equation, transient imaging, small inclusion, full-asymptotic expansions, polarization tensors, boundary integral equation, Fourier transform.

more information from a frequency point of view, they are expected to give a better performance in target classification if treated directly in the time domain.

We consider the case of a bat which is not moving and is sending a signal in time. This problem has to be evaluated in the frequency-domain first. We derive an asymptotic expansion for the perturbed wave in terms of the frequency-dependent polarization tensors by taking care of the frequency of the remainder associated to the asymptotic formulas. The target has to be small compared to the wavelength. Note that high frequencies correspond exactly to small wavelengths. The basic idea is then to truncate the high frequencies, as in [7]. The threshold of the truncation is determined by the size of the target. The truncation is not just a mathematical assumption: CF (constant frequency) bats cannot hear all the frequencies outside a certain range of finite values [20]. By truncating the high frequencies of the Fourier transform of the asymptotic expansion, we recover the time-domain formulae. These time-dependent asymptotic expansions are written in terms of the new concept of time-dependent polarization tensors.

To be more precise, suppose that the target D is of the form

$$D = \epsilon B + z,$$

where B is a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, containing the origin and ϵ is the order of magnitude of the diameter of the target. A cylindrical or spherical wave

$$U_y(x,t) := \begin{cases} \frac{\delta(t-|x-y|)}{4\pi|x-y|} & d = 3, \\ \frac{H(t-|x-y|)}{2\pi\sqrt{t^2-|x-y|^2}} & d = 2, \end{cases}$$

is generated by a point source located at y far away from z. When this wave hits the target D, it is perturbed. We will derive complete asymptotic expansions of this perturbation far away from the target as $\epsilon \to 0$. In fact, we will derive asymptotic expansions of the perturbation $u_y - U_y$ after having the high-frequency component truncated, where u_y is the solution to

$$\begin{cases} \partial_t^2 u_y - \nabla \cdot (\chi(\mathbb{R}^d \setminus \bar{D}) + k\chi(D))\nabla u_y = \delta_{x=y}\delta_{t=0} & \text{in } \mathbb{R}^d \times (0,\infty), \\ u_y(x,t) = 0 & \text{for } x \in \mathbb{R}^d \text{ and } t \ll 0. \end{cases} \tag{1.1}$$

The leading-order term in this asymptotic formula has been derived by Ammari et al. [7] for the three-dimensional case. The proof of our asymptotic expansion is different, is written for dimensions $d = 2, 3$ and is complete. It is based on layer potential techniques and the decomposition formula

$$v_y(x,\omega) = \begin{cases} V_y(x,\omega) + S_D^{\omega}[\psi](x), & x \in \mathbb{R}^d \setminus \bar{D}, \\ S_D^{\frac{\omega}{\sqrt{k}}}[\phi](x), & x \in D, \end{cases}$$

of the solution v_y to the Helmholtz equation associated (1.1). Here the background solution V_y is defined as follows:

$$V_y(x,\omega) := \begin{cases} \frac{e^{i\omega|x-y|}}{4\pi|x-y|} & d = 3, \\ \frac{i}{4}H_0^{(1)}(\omega|x-y|) & d = 2, \end{cases}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. One of the main achievements of this paper is a rigorous derivation of a complete asymptotic expansion of v_y with careful and precise estimates of the dependence with respect to the frequency of the remainders, see Theorem 3.3 and

Theorem 6.4. Note that these estimates are much more involved in two dimensions than in three dimensions because of the logarithmic behavior of the Hankel function at the origin [1, 18, 19, 22].

Finally, we recover the time-domain formulas by truncating the high frequencies of the Fourier transform of the asymptotic expansion of v_y. Note that the polarization tensors introduced by Ammari et al. in [6] are not defined for any frequency. Therefore, the threshold of the truncation is determined by the diameter of the target and is of order $O(\epsilon^{-\alpha})$ for $0 \leq \alpha < 1$, that is less than the lowest Dirichlet eigenvalue for $-\Delta$ on D. The time-dependent asymptotic expansions allows us to introduce the main concept of this paper, that is the concept of time-dependent polarization tensors, see Theorem 4.2 and Theorem 7.2. They are the truncated Fourier transform of the frequency-dependent polarization tensors.

This paper is organized as follows. In Section 2 we consider the Helmholtz equation in three dimensions. We derive in Section 3 complete asymptotic formulas for the three-dimensional Helmholtz equation and estimate the dependence of the remainders in these formulas with respect to the frequency. Based on these estimates, we obtain in Section 4 formulas for the three-dimensional transient wave equation that are valid after truncating the high-frequency components. They are written in terms of the new concept of three-dimensional time-dependent polarization tensors. In Section 5 we consider the Helmholtz equation in two dimensions. In Section 6 we derive complete asymptotic formulas for the two-dimensional Helmholtz equation and estimate the dependence of the remainders in these formulas with respect to the frequency. Note that these estimates are more involved in two dimensions than in three dimensions because of the behavior of the Hankel function. Based on these estimates, we obtain in Section 7 asymptotic formulas for the two-dimensional transient wave equation that are valid after truncating the high-frequency components. As in the three-dimensional case, these formulas are written in terms of the time-dependent polarization tensors.

2. The three-dimensional case: Preliminary results

Let $D \subset \mathbb{R}^3$, $D = z + \epsilon B$, $0 < \epsilon < 1$, $|B| = 1$. Denote by V_y the time-harmonic wave generated at $y \in \mathbb{R}^3 \setminus \bar{D}$, with $\mathrm{dist}(y, D) \geq c_0 > 0$ (we assume that the inclusion D and the point z are away from the source y), and operating frequency $\omega \geq 0$:

$$V_y(x, \omega) := \Gamma_\omega(x, y) = \frac{e^{i\omega|x-y|}}{4\pi|x-y|},$$

with $x \neq y$. Note that $\Gamma_\omega(x, y)$ satisfies

$$(\Delta_x + \omega^2)\Gamma_\omega(x, y) = -\delta_y(x),$$

and $|\Gamma_\omega(x, y)|$ does not depend on ω.

Let $k > 0$, $k \neq 1$. The pertubed field v_y is solution to

$$\nabla \cdot (\chi(\mathbb{R}^3 \setminus \bar{D}) + k\chi(D))\nabla v_y + \omega^2 v_y = -\delta_y, \tag{2.1}$$

and satisfies the Sommerfeld radiation condition. Equation (2.1) can be written as follows

$$\begin{cases} \Delta v_y + \omega^2 v_y = -\delta_y, & \mathbb{R}^3 \setminus \bar{D}, \\ \Delta v_y + \frac{\omega^2}{k} v_y = 0, & D, \\ v_y|_+ = v_y|_-, & \partial D, \\ \nu \cdot \nabla v_y|_+ = k\nu \cdot \nabla v_y|_-, & \partial D, \\ v_y \text{ satisfies the Sommerfeld radiation condition.} & \end{cases}$$

Note that v_y can be represented as follows [8]:

$$v_y(x,\omega) = \begin{cases} V_y(x,\omega) + S_D^{\omega}[\psi](x), & x \in \mathbb{R}^3 \setminus \bar{D}, \\ S_D^{\frac{\omega}{\sqrt{k}}}[\phi](x), & x \in D, \end{cases} \tag{2.2}$$

where $(\phi,\psi) \in L^2(\partial D) \times L^2(\partial D)$ is the unique solution of

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi] - S_D^{\omega}[\psi] = V_y, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi]|_- - \nu \cdot \nabla S_D^{\omega}[\psi]|_+ = \nu \cdot \nabla V_y, \end{cases} \quad \text{on } \partial D,$$

provided that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. Here, the single-layer potential is given by

$$S_D^{\omega}[\phi](x) = \int_{\partial D} \Gamma_\omega(x-s)\phi(s)\, d\sigma(s) \quad \text{for } \phi \in L^2(\partial D).$$

3. Asymptotic expansion for the three-dimensional frequency-dependent equation

Suppose that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. In addition, suppose that $\omega \in (0,\epsilon^{-\alpha})$, with $0<\alpha<1$. Note that there exists $\epsilon_0 > 0$ such that, for ϵ sufficiently small, $\epsilon\omega \le \epsilon_0 < 1$, i.e., $\epsilon\omega$ can be taken arbitrarily small. We need the following result.

