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Weak convergence rates for temporal numerical approximations of stochastic wave equations with multiplicative noise

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Abstract

In numerical analysis for stochastic partial differential equations one distinguishes between weak and strong convergence rates. Often the weak convergence rate is twice the strong convergence rate. However, there is no standard way to prove this: to obtain optimal weak convergence rates for stochastic partial differential equations requires specially tailored techniques, especially if the noise is multiplicative. In this work we establish weak convergence rates for temporal discretisations of stochastic wave equations with multiplicative noise, in particular, for the hyperbolic Anderson model. The weak convergence rates we obtain are indeed twice the known strong rates. To the best of our knowledge, our findings are the first in the scientific literature which provide essentially sharp weak convergence rates for temporal discretisations of stochastic wave equations with multiplicative noise. Key ideas of our proof are a sophisticated splitting of the error and applications of the recently introduced mild Itô formula.

Contents

1 Introduction

Stochastic partial differential equations (SPDEs) are used to model various evolutionary processes subject to random forces. For example, stochastic wave equations may model the motion of a strand of DNA in a liquid or heat flow around a ring; see, e.g., [Dal09, Tho12]. In general the solution to an SPDE cannot be given explicitly, whence it is desirable to prove convergence rates for numerical approximations. Here one distinguishes strong convergence rates, i.e., with respect to the strong (mean square) error, and weak convergence rates, i.e., with respect to the stochastic weak error. Typically, the convergence rate for the weak error is twice the convergence rate for the strong error. However, there does not exist a straightforward way to establish this. Moreover, non-trivial exceptions to this rule exist; see, e.g., [Alf05, HJ18].

For both parabolic and hyperbolic semilinear SPDEs strong convergence is by now well-understood. In particular, strong convergence rates for numerical approximations of stochastic wave equations have been established in, e.g., [ACLW16, CLS13, CQS16, KLL13, KLS10, QSSS06, Wal06, Wan15, WGT14].

Establishing optimal weak convergence rates for both hyperbolic and parabolic SPDEs is currently active field of research; see, e.g., [AHJK18, AJK, AJKW17, AKL16, AL16, ALa, ALb, Bré12, Bré14, Bré, BD, BG, BK17, CJK, CH18, CH, dBD06, Deb11, DP09, GKL09, HM, Hau03, Hau10, HJK, JdNJW, JK, Kop14, KLL13, KLL12, KLS15, KP14, Kru14, Lin12, LS13, Sha03, Wan15, Wan16, WG13]. Arguably, the most relevant basic SPDEs are the parabolic and hyperbolic Anderson model, i.e., the heat equation with multiplicative noise and the wave equation with multiplicative noise. However, establishing optimal weak convergence rates for SPDEs with multiplicative noise is challenging. Indeed, of the articles cited above only [BD, CJK, CH18, dBD06, Deb11, HJK, JdNJW, JK] provide weak rates for SPDEs with multiplicative noise. Roughly speaking, there are two successful approaches to obtain optimal weak convergence rates for parabolic SPDEs with multiplicative noise. One is based on regularity results for the corresponding Kolmogorov equation and Malliavin calculus; see, e.g., [BD, Deb11]. The other is based on more elementary regularity results of the Kolmogorov equation and the mild Itô formula; see, e.g., [CJK, HJK, JK].

No successful approach for proving optimal weak convergence rates has been developed yet for temporal discretisations of hyperbolic SPDEs with multiplicative noise. Indeed, the two approaches mentioned above are not applicable as they rely strongly on the smoothing effect of the semigroup. In this work we tackle this problem and develop a technique that allows one to establish optimal weak convergence rates for hyperbolic SPDEs with multiplicative noise. A special case of our main result is presented in the following theorem.

Theorem 1.1. *Let* $T, \vartheta \in (0, \infty)$, $b_0, b_1 \in \mathbb{R}$, $H = L^2((0,1);\mathbb{R})$, *let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ *be a filtered probability space which fulfills the usual conditions, let* $(W_t)_{t\in[0,T]}$ *be an* id_H*-cylindrical* $(\mathbb{F}_t)_{t\in[0,T]}$*-Wiener process, let* $A\colon D(A) \subseteq H \to H$ *be the Dirichlet Laplacian on* H*, let* $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$*, be a family of interpolation spaces associated to* $-A$*, let* $\mathbf{H}_0 = H_0 \times H_{-1/2}$, $\mathbf{H}_1 = H_{1/2} \times H_0$*, let* $\mathbf{A}\colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ *be the linear operator which satisfies that* $D(\mathbf{A}) = \mathbf{H}_1$ *and* $[\forall\, (v,w) \in D(\mathbf{A})\colon \mathbf{A}(v,w) = (w, \vartheta A v)]$*, let* $\xi \in \mathcal{L}^6(\mathbb{P}|_{\mathbb{F}_0}; \mathbf{H}_1)$, $\varphi \in C^4(\mathbf{H}_0, \mathbb{R})$ *satisfy that* $\sup_{k\in\{1,2,3,4\},\, x\in\mathbf{H}_0} \|\varphi^{(k)}(x)\|_{L^{(k)}(\mathbf{H}_0,\mathbb{R})} < \infty$*, let* $\mathbf{B}\colon \mathbf{H}_0 \to L_2(H, \mathbf{H}_0)$ *be the function which satisfies for every* $(v,w) \in \mathbf{H}_0$, $u \in H$ *that* $\mathbf{B}(v,w)u = \big(0, (b_0 + b_1 v)u\big)$*, let* $X\colon [0,T] \times \Omega \to \mathbf{H}_0$ *be an* $(\mathbb{F}_t)_{t\in[0,T]}$*-predictable stochastic process which satisfies for every* $t \in [0,T]$ *that* $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s\|^2_{\mathbf{H}_0}\big] < \infty$ *and*

$$X_t = e^{t\mathbf{A}}\xi + \int_0^t e^{(t-s)\mathbf{A}} \mathbf{B}(X_s)\, \mathrm{d}W_s, \tag{1}$$

and let $Y^N\colon \{0,1,2,\dots,N\} \times \Omega \to \mathbf{H}_0$, $N \in \mathbb{N}$*, be the stochastic processes which* *satisfy for all* $N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ *that* $Y_0^N = \xi$ *and*

$$Y_n^N = e^{(T/N)\mathbf{A}}\left(Y_{n-1}^N + \int_{(n-1)T/N}^{nT/N} \mathbf{B}(Y_{n-1}^N)\,\mathrm{d}W_s\right). \tag{2}$$

Then it holds for all $\varepsilon \in (0, \infty)$ *that* $\sup_{N\in\mathbb{N}} \big(N^{1-\varepsilon} \left|\mathbb{E}\big[\varphi\big(X_T\big)\big] - \mathbb{E}\big[\varphi\big(Y_N^N\big)\big]\right|\big) < \infty$.

Note that we obtain rate of convergence 1^-, which is indeed twice the known strong rate. Theorem 1.1 is an immediate consequence of Corollary 4.6 below. Corollary 4.6 follows from Theorem 3.10, which is the main result of this article. Indeed, Theorem 3.10 establishes an upper bound for the weak error of a temporal discretisation of a hyperbolic SPDE with multiplicative noise. Similar as in the parabolic case, a key ingredient of the proof of this upper bound is the mild Itô formula developed in [DPJR]. However, for parabolic SPDEs the mild Itô formula is used to insert the semigroup in an appropriate place so the smoothing property can be exploited. Here, however, the mild Itô formula is used to rewrite certain terms in the error as integrals over an interval of length $\mathcal{O}(N^{-1})$. The use of the mild Itô formula is crucial: if one would apply the 'classical' Itô formula, then one would obtain a term involving an unbounded operator. Although the underlying semigroup does not does not enjoy a smoothing property as in the parabolic case, by using the mild Itô formula one can avoid the appearance of an unbounded operator and thus the roughing effect accompanied by it. Another key ingredient of the proof is an elegant decomposition of the error into terms that can be treated using this mild Itô formula approach, and terms that can be dealt with in a relatively straightforward manner; see (84)–(86) in the proof of Theorem 3.10. It is to be expected that this method of proof can also be applied to other types of temporal discretisations, as well as to spatial discretisations such as the finite element method. Moreover, although we consider the Hilbert space setting in this work, our approach can be extended in a straightforward way to the Banach space setting; see [CJKP]. This would allow one to prove optimal weak rates for more general semilinear drift and diffusion coefficients; see [HJK] for analogous results for parabolic SPDEs. For completeness we note that optimal weak convergence rates for spatial spectral Galerkin approximations of stochastic wave equations have been established in [JdNJW18, JdNJW]. The approach taken in [JdNJW18, JdNJW] essentially relies on the specific structure of the spatial spectral Galerkin approximations and can thus neither be extended to temporal approximations nor to other more complicated spatial approximations such as the finite element method.

The remainder of this article is structured as follows. Section 2.1 recalls a well-known existence and uniqueness result for semilinear SDEs in Hilbert spaces and

Section 2.2 provides regularity results for the associated Kolomogorov equations. Further preparatory lemmas are collected in Section 2.3. Section 3.1 presents the general setting for our convergence results. Section 3.2 collects some properties of the wave semigroup. Theorem 3.10 in Section 3.5 establishes upper bounds for the weak error of a temporal discretisation. This combined with the uniform moment bounds obtained in Section 3.3 and the strong convergence of the Galerkin approximations proven in Section 3.4 establishes the weak convergence rates of the temporal discretisations, see Corollary 3.11 below. In Section 4.2 we collect some results on multipliers on Sobolev-Slobodeckij spaces, which we use in Section 4.3 to verify that Corollary 3.11 implies Corollary 4.6. Recall that Corollary 4.6 implies Theorem 1.1.

1.1 Setting

Throughout this article we shall frequently use the following setting.

Setting 1.2. *For every pair of* $\mathbb{R}$*-Hilbert spaces* $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ *and* $(W, \langle \cdot, \cdot \rangle_W, \|\cdot\|_W)$ *let* $(L_2(V, W), \langle \cdot, \cdot \rangle_{L_2(V,W)}, \|\cdot\|_{L_2(V,W)})$ *be the* $\mathbb{R}$*-Hilbert space of Hilbert-Schmidt operators from* V *to* W*, for every* $k \in \mathbb{N}$ *and every pair of* $\mathbb{R}$*-Banach spaces* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *let* $(\mathrm{Lip}(V, W), \|\cdot\|_{\mathrm{Lip}(V,W)})$ *be the* $\mathbb{R}$*-Banach space of Lipschitz continuous mappings from* V *to* W *and let* $(C^k_{\mathrm{b}}(V, W), \|\cdot\|_{C^k_{\mathrm{b}}(V,W)})$ *be the* $\mathbb{R}$*-Banach space of* k*-times continuously Fréchet differentiable functions from* V *to* W *with globally bounded derivatives, for every measure space* $(\Omega, \mathcal{F}, \mu)$*, every measurable space* (S, Σ)*, and every function* $f \colon \Omega \to S$ *let* $[f]_{\mu,\Sigma}$ *be the set given by*

$$[f]_{\mu,\Sigma} = \left\{ g \colon \Omega \to S \colon \begin{bmatrix} [\exists\, A \in \mathcal{F} \colon (\mu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A)] \\ \text{and } [\forall\, A \in \Sigma \colon g^{-1}(A) \in \mathcal{F}] \end{bmatrix} \right\}, \tag{3}$$

let $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ *be a separable* $\mathbb{R}$*-Hilbert space, let* $\mathbb{U} \subseteq U$ *be an orthonormal basis of* U*, let* $T \in (0, \infty)$*, let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ *be a filtered probability space which fulfills the usual conditions, and let* $(W_t)_{t \in [0,T]}$ *be an* id_U*-cylindrical* $(\mathbb{F}_t)_{t \in [0,T]}$*-Wiener process.*

2 Preliminaries

2.1 Stochastic differential equations Hilbert spaces

The existence and uniqueness result in Theorem 2.1 below is essentially well known in the literature (cf., for example, Da Prato & Zabczyk [DPZ92, Theorem 7.4]).

Theorem 2.1. *Assume Setting 1.2, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be a separable* $\mathbb{R}$*-Hilbert space, let* $S\colon [0,\infty) \to L(H)$ *be a strongly continuous semigroup, and let* $p \in [2,\infty)$*,* $F \in \operatorname{Lip}(H,H)$*,* $B \in \operatorname{Lip}(H, L_2(U,H))$*,* $\xi \in \mathcal{L}^p(\mathbb{P}|_{\mathbb{F}_0}; H)$*. Then there exists an up to modifications unique* $(\mathbb{F}_t)_{t\in[0,T]}$*-predictable stochastic process* $X\colon [0,T]\times\Omega \to H$ *which satisfies for every* $t \in [0,T]$ *that* $\sup_{s\in[0,T]} \mathbb{E}[\|X_s\|_H^p] < \infty$ *and*

$$[X_t]_{\mathbb{P},\mathcal{B}(H)} = [S_t\xi]_{\mathbb{P},\mathcal{B}(H)} + \int_0^t S_{t-s}F(X_s)\,\mathrm{d}s + \int_0^t S_{t-s}B(X_s)\,\mathrm{d}W_s. \tag{4}$$

2.2 Kolmogorov equations in Hilbert spaces

Lemma 2.2. *Assume Setting 1.2, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be a non-trivial separable* $\mathbb{R}$*-Hilbert space, for every* $A \in L(H)$*,* $F \in C^1_{\mathrm{b}}(H,H)$*,* $B \in C^1_{\mathrm{b}}(H, L_2(U,H))$*,* $x \in H$ *let* $X^{A,F,B,x}\colon [0,T]\times\Omega \to H$ *be an* $(\mathbb{F}_t)_{t\in[0,T]}$*-predictable stochastic process which satisfies for every* $t\in[0,T]$ *that* $\sup_{s\in[0,T]}\mathbb{E}\big[\|X_s^{A,F,B,x}\|_H^2\big] < \infty$ *and*

$$[X_t^{A,F,B,x}]_{\mathbb{P},\mathcal{B}(H)} = [e^{tA}x]_{\mathbb{P},\mathcal{B}(H)} + \int_0^t e^{(t-s)A}F(X_s^{A,F,B,x})\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s^{A,F,B,x})\,\mathrm{d}W_s, \tag{5}$$

and for every $A \in L(H)$*,* $F \in C^1_{\mathrm{b}}(H,H)$*,* $B \in C^1_{\mathrm{b}}(H, L_2(U,H))$*,* $\varphi \in C^1_{\mathrm{b}}(H,\mathbb{R})$ *let* $v^{A,F,B,\varphi}\colon [0,T]\times H \to \mathbb{R}$ *be the function which satisfies for every* $t\in[0,T]$*,* $x\in H$ *that* $v^{A,F,B,\varphi}(t,x) = \mathbb{E}[\varphi(X^{A,F,B,x}_{T-t})]$*. Then*

(i) it holds for every $A \in L(H)$*,* $F \in C^2_{\mathrm{b}}(H,H)$*,* $B \in C^2_{\mathrm{b}}(H, L_2(U,H))$*,* $\varphi \in C^2_{\mathrm{b}}(H,\mathbb{R})$*,* $t\in[0,T]$*,* $x\in H$ *that* $v^{A,F,B,\varphi} \in C^{1,2}([0,T]\times H,\mathbb{R})$ *and*

$$\begin{aligned} &\big(\tfrac{\partial}{\partial t}v^{A,F,B,\varphi}\big)(t,x) + \big(\tfrac{\partial}{\partial x}v^{A,F,B,\varphi}\big)(t,x)(Ax+F(x)) \\ &+ \tfrac{1}{2}\sum_{u\in\mathbb{U}}\big(\tfrac{\partial^2}{\partial x^2}v^{A,F,B,\varphi}\big)(t,x)(B(x)u, B(x)u) = 0, \end{aligned} \tag{6}$$

(ii) it holds for every $k\in\mathbb{N}$*,* $A\in L(H)$*,* $F\in C^k_{\mathrm{b}}(H,H)$*,* $B\in C^k_{\mathrm{b}}(H,L_2(U,H))$*,* $\varphi\in C^k_{\mathrm{b}}(H,\mathbb{R})$*,* $t\in[0,T]$ *that* $(H\ni x\mapsto v^{A,F,B,\varphi}(t,x)\in\mathbb{R}) \in C^k_{\mathrm{b}}(H,\mathbb{R})$*, and*

(iii) it holds for every $k\in\mathbb{N}$*,* $c\in(0,\infty)$ *that*

$$\sup\left\{\frac{\|(\frac{\partial^k}{\partial x^k}v^{A,F,B,\varphi})(t,x)\|_{L^{(k)}(H,\mathbb{R})}}{\|\varphi\|_{C^k_{\mathrm{b}}(H,\mathbb{R})}} : \begin{array}{c} t\in[0,T],\, x\in H,\, \varphi\in C^k_{\mathrm{b}}(H,\mathbb{R})\backslash\{0\},\, A\in L(H), \\ F\in C^k_{\mathrm{b}}(H,H),\, B\in C^k_{\mathrm{b}}(H,L_2(U,H)) \text{ with} \\ \sup_{t\in[0,T]}\|e^{tA}\|_{L(H)} + \|F\|_{C^k_{\mathrm{b}}(H,H)} + \|B\|_{C^k_{\mathrm{b}}(H,L_2(U,H))} \leq c \end{array}\right\} < \infty. \tag{7}$$

Proof of Lemma 2.2. Throughout this proof for every set S let $|S| \in \{0,1,2,\ldots\} \cup \{\infty\}$ be the cardinality of S, for every set S let $\mathcal{P}(S)$ be the power set of S, and for every $j \in \mathbb{N}$ let Π_j be the set given by

$$\Pi_j = \big\{\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})\colon \\ [\emptyset \notin \mathcal{S}] \wedge [\forall S_1, S_2 \in \mathcal{S}\colon S_1 \neq S_2 \Rightarrow S_1 \cap S_2 = \emptyset] \wedge [\cup_{S\in\mathcal{S}} S = \{1,2,\ldots,j\}]\big\} \quad (8)$$

(the set of all partitions of $\{1,\ldots,j\}$). Observe that, e.g., Da Prato & Zabczyk [DPZ92, Theorem 9.16] (cf., for example, also Harms & Müller [HM, item (ii) in Lemma 2.2]) establishes item (i). Moreover, note that Andersson et al. [AHJK18, item (ii) in Lemma 3.2] implies item (ii). Next observe that Andersson et al. [AJKW17, item (ix) in Theorem 2.1] demonstrates that for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_{\mathrm{b}}(H,H)$, $B \in C^k_{\mathrm{b}}(H, L_2(U,H))$, $t \in [0,T]$, $p \in [1,\infty)$ it holds that

$$\big(H \ni x \mapsto [X_t^{A,F,B,x}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P};H)\big) \in C^k_{\mathrm{b}}(H, L^p(\mathbb{P};H)). \quad (9)$$

In addition, note that Andersson et al. [AJKW17, items (i)–(ii) and items (ix)–(x) in Theorem 2.1] (with $\alpha = \beta = \delta_1 = \ldots = \delta_k = 0$ in the notation of [AJKW17, item (ii) in Theorem 2.1]) ensures that for every $k \in \mathbb{N}$, $c \in (0,\infty)$, $p \in [1,\infty)$ it holds that

$$\sup\left\{ \frac{\|\frac{\partial^k}{\partial x^k}[X_t^{A,F,B,x}]_{\mathbb{P},\mathcal{B}(H)}\|_{L^{(k)}(H,L^p(\mathbb{P};H))}}{t^{1/2\,\mathbb{1}_{[2,\infty)}(k)}} \colon \begin{array}{c} t\in(0,T],\, x\in H,\, A\in L(H), \\ F\in C^k_{\mathrm{b}}(H,H),\, B\in C^k_{\mathrm{b}}(H,L_2(U,H)) \text{ with} \\ \sup_{t\in[0,T]} \|e^{tA}\|_{L(H)} + \|F\|_{C^k_{\mathrm{b}}(H,H)} + \|B\|_{C^k_{\mathrm{b}}(H,L_2(U,H))} \leq c \end{array} \right\} \\ < \infty. \quad (10)$$

Moreover, observe that Andersson et al. [AHJK18, item (v) in Lemma 3.2] (with $\alpha = \beta = \delta_1 = \ldots = \delta_k = 0$ in the notation of [AHJK18, item (v) in Lemma 3.2]) proves that for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_{\mathrm{b}}(H,H)$, $B \in C^k_{\mathrm{b}}(H, L_2(U,H))$, $\varphi \in C^k_{\mathrm{b}}(H,\mathbb{R})$ it holds that

$$\sup_{t\in[0,T),\, x\in H} \big\|\big(\tfrac{\partial^k}{\partial x^k} v^{A,F,B,\varphi}\big)(t,x)\big\|_{L^{(k)}(H,\mathbb{R})} \leq |\max\{1,T\}|^{1/2\lfloor k/2\rfloor}\, \|\varphi\|_{C^k_{\mathrm{b}}(H,\mathbb{R})} \\ \cdot \left[\sum_{\pi\in\Pi_k} \left(\prod_{I\in\pi} \left[\sup_{t\in(0,T],\, x\in H} \frac{\|\frac{\partial^{|I|}}{\partial x^{|I|}}[X_t^{A,F,B,x}]_{\mathbb{P},\mathcal{B}(H)}\|_{L^{(|I|)}(H,L^{|\pi|}(\mathbb{P};H))}}{t^{1/2\mathbb{1}_{[2,\infty)}(|I|)}}\right]\right)\right]. \quad (11)$$

Next note that for every $k \in \mathbb{N}$, $\varphi \in C^k_{\mathrm{b}}(H,\mathbb{R})$ it holds that

$$\|\varphi\|_{C^k_{\mathrm{b}}(H,\mathbb{R})} = |\varphi(0)| + \left[\sum_{j=1}^{k} \left(\sup_{x\in H} \|\varphi^{(j)}(x)\|_{L^{(j)}(H,\mathbb{R})}\right)\right]. \quad (12)$$

Therefore, we obtain that for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_{\mathrm{b}}(H,H)$, $B \in C^k_{\mathrm{b}}(H, L_2(U,H))$, $\varphi \in C^k_{\mathrm{b}}(H,\mathbb{R})$ it holds that

$$\sup_{x\in H} \big\| \big(\tfrac{\partial^k}{\partial x^k} v^{A,F,B,\varphi} \big)(T,x) \big\|_{L^{(k)}(H,\mathbb{R})} = \sup_{x\in H} \|\varphi^{(k)}(x)\|_{L^{(k)}(H,\mathbb{R})} \leq \|\varphi\|_{C^k_{\mathrm{b}}(H,\mathbb{R})}. \tag{13}$$

Combining this, (10), and (11) establishes item (iii). The proof of Lemma 2.2 is thus completed. □

2.3 Preparatory lemmas

The next result, Lemma 2.3 below, is frequently used throughout this article.

