Analytic Regularity for the incompressible Navier-Stokes Equations in polygons

C. Marcati and Ch. Schwab

ANALYTIC REGULARITY FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN POLYGONS

CARLO MARCATI[*] AND CHRISTOPH SCHWAB[*]

Abstract. In a plane polygon P with straight sides, we prove analytic regularity of the Leray-Hopf solution of the stationary, viscous, and incompressible Navier-Stokes equations. We assume small data, analytic volume force and no-slip boundary conditions. Analytic regularity is quantified in so-called countably normed, corner-weighted spaces with homogeneous norms. Implications of this analytic regularity include exponential smallness of Kolmogorov N-widths of solutions, exponential convergence rates of mixed hp-discontinuous Galerkin finite element and spectral element discretizations and of model order reduction techniques.

Key words. Navier-Stokes equations, analytic regularity, conical singularities, weighted Sobolev spaces.

AMS subject classifications. 35Q30, 76N10, 35A20

Introduction. The mathematical theory of regularity of solutions to the viscous, incompressible Navier-Stokes equations has attracted enormous attention during the past decades. Besides elucidating subtle scientific issues on their validity in particular physical settings, and properties of their solutions for small viscosity, mathematical regularity results on their solutions have immediate consequences in a wide range of applications, from aerodynamics to physiological flows as, e.g., in hemodynamics.

The precise mathematical characterization of solution regularity in function spaces of Sobolev or Besov type is moreover essential in the numerical analysis of approximation schemes for the efficient numerical solution of these equations. We mention only finite element, finite volume, hp- and spectral element methods and the more recent model order reduction (MOR) and reduced basis (RB) methods [22, 37].

In a plane polygonal domain P, we consider the stationary Stokes and Navier-Stokes equations modeling viscous, incompressible flow, i.e.

$$\begin{aligned}
-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } P, \\
\nabla\cdot\boldsymbol{u} &= 0 \text{ in } P, \\
\boldsymbol{u} &= \boldsymbol{0} \text{ on } \partial P.
\end{aligned}$$

(0.1)

Here, the *source term $\boldsymbol{f}$ is assumed analytic in weighted Sobolev spaces in P*, and, together with the viscosity ν, it is supposed to satisfy a smallness assumption in $L^2(P)$ which ensures uniqueness of weak (Leray-Hopf) solutions. We establish analytic estimates on the Leray-Hopf solution of (0.1) in weighted Sobolev spaces in P.

For a classical analysis of the Stokes and Navier-Stokes problems, see [28, 16, 43]; the analyticity of the solutions in smooth domains – given sufficiently regular right hand sides – is a classical result [32, 15]. In our setting, though, special care has to be taken to account for the corners of the domain. Due to the corner singularities of P, solutions of the Navier-Stokes equations do not admit high order regularity in scales of classical Sobolev spaces. Regularity results on solutions of elliptic PDEs in polygonal domains P must account for the presence of corner singularities at the vertices of P. Accordingly, scales of *corner-weighted Sobolev spaces* are required which were introduced in the seminal paper [25] and are known today as Kondrat'ev spaces. They extend classical Sobolev spaces by weighting weak derivatives with powers

---

[*]Seminar of Applied Mathematics, ETH Zurich, Zürich, Switzerland (carlo.marcati@sam.math.ethz.ch, christoph.schwab@sam.math.ethz.ch).

of the distance to (sets of) isolated singular points. In the presently considered setting of a polygon P and of constant kinematic viscosity ν, the set of singular points comprises the corners of P. Since [25], Kondrat'ev spaces (with their extension to three dimensional polyhedra P where one considers also the distance from edges) have been widely used for the regularity analysis of elliptic problems in polygons and polyhedra. Among the vast literature on the regularity of solutions of elliptic boundary value problems, we mention [17, 26, 35, 27, 10, 33, 6, 7, 8, 9] for general elliptic problems and in particular [21] for analytic regularity results for the Stokes equation in a polygon as also considered in the present paper. The regularity of solutions of the incompressible Stokes and Navier-Stokes equations is also investigated from the point of view of Kondrat'ev spaces of finite order in the monographs [17, 27, 33].

In the present paper, we are in particular interested in *weighted analytic regularity*. I.e., in a-priori estimates of norms of the solution in function spaces of Kondrat'ev type of arbitrary high order in P, with quantitative control of the growth of constants with respect to the order of differentiation. We refer to (1.2) below for the definition of the space of functions with weighted analytic regularity in a plane sector. The notion of *weighted analytic regularity*, as developed in [4, 19, 20] fits well with the classical concept of analytic regularity for elliptic systems in the interior and up to analytic boundary as developed in [34]. It is also of great importance for the numerical analysis of these problems, since it is the basis for the derivation of, among others, exponentially convergent, so-called hp-finite element and spectral element algorithms, as in [19, 20, 39, 40], and tensor compression methods, see [24].

Among the articles already cited, we refer the reader to [8, 9] for a reference on weighted analytic regularity in polyhedral domains for linear elliptic boundary value problems. The steady-state, incompressible viscous Navier-Stokes equations constitute a nonlinear elliptic system with analytic (quadratic) nonlinearity seems to preclude generalizing the proof of analytic regularity shifts in scales of corner-weighted, Hilbertian Sobolev spaces. To handle the quadratic nonlinearity, in the present paper we propose to work, instead, in a non Hilbertian setting. Specifically, we consider L^s-type norms, for $s > 2$, as is done in [11] for non weighted spaces and for a different problem, and bootstrap weighted regularity by induction with respect to the differentiation order, as opposed to arguments by local analytic continuation and complex analysis as put forward e.g. in [32, 2, 34]. This requires the introduction of some preliminary regularity results, localised around each corner. We start therefore, in Section 1, by considering the case of an unbounded plane sector. First, we obtain a regularity shift result in concentric balls around the corner for the Stokes equation; then, when considering the Navier-Stokes equation, we can move the nonlinear term to the right hand side and show a regularity shift estimate for the solutions to the full nonlinear equation. The choice of L^s-type spaces with $s > 2$ deviates from earlier, "Hilbertian" arguments as developed e.g. in [2, 21]. It is essential in our treatment of the quadratic nonlinearity in the proof of the analytic regularity shift – the key result Lemma 2.1 can not be established with our strategy when $s \leq 2$, see also Remark 2.2. This weighted a-priori estimate in Lemma 2.1 serves as local induction step in the proof of analytic regularity of (Leray-Hopf solutions of) the Dirichlet problems for the viscous, incompressible Navier-Stokes equations in polygons. The proof of analytic regularity of the corresponding velocity and pressure fields is given in Section 2. There, the shift estimates for the localized problems at each corner of the polygon are combined with classical interior regularity results to obtain analytic estimates in the whole polygonal domain.

Main result. Theorem 2.6 is the main result of this paper: we state it here for the convenience of the reader. We assume that P is a polygonal domain with straight sides and with $n \in \mathbb{N}$ vertices c_i and internal angles φ_i, $i = 1, \ldots, n$. We write $r_i = |x - c_i|$, $\underline{\gamma} = \{\gamma_i\}_{i=1,\ldots,n}$ and

$$r^{\underline{\gamma}} = \prod_{i=1,\ldots,n} r_i^{\gamma_i}.$$

We denote by $\mathfrak{A}^i(\lambda)$ the Mellin symbol of the Stokes operator associated to the corner c_i, i.e., the operator $\mathfrak{L}(\vartheta, \partial_\vartheta; \lambda)$ in $(0, \varphi_i)$, see (1.7) below, with boundary operator $\mathfrak{B}(\vartheta, \partial_\vartheta; \lambda)_{|_{\vartheta=0,\varphi_i}}$ as introduced below in Section 1.2. Let $\Lambda_{\min}^i$ be the real part of the eigenvalue with smallest positive real part of $\mathfrak{A}^i(\lambda)$, and define $\underline{\Lambda} = \{\Lambda_{\min}^i\}_{i=1}^n > 0$. We remark that for each $i = 1, \ldots, n$, the value of $\Lambda_{\min}^i$ only depends on the corresponding internal angle φ_i of the polygon P.

The main result of this paper is then given by the following statement; note that the inequality on $\underline{\gamma}$ in the hypothesis holds componentwise.

THEOREM 0.1. *Let $2 < s < \infty$, $\underline{\gamma} \in \mathbb{R}^n$ such that $\max(-\underline{\Lambda}, -1) < \underline{\gamma} - 2/s < \min(\underline{\Lambda}, 1)$. Let $(\boldsymbol{u}, p)$ be Leray-Hopf solutions to the stationary, viscous, and incompressible Navier-Stokes equation in P (0.1) with source term $\boldsymbol{f} \in \boldsymbol{H}^{-1}(P)$ such that the solution $(\boldsymbol{u}, p)$ is unique and that there exist $C_f, A_f > 0$ such that*

$$\|r^{|\alpha|+2-\underline{\gamma}} \partial^\alpha \boldsymbol{f}\|_{L^s(P)} \leq C_f A_f^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2.$$

Then, there exist two positive constants C and A such that for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$:

$$\|r^{|\alpha|-\underline{\gamma}} \partial^\alpha \boldsymbol{u}\|_{L^s(P)} \leq C A^{|\alpha|} |\alpha|!, \qquad \|r^{|\alpha|-\underline{\gamma}} \partial^\alpha p\|_{L^s(P)} \leq C A^{|\alpha|} |\alpha|!.$$

Extensions and consequences. The analysis presented here is done in the framework of homogeneous Dirichlet boundary conditions, and it holds without modification as long as homogeneous Dirichlet conditions for the velocity field are imposed on at least one of the edges abutting on every corner. If one wants to deal with boundary conditions where the value of the velocity field is not prescribed at the corner – as, for example, for Neumann-type boundary conditions – the homogeneous weighted spaces have to be replaced by their non homogeneous version, as in the analytic regularity shift for the Stokes system, for example, in [18]. Non homogeneous spaces have a different weight structure than homogeneous ones, and they contain functions with non trivial Taylor expansion at the corners. For an overview of the relationship between homogeneous and non homogeneous spaces for isolated point singularities, see [26, 7]. The analysis presented here can also constitute part of the basis for the proof of weighted analytic regularity of the solutions to the Navier-Stokes equation in three dimensional polyhedra. The spaces involved in this case are more complicated due to the presence of edge singularities, but most of the regularity analysis in weighted spaces has been done in [33]. See also [12] for an analysis of regularity in classical Sobolev spaces. Furthermore, the analysis in Kondrat'ev spaces can be instrumental in other types of regularity analyses, for example using weighted spaces to determine the regularity in Besov spaces of the solutions to the Navier-Stokes equation, see, e.g., [13]. The analytic regularity shown here allows to establish exponential convergence of several classes of numerical schemes, such as the above-mentioned hp-finite element methods [38], tensor compression methods [42, 3, 5] and Model Order Reduction methods [44, 29, 14] for the numerical approximation of the viscous, incompressible Navier-Stokes equations.

1. The Stokes equation in an infinite sector. In this section we consider the Stokes and Navier-Stokes equations in an unbounded, plane sector K. Starting from an elliptic regularity shift theorem in the plane sector, we derive, in Proposition 1.8, a localised regularity shift theorem in concentric balls around the corner. The constants in the estimate are explicit with respect to the distance between the balls, as the careful tracking of the constants will be important for the analysis of the Navier-Stokes equation.

