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DOMAIN UNCERTAINTY QUANTIFICATION IN COMPUTATIONAL ELECTROMAGNETICS

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ABSTRACT. We study the numerical approximation of time-harmonic, electromagnetic fields inside a lossy cavity of uncertain geometry. Key assumptions are a possibly high-dimensional parametrization of the uncertain geometry along with a suitable transformation to a fixed, nominal domain. This uncertainty parametrization results in families of countably-parametric, Maxwell-like cavity problems that are posed in a single domain, with inhomogeneous coefficients that possess finite, possibly low spatial regularity, but exhibit holomorphic parametric dependence in the differential operator. Our computational scheme is composed of a sparse-grid interpolation in the high-dimensional parameter domain and an $\boldsymbol{H}(\mathbf{curl})$-conforming edge element discretization of the parametric problem in the nominal domain. As a stepping-stone in the analysis, we derive a novel Strang-type lemma for Maxwell-like problems in the nominal domain which is of independent interest. Moreover, we accommodate arbitrary small Sobolev regularity of the electric field and also cover uncertain isotropic constitutive or material laws. The shape holomorphy and edge-element consistency error analysis for the nominal problem are shown to imply convergence rates for Multi-level Monte-Carlo and for Quasi-Monte Carlo integration in UQ for Computational Electromagnetics. They also imply expression rate estimates for deep ReLU networks of shape-to-solution maps in this setting. Finally, our computational experiments confirm the presented theoretical results.

1. Introduction and Summary of Main Results

The efficient numerical simulation of PDEs with uncertain input data and/or uncertain solutions has received considerable attention in recent years, giving rise to the discipline of Uncertainty Quantification (UQ). In the present work, we consider the general setting of domain UQ in computational electromagnetics, through the case of efficient numerical computation for quantities of interest (QoIs) arising from electromagnetic (EM) fields in a lossy cavity of uncertain shape with large amplitude shape variations. This setting is of interest when studying metallic metamaterials [40, 1], deep gratings for thermovoltaic cells [6, 32], and even for non-destructive testing applications [39]. Due to stringent performance requirements in all these situations, robust design calls for efficient numerical tools capable of assessing quantitatively the effects of shape randomness in the QoIs.

In our previous work [21], we developed a fast numerical scheme to quantify computationally *domain uncertainty* in the exterior scattering of time-harmonic electromagnetic (EM) waves at obstacles with uncertain (resp. unknown) shape. The approach in [21] was based on the assumption of an unknown and random, small amplitude deviation from a known, nominal shape, leading to the so-called *first-order, second-moment* (FoSM) technique [34, 20]. The resulting deterministic second order statistical moment –or two-point correlation function– of the random response map is defined on the tensor product of the space with itself, i.e. it is deterministic, but high-dimensional. The curse of dimensionality was overcome in [21] by performing a *sparse tensor product Galerkin*

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boundary element discretization of the corresponding integral equation formulation of the scattering problem on the boundary of the nominal scatterer. Small amplitude domain perturbations appear in several applications, so that the numerical approach proposed in [21] is correspondingly applicable [31]. In several cases, however, large domain uncertainty are encountered and the FoSM approach is bound to incur large numerical errors. Hence, the FoSM linearization must be replaced by formulations which do not depend on smallness assumptions of the deviation from the nominal geometry.

In [22], we investigated large amplitude domain perturbations. We proved that the response maps of time-harmonic EM fields scattered by either perfectly conducting or dielectric bounded obstacles depend holomorphically on the shape deformation. This implied, in particular, that parametric responses obtained from an affine parametrization of the shape-variation are holomorphic-parametric with controlled sizes of domains of holomorphy. As was recently proved, this leads to dimension-independent rates of convergence of so-called *stochastic collocation* or *sparse grid* schemes (*cf.* [10] and references therein) and corresponding rates of convergence of Smolyak quadratures (*cf.* [37, 35]), and Quasi-Monte Carlo (QMC for short) quadratures [12] and inverse computational uncertainty quantification [11]. Sparsity results analogous to the ones in [22] have recently been established for other types of forward PDE problems ranging from elastic wave propagation [26] to general elliptic operators [4, 19].

In the present paper, we prove that the shape holomorphy of the response map for the lossy EM cavity problem implies dimension-independent convergence rates of several types of high-dimensional approximation methods. We discuss these methods and state several quadrature and collocation error bounds for the parametric forward models in Section 5, for independent interest and for comparison purposes also for Monte Carlo (MC) sampling.

More specifically, we develop the complete numerical analysis of edge finite element (FE) discretizations for parametric families of solutions of Maxwell equations as arise in domain UQ in computational EM. We pull back domain realizations to a fixed, nominal domain $\widehat{D}$. Even when the domain is occupied by homogeneous media, this pullback entails the numerical solution of a Maxwell-like PDE with non-homogeneous coefficients introduced by the domain-pullback, with possibly low regularity depending on the smoothness of the domain transformations. This, in turn, necessitates the analysis of the impact of numerical integration errors on the edge-element convergence in the nominal domain, via a Strang-type lemma argument which is of interest in its own right. Due to the use of numerical quadrature, the quadrature error analysis in the Strang-type perturbation argument requires the domain transformations T to belong to C^1 piecewise, in order for point evaluations of the Jacobian to be well defined. However, we point out that all function spaces C^1 and C^0 in Section 4.2.2 could be replaced by the larger spaces $W^{1,\infty}$ and L^∞. This gives slightly more general results (*cf.* [22]).

The main benefit of pulling back onto the nominal domain consists in *obviating the need for remeshing the computational domain: indeed, the presently proposed numerical scheme will allow the use of one common FE mesh in $\widehat{D}$ for all domain realizations* that arise in various computational domain UQ algorithms (MC, stochastic collocation, QMC integration, MCMC, etc.). Let us remark that a corresponding approach in the widely used integral-equation based numerical methods seems infeasible as domain mappings render homogeneous differential operators inhomogeneous thereby complicating writing fundamental solutions in nominal coordinates.

The outline of this article is as follows. In Section 2, we introduce the model problem of time-harmonic Maxwell equations in a lossy cavity. In Section 3, we introduce an edge element discretization for the Maxwell-like problem resulting from the pullback from the physical domain into the (fixed) nominal domain and establish a Strang-type lemma bound for the impact of quadrature error on the FE error. This result, which seems to be novel and of independent interest, is crucial in the error analysis of the proposed scheme: being formulated in the nominal domain, it avoids remeshing multiple instances of the cavity shape in the course of parametric sampling. Low spatial regularity of the domain maps is admissible in the Strang consistency analysis.

Section 4 then states and proves the second principal result of this paper: the holomorphic dependence of the numerical solution on the cavity shape. Based on the results in Sections 4 and 3, in Section 5 we discuss convergence rates for several sampling methods in the parameter space, which access the uncertain parametric inputs in a nonintrusive fashion. These are mainly recapitulated from abstract results in the references [35, 37, 36], but appear to be new in the present context.

Section 6 contains a set of numerical experiments on a three-dimensional geometry which confirm the theoretical results. Concluding remarks are provided in Section 7 along with outlines of several further directions which directly follow from the present analysis: Bayesian shape inversion, and the analysis and implementation of corresponding *multilevel* algorithms. Their detailed development will be presented in forthcoming work.

2. Lossy cavity electromagnetic problem

We briefly set the notation used throughout, introduce domain transformations and state the mathematical framework for the Maxwell cavity problem.

2.1. Notation.

Let $d \in \{1, 2, 3\}$ and $O \subseteq \mathbb{R}^d$ denote generically an open, bounded Lipschitz domain. For $m \in \mathbb{N}_0$, $C^m(O; \mathbb{K})$ denotes the space of continuous, functions from O to a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ that are m-times continuously differentiable in $\overline{O}$. When $\mathbb{K} = \mathbb{R}$, we write $C^m(O)$. For the space of infinitely continuously differentiable functions in $\overline{O}$, we write $C^\infty(O)$. By $C_0^m(O)$ we denote the space of compactly supported C^m-functions in $\overline{O}$. We write $L^p(O)$ for the (Banach) space of p-integrable functions on O. Boldface symbols for functional spaces represent vector-valued counterparts, e.g., $\boldsymbol{L}^2(O)$ is the space of vector-valued functions with d components in $L^2(O)$. For the L^2-inner product on O we write $(\cdot, \cdot)_O$. If it will not cause any confusion, the subscript indicating the underlying domain is omitted.

Dual spaces are defined in standard fashion with duality products denoted by angular brackets $\langle \cdot, \cdot \rangle_O$. Moreover, for any real or complex Banach space X, we write X^* for the space of all bounded antilinear mappings from X to $\mathbb{C}$. The space of bounded linear mappings between X and Y is denoted by $\mathcal{L}(X; Y)$.

Generally, an over-line will denote either complex conjugation (when the line is drawn above a complex valued function) or the closure of a set under a certain norm (when the line is drawn above a set). When the meaning of an over-line is not clear from context, it will be accompanied by a comment explaining its interpretation.

For non-negative $s \in \mathbb{R}$ and $p \geq 1$, we shall use standard Sobolev spaces $W^{s,p}(O)$ as defined in [24, Chapter 3]. In the case of $p = 2$, we use the standard notation $H^s(O)$, of complex-valued scalar functions with the customary convention $H^0(O) \equiv L^2(O)$. $H^s(O)$-norms will be written as $\|\cdot\|_{s,O}$ while semi-norms as $|\cdot|_{s,O}$ both with obvious extension to vector quantities.

Finally, we will denote scalars in simple typeface, vector fields with boldface. Euclidean norms in $\mathbb{R}^d$ are denoted by $\|\cdot\|_{\mathbb{R}^d}$, while induced matrix norms in $\mathbb{R}^{d \times d}$ are written as $\|\cdot\|_{\mathbb{R}^{d \times d}}$. The spaces of continuous functions mapping to $\mathbb{R}^d$ or $\mathbb{R}^{d \times d}$ are denoted by $C^0(O; \mathbb{R}^d)$ and $C^0(O; \mathbb{R}^{d \times d})$ with the norms

$$\|f\|_{C^0(O;\mathbb{R}^d)} := \sup_{\mathbf{x} \in O} \|f(\mathbf{x})\|_{\mathbb{R}^d}, \qquad \|\mathbf{B}\|_{C^0(O;\mathbb{R}^{d \times d})} := \sup_{\mathbf{x} \in O} \|\mathbf{B}(\mathbf{x})\|_{\mathbb{R}^{d \times d}}.$$

For complex valued functions a similar notation is used.

2.2. Functional spaces.

Let now $D \subset \mathbb{R}^3$ be an open, bounded Lipschitz domain with simply connected boundary surface ∂D. Its complement is denoted by $D^c := \mathbb{R}^3 \setminus \overline{D}$. We recall the standard vectorial spaces to formulate Maxwell problems:

$$\begin{aligned}
\boldsymbol{H}(\mathbf{curl}; D) &:= \left\{ \mathbf{U} \in \boldsymbol{L}^2(D) \; : \; \mathbf{curl}\, \mathbf{U} \in \boldsymbol{L}^2(D) \right\}, \\
\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D) &:= \left\{ \mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) \mid \mathbf{curl}\,\mathbf{curl}\, \mathbf{U} \in \boldsymbol{L}^2(D) \right\}, \\
\boldsymbol{H}(\mathrm{div}; D) &:= \left\{ \mathbf{U} \in \boldsymbol{L}^2(D) \; : \; \mathrm{div}\, \mathbf{U} \in L^2(D) \right\},
\end{aligned}$$

as well as extensions to more regular spaces $\boldsymbol{H}^s(\mathbf{curl};D)$ and $\boldsymbol{H}^s(\mathrm{div};D)$ for $s>0$, with norms:

$$\|\mathbf{U}\|_{\boldsymbol{H}^s(\mathbf{curl};D)} := \|\mathbf{curl}\,\mathbf{U}\|_{s,D} + \|\mathbf{U}\|_{s,D}\,, \qquad \|\mathbf{U}\|_{\boldsymbol{H}^s(\mathrm{div};D)} := \|\mathrm{div}\,\mathbf{U}\|_{s,D} + \|\mathbf{U}\|_{s,D}\,.$$

As customary, for a Lipschitz surface ∂D one deals with the trace spaces:

$$\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial D) := \{\mathbf{U} \in \boldsymbol{H}^{-\frac{1}{2}}(\partial D) : \mathbf{U}\cdot\mathbf{n} = 0,\ \mathrm{div}_{\partial D}\,\mathbf{U} \in H^{-\frac{1}{2}}(\partial D)\}\,,$$

$$\boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial D) := \{\mathbf{U} \in \boldsymbol{H}^{-\frac{1}{2}}(\partial D) : \mathbf{U}\cdot\mathbf{n} = 0,\ \mathrm{curl}_{\partial D}\,\mathbf{U} \in H^{-\frac{1}{2}}(\partial D)\},$$

endowed with their respective graph norms. The outward normal vector $\mathbf{n}$ points from D to D^c and $\mathrm{div}_{\partial D}$, $\mathrm{curl}_{\partial D}$ shall refer to surface divergence and scalar surface curl operators, respectively (*cp.* Chap. 2.5 in [27] and [2]).

Definition 2.1. For $\mathbf{U} \in \boldsymbol{C}^\infty(\overline{D})$, we define tangential Dirichlet and Neumann traces by

$$\gamma_{\mathrm{D}}\mathbf{U} := \mathbf{n}\times(\mathbf{U}\times\mathbf{n})|_{\partial D} \qquad \text{and} \qquad \gamma_{\mathrm{N}}\mathbf{U} := (\mathbf{n}\times\mathbf{curl}\,\mathbf{U})|_{\partial D}\,,$$

respectively. The *flipped tangential trace* $\gamma_{\mathrm{D}}^{\times}$ is $\gamma_{\mathrm{D}}^{\times}\mathbf{U} := (\mathbf{n}\times\mathbf{U})|_{\partial D}$. We also define the normal trace operator by $\gamma_{\mathbf{n}}\mathbf{U} := (\mathbf{U}\cdot\mathbf{n})|_{\partial D}$.

The trace operators γ_{D} and $\gamma_{\mathrm{D}}^{\times}$ can be extended to linear and continuous operators from $\boldsymbol{H}(\mathbf{curl};D)$ to $\boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial D)$ and $\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial D)$, respectively. Likewise, $\gamma_{\mathrm{N}} : \boldsymbol{H}(\mathbf{curl}\,\mathbf{curl};D) \to \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial D)$ and $\gamma_{\mathbf{n}} : \boldsymbol{H}(\mathrm{div};D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ continuously [2, 25]. Moreover, the traces γ_{D}, $\gamma_{\mathrm{D}}^{\times}$, $\gamma_{\mathbf{n}}$ and γ_{N} admit linear and continuous right inverses. With the trace operators $\gamma_{\mathrm{D}}^{\times}$ and $\gamma_{\mathbf{n}}$, we define

$$\begin{aligned}\boldsymbol{H}_0(\mathbf{curl};D) &:= \{\mathbf{U}\in\boldsymbol{H}(\mathbf{curl};D) \;:\; \gamma_{\mathrm{D}}^{\times}\mathbf{U} = \mathbf{0} \text{ on } \partial D\}\,,\\ \boldsymbol{H}_0(\mathrm{div};D) &:= \{\mathbf{U}\in\boldsymbol{H}(\mathrm{div};D) \;:\; \gamma_{\mathbf{n}}\mathbf{U} = \mathbf{0} \text{ on } \partial D\}\,.\end{aligned}$$

By continuity of $\gamma_{\mathrm{D}}^{\times}$, $\boldsymbol{H}_0(\mathbf{curl};D)$ is a closed subspace of $\boldsymbol{H}(\mathbf{curl};D)$. Analogously, $\boldsymbol{H}_0(\mathrm{div};D)$ is a closed subspace of $\boldsymbol{H}(\mathrm{div};D)$. Finally, for $\mathbf{U}$ and $\mathbf{V}\in\boldsymbol{H}(\mathbf{curl},D)$, where D is again bounded and Lipschitz, there holds [2, Eq. (27)]:

$$(\mathbf{U},\mathbf{curl}\,\mathbf{V})_D - (\mathbf{curl}\,\mathbf{U},\mathbf{V})_D = -\langle\gamma_{\mathrm{D}}^{\times}\mathbf{U},\gamma_{\mathrm{D}}\mathbf{V}\rangle_{\partial D} = \langle\gamma_{\mathrm{D}}\mathbf{U},\gamma_{\mathrm{D}}^{\times}\mathbf{V}\rangle_{\partial D}\,, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{\partial D}$ denotes the $\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial D)$ dual product since (*cf.* Theorem 2 in [3])

$$\left(\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial D)\right)^* = \boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial D)\,.$$

2.3. **Admissible domain transformations.** The domain $\widehat{D}\subseteq\mathbb{R}^3$ is henceforth referred to as the *nominal domain*. For a given transformation $T:\widehat{D}\to\mathbb{R}^3$ we call $D_T := T(\widehat{D})$ the *physical domain*. The *set of admissible domain transformations* will be denoted by $\mathfrak{T}$.

Assumption 2.2. The *nominal domain* $\widehat{D}\subseteq\mathbb{R}^3$ is a bounded, polyhedral Lipschitz domain and $\mathfrak{T}\subseteq\boldsymbol{C}^1(\widehat{D})$ is a compact set. For every $T\in\mathfrak{T}$, $D_T := T(\widehat{D})$ is a bounded Lipschitz domain and $T:\widehat{D}\to D_T$ is bijective and bi-Lipschitz. Moreover, there is $\vartheta\in(0,1)$ such that for every $T\in\mathfrak{T}$, it holds

$$\vartheta < \inf_{\widehat{\mathbf{x}}\in\widehat{D}}\det(dT(\widehat{\mathbf{x}})),\ \sup_{\widehat{\mathbf{x}}\in\widehat{D}}\det(dT(\widehat{\mathbf{x}})),\ \|T\|_{C^1(\widehat{D};\mathbb{R}^3)},\ \|T^{-1}\|_{C^1(D_T;\mathbb{R}^3)} < \vartheta^{-1}. \tag{2.2}$$

For a given function $\mathbf{U}$ over D_T we introduce the pullback

$$\Phi_T(\mathbf{U}) := dT^\top(\mathbf{U}\circ T) \tag{2.3}$$

wherein $dT:\widehat{D}\to\mathbb{R}^{3\times3}$ denotes the Jacobian of T. We will repeatedly use the following result.

Lemma 2.3 (Lemma 2.2 in [22]). *Let $\widehat{D}$, D_T, T be as in Assumption 2.2. Then, the map Φ_T in* (2.3) *admits a bounded extension from $\boldsymbol{H}(\mathbf{curl};D_T)\to\boldsymbol{H}(\mathbf{curl};\widehat{D})$ such that this extension is an isomorphism. The same result holds in $\boldsymbol{H}_0(\mathbf{curl};\cdot)$. Furthermore, in $\boldsymbol{L}^2(\widehat{D})$ there holds*

$$\mathbf{curl}(\Phi_T(\mathbf{U})) = (\det(dT))dT^{-1}((\mathbf{curl}\,\mathbf{U})\circ T)\,. \tag{2.4}$$

2.4. **Maxwell Equations.** We consider the EM cavity problem for a time-harmonic dependence $e^{\imath\omega t}$ with circular frequency $\omega > 0$ and $\imath^2 = -1$. Losses are represented by the domain conductivity $\sigma(\mathbf{x})$ while dielectric permittivity and magnetic permeability are denoted by $\varepsilon(\mathbf{x})$ and $\mu(\mathbf{x})$, respectively. To keep notation succinct, we introduce the space-dependent quantity $\Lambda(\mathbf{x}) := \omega^2\epsilon(\mathbf{x}) - \imath\omega\sigma(\mathbf{x})$.

Assumption 2.4 (Material properties)*.* There exists an open bounded set $D_H \subseteq \mathbb{R}^3$ – the *hold-all domain*– such that, for every $T \in \mathfrak{T}$ as in Assumption 2.2, the closure $\overline{D}_T \subseteq \mathbb{R}^3$ of D_T is contained in D_H. Moreover, $\mu \in C^0(D_H;\mathbb{C})$, $\Lambda \in C^0(D_H;\mathbb{C})$ and there exists $\theta \in [0, 2\pi)$ such that (*cf.* [17])

$$\inf_{\mathbf{x}\in D_H} \operatorname{Re}\left(e^{\imath\theta}(\mu(\mathbf{x}))^{-1}\right) =: \mu_b > 0, \qquad \inf_{\mathbf{x}\in D_H} \operatorname{Re}(-e^{\imath\theta}\Lambda(\mathbf{x})) =: \Lambda_b > 0. \tag{2.5}$$

Finally, we assume that the source current density $\mathbf{J} \in C^0(D_H;\mathbb{C}^3)$.

In this setting, we write $\mathbf{E}$ and $\mathbf{H}$ for the complex-valued electric and magnetic fields, respectively. Letting $(D_T)_{T\in\mathfrak{T}}$ be the family of Lipschitz domains in Assumption 2.2, for every $T \in \mathfrak{T}$ Maxwell equations in the domain D_T read

$$\begin{aligned} \mathbf{curl}\,\mathbf{E}_T + \imath\omega\mu\mathbf{H}_T &= \mathbf{0} \\ (\imath\omega\varepsilon + \sigma)\mathbf{E}_T - \mathbf{curl}\,\mathbf{H}_T &= -\imath\omega\mathbf{J}. \end{aligned}$$

This can be reduced to

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E}_T - \Lambda\mathbf{E}_T = -\imath\omega\mathbf{J}. \tag{2.6}$$

If μ is constant, one obtains

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_T - \kappa^2\mathbf{E}_T = -\imath\omega\mu\mathbf{J},$$

where we have defined the complex wavenumber (or propagation constant) $\kappa^2 := \mu\Lambda$, or equivalently, $\kappa^2 = \omega^2\mu\varepsilon - \imath\omega\mu\sigma$. After imposing homogenous perfect electrical conductor (PEC) boundary conditions

$$\gamma_{\mathrm{D}}^{\times}\mathbf{E} = \mathbf{0}$$

on the surface ∂D_T, we arrive at our problem of interest:

Problem 2.5 (Lossy cavity problem)**.** Under Assumptions 2.2 and 2.4, for every $T \in \mathfrak{T}$ we seek $\mathbf{E}_T \in \boldsymbol{H}_0(\mathbf{curl}; D_T)$ such that (2.6) holds.