Lemma 2.3. *Assume Setting 1.2, let* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *be* $\mathbb{R}$*-Banach spaces, and let* $F \in \mathrm{Lip}(V,W)$*. Then*

(i) it holds for every $v \in V$ *that* $\|F(v)\|_W \leq \|F\|_{\mathrm{Lip}(V,W)} \max\{1, \|v\|_V\}$ *and*

(ii) it holds for every $p \in [1,\infty)$, $\xi \in \mathcal{L}^p(\mathbb{P}; V)$ *that*

$$\|F(\xi)\|_{\mathcal{L}^p(\mathbb{P};W)} \leq \|F\|_{\mathrm{Lip}(V,W)} \, (1 + \|\xi\|_{\mathcal{L}^p(\mathbb{P};V)}). \tag{14}$$

Proof of Lemma (2.3)*.* Observe that the fact that

$$\|F\|_{\mathrm{Lip}(V,W)} = \|F(0)\|_W + \sup\Big(\Big\{ \tfrac{\|F(x)-F(y)\|_W}{\|x-y\|_V} : x,y \in V, x \neq y \Big\} \cup \{0\} \Big) \tag{15}$$

implies that for every $v \in V$ it holds that

$$\begin{aligned} \|F(v)\|_W &\leq \|F(v) - F(0)\|_W + \|F(0)\|_W \\ &\leq \Big[\tfrac{\|F(v)-F(0)\|_W}{\max\{1,\|v\|_V\}} \Big] \max\{1, \|v\|_V\} + \|F(0)\|_W \max\{1, \|v\|_V\} \\ &= \Big[\|F(0)\|_W + \tfrac{\|F(v)-F(0)\|_W}{\max\{1,\|v\|_V\}} \Big] \max\{1, \|v\|_V\} \\ &\leq \|F\|_{\mathrm{Lip}(V,W)} \max\{1, \|v\|_V\}. \end{aligned} \tag{16}$$

This establishes item (i). Moreover, note that item (i) implies that for every $p \in [1,\infty)$, $\xi \in \mathcal{L}^p(\mathbb{P};V)$ it holds that $\|F(\xi)\|_{\mathcal{L}^p(\mathbb{P};W)} \leq \big\| \, \|F\|_{\mathrm{Lip}(V,W)} \, (1+\|\xi\|_V) \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \leq \|F\|_{\mathrm{Lip}(V,W)}(1 + \|\xi\|_{\mathcal{L}^p(\mathbb{P};V)})$. This proves item (ii). The proof of Lemma 2.3 is thus completed. □

Lemma 2.4. *It holds*

(i) that $\sup_{\alpha\in[0,2],t\in(0,\infty)}(t^{-\alpha}|1-\cos(t)|) = 2$ *and*

(ii) that $\inf_{\alpha\in\mathbb{R}\setminus[0,2]}\sup_{t\in(0,\infty)}(t^{-\alpha}|1-\cos(t)|) = \infty$.

Proof of Lemma 2.4. First, note that for every $\alpha \in [0,2], t \in (1,\infty)$ it holds that

$$t^{-\alpha}|1-\cos(t)| \le 2t^{-\alpha} \le 2. \tag{17}$$

Next observe that the fundamental theorem of calculus assures that for every $t_1 \in (0,\infty)$, $t_2 \in (t_1,\infty)$ it holds that

$$\begin{aligned}
&(t_2)^{-2}|1-\cos(t_2)| = (t_2)^{-2}(1-\cos(t_2)) \\
&= (t_1)^{-2}(1-\cos(t_1)) + \int_{t_1}^{t_2}\left(\frac{\sin(s)s^2-(1-\cos(s))2s}{s^4}\right)ds \\
&= (t_1)^{-2}(1-\cos(t_1)) + \int_{t_1}^{t_2}\left(\frac{\sin(s)s-(1-\cos(s))2}{s^3}\right)ds.
\end{aligned} \tag{18}$$

In addition, note that the fundamental theorem of calculus implies that for every $s \in (0,\pi)$ it holds that

$$\begin{aligned}
&\sin(s)s - (1-\cos(s))2 \\
&= \int_0^s (\cos(u)u + \sin(u) - 2\sin(u))\,du = \int_0^s (\cos(u)u - \sin(u))\,du \\
&= \int_0^s\int_0^u (-\sin(r)r + \cos(r) - \cos(r))\,dr\,du = \int_0^s\int_0^u (-\sin(r)r)\,dr\,du \le 0.
\end{aligned} \tag{19}$$

This and (18) demonstrate that the function $[(0,\pi) \ni t \mapsto t^{-2}|1-\cos(t)| \in (0,\infty)]$ is monotonically decreasing, i.e., that for every $t_1 \in (0,\pi)$, $t_2 \in (t_1,\pi)$ it holds that

$$(t_1)^{-2}|1-\cos(t_1)| \ge (t_2)^{-2}|1-\cos(t_2)|. \tag{20}$$

Moreover, note that the fundamental theorem of calculus proves that for every $t \in (0,\infty)$ it holds that

$$\begin{aligned}
&t^{-2}|1-\cos(t)| = t^{-2}(\cos(0)-\cos(t)) \\
&= -t^{-2}\left[\int_0^t (-\sin(s))\,ds\right] = t^{-2}\left[\int_0^t \sin(s)\,ds\right] = t^{-2}\left[\int_0^t\int_0^s \cos(u)\,du\,ds\right] \\
&= t^{-2}\left[\int_0^t\int_0^s 1\,du\,ds\right] + t^{-2}\left[\int_0^t\int_0^s (\cos(u)-\cos(0))\,du\,ds\right] \\
&= \tfrac{1}{2} + t^{-2}\left[\int_0^t\int_0^s\int_0^u (-\sin(r))\,dr\,du\,ds\right].
\end{aligned} \tag{21}$$

Hence, we obtain that for every $t \in (0, \infty)$ it holds that

$$\begin{aligned} \left| t^{-2} |1 - \cos(t)| - \tfrac{1}{2} \right| &= t^{-2} \left| \int_0^t \int_0^s \int_0^u \sin(r) \, dr \, du \, ds \right| \\ &\leq t^{-2} \left[\int_0^t \int_0^s \int_0^u 1 \, dr \, du \, ds \right] = t^{-2} \left[\tfrac{t^3}{3!} \right] = \tfrac{t}{6}. \end{aligned} \tag{22}$$

Therefore, we obtain that

$$\limsup_{t \searrow 0} \left| t^{-2} |1 - \cos(t)| - \tfrac{1}{2} \right| = 0. \tag{23}$$

Combining this and (20) ensures that for every $\alpha \in [0, 2], t \in (0, 1]$ it holds that

$$t^{-\alpha} |1 - \cos(t)| \leq t^{-2} |1 - \cos(t)| \leq \tfrac{1}{2}. \tag{24}$$

In addition, note that $\sup_{\alpha \in [0,2], t \in (0,\infty)} (t^{-\alpha} |1 - \cos(t)|) \geq \pi^{-0} |1 - \cos(\pi)| = |1 + 1| = 2$. Combining this, (17), and (24) establishes item (i). Furthermore, observe that for every $\alpha \in (-\infty, 0)$ it holds that

$$\limsup_{t \to \infty} (t^{-\alpha} |1 - \cos(t)|) = \infty. \tag{25}$$

In addition, note that (23) shows that for every $\alpha \in (2, \infty)$ it holds that

$$\begin{aligned} &\limsup_{t \searrow 0} \left(t^{-\alpha} |1 - \cos(t)| \right) = \limsup_{t \searrow 0} \left(t^{-(\alpha - 2)} \left[t^{-2} |1 - \cos(t)| \right] \right) \\ &\geq \left[\limsup_{t \searrow 0} t^{-(\alpha-2)} \right] \left[\liminf_{t \searrow 0} \left(t^{-2} |1 - \cos(t)| \right) \right] = \tfrac{1}{2} \left[\limsup_{t \searrow 0} t^{-(\alpha-2)} \right] = \infty. \end{aligned} \tag{26}$$

Combining this and (25) establishes item (ii). The proof of Lemma 2.4 is thus completed. □

Lemma 2.5. *Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, for every $p \in [1, \infty)$ let $(L_p(U, H), \|\cdot\|_{L_p(U,H)})$ be the $\mathbb{R}$-Banach space of Schatten-p operators from U to H, and let $r \in (0, \infty)$, $T \in L^{(2)}(H, \mathbb{R})$, $A \in L_{1+r}(U, H)$, $B \in L_{1+1/r}(U, H)$. Then it holds that*

$$\sum_{u \in \mathbb{U}} |T(Au, Bu)| \leq \|T\|_{L^{(2)}(H, \mathbb{R})} \, \|A\|_{L_{1+r}(U,H)} \, \|B\|_{L_{1+1/r}(U,H)} \, . \tag{27}$$

Proof of Lemma 2.5. Throughout this proof for every $p \in [1, \infty)$ let $(L_p(H,U), \|\cdot\|_{L_p(H,U)})$ be the $\mathbb{R}$-Banach space of Schatten-p operators from H to U and for every $p \in [1, \infty)$ let $(L_p(U), \|\cdot\|_{L_p(U)})$ be the $\mathbb{R}$-Banach space of Schatten-p operators from U to U. Observe that the Riesz representation theorem ensures that there exists a unique $T_0 \in L(H)$ such that for every $x, y \in H$ it holds that

$$\langle T_0 x, y\rangle_H = T(x,y) \qquad \text{and} \qquad \|T_0\|_{L(H)} = \|T\|_{L^{(2)}(H,\mathbb{R})}. \tag{28}$$

Therefore, we obtain that

$$\sum_{u\in\mathbb{U}} |T(Au, Bu)| = \sum_{u\in\mathbb{U}} |\langle T_0 Au, Bu\rangle_H| = \sum_{u\in\mathbb{U}} |\langle B^* T_0 Au, u\rangle_U|. \tag{29}$$

Next note that, e.g., Meise & Vogt [MV97, item 6. in Lemma 16.6 and item 2. in Lemma 16.7] ensures that $\|B^*\|_{L_{1+1/r}(H,U)} = \|B\|_{L_{1+1/r}(U,H)}$ and $\|T_0 A\|_{L_{1+r}(U,H)} \leq \|T_0\|_{L(H)} \|A\|_{L_{1+r}(U,H)}$. Combining the Hölder inequality for Schatten norms (see, e.g., Dunford & Schwartz [DS63, item (c) in Lemma XI.9.14]) and (28) hence establishes that $B^* T_0 A \in L_1(U)$ and

$$\|B^* T_0 A\|_{L_1(U)} \leq \|B\|_{L_{1+1/r}(U,H)} \|T\|_{L^{(2)}(H,\mathbb{R})} \|A\|_{L_{1+r}(U,H)}. \tag{30}$$

Furthermore, note that for all sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subseteq \mathbb{U}$ with $\sum_{k\in\mathbb{N}} \|x_k\|_U \|y_k\|_U < \infty$ and $B^* T_0 A = \sum_{k\in\mathbb{N}} \langle \cdot\,, x_k\rangle_U y_k$ it holds that

$$\begin{aligned} \sum_{u\in\mathbb{U}} |\langle B^* T_0 Au, u\rangle_U| &= \sum_{u\in\mathbb{U}} \left| \left\langle \sum_{k\in\mathbb{N}} \langle u, x_k\rangle_U y_k, u \right\rangle_U \right| = \sum_{u\in\mathbb{U}} \left| \sum_{k\in\mathbb{N}} \langle u, x_k\rangle_U \langle y_k, u\rangle_U \right| \\ &\leq \sum_{k\in\mathbb{N}} \sum_{u\in\mathbb{U}} \left| \langle u, x_k\rangle_U \langle y_k, u\rangle_U \right| \leq \sum_{k\in\mathbb{N}} \|x_k\|_U \|y_k\|_U. \end{aligned} \tag{31}$$

The fact that

$$\|B^* T_0 A\|_{L_1(U)} = \inf\left\{ \sum_{k\in\mathbb{N}} \|x_k\|_U \|y_k\|_U : {\scriptstyle (x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subseteq \mathbb{U} \text{ with } \atop \scriptstyle \sum_{k\in\mathbb{N}} \|x_k\|_U \|y_k\|_U < \infty \text{ and } B^* T_0 A = \sum_{k\in\mathbb{N}} \langle \cdot, x_k\rangle_U y_k} \right\} \tag{32}$$

therefore implies that $\sum_{u\in\mathbb{U}} |\langle B^* T_0 Au, u\rangle_U| \leq \|B^* T_0 A\|_{L_1(U)}$. Combining this, (29), and (30) establishes (27). The proof of Lemma 2.5 is thus completed. $\square$

3 Weak convergence rates for temporal numerical approximations of semilinear stochastic wave equations

3.1 Setting

Throughout this section we shall frequently use the following setting.

Setting 3.1. *Assume Setting 1.2, let* $\gamma \in (0,\infty)$, $\beta \in (\gamma/2, \gamma] \cap [\gamma - 1/2, \gamma]$, $\rho \in [0, 2(\gamma-\beta)]$, *let* $(H, \langle \cdot,\cdot \rangle_H, \|\cdot\|_H)$ *be a non-trivial separable* $\mathbb{R}$*-Hilbert space, let* $\mathbb{H} \subseteq H$ *be an orthonormal basis of* H, *let* $\lambda \colon \mathbb{H} \to \mathbb{R}$ *be a function which satisfies that* $\sup_{h\in\mathbb{H}} \lambda_h < 0$ *and* $\sum_{h\in\mathbb{H}} |\lambda_h|^{-\beta} < \infty$, *let* $A \colon D(A) \subseteq H \to H$ *be the linear operator which satisfies that* $D(A) = \big\{ v \in H \colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty \big\}$ *and* $[\forall\, v \in D(A)\colon\ Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h]$, *let* $(H_r, \langle \cdot,\cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$, *let* $(\mathbf{H}_r, \langle \cdot,\cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, *be the family of* $\mathbb{R}$*-Hilbert spaces which satisfies for every* $r \in \mathbb{R}$ *that* $(\mathbf{H}_r, \langle \cdot,\cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = \big(H_{r/2} \times H_{r/2-1/2}, \langle \cdot,\cdot \rangle_{H_{r/2}\times H_{r/2-1/2}}, \|\cdot\|_{H_{r/2}\times H_{r/2-1/2}} \big)$, *let* $\mathbf{A} \colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ *be the linear operator which satisfies that* $D(\mathbf{A}) = \mathbf{H}_1$ *and* $[\forall\, (v,w) \in \mathbf{H}_1 \colon \mathbf{A}(v,w) = (w, Av)]$, *let* $\mathbf{\Lambda} \colon D(\mathbf{\Lambda}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ *be the linear operator which satisfies that* $D(\mathbf{\Lambda}) = \mathbf{H}_1$ *and* $[\forall\, (v,w) \in \mathbf{H}_1 \colon \mathbf{\Lambda}(v,w) = (\sum_{h\in\mathbb{H}} \langle h, v\rangle_{H_0} |\lambda_h|^{1/2} h, \sum_{h\in\mathbb{H}} \langle h, w\rangle_{H_0} |\lambda_h|^{1/2} h)]$, *let* $\varphi \in C^4_{\mathrm{b}}(\mathbf{H}_0, \mathbb{R})$, $\xi \in \mathcal{L}^2(\mathbb{P}|_{\mathbb{F}_0}; \mathbf{H}_{\max\{\rho,\gamma-\beta\}})$ *satisfy that* $\mathbb{E}[\|\xi\|^6_{\mathbf{H}_0}] < \infty$, *let* $F \in \mathrm{Lip}(\mathbf{H}_{\beta-\gamma}, \mathbf{H}_0)$, $B \in C^4_{\mathrm{b}}(\mathbf{H}_0, L_2(U, \mathbf{H}_0))$ *satisfy that* $F|_{\mathbf{H}_0} \in C^4_{\mathrm{b}}(\mathbf{H}_0, \mathbf{H}_0)$, $F|_{\mathbf{H}_\rho} \in \mathrm{Lip}(\mathbf{H}_\rho, \mathbf{H}_{2(\gamma-\beta)})$, $B|_{\mathbf{H}_\rho} \in \mathrm{Lip}(\mathbf{H}_\rho, L(U, \mathbf{H}_\gamma) \cap L_2(U, \mathbf{H}_\rho))$, *let* $\mathfrak{m} \in [1,\infty)$, $\mathfrak{c}, \mathfrak{l} \in [0,\infty)$, $\mu \colon \mathbb{U} \to (\mathbb{R}\backslash\{0\})$ *satisfy for every* $v, w \in \mathbf{H}_{\gamma-\beta}$ *that*

$$\max\big\{ \|F|_{\mathbf{H}_0}\|_{C^4_{\mathrm{b}}(\mathbf{H}_0,\mathbf{H}_0)}, \|B\|_{C^4_{\mathrm{b}}(\mathbf{H}_0, L_2(U,\mathbf{H}_0))} \big\} \leq \mathfrak{m}, \tag{33}$$

$$\sum_{u\in\mathbb{U}} |\mu_u|^2 \, \|B(v)u\|^2_{\mathbf{H}_0} \leq \mathfrak{c}^2 \max\big\{1, \|v\|^2_{\mathbf{H}_{\gamma-\beta}}\big\}, \tag{34}$$

and

$$\sum_{u\in\mathbb{U}} \tfrac{\|(B(v)-B(w))u\|^2_{\mathbf{H}_0}}{|\mu_u|^2} \leq \mathfrak{l}^2 \, \|v-w\|^2_{\mathbf{H}_{\beta-\gamma}}, \tag{35}$$

for every $I \subseteq \mathbb{H}$ *let* $P_I \colon \cup_{r\in\mathbb{R}} H_r \to \cup_{r\in\mathbb{R}} H_r$ *and* $\mathbf{P}_I \colon \cup_{r\in\mathbb{R}} \mathbf{H}_r \to \cup_{r\in\mathbb{R}} \mathbf{H}_r$ *be the functions which satisfy for every* $r \in \mathbb{R}$, $v \in H_r$, $w \in H_{r-1/2}$ *that* $P_I(v) = \sum_{h\in I} \langle |\lambda_h|^{-r} h, v\rangle_{H_r} |\lambda_h|^{-r} h$ *and* $\mathbf{P}_I(v,w) = (P_I v, P_I w)$, *let* $\lfloor \cdot \rfloor_h \colon [0,\infty) \to \mathbb{R}$, $h \in [0,\infty)$, *be the functions which satisfy for every* $h \in (0,\infty)$, $x \in [0,\infty)$ *that* $\lfloor x \rfloor_h = \max(\{0, h, 2h, 3h, \ldots\} \cap [0,x])$ *and* $\lfloor x \rfloor_0 = x$, *and for every* $I \subseteq \mathbb{H}$, $h \in [0,T]$ *let*

$Y^{h,I}\colon [0,T]\times\Omega\to\mathbf{P}_I(\mathbf{H}_0)$ *be an* $(\mathbb{F}_t)_{t\in[0,T]}$*-predictable stochastic process which satisfies for every* $t\in[0,T]$ *that* $\sup_{s\in[0,T]}\mathbb{E}\big[\|Y^{h,I}_{\lfloor s\rfloor_h}\|^2_{\mathbf{H}_0}\big]<\infty$ *and*

$$\begin{aligned}
&[Y^{h,I}_t]_{\mathbb{P},\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))}\\
&=[e^{t\mathbf{A}}\mathbf{P}_I\xi]_{\mathbb{P},\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))}+\int_0^t e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_I F(Y^{h,I}_{\lfloor s\rfloor_h})\,\mathrm{d}s+\int_0^t e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_I B(Y^{h,I}_{\lfloor s\rfloor_h})\,\mathrm{d}W_s.
\end{aligned} \tag{36}$$

Note that Setting 3.1 ensures that for every $I\subseteq\mathbb{H}$, $t\in[0,T]$ it holds that $\mathbb{P}\big(\int_0^t(\|e^{(t-s)\mathbf{A}}\mathbf{P}_I F(Y^{0,I}_s)\|_{\mathbf{H}_0}+\|e^{(t-s)\mathbf{A}}\mathbf{P}_I B(Y^{0,I}_s)\|^2_{L_2(U,\mathbf{H}_0)})\,ds<\infty\big)=1$ and

$$\begin{aligned}
&[Y^{0,I}_t]_{\mathbb{P},\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))}\\
&=[e^{t\mathbf{A}}\mathbf{P}_I\xi]_{\mathbb{P},\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))}+\int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_I F(Y^{0,I}_s)\,\mathrm{d}s+\int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_I B(Y^{0,I}_s)\,\mathrm{d}W_s
\end{aligned} \tag{37}$$

(cf., for example, Theorem 2.1 above, Lemma 3.3 below, and Jacobe de Naurois et al. [JdNJW, Remark 3.1] for sufficient conditions which ensure the existence of such a process).

3.2 Basic results for the linear wave equation

The statement and the proof of the next result, Lemma 3.2 below, can be found in, e.g., Jacobe de Naurois et al. [JdNJW, Lemma 2.4].

Lemma 3.2. *Assume Setting 3.1. Then the* $\mathbb{R}$*-Hilbert spaces* $(\mathbf{H}_r,\langle\cdot,\cdot\rangle_{\mathbf{H}_r},\|\cdot\|_{\mathbf{H}_r})$, $r\in\mathbb{R}$, *are a family of interpolation spaces associated to* $\mathbf{\Lambda}$.

The next result, Lemma 3.3 below, can be found in, e.g., Jacobe de Naurois et al. [JdNJW, Lemma 2.5]. A proof of Lemma 3.3 can be found in, e.g., Lindgren [Lin12, Section 5.3].

Lemma 3.3. *Assume Setting 3.1 and let* $\mathbf{S}\colon[0,\infty)\to L(\mathbf{H}_0)$ *be the function which satisfies for every* $t\in[0,\infty)$, $(v,w)\in\mathbf{H}_0$ *that*

$$\mathbf{S}_t(v,w)=\begin{pmatrix}\cos(t(-A)^{1/2})v+(-A)^{-1/2}\sin(t(-A)^{1/2})w,\\ -(-A)^{1/2}\sin(t(-A)^{1/2})v+\cos(t(-A)^{1/2})w\end{pmatrix}. \tag{38}$$

Then

(i) it holds that $\mathbf{S}\colon[0,\infty)\to L(\mathbf{H}_0)$ *is a strongly continuous semigroup of bounded linear operators on* $\mathbf{H}_0$ *and*

(ii) it holds that $\mathbf{A}\colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ *is the generator of* $\mathbf{S}$.

The statement and the proof of the next result, Lemma 3.4 below, can be found in, e.g., Jacobe de Naurois et al. [JdNJW, Lemma 2.6].

Lemma 3.4. *Assume Setting 3.1. Then* $\sup_{t\in[0,\infty)} \|e^{t\mathbf{A}}\|_{L(\mathbf{H}_0)} = 1$.

The next result, Lemma 3.5 below, provides another useful estimate for the semigroup $(e^{t\mathbf{A}})_{t\in[0,\infty)}$ generated by the operator $\mathbf{A}\colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ from Setting 3.1 (cf., e.g., Kovacs et al. [KLL13, Lemma 4.4] for a similar result).