Lemma 3.1. *Let $D = \epsilon B + z$, $|B| = 1$ and $D \subset \mathbb{R}^3$. Denote by $(\phi,\psi) \in L^2(\partial D) \times L^2(\partial D)$ the unique solution of the following system:*

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi] - S_D^{\omega}[\psi] = F, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi]|_- - \nu \cdot \nabla S_D^{\omega}[\psi]|_+ = G, \end{cases} \quad \text{on } \partial D. \tag{3.1}$$

Suppose there exists $\epsilon_0 > 0$ such that $\epsilon\omega \le \epsilon_0 < 1$. We have

$$\|\phi\|_{L^2(\partial D)} + \|\psi\|_{L^2(\partial D)} \le C(\epsilon^{-1}\|F\|_{L^2(\partial D)} + \|\nabla F\|_{L^2(\partial D)} + \|G\|_{L^2(\partial D)}), \tag{3.2}$$

where C does not depend on F, G, ϵ and ω.

Proof. Let

$$\tilde{\phi}(\tilde{x}) = \phi(\epsilon\tilde{x} + z), \quad \tilde{x} \in \partial B,$$

and define $\tilde{\psi}$, $\tilde{F}$ and $\tilde{G}$ likewise. After a change of variables, (3.1) becomes

$$\begin{cases} S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}] - S_B^{\epsilon\omega}[\tilde{\psi}] = \epsilon^{-1}\tilde{F}, \\ k\nu \cdot \nabla S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}]|_- - \nu \cdot \nabla S_B^{\epsilon\omega}[\tilde{\psi}]|_+ = \tilde{G}, \end{cases} \quad \text{on } \partial B.$$

Define an operator $T : L^2(\partial B) \times L^2(\partial B) \to H^1(\partial B) \times L^2(\partial B)$ by

$$T(\tilde{\phi},\tilde{\psi}) := (S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}] - S_B^{\epsilon\omega}[\tilde{\psi}], k\nu \cdot \nabla S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}]|_- - \nu \cdot \nabla S_B^{\epsilon\omega}[\tilde{\psi}]|_+).$$

We then decompose T as

$$T = T_0 + T_\epsilon,$$

where

$$T_0(\tilde{\phi}, \tilde{\psi}) := (S_B^0[\tilde{\phi}] - S_B^0[\tilde{\psi}], k\nu \cdot \nabla S_B^0[\tilde{\phi}]|_- - \nu \cdot \nabla S_B^0[\tilde{\psi}]|_+),$$

and

$$T_\epsilon := T - T_0.$$

For $\epsilon\omega < \epsilon_0$, with ϵ_0 small enough, we have

$$\|S_B^{\epsilon\omega}[\tilde{\phi}] - S_B^0[\tilde{\phi}]\|_{H^1(\partial B)} \leq C\epsilon\omega\|\tilde{\phi}\|_{L^2(\partial B)},$$

$$\|\nu \cdot \nabla S_B^{\epsilon\omega}[\tilde{\phi}]|_+ - \nu \cdot \nabla S_B^0[\tilde{\phi}]|_+\|_{L^2(\partial B)} \leq C\epsilon\omega\|\tilde{\phi}\|_{L^2(\partial B)},$$

where C does not depend on ϵ and ω. Therefore

$$\|T_\epsilon(\tilde{\phi}, \tilde{\psi})\|_{H^1(\partial B)\times L^2(\partial B)} \leq C\epsilon\omega(\|\tilde{\phi}\|_{L^2(\partial B)} + \|\tilde{\psi}\|_{L^2(\partial B)}).$$

Since T_0 is invertible [13, 16], T is invertible for $\epsilon\omega$ small enough and

$$T^{-1} = T_0^{-1} + E,$$

where the operator E satisfies

$$\|E(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{L^2(\partial B)\times L^2(\partial B)} \leq C\epsilon\omega\|(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{H^1(\partial B)\times L^2(\partial B)},$$

with C being independent of $\tilde{F}$, $\tilde{G}$, ϵ and ω. Finally, we have

$$(\tilde{\phi}, \tilde{\psi}) = T^{-1}(\epsilon^{-1}\tilde{F}, \tilde{G}) = T_0^{-1}(\epsilon^{-1}\tilde{F}, \tilde{G}) + E(\epsilon^{-1}\tilde{F}, \tilde{G}).$$

Assuming $\epsilon\omega$ small enough, it follows that

$$\|(\tilde{\phi}, \tilde{\psi})\|_{L^2(\partial B)\times L^2(\partial B)} \leq C\|(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{H^1(\partial B)\times L^2(\partial B)}.$$

where C does not depend on $\tilde{F}$, $\tilde{G}$, ϵ and ω. By scaling back, (3.2) holds true. □

Since $\operatorname{dist}(y, D) \geq c_0 > 0$, the function $V_y(x, \omega)$ is smooth for $x \in \bar{D}$. Then, for $x \in \partial D$ and z away from y, the following function is well defined:

$$V_{y,n}(x, \omega) := \sum_{|i|=0}^{n} \frac{\partial_z^i V_y(z, \omega)}{i!}(x - z)^i.$$

Denote by $(\phi_n, \psi_n) \in L^2(\partial D) \times L^2(\partial D)$ the unique solution of the following system:

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi_n] - S_D^\omega[\psi_n] = V_{y,n+1}, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi_n]|_- - \nu \cdot \nabla S_D^\omega[\psi_n]|_+ = \nu \cdot \nabla V_{y,n+1}, \end{cases} \quad \text{on } \partial D. \tag{3.3}$$

Then $(\phi - \phi_n, \psi - \psi_n)$ is the unique solution of

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi - \phi_n] - S_D^{\omega}[\psi - \psi_n] = V_y - V_{y,n+1}, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi - \phi_n]|_- - \nu \cdot \nabla S_D^{\omega}[\psi - \psi_n]|_+ = \nu \cdot \nabla (V_y - V_{y,n+1}), \end{cases} \quad \text{on } \partial D.$$

By Lemma 3.1, we have

$$\|\phi - \phi_n\|_{L^2(\partial D)} + \|\psi - \psi_n\|_{L^2(\partial D)} \leq C(\epsilon^{-1}\|V_y - V_{y,n+1}\|_{L^2(\partial D)} + \|\nabla(V_y - V_{y,n+1})\|_{L^2(\partial D)}),$$

where C does not depend on ϵ and ω. By Cauchy-Schwartz, we have

$$\|V_y - V_{y,n+1}\|_{L^2(\partial D)} = \left(\int_{\partial D} |V_y - V_{y,n+1}|^2 \right)^{1/2} \leq |\partial D|^{1/2} \|V_y - V_{y,n+1}\|_{L^\infty(\partial D)}.$$