1.1. Notation and setting. Let r and ϑ be the polar coordinates of a point $x \in \mathbb{R}^2$, and, for an aperture angle $\varphi \in (0, 2\pi) \setminus \{\pi\}$, we consider the plane sector

$$(1.1) \qquad K = \left\{ x \in \mathbb{R}^2 : x = r(\sin(\vartheta), \cos(\vartheta))^\top, 0 < r < \infty, \vartheta \in (0, \varphi) \right\}.$$

We exclude the case of a domain with a slit, i.e., $\varphi = 2\pi$, in order to render K Lipschitz in a neighborhood of the origin. For integer $\ell \geq 0$, $\gamma \in \mathbb{R}$, and for $1 < s < \infty$, the homogeneous, weighted Sobolev spaces $\mathcal{K}^{\ell,s}_\gamma(K)$ are

$$\mathcal{K}^{\ell,s}_\gamma(K) = \left\{ v \in L^s_{\mathrm{loc}}(K) : r^{|\alpha|-\gamma} \partial^\alpha u \in L^s(Q), \ \forall \alpha, |\alpha| \leq \ell \right\}$$

with norm

$$\|w\|_{\mathcal{K}^{\ell,s}_\gamma(K)} = \left(\sum_{|\alpha| \leq \ell} \|r^{|\alpha|-\gamma} \partial^\alpha w\|^s_{L^s(K)} \right)^{1/s}.$$

We will also use the seminorm

$$|w|_{\mathcal{K}^{\ell,s}_\gamma(K)} = \sum_{|\alpha|=\ell} \|r^{|\alpha|-\gamma} \partial^\alpha w\|_{L^s(K)}.$$

For $\ell \geq 1$, the spaces $\mathcal{K}^{\ell-1/s,s}_{\gamma-1/s}(\partial K \setminus \{0\})$ are defined as spaces of traces of functions in $\mathcal{K}^{\ell,s}_\gamma(K)$. We denote

$$\mathcal{K}^{\infty,s}_\gamma(K) = \bigcap_{\ell \in \mathbb{N}} \mathcal{K}^{\ell,s}_\gamma(K)$$

and use boldface characters for two dimensional vectors of functions and for their spaces. For constants $A, C > 0$, we write

$$(1.2) \qquad \mathcal{K}^{\varpi,s}_\gamma(K; A, C) = \left\{ v \in \mathcal{K}^{\infty,s}_\gamma(K) : \|v\|_{\mathcal{K}^{\ell,s}_\gamma(K)} \leq CA^\ell \ell!, \text{ for all } \ell \in \mathbb{N} \right\}.$$

With these sets we associate

$$(1.3) \qquad\qquad \mathcal{K}^{\varpi,s}_\gamma(K) = \bigcup_{A,C>0} \mathcal{K}^{\varpi,s}_\gamma(K; A, C),$$

i.e.,

$$\mathcal{K}^{\varpi,s}_\gamma(K) = \left\{ v \in \mathcal{K}^{\infty,s}_\gamma(K) : \text{ex. } A, C > 0 \text{ s.t. } \forall \ell \in \mathbb{N} : \ \|v\|_{\mathcal{K}^{\ell,s}_\gamma(K)} \leq CA^\ell \ell! \right\}.$$

Furthermore, given two functions or operators F and G, $[\cdot, \cdot]$ denotes the commutator

$$[F, G] = FG - GF.$$

We write $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ is the set of the natural numbers. For two multi indices $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$, we write $\alpha! = \alpha_1! \alpha_2!$, $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$, and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

Finally, recall e.g. from [23]

$$\sum_{\substack{|\beta|=n \\ \beta \leq \alpha}} \binom{\alpha}{\beta} = \binom{|\alpha|}{n}.$$

1.2. The Stokes equation in a sector. In the infinite sector K, we consider the Stokes system subject to volume force $\boldsymbol{f}$, with unit viscosity and with homogeneous Dirichlet boundary conditions

(1.4)
$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } K, \\ \nabla \cdot \boldsymbol{u} &= g \text{ in } K, \\ \boldsymbol{u} &= \boldsymbol{0} \text{ on } \partial K \setminus \{0\}, \end{aligned}$$

which we express symbolically as

(1.5)
$$\mathcal{L} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ g \end{pmatrix}.$$

With $(r, \vartheta) \in (0, \infty) \times (0, \varphi)$ we denote polar coordinates in the sector K. For $\lambda \in \mathbb{C}$, we write

(1.6)
$$\boldsymbol{u} = r^\lambda \boldsymbol{U}(\vartheta) = r^\lambda \begin{pmatrix} U_r(\vartheta) \\ U_\vartheta(\vartheta) \end{pmatrix} \quad \text{and} \quad p = r^{\lambda-1} P(\vartheta),$$

where U_r and U_ϑ are respectively the radial and angular components of $\boldsymbol{U}$. We introduce the parameter-dependent differential operator $\mathfrak{L}(\vartheta, \partial_\vartheta; \lambda)$ acting on $(\boldsymbol{U}, P)$ in (1.6) via

$$\mathfrak{L}(\vartheta, \partial_\vartheta; \lambda) \begin{pmatrix} \boldsymbol{U} \\ P \end{pmatrix} = \begin{pmatrix} r^{2-\lambda} & & \\ & r^{2-\lambda} & \\ & & r^{1-\lambda} \end{pmatrix} \mathcal{L} \begin{pmatrix} r^\lambda \boldsymbol{U}(\vartheta) \\ r^{\lambda-1} P(\vartheta) \end{pmatrix}$$

We find (see also [27, Section 5.1, Eqn. (5.1.4)])

(1.7)
$$\mathfrak{L}(\vartheta, \partial_\vartheta; \lambda) = \begin{pmatrix} 1 - \lambda^2 - \partial_\vartheta^2 & 2\partial_\vartheta & \lambda - 1 \\ -2\partial_\vartheta & 1 - \lambda^2 - \partial_\vartheta^2 & \partial_\vartheta \\ \lambda + 1 & \partial_\vartheta & 0 \end{pmatrix},$$

and the corresponding Dirichlet boundary operator

$$\mathfrak{B}(\vartheta, \partial_\vartheta; \lambda)_{|\vartheta=0, \varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We denote by $\mathfrak{A}(\lambda)$ the parametric operator pencil associated to the problem given by the parametric differential operator $\mathfrak{L}(\vartheta, \partial_\vartheta; \lambda)$ in $\vartheta \in (0, \varphi)$ with Dirichlet boundary operator $\mathfrak{B}(\vartheta, \partial_\vartheta; \lambda)_{|_{\vartheta=0,\varphi}}$. We recall from [33, Theorems 1.2.5 to 1.2.8] the following two results on regularity shifts in corner-weighted spaces of finite order.

Theorem 1.1. *Let $\ell \geq 0$ be an integer, $1 < s < \infty$, $\gamma \in \mathbb{R}$ and let*

$$\boldsymbol{f} \in \boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(K), \qquad g \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K).$$

Suppose furthermore that the line $\mathrm{Re}\,\lambda = \gamma - 2/s$ does not contain eigenvalues of $\mathfrak{A}(\lambda)$. Then, there exist unique solutions $\boldsymbol{u}$, p to (1.4), with

$$\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(K), \qquad p \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K)$$

and there exists $C > 0$ (possibly dependent on ℓ but independent of $\boldsymbol{u}$ and p) such that

$$(1.8) \qquad \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(K)} + \|p\|_{\mathcal{K}_{\gamma-1}^{\ell+1,s}(K)} \leq C \left\{ \|\boldsymbol{f}\|_{\boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(K)} + \|g\|_{\mathcal{K}_{\gamma-1}^{\ell+1,s}(K)} \right\}.$$

Theorem 1.2. *Let $\ell \geq 0$ be an integer, $1 < s < \infty$, and let $\boldsymbol{u}, p$ be solutions to (1.4), where*

$$\boldsymbol{f} \in \boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(K), \qquad g \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K).$$

Suppose that

$$\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_{\gamma}^{\ell+1,s}(K), \qquad p \in \mathcal{K}_{\gamma-1}^{\ell,s}(K), \qquad \eta \boldsymbol{u} \in \boldsymbol{W}^{2,s}(K), \qquad \eta p \in W^{1,s}(K),$$

for all $\eta \in C_0^\infty(\overline{K} \setminus \{0\})$.

Then $\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(K)$, $p \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K)$ and there exists a constant $C > 0$ (depending on ℓ, s and on γ) such that

$$(1.9) \quad \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(K)} + \|p\|_{\mathcal{K}_{\gamma-1}^{\ell+1,s}(K)}$$

$$\leq C \left\{ \|\boldsymbol{f}\|_{\boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(K)} + \|g\|_{\mathcal{K}_{\gamma-1}^{\ell+1,s}(K)} + \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{K}}_{\gamma}^{\ell+1,s}(K)} + \|p\|_{\mathcal{K}_{\gamma-1}^{\ell,s}(K)} \right\}.$$

Remark 1.3. The theorems in [33] are stated with the operator pencil

$$(1.10)$$
$$\begin{pmatrix} -\lambda^2 - \partial_\vartheta^2 & 0 & (\lambda-1)\cos\vartheta - \sin\vartheta\partial_\vartheta \\ 0 & -\lambda^2 - \partial_\vartheta^2 & (\lambda-1)\sin\vartheta + \cos\vartheta\partial_\vartheta \\ -(\lambda-1)\cos\vartheta + (\cos\vartheta)^2\partial_\vartheta & (\lambda-1)\sin\vartheta + (\sin\vartheta)^2\partial_\vartheta & 0 \end{pmatrix}$$

coming from the individual transformation in polar coordinates of the terms of the original matrix operator followed by multiplication by r^2 and substitution of $r\partial_r$ with λ. This symbol would therefore be applied to the Cartesian components u_x, u_y of the Stokes flow.

The symbol (1.7) results instead from the transformation of the Stokes equation into plane polar coordinates, i.e., it is applied to the polar components u_r and u_ϑ of the velocity field $\boldsymbol{u}$. Since the Cartesian components u_x and u_y of $\boldsymbol{u}$ can be obtained through a rotation of the polar components, the eigenvalues of the symbol $\mathfrak{A}(\lambda)$ coincide [21, Theorem 4.15].
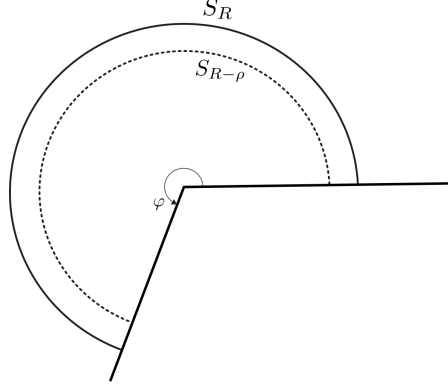
FIG. 1. *The sectors S_R, $S_{R-\rho}$.*

Denote now by $\Lambda_{\min}$ the real part of the eigenvalue with smallest positive real part of the pencil $\mathfrak{A}(\lambda)$. Note that $\Lambda_{\min} > 1/2$ and that the strip $-\Lambda_{\min} < \operatorname{Re}\lambda < \Lambda_{\min}$ does not contain eigenvalues of $\mathfrak{A}(\lambda)$. Its existence is proved, for example, in [27, Section 5.1].

Remark 1.4. Theorem 1.1 implies that, if $-\Lambda_{\min} < \gamma - 2/s < \Lambda_{\min}$, $\boldsymbol{f} \in \mathcal{K}_{\gamma-2}^{\infty,s}(K)$, and $g \in \mathcal{K}_{\gamma-1}^{\infty,s}(K)$ then

$$\boldsymbol{u} \in \mathcal{K}_\gamma^{\infty,s}(K) \qquad p \in \mathcal{K}_{\gamma-1}^{\infty,s}(K).$$

Remark 1.5. The $\ell = 0$ version of inequalities (1.8) and (1.9) read, respectively,

(1.11)
$$\sum_{|\alpha|\leq 2} \|r^{|\alpha|-\gamma}\partial^\alpha \boldsymbol{u}\|_{L^s(K)} + \sum_{|\alpha|\leq 1} \|r^{|\alpha|-\gamma+1}\partial^\alpha p\|_{L^s(K)}$$
$$\leq C\left\{\|r^{2-\gamma}\boldsymbol{f}\|_{L^s(K)} + \sum_{|\alpha|\leq 1} \|r^{|\alpha|-\gamma+1}\partial^\alpha g\|_{L^s(K)}\right\}$$

and
(1.12)
$$\sum_{|\alpha|\leq 2} \|r^{|\alpha|-\gamma}\partial^\alpha \boldsymbol{u}\|_{L^s(K)} + \sum_{|\alpha|\leq 1} \|r^{|\alpha|-\gamma+1}\partial^\alpha p\|_{L^s(K)}$$
$$\leq C\left\{\|r^{2-\gamma}\boldsymbol{f}\|_{L^s(K)} + \sum_{|\alpha|\leq 1} \|r^{|\alpha|-\gamma+1}\partial^\alpha g\|_{L^s(K)} + \sum_{|\alpha|\leq 1} \|r^{|\alpha|-\gamma}\partial^\alpha \boldsymbol{u}\|_{L^s(K)} \right.$$
$$\left. + \|r^{1-\gamma}p\|_{L^s(K)}\right\}.$$

For a radius $0 < R < 1$, an integer j, and a scalar ρ such that $R - j\rho > 0$, we introduce the sectors

(1.13)
$$S_{R-j\rho} = B_{R-j\rho} \cap K = \{x \in K : |x| < R - j\rho\},$$

as shown in Figure 1. Let $\widehat{\chi} : \mathbb{R} \to [0,1]$ be a smooth step function such that

$$\widehat{\chi} \in C^\infty(\mathbb{R}), \qquad \widehat{\chi}_{|(-\infty,0]} = 1, \qquad \widehat{\chi}_{|[1,\infty)} = 0.$$

There exists a constant $C_\chi \geq 1$ such that, for all $x \in \mathbb{R}$,

$$(1.14) \qquad |(\partial^\alpha \widehat{\chi})(x)| \leq C_\chi, \quad \forall \alpha \in \mathbb{N}_0 \text{ such that } |\alpha| \leq 2.$$

For all $j \in \mathbb{N}_0$ and all $\rho > 0$ such that $R - j\rho \geq R/2$, we can then introduce a smooth cutoff function $\chi : \mathbb{R}^2 \to [0,1]$, such that $\chi \in C_0^\infty(B_{R-j\rho})$, defined, for all $x \in \mathbb{R}^2$, as

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq R - (j+1)\rho, \\ \widehat{\chi}\left(\dfrac{|x| - (R - (j+1)\rho)}{\rho}\right) & \text{if } R - (j+1)\rho < |x| < R - j\rho, \\ 0 & \text{if } |x| \geq R - j\rho. \end{cases}$$

By algebraic manipulations, (1.14), and since $R - j\rho \geq R/2$, it can be shown that for all $x \in \mathbb{R}^2$ and all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq 2$, there holds

$$(1.15) \qquad |(\partial^\alpha \chi)(x)| \leq \left(\frac{4}{R} + 1\right) C_\chi \rho^{-|\alpha|}.$$

We introduce several auxiliary lemmas.