Lemma 3.5. *Assume Setting 3.1 and let* $\alpha \in [0,1], t \in (0,\infty)$. *Then*

$$t^{-\alpha}\|\mathbf{\Lambda}^{-\alpha}(\mathrm{id}_{\mathbf{H}_0} - e^{t\mathbf{A}})\|_{L(\mathbf{H}_0)} \leq \sqrt{2}\left[\sup_{s\in(0,\infty)} (s^{-\alpha}|1-e^{\mathbf{i}s}|)\right] \leq 2^{3/2}. \tag{39}$$

Proof of Lemma 3.5. First, observe that for every $s \in (0,\infty)$ it holds that

$$\begin{aligned} s^{-\alpha}|1-e^{\mathbf{i}s}| &= s^{-\alpha}\big[|1-e^{\mathbf{i}s}|^2\big]^{1/2} = s^{-\alpha}\big[|\mathrm{Re}(1-e^{\mathbf{i}s})|^2 + |\mathrm{Im}(1-e^{\mathbf{i}s})|^2\big]^{1/2} \\ &= s^{-\alpha}\big[|1-\cos(s)|^2 + |\sin(s)|^2\big]^{1/2} \\ &= s^{-\alpha}\big[1 - 2\cos(s) + |\cos(s)|^2 + |\sin(s)|^2\big]^{1/2} \\ &= s^{-\alpha}\big[2-2\cos(s)\big]^{1/2} = \sqrt{2}\big(s^{-\alpha}|1-\cos(s)|^{1/2}\big). \end{aligned} \tag{40}$$

In addition, note that Lemma 3.3 implies that for every $(v,w) \in \mathbf{H}_0$ it holds that

$$\begin{aligned} &\mathbf{\Lambda}^{-\alpha}(\mathrm{id}_{\mathbf{H}_0} - e^{t\mathbf{A}})(v,w) \\ &= \mathbf{\Lambda}^{-\alpha}\begin{pmatrix} (\mathrm{id}_H - \cos(t(-A)^{1/2}))v - (-A)^{-1/2}\sin(t(-A)^{1/2})w, \\ (-A)^{1/2}\sin(t(-A)^{1/2})v + (\mathrm{id}_{H_{-1/2}} - \cos(t(-A)^{1/2}))w \end{pmatrix} \\ &= \begin{pmatrix} (-A)^{-\alpha/2}(\mathrm{id}_H - \cos(t(-A)^{1/2}))v - (-A)^{-(1+\alpha)/2}\sin(t(-A)^{1/2})w, \\ (-A)^{(1-\alpha)/2}\sin(t(-A)^{1/2})v + (-A)^{-\alpha/2}(\mathrm{id}_{H_{-1/2}} - \cos(t(-A)^{1/2}))w \end{pmatrix}. \end{aligned} \tag{41}$$

Hence, we obtain that for every $(v,w) \in \mathbf{H}_0$ it holds that

$$\begin{aligned} &t^{-\alpha}\|\mathbf{\Lambda}^{-\alpha}(\mathrm{id}_{\mathbf{H}_0} - e^{t\mathbf{A}})(v,w)\|_{\mathbf{H}_0} \\ &= t^{-\alpha}\bigg[\Big\|(-A)^{-\alpha/2}\big(\mathrm{id}_H - \cos(t(-A)^{1/2})\big)v - (-A)^{-(1+\alpha)/2}\sin(t(-A)^{1/2})w\Big\|_H^2 \\ &\quad + \Big\|(-A)^{(1-\alpha)/2}\sin(t(-A)^{1/2})v + (-A)^{-\alpha/2}\big(\mathrm{id}_{H_{-1/2}} - \cos(t(-A)^{1/2})\big)w\Big\|_{H_{-1/2}}^2\bigg]^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \Bigg[\sum_{h\in\mathbb{H}} t^{-2\alpha}|\lambda_h|^{-\alpha} \left| \left(1-\cos(t|\lambda_h|^{1/2})\right) \langle h, v\rangle_H + \sin(t|\lambda_h|^{1/2}) \langle |\lambda_h|^{1/2}\, h, w\rangle_{H_{-1/2}} \right|^2 \\
&\quad + \sum_{h\in\mathbb{H}} t^{-2\alpha}\, |\lambda_h|^{-\alpha} \left| \sin(t|\lambda_h|^{1/2})\, \langle h, v\rangle_H + \left(1-\cos(t|\lambda_h|^{1/2})\right) \langle |\lambda_h|^{1/2}\, h, w\rangle_{H_{-1/2}} \right|^2 \Bigg]^{1/2} \\
&\leq \sqrt{2} \Bigg[\sum_{h\in\mathbb{H}} t^{-2\alpha}|\lambda_h|^{-\alpha} \left(\left|1-\cos(t|\lambda_h|^{1/2})\right|^2 + \left|\sin(t|\lambda_h|^{1/2})\right|^2 \right) |\langle h, v\rangle_H|^2 \\
&\quad + \sum_{h\in\mathbb{H}} t^{-2\alpha}|\lambda_h|^{-\alpha} \left(\left|1-\cos(t|\lambda_h|^{1/2})\right|^2 + \left|\sin(t|\lambda_h|^{1/2})\right|^2 \right) \left|\langle |\lambda_h|^{1/2}\, h, w\rangle_{H_{-1/2}}\right|^2 \Bigg]^{1/2} .
\end{aligned}
\tag{42}
$$

This shows that for every $(v, w) \in \mathbf{H}_0$ it holds that

$$
\begin{aligned}
& t^{-\alpha} \|\mathbf{\Lambda}^{-\alpha}(\mathrm{id}_{\mathbf{H}_0} - e^{t\mathbf{A}})(v, w)\|_{\mathbf{H}_0} \\
&\leq \sqrt{2} \left[\sup_{h\in\mathbb{H}} \left\{ t^{-2\alpha}|\lambda_h|^{-\alpha} \left(\left|1-\cos(t|\lambda_h|^{1/2})\right|^2 + \left|\sin(t|\lambda_h|^{1/2})\right|^2 \right) \right\} \right]^{1/2} \\
&\quad \cdot \left[\left(\sum_{h\in\mathbb{H}} |\langle h, v\rangle_H|^2 \right) + \left(\sum_{h\in\mathbb{H}} \left|\langle |\lambda_h|^{1/2} h, v\rangle_{H_{-1/2}}\right|^2 \right) \right]^{1/2} \\
&= \sqrt{2} \left[\sup_{h\in\mathbb{H}} \left\{ t^{-2\alpha}|\lambda_h|^{-\alpha} \left(\left|1-\cos(t|\lambda_h|^{1/2})\right|^2 + \left|\sin(t|\lambda_h|^{1/2})\right|^2 \right) \right\} \right]^{1/2} \|(v, w)\|_{\mathbf{H_0}} \\
&= \sqrt{2} \left[\sup_{h\in\mathbb{H}} \left\{ (t|\lambda_h|^{1/2})^{-\alpha} \left(\left|1-\cos(t|\lambda_h|^{1/2})\right|^2 + \left|\sin(t|\lambda_h|^{1/2})\right|^2 \right)^{1/2} \right\} \right] \|(v, w)\|_{\mathbf{H_0}}.
\end{aligned}
\tag{43}
$$

Combining this and (40) demonstrates that for every $(v, w) \in \mathbf{H}_0$ it holds that

$$
\begin{aligned}
& t^{-\alpha} \|\mathbf{\Lambda}^{-\alpha}(\mathrm{id}_{\mathbf{H}_0} - e^{t\mathbf{A}})(v, w)\|_{\mathbf{H}_0} \\
&\leq \sqrt{2} \left[\sup_{s\in(0,\infty)} \left\{ s^{-\alpha} \left(|1-\cos(s)|^2 + |\sin(s)|^2 \right)^{1/2} \right\} \right] \|(v, w)\|_{\mathbf{H}_0} \\
&= 2 \left[\sup_{s\in(0,\infty)} \left(s^{-\alpha} |1-\cos(s)|^{1/2} \right) \right] \|(v, w)\|_{\mathbf{H}_0} = \sqrt{2} \left[\sup_{s\in(0,\infty)} \left(s^{-\alpha} |1 - e^{\mathbf{i}s}| \right) \right] \|(v, w)\|_{\mathbf{H}_0} .
\end{aligned}
\tag{44}
$$

Next note that Lemma 2.4 ensures that

$$2\left[\sup_{s\in(0,\infty)}\left(s^{-\alpha}|1-\cos(s)|^{1/2}\right)\right]=2\left[\sup_{s\in(0,\infty)}\left(s^{-2\alpha}|1-\cos(s)|\right)\right]^{1/2}\leq 2\big[2\big]^{1/2}=2^{3/2}. \tag{45}$$

This and (44) establish (39). The proof of Lemma 3.5 is thus completed. □

Lemma 3.6. *Assume Setting 3.1 and let $I\subseteq\mathbb{H}$ be finite. Then it holds for every $t\in[0,\infty)$ that $\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0}\in L(\mathbf{H}_0)$ and*

$$e^{t(\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0})}=e^{t\mathbf{A}}\mathbf{P}_I|_{\mathbf{H}_0}+\mathbf{P}_{\mathbb{H}\setminus I}|_{\mathbf{H}_0}. \tag{46}$$

Proof of Lemma 3.6. First, note that the finiteness of $I\subseteq\mathbb{H}$ ensures that for every $x\in\mathbf{H}_0$ it holds that $\mathbf{P}_Ix\in\mathbf{H}_1=D(\mathbf{A})$ and $\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0}\in L(\mathbf{H}_0)$. This and Lemma 3.3 imply that for every $s,t\in[0,\infty)$, $x\in\mathbf{H}_0$ it holds that

$$\begin{aligned}&(e^{s\mathbf{A}}\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})(e^{t\mathbf{A}}\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})x\\&=e^{(s+t)\mathbf{A}}\mathbf{P}_Ix+e^{s\mathbf{A}}\mathbf{P}_I\mathbf{P}_{\mathbb{H}\setminus I}x+e^{t\mathbf{A}}\mathbf{P}_I\mathbf{P}_{\mathbb{H}\setminus I}x+(\mathbf{P}_{\mathbb{H}\setminus I})^2x\\&=(e^{(s+t)\mathbf{A}}\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})x\end{aligned} \tag{47}$$

and

$$\begin{aligned}&\limsup_{h\searrow 0}\left\|\tfrac{1}{h}\big[(e^{h\mathbf{A}}\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})x-x\big]-\mathbf{A}\mathbf{P}_Ix\right\|_{\mathbf{H}_0}\\&=\limsup_{h\searrow 0}\left\|\tfrac{1}{h}\big[(e^{h\mathbf{A}}\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})x-(\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})x\big]-\mathbf{A}\mathbf{P}_Ix\right\|_{\mathbf{H}_0}\\&=\limsup_{h\searrow 0}\left\|\tfrac{1}{h}\big[e^{h\mathbf{A}}\mathbf{P}_Ix-\mathbf{P}_Ix\big]-\mathbf{A}\mathbf{P}_Ix\right\|_{\mathbf{H}_0}=0.\end{aligned} \tag{48}$$

Moreover, observe that for every $x\in\mathbf{H}_0$ it holds that $(e^{0\mathbf{A}}\mathbf{P}_I+\mathbf{P}_{\mathbb{H}\setminus I})x=x$. Combining this, (47), and (48) demonstrates that $\big(e^{t\mathbf{A}}\mathbf{P}_I|_{\mathbf{H}_0}+\mathbf{P}_{\mathbb{H}\setminus I}|_{\mathbf{H}_0}\big)_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathbf{H}_0$ with generator $\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0}\in L(\mathbf{H}_0)$. Hence, we obtain that $(e^{t(\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0})})_{t\geq 0}=\big(e^{t\mathbf{A}}\mathbf{P}_I|_{\mathbf{H}_0}+\mathbf{P}_{\mathbb{H}\setminus I}|_{\mathbf{H}_0}\big)_{t\geq 0}$. The proof of Lemma 3.6 is thus completed. □

3.3 A priori bounds for the numerical approximations

Lemma 3.7. *Assume Setting 3.1. Then*

(i) it holds for every $h,t\in[0,T]$, $I\subseteq\mathbb{H}$ that $\mathbb{P}(Y_t^{h,I}\in\mathbf{H}_{\max\{\rho,\gamma-\beta\}})=1$,

(ii) it holds that $\sup_{h,t\in[0,T],\, I\subseteq\mathbb{H}} \mathbb{E}\big[\|Y_t^{h,I}\|_{\mathbf{H}_0}^6\big] < \infty$, *and*

(iii) it holds that $\sup_{h,t\in[0,T],\, I\subseteq\mathbb{H}} \mathbb{E}\big[\|Y_t^{h,I}\|_{\mathbf{H}_{\max\{\rho,\gamma-\beta\}}}^2\big] < \infty$.

Proof of Lemma 3.7. First, note that

$$\begin{aligned} &\|F|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho,\mathbf{H}_{\max\{\rho,\gamma-\beta\}})} \\ &\leq \|\mathbf{\Lambda}^{\max\{\rho,\gamma-\beta\}-2(\gamma-\beta)}\|_{L(\mathbf{H}_0)}\|F|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho,\mathbf{H}_{2(\gamma-\beta)})} < \infty. \end{aligned} \tag{49}$$

Theorem 2.1 hence implies that there exist up to modifications unique $(\mathbb{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^I\colon [0,T]\times\Omega\to\mathbf{H}_\rho$, $I\subseteq\mathbb{H}$, which satisfy for every $I\subseteq\mathbb{H}$, $t\in[0,T]$ that $\sup_{s\in[0,T]}\mathbb{E}\big[\|X_s^I\|_{\mathbf{H}_0}^6+\|X_s^I\|_{\mathbf{H}_\rho}^2\big]<\infty$ and

$$\begin{aligned} &[X_t^I]_{\mathbb{P},\mathcal{B}(\mathbf{H}_\rho)} \\ &= [e^{t\mathbf{A}}\mathbf{P}_I\xi]_{\mathbb{P},\mathcal{B}(\mathbf{H}_\rho)} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_I F(X_s^I)\,\mathrm{d}s + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_I B(X_s^I)\,\mathrm{d}W_s. \end{aligned} \tag{50}$$

Combining this, (36), and Theorem 2.1 ensures that for every $I\subseteq\mathbb{H}$, $t\in[0,T]$ it holds that $\mathbb{P}(X_t^I = Y_t^{0,I}) = 1$. The fact that for every $I\subseteq\mathbb{H}$ it holds that $\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t^I\|_{\mathbf{H}_0}^6+\|X_t^I\|_{\mathbf{H}_\rho}^2\big]<\infty$ hence implies that for every $I\subseteq\mathbb{H}$ it holds that

$$\sup_{t\in[0,T]}\mathbb{E}\big[\|Y_t^{0,I}\|_{\mathbf{H}_0}^6+\|Y_t^{0,I}\|_{\mathbf{H}_\rho}^2\big]<\infty. \tag{51}$$

Next note that Da Prato & Zabczyk [DPZ92, Lemma 7.7], item (ii) in Lemma 2.3, Lemma 3.4, and (36) prove that for every $\delta,\theta\in[0,\infty)$, $p\in[2,\infty)$, $h,t\in[0,T]$, $I\subseteq\mathbb{H}$ it holds that

$$\begin{aligned} &\big(\mathbb{E}\big[\|Y_t^{h,I}\|_{\mathbf{H}_\theta}^p\big]\big)^{1/p} \\ &\leq \big(\mathbb{E}\big[\|e^{t\mathbf{A}}\mathbf{P}_I\xi\|_{\mathbf{H}_\theta}^p\big]\big)^{1/p} + \int_0^t \Big(\mathbb{E}\Big[\Big\|e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_I F\big(Y_{\lfloor s\rfloor_h}^{h,I}\big)\Big\|_{\mathbf{H}_\theta}^p\Big]\Big)^{1/p} ds \\ &\quad + \sqrt{\tfrac{p(p-1)}{2}}\left(\int_0^t \Big(\mathbb{E}\Big[\Big\|e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_I B\big(Y_{\lfloor s\rfloor_h}^{h,I}\big)\Big\|_{L_2(U,\mathbf{H}_\theta)}^p\Big]\Big)^{2/p} ds\right)^{1/2} \\ &\leq \big(\mathbb{E}\big[\|\xi\|_{\mathbf{H}_\theta}^p\big]\big)^{1/p} + \|F|_{\mathbf{H}_\delta}\|_{\mathrm{Lip}(\mathbf{H}_\delta,\mathbf{H}_\theta)}\int_0^t \big(1+\big(\mathbb{E}\big[\|Y_{\lfloor s\rfloor_h}^{h,I}\|_{\mathbf{H}_\delta}^p\big]\big)^{1/p}\big)\,ds \\ &\quad + \sqrt{\tfrac{p(p-1)}{2}}\|B|_{\mathbf{H}_\delta}\|_{\mathrm{Lip}(\mathbf{H}_\delta,L_2(U,\mathbf{H}_\theta))}\left(\int_0^t \big(1+\big(\mathbb{E}\big[\|Y_{\lfloor s\rfloor_h}^{h,I}\|_{\mathbf{H}_\delta}^p\big]\big)^{1/p}\big)^2 ds\right)^{1/2} \\ &\leq \big(\mathbb{E}\big[\|\xi\|_{\mathbf{H}_\theta}^p\big]\big)^{1/p} + \left(\sqrt{t} + \left(\int_0^t \big(\mathbb{E}\big[\|Y_{\lfloor s\rfloor_h}^{h,I}\|_{\mathbf{H}_\delta}^p\big]\big)^{2/p} ds\right)^{1/2}\right) \end{aligned} \tag{52}$$

$$\cdot \left(\sqrt{t} \| F|_{\mathbf{H}_\delta} \|_{\mathrm{Lip}(\mathbf{H}_\delta, \mathbf{H}_\theta)} + \sqrt{\tfrac{p(p-1)}{2}} \| B|_{\mathbf{H}_\delta} \|_{\mathrm{Lip}(\mathbf{H}_\delta, L_2(U, \mathbf{H}_\theta))} \right).$$

Moreover, note that

$$\begin{aligned} & \| B|_{\mathbf{H}_\rho} \|_{\mathrm{Lip}(\mathbf{H}_\rho, L_2(U, \mathbf{H}_{\max\{\rho, \gamma-\beta\}}))} \\ & \leq \max\big\{ \| B|_{\mathbf{H}_\rho} \|_{\mathrm{Lip}(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))}, \, \| \mathbf{\Lambda}^{-\beta} \|_{L_2(\mathbf{H}_0)} \| B|_{\mathbf{H}_\rho} \|_{\mathrm{Lip}(\mathbf{H}_\rho, L(U, \mathbf{H}_\gamma))} \big\} < \infty. \end{aligned} \tag{53}$$

This, (49), (51), and (52) (with $p = 2$, $\delta = \rho$, $\theta = \max\{\rho, \gamma - \beta\}$ in the notation of (52)) ensure that for every $I \subseteq \mathbb{H}$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\big[\| Y_t^{0,I} \|^2_{\mathbf{H}_{\max\{\rho, \gamma-\beta\}}} \big] < \infty. \tag{54}$$

In addition, note that (52) (with $p = 6$, $\delta = 0$, $\theta = 0$ in the notation of (52)) implies that for every $h \in (0, T]$, $I \subseteq \mathbb{H}$, $k \in \mathbb{N}_0 \cap [0, T/h)$ it holds that

$$\begin{aligned} & \sup_{t \in (kh, (k+1)h] \cap [0,T]} \big(\mathbb{E}\big[\| Y_t^{h,I} \|^6_{\mathbf{H}_0} \big] \big)^{1/6} \\ & \leq \big(\mathbb{E}\big[\| \xi \|^6_{\mathbf{H}_0} \big] \big)^{1/6} + \sqrt{(k+1)h} \Big(1 + \sup_{j \in \{0,1,\dots,k\}} \big(\mathbb{E}\big[\| Y_{jh}^{h,I} \|^6_{\mathbf{H}_0} \big] \big)^{1/6} \Big) \\ & \quad \cdot \Big(\sqrt{(k+1)h} \| F|_{\mathbf{H}_0} \|_{\mathrm{Lip}(\mathbf{H}_0, \mathbf{H}_0)} + \sqrt{15} \| B|_{\mathbf{H}_0} \|_{\mathrm{Lip}(\mathbf{H}_0, L_2(U, \mathbf{H}_0))} \Big). \end{aligned} \tag{55}$$

Hence, we obtain that for every $h \in (0, T]$, $I \subseteq \mathbb{H}$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\big[\| Y_t^{h,I} \|^6_{\mathbf{H}_0} \big] < \infty. \tag{56}$$

Moreover, observe that (52) (with $p = 2$, $\delta = \rho$, $\theta = \rho$ in the notation of (52)) implies that for every $h \in (0, T]$, $I \subseteq \mathbb{H}$, $k \in \mathbb{N}_0 \cap [0, T/h)$ it holds that

$$\begin{aligned} & \sup_{t \in (kh, (k+1)h] \cap [0,T]} \big(\mathbb{E}\big[\| Y_t^{h,I} \|^2_{\mathbf{H}_\rho} \big] \big)^{1/2} \\ & \leq \big(\mathbb{E}\big[\| \xi \|^2_{\mathbf{H}_\rho} \big] \big)^{1/2} + \sqrt{(k+1)h} \Big(1 + \sup_{j \in \{0,1,\dots,k\}} \big(\mathbb{E}\big[\| Y_{jh}^{h,I} \|^2_{\mathbf{H}_\rho} \big] \big)^{1/2} \Big) \\ & \quad \cdot \Big(\sqrt{(k+1)h} \| F|_{\mathbf{H}_\rho} \|_{\mathrm{Lip}(\mathbf{H}_\rho, \mathbf{H}_\rho)} + \| B|_{\mathbf{H}_\rho} \|_{\mathrm{Lip}(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))} \Big). \end{aligned} \tag{57}$$

Hence, we obtain that for every $h \in (0, T]$, $I \subseteq \mathbb{H}$ it holds that $\sup_{t \in [0,T]} \mathbb{E}[\| Y_t^{h,I} \|^2_{\mathbf{H}_\rho}] < \infty$. Combining this, (49), (52) (with $p = 2$, $\delta = \rho$, $\theta = \max\{\rho, \gamma - \beta\}$ in the notation of (52)), and (53) implies that for every $h \in (0, T]$, $I \subseteq \mathbb{H}$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\big[\| Y_t^{h,I} \|^2_{\mathbf{H}_{\max\{\rho, \gamma-\beta\}}} \big] < \infty. \tag{58}$$

This and (54) establish item (i). Next note that (52) ensures that for every $p \in [2, \infty)$, $\delta \in [0, \infty)$, $I \subseteq \mathbb{H}$, $h, t \in [0, T]$ it holds that

$$\begin{aligned} &\sup_{s \in [0,t]} \big(\mathbb{E}\big[\|Y_s^{h,I}\|_{\mathbf{H}_\delta}^p\big]\big)^{2/p} \\ &\leq 2\Big(\big(\mathbb{E}\big[\|\xi\|_{\mathbf{H}_\delta}^p\big]\big)^{1/p} + T\|F|_{\mathbf{H}_\delta}\|_{\operatorname{Lip}(\mathbf{H}_\delta,\mathbf{H}_\delta)} + \sqrt{\tfrac{p(p-1)T}{2}}\|B|_{\mathbf{H}_\delta}\|_{\operatorname{Lip}(\mathbf{H}_\delta,L_2(U,\mathbf{H}_\delta))}\Big)^2 \\ &\quad + 2\Big(\sqrt{T}\|F|_{\mathbf{H}_\delta}\|_{\operatorname{Lip}(\mathbf{H}_\delta,\mathbf{H}_\delta)} + \sqrt{\tfrac{p(p-1)}{2}}\|B|_{\mathbf{H}_\delta}\|_{\operatorname{Lip}(\mathbf{H}_\delta,L_2(U,\mathbf{H}_\delta))}\Big)^2 \\ &\quad \cdot \int_0^t \sup_{u \in [0,s]} \big(\mathbb{E}\big[\|Y_u^{h,I}\|_{\mathbf{H}_\delta}^p\big]\big)^{2/p}\, ds. \end{aligned} \tag{59}$$

Gronwall's inequality, (51), and (56) hence imply that

$$\begin{aligned} &\sup_{h,t \in [0,T],\, I \subseteq \mathbb{H}} \big(\mathbb{E}\big[\|Y_t^{h,I}\|_{\mathbf{H}_0}^6\big]\big)^{1/3} \\ &\leq 2\Big(\big(\mathbb{E}\big[\|\xi\|_{\mathbf{H}_0}^6\big]\big)^{1/6} + T\|F|_{\mathbf{H}_0}\|_{\operatorname{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{15T}\|B\|_{\operatorname{Lip}(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\Big)^2 \\ &\quad \cdot \exp\Big(2T\Big(\sqrt{T}\|F|_{\mathbf{H}_0}\|_{\operatorname{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{15}\|B\|_{\operatorname{Lip}(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\Big)^2\Big) < \infty. \end{aligned} \tag{60}$$

This establishes item (ii). In the next step we observe that Gronwall's inequality, (49), (51), (53), (58), and (59) imply that

$$\begin{aligned} &\sup_{h,t \in [0,T],\, I \subseteq \mathbb{H}} \mathbb{E}\big[\|Y_t^{h,I}\|_{\mathbf{H}_\rho}^2\big] \\ &\leq 2\Big(\big(\mathbb{E}\big[\|\xi\|_{\mathbf{H}_\rho}^2\big]\big)^{1/2} + T\|F|_{\mathbf{H}_\rho}\|_{\operatorname{Lip}(\mathbf{H}_\rho,\mathbf{H}_\rho)} + \sqrt{T}\|B|_{\mathbf{H}_\rho}\|_{\operatorname{Lip}(\mathbf{H}_\rho,L_2(U,\mathbf{H}_\rho))}\Big)^2 \\ &\quad \cdot \exp\Big(2T\Big(\sqrt{T}\|F|_{\mathbf{H}_\rho}\|_{\operatorname{Lip}(\mathbf{H}_\rho,\mathbf{H}_\rho)} + \|B|_{\mathbf{H}_\rho}\|_{\operatorname{Lip}(\mathbf{H}_\rho,L_2(U,\mathbf{H}_\rho))}\Big)^2\Big) < \infty. \end{aligned} \tag{61}$$

Combining this, (49), (52) (with $p = 2$, $\delta = \rho$, $\theta = \max\{\rho, \gamma - \beta\}$ in the notation of (52)), and (53) establishes item (iii). The proof of Lemma 3.7 is thus completed. □

3.4 Upper bounds for the strong approximation errors

The statement and the proof of the next result, Proposition 3.8 below, is a minor modification of the statement and the proof of Jacobe de Naurois et al. [JdNJW, Lemma 3.3].