Remark 2.6*.* If $\sigma = 0$ and $\mathbf{J} = \mathbf{0}$, we obtain an eigenvalue problem. We do not elaborate on this case, but note in passing that most of the ensuing results and techniques apply also to this case.

2.5. **Variational formulation and well-posedness.** Let in the following $(D_T)_{T\in\mathfrak{T}}$ be the family of Lipschitz domains in Assumption 2.2.

2.5.1. *Physical domain* D_T*.* To present the T-dependent weak formulation of Problem 2.5, we introduce the sesquilinear form $\mathsf{a}_T(\cdot,\cdot) : \boldsymbol{H}_0(\mathbf{curl}; D_T) \times \boldsymbol{H}_0(\mathbf{curl}; D_T) \to \mathbb{C}$ and the antilinear form $\mathsf{f}_T(\cdot) : \boldsymbol{H}_0(\mathbf{curl}; D_T) \to \mathbb{C}$ via

$$\mathsf{a}_T(\mathbf{U},\mathbf{V}) := \int_{D_T} \mu^{-1}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\overline{\mathbf{V}} - \Lambda\mathbf{U}\cdot\overline{\mathbf{V}}\;\,\mathrm{d}\mathbf{x}, \;\; \mathsf{f}_T(\mathbf{V}) := -\imath\omega\int_{D_T} \mathbf{J}\cdot\overline{\mathbf{V}}\;\,\mathrm{d}\mathbf{x}.$$

Integrating by parts in (2.6), we obtain the following family of variational problems depending on $T \in \mathfrak{T}$.

Problem 2.7 (Physical domain variational problem)**.** Under Assumptions 2.2 and 2.4, for every $T \in \mathfrak{T}$ we seek $\mathbf{E}_T \in \boldsymbol{H}_0(\mathbf{curl}; D_T)$ such that

$$\mathsf{a}_T(\mathbf{E}_T,\mathbf{V}) = \mathsf{f}_T(\mathbf{V}) \qquad \forall\, \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; D_T). \tag{2.7}$$

Theorem 2.8. *Under Assumption 2.2, for all transformations $T \in \mathfrak{T}$ the perturbed sesquilinear form $\mathsf{a}_T(\cdot,\cdot)$ is $\boldsymbol{H}_0(\mathbf{curl}; D_T)-$elliptic and Problem 2.7 is well posed.*

Proof. We argue along the lines of Theorem 3.2 in [17]. For arbitrary $\mathbf{E} \in \boldsymbol{H}_0(\mathbf{curl}; D_T)$, write

$$|\mathsf{a}_T(\mathbf{E},\mathbf{E})| \geq \operatorname{Re}\left\{e^{\imath\theta}\,\mathsf{a}_T(\mathbf{E},\mathbf{E})\right\} = \operatorname{Re}\left\{e^{\imath\theta}\int_{D_T}\left(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}\cdot\mathbf{curl}\,\overline{\mathbf{E}} - \Lambda\mathbf{E}\cdot\overline{\mathbf{E}}\right)\mathrm{d}\mathbf{x}\right\}$$
$$\geq \min\{\mu_b,\Lambda_b\}\,\|\mathbf{E}\|^2_{\boldsymbol{H}(\mathbf{curl};D_T)}\,.$$

I.e., $\mathsf{a}_T(\cdot,\cdot)$ is elliptic uniformly with respect to $T\in\mathfrak{T}$. By the continuity of the sesquilinear and antilinear forms in (2.7), the Lax-Milgram theorem implies well-posedness. □

2.5.2. *Nominal domain* $\widehat{D}$. As in [22], we rewrite (2.7) as a variational problem on the nominal domain $\widehat{D}$. Denote by $\Phi_T : \boldsymbol{H}_0(\mathbf{curl}; D_T) \to \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ the isomorphism (2.3) from Lemma 2.3. Since Φ_T is an isomorphism, for every $T \in \mathfrak{T}$, $\mathbf{E}_T \in \boldsymbol{H}_0(\mathbf{curl}; D_T)$ solves (2.7) iff $\widehat{\mathbf{E}}_T = \Phi_T(\mathbf{E}_T) \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ solves

$$\mathsf{a}_T(\Phi_T^{-1}(\widehat{\mathbf{E}}_T),\Phi_T^{-1}(\widehat{\mathbf{V}})) = \mathsf{f}_T(\Phi_T^{-1}(\widehat{\mathbf{V}})) \qquad \forall\,\widehat{\mathbf{V}} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D}).$$

In case either of those problems has a unique solution, the same holds for the other. This leads us to introduce the sesquilinear form $\widehat{\mathsf{a}}_T(\cdot,\cdot) : \boldsymbol{H}_0(\mathbf{curl}; \widehat{D}) \times \boldsymbol{H}_0(\mathbf{curl}; \widehat{D}) \to \mathbb{C}$ and the antilinear form $\widehat{\mathsf{f}}_T(\cdot) : \boldsymbol{H}_0(\mathbf{curl}; \widehat{D}) \to \mathbb{C}$ defined by

$$\widehat{\mathsf{a}}_T(\widehat{\mathbf{U}},\widehat{\mathbf{V}}) := \mathsf{a}_T(\Phi_T^{-1}(\widehat{\mathbf{U}}),\Phi_T^{-1}(\widehat{\mathbf{V}})) \qquad \text{and} \qquad \widehat{\mathsf{f}}_T(\widehat{\mathbf{V}}) := \mathsf{f}_T(\Phi_T^{-1}(\widehat{\mathbf{V}})).$$

We can now state the *nominal variational problem*:

Problem 2.9 (Nominal variational problem)**.** Under Assumptions 2.2 and 2.4, for every $T \in \mathfrak{T}$ we seek $\widehat{\mathbf{E}}_T \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ such that

$$\widehat{\mathsf{a}}_T(\widehat{\mathbf{E}}_T,\widehat{\mathbf{V}}) = \widehat{\mathsf{f}}_T(\widehat{\mathbf{V}}) \qquad \forall\,\widehat{\mathbf{V}} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D}). \tag{2.8}$$

In our present setting, where $\mathbf{J} \in C^0(D_H;\mathbb{C}^3) \hookrightarrow L^2(D_H;\mathbb{C}^3)$, the sesquilinear form $\mathsf{a}_T(\cdot,\cdot)$ and the antilinear form $\mathsf{f}_T(\cdot)$ can be represented as integrals over the physical domain D_T. Transforming those integrals to $\widehat{D}$, one obtains concrete expressions for $\widehat{\mathsf{a}}_T(\cdot,\cdot)$ and $\widehat{\mathsf{f}}_T(\cdot)$. Similar as in [22], with

$$\mu_T := \mu\circ T, \qquad \Lambda_T := \Lambda\circ T, \qquad \mathbf{J}_T := \mathbf{J}\circ T, \tag{2.9}$$

we find

$$\widehat{\mathsf{a}}_T(\widehat{\mathbf{U}},\widehat{\mathbf{V}}) = \int_{\widehat{D}} \det(dT)^{-1}{\mu_T}^{-1}dT\,\mathbf{curl}\,\widehat{\mathbf{U}}\cdot dT\,\mathbf{curl}\,\overline{\widehat{\mathbf{V}}} - \Lambda_T\det(dT)dT^{-\top}\widehat{\mathbf{U}}\cdot dT^{-\top}\overline{\widehat{\mathbf{V}}}\,\mathrm{d}\widehat{\mathbf{x}} \tag{2.10}$$

and

$$\widehat{\mathsf{f}}_T(\widehat{\mathbf{V}}) = -\imath\omega\int_{\widehat{D}}\det(dT)\mathbf{J}_T\cdot dT^{-\top}\overline{\widehat{\mathbf{V}}}\,\mathrm{d}\widehat{\mathbf{x}}. \tag{2.11}$$

where we have used that $\det dT(\widehat{\mathbf{x}}) \geq 0$ for all $\widehat{\mathbf{x}} \in \widehat{D}$ by (2.2). We point out that $\mathbf{J}_T = \mathbf{J}\circ T \in \boldsymbol{L}^2(\widehat{D})$ since $\mathbf{J} \in \boldsymbol{L}^2(D_H)$, $D_H \supseteq T(\widehat{D})$, and $T : \widehat{D} \to D_T$ is bi-Lipschitz, see for instance [23, Lemma 5.7.2] for a proof.

Proposition 2.10. *Let Assumptions 2.2 and 2.4 be satisfied. Then, there exists* $\alpha > 0$ *and* $\theta \in [0, 2\pi)$ *in* (2.5)*, such that, for all* $T \in \mathfrak{T}$*, it holds*

$$\operatorname{Re}\left(e^{\imath\theta}\,\widehat{\mathsf{a}}_T(\widehat{\mathbf{U}},\widehat{\mathbf{U}})\right) \geq \alpha\|\widehat{\mathbf{U}}\|^2_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} \qquad \forall\,\widehat{\mathbf{U}} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D}). \tag{2.12}$$

Furthermore, there exists a constant $C > 0$ *such that, for every* $T \in \mathfrak{T}$ *and for every* $\widehat{\mathbf{U}} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ *and* $\widehat{\mathbf{V}} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$*, it holds*

$$|\,\widehat{\mathsf{a}}_T(\widehat{\mathbf{U}},\widehat{\mathbf{V}})| \leq C\|\widehat{\mathbf{U}}\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})}\|\widehat{\mathbf{V}}\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})}.$$

Thus, Problem 2.9 is well posed for every $T \in \mathfrak{T}$.

Proof. Fix $T \in \mathfrak{T}$. To see (2.12), note that for every $\boldsymbol{\zeta} \in \mathbb{C}^3$ and every $\widehat{\mathbf{x}} \in \widehat{D}$ by (2.2), it holds

$$\boldsymbol{\zeta}^\top dT^\top(\widehat{\mathbf{x}}) dT(\widehat{\mathbf{x}})\overline{\boldsymbol{\zeta}} = \|dT(\widehat{\mathbf{x}})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq \|dT^{-1}(\widehat{\mathbf{x}})\|^{-2}_{\mathbb{C}^{3\times 3}} \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq \vartheta^2 \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},$$

where we used $\|dT^{-1}(\widehat{\mathbf{x}})\|_{\mathbb{C}^{3\times 3}} \|dT(\widehat{\mathbf{x}})\boldsymbol{\zeta}\|_{\mathbb{C}^3} \geq \|\boldsymbol{\zeta}\|_{\mathbb{C}^3}$ and that $\det(dT(\widehat{\mathbf{x}}))^{-1} \geq \vartheta$ for all $\mathbf{x} \in \widehat{D}$ by (2.2). Thus, by (2.5) and (2.10), for every $\widehat{\mathbf{U}} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$, one has

$$\operatorname{Re}(e^{\imath\theta}\, \widehat{\mathsf{a}}_T(\widehat{\mathbf{U}}, \widehat{\mathbf{U}})) \geq \mu_b \vartheta^3 \|\mathbf{curl}\, \widehat{\mathbf{U}}\|^2_{L^2(\widehat{D};\mathbb{C}^3)} + \Lambda_b \vartheta^3 \|\widehat{\mathbf{U}}\|^2_{L^2(\widehat{D};\mathbb{C}^3)},$$

which shows (2.12) with $\alpha := \vartheta^3 \min\{\mu_b, \Lambda_b\}$. Similarly, one shows the existence of the uniform continuity constant C. The complex Lax-Milgram theorem implies that there exists a unique solution of (2.8) that depends continuously on $\widehat{\mathsf{f}}_T(\cdot) \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})^*$. $\square$

3. Discrete Approximation and Strang Lemma

In the last section, we showed that, for any admissible domain transformation $T \in \mathfrak{T}$, the pullback to the nominal domain $\widehat{D}$ problem (Problem 2.9) is well posed. In the ensuing uncertainty quantification, we shall obtain a sparse approximation of the *shape-to-solution map* $T \mapsto \widehat{\mathbf{E}}_T$. Naturally, the corresponding field $\widehat{\mathbf{E}}_T$ must be approximated numerically. To this end, and in order to generate computational meshes only once, we discretize the pullback variational problem (2.8) using edge elements following Monk [25], and verify the well-posedness of the discrete problem building on the work by A. Ern and J.-L. Guermond [15, 17, 16]. As the sesquilinear form $\widehat{\mathsf{a}}_T(\cdot,\cdot)$ defined in (2.10) has in general non-constant and non-smooth coefficients –depending on the spatial regularity of the admissible transformation $T \in \mathfrak{T}$–, numerical integration will play a crucial role in the development of viable algorithms. Specifically, a quadrature strategy needs to be devised with certified consistency order with respect to the choice of edge-element discretization of (2.8), *valid uniformly with respect to* $T \in \mathfrak{T}$. This is the purpose of the present section, which is of interest in its own right containing a quadrature error analysis for edge element discretizations of time-harmonic Maxwell equations (*cf.* Theorem 3.17). For ease of exposition, some of the more technical proofs in this section will be provided in Appendix A.

3.1. Finite elements.

Let us consider a shape-regular sequence of affine meshes $\{\tau_h\}_{h>0}$, constructed from disjoint, matching tetrahedrons that cover $\widehat{D}$ exactly. We will assume that the tetrahedrons from the mesh, i.e. $K \in \tau_h$, are constructed from a single reference tetrahedron $\breve{K}$ through affine, bijective transformations $T_K : \breve{K} \to K$. For each element $K \in \tau_h$, h_K denotes its diameter and we define $h := \max_{K\in\tau_h} h_K$. We stress the difference between mappings T_K and T: the first one maps the reference tetrahedron $\breve{K}$ to elements on a discretization τ_h of $\widehat{D}$, whereas the second relates physical and nominal domains, D_T and $\widehat{D}$, respectively.

Assumption 3.1 (Assumptions on $\{\tau_h\}_{h>0}$). The sequence of affine, shape regular meshes $\{\tau_h\}_{h>0}$ cover $\widehat{D}$ exactly, and the τ_h are uniformly quasi-uniform in the sense of [14, Definition 1.140].

For $q = 1, 3$ and $k \in \mathbb{N}$, let $\mathbb{P}_k(\Omega; \mathbb{R}^q)$ be the space of functions over a measurable domain Ω with polynomials of degree at most k in their q components. We also introduce $\widetilde{\mathbb{P}}_k(\Omega; \mathbb{R}^q)$ as the space of elements of $\mathbb{P}_k(\Omega; \mathbb{R}^q)$ of degree exactly k in their q components. The spaces $\mathbb{P}_k(\Omega; \mathbb{C}^q)$ and $\widetilde{\mathbb{P}}_k(\Omega; \mathbb{C}^q)$ are defined analogously. As in [25], we consider finite elements in the sense of Ciarlet as triples (K, P_K, Σ_K), with $K \in \tau_h$, P_K a space of functions over K, and Σ_K a set of linear functionals acting on P_K. We will use the elements (K, P^g_K, Σ^g_K), $(K, \boldsymbol{P}^c_K, \Sigma^c_K)$, and $(K, \boldsymbol{P}^d_K, \Sigma^d_K)$ as defined in [25, Chapter 5], and corresponding to grad-, curl-, div-conforming elements, respectively. Specifically, $P^g_K := \mathbb{P}_k(K; \mathbb{C})$, $P^b_K := \mathbb{P}_{k-1}(K; \mathbb{C})$ and curl and div-conforming reference elements are

$$\boldsymbol{P}^c_K := \mathbb{P}_{k-1}(K; \mathbb{C}^3) \oplus \{\boldsymbol{p} \in \widetilde{\mathbb{P}}_k(K, \mathbb{C}^3) \;\; : \mathbf{x}\cdot\boldsymbol{p} = 0\},$$
$$\boldsymbol{P}^d_K := \mathbb{P}_{k-1}(K; \mathbb{C}^3) \oplus \{\boldsymbol{x}p \;\; : \;\; p \in \widetilde{\mathbb{P}}_{k-1}(K, \mathbb{C})\}.$$

For the sake of brevity, we shall not give explicit formulas for the elements of Σ_K^g, Σ_K^c, or Σ_K^d, which shall be referred to as degrees of freedom. We do, however, indicate their respective domains, i.e. the sets of functions where they are well defined (*cf.* [25, Chapter 5]):

$$\begin{aligned} V^g(K) &:= \left\{ v \ : \ v \in H^s(K) \text{ and } \nabla v \in \boldsymbol{H}^{s-\frac{1}{2}}(K) \text{ for some } s > \frac{3}{2} \right\}, \\ \boldsymbol{V}^c(K) &:= \left\{ \mathbf{W} \ : \ \mathbf{W} \in \boldsymbol{H}^{s-1}(K) \text{ and } \mathbf{curl}\,\mathbf{W} \in \boldsymbol{L}^p(K) \text{ for some } s > \frac{3}{2},\ p > 2 \right\}, \\ \boldsymbol{V}^d(K) &:= \left\{ \mathbf{W} \ : \ \mathbf{W} \in H^{s-1}(K) \text{ for some } s > \frac{3}{2} \right\}. \end{aligned} \tag{3.1}$$

These definitions extend obviously to the reference element $\breve{K}$. Note also that the Sobolev regularity $s > \frac{3}{2}$ implies that elements of these spaces can be assumed to be continuous on $\overline{K}$. Discrete spaces on the mesh τ_h are then constructed from the spaces defined on each element,

$$\begin{aligned} P^g(\tau_h) &:= \left\{ v_h \in H^1(\widehat{D}) \ : \ v_h|_K \in P_K^g \right\}, \\ \boldsymbol{P}^c(\tau_h) &:= \left\{ \mathbf{V}_h \in \boldsymbol{H}\left(\mathbf{curl}; \widehat{D}\right) \ : \ \mathbf{V}_h|_K \in \boldsymbol{P}_K^c,\ \forall\, K \in \tau_h \right\}, \\ \boldsymbol{P}^d(\tau_h) &:= \left\{ \mathbf{V}_h \in \boldsymbol{H}(\mathrm{div}; \widehat{D}) \ : \ \mathbf{V}_h|_K \in \boldsymbol{P}_K^d,\ \forall\, K \in \tau_h \right\}, \\ P^b(\tau_h) &:= \left\{ v_h \in L^2(\widehat{D}) \ : \ v_h|_K \in P_K^b,\ \forall\, K \in \tau_h \right\}. \end{aligned}$$

Homogeneous essential boundary conditions are accounted for by considering subspaces of the above discrete spaces that satisfy said conditions:

$$\begin{aligned} P_0^g(\tau_h) &:= P^g(\tau_h) \cap H_0^1(\widehat{D}), \\ \boldsymbol{P}_0^c(\tau_h) &:= \boldsymbol{P}^c(\tau_h) \cap \boldsymbol{H}_0(\mathbf{curl}; \widehat{D}), \\ \boldsymbol{P}_0^d(\tau_h) &:= \boldsymbol{P}^d(\tau_h) \cap \boldsymbol{H}_0(\mathrm{div}; \widehat{D}). \end{aligned}$$

The discrete spaces may be equivalently defined via the following pullbacks over functions defined on an element $K \in \tau_h$, that shall also prove useful for the coming error analysis:

$$\begin{aligned} \psi_K^g(v) &:= v \circ T_K, \\ \psi_K^c(\mathbf{V}) &:= \mathbb{J}_K^\top(\mathbf{V} \circ T_K), \\ \psi_K^d(\mathbf{V}) &:= \det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\mathbf{V} \circ T_K), \\ \psi_K^b(v) &:= \det(\mathbb{J}_K)(v \circ T_K), \end{aligned} \tag{3.2}$$

where $\mathbb{J}_K$ is the Jacobian matrix of T_K, while v and $\mathbf{V}$ belong to the space signaled by the superscript. These mappings commute with the corresponding differential operators, i.e.

$$\nabla \psi_K^g(v) = \psi_K^c(\nabla v), \quad \mathbf{curl}\, \psi_K^c(\mathbf{V}) = \psi_K^d(\mathbf{curl}\,\mathbf{V}), \quad \mathrm{div}\, \psi_K^d(\mathbf{V}) = \psi_K^b(\mathrm{div}\,\mathbf{V}),$$

hold for functions on any $K \in \tau_h$ with well defined gradient, curl or divergence, respectively [25, Chapter 3.9]. Furthermore, they leave the corresponding finite element spaces unchanged so that, for all $K \in \tau_h$,

$$\psi_K^j : P_K^j \to P_{\breve{K}}^j \quad \text{and} \quad (\psi_K^j)^{-1} : P_{\breve{K}}^j \to P_K^j \quad \forall\, j \in \{g, c, d, b\}.$$

Under Assumption 3.1, there are uniform constants $c^\sharp$ and $c^\flat$ such that (*cf.* [14, Lemma 1.100])

$$|\det(\mathbb{J}_K)| = |K| \left| \breve{K} \right|^{-1}, \qquad \|\mathbb{J}_K\|_{\mathbb{R}^{3\times 3}} \le c^\sharp h, \qquad \left\| \mathbb{J}_K^{-1} \right\|_{\mathbb{R}^{3\times 3}} \le c^\flat h^{-1}. \tag{3.3}$$

The linear mappings defined in (3.2) may be summarized as

$$\psi_K^j(v) = \mathbb{A}_K^j(v \circ T_K) \quad \forall\, j \in \{g, b\}, \qquad \psi_K^j(\mathbf{V}) = \mathbb{A}_K^j(\mathbf{V} \circ T_K) \quad \forall\, j \in \{c, d\}, \tag{3.4}$$

where $\mathbb{A}_K^g = 1$, $\mathbb{A}_K^c = \mathbb{J}_K^\top$, $\mathbb{A}_K^d = \det(\mathbb{J}_K)\mathbb{J}_K^{-1}$ and $\mathbb{A}_K^b = \det(\mathbb{J}_K)$. Then, the mappings ψ_K satisfy the following properties for all $l \in \mathbb{N}$, $p \in [1,\infty]$:

$$\begin{aligned} \left|\psi_K^j\right|_{\mathcal{L}(W^{l,p}(K);W^{l,p}(\breve{K}))} &\leq c|\mathbb{A}_K^j| \, \|\mathbb{J}_K\|_{\mathbb{R}^{3\times3}}^l \, |\det(\mathbb{J}_K)|^{-\frac{1}{p}} \quad \forall\, j \in \{g,b\}, \\ \left|\psi_K^j\right|_{\mathcal{L}(\boldsymbol{W}^{l,p}(K);\boldsymbol{W}^{l,p}(\breve{K}))} &\leq c \left\|\mathbb{A}_K^j\right\|_{\mathbb{R}^{3\times3}} \|\mathbb{J}_K\|_{\mathbb{R}^{3\times3}}^l \, |\det(\mathbb{J}_K)|^{-\frac{1}{p}} \quad \forall\, j \in \{c,d\}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \left|(\psi_K^j)^{-1}\right|_{\mathcal{L}(W^{l,p}(\breve{K});W^{l,p}(K))} &\leq c|\mathbb{A}_K^j|^{-1} \left\|\mathbb{J}_K^{-1}\right\|_{\mathbb{R}^{3\times3}}^l |\det(\mathbb{J}_K)|^{\frac{1}{p}} \quad \forall\, j \in \{g,b\}, \\ \left|(\psi_K^j)^{-1}\right|_{\mathcal{L}(\boldsymbol{W}^{l,p}(\breve{K});\boldsymbol{W}^{l,p}(K))} &\leq c \left\|(\mathbb{A}_K^j)^{-1}\right\|_{\mathbb{R}^{3\times3}} \left\|\mathbb{J}_K^{-1}\right\|_{\mathbb{R}^{3\times3}}^l |\det(\mathbb{J}_K)|^{\frac{1}{p}} \quad \forall\, j \in \{c,d\}, \end{aligned} \tag{3.6}$$

for all $K \in \tau_h$, and $c > 0$ independent of K and h (*cf.* [14, Lemma 1.101]). We shall also require the following inverse inequality. In both, (3.5) and (3.6), the expression $\frac{1}{p}$ is understood as zero when $p = \infty$.