LEMMA 1.6. *Let $\ell, j \in \mathbb{N}_0$, $1 < s < \infty$, $\gamma \in \mathbb{R}$, $0 < R_0 \leq R < 1$, $\rho \in \mathbb{R}$ such that $R - j\rho \geq R/2$, and $v \in \mathcal{K}_\gamma^{\ell,s}(S_{R-j\rho})$. Then there exists $C > 0$ such that, for any $|\beta| = \ell$,*

$$\sum_{|\alpha|=1} \|r^{1-\gamma}\left[\chi r^\ell, \partial^\alpha\right]\partial^\beta v\|_{L^s(S_{R-(j+1)\rho})} \leq C(\ell + \rho^{-1})\|r^{\ell-\gamma}\partial^\beta v\|_{L^s(S_{R-j\rho})},$$

with C dependent only on C_χ in (1.15) and on R_0.

Proof. For all $x \in \mathbb{R}^2$ and $|\alpha| = 1$, there holds $|\partial^\alpha(r(x)^\ell)| \leq \ell r(x)^{\ell-1}$. By elementary manipulations, we have, for all multiindices $|\beta| = \ell$ and $|\alpha| = 1$,

$$\|r^{1-\gamma}\left[\chi r^\ell, \partial^\alpha\right]\partial^\beta v\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq \|\partial^\alpha \chi\|_{L^\infty(S_R)}\|r^{\ell+1-\gamma}\partial^\beta v\|_{L^s(S_{R-j\rho})} + \ell\|r^{\ell-\gamma}\partial^\beta v\|_{L^s(S_{R-j\rho})}.$$

Using (1.15) gives the assertion. $\qquad\qquad\square$

LEMMA 1.7. *Let $\ell, j \in \mathbb{N}_0$, $1 < s < \infty$, $\gamma \in \mathbb{R}$, $0 < R_0 \leq R < 1$, $\rho \in \mathbb{R}$ such that $R - j\rho \geq R/2$, and $v \in \mathcal{K}_\gamma^{\ell+1,s}(S_{R-j\rho})$. Then there exists $C > 0$ such that, for any $|\beta| = \ell$,*

$$\sum_{|\alpha|=2} \|r^{2-\gamma}\left[\chi r^\ell, \partial^\alpha\right]\partial^\beta v\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq C \sum_{|\delta|=0,1} \left(\ell + \rho^{-1}\right)^{2-|\delta|}\|r^{\ell+|\delta|-\gamma}\partial^{\beta+\delta}v\|_{L^s(S_{R-j\rho})}.$$

The constant C depends only on C_χ in (1.15) and on R_0.

Proof. The proof follows the same lines as the proof of the last lemma. We have, for all multiindices $|\beta| = \ell$ and $|\alpha| = 2$

$$\|r^{2-\gamma}\left[\chi r^\ell, \partial^\alpha\right]\partial^\beta v\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq \|r^{2-\gamma}\partial^\alpha\left(\chi r^\ell\right)\partial^\beta v\|_{L^s(S_{R-(j+1)\rho})} + 2\sum_{|\delta|=1:\delta\leq\alpha} \|r^{2-\gamma}\partial^\delta\left(\chi r^\ell\right)\partial^{\beta+\alpha-\delta}v\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq \left(C\rho^{-2} + C\ell\rho^{-1} + \ell^2\right)\|r^{\ell-\gamma}\partial^\beta v\|_{L^s(S_{R-j\rho})}$$
$$\quad + 2\sum_{|\delta|=1:\delta\leq\alpha} \left(C\rho^{-1} + \ell\right)\|r^{\ell+1-\gamma}\partial^{\beta+\alpha-\delta}v\|_{L^s(S_{R-j\rho})},$$

hence the assertion follows. □

We now give a localised and explicit version of Theorem 1.1.

PROPOSITION 1.8. *Let* $\ell \in \mathbb{N}$, $1 < s < \infty$, $0 < R < 1$, *and* $-\Lambda_{\min} < \gamma - 2/s < \Lambda_{\min}$. *Suppose that*

$$(1.16) \qquad \boldsymbol{f} \in \boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(K), \qquad g \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K).$$

Then, $\boldsymbol{u}$, p *are the unique solutions to the Stokes system* (1.4)*, with* $\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(K)$, $p \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K)$*, and there exists* $C > 0$ *such that for all* $j \in \mathbb{N}$ *such that* $j \geq \ell$*, and for all* $\rho \in (0, R/(2j)]$,

$$(1.17)$$

$$|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(S_{R-(j+1)\rho})} + |p|_{\mathcal{K}_{\gamma-1}^{\ell+1,s}(S_{R-(j+1)\rho})} \leq C \Bigg\{ |\boldsymbol{f}|_{\boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(S_{R-j\rho})} + \sum_{n \leq 1} |g|_{\mathcal{K}_{\gamma-1}^{\ell+n,s}(S_{R-j\rho})}$$

$$+ \rho^{-1}|p|_{\mathcal{K}_{\gamma-1}^{\ell,s}(S_{R-j\rho})} + \sum_{n \leq 1} \rho^{n-2}|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}_{\gamma}^{\ell+n,s}(S_{R-j\rho})} \Bigg\}.$$

Proof. Fix $\ell \in \mathbb{N}$, the integer $j \geq \ell$, and assume (1.16). Then it follows from Theorem 1.1 that

$$(1.18) \qquad \boldsymbol{u} \in \boldsymbol{\mathcal{K}}_{\gamma}^{\ell+2,s}(K), \qquad p \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(K).$$

Consider $\beta \in \mathbb{N}_0^2$ such that $|\beta| = \ell$ and write $\boldsymbol{w} = \chi r^{|\beta|}\partial^\beta \boldsymbol{u}$ and $q = \chi r^{|\beta|}\partial^\beta p$. Then,

$$-\Delta \boldsymbol{w} + \nabla q = \left[-\Delta, \chi r^{|\beta|} \right] \partial^\beta \boldsymbol{u} + \left[\nabla, \chi r^{|\beta|} \right] \partial^\beta p + \chi r^{|\beta|}\partial^\beta(-\Delta \boldsymbol{u} + \nabla p)$$

$$= \left[-\Delta, \chi r^{|\beta|} \right] \partial^\beta \boldsymbol{u} + \left[\nabla, \chi r^{|\beta|} \right] \partial^\beta p + \chi r^{|\beta|}\partial^\beta \boldsymbol{f}$$

hence (1.16), (1.18), and Lemmas 1.6 and 1.7 give $-\Delta \boldsymbol{w} + \nabla q \in \boldsymbol{\mathcal{K}}_{\gamma-2}^{0,s}(K)$. Then,

$$\nabla \cdot \boldsymbol{w} = \left(\nabla(\chi r^{|\beta|}) \right) \cdot (\partial^\beta \boldsymbol{u}) + \chi r^{|\beta|}\partial^\beta g \in \mathcal{K}_{\gamma-1}^{1,s}(K).$$

We can therefore apply Theorem 1.1 with $\ell = 0$. Inequality (1.11) implies that there exists $C > 0$ independent of $\boldsymbol{w}$ and q such that

$$(1.19) \quad \sum_{|\alpha| \leq 2} \|r^{|\alpha|-\gamma}\partial^\alpha \boldsymbol{w}\|_{L^s(K)} + \sum_{|\alpha| \leq 1} \|r^{|\alpha|-\gamma+1}\partial^\alpha q\|_{L^s(K)}$$

$$\leq C \Bigg\{ \|r^{2-\gamma} (-\Delta \boldsymbol{w} + \nabla q) \|_{L^s(K)} + \sum_{|\alpha| \leq 1} \|r^{|\alpha|-\gamma+1}\partial^\alpha \nabla \cdot \boldsymbol{w}\|_{L^s(K)} \Bigg\}.$$

By algebraic manipulation and the triangle inequality, the following inequalities hold, for $|\beta| = \ell$

$$(1.20) \quad \sum_{|\alpha|=2} \|r^{|\beta|+2-\gamma}\partial^{\alpha+\beta}\boldsymbol{u}\|_{L^s(S_{R-(j+1)\rho})}$$

$$\leq \sum_{|\alpha|=2} \|r^{2-\gamma}\left[\chi r^{|\beta|}, \partial^\alpha\right]\partial^\beta \boldsymbol{u}\|_{L^s(S_{R-(j+1)\rho})} + \sum_{|\alpha|=2} \|r^{2-\gamma}\partial^\alpha \boldsymbol{w}\|_{L^s(S_{R-(j+1)\rho})}$$

and

(1.21)
$$\sum_{|\alpha|=1} \|r^{|\beta|+2-\gamma} \partial^{\alpha+\beta} p\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq \sum_{|\alpha|=1} \|r^{2-\gamma} \left[\chi r^{|\beta|}, \partial^\alpha\right] \partial^\beta p\|_{L^s(S_{R-(j+1)\rho})} + \sum_{|\alpha|=1} \|r^{2-\gamma} \partial^\alpha q\|_{L^s(S_{R-(j+1)\rho})}.$$

We can estimate the terms with the commutators in (1.20) and (1.21): Lemmas 1.6 and 1.7 give

(1.22)
$$\sum_{|\alpha|=2} \|r^{2-\gamma} \left[\chi r^{|\beta|}, \partial^\alpha\right] \partial^\beta \boldsymbol{u}\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq C \sum_{|\alpha|\leq 1} \left(|\beta| + \rho^{-1}\right)^{2-|\alpha|} \|r^{|\beta|+|\alpha|-\gamma} \partial^{\beta+\alpha} \boldsymbol{u}\|_{L^s(S_{R-j\rho})},$$

and
(1.23)
$$\sum_{|\alpha|=1} \|r^{2-\gamma} \left[\chi r^{|\beta|}, \partial^\alpha\right] \partial^\beta p\|_{L^s(S_{R-(j+1)\rho})} \leq C \left(|\beta| + \rho^{-1}\right) \|r^{|\beta|-\gamma+1} \partial^\beta p\|_{L^s(S_{R-j\rho})}$$

with constant C independent of $|\beta|$, ρ, $\boldsymbol{u}$, and p.

Now, since $\rho \leq R/(2j)$ and $j \geq \ell$, then $|\beta| \leq \rho^{-1}$. In addition, from the inequalities (1.20)–(1.23) and using (1.19), we obtain

(1.24)
$$\sum_{|\alpha|=2} \|r^{|\beta|+|\alpha|-\gamma} \partial^{\beta+\alpha} \boldsymbol{u}\|_{L^s(S_{R-(j+1)\rho})} + \sum_{|\alpha|=1} \|r^{|\beta|+|\alpha|-\gamma+1} \partial^{\beta+\alpha} p\|_{L^s(S_{R-(j+1)\rho})}$$
$$\leq C \Bigg\{ \|r^{2-\gamma} \left(-\Delta \boldsymbol{w} + \nabla q\right)\|_{L^s(S_{R-j\rho})} + \sum_{|\alpha|\leq 1} \|r^{|\alpha|-\gamma+1} \partial^\alpha \nabla \cdot \boldsymbol{w}\|_{L^s(S_{R-j\rho})}$$
$$+ \rho^{-1} \|r^{|\beta|-\gamma+1} \partial^\beta p\|_{L^s(S_{R-j\rho})} + \sum_{|\alpha|\leq 1} \rho^{|\alpha|-2} \|r^{|\beta|+|\alpha|-\gamma} \partial^{\beta+\alpha} \boldsymbol{u}\|_{L^s(S_{R-j\rho})} \Bigg\}.$$

Using the commutators as before, the definition of $\boldsymbol{w}$ and q, Lemmas 1.6 and 1.7, and the inequality $|\beta| \leq \rho^{-1}$ we can show that

$$\|r^{2-\gamma} \left(-\Delta \boldsymbol{w} + \nabla q\right)\|_{L^s(S_{R-j\rho})}$$
$$\leq \|r^{2-\gamma} \left[\Delta, r^{|\beta|}\right] \partial^\beta \boldsymbol{u}\|_{L^s(S_{R-j\rho})} + \|r^{2-\gamma} \left[\nabla, r^{|\beta|}\right] \partial^\beta p\|_{L^s(S_{R-j\rho})}$$
$$+ \|r^{|\beta|-\gamma+1} \partial^\beta \left(-\Delta \boldsymbol{u} + \nabla p\right)\|_{L^s(S_{R-j\rho})}$$
$$\leq C \sum_{|\alpha|\leq 1} \rho^{|\alpha|-2} \|r^{|\beta|+|\alpha|-\gamma} \partial^{\beta+\alpha} \boldsymbol{u}\|_{L^s(S_{R-j\rho})} + C\rho^{-1} \|r^{|\beta|-\gamma+1} \partial^\beta p\|_{L^s(S_{R-j\rho})}$$
$$+ \|r^{2+|\beta|-\gamma} \partial^\beta \boldsymbol{f}\|_{L^s(S_{R-j\rho})}$$

and, for all $|\alpha| \leq 1$

$$\|r^{2-\gamma}\partial^\alpha \nabla \cdot \boldsymbol{w}\|_{L^s(S_{R-j\rho})}$$

$$\leq \sum_{|\delta|=1} \|r^{2-\gamma}\partial^\alpha \left[\partial^\delta, r^{|\beta|}\right] \partial^\beta \boldsymbol{u}\|_{L^s(S_{R-j\rho})} + \|r^{|\beta|-\gamma+2}\partial^{\beta+\alpha}\nabla \cdot \boldsymbol{u}\|_{L^s(S_{R-j\rho})}$$

$$\leq C\rho^{-1}\|r^{|\beta|-\gamma+1}\partial^{\beta+\alpha}\boldsymbol{u}\|_{L^s(S_{R-j\rho})} + \|r^{|\beta|-\gamma+2}\partial^{\beta+\alpha}g\|_{L^s(S_{R-j\rho})}.$$