Proposition 3.8. *Assume Setting 3.1 and let $h \in [0,T]$, $I, J \subseteq \mathbb{H}$. Then it holds that*

$$\begin{aligned} &\sup_{t\in[0,T]} \|Y_t^{h,J} - Y_t^{h,I}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} \\ &\leq \sqrt{2}\exp\Big(\big(T\|\mathbf{P}_{I\cap J}F|_{\mathbf{H}_0}\|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{T}\,\|\mathbf{P}_{I\cap J}B\|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)}\big)^2\Big) \\ &\quad\cdot\left[\sup_{t\in[0,T]} \|\mathbf{P}_{I\setminus J}Y_t^{h,I} - \mathbf{P}_{J\setminus I}Y_t^{h,J}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}\right]. \end{aligned} \tag{62}$$

Proof of Proposition 3.8. First, note that item (ii) in Lemma 3.7 implies that

$$\sup_{s\in[0,T]} \|Y_s^{h,I} - Y_s^{h,J}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} \leq \sup_{s\in[0,T]} \|Y_s^{h,I}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} + \sup_{s\in[0,T]} \big\|Y_s^{h,J}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} < \infty. \tag{63}$$

Moreover, observe that Lemma 3.4 ensures that for every $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned} &\big\|e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}\big(F(Y_{\lfloor s\rfloor_h}^{h,I}) - F(Y_{\lfloor s\rfloor_h}^{h,J})\big)\big\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} \\ &\leq \|\mathbf{P}_{I\cap J}F|_{\mathbf{H}_0}\|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)}\left[\sup_{u\in[0,s]} \|Y_u^{h,I} - Y_u^{h,J}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}\right] \end{aligned} \tag{64}$$

and

$$\begin{aligned} &\big\|e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}\big(B(Y_{\lfloor s\rfloor_h}^{h,I}) - B(Y_{\lfloor s\rfloor_h}^{h,J})\big)\big\|_{\mathcal{L}^2(\mathbb{P};L_2(U,\mathbf{H}_0))} \\ &\leq \|\mathbf{P}_{I\cap J}B\|_{\mathrm{Lip}(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\left[\sup_{u\in[0,s]} \|Y_u^{h,I} - Y_u^{h,J}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}\right]. \end{aligned} \tag{65}$$

Combining (33), (36), (63), and Jentzen & Kurniawan [JK, Corollary 3.1] (with $H = \mathbf{H}_0$, $p = 2$, $\vartheta = 0$, $\mathbf{y} = \|\mathbf{P}_{I\cap J}F|_{\mathbf{H}_0}\|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)}$, $\mathbf{z} = \|\mathbf{P}_{I\cap J}B\|_{\mathrm{Lip}(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}$, $X_t = Y_t^{h,I}$, $\bar{X}_t = Y_t^{h,J}$, $Y_s^t = e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}F(Y_{\lfloor s\rfloor_h}^{h,I})$, $\bar{Y}_s^t = e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}F(Y_{\lfloor s\rfloor_h}^{h,J})$, $Z_s^t = e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}B(Y_{\lfloor s\rfloor_h}^{h,I})$, $\bar{Z}_s^t = e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}B(Y_{\lfloor s\rfloor_h}^{h,J})$ for $t \in [0,T]$, $s \in [0,t]$ in the notation of [JK, Corollary 3.1]) therefore establishes that

$$\begin{aligned} &\sup_{t\in[0,T]} \|Y_t^{h,I} - Y_t^{h,J}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} \\ &\leq \sqrt{2}\exp\Big(\big(T\|\mathbf{P}_{I\cap J}F|_{\mathbf{H}_0}\|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{T}\|\mathbf{P}_{I\cap J}B\|_{\mathrm{Lip}(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\big)^2\Big) \\ &\quad\cdot\Bigg[\sup_{t\in[0,T]} \bigg\|[Y_t^{h,I}]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} - \bigg(\int_0^t e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}F(Y_{\lfloor s\rfloor_h}^{h,I})\,\mathrm{d}s \\ &\qquad + \int_0^t e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}B(Y_{\lfloor s\rfloor_h}^{h,I})\,\mathrm{d}W_s\bigg) + \bigg(\int_0^t e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{P}_{I\cap J}F(Y_{\lfloor s\rfloor_h}^{h,J})\,\mathrm{d}s \end{aligned}$$

$$+ \int_0^t e^{(t-\lfloor s \rfloor_h)\mathbf{A}} \mathbf{P}_{I\cap J} B(Y^{h,J}_{\lfloor s \rfloor_h}) \, \mathrm{d}W_s \Big) - [Y^{h,J}_t]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} \Big\|_{L^2(\mathbb{P};\mathbf{H}_0)} \Big]$$

$$= \sqrt{2} \exp\Big(\big(T \| \mathbf{P}_{I\cap J} F|_{\mathbf{H}_0} \|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{T} \| \mathbf{P}_{I\cap J} B \|_{\mathrm{Lip}(\mathbf{H}_0, L_2(U,\mathbf{H}_0))} \big)^2 \Big) \tag{66}$$

$$\cdot \Big[\sup_{t\in[0,T]} \Big\| [Y^{h,I}_t]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} - \mathbf{P}_J \Big([e^{t\mathbf{A}} \mathbf{P}_I \xi]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)}$$

$$+ \int_0^t e^{(t-\lfloor s \rfloor_h)\mathbf{A}} \mathbf{P}_I F(Y^{h,I}_{\lfloor s \rfloor_h}) \, \mathrm{d}s + \int_0^t e^{(t-\lfloor s \rfloor_h)\mathbf{A}} \mathbf{P}_I B(Y^{h,I}_{\lfloor s \rfloor_h}) \, \mathrm{d}W_s \Big)$$

$$+ \mathbf{P}_I \Big([e^{t\mathbf{A}} \mathbf{P}_J \xi]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} + \int_0^t e^{(t-\lfloor s \rfloor_h)\mathbf{A}} \mathbf{P}_J F(Y^{h,J}_{\lfloor s \rfloor_h}) \, \mathrm{d}s$$

$$+ \int_0^t e^{(t-\lfloor s \rfloor_h)\mathbf{A}} \mathbf{P}_J B(Y^{h,J}_{\lfloor s \rfloor_h}) \, \mathrm{d}W_s \Big) - [Y^{h,J}_t]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} \Big\|_{L^2(\mathbb{P};\mathbf{H}_0)} \Big]$$

$$= \sqrt{2} \exp\Big(\big(T \| \mathbf{P}_{I\cap J} F|_{\mathbf{H}_0} \|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{T} \| \mathbf{P}_{I\cap J} B \|_{\mathrm{Lip}(\mathbf{H}_0, L_2(U,\mathbf{H}_0))} \big)^2 \Big)$$

$$\cdot \Big[\sup_{t\in[0,T]} \| \mathbf{P}_{I\setminus J} Y^{h,I}_t - \mathbf{P}_{J\setminus I} Y^{h,J}_t \|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} \Big].$$

The proof of Proposition 3.8 is thus completed. ◻

The proof of the next result, Corollary 3.9 below, is a minor modification of the third step in the proof of Jacobe de Naurois et al. [JdNJW, Lemma 3.7].

Corollary 3.9. *Assume Setting 3.1, let $h \in [0,T]$, and let $I_n \subseteq \mathbb{H}$, $n \in \mathbb{N}$, satisfy that $\cup_{n\in\mathbb{N}} I_n = \mathbb{H}$ and $[\forall\, n \in \mathbb{N} \colon I_n \subseteq I_{n+1}]$. Then*

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\| Y^{h,\mathbb{H}}_t - Y^{h,I_n}_t \|^2_{\mathbf{H}_0} \big] = 0. \tag{67}$$

Proof of Corollary 3.9. Observe that Proposition 3.8, (36), Lemma 3.4, Minkowski's integral inequality, and Itô's isometry imply that for every $n \in \mathbb{N}$ it holds that

$$\sup_{t\in[0,T]} \| Y^{h,\mathbb{H}}_t - Y^{h,I_n}_t \|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}$$

$$\le \sqrt{2} \exp\Big(\big(T \| \mathbf{P}_{I_n} F|_{\mathbf{H}_0} \|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{T} \| \mathbf{P}_{I_n} B \|_{\mathrm{Lip}(\mathbf{H}_0, L_2(U,\mathbf{H}_0))} \big)^2 \Big)$$

$$\cdot \Big[\sup_{t\in[0,T]} \big\| \mathbf{P}_{\mathbb{H}\setminus I_n} Y^{h,\mathbb{H}}_t \big\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} \Big] \tag{68}$$

$$\le \sqrt{2} \exp\Big(\big(T \| \mathbf{P}_{I_n} F|_{\mathbf{H}_0} \|_{\mathrm{Lip}(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{T} \| \mathbf{P}_{I_n} B \|_{\mathrm{Lip}(\mathbf{H}_0, L_2(U,\mathbf{H}_0))} \big)^2 \Big)$$

$$\cdot \Bigg[\|\mathbf{P}_{\mathbb{H}\setminus I_n}\xi\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} + \int_0^T \big\|\mathbf{P}_{\mathbb{H}\setminus I_n} F(Y^{h,\mathbb{H}}_{\lfloor s \rfloor_h})\big\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}\, ds \\ + \Big(\int_0^T \big\|\mathbf{P}_{\mathbb{H}\setminus I_n} B(Y^{h,\mathbb{H}}_{\lfloor s \rfloor_h})\big\|^2_{\mathcal{L}^2(\mathbb{P};L_2(U,\mathbf{H}_0))}\, ds \Big)^{1/2} \Bigg].$$

This, item (ii) in Lemma 2.3, item (ii) in Lemma 3.7, and Lebesgue's theorem of dominated convergence ensure that

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \|Y^{h,\mathbb{H}}_t - Y^{h,I_n}_t\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} = 0. \tag{69}$$

The proof of Corollary 3.9 is thus completed. □

3.5 Upper bounds for the weak approximation errors

Theorem 3.10. *Assume Setting 3.1, let $h \in (0,T]$, for every finite $I \subseteq \mathbb{H}$ and every $x \in \mathbf{P}_I(\mathbf{H}_0)$ let $X^{I,x} \colon [0,T]\times\Omega \to \mathbf{P}_I(\mathbf{H}_0)$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-predictable stochastic process which satisfies for every $t\in[0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}\big[\|X^{I,x}_s\|^2_{\mathbf{H}_0}\big] < \infty$ and*

$$\big[X^{I,x}_t\big]_{\mathbb{P},\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))} = \big[e^{t\mathbf{A}}x\big]_{\mathbb{P},\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_I F\big(X^{I,x}_s\big)\,\mathrm{d}s \\ + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_I B\big(X^{I,x}_s\big)\,\mathrm{d}W_s, \tag{70}$$

and for every finite $I\subseteq\mathbb{H}$ let $v^I\colon [0,T]\times\mathbf{P}_I(\mathbf{H}_0)\to\mathbb{R}$ be the function which satisfies for every $t\in[0,T], x\in\mathbf{P}_I(\mathbf{H}_0)$ that $v^I(t,x) = \mathbb{E}[\varphi(X^{I,x}_{T-t})]$. Then

(i) it holds for every $t\in[0,T]$ and every finite $I\subseteq\mathbb{H}$ that $(\mathbf{P}_I(\mathbf{H}_0)\ni x\mapsto v^I(t,x)\in\mathbb{R})\in C^4(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})$,

(ii) it holds that

$$\sup_{\substack{I\subseteq\mathbb{H},\\ I\text{ is finite}}} \max_{k\in\{1,2,3,4\}} \sup_{t\in[0,T]} \sup_{x\in\mathbf{P}_I(\mathbf{H}_0)} \|(\tfrac{\partial^k}{\partial x^k} v^I)(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} < \infty, \tag{71}$$

and

(iii) it holds for every finite $I \subseteq \mathbb{H}$ that

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi\big(Y_T^{0,I}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,I}\big)\big]\big| \leq 6 \max\{T, T^{2-2(\gamma-\beta)}\}\, h^{2(\gamma-\beta)} \\
&\cdot \left[\max_{k\in\{1,2,3,4\}} \sup_{t\in[0,T]} \sup_{x\in\mathbf{P}_I(\mathbf{H}_0)} \|(\tfrac{\partial^k}{\partial x^k} v^I)(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\right] \\
&\cdot \Bigg[\|F|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho,\mathbf{H}_{2(\gamma-\beta)})} + 3\,\mathfrak{m}^4 + \|F\|_{\mathrm{Lip}(\mathbf{H}_{\beta-\gamma},\mathbf{H}_0)} \\
&\quad + 2\Big[\sum_{h\in\mathbb{H}} |\lambda_h|^{-\beta}\Big] \|B|_{\mathbf{H}_\rho}\|^2_{\mathrm{Lip}(\mathbf{H}_\rho, L(U,\mathbf{H}_\gamma))} + \mathfrak{l}\,\mathfrak{c}\Bigg] \\
&\cdot \Big[\max\big\{\|\mathbf{\Lambda}^{\rho-\max\{\rho,\gamma-\beta\}}\|^2_{L(\mathbf{H}_0)}, \|\mathbf{\Lambda}^{\gamma-\beta-\max\{\rho,\gamma-\beta\}}\|^2_{L(\mathbf{H}_0)}\big\}\Big] \\
&\cdot \left[1 + \sup_{t\in[0,T]} \mathbb{E}\Big[\|Y_t^{h,I}\|^2_{\mathbf{H}_{\max\{\rho,\gamma-\beta\}}} + \|Y_t^{h,I}\|^4_{\mathbf{H}_0}\Big]\right] < \infty.
\end{aligned}
\tag{72}
$$

Proof of Theorem 3.10. Throughout this proof let $\delta\colon [0,\infty) \to [0,h)$ be the function which satisfies for every $x \in [0,\infty)$ that $\delta(x) = x - \lfloor x \rfloor_h$, for every $p \in [1,\infty)$ and every $\mathbb{R}$-Hilbert space $(W, \langle\cdot,\cdot\rangle_W, \|\cdot\|_W)$ let $(L_p(U,W), \|\cdot\|_{L_p(U,W)})$ be the $\mathbb{R}$-Banach space of Schatten-p operators from U to W and let $(L_p(W), \|\cdot\|_{L_p(W)})$ be the $\mathbb{R}$-Banach space of Schatten-p operators from W to W, for every finite $I \subseteq \mathbb{H}$ let $v_{1,0}^I\colon [0,T]\times\mathbf{P}_I(\mathbf{H}_0) \to \mathbb{R}$ be the function which satisfies for every $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ that $v_{1,0}^I(t,x) = (\frac{\partial}{\partial t} v^I)(t,x)$, for every finite $I \subseteq \mathbb{H}$ and every $i \in \{1,2,3,4\}$ let $v_{0,i}^I\colon [0,T]\times\mathbf{P}_I(\mathbf{H}_0) \to L^{(i)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})$ be the function which satisfies for every $t \in [0,T]$, $x_0, x_1, \dots, x_i \in \mathbf{P}_I(\mathbf{H}_0)$ that

$$
v_{0,i}^I(t,x_0)(x_1\dots,x_i) = \big(\tfrac{\partial^i}{\partial x_0^i} v^I(t,x_0)\big)(x_1,\dots,x_i), \tag{73}
$$

for every finite $I \subseteq \mathbb{H}$ let $\varphi^I, \psi^I\colon [0,T]\times\mathbf{P}_I(\mathbf{H}_0) \to \mathbb{R}$ be the functions which satisfy for every $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ that $\varphi^I(t,x) = v_{0,1}^I(t,x)(\mathbf{P}_I F(x))$ and $\psi^I(t,x) = \sum_{u\in\mathbb{U}} v_{0,2}^I(t,x)(\mathbf{P}_I B(x)u, \mathbf{P}_I B(x)u)$, and for every finite $I \subseteq \mathbb{H}$ and every $i \in \{1,2\}$ let $\varphi_{0,i}^I, \psi_{0,i}^I\colon [0,T]\times\mathbf{P}_I(\mathbf{H}_0) \to L^{(i)}(\mathbf{H}_0,\mathbb{R})$ be the functions which satisfy for every $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ that $\varphi_{0,i}^I(t,x) = (\frac{\partial^i}{\partial x^i}\varphi^I)(t,x)$ and $\psi_{0,i}^I(t,x) = (\frac{\partial^i}{\partial x^i}\psi^I)(t,x)$. Observe that Lemma 3.6 and (70) ensure that for every finite $I \subseteq \mathbb{H}$ and every $x \in \mathbf{P}_I(\mathbf{H}_0)$, $t \in [0,T]$ it holds that

$$
\begin{aligned}
\big[X_t^{I,x}\big]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} = \big[e^{t(\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0})}x\big]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} &+ \int_0^t e^{(t-s)(\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0})}\mathbf{P}_I F\big(X_s^{I,x}\big)\,\mathrm{d}s \\
&+ \int_0^t e^{(t-s)(\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0})}\mathbf{P}_I B\big(X_s^{I,x}\big)\,\mathrm{d}W_s.
\end{aligned}
\tag{74}
$$

Next note that (33), Lemma 3.4, and Lemma 3.6 ensure that

$$\sup_{\substack{I \subseteq \mathbb{H}, \\ I \text{ is finite}}} \left[\sup_{t \in [0,T]} \left\| e^{t(\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0})} \right\|_{L(\mathbf{H}_0)} + \|\mathbf{P}_I F|_{\mathbf{H}_0}\|_{C^4_{\mathrm{b}}(\mathbf{H}_0,\mathbf{H}_0)} + \|\mathbf{P}_I B\|_{C^4_{\mathrm{b}}(\mathbf{H}_0, L_2(U,\mathbf{H}_0))} \right] < \infty. \tag{75}$$

Combining this, (74), the fact that for every finite $I \subseteq \mathbb{H}$ it holds that $\mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0} \in L(\mathbf{H}_0)$, item (ii) in Lemma 2.2 (with $H = \mathbf{H}_0$, $k = 4$, $A = \mathbf{A}\mathbf{P}_I|_{\mathbf{H}_0}$, $F = \mathbf{P}_I F|_{\mathbf{H}_0}$, $B = \mathbf{P}_I B$, $\varphi = \varphi$, $v^{A,F,B,\varphi}(t,x) = v^I(t,x)$ for $I \in \{J \subseteq \mathbb{H} \colon J \text{ is a finite set}\}$, $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ in the notation of item (ii) in Lemma 2.2), and item (iii) in Lemma 2.2 (with $H = \mathbf{H}_0$ in the notation of item (iii) in Lemma 2.2) establishes items (i) and (ii). It thus remains to prove item (iii). For this, observe that for every finite $I \subseteq \mathbb{H}$ it holds that $\mathbb{E}[\varphi(Y^{h,I}_T)] = \mathbb{E}[v^I(T, Y^{h,I}_T)]$ and

$$\begin{aligned} \mathbb{E}\big[\varphi\big(Y^{0,I}_T\big)\big] &= \mathbb{E}\big[\mathbb{E}\big(\varphi\big(Y^{0,I}_T\big)\big|\mathbb{F}_0\big)\big] = \mathbb{E}\big[v^I\big(0, Y^{0,I}_0\big)\big] \\ &= \mathbb{E}\big[v^I\big(0, \mathbf{P}_I\xi\big)\big] = \mathbb{E}\big[v^I\big(0, Y^{h,I}_0\big)\big]. \end{aligned} \tag{76}$$

Moreover, note that for every finite $I \subseteq \mathbb{H}$ it holds that $(e^{t\mathbf{A}}|_{\mathbf{P}_I(\mathbf{H}_0)})_{t\in[0,\infty)} \subseteq L(\mathbf{P}_I(\mathbf{H}_0))$ is a strongly continuous semigroup with generator $\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)} \in L(\mathbf{P}_I(\mathbf{H}_0))$. This and (36) imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\begin{aligned} &\big[Y^{h,I}_t\big]_{\mathbb{P},\,\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))} \\ &= [e^{t(\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)})}\mathbf{P}_I\xi]_{\mathbb{P},\,\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))} + \int_0^t e^{(t-\lfloor s\rfloor_h)(\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)})}\mathbf{P}_I F(Y^{h,I}_{\lfloor s\rfloor_h})\,\mathrm{d}s \\ &\quad + \int_0^t e^{(t-\lfloor s\rfloor_h)(\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)})}\mathbf{P}_I B(Y^{h,I}_{\lfloor s\rfloor_h})\,\mathrm{d}W_s \\ &= \big[\mathbf{P}_I\xi\big]_{\mathbb{P},\,\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))} + \int_0^t \big((\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)})Y^{h,I}_s + e^{\delta(s)(\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)})}\mathbf{P}_I F\big(Y^{h,I}_{\lfloor s\rfloor_h}\big)\big)\,\mathrm{d}s \\ &\quad + \int_0^t e^{\delta(s)(\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)})}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor s\rfloor_h}\big)\,\mathrm{d}W_s \\ &= \big[\mathbf{P}_I\xi\big]_{\mathbb{P},\,\mathcal{B}(\mathbf{P}_I(\mathbf{H}_0))} + \int_0^t \big(\mathbf{A}Y^{h,I}_s + e^{\delta(s)\mathbf{A}}\mathbf{P}_I F\big(Y^{h,I}_{\lfloor s\rfloor_h}\big)\big)\,\mathrm{d}s \\ &\quad + \int_0^t e^{\delta(s)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor s\rfloor_h}\big)\,\mathrm{d}W_s. \end{aligned} \tag{77}$$