Lemma 3.2 (Lemma 1.138 in [14])**.** *Let $K \in \tau_h$, $p \in [1,\infty]$ and $m, l \in \mathbb{N}_0$ such that $m \leq l$. Then under Assumption 3.1, for φ in either P_K^g or P_K^b and $\boldsymbol{\varphi}$ in either $\boldsymbol{P}_K^c$ or $\boldsymbol{P}_K^d$,*

$$\begin{aligned} \|\varphi\|_{W^{l,p}(K)} &\leq ch^{m-l} \|\varphi\|_{W^{m,p}(K)}, \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{l,p}(K)} &\leq ch^{m-l} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{m,p}(K)}, \end{aligned}$$

for a positive constant c independent of K and h.

3.2. **Interpolation Operators.** For exposition purposes, we briefly recall the construction of canonical local and global interpolation operators on finite elements. However, as these require minimal regularity conditions, we resort to the local quasi-interpolation operators developed in [16] for arbitrary low regularity.

For any of the finite elements (K, P_K, Σ_K) described in Section 3.1, the canonical interpolation operator is defined as

$$\mathcal{I}_K(v) := \sum_{n=1}^N \sigma_n(v)\varphi_n,$$

with $N \in \mathbb{N}$, $\Sigma_K = \{\sigma_n\}_{n=1}^N$, and $P_K = \operatorname{span}\{\varphi_n\}_{n=1}^N$, where the basis $\{\varphi_n\}_{n=1}^N$ has the Kronecker property

$$\sigma_n(\varphi_m) = \delta_{mn}. \tag{3.7}$$

We denote by $\mathcal{I}_K^g$, $\mathcal{I}_K^c$, and $\mathcal{I}_K^d$ the interpolators for the finite elements associated with each superscript, which are well defined over $V^g(K)$, $\boldsymbol{V}^c(K)$, and $\boldsymbol{V}^d(K)$, respectively. We also introduce $\mathcal{I}_K^b$ as the L^2-projection onto P_K^b. These local interpolators commute with the linear maps in (3.2), i.e.

$$\psi_K^j \circ \mathcal{I}_K^j = \mathcal{I}_{\breve{K}}^j \circ \psi_K^j \quad \forall\, j \in \{g,c,d,b\}, \ \forall\, K \in \tau_h. \tag{3.8}$$

Global interpolation operators are built from the local interpolators defined above as described in [25, Chapter 5]. For the ensuing proof of Theorem 3.17, we *require local quasi-interpolation operators that are defined for functions with low regularity.* We introduce one such operator, developed by Ern and Guermond [16]. Its construction, for any finite element in the reference tetrahedron $(\breve{K}, \breve{P}, \breve{\Sigma})$ (*cf.* Section 3.1) with $\breve{P} \subset \mathbb{P}_k(\breve{K};\mathbb{R})$ for $k \in \mathbb{N}$, follows by considering the elements of $\breve{\Sigma}$ as functionals over $\breve{P}$ equipped with the L^2-norm. Then, by the Riesz representation theorem, for each $\sigma \in \breve{\Sigma}$ there exists $\rho \in \breve{P}$ so that

$$\sigma(v) = \int_{\breve{K}} \rho v \, \mathrm{d}\mathbf{x} \qquad \forall\, v \in \breve{P}.$$

Since $\rho \in \breve{P} \subset L^\infty(\breve{K})$, σ may be extended to act on $v \in L^1(\breve{K})$ as

$$\sigma^\#(v) := \int\limits_{\breve{K}} \rho v \, \mathrm{d}\mathbf{x}.$$

If $\breve{\Sigma} = \{\sigma_n\}_{n=1}^N$ for $N \in \mathbb{N}$ and $\breve{P} = \mathrm{span}\{\varphi_n\}_{n=1}^N$, where the basis $\{\varphi_n\}_{n=1}^N$ is such that (3.7) holds, the quasi-interpolation operator on $\breve{K}$ is defined as (*cf.* [16, Eqn. (3.2)])

$$\mathcal{I}_{\breve{K}}^\#(v) := \sum_{n=1}^N \sigma_n^\#(v)\varphi_n \qquad \forall\, v \in L^1(\breve{K}). \tag{3.9}$$

Remark 3.3. Note that the quasi-interpolation operator in (3.9) requires only that the interpolated function be in $L^1(\breve{K})$. This will hold for all finite elements, including the grad-, curl- and div-conforming finite elements, whereas the canonical interpolation operator would require the interpolated functions to belong to the spaces in (3.1), i.e. to possess some minimum regularity.

The composition of $\mathcal{I}_{\breve{K}}^\#$ with pullbacks $\psi_K : W^{l,p}(K) \to W^{l,p}(\breve{K})$ – such as those in (3.2) – allows for the construction of quasi-interpolation operators $\mathcal{I}_K^\#$ on any $K \in \tau_h$. The same process, with analogous results, holds for $\breve{P} \subset \mathbb{P}_k(K;\mathbb{R}^3)$ and complex valued finite element spaces. Lemma 3.4 provides properties of these operators that will be used in the proof of Theorem 3.17. We refer to [16, Section 3] for more details on the construction of the quasi-interpolation operators and their properties.

Lemma 3.4 (Proposition 3.1 and Remark 5.2 in [16])**.** *Let* $p \in [1,\infty]$, $K \in \tau_h$, *and* $\{(K, P_K, \Sigma_K)\}_{K\in\tau_h}$ *be any of the finite elements described in Section 3.1. There exist local quasi-interpolation operators* $\{\mathcal{I}_K^\# : K \in \tau_h\}$, *as defined in* [16, Section 3], *such that for every* $K \in \tau_h$

(i) $\left\|\mathcal{I}_K^\#(v)\right\|_{L^p(K)} \le c\,\|v\|_{L^p(K)}$ *for every* $v \in L^p(K)$ *and* $c > 0$,

(ii) $\mathcal{I}_K^\#(v) = v$ *for all* $v \in P_K$, *and*

(iii) *if* $p < \infty$ *and* $k \ge 1$ *is the largest integer such that* $\mathbb{P}^{k-1}(K,\mathbb{C}) \subset P_K$, $r \in [0,k]$ *and* $m \in \mathbb{N}$ *with* $m \le r$, *then there exists* $c > 0$ *independent of* K *and* h *so that*

$$\left|v - \mathcal{I}_K^\#(v)\right|_{W^{m,p}(K)} \le ch^{r-m}\,|v|_{W^{r,p}(K)} \quad \forall\, v \in W^{r,p}(K). \tag{3.10}$$

An analogous result holds when the elements of P_K *are vector-valued. Furthermore, if* $\mathcal{I}_K^{\#,g}$, $\mathcal{I}_K^{\#,c}$ *and* $\mathcal{I}_K^{\#,d}$ *are the quasi-interpolation operators associated with the grad-, curl- and div-conforming finite elements, then* (3.8) *holds replacing* $\mathcal{I}$ *with* $\mathcal{I}^\#$.

From here on, we fix $k \in \mathbb{N}$ as the polynomial degree for the finite elements introduced previously, so that

$$\mathbb{P}^{k-1}(K;\mathbb{C}^3) \subset \boldsymbol{P}_K^c,\ \boldsymbol{P}_K^d \subset \mathbb{P}^k(K;\mathbb{C}^3), \qquad P_K^g = \mathbb{P}^k(K;\mathbb{C}), \qquad P_K^b = \mathbb{P}^{k-1}(K;\mathbb{C}). \tag{3.11}$$

Using the error estimate in (3.10) and the result in Lemma 3.2, we are able to prove the following result regarding the stability of the quasi-interpolation operator.

Lemma 3.5. *Let* $K \in \tau_h$, *and* $\mathcal{I}_K^\#$ *denote either* $\mathcal{I}_K^{\#,c}$ *or* $\mathcal{I}_K^{\#,d}$. *Then, for all* $m \in \mathbb{N}$ *with* $m \le k$, *and* $p \in [1,\infty]$ *there exists a constant* $C > 0$ *independent of* K *and* h *such that*

$$\left\|\mathcal{I}_K^\#(\mathbf{V})\right\|_{\boldsymbol{W}^{m,p}(K)} \le C\,\|\mathbf{V}\|_{\boldsymbol{W}^{m,p}(K)} \quad \forall\, \mathbf{V} \in \boldsymbol{W}^{m,p}(K), \tag{3.12}$$

$$\left\|\mathcal{I}_K^\#(\mathbf{V})\right\|_{\boldsymbol{W}^{m,p}(K)} \le Ch^{-1}\,\|\mathbf{V}\|_{\boldsymbol{W}^{m-1,p}(K)} \quad \forall\, \mathbf{V} \in \boldsymbol{W}^{m-1,p}(K). \tag{3.13}$$

Furthermore, for $r \in \mathbb{R}$ *such that* $m-1 < r < m$, *there exists* $C > 0$ *independent of* K *and* h, *but dependent on* r *such that*

$$\left\|\mathcal{I}_K^\#(\mathbf{V})\right\|_{m,K} \le Ch^{r-m}\,\|\mathbf{V}\|_{r,K} \quad \forall\, \mathbf{V} \in \boldsymbol{H}^r(K). \tag{3.14}$$

Proof. Equation (3.12) is consequence of the error estimate in (3.10) and the triangle inequality. The result (3.13) follows from applying Lemma 3.2 and (3.12), in that order. Finally, (3.14) follows from both (3.12) and (3.13) by real interpolation between Sobolev spaces (*cf.* [33, Lemma 22.3]). □

3.3. **Discrete problem.** We are now ready to discretize Problem 2.9.

Problem 3.6 (Discrete variational problem)**.** Find $\widehat{\mathbf{E}}_{T,h} \in \boldsymbol{P}_0^c(\tau_h)$ such that,

$$\widehat{\mathsf{a}}_T(\widehat{\mathbf{E}}_{T,h}, \widehat{\mathbf{V}}_h) = \widehat{\mathsf{f}}_T(\widehat{\mathbf{V}}_h) \quad \forall\, \widehat{\mathbf{V}}_h \in \boldsymbol{P}_0^c(\tau_h), \tag{3.15}$$

wherein the sesquilinear and antilinear form are those given in (2.10) and (2.11).

The conformity of the discrete space $\boldsymbol{P}_0^c(\tau_h)$ in $\boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ ensures the existence and uniqueness of solutions for the variational problem above. Furthermore, we get the following convergence result. Recall $k \in \mathbb{N}$ as the polynomial degree for the finite element spaces defined in Section 3.

Theorem 3.7 (Theorem 3.3 in [17])**.** *Let $\widehat{\mathbf{E}}_T$, and $\widehat{\mathbf{E}}_{T,h}$ be the solutions of Problems 2.9, and 3.6, respectively. Then, if $\widehat{\mathbf{E}}_T \in \boldsymbol{H}^r(\mathbf{curl}; \widehat{D})$ for some $r \in (0, k]$, there holds for all $T \in \mathfrak{T}$*

$$\left\| \widehat{\mathbf{E}}_T - \widehat{\mathbf{E}}_{T,h} \right\|_{\boldsymbol{H}\left(\mathbf{curl}; \widehat{D}\right)} \leq C_{\mathfrak{T}} h^r \left(\left| \widehat{\mathbf{E}}_T \right|_{\boldsymbol{H}^r(\widehat{D})} + \left| \mathbf{curl}\, \widehat{\mathbf{E}}_T \right|_{\boldsymbol{H}^r(\widehat{D})} \right),$$

for a positive constant $C_{\mathfrak{T}}$, that may be chosen independently of $T \in \mathfrak{T}$, but depends on the set $\mathfrak{T}$.

Remark 3.8. Note that we do not require any minimum smoothness for solutions of Problem 2.9 to get a convergence rate, whereas classical results (*cf.* [25, Thm 5.41]) require some minimum regularity assumptions in order to ensure well-posedness of the classical interpolation operators. This is avoided by introducing quasi-interpolation operators $\mathcal{I}^{\#}$ [16], and a quasi-optimal commuting projection [15].

3.4. **Numerical integration.** We now consider the numerical computation of the integrals for the linear system arising in (3.15).

3.4.1. *Quadrature on mesh elements.* Let $\breve{K}$ be the reference tetrahedron and let $\{\breve{\boldsymbol{b}}_l\}_{l=1}^L \subseteq \breve{K}$, and $\{\breve{w}_l\}_{l=1}^L \subseteq \mathbb{R}\backslash\{0\}$ be sets of reference quadrature nodes and weights for some fixed $L \in \mathbb{N}$. We define a quadrature rule $Q : C^0(\breve{K}; \mathbb{C}) \to \mathbb{C}$ by

$$Q(f) := \sum_{l=1}^{L} \breve{w}_l f(\breve{\boldsymbol{b}}_l). \tag{3.16}$$

For an element $K \in \tau_h$ and the corresponding affine element map $T_K : \breve{K} \to K$, this yields a quadrature rule $Q_K : C^0(K; \mathbb{C}) \to \mathbb{C}$ via

$$Q_K(f) := \sum_{l=1}^{L} w_{l,K} f(\boldsymbol{b}_{l,K}) \qquad \text{where} \qquad w_{l,K} := |\det(\mathbb{J}_K)|\, \breve{w}_l, \quad \boldsymbol{b}_{l,K} := T_K(\breve{\boldsymbol{b}}_l). \tag{3.17}$$

To establish coercivity, we will make use of the following assumption also used in [9].

Assumption 3.9. It holds $\breve{w}_l > 0$ for all $l = 1, \dots, L$ in (3.16), and at least one of the following conditions: either $\{\breve{\boldsymbol{b}}_l\}_{l=1}^L$ is $\mathbb{P}_k(\breve{K}; \mathbb{C})$-unisolvent, or $Q(p) = \int_{\breve{K}} p(\mathbf{x})\, \mathrm{d}\mathbf{x}$ for all $p \in \mathbb{P}_{2k}(\breve{K}; \mathbb{C})$, i.e. the quadrature is exact on $\mathbb{P}_{2k}(\breve{K}; \mathbb{C})$.

Remark 3.10. Assumption 3.9 implies that

$$\sum_{l=1}^{L} \breve{w}_l |p(\breve{\boldsymbol{b}}_l)|^2 > 0 \qquad \forall\, 0 \neq p \in \mathbb{P}_k(\breve{K}; \mathbb{C}).$$

Hence $\left(\sum_{l=1}^{L} \breve{w}_l |p(\breve{\boldsymbol{b}}_l)|^2 \right)^{1/2}$ defines a norm on $\mathbb{P}_k(\breve{K}; \mathbb{C})$ in this case.

3.4.2. *Fully discrete problem.* With Q_K as in (3.17) we introduce a fully discrete variant of the sesquilinear form $\widehat{\mathsf{a}}_T(\cdot,\cdot)$ and the antilinear functional $\widehat{\mathsf{f}}_T(\cdot)$ in (2.10) and (2.11) as follows. For a mesh τ_h on $\widehat{D}$ let

$$\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{U}}_h,\widehat{\mathbf{V}}_h) := \sum_{K\in\tau_h} Q_K\left(\det(dT)^{-1}\mu_T^{-1}dT\,\mathbf{curl}\,\widehat{\mathbf{U}}_h\cdot dT\,\mathbf{curl}\,\overline{\widehat{\mathbf{V}}}_h - \det(dT)\Lambda_T dT^{-\top}\widehat{\mathbf{U}}_h\cdot dT^{-\top}\overline{\widehat{\mathbf{V}}}_h\right) \quad (3.18)$$

and

$$\tilde{\mathsf{f}}_{T,h}(\widehat{\mathbf{V}}_h) := -\imath\omega\sum_{K\in\tau_h} Q_K\left(\det(dT)\mathbf{J}_T\cdot dT^{-\top}\overline{\widehat{\mathbf{V}}}\right).$$

Here, all occurring functions, including derivatives of the transformation T, are assumed to be continuous on each $K\in\tau_h$ so that point evaluations are well-defined. This leads to the following variant of Problem 3.6.

Problem 3.11 (Fully discrete problem)**.** Find $\widetilde{\mathbf{E}}_{T,h}\in \boldsymbol{P}_0^c(\tau_h)$ such that,

$$\tilde{\mathsf{a}}_{T,h}(\widetilde{\mathbf{E}}_{T,h},\widehat{\mathbf{V}}_h) = \tilde{\mathsf{f}}_{T,h}(\widehat{\mathbf{V}}_h) \quad \forall\,\widehat{\mathbf{V}}_h\in\boldsymbol{P}_0^c(\tau_h). \quad (3.19)$$

3.4.3. *Ellipticity of the discrete sesquilinear form.* As is customary in finite element quadrature error analysis (see, e.g., [9]), we show that, under certain conditions for the quadrature scheme, the perturbed form $\tilde{\mathsf{a}}_{T,h}(\cdot,\cdot)$ is $\boldsymbol{P}_0^c(\tau_h)$-elliptic. The following lemma, which constitutes part of the proof, is formulated as a separate result as it will be required again subsequently. Its proof is given in Appendix A.

Lemma 3.12. *Let $\breve{K}\subseteq\mathbb{R}^3$ be the reference tetrahedron. Recall $k\in\mathbb{N}$ as the polynomial degree for the finite element spaces, so that* (3.11) *holds. Let $Q: C^0(\breve{K};\mathbb{C})\to\mathbb{C}$ be a quadrature rule as in* (3.16)*.*

(i) *There exists $C_1>0$ such that for every affine bijective element map $T_K:\breve{K}\to K$, for every $\mathbf{B}\in C^0(K;\mathbb{C}^{3\times3})$ and for all $\mathbf{U}$, $\mathbf{V}\in\mathbb{P}_k(K;\mathbb{C}^3)$ it holds (see* (3.17) *for Q_K)*

$$|Q_K(\mathbf{B}\mathbf{U}\cdot\mathbf{V})| \le C_1\|\mathbf{B}\|_{C^0(K;\mathbb{C}^{3\times3})}\|\mathbf{U}\|_{\boldsymbol{L}^2(K)}\|\mathbf{V}\|_{\boldsymbol{L}^2(K)}.$$

(ii) *Additionally, let Assumption 3.9 be satisfied. Then, there exists $C_2>0$ independent of h and $K\in\tau_h$, such that for every $\mathbf{B}\in C^0(K;\mathbb{C}^{3\times3})$ such that*

$$\inf_{\mathbf{x}\in K}\inf_{\|\boldsymbol{\zeta}\|_{\mathbb{C}^3}=1}\mathrm{Re}(\boldsymbol{\zeta}^\top\mathbf{B}(\mathbf{x})^\top\overline{\boldsymbol{\zeta}}) =: \gamma(\mathbf{B}) > 0, \quad (3.20)$$

and for all $\mathbf{U}\in\mathbb{P}_k(K;\mathbb{C}^3)$ it holds

$$\mathrm{Re}(Q_K(\mathbf{B}\mathbf{U}\cdot\overline{\mathbf{U}})) \ge C_2\gamma(\mathbf{B})\|\mathbf{U}\|^2_{\boldsymbol{L}^2(K)}.$$

We are now in position to verify the coercivity of the perturbed discrete sesquilinear form $\tilde{\mathsf{a}}_{T,h}(\cdot,\cdot)$ in (3.18). In the following theorem, $\theta\in[0,2\pi)$ is as in (2.5), in particular, independent of h.

