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ON THE PERFORMANCE OF THE EULER-MARUYAMA SCHEME FOR SDES WITH DISCONTINUOUS DRIFT COEFFICIENT

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ABSTRACT. Recently a lot of effort has been invested to analyze the L_p-error of the Euler-Maruyama scheme in the case of stochastic differential equations (SDEs) with a drift coefficient that may have discontinuities in space. For scalar SDEs with a piecewise Lipschitz drift coefficient and a Lipschitz diffusion coefficient that is non-zero at the discontinuity points of the drift coefficient so far only an L_p-error rate of at least $1/(2p)-$ has been proven. In the present paper we show that under the latter conditions on the coefficients of the SDE the Euler-Maruyama scheme in fact achieves an L_p-error rate of at least $1/2$ for all $p \in [1, \infty)$ as in the case of SDEs with Lipschitz coefficients.

1. Introduction

Consider an autonomous stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t \in [0,1], \\ X_0 &= x_0 \end{aligned} \tag{1}$$

with deterministic initial value $x_0 \in \mathbb{R}$, drift coefficient $\mu\colon \mathbb{R} \to \mathbb{R}$, diffusion coefficient $\sigma\colon \mathbb{R} \to \mathbb{R}$ and 1-dimensional driving Brownian motion W. If (1) has a unique strong solution X then a classical numerical approach for approximating X_1 based on n observations of W is provided by the Euler-Maruyama scheme given by $\widehat{X}_{n,0} = x_0$ and

$$\widehat{X}_{n,(i+1)/n} = \widehat{X}_{n,i/n} + \mu(\widehat{X}_{n,i/n}) \cdot 1/n + \sigma(\widehat{X}_{n,i/n}) \cdot (W_{(i+1)/n} - W_{i/n})$$

for $i \in \{0, \dots, n-1\}$.

It is well-known that if the coefficients μ and σ are Lipschitz continuous then for all $p \in [1, \infty)$ the Euler-Maruyama scheme at the final time achieves an L_p-error rate of at least $1/2$ in terms of the number n of observations of W, i.e. for all $p \in [1, \infty)$ there exists $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E}\big[|X_1 - \widehat{X}_{n,1}|^p\big]\big)^{1/p} \le \frac{c}{\sqrt{n}}. \tag{2}$$

In this article we study the L_p-error of $\widehat{X}_{n,1}$ in the case when the drift coefficient μ may have finitely many discontinuity points. More precisely, we assume that the drift coefficient μ is piecewise Lipschitz continuous in the sense that

(A1) there exist $k \in \mathbb{N}_0$ and $\xi_0, \dots, \xi_{k+1} \in [-\infty, \infty]$ with $-\infty = \xi_0 < \xi_1 < \ldots < \xi_k < \xi_{k+1} = \infty$ such that μ is Lipschitz continuous on the interval (ξ_{i-1}, ξ_i) for all $i \in \{1, \dots, k+1\}$,

and we assume that the diffusion coefficient σ is Lipschitz continuous and non-zero at the potential discontinuity points of μ, i.e.

(A2) σ is Lipschitz continuous on $\mathbb{R}$ and $\sigma(\xi_i) \neq 0$ for all $i \in \{1, \dots, k\}$.

Note that under the assumptions (A1) and (A2) the equation (1) has a unique strong solution, see [14, Theorem 2.2].

Numerical approximation of SDEs with a drift coefficient that is discontinuous in space has gained a lot of interest in recent years, see [4, 5] for results on convergence in probability and almost sure convergence of the Euler-Maruyama scheme and [3, 7, 14, 15, 16, 23, 24, 25, 26] for results on L_p-approximation. In particular, in [16, 24, 25, 26] the L_p-error of the Euler-Maruyama scheme has been studied for such SDEs. The most far going results in the latter four articles provide for the one-dimensional SDE (1) under the assumptions (A1) and (A2)

(i) an L_1-error rate of at least $1/2$ for $\widehat{X}_{n,1}$ if, additionally to (A1) and (A2), the coefficients μ and σ are bounded, μ is integrable on $\mathbb{R}$ or one-sided Lipschitz continuous, and σ is bounded away from zero, see [24, 25],

(ii) an L_1-error rate of at least $1/2-$ for $\widehat{X}_{n,1}$ if, additionally to (A1) and (A2), the coefficients μ and σ are bounded and σ is bounded away from zero, see [25],

(iii) an L_2-error rate of at least $1/4-$ for $\widehat{X}_{n,1}$, if, additionally to (A1) and (A2), the coefficients μ and σ are bounded, see [16].