In addition, observe that item (i) in Lemma 2.2 ensures that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ it holds that

$$v^I \in C^{1,2}\big([0,T] \times \mathbf{P}_I(\mathbf{H}_0), \mathbb{R}\big) \tag{78}$$

and

$$v^I_{1,0}(t,x) = -v^I_{0,1}(t,x)\big(\mathbf{A}x + \mathbf{P}_I F(x)\big) - \tfrac{1}{2}\sum_{u\in\mathbb{U}} v^I_{0,2}(t,x)\big(\mathbf{P}_I B(x)u, \mathbf{P}_I B(x)u\big). \quad (79)$$

Combining item (ii), the fact that for every finite $I \subseteq \mathbb{H}$ it holds that $\mathbf{A}|_{\mathbf{P}_I(\mathbf{H}_0)} \in L(\mathbf{P}_I(\mathbf{H}_0))$, item (ii) in Lemma 2.3, Lemma 3.4, and item (ii) in Lemma 3.7 therefore proves that

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T \big|v^I_{1,0}\big(t,Y^{h,I}_t\big)\big|\,dt + \int_0^T \big|v^I_{0,1}\big(t,Y^{h,I}_t\big)\big(\mathbf{A}Y^{h,I}_t + e^{\delta(t)\mathbf{A}}\mathbf{P}_I F\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)\big)\big|\,dt \\ + \tfrac{1}{2}\sum_{u\in\mathbb{U}}\int_0^T \big|v^I_{0,2}\big(t,Y^{h,I}_t\big)\big(e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)u, e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)u\big)\big|\,dt\bigg] < \infty.\end{aligned} \quad (80)$$

Furthermore, note that Itô's isometry, item (ii), item (ii) in Lemma 2.3, Lemma 3.4, and item (ii) in Lemma 3.7 imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\mathbb{E}\bigg[\Big|\int_0^T v^I_{0,1}\big(t,Y^{h,I}_t\big)e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)\,\mathrm{d}W_t\Big|^2\bigg] < \infty. \quad (81)$$

Hence, we obtain that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\mathbb{E}\bigg[\int_0^T v^I_{0,1}\big(t,Y^{h,I}_t\big)e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)\,\mathrm{d}W_t\bigg] = 0. \quad (82)$$

The Itô formula, (76)–(78), and (80) therefore imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\begin{aligned}&\big|\mathbb{E}\big[\varphi\big(Y^{h,I}_T\big) - \varphi\big(Y^{0,I}_T\big)\big]\big| \\ &= \big|\mathbb{E}\big[v^I\big(T,Y^{h,I}_T\big) - v^I\big(0,Y^{h,I}_0\big)\big]\big| \\ &= \bigg|\mathbb{E}\bigg[\int_0^T v^I_{1,0}\big(t,Y^{h,I}_t\big)\,\mathrm{d}t + \int_0^T v^I_{0,1}\big(t,Y^{h,I}_t\big)\big(\mathbf{A}Y^{h,I}_t + e^{\delta(t)\mathbf{A}}\mathbf{P}_I F\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)\big)\,\mathrm{d}t \\ &\quad + \tfrac{1}{2}\sum_{u\in\mathbb{U}}\int_0^T v^I_{0,2}\big(t,Y^{h,I}_t\big)\big(e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)u, e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t\rfloor_h}\big)u\big)\,\mathrm{d}t\bigg]\bigg|\end{aligned} \quad (83)$$

Combining this and (79) demonstrates that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi\big(Y_T^{h,I}\big) - \varphi\big(Y_T^{0,I}\big)\big]\big| \\
&= \bigg|\mathbb{E}\bigg[\int_0^T v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(e^{\delta(t)\mathbf{A}}\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big) - \mathbf{P}_I F\big(Y_t^{h,I}\big)\big)\,\mathrm{d}t \\
&\quad + \tfrac{1}{2}\sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\big(e^{\delta(t)\mathbf{A}}\mathbf{P}_I B(Y_{\lfloor t \rfloor_h}^{h,I})u, e^{\delta(t)\mathbf{A}}\mathbf{P}_I B(Y_{\lfloor t \rfloor_h}^{h,I})u\big)\,\mathrm{d}t \\
&\quad - \tfrac{1}{2}\sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I B\big(Y_t^{h,I}\big)u, \mathbf{P}_I B\big(Y_t^{h,I}\big)u\big)\,\mathrm{d}t\bigg]\bigg|.
\end{aligned} \tag{84}
$$

Moreover, observe that the triangle inequality ensures that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\bigg|\mathbb{E}\bigg[\int_0^T v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(e^{\delta(t)\mathbf{A}}\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big) - \mathbf{P}_I F\big(Y_t^{h,I}\big)\big)\,\mathrm{d}t\bigg]\bigg| \\
&\leq \bigg|\mathbb{E}\bigg[\int_0^T v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\big(e^{\delta(t)\mathbf{A}} - \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\,\mathrm{d}t\bigg]\bigg| \\
&\quad + \bigg|\mathbb{E}\bigg[\int_0^T \Big(v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) - v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\Big)\,\mathrm{d}t\bigg]\bigg| \\
&\quad + \bigg|\mathbb{E}\bigg[\int_0^T v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big) - \mathbf{P}_I F\big(e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\,\mathrm{d}t\bigg]\bigg| \\
&\quad + \bigg|\mathbb{E}\bigg[\int_0^T \Big(v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) - v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I F\big(Y_t^{h,I}\big)\big)\Big)\,\mathrm{d}t\bigg]\bigg|.
\end{aligned} \tag{85}
$$

Furthermore, note that the triangle inequality implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\bigg|\mathbb{E}\bigg[\sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\big(e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u, e^{\delta(t)\mathbf{A}}\mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t \\
&\quad - \sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I B\big(Y_t^{h,I}\big)u, \mathbf{P}_I B\big(Y_t^{h,I}\big)u\big)\,\mathrm{d}t\bigg]\bigg| \\
&\leq \bigg|\mathbb{E}\bigg[\sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big) \\
&\qquad \big(\big(e^{\delta(t)\mathbf{A}} - \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u, \big(e^{\delta(t)\mathbf{A}} + \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t\bigg]\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\mathbb{E}\left[\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t,Y_t^{h,I}\big)\big(\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t\right.\right. \\
&\quad\left.\left.-\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t,e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)\big(\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t\right]\right| \\
&+\left|\mathbb{E}\left[\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t,e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)\right.\right. \\
&\quad\left.\left.\big(\big[\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)-\mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)\big]u,\big[\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)+\mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)\big]u\big)\,\mathrm{d}t\right]\right| \\
&+\left|\mathbb{E}\left[\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t,e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)\big(\mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)u,\mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}}Y_{\lfloor t\rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t\right.\right. \\
&\quad\left.\left.-\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t,Y_t^{h,I}\big)\big(\mathbf{P}_I B\big(Y_t^{h,I}\big)u,\mathbf{P}_I B\big(Y_t^{h,I}\big)u\big)\,\mathrm{d}t\right]\right|.
\end{aligned}
\tag{86}
$$

In the next step we estimate the first term on the right-hand side of (85). Note that item (i) in Lemma 2.3 and Lemma 3.5 imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\left|\mathbb{E}\left[\int_0^T v_{0,1}^I\big(t,Y_t^{h,I}\big)\big(\big(e^{\delta(t)\mathbf{A}}-\mathrm{id}_{\mathbf{H}_0}\big)\mathbf{P}_I F\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big)\,\mathrm{d}t\right]\right| \\
&\leq T\left[\sup_{t\in[0,T],\,x\in\mathbf{P}_I(\mathbf{H}_0)}\Big(\|v_{0,1}^I(t,x)\|_{L(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\big\|\mathbf{\Lambda}^{2(\beta-\gamma)}\,\big(e^{\delta(t)\mathbf{A}}-\mathrm{id}_{\mathbf{H}_0}\big)\,\big\|_{L(\mathbf{H}_0)}\Big)\right] \\
&\quad\cdot\|F|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho,\mathbf{H}_{2(\gamma-\beta)})}\left[\sup_{t\in[0,T]}\mathbb{E}\big[\max\{1,\|Y_{\lfloor t\rfloor_h}^{h,I}\|_{\mathbf{H}_\rho}\}\big]\right] \\
&\leq 2^{3/2}h^{2(\gamma-\beta)}T\,\|F|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho,\mathbf{H}_{2(\gamma-\beta)})}\left[\sup_{t\in[0,T],\,x\in\mathbf{P}_I(\mathbf{H}_0)}\|v_{0,1}^I(t,x)\|_{L(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\right] \\
&\quad\cdot\left[\sup_{t\in[0,T]}\mathbb{E}\big[\max\{1,\|Y_{\lfloor t\rfloor_h}^{h,I}\|_{\mathbf{H}_\rho}\}\big]\right].
\end{aligned}
\tag{87}
$$

In the next step we estimate the second term on the right-hand side of (85). Note that item (i), (36), and the mild Itô formula in Da Prato et al. [DPJR, Corollary 1] (with $\mathbb{I} = [\lfloor t\rfloor_h, t]$, $H = (\mathbf{P}_I(\mathbf{H}_0))^2$, $\hat{H} = (\mathbf{P}_I(\mathbf{H}_0))^2$, $\check{H} = (\mathbf{P}_I(\mathbf{H}_0))^2$, $U_0 = U$, $X_s = (Y_s^{h,I}, \mathbf{P}_I F(Y_{\lfloor t\rfloor_h}^{h,I}))$, $S_{r,s} = [(\mathbf{P}_I(\mathbf{H}_0))^2 \ni (x_1,x_2) \mapsto (e^{(s-r)\mathbf{A}}x_1, x_2) \in (\mathbf{P}_I(\mathbf{H}_0))^2]$, $Y_s = (e^{\delta(s)\mathbf{A}}\mathbf{P}_I F(Y_{\lfloor t\rfloor_h}^{h,I}), 0)$, $Z_s u = (e^{\delta(s)\mathbf{A}}\mathbf{P}_I B(Y_{\lfloor t\rfloor_h}^{h,I})u, 0)$, $V = \mathbb{R}$, $\varphi = [(\mathbf{P}_I(\mathbf{H}_0))^2 \ni (x_1,x_2) \mapsto v_{0,1}^I(t,x_1)\,x_2 \in \mathbb{R}]$ for $t \in [0,T]$, $I \in \{J \subseteq \mathbb{H}\colon J \text{ is a finite set}\}$, $s \in$

$[\lfloor t \rfloor_h, t]$, $r \in [\lfloor t \rfloor_h, s]$, $u \in U$ in the notation of [DPJR, Corollary 1]) imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\begin{aligned}
&\big[v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) - v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\big]_{\mathbb{P}, \mathcal{B}(\mathbb{R})} \\
&= \int_{\lfloor t \rfloor_h}^t v_{0,2}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big), e^{\delta(t)\mathbf{A}} \mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\, \mathrm{d}s \\
&\quad + \int_{\lfloor t \rfloor_h}^t v_{0,2}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big), e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\, \mathrm{d}W_s \\
&\quad + \tfrac{1}{2} \sum_{u \in \mathbb{U}} \int_{\lfloor t \rfloor_h}^t v_{0,3}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big) \\
&\qquad\qquad \big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big), e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big) u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big) u\big)\, \mathrm{d}s.
\end{aligned} \tag{88}$$

Moreover, observe that Itô's isometry, item (ii), item (i) in Lemma 2.3, Lemma 3.4, and item (ii) in Lemma 3.7 ensure that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\mathbb{E}\bigg[\Big|\int_{\lfloor t \rfloor_h}^t v_{0,2}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big), e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\, \mathrm{d}W_s\Big|^2\bigg] < \infty. \tag{89}$$

This, item (i) in Lemma 2.3, Lemma 3.4, (33), and (88) imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg[\int_0^T \Big(v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) - v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\Big)\, \mathrm{d}t\bigg]\bigg| \\
&\leq hT \sup\Big\{\|v_{0,k}^I(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0),\, t \in [0,T],\, k \in \{2,3\}}\Big\} \|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0), \mathbf{H}_0)} \\
&\quad \cdot \big(\|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0), \mathbf{H}_0)} + \tfrac{1}{2}\|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0), L_2(U, \mathbf{H}_0))}\big) \\
&\quad \cdot \bigg[\sup_{t \in [0,T]} \mathbb{E}\big[\max\{1, \|Y_{\lfloor t \rfloor_h}^{h,I}\|^3_{\mathbf{H}_0}\}\big]\bigg] \\
&\leq \tfrac{3}{2} hT \mathfrak{m}^3 \sup\Big\{\|v_{0,k}^I(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0),\, t \in [0,T],\, k \in \{2,3\}}\Big\} \\
&\quad \cdot \bigg[\sup_{t \in [0,T]} \mathbb{E}\big[\max\{1, \|Y_{\lfloor t \rfloor_h}^{h,I}\|^3_{\mathbf{H}_0}\}\big]\bigg].
\end{aligned} \tag{90}$$

In the next step we estimate the third term on the right-hand side of (85). Note that Lemma 3.5 implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\begin{aligned}
&\left|\mathbb{E}\Big[\int_0^T v^I_{0,1}\big(t, e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big(\mathbf{P}_I F\big(Y^{h,I}_{\lfloor t \rfloor_h}\big) - \mathbf{P}_I F\big(e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big)\, \mathrm{d}t\Big]\right| \\
&\leq \Bigg[\sup_{t \in [0,T],\, x \in \mathbf{P}_I(\mathbf{H}_0)} \|v^I_{0,1}(t,x)\|_{L(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\Bigg] \|F\|_{\mathrm{Lip}(\mathbf{H}_{\beta-\gamma},\mathbf{H}_0)} \\
&\quad \cdot \int_0^T \mathbb{E}\Big[\big\|\mathbf{\Lambda}^{\beta-\gamma}\left[\mathrm{id}_{\mathbf{H}_0} - e^{\delta(t)\mathbf{A}}\right] Y^{h,I}_{\lfloor t \rfloor_h}\big\|_{\mathbf{H}_0}\Big]\, dt \\
&\leq 2^{3/2} h^{2(\gamma-\beta)} T \; \|F\|_{\mathrm{Lip}(\mathbf{H}_{\beta-\gamma},\mathbf{H}_0)} \Bigg[\sup_{t \in [0,T],\, x \in \mathbf{P}_I(\mathbf{H}_0)} \|v^I_{0,1}(t,x)\|_{L(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\Bigg] \\
&\quad \cdot \Bigg[\sup_{t \in [0,T]} \mathbb{E}\big[\|Y^{h,I}_{\lfloor t \rfloor_h}\|_{\mathbf{H}_{\gamma-\beta}}\big]\Bigg].
\end{aligned} \tag{91}$$

In the next step we estimate the fourth term on the right-hand side of (85). Note that item (i) implies that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\big(\mathbf{P}_I(\mathbf{H}_0) \ni x \mapsto \varphi^I(t,x) \in \mathbb{R}\big) \in C^2(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R}). \tag{92}$$

In addition, observe that item (i) ensures that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $x, v_1, v_2 \in \mathbf{P}_I(\mathbf{H}_0)$ it holds that

$$\varphi^I_{0,1}(t,x)(v_1) = v^I_{0,2}(t,x)(\mathbf{P}_I F(x), v_1) + v^I_{0,1}(t,x)(\mathbf{P}_I F'(x)(v_1)) \tag{93}$$

and

$$\begin{aligned}
\varphi^I_{0,2}(t,x)(v_1,v_2) &= v^I_{0,3}(t,x)(\mathbf{P}_I F(x), v_1, v_2) + v^I_{0,2}(t,x)(\mathbf{P}_I F'(x)(v_2), v_1) \\
&\quad + v^I_{0,2}(t,x)(\mathbf{P}_I F'(x)(v_1), v_2) + v^I_{0,1}(t,x)(\mathbf{P}_I F''(x)(v_1,v_2)).
\end{aligned} \tag{94}$$

This and item (i) in Lemma 2.3 imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ it holds that

$$\begin{aligned}
\|\varphi^I_{0,1}(t,x)\|_{L(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} &\leq 2 \sup\Big\{\|v^I_{0,k}(s,y)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R}))} \colon {\scriptstyle y \in \mathbf{P}_I(\mathbf{H}_0), \atop s \in [0,T],\, k \in \{1,2\}}\Big\} \\
&\quad \cdot \|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{C^1_{\mathrm{b}}(\mathbf{P}_I(\mathbf{H}_0),\mathbf{H}_0)} \max\{1, \|x\|_{\mathbf{H}_0}\},
\end{aligned} \tag{95}$$

and

$$\begin{aligned}
\|\varphi^I_{0,2}(t,x)\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} &\leq 4 \sup\Big\{\|v^I_{0,k}(s,y)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R}))} \colon {\scriptstyle y \in \mathbf{P}_I(\mathbf{H}_0), \atop s \in [0,T],\, k \in \{1,2,3\}}\Big\} \\
&\quad \cdot \|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{C^2_{\mathrm{b}}(\mathbf{P}_I(\mathbf{H}_0),\mathbf{H}_0)} \max\{1, \|x\|_{\mathbf{H}_0}\}.
\end{aligned} \tag{96}$$

Next observe that (36), (92), and the mild Itô formula in Da Prato et al. [DPJR, Corollary 1] (with $\mathbb{I} = [\lfloor t \rfloor_h, t]$, $H = \mathbf{P}_I(\mathbf{H}_0)$, $\hat{H} = \mathbf{P}_I(\mathbf{H}_0)$, $\check{H} = \mathbf{P}_I(\mathbf{H}_0)$, $U_0 = U$, $X_s = Y_s^{h,I}$, $S_{r,s} = e^{(s-r)\mathbf{A}}|_{\mathbf{P}_I(\mathbf{H}_0)}$, $Y_s = e^{\delta(s)\mathbf{A}}\mathbf{P}_I F(Y_{\lfloor t \rfloor_h}^{h,I})$, $Z_s = e^{\delta(s)\mathbf{A}}\mathbf{P}_I B(Y_{\lfloor t \rfloor_h}^{h,I})$, $V = \mathbb{R}$, $\varphi = \varphi^I(t, \cdot)$ for $t \in [0,T]$, $I \in \{J \subseteq \mathbb{H} \colon J \text{ is a finite set}\}$, $s \in [\lfloor t \rfloor_h, t]$, $r \in [\lfloor t \rfloor_h, s]$ in the notation of [DPJR, Corollary 1]) imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\big[v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) - v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I F\big(Y_t^{h,I}\big)\big)\big]_{\mathbb{P}, \mathcal{B}(\mathbb{R})} \\
&= - \int_{\lfloor t \rfloor_h}^t \varphi_{0,1}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) \, \mathrm{d}s \\
&\quad - \int_{\lfloor t \rfloor_h}^t \varphi_{0,1}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) \, \mathrm{d}W_s \\
&\quad - \tfrac{1}{2} \sum_{u \in \mathbb{U}} \int_{\lfloor t \rfloor_h}^t \varphi_{0,2}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u\big) \, \mathrm{d}s.
\end{aligned}
\tag{97}
$$

Moreover, note that Itô's isometry, item (ii), item (i) in Lemma 2.3, Lemma 3.4, item (ii) in Lemma 3.7, and (95) assure that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$
\mathbb{E}\bigg[\Big| \int_{\lfloor t \rfloor_h}^t \varphi_{0,1}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) \, \mathrm{d}W_s \Big|^2\bigg] < \infty. \tag{98}
$$

Combining this, item (ii) in Lemma 2.3, Lemma 3.4, (33), (95), (96), and (97) implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\bigg|\mathbb{E}\bigg[\int_0^T v_{0,1}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I F\big(e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big) - v_{0,1}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I F\big(Y_t^{h,I}\big)\big) \, \mathrm{d}t\bigg]\bigg| \\
&\leq 4hT \sup\Big\{\|v_{0,k}^I(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0), \mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0), \atop t \in [0,T],\, k \in \{1,2,3\}}\Big\} \|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{C_b^2(\mathbf{P}_I(\mathbf{H}_0), \mathbf{H}_0)} \\
&\quad \cdot \big[\|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0), \mathbf{H}_0)} + \tfrac{1}{2}\|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0), L_2(U, \mathbf{H}_0))}\big] \\
&\quad \cdot \bigg[\sup_{t \in [0,T]} \mathbb{E}\big[\max\{1, \|Y_{\lfloor t \rfloor_h}^{h,I}\|^3_{\mathbf{H}_0}\}\big]\bigg] \\
&\leq 6hT\mathfrak{m}^3 \sup\Big\{\|v_{0,k}^I(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0), \mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0), \atop t \in [0,T],\, k \in \{1,2,3\}}\Big\} \\
&\quad \cdot \bigg[\sup_{t \in [0,T]} \mathbb{E}\big[\max\{1, \|Y_{\lfloor t \rfloor_h}^{h,I}\|^3_{\mathbf{H}_0}\}\big]\bigg].
\end{aligned}
\tag{99}
$$

In the next step we estimate the first term on the right-hand side of (86). Note that Lemma 2.5 implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\left|\mathbb{E}\left[\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\Big(\big(e^{\delta(t)\mathbf{A}} - \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u, \big(e^{\delta(t)\mathbf{A}} + \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u\Big)\,\mathrm{d}t\right]\right| \\
&\leq \mathbb{E}\bigg[\int_0^T \big\|v_{0,2}^I\big(t, Y_t^{h,I}\big)\big\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\big\|\big(\mathrm{id}_{\mathbf{H}_0} + e^{\delta(t)\mathbf{A}}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big\|_{L_{2\beta/\gamma}(U,\mathbf{H}_0)} \\
&\qquad \cdot \big\|\big(\mathrm{id}_{\mathbf{H}_0} - e^{\delta(t)\mathbf{A}}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big\|_{L_{2\beta/(2\beta-\gamma)}(U,\mathbf{H}_0)}\,\mathrm{d}t\bigg].
\end{aligned}
\tag{100}
$$

Moreover, observe that $\mathbf{\Lambda}$ and $\mathbf{P}_I$ commute. This, item (i) in Lemma 2.3, and Lemma 3.4 imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\big\|\big(\mathrm{id}_{\mathbf{H}_0} + e^{\delta(t)\mathbf{A}}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big\|_{L_{2\beta/\gamma}(U,\mathbf{H}_0)} \\
&\leq \|\,\mathrm{id}_{\mathbf{H}_0} + e^{\delta(t)\mathbf{A}}\|_{L(\mathbf{H}_0)}\|\mathbf{\Lambda}^{-\gamma}\|_{L_{2\beta/\gamma}(\mathbf{H}_0)}\big\|\mathbf{\Lambda}^{\gamma}\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big\|_{L(U,\mathbf{H}_0)} \\
&\leq 2\|\mathbf{\Lambda}^{-2\beta}\|_{L_1(\mathbf{H}_0)}^{\gamma/(2\beta)}\|B|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho, L(U,\mathbf{H}_\gamma))}\max\big\{1, \|Y_{\lfloor t\rfloor_h}^{h,I}\|_{\mathbf{H}_\rho}\big\}.
\end{aligned}
\tag{101}
$$