Theorem 3.13. *Let $k\in\mathbb{N}$ be as in* (3.11)*, and let $Q: C^0(\breve{K};\mathbb{C})\to\mathbb{C}$ be a quadrature rule as in* (3.16) *such that Assumption 3.9 holds. Suppose that $\mathfrak{T}\subseteq C^1(\widehat{D};\mathbb{R}^3)$ satisfies Assumption 2.2 and the data satisfies Assumption 2.4. Then, there exists $\tilde{\alpha}>0$ independent of h, such that, for all $T\in\mathfrak{T}$ and all meshes τ_h on $\widehat{D}$, it holds*

$$\forall\,\widehat{\mathbf{U}}_h\in\boldsymbol{P}_0^c(\tau_h):\qquad \mathrm{Re}\left\{e^{\imath\theta}\,\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{U}}_h,\widehat{\mathbf{U}}_h)\right\} \ge \tilde{\alpha}\|\widehat{\mathbf{U}}_h\|^2_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})}. \quad (3.21)$$

Proof. Set

$$\mathbf{M}_1(T) := \det(dT)^{-1}\mu_T dT^\top dT \qquad\text{and}\qquad \mathbf{M}_2(T) := -\det(dT)^{-1}\Lambda_T dT^{-1}dT^{-\top}.$$

By (3.18)

$$\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{U}}_h,\widehat{\mathbf{U}}_h) = \sum_{K\in\tau_h} Q_K(\mathbf{M}_1(T)\,\mathbf{curl}\,\widehat{\mathbf{U}}_h\cdot\overline{\mathbf{curl}\,\widehat{\mathbf{U}}_h}) + \sum_{K\in\tau_h} Q_K(\mathbf{M}_2(T)\widehat{\mathbf{U}}_h\cdot\overline{\widehat{\mathbf{U}}_h}). \quad (3.22)$$

By (2.2) and (2.5)

$$\begin{aligned}\inf_{\widehat{\mathbf{x}}\in\widehat{D}}\inf_{\|\boldsymbol{\zeta}\|_{\mathbb{C}^3}=1}\operatorname{Re}(e^{\imath\theta}\boldsymbol{\zeta}^\top\mathbf{M}_1(T)^\top\overline{\boldsymbol{\zeta}}) &\geq \inf_{\widehat{\mathbf{x}}\in\widehat{D}}\inf_{\|\boldsymbol{\zeta}\|_{\mathbb{C}^3}=1}\det dT(\widehat{\mathbf{x}})^{-1}\operatorname{Re}(e^{\imath\theta}\mu_T(\widehat{\mathbf{x}}))\|dT(\widehat{\mathbf{x}})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}\\ &\geq \inf_{\widehat{\mathbf{x}}\in\widehat{D}}\inf_{\|\boldsymbol{\zeta}\|_{\mathbb{C}^3}=1}\vartheta\mu_b\|dT(\widehat{\mathbf{x}})^{-1}\|^{-2}_{\mathbb{C}^3}\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}\\ &\geq \vartheta^3\mu_b\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},\end{aligned}$$

where we used $\|dT^{-1}(\widehat{\mathbf{x}})\|_{\mathbb{C}^{3\times 3}}\|dT(\widehat{\mathbf{x}})\boldsymbol{\zeta}\|_{\mathbb{C}^3}\geq\|\boldsymbol{\zeta}\|_{\mathbb{C}^3}$. Similarly

$$\inf_{\widehat{\mathbf{x}}\in\widehat{D}}\inf_{\|\boldsymbol{\zeta}\|_{\mathbb{C}^3}=1}\operatorname{Re}(e^{\imath\theta}\boldsymbol{\zeta}^\top\mathbf{M}_2(T)^\top\overline{\boldsymbol{\zeta}})\geq\vartheta^3\Lambda_b\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}.$$

Applying Lemma 3.12 (ii) to (3.22), we obtain (3.21) with $\tilde{\alpha}=C_2\vartheta^3\min\{\mu_b,\Lambda_b\}$. □

3.4.4. *Strang Lemma.* The preceding theorem enables us to establish a suitable form of the first Strang Lemma.

Lemma 3.14 (Theorem 4.1.1 in [9])**.** *Let* A_1 *and* A_2 *be defined, for* $\widehat{\mathbf{U}}\in\boldsymbol{H}\left(\mathbf{curl};\widehat{D}\right)$*, as*

$$\mathrm{A}_1(\widehat{\mathbf{U}}):=\inf_{\widehat{\mathbf{U}}_h\in\boldsymbol{P}^c_0(\tau_h)}\left\{\left\|\widehat{\mathbf{U}}-\widehat{\mathbf{U}}_h\right\|_{\boldsymbol{H}\left(\mathbf{curl};\widehat{D}\right)}+\sup_{\substack{\widehat{\mathbf{W}}_h\in\boldsymbol{P}^c_0(\tau_h)\\ \widehat{\mathbf{W}}_h\neq 0}}\frac{\left|\widehat{\mathsf{a}}_T(\widehat{\mathbf{U}}_h,\widehat{\mathbf{W}}_h)-\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{U}}_h,\widehat{\mathbf{W}}_h)\right|}{\left\|\widehat{\mathbf{W}}_h\right\|_{\boldsymbol{H}\left(\mathbf{curl};\widehat{D}\right)}}\right\},$$

$$\mathrm{A}_2:=\sup_{\substack{\widehat{\mathbf{W}}_h\in\boldsymbol{P}^c_0(\tau_h)\\ \widehat{\mathbf{W}}_h\neq 0}}\frac{\left|\widehat{\mathsf{f}}_T(\widehat{\mathbf{W}}_h)-\tilde{\mathsf{f}}_{T,h}(\widehat{\mathbf{W}}_h)\right|}{\left\|\widehat{\mathbf{W}}_h\right\|_{\boldsymbol{H}\left(\mathbf{curl};\widehat{D}\right)}}.$$

Then, under Assumptions 2.2, 2.4 and 3.9, there exists a constant $C>0$ *independent of the space* $\boldsymbol{P}^c_0(\tau_h)$ *such that, for every* $T\in\mathfrak{T}$

$$\left\|\widehat{\mathbf{E}}_T-\widetilde{\mathbf{E}}_{T,h}\right\|_{\boldsymbol{H}\left(\mathbf{curl};\widehat{D}\right)}\leq C(\mathrm{A}_1(\widehat{\mathbf{E}}_T)+\mathrm{A}_2),$$

where $\widehat{\mathbf{E}}_T$ *is the unique solution to Problem 2.9, and* $\widetilde{\mathbf{E}}_{T,h}$ *is the unique solution to Problem 3.11.*

Consequently, for given $\widehat{\mathbf{U}}_h$, $\widehat{\mathbf{W}}_h\in\boldsymbol{P}^c_0(\tau_h)$, we need to bound the consistency errors:

$$\left|\widehat{\mathsf{a}}_T(\widehat{\mathbf{U}}_h,\widehat{\mathbf{W}}_h)-\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{U}}_h,\widehat{\mathbf{W}}_h)\right|, \tag{3.23}$$

$$\left|\widehat{\mathsf{f}}_T(\widehat{\mathbf{W}}_h)-\tilde{\mathsf{f}}_{T,h}(\widehat{\mathbf{W}}_h)\right|. \tag{3.24}$$

This allows, through an application of the first Strang Lemma, to prove the same order of convergence shown in Theorem 3.7 for the solution of Problem 3.11. We continue by proving a local estimate for the terms in (3.23). We require the following auxiliary Lemma, proved in Appendix A.

Lemma 3.15. *Let* $m\in\mathbb{N}$*, and* $\mathbf{B}(\widehat{\mathbf{x}})=(B_{i,j}(\widehat{\mathbf{x}}))^3_{i,j=1}\in\mathbb{C}^{3\times 3}$ *be such that* $B_{i,j}\in W^{m,\infty}(\widehat{D})$*, with* $i,j\in\{1,2,3\}$*. If the quadrature scheme,* $Q_{\breve{K}}(\cdot)$*, constructed from the nodes* $\{\breve{\boldsymbol{b}}_l\}^L_{l=1}$ *and weights* $\{\breve{w}_l\}^L_{l=1}$ *is exact on* $\mathbb{P}_{k+m-1}(\breve{K};\mathbb{C})$*, then the local quadrature error,*

$$\mathcal{E}_K(\mathbf{B}\widehat{\mathbf{U}}_h,\widehat{\mathbf{W}}_h):=\int_K\left(\mathbf{B}\widehat{\mathbf{U}}_h\right)\cdot\widehat{\mathbf{W}}_h\,\mathrm{d}\mathbf{x}-\sum_{l=1}^L w_{K,l}\left(\mathbf{B}(\boldsymbol{b}_{K,l})\widehat{\mathbf{U}}_h(\boldsymbol{b}_{K,l})\right)\cdot\widehat{\mathbf{W}}_h(\boldsymbol{b}_{K,l}),$$

is such that, for all $\widehat{\mathbf{U}}_h$ *and* $\widehat{\mathbf{W}}_h$ *in either* $\boldsymbol{P}^c_0(\tau_h)$ *or* $\boldsymbol{P}^d_0(\tau_h)$*, and for all* $K\in\tau_h$*, the following bound holds*

$$\left|\mathcal{E}_K(\mathbf{B}\widehat{\mathbf{U}}_h,\widehat{\mathbf{W}}_h)\right|\leq CC_{\mathbf{B}}h^m\left\|\widehat{\mathbf{U}}_h\right\|_{m,K}\left\|\widehat{\mathbf{W}}_h\right\|_{0,K},$$

for a positive constant C *independent of* h*,* K*, and* $\mathbf{B}$ *and with* $C_{\mathbf{B}}:=\sum^3_{i,j=1}\|B_{i,j}\|_{W^{m,\infty}(K)}$*.*

We can derive a similar estimate for the perturbed right-hand side (3.24). It follows from a slight modification of [9, Theorem 4.1.5], upon noticing that for $\widehat{\mathbf{W}}_h \in \boldsymbol{P}_0^c(\tau_h)$ the integrand in $\widehat{\mathsf{f}}_T(\widehat{\mathbf{W}}_h)$ may be written as

$$\det(dT)\mathbf{J}_T \cdot dT^{-\top}\overline{\widehat{\mathbf{W}}}_h = \det(dT)dT^{-1}\mathbf{J}_T \cdot \overline{\widehat{\mathbf{W}}}_h \,.$$

Lemma 3.16. *Let $m \in \mathbb{N}$. If the quadrature scheme constructed from the nodes $\{\breve{\boldsymbol{b}}_l\}_{l=1}^L$ and weights $\{\breve{w}_l\}_{l=1}^L$ is exact on $\mathbb{P}_{k+m-1}(\breve{K};\mathbb{C})$, and for some $p > 2$ such that $m - \frac{3}{p} > 0$, it holds*

$$\det(dT)dT^{-1}\mathbf{J}_T \in \boldsymbol{W}^{m,p}(\widehat{D}), \tag{3.25}$$

then

$$\left|\widehat{\mathsf{f}}_T(\widehat{\mathbf{W}}_h) - \tilde{\mathsf{f}}_{T,h}(\widehat{\mathbf{W}}_h)\right| \leq h^m \left|\widehat{D}\right|^{\frac{1}{2}-\frac{1}{p}} \left\|\widehat{\mathbf{W}}_h\right\|_{\boldsymbol{L}^2(\widehat{D})} \left\|\det(dT)dT^{-1}\mathbf{J}_T\right\|_{\boldsymbol{W}^{m,p}(\widehat{D})},$$

for all $\widehat{\mathbf{W}}_h \in \boldsymbol{P}_0^c(\tau_h)$.

As in [9, Thm 4.1.6], the previous results yield a convergence rate of $\mathcal{O}(h^r)$ for the numerical solution computed with quadrature schemes satisfying the conditions imposed by the various results above, whenever the solution of Problem 2.9 belongs to $\boldsymbol{H}^r(\mathbf{curl};\widehat{D})$ for $r > 0$. In the following, we shall denote $\lceil r \rceil$ as the unique integer such that $\lceil r \rceil - 1 < r \leq \lceil r \rceil$.

Theorem 3.17. *Let $r,\, p \in \mathbb{R}$ and $m \in \mathbb{N}$ be such that $0 < r \leq k$, $\lceil r \rceil \leq m$, and $p > \max\left(2, \frac{3}{m}\right)$. Also assume that*

$$\sum_{i,j,n=1}^{3} \left\|\det(dT)\Lambda_T(dT_{j,n}^{-\top} dT_{j,i}^{-\top})\right\|_{W^{\lceil r \rceil,\infty}(\widehat{D})} < \infty, \qquad \sum_{i,j,n=1}^{3} \left\|\frac{dT_{j,n}dT_{j,i}}{\det(dT)\mu_T}\right\|_{W^{\lceil r \rceil,\infty}(\widehat{D})} < \infty \tag{3.26}$$

and (3.25) hold. Then, if $\widehat{\mathbf{E}}_T$ and $\widetilde{\mathbf{E}}_{T,h}$ are the unique solutions of Problems 2.9 and 3.11, and the quadrature scheme used for the computation of $\tilde{\mathsf{a}}_{T,h}(\cdot,\cdot)$ and $\tilde{\mathsf{f}}_{T,h}(\cdot)$ is such that Assumption 3.9 holds, and the quadrature is exact on $\mathbb{P}_{k+m-1}(\breve{K};\mathbb{C})$, there exists a positive constant C_T – independent of h, but depending on $T \in \mathfrak{T}$ – such that

$$\left\|\widehat{\mathbf{E}}_T - \widetilde{\mathbf{E}}_{T,h}\right\|_{\boldsymbol{H}\left(\mathbf{curl};\widehat{D}\right)} \leq C_T h^r \left\|\widehat{\mathbf{E}}_T\right\|_{\boldsymbol{H}^r(\mathbf{curl},\widehat{D})}$$

as h tends to 0, whenever $\widehat{\mathbf{E}}_T \in \boldsymbol{H}^r(\mathbf{curl},\widehat{D})$.

Proof. Let $\widehat{\mathbf{E}}_{T,h} \in \boldsymbol{P}_0^c(\tau_h)$ be the unique solution of Problem 3.6. By direct application of Lemma 3.15, since $k + \lceil r \rceil - 1 \leq k + m - 1$, and therefore the quadrature used here satisfies the conditions in Lemma 3.15, we can write

$$\begin{aligned}\left|\widehat{\mathsf{a}}_T(\widehat{\mathbf{E}}_{T,h},\widehat{\mathbf{W}}_h) - \tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{E}}_{T,h},\widehat{\mathbf{W}}_h)\right| \leq CC_1 h^{\lceil r \rceil} &\left(\sum_{K\in\tau_h} \left\|\mathbf{curl}\,\widehat{\mathbf{E}}_{T,h}\right\|_{\lceil r \rceil,K} \left\|\mathbf{curl}\,\widehat{\mathbf{W}}_h\right\|_{0,K}\right)\\ &+CC_2 h^{\lceil r \rceil} \sum_{K\in\tau_h} \left\|\widehat{\mathbf{E}}_{T,h}\right\|_{\lceil r \rceil,K} \left\|\widehat{\mathbf{W}}_h\right\|_{0,K},\end{aligned}$$

for a positive constant C independent of T and h, and where C_1 and C_2 depend on the terms in (3.26). We shall find an estimate for $\left\|\widehat{\mathbf{E}}_{T,h}\right\|_{\lceil r \rceil,K}$,

$$\begin{aligned}\left\|\widehat{\mathbf{E}}_{T,h}\right\|_{\lceil r \rceil,K} &\leq \left\|\widehat{\mathbf{E}}_{T,h} - \mathcal{I}_K^{\#,c}(\widehat{\mathbf{E}}_T)\right\|_{\lceil r \rceil,K} + \left\|\mathcal{I}_K^{\#,c}(\widehat{\mathbf{E}}_T)\right\|_{\lceil r \rceil,K}\\ &\leq ch^{-\lceil r \rceil} \left\|\widehat{\mathbf{E}}_{T,h} - \mathcal{I}_K^{\#,c}(\widehat{\mathbf{E}}_T)\right\|_{0,K} + ch^{r-\lceil r \rceil} \left\|\widehat{\mathbf{E}}_T\right\|_{r,K}\\ &\leq ch^{-\lceil r \rceil} \left\|\widehat{\mathbf{E}}_{T,h} - \widehat{\mathbf{E}}_T\right\|_{0,K} + ch^{r-\lceil r \rceil} \left\|\widehat{\mathbf{E}}_T\right\|_{r,K}\end{aligned}$$

where we have used the inverse inequality from Lemma 3.2, the invariance of $\boldsymbol{P}^c(K)$ under $\mathcal{I}_K^{\#,c}$, and (3.14), so $c > 0$ does not depend on h nor K. We can obtain a similar bound for $\left\|\mathbf{curl}\,\widehat{\mathbf{E}}_{T,h}\right\|_{\lceil r\rceil,K}$ by the exact same proceeding, replacing $\mathcal{I}^{\#,c}$ with $\mathcal{I}^{\#,d}$,

$$\left\|\mathbf{curl}\,\widehat{\mathbf{E}}_{T,h}\right\|_{\lceil r\rceil,K} \leq ch^{-\lceil r\rceil}\left\|\mathbf{curl}\left(\widehat{\mathbf{E}}_{T,h}-\widehat{\mathbf{E}}_T\right)\right\|_{0,K} + ch^{r-\lceil r\rceil}\left\|\mathbf{curl}\,\widehat{\mathbf{E}}_T\right\|_{r,K},$$

where all constants are independent of K and h. Then,

$$\begin{aligned}\left|\hat{\mathsf{a}}_T(\widehat{\mathbf{E}}_{T,h},\widehat{\mathbf{W}}_h)-\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{E}}_{T,h},\widehat{\mathbf{W}}_h)\right| &\leq C_T\left(h^r\left\|\widehat{\mathbf{E}}_T\right\|_{\boldsymbol{H}^r(\mathbf{curl};\widehat{D})}+\left\|\widehat{\mathbf{E}}_{T,h}-\widehat{\mathbf{E}}_T\right\|_{\boldsymbol{H}(\mathbf{curl};\widehat{D})}\right)\left\|\widehat{\mathbf{W}}\right\|_{\boldsymbol{H}(\mathbf{curl};\widehat{D})},\\ &\leq C_T h^r\left\|\widehat{\mathbf{E}}_T\right\|_{\boldsymbol{H}^r(\mathbf{curl};\widehat{D})}\left\|\widehat{\mathbf{W}}\right\|_{\boldsymbol{H}(\mathbf{curl};\widehat{D})},\end{aligned}$$

where we have applied Theorem 3.17 to bound the approximation error. The conditions on the quadrature rule ensure, by direct application of Lemma 3.16,

$$\begin{aligned}\left|\hat{\mathsf{f}}_T(\widehat{\mathbf{W}}_h)-\tilde{\mathsf{f}}_{T,h}(\widehat{\mathbf{W}}_h)\right| &\leq h^m\left|\widehat{D}\right|^{\frac{1}{2}-\frac{1}{p}}\left\|\widehat{\mathbf{W}}_h\right\|_{\boldsymbol{L}^2(\widehat{D})}\left(\sum_{K\in\tau_h}\left\|\det(dT)dT^{-1}\mathbf{J}_T\right\|^p_{\boldsymbol{W}^{m,p}(K)}\right)^{1/p}\\ &\leq h^r\left|\widehat{D}\right|^{\frac{1}{2}-\frac{1}{p}}\left\|\widehat{\mathbf{W}}_h\right\|_{\boldsymbol{L}^2(\widehat{D})}\left\|\det(dT)dT^{-1}\mathbf{J}_T\right\|_{\boldsymbol{W}^{m,p}(\widehat{D})}\end{aligned}$$

for all $\widehat{\mathbf{W}}_h \in \boldsymbol{P}_0^c(\tau_h)$ and $h \in (0,1)$. The first Strang Lemma (Lemma 3.14) combined with the error bound in Theorem 3.7 yield the claimed error estimate. □

4. Shape Holomorphy

In this section we show that the solution to the pullback Maxwell problem depends holomorphically on the domain transformation T. In order to do so, we require smoothness of the data as stated in the following assumption which will replace Assumption 2.4 in the current section.

Assumption 4.1 (Material properties). There exists an open set $D_H \subseteq \mathbb{C}^3$ (the *hold-all domain*) such that for every $T \in \mathfrak{T}$ as in Assumption 2.2 it holds that the closure $\overline{D}_T \subseteq \mathbb{R}^3$ of D_T is contained in D_H. Moreover, $\mu : D_H \to \mathbb{C}$, $\Lambda : D_H \to \mathbb{C}$ and $\mathbf{J} : D_H \to \mathbb{C}^3$ in (2.6) are holomorphic functions and there exists $\theta \in [0, 2\pi)$ such that (2.5) holds.