We add that the proof techniques in [16] can readily be adapted to show that the Euler-Maruyama scheme at the final time $\widehat{X}_{n,1}$ achieves an L_p-error rate of at least $1/(2p)-$ for all $p \in [1, \infty)$ if the coefficients μ and σ are bounded and satisfy the assumptions (A1) and (A2), see the discussion at the beginning of Section 3. Furthermore, in [23, Remark 4.2] it is stated that the proof techniques in [16] could be modified to cover the case of unbounded coefficients μ and σ as well.

To summarize, under the assumptions (A1) and (A2) it was only known up to now that the Euler-Maruyama scheme at the final time achieves an L_p-error rate of at least $1/(2p)-$ for all $p \in [1, \infty)$, and it was a challenging question whether these error bounds can be improved, and if so, whether under the assumptions (A1) and (A2) the Euler-Maruyama scheme at the final time even achieves an L_p-error rate of at least $1/2$ for all $p \in [1, \infty)$ as it is the case for SDEs with Lipschitz continuous coefficients, see (2).

Note that the recent literature on numerical approximation of SDEs contains a number of examples of SDEs with coefficients that are not Lipschitz continuous and such that the Euler-Maruyama scheme at the final time does not achieve an L_p-error rate of $1/2$, see [2, 6, 9, 11, 12, 22, 29]. Furthermore, in [3] numerical studies are carried out for a number of SDEs (1) with a discontinuous μ satisfying (A1) and $\sigma = 1$, and for several of these SDEs an empirical L_2-error rate significantly smaller than $1/2$ is observed for the Euler-Maruyama scheme at the final time.

However, regardless of the latter negative findings it turns out that under the assumptions (A1) and (A2) the Euler-Maruyama scheme at the final time $\widehat{X}_{n,1}$ in fact satisfies (2) for all $p \in [1, \infty)$. This estimate is an immediate consequence of our main result, Theorem 1, which states that under the assumptions (A1) and (A2) the maximum error of the time-continuous Euler-Maruyama scheme achieves at least the rate $1/2$ in the p-th mean sense, for all $p \in [1, \infty)$, see Section 2.

We add that in [14, 15] a numerical method for approximating X_1 is constructed that is based on a suitable transformation of the solution X of (1) and achieves an L_2-error rate of at least $1/2$ in terms of the number of observations of W under the assumptions (A1) and (A2). Furthermore, in [23] an adaptive Euler-Maruyama scheme is constructed, which achieves at the final time an L_2-error rate of at least $1/2-$ in terms of the average number of observations of W under the assumptions (A1) and (A2). However, in contrast to the classical Euler-Maruyama scheme, an implementation of either of the latter two methods requires the knowledge of the points of discontinuity of μ.

In this paper we furthermore consider the piecewise linear interpolation $\overline{X}_n = (\overline{X}_{n,t})_{t\in[0,1]}$ of the Euler-Maruyama scheme $(\widehat{X}_{n,i/n})_{i=0,\dots,n}$ and we study the performance of $\overline{X}_n$ globally on $[0,1]$. Using Theorem 1 we show that if the assumptions (A1) and (A2) are satisfied then for all $p \in [1,\infty)$ and all $q \in [1,\infty]$ there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E}\big[\|X - \overline{X}_n\|_q^p\big]\big)^{1/p} \le \begin{cases} c/\sqrt{n}, & \text{if } q < \infty, \\ c\sqrt{\ln(n+1)}/\sqrt{n}, & \text{if } q = \infty, \end{cases} \tag{3}$$

where $\|\cdot\|_q$ denotes the L_q-norm on the space of real-valued, continuous functions on $[0,1]$, see Theorem 2.

Our results provide upper error bounds for the Euler-Maruyama scheme at the final time $\widehat{X}_{n,1}$ and the piecewise linear interpolation $\overline{X}_n$ of the Euler-Maruyama scheme in terms of the number n of observations of the driving Brownian motion W that are used. It is natural to ask whether these bounds are asymptotically sharp or whether there exist alternative algorithms based on n observations of W that achieve under the assumptions (A1) and (A2) better rates of convergence in terms of the number n. For the error criteria considered in (3) the answer to this question is already known. The corresponding error rates can not be improved in general, see [8, 10, 20] for the case $q \in [1,\infty)$ and [8, 19] for the case $q = \infty$. For the L_p-approximation of X_1 the question is open up to now. For this problem it is so far only known that under the assumptions (A1) and (A2) it is impossible to obtain an L_p-error rate better than 1 in general, see [8, 21]. Whether or not there exists an algorithm that approximates X_1 under the assumptions (A1) and (A2) with an L_p-error rate better than $1/2$ in terms of the number of observations of W remains a challenging question.