In addition, note that item (i) in Lemma 2.3 and Lemma 3.5 imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\big\|\big(\mathrm{id}_{\mathbf{H}_0} - e^{\delta(t)\mathbf{A}}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big\|_{L_{2\beta/(2\beta-\gamma)}(U,\mathbf{H}_0)} \\
&\leq \big\|\mathbf{\Lambda}^{2(\beta-\gamma)}\,\big(\mathrm{id}_{\mathbf{H}_0} - e^{\delta(t)\mathbf{A}}\big)\big\|_{L(\mathbf{H}_0)}\,\|\mathbf{\Lambda}^{\gamma-2\beta}\|_{L_{2\beta/(2\beta-\gamma)}(\mathbf{H}_0)}\big\|\mathbf{\Lambda}^{\gamma}\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)\big\|_{L(U,\mathbf{H}_0)} \\
&\leq 2^{3/2}h^{2(\gamma-\beta)}\|\mathbf{\Lambda}^{-2\beta}\|_{L_1(\mathbf{H}_0)}^{(2\beta-\gamma)/(2\beta)}\|B|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho, L(U,\mathbf{H}_\gamma))}\max\big\{1, \|Y_{\lfloor t\rfloor_h}^{h,I}\|_{\mathbf{H}_\rho}\big\}.
\end{aligned}
\tag{102}
$$

Combining (100), (101), and (102) ensures that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\left|\mathbb{E}\left[\sum_{u\in\mathbb{U}}\int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\big(\big(e^{\delta(t)\mathbf{A}} - \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u, \big(e^{\delta(t)\mathbf{A}} + \mathrm{id}_{\mathbf{H}_0}\big)\,\mathbf{P}_I B\big(Y_{\lfloor t\rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t\right]\right| \\
&\leq 2^{5/2}h^{2(\gamma-\beta)}T\,\|\mathbf{\Lambda}^{-2\beta}\|_{L_1(\mathbf{H}_0)}\|B|_{\mathbf{H}_\rho}\|^2_{\mathrm{Lip}(\mathbf{H}_\rho, L(U,H_\gamma))} \\
&\quad \cdot \Bigg[\sup_{t\in[0,T],\, x\in\mathbf{P}_I(\mathbf{H}_0)}\|v_{0,2}^I(t,x)\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\Bigg]\Bigg[\sup_{t\in[0,T]}\mathbb{E}\big[\max\big\{1, \big\|Y_{\lfloor t\rfloor_h}^{h,I}\big\|^2_{\mathbf{H}_\rho}\big\}\big]\Bigg].
\end{aligned}
\tag{103}
$$

In the next step we estimate the second term on the right-hand side of (86). Note that item (i), (36), and the mild Itô formula in Da Prato et al. [DPJR, Corollary 1] (with $\mathbb{I} = [\lfloor t\rfloor_h, t]$, $H = (\mathbf{P}_I(\mathbf{H}_0))^3$, $\hat{H} = (\mathbf{P}_I(\mathbf{H}_0))^3$, $\check{H} = (\mathbf{P}_I(\mathbf{H}_0))^3$, $U_0 = U$, $X_s = (Y_s^{h,I},$

$\mathbf{P}_I B(Y^{h,I}_{\lfloor t \rfloor_h})u, \mathbf{P}_I B(Y^{h,I}_{\lfloor t \rfloor_h})u)$, $S_{r,s} = [(\mathbf{P}_I(\mathbf{H}_0))^3 \ni (x_1, x_2, x_3) \mapsto (e^{(s-r)\mathbf{A}} x_1, x_2, x_3) \in (\mathbf{P}_I(\mathbf{H}_0))^3]$, $Y_s = (e^{\delta(s)\mathbf{A}} \mathbf{P}_I F(Y^{h,I}_{\lfloor t \rfloor_h}), 0, 0)$, $Z_s u' = (e^{\delta(s)\mathbf{A}} \mathbf{P}_I B(Y^{h,I}_{\lfloor t \rfloor_h}) u', 0, 0)$, $V = \mathbb{R}$, $\varphi = [(\mathbf{P}_I(\mathbf{H}_0))^3 \ni (x_1, x_2, x_3) \mapsto v^I_{0,2}(t, x_1)(x_2, x_3) \in \mathbb{R}]$ for $t \in [0,T]$, $I \in \{J \subseteq \mathbb{H} \colon J \text{ is a finite set}\}$, $s \in [\lfloor t \rfloor_h, t]$, $r \in [\lfloor t \rfloor_h, s]$, $u, u' \in U$ in the notation of [DPJR, Corollary 1]) ensure that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $u \in \mathbb{U}$ it holds that

$$\begin{aligned}
&\Big[v^I_{0,2}\big(t, Y^{h,I}_t\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u\big) \\
&\quad - v^I_{0,2}\big(t, e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u\big)\Big]_{\mathbb{P}, \mathcal{B}(\mathbb{R})} \\
&= \int_{\lfloor t \rfloor_h}^t v^I_{0,3}\big(t, e^{(t-s)\mathbf{A}} Y^{h,I}_s\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I F\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)\big) \, \mathrm{d}s \\
&+ \int_{\lfloor t \rfloor_h}^t v^I_{0,3}\big(t, e^{(t-s)\mathbf{A}} Y^{h,I}_s\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)\big) \, \mathrm{d}W_s \\
&+ \tfrac{1}{2} \sum_{u' \in \mathbb{U}} \int_{\lfloor t \rfloor_h}^t v^I_{0,4}\big(t, e^{(t-s)\mathbf{A}} Y^{h,I}_s\big) \\
&\qquad \big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u', e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u'\big) \, \mathrm{d}s.
\end{aligned} \tag{104}$$

Moreover, observe that Itô's isometry, item (ii), item (i) in Lemma 2.3, Lemma 3.4, and item (ii) in Lemma 3.7 ensure that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $u \in \mathbb{U}$ it holds that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg| \int_{\lfloor t \rfloor_h}^t v^I_{0,3}\big(t, e^{(t-s)\mathbf{A}} Y^{h,I}_s\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)\big) \, \mathrm{d}W_s \bigg|^2\bigg] \\
&\qquad < \infty.
\end{aligned} \tag{105}$$

Combining this, item (i) in Lemma 2.3, Lemma 3.4, (33), and (104) implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\begin{aligned}
&\Bigg| \mathbb{E}\Bigg[\sum_{u \in \mathbb{U}} \int_0^T \Big(v^I_{0,2}\big(t, Y^{h,I}_t\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u\big) \\
&\qquad - v^I_{0,2}\big(t, e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u\big)\Big) \, \mathrm{d}t \Bigg] \Bigg| \\
&= \Bigg| \sum_{u \in \mathbb{U}} \int_0^T \mathbb{E}\Big[v^I_{0,2}\big(t, Y^{h,I}_t\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u\big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - v^I_{0,2}\big(t, e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big(\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u, \mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big)u\big)\Big]\,\mathrm{d}t\Bigg| \\
&\leq hT \sup\Big\{\|v^I_{0,k}(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x\in\mathbf{P}_I(\mathbf{H}_0), \atop t\in[0,T],\, k\in\{3,4\}}\Big\} \|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0),L_2(U,\mathbf{H}_0))} \\
&\quad \cdot \big(\|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0),\mathbf{H}_0)} + \tfrac{1}{2}\|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0),L_2(U,\mathbf{H}_0))}\big) \\
&\quad \cdot \Big[\sup_{t\in[0,T]} \mathbb{E}\big[\max\{1, \|Y^{h,I}_{\lfloor t \rfloor_h}\|^4_{\mathbf{H}_0}\}\big]\Big] \qquad (106)\\
&\leq \tfrac{3}{2} hT \mathfrak{m}^4 \sup\Big\{\|v^I_{0,k}(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x\in\mathbf{P}_I(\mathbf{H}_0), \atop t\in[0,T],\, k\in\{3,4\}}\Big\} \\
&\quad \cdot \Big[\sup_{t\in[0,T]} \mathbb{E}\big[\max\{1, \|Y^{h,I}_{\lfloor t \rfloor_h}\|^4_{\mathbf{H}_0}\}\big]\Big].
\end{aligned}
$$

In the next step we estimate the third term on the right-hand side of (86). Note that the Cauchy-Schwarz inequality, Lemma 3.5, (34), and (35) imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\Bigg|\mathbb{E}\Bigg[\sum_{u\in\mathbb{U}} \int_0^T v^I_{0,2}\big(t, e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big) \\
&\quad \big(\big[\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big) - \mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big]u, \big[\mathbf{P}_I B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big) + \mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big]u\big)\,\mathrm{d}t\Bigg]\Bigg| \\
&\leq \Big[\sup_{t\in[0,T],\, x\in\mathbf{P}_I(\mathbf{H}_0)} \|v^I_{0,2}(t,x)\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\Big] \\
&\quad \cdot \mathbb{E}\Bigg[\int_0^T \Big(\sum_{u\in\mathbb{U}} |\mu_u|^{-2} \big\|\big[B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big) - B\big(e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big]u\big\|^2_{\mathbf{H}_0}\Big)^{1/2} \\
&\qquad \cdot \Big(\sum_{u\in\mathbb{U}} |\mu_u|^{2} \big\|\big[B\big(Y^{h,I}_{\lfloor t \rfloor_h}\big) + B\big(e^{\delta(t)\mathbf{A}} Y^{h,I}_{\lfloor t \rfloor_h}\big)\big]u\big\|^2_{\mathbf{H}_0}\Big)^{1/2} \mathrm{d}t\Bigg] \qquad (107)\\
&\leq 2\,\mathfrak{l}\,\mathfrak{c}\Big[\sup_{t\in[0,T],\, x\in\mathbf{P}_I(\mathbf{H}_0)} \|v^I_{0,2}(t,x)\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\Big] \\
&\quad \cdot \int_0^T \big\|\mathbf{\Lambda}^{2(\beta-\gamma)}\big(\mathrm{id}_{\mathbf{H}_0} - e^{\delta(t)\mathbf{A}}\big)\big\|_{L(\mathbf{H}_0)} \mathbb{E}\big[\max\{1, \|Y^{h,I}_{\lfloor t \rfloor_h}\|^2_{\mathbf{H}_{\gamma-\beta}}\}\big]\,dt \\
&\leq 2^{5/2} h^{2(\gamma-\beta)} T\,\mathfrak{l}\,\mathfrak{c}\Big[\sup_{t\in[0,T],\, x\in\mathbf{P}_I(\mathbf{H}_0)} \|v^I_{0,2}(t,x)\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})}\Big] \\
&\quad \cdot \Big[\sup_{t\in[0,T]} \mathbb{E}\big[\max\{1, \|Y^{h,I}_{\lfloor t \rfloor_h}\|^2_{\mathbf{H}_{\gamma-\beta}}\}\big]\Big].
\end{aligned}
$$

In the next step we estimate the final term on the right-hand side of (86). Note that item (i) implies that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\big(\mathbf{P}_I(\mathbf{H}_0) \ni x \mapsto \psi^I(t,x) \in \mathbb{R}\big) \in C^2(\mathbf{P}_I(\mathbf{H}_0), \mathbb{R}). \tag{108}$$

In addition, observe that item (i) ensures that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $x, v_1, v_2 \in \mathbf{P}_I(\mathbf{H}_0)$ it holds that

$$\begin{aligned} \psi^I_{0,1}(t,x)(v_1) &= \sum_{u\in\mathbb{U}} v^I_{0,3}(t,x)\left(\mathbf{P}_I B(x)u, \mathbf{P}_I B(x)u, v_1\right) \\ &\quad + 2\sum_{u\in\mathbb{U}} v^I_{0,2}(t,x)\left(\mathbf{P}_I B'(x)(v_1)u, \mathbf{P}_I B(x)u\right), \end{aligned} \tag{109}$$

and

$$\begin{aligned} \psi^I_{0,2}(t,x)(v_1,v_2) &= \sum_{u\in\mathbb{U}} v^I_{0,4}(t,x)\left(\mathbf{P}_I B(x)u, \mathbf{P}_I B(x)u, v_1, v_2\right) \\ &\quad + 2\sum_{u\in\mathbb{U}} v^I_{0,3}(t,x)\left(\mathbf{P}_I B'(x)(v_2)u, \mathbf{P}_I B(x)u, v_1\right) \\ &\quad + 2\sum_{u\in\mathbb{U}} v^I_{0,3}(t,x)\left(\mathbf{P}_I B'(x)(v_1)u, \mathbf{P}_I B(x)u, v_2\right) \\ &\quad + 2\sum_{u\in\mathbb{U}} v^I_{0,2}(t,x)\left(\mathbf{P}_I B''(x)(v_1,v_2)u, \mathbf{P}_I B(x)u\right) \\ &\quad + 2\sum_{u\in\mathbb{U}} v^I_{0,2}(t,x)\left(\mathbf{P}_I B'(x)(v_1)u, \mathbf{P}_I B'(x)(v_2)u\right). \end{aligned} \tag{110}$$

Combining this and item (i) in Lemma 2.3 shows that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$, $x \in \mathbf{P}_I(\mathbf{H}_0)$ it holds that

$$\begin{aligned} \|\psi^I_{0,1}(t,x)\|_{L(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} &\leq 3\sup\Big\{\|v^I_{0,k}(s,y)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R}))} \colon {\scriptstyle y\in\mathbf{P}_I(\mathbf{H}_0), \atop s\in[0,T],\, k\in\{2,3\}}\Big\} \\ &\quad \cdot \|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{C^1_{\mathrm{b}}(\mathbf{P}_I(\mathbf{H}_0),L_2(U,H_0))} \max\big\{1, \|x\|^2_{\mathbf{H}_0}\big\} \end{aligned} \tag{111}$$

and

$$\begin{aligned} \|\psi^I_{0,2}(t,x)\|_{L^{(2)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} &\leq 9\sup\Big\{\|v^I_{0,k}(s,y)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R}))} \colon {\scriptstyle y\in\mathbf{P}_I(\mathbf{H}_0), \atop s\in[0,T],\, k\in\{2,3,4\}}\Big\} \\ &\quad \cdot \|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{C^2_{\mathrm{b}}(\mathbf{P}_I(\mathbf{H}_0),L_2(U,H_0))} \max\big\{1, \|x\|^2_{\mathbf{H}_0}\big\}. \end{aligned} \tag{112}$$

In addition, observe that (36), (108), and the mild Itô formula in Da Prato et al. [DPJR, Corollary 1] (with $\mathbb{I} = [\lfloor t \rfloor_h, t]$, $H = \mathbf{P}_I(\mathbf{H}_0)$, $\hat{H} = \mathbf{P}_I(\mathbf{H}_0)$, $\check{H} = \mathbf{P}_I(\mathbf{H}_0)$, $U_0 = U$, $X_s = Y^{h,I}_s$, $S_{r,s} = e^{(s-r)\mathbf{A}}|_{\mathbf{P}_I(\mathbf{H}_0)}$, $Y_s = e^{\delta(s)\mathbf{A}}\mathbf{P}_I F(Y^{h,I}_{\lfloor t \rfloor_h})$, $Z_s = e^{\delta(s)\mathbf{A}}\mathbf{P}_I B(Y^{h,I}_{\lfloor t \rfloor_h})$, $V = \mathbb{R}$, $\varphi = \psi^I(t,\cdot)$ for $t \in [0,T]$, $I \in \{J \subseteq \mathbb{H} \colon J \text{ is a finite set}\}$,

$s \in [\lfloor t \rfloor_h, t]$, $r \in [\lfloor t \rfloor_h, s]$ in the notation of [DPJR, Corollary 1]) imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\begin{aligned}
&\big[\psi^I\big(t, Y_t^{h,I}\big) - \psi^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big]_{\mathbb{P},\mathcal{B}(\mathbb{R})} = \int_{\lfloor t \rfloor_h}^t \psi_{0,1}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} F\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\,\mathrm{d}s \\
&\quad + \int_{\lfloor t \rfloor_h}^t \psi_{0,1}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\,\mathrm{d}W_s \qquad (113) \\
&\quad + \tfrac{1}{2} \sum_{u \in \mathbb{U}} \int_{\lfloor t \rfloor_h}^t \psi_{0,2}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u, e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}s
\end{aligned}$$

Next note that Itô's isometry, item (ii), item (i) in Lemma 2.3, Lemma 3.4, item (ii) in Lemma 3.7, and (111) prove that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\mathbb{E}\left[\Big|\int_{\lfloor t \rfloor_h}^t \psi_{0,1}^I\big(t, e^{(t-s)\mathbf{A}} Y_s^{h,I}\big)\big(e^{\delta(t)\mathbf{A}} \mathbf{P}_I B\big(Y_{\lfloor t \rfloor_h}^{h,I}\big)\big)\,\mathrm{d}W_s\Big|^2\right] < \infty. \qquad (114)$$

Combining this, item (i) in Lemma 2.3, Lemma 3.4, (33), (111), (112), and (113) ensures that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0,T]$ it holds that

$$\begin{aligned}
&\big|\mathbb{E}\big[\psi^I\big(t, Y_t^{h,I}\big) - \psi^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big]\big| \\
&\leq h \sup\Big\{\|v_{0,k}^I(s,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0),\ s \in [0,T],\, k \in \{2,3,4\}}\Big\} \|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{C_{\mathrm{b}}^2(\mathbf{P}_I(\mathbf{H}_0), L_2(U,\mathbf{H}_0))} \\
&\quad \cdot \big(3\|F|_{\mathbf{P}_I(\mathbf{H}_0)}\|_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0),\mathbf{H}_0)} + \tfrac{9}{2}\|B|_{\mathbf{P}_I(\mathbf{H}_0)}\|^2_{\mathrm{Lip}(\mathbf{P}_I(\mathbf{H}_0),L_2(U,\mathbf{H}_0))}\big) \\
&\quad \cdot \Big[\sup_{s \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_{\lfloor s \rfloor_h}^{h,I}\|_{\mathbf{H}_0}^4\big\}\big]\Big] \qquad (115) \\
&\leq \tfrac{15}{2} h\, \mathfrak{m}^4 \sup\Big\{\|v_{0,k}^I(s,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0),\ s \in [0,T],\, k \in \{2,3,4\}}\Big\} \\
&\quad \cdot \Big[\sup_{s \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_{\lfloor s \rfloor_h}^{h,I}\|_{\mathbf{H}_0}^4\big\}\big]\Big].
\end{aligned}$$

This implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$\begin{aligned}
&\Bigg|\mathbb{E}\Bigg[\sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big(\mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)u, \mathbf{P}_I B\big(e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)u\big)\,\mathrm{d}t \\
&\quad - \sum_{u \in \mathbb{U}} \int_0^T v_{0,2}^I\big(t, Y_t^{h,I}\big)\big(\mathbf{P}_I B\big(Y_t^{h,I}\big)u, \mathbf{P}_I B\big(Y_t^{h,I}\big)u\big)\,\mathrm{d}t\Bigg]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_0^T \mathbb{E}\big[\psi^I\big(t, Y_t^{h,I}\big) - \psi^I\big(t, e^{\delta(t)\mathbf{A}} Y_{\lfloor t \rfloor_h}^{h,I}\big)\big] \, dt \right| \\
&\leq \tfrac{15}{2} h T \mathfrak{m}^4 \sup\Big\{ \|v_{0,k}^I(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0), \atop t \in [0,T],\, k \in \{2,3,4\}} \Big\} \\
&\quad \cdot \Big[\sup_{t \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_{\lfloor t \rfloor_h}^{h,I}\|_{\mathbf{H}_0}^4\big\}\big] \Big].
\end{aligned} \tag{116}
$$

Combining (84), (85), (86), (87), (90), (91), (99), (103), (106), (107), and (116) ensures that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi\big(Y_T^{0,I}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,I}\big)\big]\big| \leq \max\{h, h^{2(\gamma-\beta)}\} T \\
&\cdot \sup\Big\{ \|v_{0,k}^I(t,x)\|_{L^{(k)}(\mathbf{P}_I(\mathbf{H}_0),\mathbb{R})} \colon {\scriptstyle x \in \mathbf{P}_I(\mathbf{H}_0), \atop t \in [0,T],\, k \in \{1,2,3,4\}} \Big\} \\
&\quad \cdot \bigg(2^{3/2} \|F|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho, \mathbf{H}_{2(\gamma-\beta)})} \Big[\sup_{t \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_t^{h,I}\|_{\mathbf{H}_\rho}\big\}\big] \Big] \\
&\qquad + 17\, \mathfrak{m}^4 \Big[\sup_{t \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_t^{h,I}\|_{\mathbf{H}_0}^4\big\}\big] \Big] \\
&\qquad + 2^{3/2} \|F\|_{\mathrm{Lip}(\mathbf{H}_{\beta-\gamma}, \mathbf{H}_0)} \Big[\sup_{t \in [0,T]} \mathbb{E}\big[\|Y_t^{h,I}\|_{\mathbf{H}_{\gamma-\beta}}\big] \Big] \\
&\qquad + 2^{5/2} \big\|\mathbf{\Lambda}^{-2\beta}\big\|_{L_1(\mathbf{H}_0)} \|B|_{\mathbf{H}_\rho}\|_{\mathrm{Lip}(\mathbf{H}_\rho, L(U,H_\gamma))}^2 \Big[\sup_{t \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_t^{h,I}\|_{\mathbf{H}_\rho}^2\big\}\big] \Big] \\
&\qquad + 2^{5/2} \mathfrak{l}\, \mathfrak{c} \Big[\sup_{t \in [0,T]} \mathbb{E}\big[\max\big\{1, \|Y_t^{h,I}\|_{\mathbf{H}_{\gamma-\beta}}^2\big\}\big] \Big] \bigg).
\end{aligned} \tag{117}
$$

Combining this, the estimates $\max\{h, h^{2(\gamma-\beta)}\} \leq \max\{T^{1-2(\gamma-\beta)}, 1\} h^{2(\gamma-\beta)}$ and $2^{5/2} \leq 6$, the fact that $\|\mathbf{\Lambda}^{-2\beta}\|_{L_1(\mathbf{H}_0)} = 2\big[\sum_{h \in \mathbb{H}} |\lambda_h|^{-\beta}\big]$, and items (ii)–(iii) in Lemma 3.7 establishes item (iii). The proof of Theorem 3.10 is thus completed. □

Corollary 3.11. *Assume Setting 3.1. Then*

$$
\sup_{h \in (0,T]} \Big(h^{2(\beta-\gamma)} \big|\mathbb{E}\big[\varphi\big(Y_T^{0,\mathbb{H}}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,\mathbb{H}}\big)\big]\big| \Big) < \infty. \tag{118}
$$

Proof of Corollary 3.11. Throughout this proof let $I_n \subseteq \mathbb{H}$, $n \in \mathbb{N}$, be a non-decreasing sequence of finite sets which satisfies that $\cup_{n \in \mathbb{N}} I_n = \mathbb{H}$. Observe that the triangle inequality implies that for every $n \in \mathbb{N}$, $h \in (0,T]$ it holds that

$$\begin{aligned}
&\big|\mathbb{E}\big[\varphi\big(Y_T^{0,\mathbb{H}}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,\mathbb{H}}\big)\big]\big| \\
&\leq \big|\mathbb{E}\big[\varphi\big(Y_T^{0,\mathbb{H}}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{0,I_n}\big)\big]\big| + \big|\mathbb{E}\big[\varphi\big(Y_T^{0,I_n}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,I_n}\big)\big]\big| \\
&\quad + \big|\mathbb{E}\big[\varphi\big(Y_T^{h,I_n}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,\mathbb{H}}\big)\big]\big| \\
&\leq \left[\sup_{x \in \mathbf{H}_0} \|\varphi'(x)\|_{L(\mathbf{H}_0,\mathbb{R})}\right] \big(\|Y_T^{0,\mathbb{H}} - Y_T^{0,I_n}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} + \|Y_T^{h,\mathbb{H}} - Y_T^{h,I_n}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}\big) \\
&\quad + \big|\mathbb{E}\big[\varphi\big(Y_T^{0,I_n}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,I_n}\big)\big]\big|.
\end{aligned} \tag{119}$$

Furthermore, note that Corollary 3.9 ensures that for every $h \in (0,T]$ it holds that

$$\limsup_{n\to\infty} \big(\|Y_T^{0,\mathbb{H}} - Y_T^{0,I_n}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)} + \|Y_T^{h,\mathbb{H}} - Y_T^{h,I_n}\|_{\mathcal{L}^2(\mathbb{P};\mathbf{H}_0)}\big) = 0. \tag{120}$$

Moreover, observe that items (ii)–(iii) in Lemma 3.7 and items (ii)–(iii) in Theorem 3.10 imply that

$$\sup_{h\in(0,T]} \left(h^{2(\beta-\gamma)}\left[\limsup_{n\to\infty} \big|\mathbb{E}\big[\varphi\big(Y_T^{0,I_n}\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^{h,I_n}\big)\big]\big|\right]\right) < \infty. \tag{121}$$

This, (119), and (120) imply (118). The proof of Corollary 3.11 is thus completed. ☐

4 Weak convergence rates for temporal numerical approximations of the hyperbolic Anderson model

4.1 Setting

Throughout this section we shall frequently use the following setting.