Summing (1.24) over all $|\beta| = \ell$ and using the last two inequalities,

$$|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}_\gamma^{\ell+2,s}(S_{R-(j+1)\rho})} + |p|_{\mathcal{K}_{\gamma-1}^{\ell+1,s}(S_{R-(j+1)\rho})} \leq C\Bigg\{|\boldsymbol{f}|_{\boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(S_{R-j\rho})} + \sum_{n\leq 1}|g|_{\mathcal{K}_{\gamma-1}^{\ell+n,s}(S_{R-j\rho})}$$

$$+ \rho^{-1}|p|_{\mathcal{K}_{\gamma-1}^{\ell,s}(S_{R-j\rho})} + \sum_{n\leq 1}\rho^{n-2}|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}_\gamma^{\ell+n,s}(S_{R-j\rho})}\Bigg\}.$$

$\square$

PROPOSITION 1.9. *Let $\ell \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, $1 < s < \infty$, $0 < R < 1$, $j \in \mathbb{N}_0$ such that $j \geq \ell$, and $\rho \in (0, R/(2j)]$. Let $\boldsymbol{u}$, p be solutions to the Stokes system (1.4), with*

$$(1.25) \qquad \boldsymbol{f} \in \boldsymbol{\mathcal{K}}_{\gamma-2}^{\ell,s}(S_{R-j\rho}), \qquad g \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(S_{R-j\rho}).$$

Suppose that

$$\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_\gamma^{\ell+1,s}(S_{R-j\rho}), \quad p \in \mathcal{K}_{\gamma-1}^{\ell,s}(S_{R-j\rho}), \quad \eta\boldsymbol{u} \in \boldsymbol{W}^{2,s}(S_{R-j\rho}), \quad \eta p \in W^{1,s}(S_{R-j\rho}),$$

for all $\eta \in C_0^\infty(\bar{S}_{R-j\rho} \setminus \{0\})$. Then, $\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_\gamma^{\ell+2,s}(S_{R-(j+1)\rho})$, $p \in \mathcal{K}_{\gamma-1}^{\ell+1,s}(S_{R-(j+1)\rho})$, and there exists $C > 0$ independent of ℓ, j, ρ, such that (1.17) holds.

Proof. The proof follows the same steps as the proof of Proposition 1.8, with (1.12) replacing (1.11) in (1.19). $\square$

1.3. Analytic regularity shift for the quadratic nonlinearity in a sector. To handle the regularity shift for the quadratic nonlinearity in the weighted scales $\mathcal{K}_\gamma^{\ell,s}$, we apply a technique inspired by [11] and used also in [31], for different problems but similar nonlinearities. We derive interpolation inequalities in weighted spaces and use them to bound the quadratic nonlinearity in the momentum transport term of the Navier-Stokes system in the homogeneous, corner-weighted norms.

1.3.1. Induction hypotheses. For $1 < s < \infty$, $\gamma \in \mathbb{R}$, $j \in \mathbb{N}_0$, $0 < R < 1$ and $C_{\boldsymbol{u}}, A_{\boldsymbol{u}}, C_p, A_p > 0$, we introduce two statements on the boundedness of high order derivatives of weak solutions with quantitative control of their size in weighted Kondrat'ev spaces. We shall say that $H_{\boldsymbol{u}}(s, \gamma, j, C_{\boldsymbol{u}}, A_{\boldsymbol{u}})$ *holds in S_R* if for all $\rho \in (0, R/(2j)]$,

$$(1.26\text{a}) \qquad |\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}_\gamma^{\ell,s}(S_{R-j\rho})} \leq C_{\boldsymbol{u}} A_{\boldsymbol{u}}^\ell (j\rho)^{-\ell}\ell^\ell \text{ for all } \ell = 0, \dots, j$$

and we shall say that $H_p(s, \gamma, j, C_p, A_p)$ *holds in S_R* if for all $\rho \in (0, R/(2j)]$,

$$(1.26\text{b}) \qquad |p|_{\mathcal{K}_{\gamma-1}^{\ell-1,s}(S_{R-j\rho})} \leq C_p A_p^\ell (j\rho)^{-\ell}\ell^\ell \text{ for all } \ell = 1, \dots, j.$$

For $k \in \mathbb{N}$, we state the induction hypothesis succinctly as

$$(1.27) \quad H_{\boldsymbol{u},p}(s, \gamma, k, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p}) : \quad \begin{cases} H_{\boldsymbol{u}}(s, \gamma, j, C_{\boldsymbol{u}}, A_{\boldsymbol{u}}) \text{ holds for } j = 0, \dots, k \\ H_p(s, \gamma, j, C_p, A_p) \text{ holds for } j = 1, \dots, k. \end{cases}$$

1.3.2. Weighted interpolation estimates. We introduce interpolation estimates for weighted norms in sectors near the origin, analogous to the estimates used to establish regularity in classical Sobolev spaces, see, e.g., [1]. The derivation of these estimates is based on the homogeneity of the weighted norms (which is a consequence of the homogeneous Dirichlet boundary conditions), a dyadic decomposition of the domain around the point singularity combined with an homothety to a reference annulus of unit size.

LEMMA 1.10. *Let $0 < D_0 \leq D \leq 1$ and let S_D be defined as in (1.13), $\beta \in \mathbb{N}_0^2$ such that $|\beta| > 0$, $v \in \mathcal{K}_\gamma^{|\beta|+1,s}(S_D), \gamma - 2/s > \xi - 2/t$, and $2 < s \leq t \leq \infty$. Then, the following "interpolation" estimate holds*

(1.28)
$$\|r^{-\xi+|\beta|}\partial^\beta v\|_{L^t(S_D)} \leq C\|r^{|\beta|-\gamma}\partial^\beta v\|_{L^s(S_D)}^{1-\vartheta} \sum_{|\alpha|\leq 1} |\beta|^{(1-|\alpha|)\vartheta}\|r^{|\beta|+|\alpha|-\gamma}\partial^{\beta+\alpha}v\|_{L^s(S_D)}^\vartheta$$

with $\vartheta = 2/s - 2/t$ and for C dependent on s, t, γ, ξ and D_0, but independent of $|\beta|$, and of v.

Proof. We consider the case where $D = 1$, the general one will follow by scaling and the fact that $D \geq D_0$. For $j \in \mathbb{N}_0$, we introduce the dyadic annuli

$$V^j = \{x \in \mathbb{R}^2 : 2^{-j-1} < |x| < 2^{-j}\}$$

and their sectorial intersections

$$S^j = S_1 \cap V^j.$$

We also introduce the homothetic dilation $\Phi_j : V^j \to V^0 : x \mapsto 2^j x$ and denote with a hat the scaled quantities, e.g., $\hat{v} = v \circ \Phi_j^{-1}$. Then, for all $j \in \mathbb{N}$,

(1.29)
$$\|r^{-\xi+|\beta|}\partial^\beta v\|_{L^t(S^j)} \leq 2^{-j(-\xi+2/t)}\|\hat{r}^{-\xi+|\beta|}\hat{\partial}^\beta \hat{v}\|_{L^t(S^0)}.$$

On S^0 we have $1/2 < \hat{r} < 1$, then,

(1.30)
$$\|\hat{r}^{-\xi+|\beta|}\hat{\partial}^\beta \hat{v}\|_{L^t(S^0)} \leq 2^{-|\xi|}\|\hat{r}^{|\beta|}\hat{\partial}^\beta \hat{v}\|_{L^t(S^0)}.$$

Furthermore, since S^0 satisfies the cone condition [1], there holds the interpolation estimate

$$\|v\|_{L^t(S^0)} \leq C\|v\|_{L^s(S^0)}^\vartheta \|v\|_{W^{1,s}(S^0)}^{1-\vartheta},$$

valid for all $v \in W^{1,s}(S^0)$, with $\vartheta = 2/s - 2/t \in [0,1)$. From (1.29), (1.30) and from this interpolation estimate we obtain

$$\|r^{-\xi+|\beta|}\partial^\beta v\|_{L^t(S^j)} \leq C2^{-j(-\xi+2/t)}\|\hat{r}^{|\beta|}\hat{\partial}^\beta \hat{v}\|_{L^s(S^0)}^{1-\vartheta} \sum_{|\alpha|\leq 1} \|\hat{\partial}^\alpha \hat{r}^{|\beta|}\hat{\partial}^\beta \hat{v}\|_{L^s(S^0)}^\vartheta.$$

By an elementary manipulation, using again the fact that $1/2 < \hat{r} < 1$, and possibly adjusting the constant C,

(1.31) $\|r^{-\xi+|\beta|}\partial^\beta v\|_{L^t(S^j)}$
$$\leq C2^{-j(\xi+2/t)}\|\hat{r}^{|\beta|-\gamma}\hat{\partial}^\beta \hat{v}\|_{L^s(S^0)}^{1-\vartheta} \sum_{|\alpha|\leq 1} |\beta|^{(1-|\alpha|)\vartheta}\|\hat{r}^{|\beta|+|\alpha|-\gamma}\hat{\partial}^\alpha\hat{\partial}^\beta \hat{v}\|_{L^s(S^0)}^\vartheta.$$

Scaling back from S^0 to S^j,

$$\|r^{-\xi+|\beta|}\partial^\beta v\|_{L^t(S^j)}$$
$$\leq C2^{-j(-\xi+2/t+\gamma-2/s)}\|r^{|\beta|-\gamma}\partial^\beta v\|_{L^s(S^j)}^{1-\vartheta}\sum_{|\alpha|\leq 1}|\beta|^{(1-|\alpha|)\vartheta}\|r^{|\beta|+|\alpha|-\gamma}\partial^\alpha\partial^\beta v\|_{L^s(S^j)}^{\vartheta}.$$

If $\gamma - 2/s > \xi - 2/t$, then, we can sum over all S^j and obtain the assertion. □

Remark 1.11. The constant C in (1.28) depends on s, t, γ, and ξ. Nonetheless, it will be used only for specific values of s, t, γ, and ξ, thus this dependence will not be relevant in the following proofs. Specifically, under the hypothesis of Lemma 1.10, we will use the inequalities
(1.32)
$$\|r^{\frac{1-\gamma}{2}+|\beta|}\partial^\beta v\|_{L^{2s}(S_D)} \leq C\|r^{|\beta|-\gamma}\partial^\beta v\|_{L^s(S_D)}^{1-\vartheta}\sum_{|\alpha|\leq 1}|\beta|^{(1-|\alpha|)\vartheta}\|r^{|\beta|+|\alpha|-\gamma}\partial^{\beta+\alpha}v\|_{L^s(S_D)}^{\vartheta}$$

valid for $\gamma - 2/s > -1$ and with $\vartheta = 1/s$, and
(1.33)
$$\|r^{1+|\beta|}\partial^\beta v\|_{L^\infty(S_D)} \leq C\|r^{|\beta|-\gamma}\partial^\beta v\|_{L^s(S_D)}^{1-\eta}\sum_{|\alpha|\leq 1}|\beta|^{(1-|\alpha|)\eta}\|r^{|\beta|+|\alpha|-\gamma}\partial^{\beta+\alpha}v\|_{L^s(S_D)}^{\eta},$$

valid for $\gamma - 2/s > -1$ and with $\eta = 2/s$.

1.3.3. Estimate of the nonlinear term. We can now give an estimate of the norm of the nonlinear term, under the assumption that the induction hypothesis holds. Specifically, this will be done in Lemma 1.15. We start by introducing three auxiliary lemmas, that will subsequently be necessary for the proof of Lemma 1.15.