In the following estimates, we assume that $\omega > 1$, since the case $\omega \leq 1$ is much easier to handle. By the definition of $V_{y,n+1}$, we have

$$\|V_y - V_{y,n+1}\|_{L^2(\partial D)} \leq C|\partial D|^{1/2} \|V_y - V_{y,n+1}\|_{L^\infty} \leq C|\partial D|^{1/2} \epsilon^{n+2} \|V_y\|_{C^{n+2}(\bar{D})},$$

and

$$\|\nabla(V_y - V_{y,n+1})\|_{L^2(\partial D)} \leq C|\partial D|^{1/2} \epsilon^{n+1} \|V_y\|_{C^{n+1}(\bar{D})}.$$

By a straightforward calculation, we observe that $\|V_y\|_{C^\ell(\bar{D})} \leq C\omega^\ell$, where C does not depend on ω. We have

$$\|V_y - V_{y,n+1}\|_{L^2(\partial D)} \leq C|\partial D|^{1/2} \epsilon^{n+2} \omega^{n+2},$$

and

$$\|\nabla(V_y - V_{y,n+1})\|_{L^2(\partial D)} \leq C|\partial D|^{1/2} \epsilon^{n+1} \omega^{n+2}.$$

We have proved that

$$\|\phi - \phi_n\|_{L^2(\partial D)} + \|\psi - \psi_n\|_{L^2(\partial D)} \leq C|\partial D|^{1/2} \epsilon^{n+1} \omega^{n+2}. \tag{3.4}$$

For $x \in \mathbb{R}^3 \setminus \bar{D}$, $x \neq y$, $\operatorname{dist}(x, D) \geq c_1 > 0$, the representation formula (2.2) yields

$$v(x, \omega) - V_y(x, \omega) = S_D^{\omega} \psi_n(x) + S_D^{\omega}[\psi - \psi_n](x).$$

Note that $\|\Gamma_\omega(x, \cdot)\|_{L^\infty(\partial D)} \leq C$, where C that does not depend on ω. By Cauchy-Schwarz and (3.4), we have

$$\begin{aligned} |S_D^{\omega}[\psi - \psi_n](x)| &\leq \left[\int_{\partial D} |\Gamma_\omega(x, s)|^2 d\sigma(s) \right]^{1/2} \|\psi - \psi_n\|_{L^2(\partial D)} \\ &\leq \|\Gamma_\omega(x, \cdot)\|_{L^\infty(\partial D)} |\partial D|^{1/2} |\partial D|^{1/2} \epsilon^{n+1} \omega^{n+2} \leq C\epsilon^{n+3} \omega^{n+2}, \end{aligned}$$

where C is independent of ϵ and ω. Then we have proved that

$$v(x) - V_y(x) = S_D^{\omega} \psi_n(x) + O(\epsilon^{n+3} \omega^{n+2}), \text{ for } x \in \mathbb{R}^3 \setminus \bar{D}, \ x \neq y. \tag{3.5}$$

For each multi-index i, define (ϕ_i, ψ_i) to be the unique solution to

$$\begin{cases} S_B^{\epsilon\omega}[\phi_i](\tilde{x}) - S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\psi_i](\tilde{x}) = \tilde{x}^i, \\ k\nu \cdot \nabla S_B^{\epsilon\omega}[\phi_i](\tilde{x})|_- - \nu \cdot \nabla S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\psi_i](\tilde{x})|_+ = \nu \cdot \nabla \tilde{x}^i, \end{cases} \quad \tilde{x} \in \partial B,$$

where $\tilde{x} = \epsilon^{-1}(x - z)$, $x \in \partial D$. The following proposition has been proved in [3].

Proposition 3.2. *We claim that*

$$\phi_n(x) = \sum_{|i|=0}^{n+1} \epsilon^{|i|-1} \frac{\partial_z^i V_y(z,\omega)}{i!} \phi_i(\epsilon^{-1}(x-z)), \tag{3.6}$$

$$\psi_n(x) = \sum_{|i|=0}^{n+1} \epsilon^{|i|-1} \frac{\partial_z^i V_y(z,\omega)}{i!} \psi_i(\epsilon^{-1}(x-z)), \tag{3.7}$$

for $x \in \partial D$ and (ϕ_n, ψ_n) defined as in (3.3).

Expansion (3.5) together with formula (3.7) yields:

$$v_y(x,\omega) - V_y(x,\omega) = \sum_{|i|=0}^{n+1} \epsilon^{|i|-1} \frac{\partial_z^i V_y(z,\omega)}{i!} S_D^\omega[\psi_i(\epsilon^{-1}(\cdot - z))](x) + O(\epsilon^{n+3}\omega^{n+2}), \tag{3.8}$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$ and $x \neq y$. Note that

$$S_D^\omega[\psi_i(\epsilon^{-1}(\cdot - z))](x) = \int_{\partial D} \Gamma_\omega(x,s)\psi_i(\epsilon^{-1}(s-z))\, d\sigma(s) = \epsilon^2 \int_{\partial B} \Gamma_\omega(x, \epsilon\tilde{s} + z)\psi_i(\tilde{s})\, d\sigma(\tilde{s}).$$

By a straightforward calculation, we observe that $\|\Gamma_\omega(x,\cdot)\|_{C^{n+2}(\bar{D})} \leq C\omega^{n+2}$, where C does not depend on ω. Therefore, for sufficiently small ϵ, we have

$$\Gamma_\omega(x, \epsilon\tilde{s} + z) = \sum_{|j|=0}^{n+1} \frac{\epsilon^{|j|}}{j!} \partial_z^j \Gamma_\omega(x,z)\tilde{s}^j + O(\epsilon^{n+2}\omega^{n+2}).$$

Finally, we get

$$S_D^\omega[\psi_i(\epsilon^{-1}(\cdot - z))](x) = \sum_{|j|=0}^{n+1} \frac{\epsilon^{|j|+2}}{j!} \partial_z^j \Gamma_\omega(x,z) \int_{\partial B} \tilde{s}^j \psi_i(\tilde{s})\, d\sigma(\tilde{s}) + O(\epsilon^{n+4}\omega^{n+2}). \tag{3.9}$$

For multi-indices i and j in $\mathbb{N}^3$, the frequency dependent polarization tensors (FDPTs) $\hat{W}_{ij} := \hat{W}_{ij}(B, \epsilon\omega, \frac{\epsilon\omega}{\sqrt{k}})$ are defined as follows [6, 8]:

$$\hat{W}_{ij} := \int_{\partial B} \tilde{s}^j \psi_i(\tilde{s})\, d\sigma(\tilde{s}). \tag{3.10}$$

We obtain the following theorem from (3.8) and (3.9).

Theorem 3.3. *Suppose that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D and $\omega \in (0, \epsilon^{-\alpha})$, with $0 < \alpha < 1$. The following asymptotic expansion holds:*

$$v_y(x,\omega) - V_y(x,\omega) = \sum_{|j|=0}^{n+1} \sum_{|i|=0}^{n-|j|+1} \frac{\epsilon^{|i|+|j|+1}}{i!\, j!} \partial_z^i V_y(z,\omega) \partial_z^j \Gamma_\omega(x,z) \hat{W}_{ij} + O(\epsilon^{n+3}(1+\omega^{n+2})), \tag{3.11}$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$ and $x \neq y$.

By using [3], one can recover from (3.11) the leading-order term of the scattered field derived in [7]

$$v_y(x,\omega) - V_y(x,\omega) = \epsilon^3 \nabla_z V_y(z,\omega) M(B) \nabla_z \Gamma_\omega(x,z) + O(\epsilon^4\omega^3),$$

where M is the polarization tensor [4] and $\omega > 1$.