4.1. Operator inversion.

We recall a few standard facts from functional analysis. Let X be a Banach space over $\mathbb{C}$, and let X^* be the space of all bounded antilinear mappings from X to $\mathbb{C}$. Moreover $\mathcal{L}(X;X^*)$ denotes the space of all continuous bounded linear mappings from X to X^*, and additionally in the following $\mathcal{L}_{\text{iso}}(X;X^*)$ stands for the space of bijective and bounded (and therefore boundedly invertible) mappings in $\mathcal{L}(X;X^*)$. As is well-known, if $A \in \mathcal{L}_{\text{iso}}(X;X^*)$ and $B \in \mathcal{L}(X;X^*)$ satisfies $\|A^{-1}(A-B)\|_{\mathcal{L}(X;X)} < 1$, then also $B \in \mathcal{L}_{\text{iso}}(X;X^*)$. Writing

$$B^{-1} = (A(I - A^{-1}(A-B)))^{-1} = (I - A^{-1}(A-B))^{-1}A^{-1}$$

and using the Neumann series expansion of $(I - A^{-1}(A-B))^{-1}$ we get

$$B^{-1} = \sum_{n\in\mathbb{N}_0}(A^{-1}(A-B))^n A^{-1} \in \mathcal{L}(X^*;X). \tag{4.1}$$

Equation (4.1) is formally a power series expansion of $B \mapsto B^{-1}$ locally around A between the Banach spaces $\mathcal{L}(X;X^*)$ and $\mathcal{L}(X^*;X)$. Due to $\|(A^{-1}(A-B))^n\|_{\mathcal{L}(X;X)} \leq \|A^{-1}(A-B)\|^n_{\mathcal{L}(X;X)}$, the series converges for all $B \in \mathcal{L}(X;X^*)$ with $\|A-B\|_{\mathcal{L}(X;X^*)} < \|A^{-1}\|^{-1}_{\mathcal{L}(X^*;X)}$ or $\|A^{-1}(A-B)\|_{\mathcal{L}(X;X)} < 1$. It is well-known that functions allowing a representation as a power series as in (4.1) are Fréchet differentiable, see for instance [5, Sect. 11.12]. We thus have the following proposition.

Proposition 4.2. *Let X be a Banach space. Then the inversion map*

$$\operatorname{inv} : \mathcal{L}_{\text{iso}}(X;X^*) \to \mathcal{L}_{\text{iso}}(X^*;X) : B \mapsto B^{-1}$$

is Fréchet differentiable. Moreover, if $A \in \mathcal{L}_{\rm iso}(X;X^*)$*, then*

$$\{B \in \mathcal{L}(X;X^*) \,:\, \|A-B\|_{\mathcal{L}(X;X^*)} < \|A^{-1}\|^{-1}_{\mathcal{L}(X^*;X)}\} \subseteq \mathcal{L}_{\rm iso}(X;X^*)$$

and for all B *in this set it holds* (4.1) *as well as*

$$\|B^{-1}\|_{\mathcal{L}(X^*;X)} \le \frac{\|A^{-1}\|_{\mathcal{L}(X^*;X)}}{1-\|A^{-1}\|_{\mathcal{L}(X^*;X)}\|A-B\|_{\mathcal{L}(X;X^*)}}. \tag{4.2}$$

4.2. **Holomorphic parameter dependence.** In Section 4.2.1 we show that the weak solution of the pullback lossy cavity problem, Problem 2.9, is (complex) Fréchet differentiable as a function of the domain transformation $T \in C^1(\widehat{D};\mathbb{C}^3)$. The same is verified for the discrete Galerkin solutions of Problem 3.11.

4.2.1. *Continuous case.* Introduce the linear operator

$$A : C^1(\widehat{D};\mathbb{C}^{3\times 3}) \times C^1(\widehat{D};\mathbb{C}^{3\times 3}) \to \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\widehat{D});\boldsymbol{H}_0(\mathbf{curl};\widehat{D})^*),$$

which maps data in the form of the PDE coefficients $\mathbf{B}_1 \in C^0(\widehat{D};\mathbb{C}^{3\times 3})$ and $\mathbf{B}_2 \in C^0(\widehat{D};\mathbb{C}^{3\times 3})$ to the differential operator $A(\mathbf{B}_1,\mathbf{B}_2) \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\widehat{D});\boldsymbol{H}_0(\mathbf{curl};\widehat{D})^*)$ defined through

$$\langle A(\mathbf{B}_1,\mathbf{B}_2)\widehat{\mathbf{U}},\widehat{\mathbf{V}}\rangle := \int_{\widehat{D}} \mathbf{B}_1\,\mathbf{curl}\,\widehat{\mathbf{U}}\cdot\overline{\mathbf{curl}\,\widehat{\mathbf{V}}}\,{\rm d}\widehat{\mathbf{x}} + \int_{\widehat{D}} \mathbf{B}_2\widehat{\mathbf{U}}\cdot\overline{\widehat{\mathbf{V}}}\,{\rm d}\widehat{\mathbf{x}}. \tag{4.3}$$

Let $T \in C^1(\widehat{D};\mathbb{R}^3)$. Observe that with

$$\begin{aligned}\mathbf{M}_1(T) &= \mu_T^{-1}\det(dT)^{-1}dT^\top dT \in C^0(\widehat{D};\mathbb{R}^{3\times 3}),\\ \mathbf{M}_2(T) &= -\Lambda_T\det(dT)dT^{-1}dT^{-\top} \in C^0(\widehat{D};\mathbb{R}^{3\times 3}),\end{aligned} \tag{4.4}$$

it holds for all $\widehat{\mathbf{U}}$, $\widehat{\mathbf{V}} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ that (*cp.* (2.10))

$$\mathsf{a}_T(\widehat{\mathbf{U}},\widehat{\mathbf{V}}) = \langle A(\mathbf{M}_1(T),\mathbf{M}_2(T))\widehat{\mathbf{U}},\overline{\widehat{\mathbf{V}}}\rangle_{\widehat{D}}, \tag{4.5}$$

and with $\mathbf{F}_T \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})^*$ defined by

$$\mathbf{F}_T(\widehat{\mathbf{V}}) := -\imath\omega\int_{\widehat{D}} dT^{-1}\mathbf{J}_T\cdot\overline{\widehat{\mathbf{V}}}\,{\rm d}\widehat{\mathbf{x}},$$

We write Problem 2.9 as operator equation:

$$A(\mathbf{M}_1(T),\mathbf{M}_2(T))\widehat{\mathbf{E}}_T = \mathbf{F}_T. \tag{4.6}$$

To show Fréchet differentiability of the weak solution $\widehat{\mathbf{E}}_T \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ of Problem 2.9 with respect to T, we start by verifying that the right-hand side $\mathbf{F}_T \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})^*$ and the coefficients $\mathbf{M}_1(T) \in C^0(\widehat{D};\mathbb{R}^{3\times 3})$ and $\mathbf{M}_2(T) \in C^0(\widehat{D};\mathbb{R}^{3\times 3})$ are Fréchet differentiable as functions of T. To this end, first note the following: with $D_H \subseteq \mathbb{C}^3$ as in Assumption 4.1, suppose that $f : D_H \to \mathbb{C}$ is a holomorphic function with uniformly bounded second derivatives and let $T : \widehat{D} \to D_H$ be such that the compact closure of $T(\widehat{D})$ is contained in D_H. Denote by

$$\nabla^2 f = \left(\frac{\partial^2 f}{\partial_{x_i}\partial_{x_j}}\right)_{i,j=1}^3 : D_H \to \mathbb{C}^{3\times 3}$$

the Hessian of f. Then, in a neighbourhood of T, $T \mapsto f\circ T \in C^0(\widehat{D})$ is Fréchet differentiable as

$$\|f\circ(T+H) - f\circ T - \nabla(f\circ T)\cdot H\|_{C^0(\widehat{D};\mathbb{C})} \le \|\nabla^2 f\|_{C^0(D_H;\mathbb{C}^{3\times 3})}\|H\|^2_{C^0(\widehat{D};\mathbb{C}^3)} \tag{4.7}$$

so that

$$\lim_{\|H\|_{C^0(\widehat{D};\mathbb{C}^3)}\to 0} \frac{\|f\circ(T+H) - f\circ T - \nabla(f\circ T)\cdot H\|_{C^0(\widehat{D};\mathbb{C})}}{\|H\|_{C^0(\widehat{D};\mathbb{C}^3)}} = 0. \tag{4.8}$$

We thus have the following lemma (*cf.* [22, Lemma 4.1] for a similar result). For its proof, we recall that the composition of Fréchet differentiable maps between Banach spaces is again Fréchet differentiable [38, Proposition 4.10].

Lemma 4.3. *Define*

$$S := \left\{ T \in C^1(\widehat{D};\mathbb{C}^3) \, : \, \inf_{\widehat{\mathbf{x}}\in\widehat{D}} |\det(dT(\widehat{\mathbf{x}}))| > 0, \; \overline{T(\widehat{D})} \subseteq D_H \right\}, \tag{4.9}$$

where $\overline{T(\widehat{D})}$ *is understood as the closure of* $T(\widehat{D})$ *in* $\mathbb{C}^3$*. Then* $S \subseteq C^1(\widehat{D};\mathbb{C}^{3\times 3})$ *is open and*

$$\mathbf{M}_1 : S \to C^0(\widehat{D};\mathbb{C}^{3\times3}) \qquad \text{and} \qquad \mathbf{M}_2 : S \to C^0(\widehat{D};\mathbb{C}^{3\times3})$$

defined in (4.4) *are Fréchet differentiable.*

Proof. We only show the statement for $\mathbf{M}_1$, since similar arguments apply to $\mathbf{M}_2$. The maps $T \mapsto dT$ and $T \mapsto dT^\top$ are bounded and linear, and thus Fréchet differentiable, as mappings from $C^1(\widehat{D};\mathbb{C}^3)$ to $C^0(\widehat{D};\mathbb{C}^{3\times3})$. By (4.8), the map $T \mapsto \mu_T = \mu \circ T$ is Fréchet differentiable from $\{T \in C^0(\widehat{D};\mathbb{C}^3) \, : \, \overline{T(\widehat{D})} \subseteq D_H\} \to C^0(\widehat{D};\mathbb{C}^3)$, and, consequently, also from $\{T \in C^1(\widehat{D};\mathbb{C}^3) \, : \, \overline{T(\widehat{D})} \subseteq D_H\} \to C^0(\widehat{D};\mathbb{C}^3))$. Letting $f : \{\mathbf{M} \in \mathbb{C}^{3\times3} \, : \, \det(\mathbf{M}) \neq 0\} \to \mathbb{C}$ in (4.8) be the rational –thus holomorphic– function $f(\mathbf{M}) = \det(\mathbf{M})^{-1}$, we find that also the composition $T \mapsto dT \mapsto \det(dT)^{-1}$ is Fréchet differentiable from S to $C^0(\widehat{D};\mathbb{C})$. Finally, the product $\prod_{j=1}^n F_j \in C^0(\widehat{D};\mathbb{C})$ of functions $F_j \in C^0(\widehat{D};\mathbb{C})$ is a bounded multilinear map, and therefore jointly differentiable as a function of $(F_j)_{j=1}^n \in C^0(\widehat{D};\mathbb{C})^n$. This shows that $T \mapsto \mathbf{M}_1(T) = \mu_T^{-1}\det(dT)^{-1} dT^\top dT \in C^0(\widehat{D};\mathbb{C}^{3\times3})$ is Fréchet differentiable for all $T \in S$. ☐

In the same fashion we obtain holomorphic dependence of the right-hand side on the transformation (also see the proof of Lemma 4.7 ahead).

Lemma 4.4. *With* S *as in* (4.9), $S \ni T \mapsto \mathbf{F}_T \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})^*$ *is Fréchet differentiable.*

In line with our results in [22, Theorem 4.1 and Theorem 4.2] where we considered a perfect conductor and a dielectric scatterer, we obtain the following theorem stating *shape holomorphy* for the lossy cavity problem.

Theorem 4.5. *Let* $\mathfrak{T} \subseteq C^1(\widehat{D};\mathbb{R}^3)$ *satisfy Assumption 2.2, and let the data satisfy Assumption 4.1. There exists an open set* $O_{\mathfrak{T}} \subseteq C^1(\widehat{D};\mathbb{C}^3)$ *containing* $\mathfrak{T}$*, and a Fréchet differentiable function* $\mathfrak{E} : O_{\mathfrak{T}} \to \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ *such that, for every* $T \in \mathfrak{T}$*, the unique solution* $\widehat{\mathbf{E}}_T \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ *of Problem 2.9 is given by* $\widehat{\mathbf{E}}_T = \mathfrak{E}(T)$*.*

Proof. Throughout this proof let $X := \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ and let $S \subseteq C^1(\widehat{D};\mathbb{C}^3)$ be as in (4.9). The solution operator $\mathfrak{E}$ mapping a transformation T to the solution $\widehat{\mathbf{E}}_T$ of (4.6) can formally be written as

$$\mathfrak{E}(T) = A(\mathbf{M}_1(T),\mathbf{M}_2(T))^{-1}\mathbf{F}_T$$

and thus as a composition of the functions:

$$\Phi_1 := \begin{cases} S \to X^* \\ T \mapsto \mathbf{F}_T, \end{cases} \qquad \Phi_2 := \begin{cases} S \to C^0(\widehat{D};\mathbb{C}^{3\times3}) \times C^0(\widehat{D};\mathbb{C}^{3\times3}) \\ T \mapsto (\mathbf{M}_1(T),\mathbf{M}_2(T)), \end{cases}$$

and

$$\Phi_3 := \begin{cases} C^0(\widehat{D};\mathbb{C}^{3\times3}) \times C^0(\widehat{D};\mathbb{C}^{3\times3}) \to \mathcal{L}(X;X^*) \\ (\mathbf{B}_1,\mathbf{B}_2) \mapsto A(\mathbf{B}_1,\mathbf{B}_2), \end{cases}$$

together with the inversion of these operators.

By Lemma 4.3 and Lemma 4.4, Φ_1 and Φ_2 are Fréchet differentiable. Fréchet differentiability of Φ_3 is an immediate consequence of the fact that $A(\mathbf{B}_1,\mathbf{B}_2) \in \mathcal{L}(X;X^*)$ in (4.3) depends linearly and boundedly on $(\mathbf{B}_1,\mathbf{B}_2) \in C^0(\widehat{D};\mathbb{C}^{3\times3}) \times C^0(\widehat{D};\mathbb{C}^{3\times3})$: we have

$$\begin{aligned} \|A(\mathbf{B}_1,\mathbf{B}_2)\|_{\mathcal{L}(X;X^*)} &= \sup_{\|\widehat{\mathbf{U}}\|_X=1} \sup_{\|\widehat{\mathbf{V}}\|_X=1} |\langle A(\mathbf{B}_1,\mathbf{B}_2)\widehat{\mathbf{U}},\widehat{\mathbf{V}}\rangle_{\widehat{D}}| \\ &= \sup_{\|\widehat{\mathbf{U}}\|_X=1} \sup_{\|\widehat{\mathbf{V}}\|_X=1} \left| \int_{\widehat{D}} \mathbf{B}_1 \,\mathbf{curl}\,\widehat{\mathbf{U}} \cdot \overline{\mathbf{curl}\,\widehat{\mathbf{V}}} \,\mathrm{d}\widehat{\mathbf{x}} + \int_{\widehat{D}} \mathbf{B}_2 \widehat{\mathbf{U}} \cdot \overline{\widehat{\mathbf{V}}} \,\mathrm{d}\widehat{\mathbf{x}} \right| \\ &\leq \|\mathbf{B}_1\|_{C^0(\widehat{D};\mathbb{C}^{3\times3})} + \|\mathbf{B}_2\|_{C^0(\widehat{D};\mathbb{C}^{3\times3})}. \end{aligned} \tag{4.10}$$

Next, let us show that $A(\mathbf{M}_1(T), \mathbf{M}_2(T)) \in \mathcal{L}_{\rm iso}(X; X^*)$ for all T in some open subset $O_{\mathfrak{T}}$ of $C^1(\widehat{D}; \mathbb{C}^3)$ containing $\mathfrak{T}$. As stated in Proposition 2.10, for every $T \in \mathfrak{T}$ the perturbed sesquilinear form $\mathsf{a}_T(\cdot,\cdot) : X \times X \to \mathbb{C}$ is coercive with constant $\alpha > 0$, and also continuous. The complex Lax-Milgram lemma implies that for every $\mathbf{F} \in X^*$ there exists a unique solution $\mathbf{E} \in X$ of $\mathsf{a}_T(\mathbf{E}, \mathbf{V}) = \langle \mathbf{F}, \mathbf{V} \rangle_{\widehat{D}}$ for all $\mathbf{V} \in X$. Moreover, we have the apriori estimate $\|\mathbf{E}\|_X \leq \alpha^{-1} \|\mathbf{F}\|_{X^*}$. Due to (4.5) this is equivalent to $A(\mathbf{M}_1(T), \mathbf{M}_2(T)) \in \mathcal{L}_{\rm iso}(X; X^*)$ and $\|A(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1}\|_{\mathcal{L}(X^*;X)} \leq \frac{1}{\alpha}$. Using (4.10) and Proposition 4.2 we conclude that, for every $T \in \mathfrak{T}$ and all $\mathbf{H}_1, \mathbf{H}_2 \in C^0(\widehat{D}; \mathbb{C}^{3\times 3})$ with

$$\|\mathbf{M}_1(T) - \mathbf{H}_1\|_{C^0(\widehat{D};\mathbb{C}^{3\times3})} + \|\mathbf{M}_2(T) - \mathbf{H}_2\|_{C^0(\widehat{D};\mathbb{C}^{3\times3})} < \alpha,$$

then $A(\mathbf{M}_1(T) + \mathbf{H}_1, \mathbf{M}_2(T) + \mathbf{H}_2) \in \mathcal{L}_{\rm iso}(X; X^*)$. Using the continuity of $\mathbf{M}_1$ and $\mathbf{M}_2$ and compactness of $\mathfrak{T}$, we can find an open set $O_{\mathfrak{T}} \subseteq C^1(\widehat{D}; \mathbb{C}^3)$ containing $\mathfrak{T}$ such that $A(\mathbf{M}_1(T), \mathbf{M}_2(T)) \in \mathcal{L}_{\rm iso}(X; X^*)$ for all $T \in O_{\mathfrak{T}}$. Decreasing $O_{\mathfrak{T}}$ if necessary, it holds $O_{\mathfrak{T}} \subseteq S$ and thus

$$O_{\mathfrak{T}} \ni T \mapsto A(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1} \in \mathcal{L}(X^*; X) \qquad \text{and} \qquad O_{\mathfrak{T}} \ni T \mapsto \mathbf{F}_T \in X^*$$

are both Fréchet differentiable by Proposition 4.2 and Lemma 4.4.

Finally, the map $\mathcal{L}(X^*; X) \times X^* \ni (B, F) \mapsto BF \in X$ is bilinear, and therefore Fréchet differentiable as a function of (B, F). We conclude that

$$\mathfrak{E} = \begin{cases} O_{\mathfrak{T}} \to \boldsymbol{H}_0(\mathbf{curl}; \widehat{D}) \\ T \mapsto A(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1} \mathbf{F}_T \end{cases}$$

is Fréchet differentiable. □

Remark 4.6. For $\mathbf{B}_1$, $\mathbf{B}_2 \in C^0(\widehat{D}; \mathbb{C}^{3\times3})$ let $A(\mathbf{B}_1, \mathbf{B}_2) \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl}; \widehat{D}); \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})^*)$ be as in (4.3). The proof of Theorem 4.5 then shows that for $F \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})^*$ the map $(\mathbf{B}_1, \mathbf{B}_2) \mapsto A(\mathbf{B}_1, \mathbf{B}_2)^{-1} F \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ is locally complex Fréchet differentiable with respect to $\mathbf{B}_1$, $\mathbf{B}_2 \in C^0(\widehat{D}; \mathbb{C}^{3\times3})$. Therefore, our analysis also covers general uncertainty in the coefficients $\mathbf{B}_1$, $\mathbf{B}_2$ of the Maxwell-type differential operator $A(\mathbf{B}_1, \mathbf{B}_2)$. The currently discussed case of a parametric domain is a specific application of uncertainty in those coefficients.

4.2.2. *Discrete case.* Unlike in [22], we now show Fréchet differentiability of the discrete Galerkin solution with respect to the transformation. For a mesh τ_h of $\widehat{D}$ and $K \in \tau_h$, let in the following $Q : C^0(\breve{K}; \mathbb{C}) \to \mathbb{C}$ and $Q_K : C^0(K; \mathbb{C}) \to \mathbb{C}$ be fixed quadrature rules as in (3.16), (3.17). Moreover, the subspace $\boldsymbol{P}_0^c(\tau_h) \subseteq \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ is considered as a Banach space equipped with the norm of $\boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$. Recall that $k \in \mathbb{N}$ denotes the polynomial degree of $\boldsymbol{P}_0^c(\tau_h)$ so that (3.11) holds.