LEMMA 1.12. *Let* $k \in \mathbb{N}$, $0 < R < 1$, *and* $2 < s < \infty$, γ *such that* $\gamma - 2/s > -1$. *Let* $\boldsymbol{u}$ *be such that* $H_{\boldsymbol{u}}(s,\gamma,k,C_{\boldsymbol{u}},A_{\boldsymbol{u}})$ *holds in* S_R. *Then, there exists* $C > 0$ *independent of* k, $A_{\boldsymbol{u}}$, *and* ρ *such that there holds, for all integer* $0 \leq \ell \leq k-1$ *and with* $\vartheta = 1/s$,

$$(1.34) \qquad \sum_{|\beta|=\ell}\|r^{\frac{1-\gamma}{2}+|\beta|}\partial^\beta\boldsymbol{u}\|_{L^{2s}(S_{R-k\rho})} \leq CA_{\boldsymbol{u}}^{\ell+\vartheta}(k\rho)^{-\ell-\vartheta}(\ell+1)^{\ell+\vartheta}.$$

Proof. Let $\boldsymbol{u} = (u_1, u_2)$ and denote $v = u_i$, for any $i = 1, 2$. We start by applying Lemma 1.10, using the bound (1.32):

$$(1.35) \quad \|r^{\frac{1-\gamma}{2}+|\beta|}\partial^\beta v\|_{L^{2s}(S_{R-k\rho})}$$
$$\leq C\|r^{|\beta|-\gamma}\partial^\beta v\|_{L^s(S_{R-k\rho})}^{1-\vartheta}\sum_{|\alpha|\leq 1}|\beta|^{(1-|\alpha|)\vartheta}\|r^{|\beta|+|\alpha|-\gamma}\partial^{\beta+\alpha}v\|_{L^s(S_{R-k\rho})}^{\vartheta},$$

with C independent of $|\beta|$ and v. Then, from (1.26a) we have that for all $1 \leq \ell \leq k-1$

$$\sum_{|\beta|=\ell}\|r^{\ell-\gamma}\partial^\beta v\|_{L^s(S_{R-k\rho})}^{1-\vartheta} \leq C_{\boldsymbol{u}}A_{\boldsymbol{u}}^{(1-\vartheta)\ell}(k\rho)^{-(1-\vartheta)\ell}\ell^{(1-\vartheta)\ell}$$

and

$$\sum_{|\beta|=\ell}\sum_{|\alpha|\leq 1}\ell^{(1-|\alpha|)\vartheta}\|r^{\ell+|\alpha|-\gamma}\partial^{\beta+\alpha}v\|_{L^s(S_{R-k\rho})}^{\vartheta}$$
$$\leq C_{\boldsymbol{u}}A_{\boldsymbol{u}}^{\vartheta\ell}(k\rho)^{-\vartheta\ell}\ell^{\vartheta(\ell+1)} + C_{\boldsymbol{u}}A_{\boldsymbol{u}}^{\vartheta(\ell+1)}(k\rho)^{-\vartheta(\ell+1)}(\ell+1)^{\vartheta(\ell+1)},$$

hence

$$\sum_{|\beta|=\ell} \|r^{\frac{1-\gamma}{2}+\ell}\partial^\beta v\|_{L^{2s}(S_{R-k\rho})} \leq CA_{\boldsymbol{u}}^{\ell+\vartheta}(k\rho)^{-\ell-\vartheta}(\ell+1)^{\ell+\vartheta},$$

which is what was to be proved.   □

LEMMA 1.13. *Under the same hypotheses as in Lemma 1.12, there exists $C > 0$ independent of k, $A_{\boldsymbol{u}}$, and ρ such that there holds for all $0 \leq \ell \leq k-2$ and with $\vartheta = 1/s$*

$$(1.36) \qquad \sum_{|\beta|=\ell} \|r^{\frac{1-\gamma}{2}+1+|\beta|}\partial^\beta \nabla \boldsymbol{u}\|_{L^{2s}(S_{R-k\rho})} \leq CA_{\boldsymbol{u}}^{\ell+1+\vartheta}(k\rho)^{-\ell-1-\vartheta}(\ell+2)^{\ell+1+\vartheta}.$$

Proof. The assertion follows directly from Lemma 1.12.   □

LEMMA 1.14. *Under the same hypotheses as in Lemma 1.12, there exists $C > 0$ independent of k, $A_{\boldsymbol{u}}$, and ρ such that there holds, for all $0 \leq \ell \leq k-1$ and with $\eta = 2/s$,*

$$(1.37) \qquad \sum_{|\beta|=\ell} \|r^{1+|\beta|}\partial^\beta \boldsymbol{u}\|_{L^\infty(S_{R-k\rho})} \leq CA_{\boldsymbol{u}}^{\ell+\eta}(k\rho)^{-\ell-\eta}(\ell+1)^{\ell+\eta}.$$

Proof. The proof follows verbatim the steps as the proof of Lemma 1.12, with (1.35) replaced by (1.33) and ϑ replaced by η.   □

The weighted seminorms of the nonlinear term in (2.1) can now be controlled, subject to the induction hypothesis.

LEMMA 1.15. *Let $0 < R < 1$, $k \in \mathbb{N}$, $2 < s < \infty$, $\gamma \in \mathbb{R}$, $\boldsymbol{u} \in \mathcal{K}_\gamma^k(S_{R-k\rho})$ be such that $\gamma - 2/s > -1$ and assume that $H_{\boldsymbol{u}}(s, \gamma, k, C_{\boldsymbol{u}}, A_{\boldsymbol{u}})$ holds in S_R.*

Then, $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \mathcal{K}_{\gamma-2}^{k-1,s}(S_{R-k\rho})$ and there exists a constant $C > 0$ independent of k and $A_{\boldsymbol{u}}$, such that for all $\rho \in (0, R/(2k)]$ and for all $\ell \in \mathbb{N}_0$ such that $\ell \leq k-1$ there holds the bound

$$(1.38) \qquad \sum_{|\beta|=\ell} \|r^{2-\gamma+|\beta|}\partial^\beta (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\|_{L^s(S_{R-k\rho})} \leq CA_{\boldsymbol{u}}^{\ell+1+2\vartheta}(k\rho)^{-\ell-1-2\vartheta}(\ell+1)^{\ell+3/2}$$

with $\vartheta = 1/s$.

Proof. Let $\boldsymbol{u} = (u_1, u_2)$ and $\ell \in \mathbb{N}$, $\beta \in \mathbb{N}_0^2$ such that $|\beta| = \ell \leq k-1$. By the Leibniz rule and the Hölder inequality, for $m, n \in \{1, 2\}$,

$$(1.39)$$
$$\|r^{2-\gamma+|\beta|}\partial^\beta u_m \partial_m u_n\|_{L^s(S_{R-k\rho})}$$
$$\leq \sum_{j=0}^{|\beta|-1} \sum_{\substack{|\alpha|=j \\ \alpha \leq \beta}} \binom{\beta}{\alpha} \|r^{\frac{1-\gamma}{2}+|\beta|-|\alpha|}\partial^{\beta-\alpha}u_m\|_{L^{2s}(S_{R-k\rho})} \|r^{\frac{1-\gamma}{2}+1+|\alpha|}\partial^\alpha \partial_m u_n\|_{L^{2s}(S_{R-k\rho})}$$
$$+ \|r^{2-\gamma+|\beta|}u_m \partial^\beta \partial_m u_n\|_{L^s(S_{R-k\rho})}.$$

We start by considering the sum in the right hand side of the above inequality. In the inequality (1.40) we will use the fact that, for two quantities $a, b : \mathbb{N}_0^2 \to \mathbb{R}$, there

holds

$$\sum_{\substack{|\beta|=\ell}}\sum_{j=0}^{\ell-1}\sum_{\substack{|\alpha|=j\\\alpha\leq\beta}} a(\alpha)b(\beta-\alpha) = \sum_{j=0}^{\ell-1}\sum_{\substack{|\beta|=\ell}}\sum_{\substack{|\alpha|=j\\\alpha\leq\beta}} a(\alpha)b(\beta-\alpha) = \sum_{j=0}^{\ell-1}\sum_{\substack{|\alpha|=j}}\sum_{\substack{|\beta|=\ell\\\beta\geq\alpha}} a(\alpha)b(\beta-\alpha)$$

$$= \sum_{j=0}^{\ell-1}\sum_{\substack{|\alpha|=j}}\sum_{\substack{|\xi|=\ell-j}} a(\alpha)b(\xi) \ .$$

With this summation identity, and due to Lemma 1.12, Lemma 1.13, and Stirling's inequality, there exists constants C_i, $i = 1, 2, 3, 4$ independent of ℓ, k, ρ and $A_{\boldsymbol{u}}$, such that

(1.40)

$$\sum_{\substack{|\beta|=\ell}}\sum_{j=0}^{\ell-1}\sum_{\substack{|\alpha|=j\\\alpha\leq\beta}} \binom{\beta}{\alpha} \|r^{\frac{1-\gamma}{2}+|\beta|-|\alpha|}\partial^{\beta-\alpha}u_m\|_{L^{2s}(S_{R-k\rho})}\|r^{\frac{1-\gamma}{2}+1+|\alpha|}\partial^{\alpha}\partial_m u_n\|_{L^{2s}(S_{R-k\rho})}$$

$$\leq \sum_{j=0}^{\ell-1}\binom{\ell}{j}\sum_{\substack{|\alpha|=j}}\sum_{\substack{|\xi|=\ell-j}} \|r^{\frac{1-\gamma}{2}+|\xi|}\partial^{\xi}u_m\|_{L^{2s}(S_{R-k\rho})}\|r^{\frac{1-\gamma}{2}+1+|\alpha|}\partial^{\alpha}\partial_m u_n\|_{L^{2s}(S_{R-k\rho})}$$

$$\leq C_1 A_{\boldsymbol{u}}^{\ell+1+2\vartheta}\rho^{-\ell-1-2\vartheta}\sum_{j=0}^{\ell-1}\binom{\ell}{j}\frac{(\ell-j+1)^{\ell-j+\vartheta}(j+2)^{j+1+\vartheta}}{k^{\ell+1+2\vartheta}}$$

$$\leq C_2 A_{\boldsymbol{u}}^{\ell+1+2\vartheta}\rho^{-\ell-1-2\vartheta}\sum_{j=0}^{\ell-1}\binom{\ell}{j}(j+1)!(\ell-j)!e^{\ell}\frac{(j+2)^{\vartheta}(\ell-j+1)^{\vartheta}}{k^{\ell+1+2\vartheta}\sqrt{(j+2)(\ell-j+1)}}$$

$$\leq C_3 A_{\boldsymbol{u}}^{\ell+1+2\vartheta}\rho^{-\ell-1-2\vartheta}\frac{\ell!e^{\ell}}{k^{\ell}}\sum_{j=0}^{\ell-1}\frac{(j+1)}{k}\frac{1}{\sqrt{(j+2)(\ell-j+1)}}$$

$$\leq C_4 A_{\boldsymbol{u}}^{\ell+1+2\vartheta}\rho^{-\ell-1-2\vartheta}\left(\frac{\ell}{k}\right)^{\ell+1}\ell^{1/2}.$$

Consider now the remaining term in (1.39). This term has to be treated differently due to the possibility that $\ell = k - 1$. There holds, using (1.33), (1.26a), and writing $\eta = 2/s$,

$$\sum_{\substack{|\beta|=\ell}} \|r^{2-\gamma+|\beta|}u_m\partial^{\beta}\partial_m u_n\|_{L^s(S_{R-k\rho})}$$

$$\leq \sum_{\substack{|\beta|=\ell}} \|ru_m\|_{L^{\infty}(S_{R-k\rho})}\|r^{1-\gamma+|\beta|}\partial^{\beta}\partial_m u_n\|_{L^s(S_{R-k\rho})}$$

$$\leq C_5 \|r^{-\gamma}u_m\|_{L^s(S_{R-k\rho})}^{1-\eta}\|u_m\|_{\mathcal{K}_{\gamma}^{1,s}(S_{R-k\rho})}^{\eta}\sum_{\substack{|\beta|=\ell}} \|r^{1-\gamma+|\beta|}\partial^{\beta}\partial_m u_n\|_{L^s(S_{R-k\rho})}$$

$$\leq C_6 A_{\boldsymbol{u}}^{\ell+1+\eta}(k\rho)^{-\ell-1-\eta}(\ell+1)^{\ell+1}.$$

with C_5, C_6 independent of $A_{\boldsymbol{u}}$, ℓ, k, and ρ. This concludes the proof. $\quad\square$

2. The Stokes and Navier-Stokes equations in a polygon P. We now consider the Navier-Stokes problem in a polygon P, with internal angles φ_i such that $0 < \varphi_i < 2\pi$, corners c_i and edges e_i, $i = 1, \ldots, n$ as depicted in Figure 2. We recall (0.1),
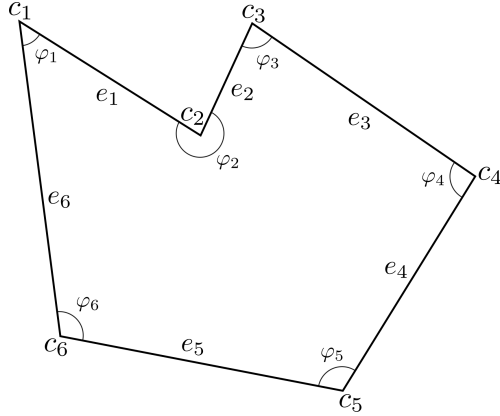
FIG. 2. *The polygon P, with internal angles φ_i, corners c_i, and edges e_i, $i = 1, \ldots, 6$.*

the incompressible Navier-Stokes equations with homogeneous Dirichlet boundary conditions

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } P,$$

(2.1)
$$\nabla\cdot\boldsymbol{u} = 0 \text{ in } P,$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial P.$$

It is well-known that for sufficiently small data, i.e. for sufficiently small $\|f\|_{\boldsymbol{H}^{-1}(P)}/\nu^2$, (2.1) admits a unique weak (Leray-Hopf) solution $(\boldsymbol{u}, p) \in H_0^1(P)^2 \times L^2(P)/\mathbb{R}$ (e.g. [16, 28, 30, 43]). For convenience, *we assume throughout what follows that $\nu = 1$ and that* $\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(P)}$ *is sufficiently small to ensure uniqueness of* $(\boldsymbol{u}, p)$. The proof of weighted analytic estimates for Leray-Hopf weak solutions of problem (2.1) consists in the combination of the estimates localised at every corner sector with classical regularity results in the interior and along the (straight) edges of the domain.