4. Three-dimensional time-dependent asymptotic expansion

Define

$$U_y(x,t) := \frac{\delta(t-|x-y|)}{4\pi|x-y|},$$

where δ is the Dirac mass at 0. U_y satisfies

$$\begin{cases} (\partial_t^2 - \Delta)U_y(x,t) = \delta_{x=y}\delta_{t=0}, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ U_y(x,t) = 0, & \text{for } x \in \mathbb{R}^3 \text{ and } t \ll 0. \end{cases}$$

For $\rho > 0$, we define the operator P_ρ on tempered distributions by

$$P_\rho[\psi](t) = \int\limits_{|\omega|\le\rho} e^{-i\omega t}\hat{\psi}(\omega)\, d\omega, \tag{4.1}$$

where $\hat{\psi}$ is the Fourier transform of ψ. The operator P_ρ truncates the high-frequency component of ψ. Since

$$\hat{U}_y(x,\omega) := \int_{\mathbb{R}} e^{i\omega t}U_y(x,t)\, dt = \frac{e^{i\omega|x-y|}}{4\pi|x-y|} = V_y(x,\omega),$$

it follows that

$$P_\rho[U_y](x,t) = \frac{\psi_\rho(t-|x-y|)}{4\pi|x-y|},$$

where

$$\psi_\rho(t) := \frac{2\sin\rho t}{t} = \int\limits_{|\omega|\le\rho} e^{-i\omega t}\, d\omega.$$

Therefore, $P_\rho[U_y]$ satisfies

$$(\partial_t^2 - \Delta)P_\rho[U_y](x,t) = \delta_{x=y}\psi_\rho(t) \text{ in } \mathbb{R}^3 \times \mathbb{R}.$$

For $u_y = u_y(x,t)$, we consider the wave equation

$$\begin{cases} \partial_t^2 u_y - \nabla\cdot(\chi(\mathbb{R}^3 \setminus \bar{D}) + k\chi(D))\nabla u_y = \delta_{x=y}\delta_{t=0} & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u_y(x,t) = 0 & \text{for } x \in \mathbb{R}^3 \text{ and } t \ll 0. \end{cases}$$

We want to derive an asymptotic expansion for $P_\rho[u_y - U_y](x,t)$. We have

$$P_\rho[u_y](x,t) = \int\limits_{|\omega|\le\rho} e^{-i\omega t}v_y(x,\omega)\, d\omega,$$

where v_y is the solution to (2.1). To introduce $P_\rho[W_{ij}](x,t)$, we must remember that the frequency-dependent $\hat{W}_{ij}$ are defined when system (3.3) has unique solution, i.e., ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. Given the n-dimensional isoperimetric inequality [10, 17]:

$$\lambda_1(D) \ge \left(\frac{1}{|D|}\right)^{2/n} C_n^{2/n} j_{n/2-1,1},$$

where $j_{m,1}$ is the first positive zero of the Bessel function J_m, C_n the volume of the n-dimensional unit ball, and $\lambda_1(D) > 0$ is the lowest eigenvalue for $-\Delta$ on D, it is sufficient to take $|\omega| < 1/\epsilon$.

Definition 4.1. For $\rho < 1/\epsilon$ and multi-indices i and j, the three-dimensional truncated time-dependent polarization tensors (TTDPTs), $P_\rho[W_{ij}]$, are defined as follows:

$$P_\rho[W_{ij}](x,t) = \int\limits_{|\omega|\le\rho} e^{-i\omega t}\hat{W}_{ij}\, d\omega, \tag{4.2}$$

where $\hat{W}_{ij}$ are the FDPTs given by (3.10).

From Theorem 3.3, we have

$$\int\limits_{|\omega|\le\rho} e^{-i\omega t}(v_y(x,\omega) - V_y(x,\omega))\, d\omega = \epsilon \sum_{|j|=0}^{n+1} \sum_{|i|=0}^{n-|j|+1} \frac{\epsilon^{|i|+|j|}}{i!\, j!} \int\limits_{|\omega|\le\rho} e^{-i\omega t}\partial_z^i V_y(z,\omega)\partial_z^j \Gamma_\omega(x,z)\hat{W}_{ij}\, d\omega$$
$$+ \int\limits_{|\omega|\le\rho} e^{-i\omega t} R(x,\omega)\, d\omega,$$

where

$$\int\limits_{|\omega|\le\rho} e^{-i\omega t} R(x,\omega)\, d\omega = O(\epsilon^{n+3}\rho^{n+3}).$$

Suppose that $\rho = O(\epsilon^{-\alpha})$ for some positive $\alpha < 1$. Then

$$\int\limits_{|\omega|\le\rho} e^{-i\omega t} R(x,\omega)\, d\omega = O\left(\epsilon^{(n+3)(1-\alpha)}\right).$$

Since

$$\int\limits_{|\omega|\le\rho} e^{-i\omega t}\partial_z^i V_y(z,\omega)\partial_z^j\Gamma_\omega(x,z)\hat{W}_{ij}(\omega;B)\, d\omega = \int_{\mathbb{R}^2} \partial_z^i P_\rho[U_y](z,t-\tau-\tau')\partial_z^j P_\rho[U_z](x,\tau)P_\rho[W_{ij}](\tau')\, d\tau\, d\tau',$$

we have proved the following theorem.

Theorem 4.2. *For $0 < \alpha < 1$, the following asymptotic expansion holds:*

$$\begin{aligned} &P_\rho[u_y](x,t) = P_\rho[U_y](x,t) \\ &+ \epsilon \sum_{|j|=0}^{n+1} \sum_{|i|=0}^{n-|j|+1} \frac{\epsilon^{|i|+|j|}}{i!\, j!} \int_{\mathbb{R}} \partial_z^j P_\rho[U_z](x,\tau) \left(\int_{\mathbb{R}} \partial_z^i P_\rho[U_y](z,t-\tau-\tau')P_\rho[W_{ij}](\tau')\, d\tau' \right) d\tau \\ &+ O\left(\epsilon^{(n+3)(1-\alpha)}\right), \end{aligned} \tag{4.3}$$

where $x \in \mathbb{R}^3 \setminus \bar{D}$, $D = \epsilon B + z$, $|B| = 1$, $P_\rho[W_{ij}]$ are the TTDPTs defined in (4.2) and $\rho = O(\epsilon^{-\alpha})$.

5. The two-dimensional case: Preliminary results

Let $D \subset \mathbb{R}^2$, $D = z + \epsilon B$, $0 < \epsilon < 1$, $|B| = 1$. Denote by V_y the time-harmonic wave generated at $y \in \mathbb{R}^2 \setminus \bar{D}$, with $\operatorname{dist}(y, D) \geq c_0 > 0$ (we assume that the inclusion D and the point z are away from the source y), and operating frequency $\omega > 0$:

$$V_y(x, \omega) := \Gamma_\omega(x, y) = \frac{i}{4} H_0^{(1)}(\omega |x - y|),$$

with $x \neq y$. Note that $\Gamma_\omega(x, y)$ satisfies

$$(\Delta + \omega^2)\Gamma_\omega(x, y) = -\delta_y(x),$$

for $x \in \mathbb{R}^2$.

Let $k > 0$, $k \neq 1$. The pertubed field v_y is solution to

$$\nabla \cdot (\chi(\mathbb{R}^2 \setminus \bar{D}) + k\chi(D))\nabla v_y + \omega^2 v_y = -\delta_y, \tag{5.1}$$

and satisfies the Sommerfeld radiation condition. Equation (5.1) can be written as follows

$$\begin{cases} \Delta v_y + \omega^2 v_y = -\delta_y, & \mathbb{R}^2 \setminus \bar{D}, \\ \Delta v_y + \frac{\omega^2}{k} v_y = 0, & D, \\ v_y|_+ = v_y|_-, & \partial D, \\ \nu \cdot \nabla v_y|_+ = k\nu \cdot \nabla v_y|_-, & \partial D, \\ v_y \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

As in the three-dimensional case, v_y can be represented as follows:

$$v_y(x, \omega) = \begin{cases} V_y(x, \omega) + S_D^\omega[\psi](x), & x \in \mathbb{R}^2 \setminus \bar{D}, \\ S_D^{\frac{\omega}{\sqrt{k}}}[\phi](x), & x \in D, \end{cases} \tag{5.2}$$

where $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ is the unique solution of

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi] - S_D^\omega[\psi] = V_y, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi]|_- - \nu \cdot \nabla S_D^\omega[\psi]|_+ = \nu \cdot \nabla V_y, \end{cases} \quad \text{on } \partial D,$$

provided that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D.