Define $A_h : C^0(\widehat{D}; \mathbb{C}^{3\times3}) \times C^0(\widehat{D}; \mathbb{C}^{3\times3}) \to \mathcal{L}(\boldsymbol{P}_0^c(\tau_h); \boldsymbol{P}_0^c(\tau_h)^*)$ via

$$\langle A_h(\mathbf{B}_1, \mathbf{B}_2) \widehat{\mathbf{U}}_h, \widehat{\mathbf{V}}_h \rangle_{\widehat{D}} = \sum_{K \in \tau_h} Q_K \left(\mathbf{B}_1 \, \mathbf{curl}\, \widehat{\mathbf{U}}_h \cdot \overline{\mathbf{curl}\, \widehat{\mathbf{V}}_h} + B_2 \widehat{\mathbf{U}}_h \cdot \overline{\widehat{\mathbf{V}}_h} \right). \tag{4.11}$$

Then, with $\mathbf{M}_1(T)$, $\mathbf{M}_2(T)$ as in (4.4) and $\tilde{\mathsf{a}}_{T,h}(\cdot,\cdot)$ as in (3.18), for all $\widehat{\mathbf{U}}_h$, $\widehat{\mathbf{V}}_h \in \boldsymbol{P}_0^c(\tau_h)$ (cp. (3.18))

$$\tilde{\mathsf{a}}_{T,h}(\widehat{\mathbf{U}}, \widehat{\mathbf{V}}) = \langle A_h(\mathbf{M}_1(T), \mathbf{M}_2(T)) \widehat{\mathbf{U}}, \widehat{\mathbf{V}} \rangle_{\widehat{D}}.$$

Recall that we denote by the constant $\omega > 0$ the circular frequency. With $\mathbf{F}_{T,h} \in \boldsymbol{P}_0^c(\tau_h)^*$ defined as (cp. (2.9))

$$\mathbf{F}_{T,h}(\widehat{\mathbf{V}}_h) := -\imath\omega \sum_{K \in \tau_h} Q_K \left(\mathbf{J}_T \cdot dT^{-\top} \widehat{\mathbf{V}}_h \right),$$

Problem 3.11 then reads

$$A_h(\mathbf{M}_1(T), \mathbf{M}_2(T)) \widetilde{\mathbf{E}}_{T,h} = \mathbf{F}_{T,h}. \tag{4.12}$$

Lemma 4.7. *Let $S \subseteq C^1(\widehat{D}; \mathbb{C}^3)$ be as in (4.9), and let Assumption 2.2 and Assumption 4.1 be satisfied. Then there exists a constant $C > 0$ such that for any mesh τ_h on $\widehat{D}$ the map $S \ni T \mapsto \mathbf{F}_{T,h} \in \boldsymbol{P}_0^c(\tau_h)^*$ is Fréchet differentiable and $\|\mathbf{F}_{T,h}\|_{\boldsymbol{P}_0^c(\tau_h)^*} \leq C \| -\imath\omega dT^{-1} \mathbf{J}_T\|_{C^0(\widehat{D};\mathbb{C}^3)}$ for all $T \in S$.*

Proof. Throughout this proof fix an (arbitrary) mesh τ_h on $\widehat{D}$ and denote $X_h := \boldsymbol{P}_0^c(\tau_h)$. With

$$\Phi_1 := \begin{cases} S \to C^0(\widehat{D};\mathbb{C}^3) \\ T \mapsto -\imath\omega dT^{-1}\mathbf{J}_T \end{cases} \qquad \text{and} \qquad \Phi_2 := \begin{cases} C^0(\widehat{D};\mathbb{C}^3) \to X_h^* \\ \boldsymbol{f} \mapsto \left(X_h \ni \widehat{\mathbf{V}}_h \mapsto \sum_{K\in\tau_h} Q_K(\boldsymbol{f}\cdot\overline{\widehat{\mathbf{V}}}_h)\right) \end{cases}$$

we can write $\mathbf{F}_{T,h} = \Phi_2 \circ \Phi_1(T) \in X_h^*$ for all $T \in S$.

By Lemma 4.4 Φ_1 is Fréchet differentiable. To show Fréchet differentiability of Φ_2 we show that it is a bounded and linear map. Linearity is clear. To see its boundedness, we compute

$$\begin{aligned} \sup_{\|\boldsymbol{f}\|_{C^0(\widehat{D};\mathbb{C}^3)}=1} \|\Phi_2(\boldsymbol{f})\|_{X_h^*} &= \sup_{\|\boldsymbol{f}\|_{C^0(\widehat{D};\mathbb{C}^3)}=1} \sup_{\|\mathbf{V}_h\|_{X_h^*}=1} \left|\langle \Phi_2(\boldsymbol{f}), \mathbf{V}_h\rangle_{\widehat{D}}\right| \\ &\le \sup_{\|\boldsymbol{f}\|_{C^0(\widehat{D};\mathbb{C}^3)}=1} \sup_{\|\mathbf{V}_h\|_{X_h^*}=1} \sum_{K\in\tau_h} \left|Q_K(\boldsymbol{f}\cdot\overline{\mathbf{V}}_h)\right|. \end{aligned} \tag{4.13}$$

Now, for every $\mathbf{V}_h \in X_h$ and every $K \in \tau_h$ (cp. (3.17))

$$\left|Q_K(\boldsymbol{f}\cdot\overline{\mathbf{V}}_h)\right| \le \left(\sum_{l=1}^L |\breve{w}_l|\right) |\det(\mathbb{J}_K)|\, \|\boldsymbol{f}\|_{C^0(K;\mathbb{C}^3)} \|\mathbf{V}_h\|_{\boldsymbol{L}^\infty(K)}. \tag{4.14}$$

Due to the finite dimension of $\mathbb{P}_k(\breve{K};\mathbb{C}^3)$ there exists $\tilde{C}$ such that $\|\boldsymbol{g}\|_{\boldsymbol{L}^\infty(\breve{K})} \le \tilde{C}\|\boldsymbol{g}\|_{\boldsymbol{L}^1(\breve{K})}$ for all $\boldsymbol{g} \in \mathbb{P}_k(\breve{K};\mathbb{C}^3)$. Thus

$$\begin{aligned} |\det(\mathbb{J}_K)|\, \|\mathbf{V}_h\|_{\boldsymbol{L}^\infty(K)} &= |\det(\mathbb{J}_K)|\, \|\mathbf{V}_h \circ T_K\|_{\boldsymbol{L}^\infty(\breve{K})} \\ &\le \tilde{C}\, |\det(\mathbb{J}_K)|\, \|\mathbf{V}_h \circ T_K\|_{\boldsymbol{L}^1(\breve{K})} \\ &= \tilde{C}\|\mathbf{V}_h\|_{\boldsymbol{L}^1(K)}. \end{aligned} \tag{4.15}$$

In all, (4.13), (4.14) and (4.15) imply

$$\begin{aligned} \|\Phi_2\|_{\mathcal{L}(C^0(\widehat{D};\mathbb{C}^3);X_h^*)} &= \sup_{\|\boldsymbol{f}\|_{C^0(\widehat{D})}=1} \|\Phi_2(\boldsymbol{f})\|_{X_h^*} \\ &\le \tilde{C}\left(\sum_{l=1}^L |\breve{w}_l|\right) \sup_{\|\mathbf{V}_h\|_{X_h}=1} \|\mathbf{V}_h\|_{\boldsymbol{L}^1(\widehat{D})} \\ &\le \tilde{C}\left(\sum_{l=1}^L |\breve{w}_l|\right) \left|\widehat{D}\right|^{\frac{1}{2}} \sup_{\|\mathbf{V}_h\|_{X_h}=1} \|\mathbf{V}_h\|_{\boldsymbol{L}^2(\widehat{D})} \\ &\le \tilde{C}\left(\sum_{l=1}^L |\breve{w}_l|\right) \left|\widehat{D}\right|^{\frac{1}{2}}. \end{aligned}$$

This shows that Φ_2 is Fréchet differentiable (because it is linear and bounded), and therefore also $S \ni T \mapsto \mathbf{F}_{T,h} = \Phi_2 \circ \Phi_1(T) \in X_h^*$ is Fréchet differentiable. Finally,

$$\|\mathbf{F}_{T,h}\|_{X_h^*} = \|\Phi_2 \circ \Phi_1(T)\|_{X_h^*} \le \|\Phi_2\|_{\mathcal{L}(C^0(\widehat{D};\mathbb{C}^3);X_h^*)} \| -\imath\omega dT^{-1}\mathbf{J}_T\|_{C^0(\widehat{D};\mathbb{C}^3)},$$

where $\|\Phi_2\|_{\mathcal{L}(C^0(\widehat{D};\mathbb{C}^3);X_h^*)} \le \tilde{C}\left(\sum_{l=1}^L |\breve{w}_l|\right)\left|\widehat{D}\right|^{\frac{1}{2}}$ is bounded by a constant independent of the mesh τ_h. □

The next theorem is a discrete version of Theorem 4.5.

Theorem 4.8. *Let $\mathfrak{T} \subseteq C^1(\widehat{D};\mathbb{R}^3)$ satisfy Assumption 2.2, let the quadrature rule $Q : C^0(\breve{K};\mathbb{C}) \to \mathbb{C}$ satisfy Assumption 3.9, and let the data satisfy Assumption 4.1. There exists an open set $O_{\mathfrak{T}} \subseteq C^1(\widehat{D};\mathbb{C}^3)$ containing $\mathfrak{T}$ and a constant $C > 0$ such that the following holds: If τ_h is a mesh on $\widehat{D}$, then there exists a Fréchet differentiable function $\mathfrak{E}_h : O_{\mathfrak{T}} \to \boldsymbol{P}_0^c(\tau_h) \subseteq \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ such that*

$$\sup_{T\in O_{\mathfrak{T}}} \|\mathfrak{E}_h(T)\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} \le C. \tag{4.16}$$

For every $T \in \mathfrak{T}$, the unique solution $\widetilde{\mathbf{E}}_{T,h} \in \boldsymbol{P}_0^c(\tau_h)$ of Problem 3.11 is given by $\widetilde{\mathbf{E}}_{T,h} = \mathfrak{E}_h(T)$.

Proof. We proceed similar as in the proof of Theorem 4.5. Fix a mesh τ_h on $\widehat{D}$. Throughout this proof S is as in (4.9) and we denote $X_h := \boldsymbol{P}_0^c(\tau_h) \subseteq \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$.

The discrete solution operator $\mathfrak{E}_h$, which maps a domain transformation T to the FE-solution $\widetilde{\mathbf{E}}_{T,h}$ of Problem 3.11, can formally be written as

$$\mathfrak{E}_h(T) = A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1}\mathbf{F}_{T,h}\ .$$

It is a composition of the functions

$$\Phi_1 := \begin{cases} S \to X_h^* \\ T \mapsto \mathbf{F}_{T,h}, \end{cases} \qquad \Phi_2 := \begin{cases} S \to C^0(\widehat{D}; \mathbb{C}^{3\times3}) \times C^0(\widehat{D}; \mathbb{C}^{3\times3}) \\ T \mapsto (\mathbf{M}_1(T), \mathbf{M}_2(T)), \end{cases}$$

and

$$\Phi_3 := \begin{cases} C^0(\widehat{D}; \mathbb{C}^{3\times3}) \times C^0(\widehat{D}; \mathbb{C}^{3\times3}) \to \mathcal{L}(X_h; X_h^*) \\ (\mathbf{B}_1, \mathbf{B}_2) \mapsto A_h(\mathbf{B}_1, \mathbf{B}_2), \end{cases}$$

together with the inversion of operators.

By Lemma 4.3 and Lemma 4.7, Φ_1 and Φ_2 are Fréchet differentiable. Fréchet differentiability of Φ_3 follows by the fact that $A_h(\mathbf{B}_1, \mathbf{B}_2) \in \mathcal{L}(X_h; X_h^*)$ in (4.11) is a bounded linear function of $(\mathbf{B}_1, \mathbf{B}_2) \in C^0(\widehat{D}; \mathbb{C}^3) \times C^0(\widehat{D}; \mathbb{C}^3)$: using Lemma 3.12 (i) and Hölder's inequality we infer

$$\begin{aligned} \|A_h(\mathbf{B}_1, \mathbf{B}_2)\|_{\mathcal{L}(X_h; X_h^*)} &\leq \sup_{\|\widehat{\mathbf{U}}_h\|_{X_h}=1} \sup_{\|\widehat{\mathbf{V}}_h\|_{X_h}=1} \sum_{K\in\tau_h} |Q_K(\mathbf{B}_1 \,\mathbf{curl}\, \widehat{\mathbf{U}}_h \cdot \overline{\mathbf{curl}\, \widehat{\mathbf{V}}_h} + \mathbf{B}_2 \widehat{\mathbf{U}}_h \cdot \widehat{\mathbf{V}}_h)| \\ &\leq C_1(\|\mathbf{B}_1\|_{C^0(\widehat{D}; \mathbb{C}^{3\times3})} + \|\mathbf{B}_2\|_{C^0(\widehat{D}; \mathbb{C}^{3\times3})}) \end{aligned} \tag{4.17}$$

with a constant C_1 that does not depend on the mesh τ_h. This shows that $A_h : C^0(\widehat{D}; \mathbb{C}^3) \times C^0(\widehat{D}; \mathbb{C}^3) \to \mathcal{L}(X_h; X_h^*)$ is bounded and linear.

We claim that there exists an open set $O_{\mathfrak{T}} \subseteq C^1(\widehat{D}; \mathbb{C}^3)$ containing $\mathfrak{T}$ and a constant $C_{\mathfrak{T}} > 0$, both independent of the mesh τ_h, such that $A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1} \in \mathcal{L}_{\mathrm{iso}}(X^*; X)$ and

$$\|A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1}\|_{\mathcal{L}(X_h^*; X_h)} \leq C_{\mathfrak{T}}$$

for all $T \in O_{\mathfrak{T}}$. As stated in Theorem 3.13, there exists $\tilde{\alpha} > 0$ such that for every $T \in \mathfrak{T}$ the perturbed sesquilinear form $\tilde{\mathsf{a}}_{T,h}(\cdot,\cdot) : X_h \times X_h \to \mathbb{C}$ is coercive with coercivity constant $\tilde{\alpha} > 0$ independent of τ_h. Therefore, for all $T \in \mathfrak{T}$, it holds

$$\|A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1}\|_{\mathcal{L}(X_h^*; X_h)} \leq \frac{1}{\tilde{\alpha}}.$$

Fix a domain transformation $T \in \mathfrak{T}$. Then for every $\mathbf{H}_1$, $\mathbf{H}_2 \in C^0(\widehat{D}; \mathbb{C}^{3\times3})$ with

$$\|\mathbf{H}_1\|_{C^0(\widehat{D}; \mathbb{C}^{3\times3})} + \|\mathbf{H}_2\|_{C^0(\widehat{D}; \mathbb{C}^{3\times3})} < \frac{\tilde{\alpha}}{2C_1}$$

there holds, by (4.17),

$$\|A_h(\mathbf{M}_1(T) + \mathbf{H}_1, \mathbf{M}_2(T) + \mathbf{H}_2) - A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))\|_{\mathcal{L}(X_h; X_h^*)} < \frac{\tilde{\alpha}}{2}.$$

Due to $\|A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1}\|_{\mathcal{L}(X_h; X_h^*)} \leq \frac{1}{\tilde{\alpha}}$, (4.2) implies that for

$$O_N := \bigcup_{T\in\mathfrak{T}} \{(\mathbf{M}_1(T), \mathbf{M}_2(T)) + (\mathbf{H}_1, \mathbf{H}_2) \,:\, \|\mathbf{H}_1\|_{C^0(\widehat{D}; \mathbb{C}^{3\times3})} + \|\mathbf{H}_2\|_{C^0(\widehat{D}; \mathbb{C}^{3\times3})} < \frac{\tilde{\alpha}}{2C_1}\}$$

it holds $\{A(\mathbf{B}_1, \mathbf{B}_2) \,:\, (\mathbf{B}_1, \mathbf{B}_2) \in O_N\} \subseteq \mathcal{L}_{\mathrm{iso}}(X_h; X_h^*)$ and

$$\sup_{(\mathbf{B}_1, \mathbf{B}_2)\in O_N} \|A(\mathbf{B}_1, \mathbf{B}_2)^{-1}\|_{\mathcal{L}(X_h^*; X_h)} \leq \frac{\tilde{\alpha}^{-1}}{1 - \frac{1}{\tilde{\alpha}}\frac{\tilde{\alpha}}{2}} = \frac{2}{\tilde{\alpha}}.$$

Continuity of $T \mapsto \mathbf{M}_1(T)$ and $T \mapsto \mathbf{M}_2(T)$ implies that there exists an open set $O_{\mathfrak{T}} \subseteq C^1(\widehat{D}; \mathbb{C}^3)$ containing the compact set $\mathfrak{T}$ such that $(\mathbf{M}_1(T), \mathbf{M}_2(T)) \in O_N$ for all $T \in O_{\mathfrak{T}}$.

In all, we conclude that, for any mesh τ_h, both

$$O_{\mathfrak{T}} \ni T \mapsto A_h(\mathbf{M}_1(T), \mathbf{M}_2(T))^{-1} \in \mathcal{L}(X_h^*; X_h) \qquad \text{and} \qquad O_{\mathfrak{T}} \ni T \mapsto \mathbf{F}_{T,h} \in X_h^*$$

are Fréchet differentiable as well as

$$\sup_{T\in O_{\mathfrak{T}}} \|A_h(\mathbf{M}_1(T),\mathbf{M}_2(T))^{-1}\|_{\mathcal{L}(X_h^*;X_h)} \le \frac{2}{\tilde{\alpha}}. \tag{4.18}$$

Finally, as in the proof of Theorem 4.5, the fact that $\mathcal{L}(X^*;X)\times X^* \ni (B,F)\mapsto BF\in X$ is bilinear and bounded, and therefore Fréchet differentiable as a function of (B,F), implies that

$$\mathfrak{E}_h = \begin{cases} O_{\mathfrak{T}} \to X_h \\ T\mapsto A_h(\mathbf{M}_1(T),\mathbf{M}_2(T))^{-1}\mathbf{F}_{T,h} \end{cases}$$

is Fréchet differentiable.

Decreasing the open superset $O_{\mathfrak{T}}\subseteq C^1(\widehat{D};\mathbb{C}^3)$ of the compact set $\mathfrak{T}$ if necessary we can assume

$$\sup_{T\in O_{\mathfrak{T}}} \|-\imath\omega dT^{-1}\mathbf{J}_T\|_{C^0(\widehat{D};\mathbb{C}^3)} < \infty.$$

Lemma 4.7 and (4.18) then give the uniform bound (4.16). □

5. High dimensional approximation

We specialize the preceding, abstract considerations for concrete, *countably-parametric* families of domain transformations. This kind of *uncertainty parametrization* will reduce the computational UQ for CEM in shape space to *high-dimensional parametric quadrature and interpolation problems.* We discuss several numerical techniques that have emerged in recent years that are able to deliver dimension-independent convergence rates.

5.1. **Domain Parametrization.** We prepare the computational domain uncertainty quantification with *uncertainty parametrization.* Specifically, we adopt a family $\mathfrak{T}$ of countably-parametric domain transformations satisfying Assumption 2.2.

Let $(T_j)_{j\in\mathbb{N}_0}\subseteq C^1(\widehat{D};\mathbb{R}^3)$ and $p\in(0,1)$ be such that

$$(\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)})_{j\in\mathbb{N}_0}\in\ell^p(\mathbb{N}_0). \tag{5.1}$$

Set $U:=[-1,1]^{\mathbb{N}}$. Define

$$T(\boldsymbol{y}) := T_0 + \sum_{j\in\mathbb{N}} y_j T_j \qquad \forall\, \boldsymbol{y}\in U.$$

Due to (5.1) we have $(\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)})_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0)$ and thus $T(\boldsymbol{y})\in C^1(\widehat{D};\mathbb{R}^3)$ is well-defined for all $\boldsymbol{y}\in U$. The set of admissible domain maps

$$\mathfrak{T} := \{T(\boldsymbol{y}) \,:\, \boldsymbol{y}\in U\}\subseteq C^1(\widehat{D};\mathbb{R}^3) \tag{5.2}$$

is compact. This follows by the compactness of the set U (equipped with the product topology as a consequence of Tychonoff's theorem) and the fact that $U\ni\boldsymbol{y}\mapsto T(\boldsymbol{y})\in C^1(\widehat{D};\mathbb{R}^3)$ is continuous, which can be verified by using that $(\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)})_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0)$. Throughout what follows, we assume $(T_j)_{j\in\mathbb{N}}$ and $\widehat{D}\subseteq\mathbb{R}^3$ to be such that Assumption 2.2 is satisfied.

Then, according to Theorem 4.5, the pullback solution $\widehat{\mathbf{E}}_{T(\boldsymbol{y})}\in\boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ of Problem 2.7 allows the representation:

$$\widehat{\mathbf{E}}_{T(\boldsymbol{y})} = \mathfrak{E}\left(T_0+\sum_{j\in\mathbb{N}} y_jT_j\right) \qquad \forall\,\boldsymbol{y}\in U, \tag{5.3}$$

where $\mathfrak{E}:O_{\mathfrak{T}}\to\boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ is Fréchet differentiable on an open superset $O_{\mathfrak{T}}\subseteq C^1(\widehat{D};\mathbb{C}^3)$ of $\mathfrak{T}$. Similarly, for a mesh τ_h on $\widehat{D}$, Theorem 4.5 gives for the discrete solution $\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h}\in\boldsymbol{P}_0^c(\tau_h)$ of Problem 3.11 that

$$\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h} = \mathfrak{E}_h\left(T_0+\sum_{j\in\mathbb{N}} y_jT_j\right) \qquad \forall\,\boldsymbol{y}\in U, \tag{5.4}$$

where $\mathfrak{E}_h : O_{\mathfrak{T}}\to\boldsymbol{P}_0^c(\tau_h)$ is Fréchet differentiable. Functions as in (5.3) or (5.4) allow fast approximation by polynomial methods, as they possess holomorphic extensions in each y_j. As we shall see in the following, the decay of $\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)}$ determines the (parameter-dimension independent) algebraic convergence rate of higher-order numerical methods.

Throughout, we denote by $\mu = \bigotimes_{j\in\mathbb{N}} \frac{\lambda}{2}$ the infinite product measure, where $\frac{\lambda}{2}$ stands for $\frac{1}{2}$ times the Lebesgue measure on $[-1,1]$. This implies that μ is a probability measure on $U=[-1,1]^{\mathbb{N}}$. Our goal in the following is to either approximate the parametric response map

$$\begin{cases} U \to \boldsymbol{H}_0(\mathbf{curl};\widehat{D}) \\ \boldsymbol{y} \mapsto \widehat{\mathbf{E}}_{T(\boldsymbol{y})} \end{cases}$$

or to compute expected responses over the ensemble of all shapes, i.e. to numerically approximate the Bochner integral

$$\int_U \widehat{\mathbf{E}}_{T(\boldsymbol{y})}\,\mathrm{d}\mu(\boldsymbol{y}) \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D}). \tag{5.5}$$

We remark that two- and higher order statistical correlations of the parametric response map will give rise to similar integrals whose numerical approximation can, likewise, be afforded with the techniques discussed subsequently, based on the present results.