To show the estimates in the corner sectors, we will again proceed by induction. Specifically, we decouple the inductive step and the starting point of our argument. First, in Lemma 2.1, we prove an analytic regularity shift estimate, starting from the induction hypothesis introduced in (1.27) and using the results obtained in Section 1.3. This will serve in the following as the local induction step for the proof of analytic regularity in polygons. Then, the base case for induction is given in Lemma 2.5, where we derive a basic regularity estimate in weighted spaces for the Leray-Hopf solution to the problem under consideration.

Theorem 2.6 constitutes then the main result of this paper, i.e., weighted analytic regularity of the Leray-Hopf solutions to the incompressible Navier-Stokes problem, in polygonal domains and with "no-slip", homogeneous Dirichlet boundary conditions.

2.1. The Navier-Stokes equation in the polygon. We start by the introduction of the setting and notation for the polygonal case. For all $i = 1, \ldots, n$ and for $0 < D < |e_i|$, where $|e_i|$ is the length of the edge e_i, we denote the sectors centered at corner c_i by
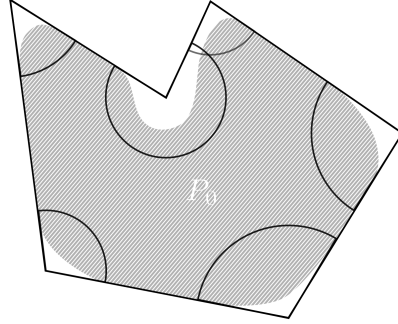
$$S_D^i = B_D(c_i) \cap P$$

FIG. 3. *Corner sectors with different radii and set P_0, denoted by the shaded part of the polygon.*

where $B_D(c_i)$ is the ball of radius D centered in c_i. We write $r_i = |x - c_i|$, $\underline{\gamma} = \{\gamma_i\}_{i=1,\dots,n}$ and

$$r^{\underline{\gamma}} = \prod_{i=1,\dots,n} r_i^{\gamma_i}.$$

For $\ell \in \mathbb{N}_0$, $1 < s < \infty$ and for $\underline{\gamma} \in \mathbb{R}^n$, the definition of the spaces $\mathcal{K}_{\underline{\gamma}}^{\ell,s}(P)$ and $\mathcal{K}_{\underline{\gamma}}^{\varpi,s}(P)$ follows directly from the one introduced for K. In the following, comparisons between n-dimensional vectors have to be interpreted elementwise, e.g., for $\underline{\gamma} = \{\gamma_i\}_{i=1,\dots,n}$ and $\underline{\eta} = \{\eta_i\}_{i=1,\dots,n}$, we write $\underline{\gamma} < \underline{\eta}$ if

$$\gamma_i < \eta_i \quad \text{for all} \quad i = 1, \dots, n.$$

Also, for $x \in \mathbb{R}$, we write $x + \underline{\gamma} = \{x + \gamma_i\}_{i=1,\dots,n}$.

Fix $R_P \in \mathbb{R}_+$ such that $R_P \leq 1$ and $R_P \leq \min_{i,j=1,\dots,n} |c_i - c_j|$. Denote by P_0 a set bounded away from the corners such that $P_0 \cup \left(\bigcup_{i=1,\dots,n} S_{R_P/2}^i \right) = P$, see Figure 3.

2.1.1. Inductive step. Using the results introduced in Sections 1, we can now show that, given a sufficiently regular right hand side, if the derivatives of a Leray-Hopf solution to the Navier-Stokes equations satisfy weighted analytic estimates up to a certain order in a sector, then by elliptic regularity in weighted spaces the estimates hold to higher order. This will be used as the inductive step in the proof of weighted analytic regularity in polygons.

LEMMA 2.1. *Let $k \in \mathbb{N}$, $i \in \{1, \dots, n\}$, $2 < s < \infty$, $\gamma \in \mathbb{R}$ such that $\gamma - 2/s > -1$. Let*

$$\boldsymbol{f} \in \mathcal{K}_{\gamma-2}^{\varpi,s}(S_{R_P}^i; C_f, A_f),$$

and let $(\boldsymbol{u}, p) \in \mathcal{K}_{\gamma}^k(S_{R_P-k\rho}^i) \times \mathcal{K}_{\gamma-1}^{k-1}(S_{R_P-k\rho}^i)$ be a solution to (2.1). Then, $C_{\boldsymbol{u}}$, $A_{\boldsymbol{u}}$, C_p, and A_p can be chosen independently of k and ρ so that the hypothesis "$H_{\boldsymbol{u},p}(s, \gamma, k, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p})$ holds in $S_{R_P}^i$" implies

$$H_{\boldsymbol{u},p}(s, \gamma, k+1, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p}) \text{ holds in } S_{R_P}^i .$$

Proof. We remind that we work under the small data hypothesis which ensures that $(\boldsymbol{u}, p)$ is unique.

Assume that $H_{\boldsymbol{u},p}(s,\gamma,k,C_{\boldsymbol{u},p},A_{\boldsymbol{u},p})$ holds in $S^i_{R_P}$. Then, for all $\rho \in (0, R_P/(2(k+1))]$, there exists $\widetilde{\rho} = (k+1)\rho/k$ such that

$$|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{\ell,s}_\gamma(S_{R_P-(k+1)\rho})} = |\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{\ell,s}_\gamma(S_{R_P-k\widetilde{\rho}})} \leq C_{\boldsymbol{u}} A^\ell_{\boldsymbol{u}}(k\widetilde{\rho})^{-\ell}\ell^\ell = C_{\boldsymbol{u}} A^\ell_{\boldsymbol{u}}((k+1)\rho)^{-\ell}\ell^\ell$$

for all $\ell = 0, \ldots, k$. Similarly,

$$|p|_{\mathcal{K}^{\ell-1,s}_{\gamma-1}(S_{R_P-(k+1)\rho})} = |p|_{\mathcal{K}^{\ell-1,s}_{\gamma-1}(S_{R_P-k\widetilde{\rho}})} \leq C_p A^\ell_p(k\widetilde{\rho})^{-\ell}\ell^\ell = C_p A^\ell_p((k+1)\rho)^{-\ell}\ell^\ell$$

for all $\ell = 1, \ldots, k$. Therefore, it remains to prove the bounds on $|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{k+1,s}_\gamma(S_{R_P-(k+1)\rho})}$ and on $|p|_{\mathcal{K}^{k,s}_{\gamma-1}(S_{R_P-(k+1)\rho})}$.

We write (2.1) as (recall that we assume $\nu = 1$)

$$-\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \text{ in } P,$$
$$\nabla\cdot\boldsymbol{u} = 0 \text{ in } P,$$
$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial P \setminus \{0\}.$$

By the hypotheses and Lemma 1.15, $\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{\mathcal{K}}^{k-1,s}_{\gamma-2}(S^i_{R_P-k\rho})$. We apply Proposition 1.9 to obtain, for all $\rho \in (0, R_P/(2(k+1))]$,

$$|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{k+1,s}_\gamma(S^i_{R_P-(k+1)\rho})} + |p|_{\mathcal{K}^{k,s}_{\gamma-1}(S^i_{R_P-(k+1)\rho})}$$

$$\leq C_1 \Bigg\{ |\boldsymbol{f}|_{\boldsymbol{\mathcal{K}}^{k-1,s}_{\gamma-2}(S^i_{R_P-k\rho})} + |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{k-1,s}_{\gamma-2}(S^i_{R_P-k\rho})}$$

$$+ \rho^{-1}|p|_{\mathcal{K}^{k-1,s}_{\gamma-1}(S^i_{R_P-k\rho})} + \sum_{n\leq 1}\rho^{n-2}|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{k+n-1,s}_\gamma(S^i_{R_P-k\rho})} \Bigg\},$$

with C_1 independent of k, ρ, $\boldsymbol{u}$, and p. Now, if $H_{\boldsymbol{u},p}(s,\gamma,k,C_{\boldsymbol{u},p},A_{\boldsymbol{u},p})$ holds in $S^i_{R_P}$ and using the estimate (1.38) of Lemma 1.15, we obtain
(2.2)
$$|\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{k+1,s}_\gamma(S^i_{R_P-(k+1)\rho})} + |p|_{\mathcal{K}^{k,s}_{\gamma-1}(S^i_{R_P-(k+1)\rho})}$$

$$\leq C_2 \Bigg\{ A^{k-1}_f(k-1)! + A^{k+2/s}_{\boldsymbol{u}}(k\rho)^{-k-2/s}k^{k+1/2} + A^k_p\rho^{-k-1} + A^k_{\boldsymbol{u}}\rho^{-k-1} \Bigg\},$$

with $C_2 \geq 1$ independent of k, ρ, $A_{\boldsymbol{u}}$, and A_p. Thus, choosing $C_{\boldsymbol{u}}, C_p \geq 1$ and $A_{\boldsymbol{u}} = A_p \geq 1$ so large that

(2.3) $$A_{\boldsymbol{u}} \geq A_f, \qquad A^{1-2/s}_{\boldsymbol{u}} \geq 4C_2,$$

and that $H_{\boldsymbol{u},p}(s,\gamma,k,C_{\boldsymbol{u},p},A_{\boldsymbol{u},p})$ holds in $S^i_{R_P}$, we obtain the assertion.   □

Remark 2.2. The proof of Lemma 2.1 shows how the argument used in the present paper does not work when $s \leq 2$. Specifically, only if $s > 2$ there exists an $A > 0$ such that the second inequality in (2.3) is verified and $(k\rho)^{-2/s} < (k\rho)^{-1}$ in (2.2).

2.1.2. Base regularity estimate. We recall two lemmas characterizing continuous imbeddings in weighted spaces from [33, Lemmas 4.1.2 and 4.1.3].

LEMMA 2.3. *Let* $1 < t < s < \infty$, $\underline{\gamma}, \underline{\eta} \in \mathbb{R}^n$, *and* $\ell \in \mathbb{N}_0$ *such that* $\underline{\gamma} - 2/s > \underline{\eta} - 2/t$. *Then,*

$$\mathcal{K}^{\ell,s}_{\underline{\gamma}}(P) \hookrightarrow \mathcal{K}^{\ell,t}_{\underline{\eta}}(P).$$

LEMMA 2.4. *Let* $1 < s \leq t < \infty$, $\underline{\gamma}, \underline{\eta} \in \mathbb{R}^n$, *and* ℓ, k *in* $\mathbb{N}$ *such that* $\ell > k$ *and* $\ell - 2/s \geq k - 2/t$, $\underline{\gamma} - 2/s \geq \underline{\eta} - 2/t$. *Then,*

$$\mathcal{K}_{\underline{\gamma}}^{\ell,s}(P) \hookrightarrow \mathcal{K}_{\underline{\eta}}^{k,t}(P).$$

We next introduce a regularity result in weighted spaces for the Navier-Stokes problem (2.1). This will constitute the base case for the inductive proof of analytic regularity. In the following we investigate the regularity in the weighted spaces $\mathcal{K}_{\gamma}^{1,s}(P)$ for $s > 2$, i.e., in the non Hilbertian setting. Our strategy to prove the base case is to prove a higher order regularity result, then use the preceding imbeddings to prove the base case.

LEMMA 2.5. *Let* $\boldsymbol{u}$, p *be a solution to* (2.1). *Suppose that there exists* $2 \leq \tilde{s}$ *and* $\widetilde{\underline{\gamma}} \in \mathbb{R}^n$ *such that*

$$\boldsymbol{f} \in \boldsymbol{H}^{-1}(P) \cap \mathcal{K}_{\widetilde{\underline{\gamma}}-2}^{0,\tilde{s}}(P).$$

Then, for all $1 < s < \infty$ *and* $\underline{\gamma} \in \mathbb{R}^n$ *such that* $\underline{\gamma} - 2/s \leq \min(2, \widetilde{\underline{\gamma}} + 1 - 2/\tilde{s})$, *there holds*

$$\boldsymbol{u} \in \mathcal{K}_{\underline{\gamma}}^{1,s}(P), \qquad p \in \mathcal{K}_{\underline{\gamma}-1}^{0,s}(P).$$

Proof. Since $\boldsymbol{u}$ is solution to (2.1), $\boldsymbol{u} \in \boldsymbol{H}_0^1(P) = \mathcal{K}_{\underline{1}}^{1,2}(P)$, where $\underline{1} = \{1,\ldots,1\} \in \mathbb{R}^n$, see [16]. Hence, by the Hölder inequality and by Sobolev imbedding, for all $1 < s_1 < 2$ and $\widetilde{\underline{\gamma}}_1 - 2/s_1 < -1$, denoting $t = (1/s_1 - 1/2)^{-1}$,

$$\|r^{-\widetilde{\underline{\gamma}}_1}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\|_{L^{s_1}(P)} \leq \|r^{-\widetilde{\underline{\gamma}}_1}\boldsymbol{u}\|_{L^t(P)}\|\nabla\boldsymbol{u}\|_{L^2(P)} < \infty,$$

i.e.,

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \mathcal{K}_{\widetilde{\underline{\gamma}}_1}^{0,s_1}(P).$$