6. Asymptotic expansion for the two-dimensional frequency-dependent equation

Suppose that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. In addition, suppose that $\omega \in (0, \epsilon^{-\alpha})$, with $0 < \alpha < 1$. Note that there exists $\epsilon_0 > 0$ such that, for ϵ sufficiently small, $\epsilon\omega \leq \epsilon_0 < 1$, i.e., $\epsilon\omega$ can be taken arbitrarily small. We need the following result.

Lemma 6.1. *Let $D = \epsilon B + z$, $|B| = 1$ and $D \subset \mathbb{R}^2$. For each $(F, G) \in H^1(\partial B) \times L^2(\partial B)$, denote by $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ the unique solution of the following system:*

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi] - S_D^{\omega}[\psi] = F, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi]|_- - \nu \cdot \nabla S_D^{\omega}[\psi]|_+ = G, \end{cases} \quad \text{on } \partial D. \tag{6.1}$$

Suppose there exists $\epsilon_0 > 0$ such that $\epsilon\omega \leq \epsilon_0 < 1$. We have

$$\|\phi\|_{L^2(\partial D)} + \|\psi\|_{L^2(\partial D)} \leq C(\epsilon^{-1}\|F\|_{L^2(\partial D)} + \|\nabla F\|_{L^2(\partial D)} + \|G\|_{L^2(\partial D)}), \tag{6.2}$$

where C does not depend on ϵ and ω.

Since the two-dimensional fundamental solutions $\Gamma_{\epsilon\omega}(x, y)$ and $\Gamma_{(\epsilon\omega)/k}(x, y)$ do not converge to $\Gamma_0(x, y) = 1/(2\pi)\log|x - y|$ as ϵ goes to zero, we need a few facts to prove Lemma 6.1. The following results are an application of [8, 14, 15].

Lemma 6.2. *Suppose that ∂B is of class $\mathcal{C}^2$. For each $(F, G, a) \in H^1(\partial B) \times L^2(\partial B) \times \mathbb{R}$, there exists a unique solution $(\phi, \psi, c) \in L^2(\partial B) \times L^2(\partial B) \times \mathbb{C}$ such that*

$$\begin{cases} S_B^0[\phi] - S_B^0[\psi] - c = F \\ k\nu \cdot \nabla S_B^0[\phi]|_- - \nu \cdot \nabla S_B^0[\psi]|_+ = G \\ \int_{\partial B}(\psi - \phi) = a, \end{cases} \quad \text{on } \partial B. \tag{6.3}$$

Proof. We set $f = \psi - \phi$. By Lemma 2.2 of [15], there exists a unique solution $(f, c) \in L^2(\partial B) \times \mathbb{C}$ such that

$$\begin{cases} S_B^0[f] + c = -F \\ \int_{\partial B} f = a, \end{cases} \quad \text{on } \partial B.$$

Therefore, ϕ and $\psi = \phi + f$ will be the unique solution of (6.3) provided

$$k\nu \cdot \nabla S_B^0[\phi]|_- = \nu \cdot \nabla S_B^0[\psi]|_+ + G,$$

or

$$(1 - k)\left(\frac{k+1}{2(k-1)}I - \mathcal{K}_B^*\right)[\phi] = \left(\frac{1}{2}I + \mathcal{K}_B^*\right)[f] + G,$$

where $\mathcal{K}_B^*$ is the Neumann-Poincaré operator. Hence (see [12])

$$\phi = \left(\frac{k+1}{2(k-1)}I - \mathcal{K}_B^*\right)^{-1}\left[\frac{1}{1-k}\left(\left(\frac{1}{2}I + \mathcal{K}_B^*\right)[f] + G\right)\right],$$

$$\psi = \left(\frac{k+1}{2(k-1)}I - \mathcal{K}_B^*\right)^{-1}\left[\frac{1}{1-k}\left(\left(\frac{1}{2}I + \mathcal{K}_B^*\right)[f] + G\right)\right] + f.$$

□

Lemma 6.3. *For each $(F, G) \in H^1(\partial B) \times L^2(\partial B)$, there exists a unique solution $(\phi, \psi) \in L^2(\partial B) \times L^2(\partial B)$, such that*

$$\begin{cases} S_B^0[\phi] + \beta_{\epsilon\omega}\int_{\partial B}\phi - S_B^0[\psi] - \beta_{\epsilon\omega}\int_{\partial B}\psi = F \\ k\nu \cdot \nabla S_B^0[\phi]|_- - \nu \cdot \nabla S_B^0[\phi]|_+ = G, \end{cases} \quad \text{on } \partial B, \tag{6.4}$$

where $\beta_{\epsilon\omega} = (1/2\pi)\ln\epsilon\omega + \gamma - i/4$, γ is the Euler constant. Moreover, for $\epsilon\omega \le \epsilon_0$, ϵ_0 sufficiently small, there exists $C > 0$ independent of F, G, ϵ and ω such that

$$\|\phi\|_{L^2(\partial B)} + \|\psi\|_{L^2(\partial B)} \le C(\|F\|_{H^1(\partial B)} + \|G\|_{L^2(\partial B)}). \tag{6.5}$$

Proof. Existence

Define (ϕ_0, ψ_0, c_0) by

$$\begin{cases} S_B^0[\phi_0] = S_B^0[\psi_0] + c_0 \\ k\nu \cdot \nabla S_B^0[\phi_0]|_- - \nu \cdot \nabla S_B^0[\psi_0]|_+ = 0 \qquad \text{on } \partial B. \\ \int_{\partial B}(\psi_0 - \phi_0) = 1, \end{cases}$$

This has a unique solution $(\phi_0, \psi_0, c_0) \in L^2(\partial B) \times L^2(\partial B) \times \mathbb{C}$ by Lemma 6.2. Obviously, there exists a constant $C_1 > 0$ independent of ϵ and ω such that

$$\|\phi_0\|_{L^2(\partial B)} + \|\psi_0\|_{L^2(\partial B)} \le C_1. \tag{6.6}$$

Define $(\phi_1, \psi_1, c_1) \in L^2(\partial B) \times L^2(\partial B) \times \mathbb{C}$ by

$$\begin{cases} S_B^0[\phi_1] = S_B^0[\psi_1] + c_1 + F \\ k\nu \cdot \nabla S_B^0[\phi_1]|_- - \nu \cdot \nabla S_B^0[\psi_1]|_+ = G \qquad \text{on } \partial B. \\ \int_{\partial B}(\psi_1 - \phi_1) = 0, \end{cases}$$