The integral (5.5) is a well-defined Bochner integral: by [37, Lemma 3.2], since $\widehat{\mathbf{E}}_{T(\boldsymbol{y})}$ allows the representation (5.3), $\widehat{\mathbf{E}}_{T(\boldsymbol{y})}$ depends continuously on $\boldsymbol{y}\in U$ (again we consider the product topology on U). As such $U \ni \boldsymbol{y} \mapsto \widehat{\mathbf{E}}_{T(\boldsymbol{y})} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ is measurable w.r.t. the Borel σ-algebras on U and $\boldsymbol{H}_0(\mathbf{curl};\widehat{D})$. The compactness of U (which holds by Tychonoff's theorem) implies $\sup_{\boldsymbol{y}\in U}\|\widehat{\mathbf{E}}_{T(\boldsymbol{y})}\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} < \infty$. Continuity also implies $\{\widehat{\mathbf{E}}_{T(\boldsymbol{y})} : \boldsymbol{y}\in U\} \subseteq \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ to be compact (and thus separable), which can be used to infer strong measurability of $\widehat{\mathbf{E}}_{T(\cdot)} : U \to \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$. For more details we also refer to [36, Appendix A].

5.2. **Monte Carlo quadrature.** Let again $U=[-1,1]^{\mathbb{N}}$ and $u : U \to \mathbb{R}$. A simple method to approximate the high-dimensional integral $\int_U u(\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y})$ which does not suffer from the curse of dimensionality, is Monte-Carlo (MC) integration. Given a sequence $(Y_i)_{i=1}^N$ of i.i.d. random variables $Y_i : \Omega \to U$ distributed according to the uniform measure μ on U, the random variable

$$Q_N(u) := \frac{1}{N}\sum_{j=1}^{N} u(Y_j) :\ \Omega \to \mathbb{R}$$

approximates in mean square and at rate $\frac{1}{2}$ the mathematical expectation, i.e.

$$\forall\, N\in\mathbb{N}: \quad \mathbb{E}\left(Q_N(u) - \int_U u(\boldsymbol{z})\,\mathrm{d}\mu(\boldsymbol{z})\right)^2 \le \frac{\mathrm{Var}(u)}{N}$$

provided $u\in L^2(U,\mu;\mathbb{R})$. Here

$$\mathrm{Var}(u) = \int_U \left(u(\boldsymbol{y}) - \left(\int_U u(\boldsymbol{z})\,\mathrm{d}\mu(\boldsymbol{z})\right)\right)^2 \mathrm{d}\mu(\boldsymbol{y}). \tag{5.6}$$

A bound of this type remains valid also for strongly μ-measurable maps $u : U \to H$ where H denotes a separable Hilbert space (e.g. [34]). In particular, and as mentioned above, for the uniform measure μ on the set U of parameter sequences, the holomorphy of the map $U \ni \boldsymbol{y} \mapsto \widehat{\mathbf{E}}_{T(\boldsymbol{y})} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ (which we showed in Theorem 4.5) implies strong measurability. This, in turn, implies $\mathrm{Var}(\|\widehat{\mathbf{E}}_{T(\cdot)}\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})}) < \infty$, thereby justifying the use of MC sampling to estimate the expectation of the solution. The MC approach in UQ for CEM was recently also analyzed in [18].

5.3. **Smolyak approximation.** Smolyak approximation –also called sparse grid approximation– refers to a family of approximation methods that allow parsimonious approximate representation of many-variate functions. For functions which exhibit sufficient sparsity, Smolyak approximation produces sparse polynomial approximations with convergence rates that are free from the curse of dimensionality. We present two variants of Smolyak algorithms, one for interpolation and another for quadrature, which are sufficiently general, comprising for example, the so-called *hyperbolic* *cross* and *total degree* approximations. We now describe these techniques based on [10] and the references there.

Let us introduce the countable set $\mathcal{F}$ of multi-indices:

$$\mathcal{F} := \{\boldsymbol{\nu} = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}} \, : \, |\boldsymbol{\nu}| < \infty\} \, .$$

For $\boldsymbol{\nu} \in \mathcal{F}$ we denote by $\operatorname{supp}(\boldsymbol{\nu}) := \{j \in \mathbb{N} \, : \, \nu_j \neq 0\}$ the *support of* $\boldsymbol{\nu}$. We call $\Lambda \subset \mathcal{F}$ *downward closed* if for all $\mathbf{0} \neq \boldsymbol{\nu} \in \Lambda$, there holds $\boldsymbol{\nu} - \boldsymbol{e}_j \in \Lambda$ for every $j \in \operatorname{supp}(\boldsymbol{\nu})$.

5.3.1. *Sparse Grid Interpolation.* The Smolyak algorithm provides a method to approximate functions by sparse grid interpolants. We briefly recall the construction. Let X denote a Banach space. Denote by $(\chi_j)_{j\in\mathbb{N}}$ a sequence of distinct interpolation points in $[-1,1]$, and let $I_n : C^0([-1,1];X) \to C^0([-1,1];X)$ be the univariate Lagrange polynomial interpolation operator

$$(I_n f)(y) = \sum_{j=0}^{n} f(\chi_j) \prod_{i\neq j} (y - \chi_i)/(\chi_j - \chi_i).$$

We assume that $\{\chi_j\}_{j\geq 0}$ is such that there exists $\tau > 0$ with

$$\forall\, n \in \mathbb{N} : \qquad \sup_{\|f\|_{L^\infty([-1,1])}\leq 1} \|I_n f\|_{L^\infty([-1,1])} \leq (1+n)^\tau. \tag{5.7}$$

Sequences $\{\chi_j\}_{j\geq 0}$ which satisfy (5.7) are known (*cf.* [8]). Additionally, let $I_{-1} : C^0([-1,1];\mathbb{R}) \to C^0([-1,1];\mathbb{R})$ be the operator mapping a function f to the constant function, i.e., $I_{-1} f \equiv 0$. For a finite downward closed set $\Lambda \subseteq \mathcal{F}$ and a Banach space X, the Smolyak interpolant $\mathbf{I}_\Lambda : C^0(U;X) \to C^0(U;X)$ is defined as

$$\mathbf{I}_\Lambda u := \sum_{\boldsymbol{\nu}\in\Lambda} \left(\bigotimes_{j\in\mathbb{N}} (I_{\nu_j} - I_{\nu_j - 1}) \right) u,$$

where $\bigotimes_{j\in\mathbb{N}} (I_{\nu_j} - I_{\nu_j-1})$ denotes the tensorized operator (*cf.* [7] for more details).

For functions of the type

$$u(\boldsymbol{y}) = \mathfrak{E} \left(T_0 + \sum_{j\in\mathbb{N}} y_j T_j \right) \qquad \boldsymbol{y} \in U, \tag{5.8}$$

where $\mathfrak{E}$ is Fréchet differentiable as a mapping between two complex Banach spaces, the Smolyak interpolant is able to approximate u at the algebraic rate $\frac{1}{p} - 1$ in terms of the number of required function evaluations of u. Here, $p \in (0,1)$ refers to the summability exponent of the sequence $(T_j)_{j\in\mathbb{N}_0}$ as in (5.1). A version of this statement can already be found in [10]. By (5.3) and (5.4), both the pullback solution $\widehat{\mathbf{E}}_{T(\boldsymbol{y})} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ of Problem 2.7 as well as the discrete Galerkin solution $\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h} \in \boldsymbol{P}_0^c(\tau_h)$ of Problem 3.11 allow the representation (5.8), which leads to the following theorem. For a proof in the present setting, see [36, Chapter 3].

Theorem 5.1. *Let* (5.7) *be satisfied. Let* $p \in (0,1)$ *and assume that* $(T_j)_{j\in\mathbb{N}_0} \subseteq C^1(\widehat{D};\mathbb{R}^3)$ *satisfies* $(\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)})_{j\in\mathbb{N}_0} \in \ell^p(\mathbb{N}_0)$. *Let* $\mathfrak{T} \subseteq C^1(\widehat{D};\mathbb{R}^3)$ *be as in* (5.2)*, and suppose that Assumption 2.2 and Assumption 4.1 are satisfied.*

Then there exists a constant $C > 0$ *and a sequence* $(\Lambda_N)_{N\in\mathbb{N}} \subseteq \mathcal{F}$ *of finite downward closed index sets such that* $|\Lambda_N| \to \infty$ *and*

(i) *for the solution* $\widehat{\mathbf{E}}_{T(\boldsymbol{y})} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{D})$ *of Problem 2.7 it holds for all* $N \in \mathbb{N}$

$$\sup_{\boldsymbol{y}\in U} \left\| \widehat{\mathbf{E}}_{T(\boldsymbol{y})} - (\mathbf{I}_{\Lambda_n} \widehat{\mathbf{E}}_{T(\cdot)})(\boldsymbol{y}) \right\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} \leq C |\Lambda_N|^{-\frac{1}{p}+1},$$

(ii) *if* τ_h *is a mesh on* $\widehat{D}$*, then for the solution* $\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h} \in \boldsymbol{P}_0^c(\tau_h)$ *of Problem 3.11 it holds for all* $N \in \mathbb{N}$

$$\sup_{\boldsymbol{y}\in U} \left\| \widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h} - (\mathbf{I}_{\Lambda_n} \widetilde{\mathbf{E}}_{T(\cdot),h})(\boldsymbol{y}) \right\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} \leq C |\Lambda_N|^{-\frac{1}{p}+1}.$$

We point out that in the above theorem $|\Lambda_N|$ coincides with the number of required function evaluations to compute the interpolant.

5.3.2. *Sparse Grid Quadrature.* Similar to Smolyak interpolation, Smolyak quadrature provides an efficient method to approximate high-dimensional integrals over U with respect to the probability measure μ on U. For a finite, downward closed set $\Lambda \subseteq \mathcal{F}$ with unisolvent interpolation operator $\mathbf{I}_\Lambda$ as defined in Section 5.3.1, the corresponding Smolyak quadrature is $\mathbf{Q}_\Lambda u := \int_U \mathbf{I}_\Lambda u(\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y})$. With the univariate interpolation points $(\chi_j)_{j\in\mathbb{N}} \subseteq [-1,1]$ from Section 5.3.1, this can be rewritten as $\mathbf{Q}_\Lambda u = \sum_{\boldsymbol{\nu}\in\Lambda} u((\chi_{\nu_j})_{j\in\mathbb{N}})\omega_{\Lambda,\boldsymbol{\nu}}$ for certain quadrature weights $\omega_{\Lambda,\boldsymbol{\nu}} \in \mathbb{R}$, which can be computed *a priori*, given Λ. In particular, $|\Lambda|$ coincides with the number of function evaluations required to compute $\mathbf{Q}_\Lambda u$. Compared to the interpolation result in Theorem 5.1, quadrature rules for functions as in (5.8) allow for improved convergence rates. A simplified version of the following statement can be found in [37]. For a proof of the result under the present assumptions, we refer again to [36, Chapter 3].

Theorem 5.2. *Additional to* (5.7) *assume that* $\chi_0 = 0$. *Let* $p \in (0,1)$ *and assume that in* (5.8), *the sequence* $(T_j)_{j\in\mathbb{N}_0} \subseteq C^1(\widehat{D};\mathbb{R}^3)$ *satisfies* $(\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)})_{j\in\mathbb{N}_0} \in \ell^p(\mathbb{N}_0)$. *Let* $\mathfrak{T} \subseteq C^1(\widehat{D};\mathbb{R}^3)$ *be as in* (5.2), *and let Assumption 2.2 and Assumption 4.1 hold.*

Then there exists a constant $C > 0$ *and a sequence* $(\Lambda_N)_{N\in\mathbb{N}} \subseteq \mathcal{F}$ *of finite downward closed index sets such that* $|\Lambda_N| \to \infty$ *and*

(i) *for the solution* $\widehat{\mathbf{E}}_{T(\boldsymbol{y})} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$ *of Problem 2.7 it holds for all* $N \in \mathbb{N}$

$$\left\| \int_U \widehat{\mathbf{E}}_{T(\boldsymbol{y})}\,\mathrm{d}\mu(\boldsymbol{y}) - \mathbf{Q}_{\Lambda_n}\widehat{\mathbf{E}}_{T(\cdot)} \right\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} \leq C|\Lambda_N|^{-\frac{2}{p}+1},$$

(ii) *if* τ_h *is a mesh on* $\widehat{D}$, *then for the solution* $\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h} \in \boldsymbol{P}_0^c(\tau_h)$ *of Problem 3.11 it holds for all* $N \in \mathbb{N}$

$$\left\| \int_U \widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h}\,\mathrm{d}\mu(\boldsymbol{y}) - \mathbf{Q}_{\Lambda_n}\widetilde{\mathbf{E}}_{T(\cdot),h} \right\|_{\boldsymbol{H}_0(\mathbf{curl};\widehat{D})} \leq C|\Lambda_N|^{-\frac{2}{p}+1}.$$

5.4. **Quasi-Monte Carlo Integration.** An alternative to Smolyak quadrature Q_Λ in Section 5.3.2 is called *High-Order QMC* (HoQMC) Quadrature. It is likewise capable of delivering dimension-independent convergence rates subject to an appropriate notion of sparsity of the parametric integrand functions. For integrand functions of the type (5.8), under condition (5.1), the parameter-to-solution maps $U \ni \boldsymbol{y} \to \widehat{\mathbf{E}}_{T(\boldsymbol{y})}, \widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h}$ satisfy the so-called $(\boldsymbol{b},\varepsilon)$-holomorphy condition, with a positive sequence $\boldsymbol{b} \in \ell^p(\mathbb{N})$ – with $p \in (0,1)$ as in condition (5.1)– and which is independent of h [37, Lemma 3.3]. This, in turn, implies the dimension-independent convergence rate $1/p > 1$ of HoQMC numerical integration methods according to [13, Proposition 4.1]. Similar results hold for computational Bayesian shape estimation; we refer to [12] and references therein.

6. Numerical Results

Based on GMSH and GETDP, we now provide numerical experiments validating our theoretical results.

6.1. **Model problem.** Let $\widehat{D} := [-1,1]^3$ and let $r > 1$, $\Theta \in (0,1)$. We introduce a family of domain transformations $T(\boldsymbol{y}) : \widehat{D} \to \mathbb{R}^3$ for $\boldsymbol{y} \in [-1,1]^{50}$ via

$$T(\boldsymbol{y}) = T_0 + \sum_{j=1}^{50} y_j T_j \in C^1(\widehat{D};\mathbb{R}^3),$$

where for $\mathbf{x} = (x_1,x_2,x_3)^\top \in \mathbb{R}^3$

$$T_0(\mathbf{x}) = \mathbf{x}, \qquad T_j(\mathbf{x}) := \Theta j^{-(r+1)} \begin{pmatrix} 0 \\ 0 \\ \sin(\pi j x_1) \end{pmatrix} \quad \forall\, j \in \mathbb{N}. \tag{6.1}$$

While $\Theta > 0$ is a scaling parameter, $r > 1$ determines the algebraic decay of the sequence $(T_j)_{j\in\mathbb{N}} \subseteq C^1(\widehat{D};\mathbb{R}^3)$, namely

$$(\|T_j\|_{C^1(\widehat{D};\mathbb{R}^3)})_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \qquad \forall\, p > \frac{1}{r}. \tag{6.2}$$

Thus (5.1) is satisfied for any $p > 1/r$. Contrary to Section 5, we consider here the simpler situation of a finite dimensional parameter $\boldsymbol{y} \in [-1,1]^{50}$, rather than an infinite dimensional parameter $\boldsymbol{y} \in U = [-1,1]^{\mathbb{N}}$. The reason is, that QMC quadrature requires the truncation of the parameter domain to finite dimension and in order to make a fair comparison between MC, QMC and Smolyak quadrature in the following, we keep the same setting for all three methods.

As explained in Section 2.5, for every $\boldsymbol{y} \in [-1,1]^{50}$ the transformation $T(\boldsymbol{y})$ yields a pullback Maxwell problem with $\boldsymbol{y}$ dependent variable coefficients on the nominal domain $\widehat{D}$. For all computations, the FEM space $\boldsymbol{P}_0^c(\tau_h) \subseteq \boldsymbol{H}(\mathbf{curl};\widehat{D})$ is fixed, i.e. we fix a regular tetrahedral mesh τ_h with 111718 elements on $\widehat{D}$, and consider on τ_h first-order curl-conforming Nédélec elements. The corresponding weak solution of Problem 3.11 for the data μ, ϵ, σ, $\omega := 1$ and

$$\mathbf{J}(\boldsymbol{x}) := \begin{pmatrix} x_3 + \imath x_2 \\ \imath x_3 \\ x_1 \end{pmatrix},$$

with the transformation $T(\boldsymbol{y}) = T$ is denoted as earlier by $\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h} \in \boldsymbol{P}_0^c(\tau_h) \subseteq \boldsymbol{H}_0(\mathbf{curl};\widehat{D})$. As a so-called quantity of interest we consider the linear functional $G \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D})'$ defined by

$$G(\mathbf{E}) := \int_{[2/3,1]\times[-1,1]^2} (E_1 + E_2 + E_3)^2 \,\mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{E} = (E_j)_{j=1}^3 \in \boldsymbol{H}_0(\mathbf{curl};\widehat{D}).$$

The goal becomes to approximate the complex valued map $\boldsymbol{y} \mapsto G(\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h}) \in \mathbb{C}$ for $\boldsymbol{y} \in U$, or its integral

$$\int_{[-1,1]^{50}} G(\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h})\,\mathrm{d}\mu_{50}(\boldsymbol{y}) \in \mathbb{C},$$

where μ_{50} stands for the product probability measure $\bigotimes_{j=1}^{50} \lambda/2$ on $[-1,1]^{50}$, where λ is the Lebesgue measure on $[-1,1]$.

6.2. **Interpolation.** Figure 1 shows the convergence of the interpolation error

$$\sup_{\boldsymbol{y}\in[-1,1]^{50}} |G(\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h}) - (\mathbf{I}_\Lambda G(\widetilde{\mathbf{E}}_{T(\cdot),h}))(\boldsymbol{y})| \tag{6.3}$$

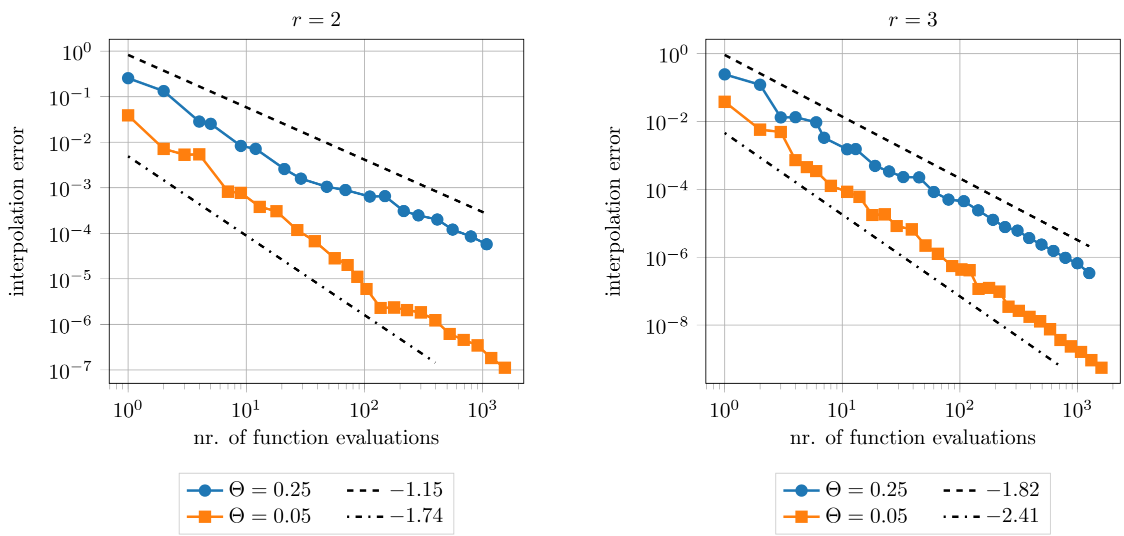
for $\Theta \in \{0.25, 0.05\}$ and $r \in \{2,3\}$ in (6.1). Here $\mathbf{I}_\Lambda$ denotes the Smolyak interpolant introduced in Section 5.3.1. As interpolation points $(\chi_j)_{j\in\mathbb{N}}$ in Section 5.3.1 we use an Re-Leja sequence as constructed in [8]. Apropriate index sets Λ are determined apriori (see [37, 35, 36]), which allows parallel evaluation of the function $\boldsymbol{y} \mapsto G(\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h})$ at all required interpolation points. The supremum in (6.3) is numerically estimated, by taking the maximum at 200 random points in $[-1,1]^{50} \times \{0\}^{\mathbb{N}} \subseteq U$.

Due to Theorem 5.1 (ii) and (6.2) we expect the convergence rate $r-1$ (up to arbitrarily small $\delta > 0$) in terms of the number of required evaluations of $G(\widetilde{\mathbf{E}}_{T(\cdot),h})$. While $\Theta > 0$ is merely a scaling parameter that should not influence the asymptotic decay of the error, in practice we observe that smaller Θ amounts to faster decay of the error. This is in accordance with our previous findings that revealed significant preasymptotic ranges of slower convergence in case $\Theta > 0$ is too large, we refer to [37, Section 5.2] for more details.

6.3. **Quadrature.** We now consider the quadrature error

$$\left| \int_{[-1,1]^{50}} G(\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h})\,\mathrm{d}\mu_{50}(\boldsymbol{y}) - \mathrm{Quad}(G(\widetilde{\mathbf{E}}_{T(\cdot),h})) \right| \tag{6.4}$$

in terms of the required functions evaluations of the integrand $G(\widetilde{\mathbf{E}}_{T(\cdot),h})$ for $r \in \{2,3\}$ and $\Theta \in \{0.25, 0.05\}$. Here Quad stands for either MC, QMC or Smolyak quadrature. In case of



(A) $r = 2$ (B) $r = 3$

FIGURE 1. Interpolation error in (6.3). The proven asymptotic convergence rate is $r - 1$ (up to arbitrarily small $\delta > 0$).