Setting 4.1. *For every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space (S, Σ), every set $\mathcal{O}$, and every function $f\colon \mathcal{O} \to S$ let $[f]_{\mu,\Sigma}$ be the set given by*

$$[f]_{\mu,\Sigma} = \left\{g\colon \Omega \to S\colon \left[\begin{smallmatrix}[\exists\, A\in\mathcal{F}\colon\, (\mu(A)=0 \;\, and\;\, \{\omega\in\Omega\cap\mathcal{O}\colon\, f(\omega)\neq g(\omega)\}\subseteq A)] \\ and\;\, [\forall\, A\in\Sigma\colon\, g^{-1}(A)\in\mathcal{F}]\end{smallmatrix}\right]\right\}, \tag{122}$$

let $\lambda\colon \mathcal{B}((0,1)) \to [0,1]$ be the Lebesgue-Borel measure on $(0,1)$, for every $r \in [0,\infty)$, $p \in (1,\infty)$ let $(W^{r,p}((0,1),\mathbb{R}), \|\cdot\|_{W^{r,p}((0,1),\mathbb{R})})$ be the Sobolev-Slobodeckij space with *smoothness parameter r and integrability parameter p, for every $r \in [0,2]$, $p \in (1,\infty)$ let $(\mathcal{W}^{r,p}((0,1),\mathbb{R}), \|\cdot\|_{\mathcal{W}^{r,p}((0,1),\mathbb{R})})$ be the $\mathbb{R}$-Banach space which satisfies that*

$$\mathcal{W}^{r,p}((0,1),\mathbb{R}) = \begin{cases} W^{r,p}((0,1),\mathbb{R}) & : r \leq 1/p \\ \left\{ f \in W^{r,p}((0,1),\mathbb{R}) \colon \left[\substack{\exists\, g \in C([0,1],\mathbb{R}) \colon (g|_{(0,1)} \in f \\ \text{and } g(0)=g(1)=0)} \right] \right\} & : r > 1/p \end{cases} \tag{123}$$

and $[\forall\, v \in \mathcal{W}^{r,p}((0,1),\mathbb{R}) \colon \|v\|_{\mathcal{W}^{r,p}((0,1),\mathbb{R})} = \|v\|_{W^{r,p}((0,1),\mathbb{R})}]$, for every $p \in (1,\infty)$ let $A_p \colon D(A_p) \subseteq L^p(\lambda;\mathbb{R}) \to L^p(\lambda;\mathbb{R})$ be the linear operator which satisfies that $D(A_p) = \mathcal{W}^{2,p}((0,1),\mathbb{R})$ and $[\forall\, h \in D(A_p) \colon A_p(h) = \Delta h]$, for every $p \in (1,\infty)$ let $(V_{r,p}, \|\cdot\|_{V_{r,p}})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A_p$, and for every $\delta \in (0,1)$ let $(C^\delta([0,1],\mathbb{R}), \|\cdot\|_{C^\delta([0,1],\mathbb{R})})$ be the space of δ-Hölder continuous functions.

Note that for every $p \in (1,\infty)$ it holds that A_p is the Dirichlet Laplacian on $L^p(\lambda;\mathbb{R})$.

4.2 Preparatory lemmas

Various results closely related to Lemmas 4.2–4.5 below are available in the literature; see, e.g., Lemarie-Rieusset & Gala [LRG06, Lemma 1] for a result closely related to Lemmas 4.3 and 4.4 below. We provide these lemmas in the exact form that we need.

Lemma 4.2. *Assume Setting 4.1. Then*

(i) it holds for every $r \in (0,1)\backslash\{1/4\}$ that $V_{r,2} \subseteq \mathcal{W}^{2r,2}((0,1),\mathbb{R})$ continuously,

(ii) it holds for every $r \in (0,1)\backslash\{1/4\}$ that $\mathcal{W}^{2r,2}((0,1),\mathbb{R}) \subseteq V_{r,2}$ continuously, and

(iii) it holds for every $p \in (1,\infty)$, $r \in (0,1)$, $s \in [0,r)$ that $V_{r,p} \subseteq \mathcal{W}^{2s,p}((0,1),\mathbb{R})$ continuously.

Proof of Lemma 4.2. First, note that, e.g., Triebel [Tri78, Theorem 1.15.3, Definition 2.3.1/1, item (d) in Theorem 2.3.2, Definition 4.2.1/1, Definition 4.3.3/2, equation (7) in Theorem 4.3.3, item (b) in Theorem 4.9.1, and item (b) in Theorem 5.5.1] (with $k = 1$, $B_1 = \mathrm{id}_{\mathbb{C}^{\{0,1\}}}$, $m_1 = 0$, $m = 2$, $p = 2$, $\theta = r$ for $r \in (0,1) \setminus \{1/4\}$ in the notation of [Tri78, Definition 4.3.3/2 and equation (7) in Theorem 4.3.3]) implies that for every $r \in (0,1)\backslash\{1/4\}$ it holds that $V_{r,2} \subseteq \mathcal{W}^{2r,2}((0,1),\mathbb{R}) \subseteq V_{r,2}$ continuously (cf. Triebel [Tri78, Definition 2.3.1/1 and Definition 4.2.1/1] for a definition of $W^{r,p}((0,1),\mathbb{C})$, $r \in [0,\infty)$, $p \in (1,\infty)$, and cf. Triebel [Tri78, Section 4.2.4, Remark 2 in Section 4.4.1, and Remark 2 in Section 4.4.2] for equivalent definitions of $W^{r,p}((0,1),\mathbb{C})$, $r \in [0,\infty)$, $p \in (1,\infty)$). This proves items (i) and (ii). Next observe that, e.g., Triebel [Tri78, Theorem 1.15.3, Definition 4.2.1/1, Definition 4.3.3/2, equation (7) in Theorem 4.3.3, items (a)–(b) in Theorem 4.6.1, item (b) in Theorem 4.9.1, item (c) in Theorem 5.4.4/1, and item (b) in Theorem 5.5.1] (with $k = 1$, $B_1 = \mathrm{id}_{\mathbb{C}^{\{0,1\}}}$, $m_1 = 0$, $m = 2$, $p = p$, $\theta = r - (\varepsilon/2)\mathbb{1}_{\{1/(2p)\}}(r)$ for $p \in (1,\infty)$, $r \in (0,1)$, $\varepsilon \in (0,r]$ in the notation of [Tri78, Definition 4.3.3/2 and equation (7) in Theorem 4.3.3]) ensures that for every $p \in (1,\infty)$, $r \in (0,1)$, $\varepsilon \in (0,r]$ it holds that $V_{r,p} \subseteq \mathcal{W}^{2(r-\varepsilon)}((0,1),\mathbb{R})$ continuously. This establishes item (iii). The proof of Lemma 4.2 is thus completed. □

Lemma 4.3. *Assume Setting 4.1 and let $r \in [0, 1/2)\backslash\{1/4\}$, $\delta \in (2r, 1)$. Then*

(i) it holds for every $f \in V_{r,2}$, $v \in C^\delta([0,1],\mathbb{R})$ that $fv \in V_{r,2}$ and

(ii) it holds that

$$\begin{aligned}&\sup\left\{\tfrac{\|fv\|_{V_{r,2}}}{\|f\|_{V_{r,2}}\|v\|_{C^\delta([0,1],\mathbb{R})}} : \substack{f\in V_{r,2}\backslash\{0\},\\ v\in C^\delta([0,1],\mathbb{R})\backslash\{0\}}\right\}\\ &\leq \tfrac{\sqrt{3}}{\sqrt{\delta-2r}}\left[\sup_{w\in V_{r,2}\backslash\{0\}}\tfrac{\|w\|_{V_{r,2}}}{\|w\|_{W^{2r,2}((0,1),\mathbb{R})}}\right]\left[\sup_{w\in V_{r,2}\backslash\{0\}}\tfrac{\|w\|_{W^{2r,2}((0,1),\mathbb{R})}}{\|w\|_{V_{r,2}}}\right] < \infty.\end{aligned}\tag{124}$$

Proof of Lemma 4.3. To prove items (i) and (ii) we distinguish between the case $r = 0$ and the case $r > 0$. We first prove items (i) and (ii) in the case $r = 0$. Observe that the fact that for every $w \in C([0,1],\mathbb{R})$ it holds that

$$\|w\|_{C^\delta([0,1],\mathbb{R})} = \sup_{x\in[0,1]}|w(x)| + \sup_{x,y\in[0,1],\, x\neq y}\left(\tfrac{|w(x)-w(y)|}{|x-y|^\delta}\right)\tag{125}$$

establishes items (i) and (ii) in the case $r = 0$. Next we prove items (i) and (ii) in the case $r > 0$. Note that item (i) in Lemma 4.2 and (23) in Jentzen & Röckner [JR12] imply that for every $f \in V_{r,2}$, $v \in C^\delta([0,1],\mathbb{R})$ it holds that $f \in \mathcal{W}^{2r,2}((0,1),\mathbb{R})$ and $fv \in \mathcal{W}^{2r,2}((0,1),\mathbb{R})$. Combining this and item (ii) in Lemma 4.2 establishes item (i) in the case $r > 0$. Moreover, observe that (23) in Jentzen & Röckner [JR12] assures that

$$\sup\left\{\tfrac{\|fv\|_{W^{2r,2}((0,1),\mathbb{R})}}{\|f\|_{W^{2r,2}((0,1),\mathbb{R})}\|v\|_{C^\delta([0,1],\mathbb{R})}} : \substack{f\in W^{2r,2}((0,1),\mathbb{R})\backslash\{0\},\\ v\in C^\delta([0,1],\mathbb{R})\backslash\{0\}}\right\} \leq \tfrac{\sqrt{3}}{\sqrt{\delta-2r}}.\tag{126}$$

Combining this and items (i)–(ii) in Lemma 4.2 establishes item (ii) in the case $r > 0$. The proof of Lemma 4.3 is thus completed. □

Lemma 4.4. *Assume Setting 4.1 and let* $r \in (0, 1/2)\backslash\{1/4\}$, $\delta \in (2r, 1)$. *Then*

$$\begin{aligned} &\sup\Big\{ \tfrac{\|fv\|_{V_{-r,2}}}{\|f\|_{V_{-r,2}} \|v\|_{C^\delta([0,1],\mathbb{R})}} : \substack{f \in L^2(\lambda;\mathbb{R})\backslash\{0\},\\ v \in C^\delta([0,1],\mathbb{R})\backslash\{0\}} \Big\} \\ &\leq \tfrac{\sqrt{3}}{\sqrt{\delta - 2r}} \left[\sup_{w \in V_{r,2}\backslash\{0\}} \tfrac{\|w\|_{V_{r,2}}}{\|w\|_{W^{2r,2}((0,1),\mathbb{R})}} \right] \left[\sup_{w \in V_{r,2}\backslash\{0\}} \tfrac{\|w\|_{W^{2r,2}((0,1),\mathbb{R})}}{\|w\|_{V_{r,2}}} \right] < \infty. \end{aligned} \tag{127}$$

Proof of Lemma 4.4. First, note that Lemma 4.3 proves that for every $u \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0,1], \mathbb{R})$ it holds that $v(-A_2)^{-r}u \in V_{r,2}$. The fact that for every $f \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0,1], \mathbb{R})$ it holds that $fv \in L^2(\lambda; \mathbb{R})$ and the self-adjointness of $L^2(\lambda; \mathbb{R}) \ni v \mapsto (-A_2)^{-r}v \in L^2(\lambda; \mathbb{R})$ therefore imply that for every $f \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0,1], \mathbb{R})$ it holds that

$$\begin{aligned} \|fv\|_{V_{-r,2}} &= \|(-A_2)^{-r}(fv)\|_{L^2(\lambda;\mathbb{R})} = \sup_{u \in L^2(\lambda;\mathbb{R})\backslash\{0\}} \tfrac{\langle (-A_2)^{-r}u, fv \rangle_{L^2(\lambda;\mathbb{R})}}{\|u\|_{L^2(\lambda;\mathbb{R})}} \\ &= \sup_{u \in L^2(\lambda;\mathbb{R})\backslash\{0\}} \tfrac{\langle (-A_2)^{r}(v(-A_2)^{-r}u), (-A_2)^{-r}f \rangle_{L^2(\lambda;\mathbb{R})}}{\|u\|_{L^2(\lambda;\mathbb{R})}} \\ &\leq \left[\sup_{u \in L^2(\lambda;\mathbb{R})\backslash\{0\}} \tfrac{\|v(-A_2)^{-r}u\|_{V_{r,2}}}{\|u\|_{L^2(\lambda;\mathbb{R})}} \right] \|f\|_{V_{-r,2}}. \end{aligned} \tag{128}$$

Combining this and Lemma 4.3 (with $v = v$, $f = (-A_2)^{-r}u$ in the notation of Lemma 4.3) establishes (127). The proof of Lemma 4.4 is thus completed. □

Lemma 4.5. *Assume Setting 4.1, for every* $\mathbb{R}$*-Hilbert space* $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ *let* $(L_2(H), \langle \cdot, \cdot \rangle_{L_2(H)}, \|\cdot\|_{L_2(H)})$ *be the* $\mathbb{R}$*-Hilbert space of Hilbert-Schmidt operators from* H *to* H, *let* $r \in (-1/4, 1/4)$, *and for every* $m \in V_{\max\{0,r\},2}$ *let* $M_m \colon D(M_m) \to L^2(\lambda; \mathbb{R})$ *be the linear operator which satisfies that* $D(M_m) = \{h \in L^2(\lambda; \mathbb{R}) \colon mh \in L^2(\lambda; \mathbb{R})\}$ *and* $[\forall\, h \in D(M_m) \colon M_m h = mh]$. *Then*

(i) it holds for every $m \in V_{\max\{0,r\},2}$, $h \in L^2(\lambda; \mathbb{R})$ *that* $(-A_2)^{-1/2}h \in D(M_m)$ *and* $M_m(-A_2)^{-1/2}h \in V_{\max\{0,r\},2}$ *and*

(ii) it holds that $\sup_{m \in V_{\max\{0,r\},2}\backslash\{0\}} \left[\frac{\|(-A_2)^r M_m (-A_2)^{-1/2}\|_{L_2(L^2(\lambda;\mathbb{R}))}}{\|m\|_{V_{r,2}}} \right] < \infty$.

Proof of Lemma 4.5. Throughout this proof let $\varrho = \max\{0, r\}$, $\varepsilon \in (0, \frac{1}{4} - \varrho)$, let $e_n \colon [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, be the functions which satisfy for every $n \in \mathbb{N}$, $x \in [0,1]$ that $e_n(x) = \sqrt{2}\sin(n\pi x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\gamma_n \colon \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be independent standard Gaussian random variables, and let $K_p \in [1, \infty]$, $p \in [1, \infty)$, be the extended real numbers which satisfy for every $p \in [1, \infty)$ that

$$K_p = \sup\left\{ \frac{\left(\mathbb{E}\left[\left|\sum_{k=1}^n \gamma_k x_k\right|^p\right]\right)^{1/p}}{\left(\mathbb{E}\left[\left|\sum_{k=1}^n \gamma_k x_k\right|^2\right]\right)^{1/2}} : n \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R}\backslash\{0\} \right\}. \tag{129}$$

Observe that the Khintchine inequalities imply that for every $p \in [1, \infty)$ it holds that $K_p < \infty$. Moreover, note that item (i) in Lemma 4.2 and the fractional Sobolev inequalities prove that for every $h \in L^2(\lambda; \mathbb{R})$, $\delta \in (0, 1/2)$ it holds that there exists a $v \in C^\delta([0,1], \mathbb{R})$ such that $(-A_2)^{-1/2} h = [v]_{\lambda, \mathcal{B}(\mathbb{R})}$. Lemma 4.3 and the fact that for every $f \in \mathcal{L}^2(\lambda; \mathbb{R})$, $v \in C^\delta([0,1], \mathbb{R})$, $v_1, v_2 \in [v]_{\lambda, \mathcal{B}(\mathbb{R})}$ it holds that

$$[v_1 f]_{\lambda, \mathcal{B}(\mathbb{R})} = [v_2 f]_{\lambda, \mathcal{B}(\mathbb{R})} \tag{130}$$

hence imply that for every $h \in L^2(\lambda; \mathbb{R})$, $m \in V_{\varrho,2}$ it holds that $M_m (-A_2)^{-1/2} h \in V_{\varrho,2}$. This establishes item (i). Furthermore, observe that for every $p \in (1, \infty)$ it holds that

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \left(\mathbb{E}\left[\left\| \sum_{k=1}^n (k\pi)^{-1} \gamma_k e_k \right\|^2_{C^{2\varrho+\varepsilon}([0,1],\mathbb{R})} \right] \right)^{1/2} \leq \sup\left\{ \frac{\|v\|_{C^{2\varrho+\varepsilon}([0,1],\mathbb{R})}}{\|[v]_{\lambda,\mathcal{B}(\mathbb{R})}\|_{V_{\varrho+\varepsilon,p}}} : \begin{smallmatrix} v \in C^2([0,1],\mathbb{R})\backslash\{0\}, \\ v(0)=v(1)=0 \end{smallmatrix} \right\} \\ &\cdot \sup_{n \in \mathbb{N}} \left(\mathbb{E}\left[\left\| \sum_{k=1}^n (k\pi)^{-1+2(\varrho+\varepsilon)} \gamma_k e_k \right\|^2_{\mathcal{L}^p(\lambda;\mathbb{R})} \right] \right)^{1/2}. \end{aligned} \tag{131}$$

In addition, note that item (iii) in Lemma 4.2 and the fractional Sobolev inequalities demonstrate that for every $p \in (\varepsilon^{-1}, \infty)$ it holds that

$$\sup\left\{ \frac{\|v\|_{C^{2\varrho+\varepsilon}([0,1],\mathbb{R})}}{\|[v]_{\lambda,\mathcal{B}(\mathbb{R})}\|_{V_{\varrho+\varepsilon,p}}} : \begin{smallmatrix} v \in C^2([0,1],\mathbb{R})\backslash\{0\}, \\ v(0)=v(1)=0 \end{smallmatrix} \right\} < \infty. \tag{132}$$

Moreover, note that Hölder's inequality, Fubini's theorem, and (129) imply that for every $p \in (\varepsilon^{-1}, \infty)$ it holds that

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \left(\mathbb{E}\left[\left\| \sum_{k=1}^n (k\pi)^{-1+2(\varrho+\varepsilon)} \gamma_k e_k \right\|^2_{\mathcal{L}^p(\lambda;\mathbb{R})} \right] \right)^{1/2} \\ &= \sup_{n \in \mathbb{N}} \left(\mathbb{E}\left[\left(\int_0^1 \left| \sum_{k=1}^n (k\pi)^{-1+2(\varrho+\varepsilon)} \gamma_k e_k(x) \right|^p dx \right)^{2/p} \right] \right)^{1/2} \\ &\leq \sup_{n \in \mathbb{N}} \left(\mathbb{E}\left[\int_0^1 \left| \sum_{k=1}^n (k\pi)^{-1+2(\varrho+\varepsilon)} \gamma_k e_k(x) \right|^p dx \right] \right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{n\in\mathbb{N}} \left(\int_0^1 \mathbb{E}\Big[\Big| \sum_{k=1}^{n} (k\pi)^{-1+2(\varrho+\varepsilon)} \gamma_k e_k(x) \Big|^p \Big] \, dx \right)^{1/p} \qquad (133) \\
&\leq K_p \sup_{n\in\mathbb{N}} \left(\int_0^1 \left(\mathbb{E}\Big[\Big| \sum_{k=1}^{n} (k\pi)^{-1+2(\varrho+\varepsilon)} \gamma_k e_k(x) \Big|^2 \Big] \right)^{p/2} dx \right)^{1/p} \\
&= K_p \sup_{n\in\mathbb{N}} \left(\int_0^1 \left(\sum_{k=1}^{n} (k\pi)^{-2+4(\varrho+\varepsilon)} |e_k(x)|^2 \right)^{p/2} dx \right)^{1/p} \\
&\leq \sqrt{2} K_p \sup_{n\in\mathbb{N}} \left(\sum_{k=1}^{n} (k\pi)^{-2+4(\varrho+\varepsilon)} \right)^{1/2} = \sqrt{2} K_p \left(\sum_{k=1}^{\infty} (k\pi)^{-2+4(\varrho+\varepsilon)} \right)^{1/2} < \infty.
\end{aligned}
$$

Next observe that for every $m \in V_{\varrho,2}$ it holds that

$$
\begin{aligned}
&\left\| (-A_2)^r M_m (-A_2)^{-1/2} \right\|_{L_2(L^2(\lambda;\mathbb{R}))}^2 = \sup_{n\in\mathbb{N}} \left(\sum_{k=1}^{n} \left\| M_m (-A_2)^{-1/2} [e_k]_{\lambda,\mathcal{B}(\mathbb{R})} \right\|_{V_{r,2}}^2 \right) \\
&= \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left\| \sum_{k=1}^{n} \gamma_k M_m (-A_2)^{-1/2} [e_k]_{\lambda,\mathcal{B}(\mathbb{R})} \right\|_{V_{r,2}}^2 \right] \qquad (134) \\
&= \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left\| m \sum_{k=1}^{n} (k\pi)^{-1} \gamma_k e_k \right\|_{V_{r,2}}^2 \right].
\end{aligned}
$$

Moreover, note that Lemma 4.3, Lemma 4.4, (131), (132), and (133) imply that

$$
\begin{aligned}
&\sup_{m\in V_{\varrho,2}\setminus\{0\}} \left\{ \frac{\sup_{n\in\mathbb{N}} \left(\mathbb{E}\left[\left\| m \sum_{k=1}^{n} (k\pi)^{-1} \gamma_k e_k \right\|_{V_{r,2}}^2 \right] \right)^{1/2}}{\|m\|_{V_{r,2}}} \right\} \\
&\leq \sqrt{\tfrac{3}{\varepsilon}} \left[\sup_{w\in V_{\varrho,2}\setminus\{0\}} \frac{\|w\|_{V_{\varrho,2}}}{\|w\|_{W^{2\varrho,2}((0,1),\mathbb{R})}} \right] \left[\sup_{w\in V_{\varrho,2}\setminus\{0\}} \frac{\|w\|_{W^{2\varrho,2}((0,1),\mathbb{R})}}{\|w\|_{V_{\varrho,2}}} \right] \qquad (135) \\
&\quad \cdot \sup_{n\in\mathbb{N}} \left(\mathbb{E}\left[\left\| \sum_{k=1}^{n} (k\pi)^{-1} \gamma_k e_k \right\|_{C^{2\varrho+\varepsilon}([0,1],\mathbb{R})}^2 \right] \right)^{1/2} < \infty.
\end{aligned}
$$