This has a unique solution $(\phi_1, \psi_1, c_1) \in L^2(\partial B) \times L^2(\partial B) \times \mathbb{C}$ by Lemma 6.2. Moreover, there exists a constant $C_2 > 0$ independent of F, G, ϵ and ω such that

$$\|\phi_1\|_{L^2(\partial B)} + \|\psi_1\|_{L^2(\partial B)} \le C_2(\|F\|_{H^1(\partial B)} + \|G\|_{L^2(\partial B)}), \tag{6.7}$$

and

$$|c_1| \le C_2(\|F\|_{H^1(\partial B)} + \|G\|_{L^2(\partial B)}). \tag{6.8}$$

These estimates are a consequence of solvability and the closed graph theorem. Let

$$(\phi, \psi) = (\phi_1, \psi_1) + p(\phi_0, \psi_0), \tag{6.9}$$

for some $p \in \mathbb{C}$ being dependent of ϵ and ω. This will satisfy (6.4) provided

$$S_B^0[\phi_1] + pS_B^0[\phi_0] + \beta_{\epsilon\omega}\int_{\partial B}\phi_1 + p\beta_{\epsilon\omega}\int_{\partial B}\phi_0 - S_B^0[\psi_1] - pS_B^0[\psi_0] - \beta_{\epsilon\omega}\int_{\partial B}\psi_1 - p\beta_{\epsilon\omega}\int_{\partial B}\psi_0 = F,$$

or

$$c_1 + F + pc_0 - p\beta_{\epsilon\omega} = F,$$

or

$$p = \frac{c_1}{\beta_{\epsilon\omega} - c_0}. \tag{6.10}$$

Note that c_0 is a complex constant but independent of ϵ and ω (or $\beta_{\epsilon\omega}$). Thus c_0 would possibly equal to one value of $\beta_{\epsilon\omega}$ which we rule out by taking $\epsilon\omega \le \epsilon_0$, with ϵ_0 being sufficiently small.

Uniqueness

Consider

$$\begin{cases} S_B^0[\phi] + \beta_{\epsilon\omega} \int_{\partial B} \phi - S_B^0[\psi] - \beta_{\epsilon\omega} \int_{\partial B} \psi = 0 \\ k\nu \cdot \nabla S_B^0[\phi]|_- - \nu \cdot \nabla S_B^0[\phi]|_+ = 0, \end{cases} \quad \text{on } \partial B. \tag{6.11}$$

Case (i): $\int_{\partial B}(\psi - \phi) = 0$

This is the homogeneous case of Lemma 6.2 and we find $(\phi, \psi) = (0, 0)$.

Case (ii): $\int_{\partial B}(\psi - \phi) \neq 0$

Define

$$\hat{\psi} = \frac{\psi}{\int_{\partial B}(\psi - \phi)},$$

$$\hat{\phi} = \frac{\phi}{\int_{\partial B}(\psi - \phi)},$$

then

$$\int_{\partial B} (\hat{\psi} - \hat{\phi}) = 1.$$

System (6.11) becomes

$$\begin{cases} S_B^0[\hat{\phi}] - S_B^0[\hat{\psi}] - \beta_{\epsilon\omega} = 0 \\ k\nu \cdot \nabla S_B^0[\hat{\phi}]|_- - \nu \cdot \nabla S_B^0[\hat{\phi}]|_+ = 0, \\ \int_{\partial B}(\hat{\psi} - \hat{\phi}) = 1, \end{cases} \quad \text{on } \partial B.$$

Lemma 6.2 for $F = G = 0$ yields $\beta_{\epsilon\omega} = c_0$, but this could not happen since c_0 is independent of ϵ and ω. Hence uniqueness is proved.

Estimate (6.5)

The estimate (6.5) is a consequence of (6.6), (6.7), (6.8), (6.9) and (6.10). We have

$$\begin{aligned} \|\phi\|_{L^2(\partial B)} + \|\psi\|_{L^2(\partial B)} &\leq \|\phi_1\|_{L^2(\partial B)} + \|\psi_1\|_{L^2(\partial B)} + |p|(\|\phi_0\|_{L^2(\partial B)} + \|\psi_0\|_{L^2(\partial B)}) \\ &\leq C_2(\|F\|_{H^1(\partial B)} + \|G\|_{L^2(\partial B)}) + C_1 C_2 (\ln \epsilon\omega)^{-1} (\|F\|_{H^1(\partial B)} + \|G\|_{L^2(\partial B)}) \\ &\leq C(\|F\|_{H^1(\partial B)} + \|G\|_{L^2(\partial B)}), \end{aligned}$$

where C is independent of F, G, ϵ and ω, for $\epsilon\omega \leq \epsilon_0$, with ϵ_0 being sufficiently small. □

We are now ready to prove Lemma 6.1.

Proof of Lemma 6.1. Let

$$\tilde{\phi}(\tilde{x}) = \phi(\epsilon\tilde{x} + z), \quad \tilde{x} \in \partial B,$$

and define $\tilde{\psi}$, $\tilde{F}$ and $\tilde{G}$ likewise. After a change of variables, (6.1) becomes

$$\begin{cases} S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}] - S_B^{\epsilon\omega}[\tilde{\psi}] = \epsilon^{-1}\tilde{F}, \\ k\nu \cdot \nabla S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}]|_- - \nu \cdot \nabla S_B^{\epsilon\omega}[\tilde{\psi}]|_+ = \tilde{G}, \end{cases} \quad \text{on } \partial B.$$

Define an operator $T : L^2(\partial B) \times L^2(\partial B) \to H^1(\partial B) \times L^2(\partial B)$ by

$$T(\tilde{\phi}, \tilde{\psi}) := (S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}] - S_B^{\epsilon\omega}[\tilde{\psi}], k\nu \cdot \nabla S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\phi}]|_- - \nu \cdot \nabla S_B^{\epsilon\omega}[\tilde{\psi}]|_+).$$

We then decompose T as

$$T = T_0 + T_\epsilon$$

where

$$T_0(\tilde{\phi}, \tilde{\psi}) := \left(S_B^0[\tilde{\phi}] + \beta_{\epsilon\omega} \int_{\partial B} \tilde{\phi} - S_B^0[\tilde{\psi}] - \beta_{\epsilon\omega} \int_{\partial B} \tilde{\psi}, k\nu \cdot \nabla S_B^0[\tilde{\phi}]|_- - \nu \cdot \nabla S_B^0[\tilde{\psi}]|_+ \right),$$

and

$$T_\epsilon := T - T_0.$$

For $\epsilon\omega < \epsilon_0$, with ϵ_0 small enough, we have [8, 14]

$$\|T_\epsilon(\tilde{\phi}, \tilde{\psi})\|_{H^1(\partial B) \times L^2(\partial B)} \leq C(\epsilon\omega)^2 (\ln \epsilon\omega)(\|\tilde{\phi}\|_{L^2(\partial B)} + \|\tilde{\psi}\|_{L^2(\partial B)}),$$

where C does not depend on ϵ and ω. By Lemma 6.3, T_0 is invertible. Therefore, T is invertible for $\epsilon\omega$ small enough and

$$T^{-1} = T_0^{-1} + E,$$

where the operator E satisfies

$$\|E(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{L^2(\partial B) \times L^2(\partial B)} \leq C(\epsilon\omega)^2 (\ln \epsilon\omega) \|(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{H^1(\partial B) \times L^2(\partial B)},$$

with C being independent of $\tilde{F}$, $\tilde{G}$, ϵ and ω. Finally, we have

$$(\tilde{\phi}, \tilde{\psi}) = T^{-1}(\epsilon^{-1}\tilde{F}, \tilde{G}) = T_0^{-1}(\epsilon^{-1}\tilde{F}, \tilde{G}) + E(\epsilon^{-1}\tilde{F}, \tilde{G}) = (\tilde{\phi}_0, \tilde{\psi}_0) + E(\epsilon^{-1}\tilde{F}, \tilde{G}).$$

Assuming $\epsilon\omega$ small enough, it follows that

$$\begin{aligned}
\|(\tilde{\phi}, \tilde{\psi})\|_{L^2(\partial B) \times L^2(\partial B)} &\leq C\|(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{H^1(\partial B) \times L^2(\partial B)} + C(\epsilon\omega)^2 (\ln \epsilon\omega) \|(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{H^1(\partial B) \times L^2(\partial B)} \\
&\leq C\|(\epsilon^{-1}\tilde{F}, \tilde{G})\|_{H^1(\partial B) \times L^2(\partial B)},
\end{aligned}$$

where C does not depend on ϵ and ω. By scaling back, (6.2) holds true.