Smolyak quadrature we use as quadrature points $(\chi_j)_{j\in\mathbb{N}}$ in Section 5.3.2 the same Re-Leja sequence provided in [8], as for interpolation.

Figure 2 displays the convergence of the error. To numerically evaluate the error, as a reference value for $\int_{[-1,1]^{50}} G(\widetilde{\mathbf{E}}_{T(\boldsymbol{y}),h})\,\mathrm{d}\mu_{50}(\boldsymbol{y})$, in all cases we took the value obtained by the Smolyak quadrature with approximately 2000 quadrature points. By Theorem 5.2 (ii), the proven asymptotic convergence rate for Smolyak quadrature is $2r - 1$. For MC quadrature, the proven rate is $1/2$ and for QMC the proven rate is r (again up to arbitrarily small $\delta > 0$), see Sections 5.2 and 5.4. We note that similar as in the case of interpolation, for Smolyak and QMC quadrature the proven rates are not necessarily observed in our numerical experiments. In particular, the convergence rate in the plotted range depends on Θ and improves slightly as Θ decreases. We point out again that this in accordance with our previous findings, and refer once more to [37, Section 5.2], where the preasymptotic convergence of the Smolyak quadrature in case of “large” Θ is investigated in more detail.

7. Concluding Remarks

The present shape-holomorphy results imply sparsity of generalized polynomial chaos (gpc) coefficient sequences in the gpc expansions of uncertain responses. This sparsity, in turn, implies dimension-independent rates of convergence for Smolyak type interpolation or stochastic collocation of the input-to-response maps. The shape-holomorphy results proved in the present paper for the forward UQ will also imply corresponding dimension-independent rates for several other computational UQ methods: as shown in [30], dimension-independent rates of ReLU-based deep learning approximations, and, as shown in [28], also of compressed sensing based approximations of the domain-to-solution map [30] are implied by forward UQ. They also imply corresponding convergence rates for sparse, deterministic (Smolyak and Quasi-Monte Carlo) quadrature schemes for Bayesian shape inversion for CEM, as explained in [29, 12]. These developments are currently in progress, and will be reported elsewhere.

Acknowledgments

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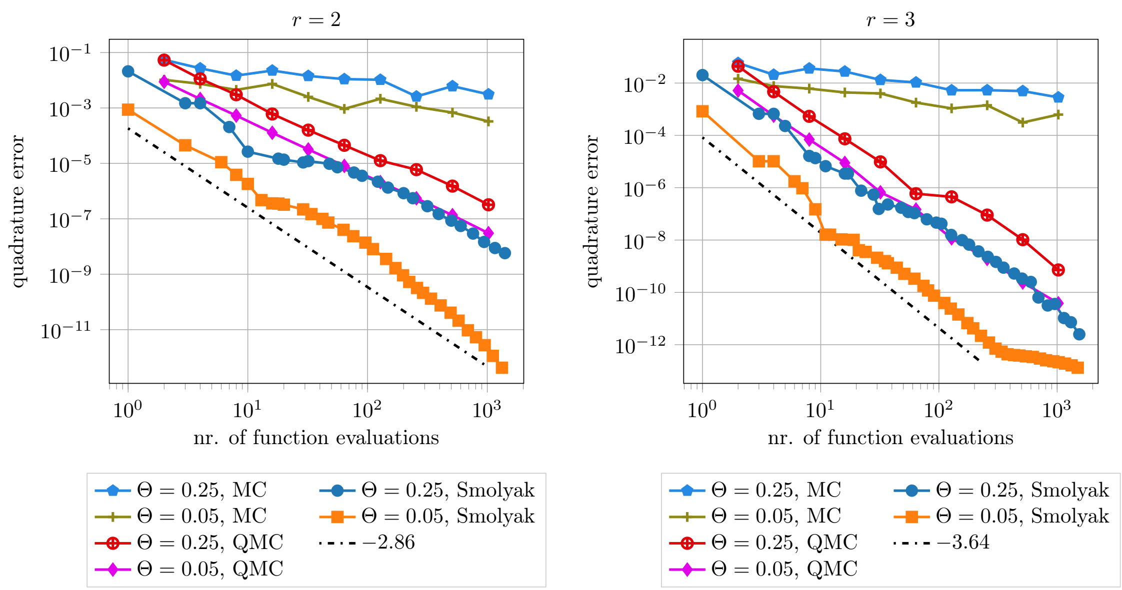

(A) $r = 2$ (B) $r = 3$

FIGURE 2. Quadrature error in (6.4). The proven asymptotic convergence rates are $1/2$ for MC, r for QMC and $2r - 1$ for Smolyak quadrature (up to arbitrarily small $\delta > 0$).

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APPENDIX A. PROOFS OF LEMMAS IN SECTION 3

A.1. **Proof of Lemma 3.12.** We start with (i). By definition of Q_K in (3.17)

$$\begin{aligned}|Q_K(\mathbf{B}\mathbf{U}\cdot\mathbf{V})| &\le \|\mathbf{B}\|_{C^0(K;\mathbb{C}^{3\times3})}\,|\det(\mathbb{J}_K)|\sum_{l=1}^{L}\left(|\breve{w}_l|\|\mathbf{U}(T_K(\breve{\boldsymbol{b}}_l))\|_{\mathbb{C}^3}\|\mathbf{V}(T_K(\breve{\boldsymbol{b}}_l))\|_{\mathbb{C}^3}\right)\\ &\le \|\mathbf{B}\|_{C^0(K;\mathbb{C}^{3\times3})}\,|\det(\mathbb{J}_K)|\left(\sum_{l=1}^{L}|\breve{w}_l|\right)\|\mathbf{U}\circ T_K\|_{C^0(\breve{K};\mathbb{C}^3)}\|\mathbf{V}\circ T_K\|_{C^0(\breve{K};\mathbb{C}^3)}.\end{aligned}$$

Due to the equivalence of all norms in finite dimensional spaces, there exists $0 < \tilde{C} < \infty$ such that for all $\boldsymbol{p} \in \mathbb{P}_k(\breve{K};\mathbb{C}^3)$

$$\|\boldsymbol{p}\|_{C^0(\breve{K})} \le \tilde{C}\|\boldsymbol{p}\|_{\boldsymbol{L}^2(K)}.$$

Since $T_K : \breve{K} \to K$ is affine it holds $\mathbf{U}\circ T_K,\ \mathbf{V}\circ T_K \in \mathbb{P}_k(\breve{K};\mathbb{C}^3)$ and we obtain

$$\begin{aligned}|Q_K(\mathbf{B}\mathbf{U}\cdot\mathbf{V})| &\le \tilde{C}^2\|\mathbf{B}\|_{C^0(K;\mathbb{C}^{3\times3})}\left(\sum_{l=1}^{L}|\breve{w}_l|\right)|\det(\mathbb{J}_K)|\,\|\mathbf{U}\circ T_K\|_{\boldsymbol{L}^2(\breve{K})}\|\mathbf{V}\circ T_K\|_{\boldsymbol{L}^2(\breve{K})}\\ &= \tilde{C}^2\|\mathbf{B}\|_{C^0(K;\mathbb{C}^{3\times3})}\left(\sum_{l=1}^{L}|\breve{w}_l|\right)\|\mathbf{U}\|_{\boldsymbol{L}^2(K)}\|\mathbf{V}\|_{\boldsymbol{L}^2(K)}\end{aligned}$$

which shows (i) with $C_1 = \tilde{C}^2\sum_{l=1}^{L}|\breve{w}_l|$.

Next let us show (ii). By Rmk. 3.10, $(\sum_{l=1}^{L} \breve{w}_l |p(\breve{\boldsymbol{b}}_l)|^2)^{1/2}$ is a norm on $\mathbb{P}_k(\breve{K};\mathbb{C})$. Thus there exists $\tilde{c} > 0$, independent of h, such that

$$\tilde{c}\|\boldsymbol{p}\|_{\boldsymbol{L}^2(\breve{K})} \leq \left(\sum_{l=1}^{L} \breve{w}_l \|\boldsymbol{p}(\breve{\boldsymbol{b}}_l)\|_{\mathbb{C}^3}^2\right)^{1/2} \qquad \forall\, \boldsymbol{p} \in \mathbb{P}_k(\breve{K};\mathbb{C}^3).$$

Now, by (3.20), one can derive

$$\begin{aligned}\operatorname{Re}(Q_K(\mathbf{BU}\cdot\overline{\mathbf{U}})) &\geq \gamma(\mathbf{B})\,|\det(\mathbb{J}_K)| \sum_{l=1}^{L} \breve{w}_l \|\mathbf{U}(T_K(\breve{\boldsymbol{b}}_l))\|_{\mathbb{C}^3}^2 \\ &\geq \gamma(\mathbf{B})\tilde{c}^2\,|\det(\mathbb{J}_K)|\,\|\mathbf{U}\circ T_K\|_{\boldsymbol{L}^2(\breve{K})}^2 \\ &= \gamma(\mathbf{B})\tilde{c}^2 \|\mathbf{U}\|_{\boldsymbol{L}^2(K)}^2,\end{aligned}$$

which shows (ii) with $C_2 = \tilde{c}^2$.

A.2. **Proof of Lemma 3.15.** For the proof of Lemma 3.15, we shall require the Bramble-Hilbert Lemma (*cf.* [9, Theorem 4.1.3]).

Lemma A.1 (Bramble-Hilbert)**.** *Let Ω be an open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. For some integer $k \geq 0$ and $p \in [1,\infty]$, let f be a continuous linear form on $W^{k+1,p}(\Omega)$ with the property that*

$$f(q) = 0, \quad \forall\, q \in \mathbb{P}_k(\Omega;\mathbb{C}).$$

Then, there exists a constant C_Ω, depending on the domain, such that

$$\forall\, v \in W^{k+1,p}(\Omega),\; |f(v)| \leq C_\Omega \|f\|_{W^{k+1,p}(\Omega)}^* |v|_{W^{k+1,p}(\Omega)},$$

where $\|\cdot\|_{W^{k+1,p}(\Omega)}^*$ is the norm in the dual space of $W^{k+1,p}(\Omega)$.

We now state a version of Lemma A.1 for vector-valued functions.

Lemma A.2. *Let Ω be an open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. For some integer $k \geq 0$ and $p \in [1,\infty]$, let $\boldsymbol{f}$ be a continuous linear form on $\boldsymbol{W}^{k+1,p}(\Omega)$ with the property that*

$$\boldsymbol{f}(\boldsymbol{q}) = 0, \quad \forall\, \boldsymbol{q} \in \mathbb{P}_k(\Omega;\mathbb{C}^3).$$

Then, there exists a constant C_Ω, depending on the domain such that

$$\forall\, \mathbf{V} \in \boldsymbol{W}^{k+1,p}(\Omega),\; |\boldsymbol{f}(\mathbf{V})| \leq C_\Omega \|\boldsymbol{f}\|_{\boldsymbol{W}^{k+1,p}(\Omega)}^* |\mathbf{V}|_{\boldsymbol{W}^{k+1,p}(\Omega)},$$

where $\|\cdot\|_{\boldsymbol{W}^{k+1,p}(\Omega)}^*$ is the norm in the dual space of $\boldsymbol{W}^{k+1,p}(\Omega)$.

Proof. Since $\boldsymbol{f}$ is a linear form, it can be written as

$$\boldsymbol{f}(\mathbf{V}) = \boldsymbol{f}\begin{pmatrix}\mathrm{V}_1\\0\\0\end{pmatrix} + \boldsymbol{f}\begin{pmatrix}0\\\mathrm{V}_2\\0\end{pmatrix} + \boldsymbol{f}\begin{pmatrix}0\\0\\\mathrm{V}_3\end{pmatrix} = f_1(\mathrm{V}_1) + f_2(\mathrm{V}_2) + f_3(\mathrm{V}_3).$$

By hypothesis, each $f_j,\; j \in \{1,2,3\}$, is a bounded linear form on $W^{k+1,p}(\Omega)$ satisfying $f_j(q) = 0$, for all $q \in \mathbb{P}_k(\Omega;\mathbb{C})$. By Lemma A.1, there exist positive constants $C_{\Omega,j}$ such that

$$|f_j(\mathrm{V}_j)| \leq C_{\Omega,j} \|f_j\|_{W^{k+1,p}(\Omega)}^* |\mathrm{V}_j|_{W^{k+1,p}(\Omega)}, \quad \forall\, \mathrm{V}_j \in W^{k+1,p}(\Omega).$$

Hence,

$$\begin{aligned}|\boldsymbol{f}(\mathbf{V})| \leq |f_1(\mathrm{V}_1)| + |f_2(\mathrm{V}_2)| + |f_3(\mathrm{V}_3)| &\leq \sum_{j=1}^{3} C_{\Omega,j} \|f_j\|_{W^{k+1,p}(\Omega)}^* |\mathrm{V}_j|_{W^{k+1,p}(\Omega)} \\ &\leq C_\Omega \|\boldsymbol{f}\|_{\boldsymbol{W}^{k+1,p}(\Omega)}^* |\mathbf{V}|_{\boldsymbol{W}^{k+1,p}(\Omega)},\end{aligned}$$

as stated. □

We are now ready to prove Lemma 3.15.

Proof of Lemma 3.15. We give only the proof for $\widehat{\mathbf{U}}_h$ and $\widehat{\mathbf{W}}_h \in \boldsymbol{P}_0^c(\tau_h)$. The case $\widehat{\mathbf{U}}_h$ and $\widehat{\mathbf{W}}_h \in \boldsymbol{P}_0^d(\tau_h)$ follows analogously.

Consider $\boldsymbol{\phi} \in \boldsymbol{W}^{m,\infty}(\breve{K})$ and $\widehat{\mathbf{W}}_h \in \breve{\boldsymbol{P}}^c$, assuming that $\left|\boldsymbol{\phi}(\breve{\boldsymbol{b}}_l)\right| \leq \|\boldsymbol{\phi}\|_{\boldsymbol{L}^\infty(\breve{K})}$ for all quadrature points $\{\breve{\boldsymbol{b}}_l\}_{l=1}^L$. Then, for some positive constant $C_{\mathcal{E}}$, depending only on $\breve{K}$ and the quadrature scheme on $\breve{K}$,

$$\left|\mathcal{E}_{\breve{K}}(\boldsymbol{\phi}, \widehat{\mathbf{W}}_h)\right| \leq C_{\mathcal{E}} \left\|\boldsymbol{\phi}\cdot \widehat{\mathbf{W}}_h\right\|_{L^\infty(\breve{K})} \leq C_{\mathcal{E}} \|\boldsymbol{\phi}\|_{\boldsymbol{L}^\infty(\breve{K})} \left\|\widehat{\mathbf{W}}_h\right\|_{\boldsymbol{L}^\infty(\breve{K})} \leq C_{\mathcal{E}} \|\boldsymbol{\phi}\|_{\boldsymbol{W}^{m,\infty}(\breve{K})} \left\|\widehat{\mathbf{W}}_h\right\|_{0,\breve{K}},$$

where the last inequality follows from the norm equivalence over the finite dimensional space $\breve{\boldsymbol{P}}^c$, and the definition of $\|\cdot\|_{\boldsymbol{W}^{m,\infty}(\breve{K})}$. Fixing $\widehat{\mathbf{W}}_h \in \breve{\boldsymbol{P}}^c$, the form $\mathcal{E}_{\breve{K}}(\cdot, \widehat{\mathbf{W}}_h)$ is linear and bounded on $\boldsymbol{W}^{m,\infty}(\breve{K})$, and vanishes on $\mathbb{P}_{m-1}(\breve{K};\mathbb{C}^3)$. By Lemma A.2, there exists a constant $C_{\breve{K}}$ such that,

$$\left|\mathcal{E}_{\breve{K}}(\boldsymbol{\phi}, \widehat{\mathbf{W}}_h)\right| \leq C_{\breve{K}} \left|\boldsymbol{\phi}\right|_{\boldsymbol{W}^{m,\infty}(\breve{K})} \left\|\widehat{\mathbf{W}}_h\right\|_{0,\breve{K}}.$$

Let $K \in \tau_h$ be an arbitrary mesh element. Then, for $\widehat{\mathbf{U}}_h$ and $\widehat{\mathbf{W}}_h$ in $\boldsymbol{P}_0^c(\tau_h)$,

$$\begin{aligned}\left|\mathcal{E}_K(\mathbf{B}\widehat{\mathbf{U}}_h, \widehat{\mathbf{W}}_h)\right| &= \left|\det(\mathbb{J}_K)\right| \left|\mathcal{E}_{\breve{K}}\left((\mathbf{B}\widehat{\mathbf{U}}_h)\circ T_K, \widehat{\mathbf{W}}_h \circ T_K\right)\right|, \\ &\leq C_{\breve{K}} \left|\det(\mathbb{J}_K)\right| \left|(\mathbf{B}\widehat{\mathbf{U}}_h)\circ T_K\right|_{W^{m,\infty}(\breve{K})} \left\|\widehat{\mathbf{W}}_h \circ T_K\right\|_{0,\breve{K}}. \end{aligned} \tag{A.1}$$

We begin by bounding $\left|(\mathbf{B}\widehat{\mathbf{U}}_h)\circ T_K\right|_{W^{m,\infty}(\breve{K})}$, using the symbol $\lesssim$, to avoid specifying constants independent of h, K, and $\mathbf{B}$. There holds

$$\begin{aligned}
\left|(\mathbf{B}\widehat{\mathbf{U}}_h)\circ T_K\right|_{W^{m,\infty}(\breve{K})} &\lesssim \sum_{i=1}^{3}\left|\sum_{j=1}^{3}(B_{i,j}\circ T_K)((\widehat{U}_h)_j \circ T_K)\right|_{W^{m,\infty}(\breve{K})} \\
&\lesssim \sum_{i,j=1}^{3}\left|(B_{i,j}\circ T_K)((\widehat{U}_h)_j \circ T_K)\right|_{W^{m,\infty}(\breve{K})} \\
&\lesssim \sum_{i,j=1}^{3}\sum_{n=0}^{m}\left|(B_{i,j}\circ T_K)\right|_{W^{m-n,\infty}(\breve{K})}\left|((\widehat{U}_h)_j \circ T_K)\right|_{W^{n,\infty}(\breve{K})} \\
&\lesssim \sum_{i,j=1}^{3}\sum_{n=0}^{m}\left|(B_{i,j}\circ T_K)\right|_{W^{m-n,\infty}(\breve{K})}\left|(\widehat{\mathbf{U}}_h \circ T_K)\right|_{\boldsymbol{W}^{n,\infty}(\breve{K})} \\
&\lesssim \sum_{n=0}^{m}\left\|\mathbb{J}_K^{-\top}\right\|_{3\times 3}\left|\psi_K^c(\widehat{\mathbf{U}}_h)\right|_{\boldsymbol{W}^{n,\infty}(\breve{K})}\sum_{i,j=1}^{3}\left|\psi_K^g(B_{i,j})\right|_{W^{m-n,\infty}(\breve{K})} &\text{(A.2)}\\
&\lesssim \sum_{n=0}^{m}\left\|\mathbb{J}_K^{-\top}\right\|_{3\times 3}\left|\psi_K^c(\widehat{\mathbf{U}}_h)\right|_{n,\breve{K}}\sum_{i,j=1}^{3}\left|\psi_K^g(B_{i,j})\right|_{W^{m-n,\infty}(\breve{K})} &\text{(A.3)}\\
&\lesssim \sum_{n=0}^{m}\left\|\mathbb{J}_K\right\|_{3\times 3}^{m}\left|\det(\mathbb{J}_K)\right|^{-\frac{1}{2}}\left|\widehat{\mathbf{U}}_h\right|_{n,K}\sum_{i,j=1}^{3}\left|(B_{i,j})\right|_{W^{m-n,\infty}(K)} &\text{(A.4)}\\
&\lesssim \left\|\mathbb{J}_K\right\|_{3\times 3}^{m}\left|\det(\mathbb{J}_K)\right|^{-\frac{1}{2}}\left\|\widehat{\mathbf{U}}_h\right\|_{m,K}\sum_{i,j=1}^{3}\left\|(B_{i,j})\right\|_{W^{m,\infty}(K)}, &\text{(A.5)}
\end{aligned}$$

where (A.2) follows from the representation (3.4) for ψ_K^c, (A.3) employs the equivalence of norms in spaces of finite dimension, and (A.4) is a consequence of (3.5). In a similar manner,

$$\left\|\widehat{\mathbf{W}}_h \circ T_K\right\|_{0,\breve{K}} \lesssim \left|\det(\mathbb{J}_K)\right|^{-\frac{1}{2}}\left\|\widehat{\mathbf{W}}_h\right\|_{0,K}. \tag{A.6}$$

Combining (A.1), (A.5), and (A.6), together with (3.3), yields

$$\begin{aligned}\left|\mathcal{E}_K(\mathbf{B}\widehat{\mathbf{U}}_h, \widehat{\mathbf{W}}_h)\right| &\lesssim |\det(\mathbb{J}_K)| \left|(\mathbf{B}\widehat{\mathbf{U}}_h)\circ T_K\right|_{\boldsymbol{W}^{m,\infty}(\check{K})} \left\|\widehat{\mathbf{W}}_h \circ T_K\right\|_{0,\check{K}} \\ &\lesssim C_{\mathbf{B}} \|\mathbb{J}_K\|_{3\times 3}^m \left\|\widehat{\mathbf{U}}_h\right\|_{m,K} \left\|\widehat{\mathbf{W}}_h\right\|_{0,K} \lesssim h^m C_{\mathbf{B}} \left\|\widehat{\mathbf{U}}_h\right\|_{m,K} \left\|\widehat{\mathbf{W}}_h\right\|_{0,K},\end{aligned}$$

from where the stated result follows. □

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