We temporarily fix s_1 and $\widetilde{\underline{\gamma}}_1$. By Lemma 2.3,

$$\boldsymbol{f} \in \mathcal{K}_{\underline{\eta}}^{0,s_1}(P) \quad \text{for all} \quad \underline{\eta} < \widetilde{\underline{\gamma}} - 2 + 2/s_1 - 2/\tilde{s}.$$

Problem (2.1) can be rewritten as

$$-\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \text{ in } P,$$
$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } P,$$
$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial P.$$

Then, by classical elliptic regularity in P_0 – see for example [16] – and using Theorem 1.1 in all the sectors $S_{R_P}^i$, $i = 1, \ldots, n$, we obtain

$$\boldsymbol{u} \in \mathcal{K}_{\underline{\gamma}_1}^{2,s_1}(P), \qquad p \in \mathcal{K}_{\underline{\gamma}_1-1}^{1,s_1}(P),$$

for all $\underline{\gamma}_1 \in \mathbb{R}^n$ such that $\underline{\gamma}_1 - 2/s_1 \leq \widetilde{\underline{\gamma}}_1 + 2$ and $\underline{\gamma}_1 - 2/s_1 < \widetilde{\underline{\gamma}} + 2/s_1 - 2/\tilde{s}$. Then, by continuous imbedding – see Lemma 2.4 – if $s_1 \leq s_2 \leq 2s_1/(2 - s_1)$ and $\underline{\gamma}_2 - 2/s_2 \leq \underline{\gamma}_1 - 2/s_1$, there holds

(2.4) $$\boldsymbol{u} \in \mathcal{K}_{\underline{\gamma}_2}^{1,s_2}(P), \qquad p \in \mathcal{K}_{\underline{\gamma}_2-1}^{0,s_2}(P).$$

Due to the arbitrariness of $1 < s_1 < 2$, (2.4) holds for all $2 \leq s_2 < \infty$, $\underline{\gamma}_2 - 2/s_2 \leq \min(2, \widetilde{\underline{\gamma}} + 1 - 2/\tilde{s})$. $\square$

2.1.3. Analytic regularity. We denote by $\mathfrak{A}^i(\lambda)$ the Mellin symbol of the Stokes operator associated to the corner c_i, i.e., the operator $\mathfrak{L}(\vartheta, \partial_\vartheta; \lambda)$ in $(0, \varphi_i)$ with boundary operator $\mathfrak{B}(\vartheta, \partial_\vartheta; \lambda)_{|\vartheta=0,\varphi_i}$ introduced in Section 1.2. The operator pencil $\mathfrak{A}^i(\lambda)$ is Fredholm and therefore admits a discrete spectrum of countably many, possibly complex eigenvalues which accumulate only at $\lambda = +\infty$. Let $\Lambda^i_{\min}$ be the real part of the eigenvalue with smallest positive real part of $\mathfrak{A}^i(\lambda)$, and let $\underline{\Lambda} = \{\Lambda^i_{\min}\}_{i=1,\ldots,n} > 0$.

The regularity in countably normed, weighted analytic spaces for the solution to the incompressible Navier-Stokes equation in the polygon P then follows by a bootstrapping argument with induction with respect to the order of differentiation in the Kondrat'ev scales, with parameter γ in one common strip of width $2\underline{\Lambda}$ determined by the spectra of $\mathfrak{A}^i(\lambda)$, $i = 1, \ldots, n$ as in finite-order regularity results in [17, 36].

THEOREM 2.6. *Let* $2 < s < \infty$, $\underline{\gamma} \in \mathbb{R}^n$ *such that* $\max(-\underline{\Lambda}, -1) < \underline{\gamma} - 2/s < \min(\underline{\Lambda}, 1)$. *Let* $(\boldsymbol{u}, p)$ *be solution to* (2.1) *with*

$$\boldsymbol{f} \in \boldsymbol{\mathcal{K}}^{\varpi,s}_{\underline{\gamma}-2}(P) \cap \boldsymbol{H}^{-1}(P).$$

Then,

$$\boldsymbol{u} \in \boldsymbol{\mathcal{K}}^{\varpi,s}_{\underline{\gamma}}(P), \qquad p \in \mathcal{K}^{\varpi,s}_{\underline{\gamma}-1}(P).$$

Proof. We rewrite for the polygon P the induction hypotheses introduced in Section 1.3. For $1 < s < \infty$, $\underline{\gamma} \in \mathbb{R}^n$, $i = 1, \ldots, n$, $C_{\boldsymbol{u}}, A_{\boldsymbol{u}}, C_p, A_p > 0$, and for $j \in \mathbb{N}_0$, we introduce two statements: we shall say that $H^i_{\boldsymbol{u}}(s, \underline{\gamma}, j, C_{\boldsymbol{u}}, A_{\boldsymbol{u}})$ holds if for all $\rho \in (0, R_P/(2j)]$,

$$(2.5a) \qquad |\boldsymbol{u}|_{\boldsymbol{\mathcal{K}}^{\ell,s}_{\gamma_i}(S^i_{R_P - j\rho})} \leq C_{\boldsymbol{u}} A^\ell_{\boldsymbol{u}} (j\rho)^{-\ell} \ell^\ell \text{ for all } \ell = 0, \ldots, j$$

and $H^i_p(s, \underline{\gamma}, j, C_p, A_p)$ holds if for all $\rho \in (0, R_P/(2j)]$,

$$(2.5b) \qquad |p|_{\mathcal{K}^{\ell-1,s}_{\gamma_i-1}(S^i_{R_P - j\rho})} \leq C_p A^\ell_p (j\rho)^{-\ell} \ell^\ell \text{ for all } \ell = 1, \ldots, j.$$

For integer k, the induction hypothesis is formulated as

$$(2.6) \qquad \begin{aligned} &H_{\boldsymbol{u},p}(s, \underline{\gamma}, k, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p}) : \\ &\left\{ \begin{array}{l} H^i_{\boldsymbol{u}}(s, \underline{\gamma}, j, C_{\boldsymbol{u}}, A_{\boldsymbol{u}}) \text{ holds for } j = 0, \ldots, k, \\ H^i_p(s, \underline{\gamma}, j, C_p, A_p) \text{ holds for } j = 1, \ldots, k, \end{array} \right. \text{ for all } i = 1, \ldots, n. \end{aligned}$$

Fix $2 < s < \infty$ and $\underline{\gamma}$ such that the hypotheses of the theorem hold. Then, by Lemma 2.5, $\boldsymbol{u} \in \boldsymbol{\mathcal{K}}^{1,s}_{\underline{\gamma}}(P)$ and $p \in \mathcal{K}^{0,s}_{\underline{\gamma}-1}(P)$, i.e., there exist $C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p}$ such that $H_{\boldsymbol{u},p}(s, \underline{\gamma}, 1, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p})$ holds. Suppose now that, for a $k \in \mathbb{N}$, $H_{\boldsymbol{u},p}(s, \underline{\gamma}, k, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p})$ holds, with $C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p}$ chosen so large that the conclusion of Lemma 2.1 holds in every corner sector. Then, by Lemma 2.1 applied to every corner sector, $H_{\boldsymbol{u},p}(s, \underline{\gamma}, k + 1, C_{\boldsymbol{u},p}, A_{\boldsymbol{u},p})$ holds. By induction, this implies that, for all $i = 1, \ldots, n$,

$$(2.7) \qquad \boldsymbol{u} \in \boldsymbol{\mathcal{K}}^{\varpi,s}_{\underline{\gamma}}(S^i_{R_P/2}), \qquad p \in \mathcal{K}^{\varpi,s}_{\underline{\gamma}-1}(S^i_{R_P/2}).$$

The set P_0 can be covered by a finite number of balls, and from classical results of interior analyticity and of analyticity in regular parts of the boundary, see, e.g., [32, 15], we derive analyticity in P_0. Since r is bounded away from zero in P_0, this implies

$$(2.8) \qquad \boldsymbol{u} \in \boldsymbol{\mathcal{K}}^{\varpi,s}_{\underline{\gamma}}(P_0), \qquad p \in \mathcal{K}^{\varpi,s}_{\underline{\gamma}-1}(P_0).$$

Combining (2.7) for $i = 1, \ldots, n$ and (2.8) implies the assertion.   □

2.2. Approximability of the solutions. We briefly outline some consequences of the weighted analytic regularity result on Kolmogorov N-widths of solution families. As is well-known, bounds on Kolmogorov N-widths of solution families of (0.1) will imply convergence rate estimates for for model order reduction and reduced basis methods for NSE. See, e.g., [29, 22] and the references there for examples and details.

We start with a corollary of Theorem 2.6, based on an embedding in weighted spaces.

COROLLARY 2.7. *Let* $2 < s < \infty$, $\underline{\gamma}_f \in \mathbb{R}^n$ *such that* $\underline{\gamma}_f - 2/s > 0$. *Let* $(\boldsymbol{u}, p)$ *be solution to* (2.1) *with* $\boldsymbol{f} \in \mathcal{K}^{\varpi,s}_{\underline{\gamma}_f-2}(P) \cap \boldsymbol{H}^{-1}(P)$. *Then there exists a weight vector* $\underline{\gamma} > 1$ *such that*

$$\boldsymbol{u} \in \mathcal{K}^{\varpi,2}_{\underline{\gamma}}(P), \qquad p \in \mathcal{K}^{\varpi,2}_{\underline{\gamma}-1}(P).$$

Proof. From Theorem 2.6, $\boldsymbol{u} \in \mathcal{K}^{\varpi,s}_{\underline{\gamma}_f}(P)$ and $p \in \mathcal{K}^{\varpi,s}_{\underline{\gamma}_f-1}(P)$. Then, let $1/q = 1/2 - 1/s$. By Hölder's inequality,

$$\|r^{|\alpha|-\underline{\gamma}}\partial^\alpha \boldsymbol{u}\|_{L^2(P)} \le C\|r^{|\alpha|-\underline{\gamma}_f}\partial^\alpha \boldsymbol{u}\|_{L^s(P)}\|r^{\underline{\gamma}_f-\underline{\gamma}}\|_{L^q(P)}.$$

Fix any $\underline{\gamma} \in \mathbb{R}^n$ such that $\underline{\gamma} < \underline{\gamma}_f - 2/s + 1$. The second norm on the right hand side of the above inequality is then bounded independently of $|\alpha|$. This implies $\boldsymbol{u} \in \mathcal{K}^{\varpi,2}_{\underline{\gamma}}(P)$. Now, since $\underline{\gamma}_f - 2/s > 0$, we obtain the first part of the assertion. The regularity of the pressure follows by the same embedding argument. □

From Theorem 2.6, we can also derive an upper bound on the approximability of the set of solutions to (2.1). To this end, given a Hilbert space X, a compact subset S of X, and $N \in \mathbb{N}$, we recall the notion of *Kolmogorov N-width of S in X* which is defined as

$$(2.9) \qquad d_N(S, X) = \inf_{\substack{X_N \subset X \\ \dim(X_N)=N}} \sup_{x \in S} \inf_{x_N \in X_N} \|x - x_N\|_X.$$

In (2.9), $X_N \subset X$ denotes any linear subspace of X of finite dimension N.

The following Lemma bounds the N-widths of the weighted analytic function spaces.

LEMMA 2.8. *Let* $C_{\boldsymbol{u}}, A_{\boldsymbol{u}}, C_p, A_p > 0$ *and* $\underline{\gamma} > 1$. *Then, there exist* $C, b > 0$ *such that*

$$d_N(\mathcal{K}^{\varpi,2}_{\underline{\gamma}}(P; C_{\boldsymbol{u}}, A_{\boldsymbol{u}}), H^1(P)) \le C \exp\left(-bN^{1/3}\right)$$

and

$$d_N(\mathcal{K}^{\varpi,2}_{\underline{\gamma}-1}(P; C_p, A_p), L^2(P)) \le C \exp\left(-bN^{1/3}\right).$$

Proof. We recall the definition of the countably normed spaces $B^\ell_{\underline{\beta}}(P)$ as, e.g., in [21]. For $\ell \in \mathbb{N}$, $\underline{\beta} \in \mathbb{R}^n$, and $c, d > 0$, a function v belongs to $B^\ell_{\underline{\beta}}(P; c, d)$ if $v \in H^{\ell-1}(P)$ and, for all $|\alpha| \ge \ell$

$$\|r^{|\alpha|+\underline{\beta}-\ell}\partial^\alpha v\|_{L^2(P)} \le cd^{|\alpha|-\ell}(|\alpha| - \ell)!.$$

The spaces $B^\ell_{\underline{\beta}}(P)$ correspond to non-homogeneous versions of the spaces $\mathcal{K}^{\varpi}_{\underline{\gamma}}(P)$, see [7], for a thorough analysis of their relationship.