Combining this and (134) establishes item (ii). The proof of Lemma 4.5 is thus completed. □

4.3 The hyperbolic Anderson model

Corollary 4.6. *For every pair of* $\mathbb{R}$*-Hilbert spaces* $(V, \langle\cdot,\cdot\rangle_V, \|\cdot\|_V)$ *and* $(W, \langle\cdot,\cdot\rangle_W, \|\cdot\|_W)$ *let* $(L_2(V,W), \langle\cdot,\cdot\rangle_{L_2(V,W)}, \|\cdot\|_{L_2(V,W)})$ *be the* $\mathbb{R}$*-Hilbert space of Hilbert-Schmidt* *operators from V to W, for every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space (S, Σ), every set $\mathcal{O}$, and every function $f\colon \mathcal{O} \to S$ let $[f]_{\mu,\Sigma}$ be the set given by*

$$[f]_{\mu,\Sigma} = \left\{ g\colon \Omega \to S\colon \left[\begin{smallmatrix} [\exists\, A\in\mathcal{F}\colon\, (\mu(A)=0 \text{ and } \{\omega\in\Omega\cap\mathcal{O}\colon\, f(\omega)\neq g(\omega)\}\subseteq A)] \\ \text{and } [\forall\, A\in\Sigma\colon\, g^{-1}(A)\in\mathcal{F}] \end{smallmatrix}\right] \right\}, \tag{136}$$

let $T, \vartheta \in (0,\infty)$, b_0, $b_1 \in \mathbb{R}$, let $\lambda\colon \mathcal{B}((0,1)) \to [0,1]$ be the Lebesgue-Borel measure on $(0,1)$, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be the $\mathbb{R}$-Hilbert space given by $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(\lambda;\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda;\mathbb{R})}, \|\cdot\|_{L^2(\lambda;\mathbb{R})})$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t\in[0,T]}$ be an id_H-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, let $(e_n)_{n\in\mathbb{N}} \subseteq H$ satisfy for every $n \in \mathbb{N}$ that $e_n = [(\sqrt{2}\sin(n\pi x))_{x\in(0,1)}]_{\lambda,\mathcal{B}(\mathbb{R})}$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies that $D(A) = \{h \in H\colon \sum_{n=1}^{\infty} |(n\pi)^2\langle e_n, h\rangle_H|^2 < \infty\}$ and $[\forall\, h \in D(A)\colon Ah = \sum_{n=1}^{\infty} -(n\pi)^2\langle e_n, h\rangle_H e_n]$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $(\mathbf{H}_r, \langle\cdot,\cdot\rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, be the family of $\mathbb{R}$-Hilbert spaces which satisfies for every $r \in \mathbb{R}$ that $(\mathbf{H}_r, \langle\cdot,\cdot\rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = \big(H_{r/2}\times H_{r/2-1/2}, \langle\cdot,\cdot\rangle_{H_{r/2}\times H_{r/2-1/2}}, \|\cdot\|_{H_{r/2}\times H_{r/2-1/2}}\big)$, let $\mathbf{A}\colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ be the linear operator which satisfies that $D(\mathbf{A}) = \mathbf{H}_1$ and $[\forall\, (v,w) \in \mathbf{H}_1\colon \mathbf{A}(v,w) = (w, \vartheta A v)]$, let $\varphi \in C^4(\mathbf{H}_0, \mathbb{R})$ satisfy that $\sup_{k\in\{1,2,3,4\},\, x\in\mathbf{H}_0} \|\varphi^{(k)}(x)\|_{L^{(k)}(\mathbf{H}_0,\mathbb{R})} < \infty$, let $\xi \in \mathcal{L}^6(\mathbb{P}|_{\mathbb{F}_0}; \mathbf{H}_{1/2})$, let $B\colon H \to L_2(H, H_{-1/2})$ be the function which satisfies for every $v \in \mathcal{L}^2(\lambda;\mathbb{R})$, $u \in C([0,1],\mathbb{R})$ that $B([v]_{\lambda,\mathcal{B}(\mathbb{R})})[u]_{\lambda,\mathcal{B}(\mathbb{R})} = [((b_0{+}b_1 v(x))u(x))_{x\in(0,1)}]_{\lambda,\mathcal{B}(\mathbb{R})}$, let $\mathbf{B}\colon \mathbf{H}_0 \to L_2(H, \mathbf{H}_0)$ be the function which satisfies for every $(v,w) \in \mathbf{H}_0$, $u \in H$ that $\mathbf{B}(v,w)u = \big(0, B(v)u\big)$, let $X\colon [0,T]\times\Omega \to \mathbf{H}_0$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-predictable stochastic process which satisfies for every $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s\|_{\mathbf{H}_0}^2\big] < \infty$ and

$$[X_t]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} = \big[e^{t\mathbf{A}}\xi\big]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}(X_s)\,\mathrm{d}W_s, \tag{137}$$

let $\lfloor\cdot\rfloor_h\colon [0,\infty) \to \mathbb{R}$, $h \in (0,T]$, be the functions which satisfy for every $h \in (0,T]$, $x \in [0,\infty)$ that

$$\lfloor x\rfloor_h = \max(\{0, h, 2h, 3h, \ldots\} \cap [0,x]), \tag{138}$$

and let $Y^h\colon [0,T]\times\Omega \to \mathbf{H}_0$, $h \in (0,T]$, be $(\mathbb{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy for every $h \in (0,T]$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}\big[\|Y_s^h\|_{\mathbf{H}_0}^2\big] < \infty$ and

$$[Y_t^h]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} = \big[e^{t\mathbf{A}}\xi\big]_{\mathbb{P},\mathcal{B}(\mathbf{H}_0)} + \int_0^t e^{(t-\lfloor s\rfloor_h)\mathbf{A}}\mathbf{B}\big(Y_{\lfloor s\rfloor_h}^h\big)\,\mathrm{d}W_s. \tag{139}$$

Then it holds for every $\varepsilon \in (0,\infty)$ that

$$\sup_{h\in(0,T]}\Big(h^{\varepsilon-1}\,\big|\mathbb{E}\big[\varphi\big(X_T\big)\big] - \mathbb{E}\big[\varphi\big(Y_T^h\big)\big]\big|\Big) < \infty. \tag{140}$$

Proof of Corollary 4.6. Throughout this proof for every $\mathbb{R}$-Hilbert space $(V, \langle\cdot,\cdot\rangle_V, \|\cdot\|_V)$ let $(L_2(V), \langle\cdot,\cdot\rangle_{L_2(V)}, \|\cdot\|_{L_2(V)})$ be the $\mathbb{R}$-Hilbert space of Hilbert-Schmidt operators from V to V, for every pair of $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ let $(\operatorname{Lip}(V,W), \|\cdot\|_{\operatorname{Lip}(V,W)})$ be the $\mathbb{R}$-Banach space of Lipschitz continuous mappings from V to W, for every $\ell \in \mathbb{N}$ and every pair of $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ let $(C^\ell_{\mathrm{b}}(V,W), \|\cdot\|_{C^\ell_{\mathrm{b}}(V,W)})$ be the $\mathbb{R}$-Banach space of ℓ-times continuously Fréchet differentiable functions from V to W with globally bounded derivatives, for every $v \in H$ let $M_v \colon D(M_v) \subseteq H \to H$ be the linear operator which satisfies that $D(M_v) = \{h \in H \colon vh \in H\}$ and $[\forall\, h \in D(M_v) \colon M_v h = vh]$, and let $\varepsilon \in (0, 2/3]$. Observe that the fact that for every $\rho \in [0, 1/4)$ it holds that $B \in \operatorname{Lip}(H, L_2(H, H_{\rho-1/2}))$, the fact that for every $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_0$ it holds that $\|(v_1,v_2)-(w_1,w_2)\|_{\mathbf{H}_0} = (\|v_1-w_1\|^2_H + \|v_2-w_2\|^2_{H_{-1/2}})^{1/2}$, the fact that for every $\rho \in [0, 1/4)$, $u \in H$ it holds that $\|\mathbf{B}(0,0)u\|_{\mathbf{H}_{2\rho}} = \|(0, B(0)u)\|_{\mathbf{H}_{2\rho}} = \|B(0)u\|_{H_{\rho-1/2}}$, and the fact that for every $\rho \in [0, 1/4)$, $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_0$, $u \in H$ it holds that

$$\begin{aligned} \|(\mathbf{B}(v_1,v_2) - \mathbf{B}(w_1,w_2))u\|_{\mathbf{H}_{2\rho}} &= \|(0, (B(v_1) - B(w_1))u)\|_{\mathbf{H}_{2\rho}} \\ &= \|(B(v_1) - B(w_1))u\|_{H_{\rho-1/2}} \end{aligned} \tag{141}$$

imply that for every $\rho \in [0, 1/4)$ it holds that $\mathbf{B} \in \operatorname{Lip}(\mathbf{H}_0, L_2(H, \mathbf{H}_{2\rho}))$. Hence, we obtain that $\mathbf{B} \in \operatorname{Lip}(\mathbf{H}_0, L_2(H, \mathbf{H}_{1/2-\varepsilon/4}))$ and

$$\mathbf{B}|_{\mathbf{H}_{1/2-\varepsilon/4}} \in \operatorname{Lip}(\mathbf{H}_{1/2-\varepsilon/4}, L_2(H, \mathbf{H}_{1/2-\varepsilon/4})). \tag{142}$$

Moreover, note that (3.77)–(3.78) in de Naurois et al. [JdNJW] and Hölder's inequality ensure that for every $\rho \in (0, 1/4)$, $v, w \in H_\rho$, $u \in H_1$ it holds that $B(v), B(w) \in L(H, H_{\rho-1/4})$ and

$$\begin{aligned} &\|(B(v) - B(w))u\|_{H_{\rho-1/4}} \\ &= \sup_{\psi \in H_1 \setminus \{0\}} \frac{|\langle \psi, (B(v) - B(w))u \rangle_H|}{\|\psi\|_{H_{(1/4)-\rho}}} = \sup_{\psi \in H_1 \setminus \{0\}} \frac{\|\psi\, b_1 (v-w) u\|_{L^1(\lambda;\mathbb{R})}}{\|\psi\|_{H_{(1/4)-\rho}}} \\ &\leq \sup_{\psi \in H_1 \setminus \{0\}} \frac{|b_1| \|\psi\|_{L^{1/(2\rho)}(\lambda;\mathbb{R})} \|v-w\|_{L^{2/(1-4\rho)}(\lambda;\mathbb{R})} \|u\|_{L^2(\lambda;\mathbb{R})}}{\|\psi\|_{H_{(1/4)-\rho}}} \\ &\leq |b_1| \left[\sup_{\psi \in H_1 \setminus \{0\}} \frac{\|\psi\|_{L^{1/(2\rho)}(\lambda;\mathbb{R})}}{\|\psi\|_{H_{(1/4)-\rho}}} \right] \left[\sup_{\zeta \in H_\rho \setminus \{0\}} \frac{\|\zeta\|_{L^{2/(1-4\rho)}(\lambda;\mathbb{R})}}{\|\zeta\|_{H_\rho}} \right] \|v-w\|_{H_\rho} \|u\|_H \\ &< \infty. \end{aligned} \tag{143}$$

This assures that for every $\rho \in (0, 1/4)$ it holds that $B|_{H_\rho} \in \operatorname{Lip}(H_\rho, L(H, H_{\rho-1/4}))$. The fact that for every $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_{2\rho}$ it holds that $\|(v_1,v_2)-(w_1,w_2)\|_{\mathbf{H}_{2\rho}} =$

$(\|v_1 - w_1\|_{H_\rho}^2 + \|v_2 - w_2\|_{H_{\rho-1/2}}^2)^{1/2}$, the fact that for every $\rho \in (0, 1/4)$, $u \in H$ it holds that $\|\mathbf{B}(0,0)u\|_{\mathbf{H}_{2\rho+1/2}} = \|(0, B(0)u)\|_{\mathbf{H}_{2\rho+1/2}} = \|B(0)u\|_{H_{\rho-1/4}}$, and the fact that for every $\rho \in (0, 1/4)$, $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_{2\rho}$, $u \in H$ it holds that

$$\begin{aligned} \|(\mathbf{B}(v_1, v_2) - \mathbf{B}(w_1, w_2))u\|_{\mathbf{H}_{2\rho+1/2}} &= \|(0, (B(v_1) - B(w_1))u)\|_{\mathbf{H}_{2\rho+1/2}} \\ &= \|(B(v_1) - B(w_1))u\|_{H_{\rho-1/4}} \end{aligned} \tag{144}$$

hence imply that for every $\rho \in (0, 1/4)$ it holds that $\mathbf{B}|_{\mathbf{H}_{2\rho}} \in \operatorname{Lip}(\mathbf{H}_{2\rho}, L(H, \mathbf{H}_{2\rho+1/2}))$. Therefore, we obtain that

$$\mathbf{B}|_{\mathbf{H}_{1/2-\varepsilon/4}} \in \operatorname{Lip}(\mathbf{H}_{1/2-\varepsilon/4}, L(H, \mathbf{H}_{1-\varepsilon/4})). \tag{145}$$

Furthermore, observe that for every $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_0$, $u \in H_1$ it holds that

$$\begin{aligned} &\mathbf{B}(v_1 + w_1, v_2 + w_2)u = (0, B(v_1 + w_1)u) = (0, (b_0 + b_1(v_1 + w_1))u) \\ &= (0, (b_0 + b_1 v_1)u) + (0, b_1 w_1 u) = (0, B(v_1)u) + (0, b_1 w_1 u) \\ &= \mathbf{B}(v_1, v_2)u + (0, b_1 w_1 u). \end{aligned} \tag{146}$$

Combining this, the fact that for every $(v_1, v_2) \in \mathbf{H}_0$ it holds that $\mathbf{B}(v_1, v_2) \in L_2(H, \mathbf{H}_0)$, and the fact that H_1 is a dense subset of H implies that for every $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_0$, $u \in H_1$ it holds that $\mathbf{B} \in C^1(\mathbf{H}_0, L_2(H, \mathbf{H}_0))$ and

$$\big[\big(B^{(1)}(v_1, v_2)\big)(w_1, w_2)\big]u = (0, b_1 w_1 u). \tag{147}$$

Hence, we obtain that for every $k \in \mathbb{N}$ it holds that

$$\mathbf{B} \in C_{\mathrm{b}}^k(\mathbf{H}_0, L_2(H, \mathbf{H}_0)). \tag{148}$$

Next observe that for every $(v_1, v_2) \in \mathbf{H}_0$ it holds that

$$\begin{aligned} &\sum_{n\in\mathbb{N}} (n\pi)^{1-\varepsilon} \|\mathbf{B}(v_1, v_2)e_n\|_{\mathbf{H}_0}^2 = \sum_{n\in\mathbb{N}} (n\pi)^{1-\varepsilon} \big\|(-A)^{-1/2}((b_0 + b_1 v_1)e_n)\big\|_H^2 \\ &= \sum_{n\in\mathbb{N}} (n\pi)^{1-\varepsilon} \big\|b_0(-A)^{-1/2}e_n + b_1(-A)^{-1/2}M_{v_1}e_n\big\|_H^2 \\ &= \sum_{n\in\mathbb{N}} \big\|b_0(-A)^{-(1+\varepsilon)/4}e_n + b_1(-A)^{-1/2}M_{v_1}(-A)^{(1-\varepsilon)/4}e_n\big\|_H^2 \\ &\leq 2|b_0|^2 \big\|(-A)^{-(1+\varepsilon)/4}\big\|_{L_2(H)}^2 + 2|b_1|^2 \left[\sum_{n\in\mathbb{N}} \big\|(-A)^{-1/2}M_{v_1}(-A)^{(1-\varepsilon)/4}e_n\big\|_H^2\right]. \end{aligned} \tag{149}$$

Moreover, note that for every $(v_1, v_2), (w_1, w_2) \in \mathbf{H}_0$ it holds that

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} (n\pi)^{\varepsilon-1} \|[\mathbf{B}(v_1, v_2) - \mathbf{B}(w_1, w_2)]e_n\|^2_{\mathbf{H}_0} \\
&= \sum_{n\in\mathbb{N}} (n\pi)^{\varepsilon-1} \left\|(-A)^{-1/2}(b_1(v_1 - w_1)e_n)\right\|^2_H \\
&= |b_1|^2 \left[\sum_{n\in\mathbb{N}} \left\|(-A)^{-1/2} M_{(v_1-w_1)}(-A)^{(\varepsilon-1)/4} e_n\right\|^2_H\right].
\end{aligned} \tag{150}$$

In addition, observe that item (i) in Lemma 4.5 ensures that for every $r \in (-1/4, 1/4)$, $v \in H_{\max\{0,r\}}$, $n \in \mathbb{N}$ it holds that $M_v(-A)^{-1/2}e_n \in H_{\max\{0,r\}}$. This and the fact that for every $v \in H$ it holds that $M_v\colon D(M_v) \subseteq H \to H$ is a symmetric linear operator imply that for every $r \in (-1/4, 1/4)$, $v \in H_{\max\{0,r\}}$ it holds that

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \left\|(-A)^{-1/2} M_v (-A)^r e_n\right\|^2_H = \sum_{m,n\in\mathbb{N}} \left|\left\langle (-A)^{-1/2} M_v(-A)^r e_n, e_m\right\rangle_H\right|^2 \\
&= \sum_{m,n\in\mathbb{N}} \left|\left\langle e_n, (-A)^r M_v (-A)^{-1/2} e_m\right\rangle_H\right|^2 = \sum_{m\in\mathbb{N}} \left\|(-A)^r M_v (-A)^{-1/2} e_m\right\|^2_H.
\end{aligned} \tag{151}$$

Lemma 4.5, (149), (150), and the fact that for every $r \in (1/4, \infty)$ it holds that $\|A^{-r}\|_{L_2(H)} < \infty$ therefore ensure that

$$\begin{aligned}
&\sup\left\{ \tfrac{\sum_{n\in\mathbb{N}} (n\pi)^{1-\varepsilon} \|\mathbf{B}(v_1,v_2)e_n\|^2_{\mathbf{H}_0}}{\max\{1, \|(v_1,v_2)\|^2_{\mathbf{H}_{(1-\varepsilon)/2}}\}} : (v_1,v_2)\in\mathbf{H}_{(1-\varepsilon)/2} \right\} \\
&\leq \sup\left\{ \tfrac{2|b_0|^2\|(-A)^{-(1+\varepsilon)/4}\|^2_{L_2(H)} + 2|b_1|^2 \|(-A)^{(1-\varepsilon)/4} M_{v_1}(-A)^{-1/2}\|^2_{L_2(H)}}{\max\{1, \|(v_1,v_2)\|^2_{\mathbf{H}_{(1-\varepsilon)/2}}\}} : (v_1,v_2)\in\mathbf{H}_{(1-\varepsilon)/2} \right\} \\
&< \infty
\end{aligned} \tag{152}$$

and

$$\begin{aligned}
&\sup\left\{ \tfrac{\sum_{n\in\mathbb{N}} (n\pi)^{\varepsilon-1} \|[\mathbf{B}(v_1,v_2) - \mathbf{B}(w_1,w_2)]e_n\|^2_{\mathbf{H}_0}}{\|(v_1,v_2)-(w_1,w_2)\|^2_{\mathbf{H}_{(\varepsilon-1)/2}}} : \begin{array}{c}(v_1,v_2),(w_1,w_2)\in\mathbf{H}_{(1-\varepsilon)/2},\\ (v_1,v_2)\neq(w_1,w_2)\end{array} \right\} \\
&= \sup\left\{ \tfrac{|b_1|^2 \|(-A)^{(\varepsilon-1)/4} M_{(v_1-w_1)}(-A)^{-1/2}\|^2_{L_2(H)}}{\|(v_1,v_2)-(w_1,w_2)\|^2_{\mathbf{H}_{(\varepsilon-1)/2}}} : \begin{array}{c}(v_1,v_2),(w_1,w_2)\in\mathbf{H}_{(1-\varepsilon)/2},\\ (v_1,v_2)\neq(w_1,w_2)\end{array} \right\} < \infty.
\end{aligned} \tag{153}$$

Combining this, (142), (145), (148), and Corollary 3.11 (with $U = H$, $\mathbb{U} = \{e_n\}_{n\in\mathbb{N}}$, $T = T$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $\gamma = 1 - \varepsilon/4$, $\beta = 1/2 + \varepsilon/4$, $\rho = 1/2 - \varepsilon/4$, $H = H$, $\mathbb{H} = \{e_n\}_{n\in\mathbb{N}}$, $[\forall\, n \in \mathbb{N}\colon \lambda_{e_n} = -\vartheta(n\pi)^2]$, $A = \vartheta A$, $\big[\forall\, r \in \mathbb{R} : \|\cdot\|_{H_r} =$

$\vartheta^r \|\cdot\|_{H_r}\big]$, $\mathbf{A} = \mathbf{A}$, $\varphi = \varphi$, $\xi = \xi$, $F = 0$, $B = \mathbf{B}$, $\big[\forall\, n \in \mathbb{N}\colon \mu_{e_n} = (n\pi)^{(1-\varepsilon)/2}\big]$, $\mathfrak{c}^2 = \max\{\vartheta^{(\varepsilon-3)/2}, \vartheta^{(\varepsilon-1)/2}\} \sup\Big\{ \frac{\sum_{n\in\mathbb{N}}(n\pi)^{1-\varepsilon}\|\mathbf{B}(v_1,v_2)e_n\|^2_{\mathbf{H}_0}}{\max\{1,\|(v_1,v_2)\|^2_{\mathbf{H}_{(1-\varepsilon)/2}}\}} \colon (v_1,v_2)\in\mathbf{H}_{(1-\varepsilon)/2} \Big\}$, $\mathfrak{l}^2 = \max\{\vartheta^{-(1+\varepsilon)/2}, \vartheta^{(1-\varepsilon)/2}\} \sup\Big\{ \frac{\sum_{n\in\mathbb{N}}(n\pi)^{\varepsilon-1}\|[\mathbf{B}(v_1,v_2)-\mathbf{B}(w_1,w_2)]e_n\|^2_{\mathbf{H}_0}}{\|(v_1,v_2)-(w_1,w_2)\|^2_{\mathbf{H}_{(\varepsilon-1)/2}}} \colon \begin{array}{c}(v_1,v_2),(w_1,w_2)\in\mathbf{H}_{(1-\varepsilon)/2},\\ (v_1,v_2)\neq(w_1,w_2)\end{array} \Big\}$, $\mathfrak{m} = \max\{\vartheta^{-1/2}, 1\}\|\mathbf{B}\|_{C_b^4(\mathbf{H}_0, L_2(H,\mathbf{H}_0))} + 1$, $\big[\forall\, h \in (0,T]\colon Y^{h,\mathbb{H}} = Y^h\big]$, $Y^{0,\mathbb{H}} = X$ in the notation of Corollary 3.11) establishes (140). The proof of Corollary 4.6 is thus completed. □

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