□

For $x \in \partial D$, z away from y, and $\omega > 0$, the following function is well defined:

$$V_{y,n}(x, \omega) := \sum_{|i|=0}^{n} \frac{\partial_z^i V_y(z, \omega)}{i!} (x - z)^i.$$

Denote by $(\phi_n, \psi_n) \in L^2(\partial D) \times L^2(\partial D)$ the unique solution of the following system:

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi_n] - S_D^{\omega}[\psi_n] = V_{y,n+1}, \\ k\nu \cdot \nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi_n]|_- - \nu \cdot \nabla S_D^{\omega}[\psi_n]|_+ = \nu \cdot \nabla V_{y,n+1}, \end{cases} \quad \text{on } \partial D.$$

Then $(\phi-\phi_n, \psi-\psi_n)$ is the unique solution of

$$\begin{cases} S_D^{\frac{\omega}{\sqrt{k}}}[\phi-\phi_n] - S_D^{\omega}[\psi-\psi_n] = V_y - V_{y,n+1}, \\ k\nu\cdot\nabla S_D^{\frac{\omega}{\sqrt{k}}}[\phi-\phi_n]|_- - \nu\cdot\nabla S_D^{\omega}[\psi-\psi_n]|_+ = \nu\cdot\nabla(V_y - V_{y,n+1}), \end{cases} \quad \text{on } \partial D.$$

By Lemma 6.1, we have

$$\|\phi-\phi_n\|_{L^2(\partial D)} + \|\psi-\psi_n\|_{L^2(\partial D)} \le C(\epsilon^{-1}\|V_y - V_{y,n+1}\|_{L^2(\partial D)} + \|\nabla(V_y - V_{y,n+1})\|_{L^2(\partial D)}),$$

where C does not depend on ϵ and ω. By definition of $V_y - V_{y,n+1}$, we have

$$\|V_y - V_{y,n+1}\|_{L^2(\partial D)} = \left(\int_{\partial D} |V_y - V_{y,n+1}|^2\right)^{1/2} \le |\partial D|^{1/2}\|V_y - V_{y,n+1}\|_{L^\infty(\partial D)}.$$

The derivation of asymptotic expansion (3.11) is more involved in two dimensions than in three dimensions because of the logarithmic singularity of the Green function [1, 8]. For $n \ge 0$, $H_n^{(1)}(\omega|z-y|)$ is bounded when $\omega \ge 1$. In particular, we have

$$H_n^{(1)}(\omega|z-y|) = O(\omega^{-1/2}).$$

For $\omega < 1$, $H_0^{(1)}(\omega|z-y|) = O(\ln\omega)$. For $n \ge 1$, we have

$$H_n^{(1)}(\omega|z-y|) = O(\omega^{-n}).$$

Moreover, recall that

$$(H_0^{(1)})'(x) = -H_1^{(1)}(x),$$

and

$$(H_n^{(1)})'(x) = \frac{1}{2}(-H_{n-1}^{(1)}(x) + H_{n+1}^{(1)}(x)).$$

In the following estimates we assume that $\omega \in (0, \epsilon^{-\alpha})$, with $0 < \alpha < 1$. By expanding in Taylor series, we obtain

$$\|V_y - V_{y,n+1}\|_{L^2(\partial D)} \le C|\partial D|^{1/2}\epsilon^{n+2}(1+\omega^{n+3/2}),$$

and

$$\|\nabla(V_y - V_{y,n+1})\|_{L^2(\partial D)} \le C|\partial D|^{1/2}\epsilon^{n+1}(1+\omega^{n+3/2}).$$

We have proved that

$$\|\phi-\phi_n\|_{L^2(\partial D)} + \|\psi-\psi_n\|_{L^2(\partial D)} \le C|\partial D|^{1/2}\epsilon^{n+1}(1+\omega^{n+3/2}). \tag{6.12}$$

For $x \in \mathbb{R}^2 \setminus \bar{D}$, $x \ne y$, $\mathrm{dist}(x, D) \ge c_1 > 0$, representation formula (5.2) yields

$$v(x,\omega) - V_y(x,\omega) = S_D^{\omega}\psi_n(x) + S_D^{\omega}[\psi-\psi_n](x).$$

By Cauchy-Schwarz, we have

$$\begin{aligned} |S_D^{\omega}[\psi-\psi_n](x)| &\le \left[\int_{\partial D} |\Gamma_\omega(x,s)|^2 d\sigma(s)\right]^{1/2} \|\psi-\psi_n\|_{L^2(\partial D)} \\ &\le \|\Gamma_\omega(x,\cdot)\|_{L^\infty(\partial D)}|\partial D|^{1/2}\|\psi-\psi_n\|_{L^2(\partial D)}. \end{aligned}$$

By (6.12),

$$|S_D^\omega[\psi - \psi_n](x)| \le C\epsilon^{n+2}(|\ln\omega| + \omega^{n+1}),$$

where C is independent of ϵ and ω.

As in the three-dimensional case, we obtain the following theorem.

Theorem 6.4. *Suppose that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on D and $\omega \in (0, \epsilon^{-\alpha})$, with $0 < \alpha < 1$. The following asymptotic expansion holds:*

$$v_y(x,\omega) - V_y(x,\omega) = \sum_{|j|=0}^{n+1} \sum_{|i|=0}^{n-|j|+1} \frac{\epsilon^{|i|+|j|}}{i!\, j!} \partial_z^i V_y(z,\omega) \partial_z^j \Gamma_\omega(x,z) \hat{W}_{ij} + R(x,\omega), \tag{6.13}$$

where

$$R(x,\omega) \le C\epsilon^{n+2}(|\ln\omega| + \omega^{n+1}), \tag{6.14}$$

for $x \in \mathbb{R}^2 \setminus \bar{D}$, $x \neq y$, and the two-dimensional FDPTs $\hat{W}_{ij}$.

7. Two-dimensional time-dependent asymptotic expansion

Define

$$U_y(x,t) := \frac{H(t - |x-y|)}{2\pi\sqrt{t^2 - |x-y|^2}},$$

where H is the Heaviside function at 0. U_y satisfies

$$\begin{cases} (\partial_t^2 - \Delta) U_y(x,t) = \delta_{x=y}\delta_{t=0}, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \\ U_y(x,t) = 0, & \text{for } x \in \mathbb{R}^3 \text{ and } t \ll 0. \end{cases}$$

For $\rho > 0$, consider the operator P_ρ on tempered distributions defined (4.1). Note that

$$P_\rho[U_y](x,t) = \int\limits_{|\omega|\le\rho} e^{-i\omega t} \left(\int_{\mathbb{R}} e^{i\omega t} U_y(x,t)\, dt \right) d\omega = \int\limits_{|\omega|\le\rho} e^{-i\omega t} \frac{i}{4} H_0^{(1)}(\omega|x-y|).$$

and satisfies

$$(\partial_t^2 - \Delta) P_\rho[U_y](x,t) = \delta_{x=y}\psi_\rho(t) \text{ in } \mathbb{R}^2 \times \mathbb{R},$$

For $u_y = u_y(x,t)$, we consider the wave equation

$$\begin{cases} \partial_t^2 u_y - \nabla \cdot (\chi(\mathbb{R}^2 \setminus \bar{D}) + k\chi(D)) \nabla u_y = \delta_{x=y}\delta_{t=0} & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u_y(x,t) = 0 & \text{for } x \in \mathbb{R}^2 \text{ and } t \ll 0. \end{cases}$$

We want to derive an asymptotic expansion for $P_\rho[u_y - U_y](x,t)$.