Let $\underline{\gamma} > 1$: we show that for all $\widetilde{A} > A$, there exists $\widetilde{C}$ such that $\mathcal{K}_{\underline{\gamma}}^{\varpi}(P; C, A) \subset B_{2-\underline{\gamma}}^2(P; \widetilde{C}, \widetilde{A})$. If $v \in \mathcal{K}_{\underline{\gamma}}^{\varpi}(P; C, A)$, one clearly has

$$(2.10) \qquad \|v\|_{H^1(P)} \leq \max(\|r^{\underline{\gamma}}\|_{L^\infty(P)}, \|r^{\underline{\gamma}-1}\|_{L^\infty(P)})\|v\|_{\mathcal{K}_{\underline{\gamma}}^1(P)},$$

hence $v \in H^1(P)$. Furthermore, by definition, there exist $C, A > 0$ such that

$$(2.11) \qquad \|r^{|\alpha|+(2-\underline{\gamma})-2}\partial^\alpha v\|_{L^2(P)} \leq CA^{|\alpha|}|\alpha|! = CA^2|\alpha|(|\alpha|-1)A^{|\alpha|-2}(|\alpha|-2)!.$$

For any $\widetilde{A} > A$ one can choose $\widetilde{C} \geq \max_{k \geq 2} A^k \widetilde{A}^{2-k} k(k-1)$ and from (2.10) and (2.11), $v \in B_{2-\underline{\gamma}}^2(P; \widetilde{C}, \widetilde{A})$. The inclusion $\mathcal{K}_{\underline{\gamma}-1}^{\varpi}(P; C, A) \subset B_{2-\underline{\gamma}}^1(P; \widetilde{C}, \widetilde{A})$ can be shown in the same fashion.

The application of [42, Theorem 3.1], resp. of [41], which contain the construction of an exponentially convergent, $H_0^1(P)^2 \times L_0^2(P)$-conforming approximation of functions in $B_{\underline{\beta}}^2(P; \widetilde{C}, \widetilde{A})^2 \times B_{\underline{\beta}}^1(P; \widetilde{C}, \widetilde{A})$ for arbitrary $\underline{\beta} \in (0, 1)$, concludes the proof.□

We denote by $\mathcal{S}$ the solution operator of (2.1), i.e., we write $(\boldsymbol{u}, p) = \mathcal{S}(\boldsymbol{f})$ if $(\boldsymbol{u}, p)$ are the solutions to (2.1) with right hand side $\boldsymbol{f}$.

As a direct consequence of Corollary 2.7 and Lemma 2.8, a second corollary to Theorem 2.6 gives bounds on the N-widths of the sets of solutions to (2.1).

COROLLARY 2.9. *Let $2 < s < \infty$, $\underline{\gamma}_f \in \mathbb{R}^n$ such that $\underline{\gamma}_f - 2/s > 0$, and $C_f, A_f > 0$. Let $(\mathcal{U}, \mathcal{P})$ be the sets of, respectively, all velocities and pressures, solutions of the Navier-Stokes equations (2.1) with right hand sides in $\mathcal{K}_{\underline{\gamma}_f-2}^{\varpi,s}(P; C_f, A_f) \cap \boldsymbol{H}^{-1}(P)$, i.e.,*

$$(\mathcal{U}, \mathcal{P}) = \mathcal{S}\left(\mathcal{K}_{\underline{\gamma}_f-2}^{\varpi,s}(P; C_f, A_f) \cap \boldsymbol{H}^{-1}(P)\right) \subset H_0^1(P)^2 \times L_0^2(P).$$

Then, $(\mathcal{U}, \mathcal{P})$ is a compact subset of $H_0^1(P)^2 \times L_0^2(P)$. Moreover, there exist $C, b > 0$ such that for all $N \in \mathbb{N}$

$$d_N(\mathcal{U} \times \mathcal{P}, \left(H_0^1(P)\right)^2 \times L_0^2(P)) \leq C \exp\left(-bN^{1/3}\right).$$

Remark 2.10. By embedding in weighted spaces and rescaling of the constants, Corollary 2.9 holds under the more usual condition that right hand sides of (2.1) belong to $\mathcal{K}_{\underline{\gamma}_f-2}^{\varpi,2}(P, C_f, A_f)$, with $\underline{\gamma}_f > 1$.

3. Conclusion. We established analytic regularity of stationary solutions of the incompressible Navier-Stokes equation in plane, polygonal domains, subject to the "no-slip", homogeneous Dirichlet boundary condition.

The analytic regularity result pertains to Leray-Hopf solutions of the viscous, incompressible Navier-Stokes equations (0.1) with sufficiently small data. We also remark that the variational bootstrapping argument used to establish (2.8) can also be used to establish analytic regularity for the solution of the Navier-Stokes equation in Sobolev spaces without weights, in compact subsets of P, and up to the boundary in domains with analytic boundary, thereby comprising a "real-valued" proof of the analytic regularity results in [32, 15] which were obtained in these references with function-theoretic arguments. The proof in the present paper proceeds, instead, by the classical arguments [34] which are based on localization, a-priori estimates and bootstrapping arguments combined with induction to control growth of constants in terms of derivative orders.

The present proof is developed for so-called "no-slip" boundary conditions (i.e. homogeneous Dirichlet boundary conditions) for the velocity field which entailed weighted norms with homogeneous corner weights. They can be extended verbatim to certain other combinations of boundary conditions at the corners. Corresponding weighted shift theorems in *finite order* weighted spaces were obtained in [36].

The presently obtained analytic regularity shifts for the (Navier-)Stokes equations will allow for *exponential convergence rates* of suitable high-order and model order reduction discretizations of these equations. See, e.g., [5, 42, 24, 3]. The detailed *a priori* analysis and exponential convergence of a mixed hp discontinuous Galerkin discretization of the Navier-Stokes equations in polygons (0.1) are developed in [38].

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, vol. 140 of Pure and Applied Mathematics, Elsevier/Academic Press, Amsterdam, second ed., 2003.

[2] I. Babuška and B. Q. Guo, *The h-p version of the finite element method for domains with curved boundaries*, SIAM J. Numer. Anal., 25 (1988), pp. 837–861.

[3] L. Beirão da Veiga, C. Lovadina, and G. Vacca, *Virtual elements for the Navier-Stokes problem on polygonal meshes*, SIAM J. Numer. Anal., 56 (2018), pp. 1210–1242.

[4] P. Bolley, M. Dauge, and J. Camus, *Régularité Gevrey pour le problème de Dirichlet dans des domaines à singularités coniques*, Comm. Partial Differential Equations, 10 (1985), pp. 391–431.

[5] B. Cockburn, G. Kanschat, and D. Schötzau, *A locally conservative LDG method for the incompressible Navier-Stokes equations*, Math. Comp., 74 (2005), pp. 1067–1095.

[6] M. Costabel, M. Dauge, and S. Nicaise, *Corner singularities and analytic regularity for linear elliptic systems*, 2010. Book in preparation.

[7] M. Costabel, M. Dauge, and S. Nicaise, *Mellin analysis of weighted Sobolev spaces with nonhomogeneous norms on cones*, in Around the research of Vladimir Maz'ya. I, vol. 11 of Int. Math. Ser. (N. Y.), Springer, New York, 2010, pp. 105–136.

[8] M. Costabel, M. Dauge, and S. Nicaise, *Analytic Regularity for Linear Elliptic Systems in Polygons and Polyhedra*, Math. Models Methods Appl. Sci., 22 (2012), pp. 1250015, 63.

[9] M. Costabel, M. Dauge, and S. Nicaise, *Weighted analytic regularity in polyhedra*, Comput. Math. Appl., 67 (2014), pp. 807–817.

[10] M. Costabel, M. Dauge, and C. Schwab, *Exponential convergence of hp-FEM for Maxwell equations with weighted regularization in polygonal domains*, Math. Models Methods Appl. Sci., 15 (2005), pp. 575–622.

[11] A. Dall'Acqua, S. Fournais, T. Østergaard Sørensen, and E. Stockmeyer, *Real analyticity away from the nucleus of pseudorelativistic Hartree–Fock orbitals*, Anal. PDE, 5 (2012), pp. 657–691.

[12] C. Ebmeyer and J. Frehse, *Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains*, Math. Ann., 319 (2001), pp. 349–381.

[13] F. Eckhardt, P. A. Cioica-Licht, and S. Dahlke, *Besov regularity for the stationary Navier-Stokes equation on bounded Lipschitz domains*, Appl. Anal., 97 (2018), pp. 466–485.

[14] L. Fick, Y. Maday, A. T. Patera, and T. Taddei, *A stabilized POD model for turbulent flows over a range of Reynolds numbers: optimal parameter sampling and constrained projection*, J. Comput. Phys., 371 (2018), pp. 214–243.

[15] Y. Giga, *Time and spatial analyticity of solutions of the Navier-Stokes equations*, Comm. Partial Differential Equations, 8 (1983), pp. 929–948.

[16] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986.

[17] P. Grisvard, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[18] B. Guo and I. Babuška, *Approximation theory for the p-version of the finite element method in three dimensions. I. Approximabilities of singular functions in the framework of the Jacobi-weighted Besov and Sobolev spaces*, SIAM J. Numer. Anal., 44 (2006), pp. 246–269.

[19] B. Guo and I. Babuška, *The h-p version of the finite element method - Part 1: The basic approximation results*, Comput. Mech., 1 (1986), pp. 21–41.

[20] B. Guo and I. Babuška, *The h-p version of the finite element method - Part 2: General results and applications*, Comput. Mech., 1 (1986), pp. 203–220.

[21] B. Guo and C. Schwab, *Analytic regularity of Stokes flow on polygonal domains in countably weighted Sobolev spaces*, J. Comput. Appl. Math., 190 (2006), pp. 487–519.

[22] J. S. HESTHAVEN, G. ROZZA, AND B. STAMM, *Certified reduced basis methods for parametrized partial differential equations*, SpringerBriefs in Mathematics, Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2016.

[23] K. KATO, *New idea for proof of analyticity of solutions to analytic nonlinear elliptic equations*, SUT J. Math., 32 (1996), pp. 157–161.

[24] V. KAZEEV AND C. SCHWAB, *Quantized tensor-structured finite elements for second-order elliptic PDEs in two dimensions*, Numer. Math., 138 (2018), pp. 133–190.

[25] V. A. KONDRAT'EV, *Boundary value problems for elliptic equations in domains with conical or angular points*, Trudy Moskov. Mat. Obšč., 16 (1967), pp. 209–292.

[26] V. A. KOZLOV, V. G. MAZ'YA, AND J. ROSSMANN, *Elliptic boundary value problems in domains with point singularities*, vol. 52 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, 1997.

[27] V. A. KOZLOV, V. G. MAZ'YA, AND J. ROSSMANN, *Spectral problems associated with corner singularities of solutions to elliptic equations*, vol. 85 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, 2001.

[28] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 2d english ed., 1969.

[29] T. LASSILA, A. MANZONI, A. QUARTERONI, AND G. ROZZA, *Model order reduction in fluid dynamics: challenges and perspectives*, in Reduced order methods for modeling and computational reduction, vol. 9 of MS&A. Model. Simul. Appl., Springer, Cham, 2014, pp. 235–273.

[30] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.

[31] Y. MADAY AND C. MARCATI, *Analyticity and hp discontinuous Galerkin approximation of nonlinear Schrödinger eigenproblems*, Tech. Report 2019-69, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2019.

[32] K. MASUDA, *On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation*, Proc. Japan Acad., 43 (1967), pp. 827–832.

[33] V. MAZ'YA AND J. ROSSMANN, *Elliptic Equations in Polyhedral Domains*, vol. 162 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, 2010.

[34] C. B. MORREY, JR. AND L. NIRENBERG, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Comm. Pure Appl. Math., 10 (1957), pp. 271–290.

[35] S. NICAISE, *Regularity of the solutions of elliptic systems in polyhedral domains*, Bull. Belg. Math. Soc. Simon Stevin, 4 (1997), pp. 411–429.

[36] M. ORLT AND A.-M. SÄNDIG, *Regularity of viscous Navier-Stokes flows in nonsmooth domains*, in Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993), vol. 167 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1995, pp. 185–201.

[37] A. QUARTERONI, A. MANZONI, AND F. NEGRI, *Reduced basis methods for partial differential equations*, vol. 92 of Unitext, Springer, Cham, 2016.

[38] D. SCHÖTZAU, C. MARCATI, AND C. SCHWAB, *Exponential convergence of mixed hp-DGFEM for the incompressible Navier-Stokes equations in $\mathbb{R}^2$*, Tech. Report 2020-15, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2020.

[39] D. SCHÖTZAU, C. SCHWAB, AND T. P. WIHLER, *hp-dGFEM for Second-Order Elliptic Problems in Polyhedra I: Stability on Geometric Meshes*, SIAM J. Numer. Anal., 51 (2013), pp. 1610–1633.

[40] D. SCHÖTZAU, C. SCHWAB, AND T. P. WIHLER, *hp-dGFEM for second order elliptic problems in polyhedra. II: Exponential convergence*, SIAM J. Numer. Anal., 51 (2013), pp. 2005–2035.

[41] D. SCHÖTZAU AND T. P. WIHLER, *Exponential convergence of mixed hp-DGFEM for Stokes flow in polygons*, Numer. Math., 96 (2003), pp. 339–361.

[42] C. SCHWAB AND M. SURI, *Mixed hp finite element methods for Stokes and non-Newtonian flow*, Comput. Methods Appl. Mech. Engrg., 175 (1999), pp. 217–241.

[43] R. TEMAM, *Navier-Stokes equations and nonlinear functional analysis*, vol. 66 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second ed., 1995.

[44] K. VEROY AND A. T. PATERA, *Certified real-time solution of the parametrized steady incompressible Navier-Stokes equations: rigorous reduced-basis a posteriori error bounds*, Internat. J. Numer. Methods Fluids, 47 (2005), pp. 773–788.