Definition 7.1. For $\rho < 1/\epsilon$ and multi-indices i and j, the two-dimensional truncated time-dependent polarization tensors (TTDPTs), $P_\rho[W_{ij}]$, are defined as follows:

$$P_\rho[W_{ij}](x,t) = \int\limits_{|\omega|\le\rho} e^{-i\omega t} \hat{W}_{ij}\, d\omega, \tag{7.1}$$

where $\hat{W}_{ij}$ are the two-dimensional FDPTs.

From Theorem 6.4, we have

$$\int_{|\omega|\leq\rho} e^{-i\omega t}(v_y(x,\omega) - V_y(x,\omega))\, d\omega = \sum_{|j|=0}^{n+1}\sum_{|i|=0}^{n-|j|+1} \frac{\epsilon^{|i|+|j|}}{i!\, j!} \int_{|\omega|\leq\rho} e^{-i\omega t}\partial_z^i V_y(z,\omega)\partial_z^j \Gamma_\omega(x,z)\hat{W}_{ij}\, d\omega + \int_{|\omega|\leq\rho} e^{-i\omega t} R(x,\omega)\, d\omega.$$

Suppose that $\rho = O(\epsilon^{-\alpha})$ for some $\alpha < 1$. Then

$$\left| \int_{|\omega|\leq\rho} e^{-i\omega t} R(x,\omega)\, d\omega \right| = O\left(\epsilon^{(n+2)(1-\alpha)}\right).$$

Since

$$\int_{|\omega|\leq\rho} e^{-i\omega t}\partial_z^i V_y(z,\omega)\partial_z^j \Gamma_\omega(x,z)\hat{W}_{ij}\, d\omega = \int_{\mathbb{R}^2} \partial_z^i P_\rho[U_y](z, t-\tau-\tau')\partial_z^j P_\rho[U_z](x,\tau) P_\rho[W_{ij}](\tau')\, d\tau\, d\tau',$$

we have proved the following theorem.

Theorem 7.2. *For $0 < \alpha < 1$, the following asymptotic expansion holds:*

$$\begin{aligned} &P_\rho[u_y](x,t) = P_\rho[U_y](x,t) \\ &+ \sum_{|j|=0}^{n+1}\sum_{|i|=0}^{n-|j|+1} \frac{\epsilon^{|i|+|j|}}{i!\, j!} \int_{\mathbb{R}} \partial_z^j P_\rho[U_z](x,\tau) \left(\int_{\mathbb{R}} \partial_z^i P_\rho[U_y](z, t-\tau-\tau') P_\rho[W_{ij}](\tau')\, d\tau' \right) d\tau \\ &+ O\left(\epsilon^{(n+2)(1-\alpha)}\right), \end{aligned} \tag{7.2}$$

where $x \in \mathbb{R}^2 \setminus \bar{D}$, $D = \epsilon B + z$, $|B| = 1$, $P_\rho[W_{ij}]$ are the TTDPTs defined in (7.1) *and $\rho = O(\epsilon^{-\alpha})$.*

Acknowledgments

The author gratefully acknowledges Prof. H. Ammari for his guidance. During the preparation of this work, the author was financially supported by a Swiss National Science Foundation grant (number 200021-172483).

References

[1] Abramowitz, M. & Stegun, I. (Eds.) (1964): *Handbook of Mathematical Functions*, National Bureau of Standards, Washington D.C.

[2] Ammari, H., Phuong Tran, M., & Wang, H. (2014): *Shape Identification and Classification in Echolocation*, SIAM J. Imaging Sci., 7 (2014), 1883–1905.

[3] Ammari, H. & Kang, H. (2004): *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Mathematics, Vol. 1846, Springer-Verlag, Berlin.

[4] Ammari, H., Kang, H., Lim, M. & Zribi, H. (2012): *The generalized polarization tensors for resolved imaging. Part I: Shape reconstruction of a conductivity inclusion*, Math. Comp., 81, 367–386.

[5] Ammari, H., Kang, H., Lee, H. & Lim, M. (2013): *Enhancement of Near-Cloaking. Part II: The Helmholtz Equation*, Comm. Math. Phys., 317, 485–502.

[6] Ammari, H., Kang, H., Kim, E. & Lee, J.-Y. (2012): *The generalized polarization tensors for resolved imaging. Part II: Shape and electromagnetic parameters reconstruction of an electromagnetic inclusion from multistatic measurements*, Math. Comp., 81, 839–860.

[7] Ammari, H., Garapon, P., Guadarrama Bustos, L. & Kang, H. (2010): *Transient anomaly imaging by the acoustic radiation force*, J. Differential Equations 249, 1579–1595.

[8] Ammari, H. & Kang, H. (2004): *Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities*, J. Math. Anal. Appl. **296**, 190–208.

[9] Ammari, H. & Wang, H. (2016): *Time-domain multiscale shape identification in electro-sensing*, Imaging, Multi-scale and High Contrast Partial Differential Equations 660, 23.

[10] Ashbaugh, M.S. & Benguria, R.D. (2007): *Isoperimetric Inequalities for Eigenvalues of the Laplacian*, in **Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday**, Proceedings of Symposia in Pure Mathematics **76**, Part 1, F. Gesztesy, P. Deift, C. Galvez, P. Perry, and W. Schlag, eds., Amer. Math. Soc., Providence, BL, RI, 105–139.

[11] Baldassari, L. (2018): *Scattering Coefficients of Inhomogeneous Objects and Their Application in Target Classification in Wave Imaging*, arXiv:1806.07841 [cs].

[12] Escauriaza, L., Fabes, E.B., & Verchota, G. (1992): *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. Amer. Math. Soc. **115**, 1069–1076.

[13] Escauriaza, L. & Seo, J.K. (1993): *Regularity properties of solutions to transmission problems*, Trans. Amer. Math. Soc. **338**, 405–430.

[14] Hariharan, S.I. & MacCamy, R.C. (1983): *Low Frequency Acoustic and Electromagnetic Scattering*, NASA Contractor Report 172203, Contract No. NAS1-17070.

[15] Hariharan, S.I. & MacCamy, R.C. (1983): *A Boundary Element Method for a Two-Dimensional Interface Problem in Electromagnetics*, Numer. Math. 42, 311–322.

[16] Kang, H. & Seo, J.K. (2000): *Recent progress in the inverse conductivity problem with single measurement*, Inverse Problems and Related Fields, CRC Press, Boca Raton, FL, 69–80.

[17] Krahn, E. (1926): *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat), 1–44.

[18] Lebedev, N.N. (1965): *Special Functions and their Applications*, Prentice-Hall, Englewood Cliffs.

[19] Matviyenko, G. (1993): *On the evaluation of Bessel functions*, Appl. Comput. Harmon. Anal. **1**, 116–135.

[20] Simmons, J. A. (1979): *Perception of echo phase information in bat sonar*, Science, **204**, 1336–38.

[21] Simmons, J. A., Fenton, M. B. & O'Farrell, M. J. (1979): *Echolocation and pursuit of prey by bats*, Science, **203**, pp. 16–21.

[22] Watson, G.N. (1944): *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge.

Department of Mathematics, ETH Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland

Email address: lorenzo.baldassari@sam.math.ethz.ch