In the present paper we have only studied scalar SDEs while the results in [15, 16, 23, 24] also cover the case of multidimensional SDEs. We believe however that our proof techniques can be extended to obtain for all $p \in [1,\infty)$ an L_p-error rate of at least $1/2$ for the Euler-Maruyama scheme at the final time in a suitable multidimensional setting as well. This will be the subject of future work.

We briefly describe the content of the paper. Our error estimates, Theorem 1 and Theorem 2, are stated in Section 2. Section 3 contains proofs of these results and a discussion on the relation of our analysis and the analysis of the Euler-Maruyama scheme carried out in [16].

2. Error estimates for the Euler-Maruyama scheme

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,1]}$, let $W\colon [0,1]\times\Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)_{t\in[0,1]}$-Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, let $x_0 \in \mathbb{R}$ and let $\mu, \sigma\colon \mathbb{R} \to \mathbb{R}$ be functions that satisfy the following two conditions.

(A1) There exist $k \in \mathbb{N}_0$ and $\xi_0, \dots, \xi_{k+1} \in [-\infty, \infty]$ with $-\infty = \xi_0 < \xi_1 < \dots < \xi_k < \xi_{k+1} = \infty$ such that μ is Lipschitz continuous on the interval (ξ_{i-1}, ξ_i) for all $i \in \{1, \dots, k+1\}$,

(A2) σ is Lipschitz continuous on $\mathbb{R}$ and $\sigma(\xi_i) \neq 0$ for all $i \in \{1, \dots, k\}$.

We consider the SDE

$$
\begin{aligned}
dX_t &= \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \quad t \in [0,1], \\
X_0 &= x_0,
\end{aligned}
\tag{4}
$$

which has a unique strong solution, see [14, Theorem 2.2].

Remark 1. Note that if in (A2) the assumption $\sigma(\xi_i) \neq 0$ for all $i \in \{1, \dots, k\}$ is violated then the existence of a strong solution of (4) can not be guaranteed anymore, see [17, Example 4.2].

For $n \in \mathbb{N}$ let $\widehat{X}_n = (\widehat{X}_{n,t})_{t\in[0,1]}$ denote the time-continuous Euler-Maruyama scheme with step-size $1/n$ associated to the SDE (4), i.e. $\widehat{X}_n$ is recursively given by $\widehat{X}_{n,0} = x_0$ and

$$
\widehat{X}_{n,t} = \widehat{X}_{n,i/n} + \mu(\widehat{X}_{n,i/n}) \cdot (t - i/n) + \sigma(\widehat{X}_{n,i/n}) \cdot (W_t - W_{i/n})
$$

for $t \in (i/n, (i+1)/n]$ and $i \in \{0, \dots, n-1\}$. We have the following error estimates for $\widehat{X}_n$.

Theorem 1. *Let $p \in [1, \infty)$. Then there exists $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$,*

$$
\big(\mathbb{E}\big[\|X - \widehat{X}_n\|_\infty^p\big]\big)^{1/p} \leq \frac{c}{\sqrt{n}}.
\tag{5}
$$

Next, we study the performance of the piecewise linear interpolation $\overline{X}_n = (\overline{X}_{n,t})_{t\in[0,1]}$ of the time-discrete Euler-Maruyama scheme $(\widehat{X}_{n,i/n})_{i=0,\dots,n}$, i.e.

$$
\overline{X}_{n,t} = (n \cdot t - i) \cdot \widehat{X}_{n,(i+1)/n} + (i + 1 - n \cdot t) \cdot \widehat{X}_{n,i/n}
$$

for $t \in [i/n, (i+1)/n]$ and $i \in \{0, \dots, n-1\}$. We have the following error estimates for $\overline{X}_n$.

Theorem 2. *Let $p \in [1, \infty)$ and $q \in [1, \infty]$. Then there exists $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$,*

$$
\big(\mathbb{E}\big[\|X - \overline{X}_n\|_q^p\big]\big)^{1/p} \leq
\begin{cases}
c/\sqrt{n}, & \text{if } q < \infty, \\
c\sqrt{\ln(n+1)}/\sqrt{n}, & \text{if } q = \infty.
\end{cases}
\tag{6}
$$

3. Proofs

Throughout this section we put

$$
\underline{t}_n = \lfloor n \cdot t \rfloor / n
$$

for every $n \in \mathbb{N}$ and every $t \in [0,1]$.

We briefly describe the structure of the proof of our main result, Theorem 1, and the relation of our analysis and the analysis of the Euler-Maruyama scheme carried out in [16]. Let $p \in [1, \infty)$. In [16] a bijection $G\colon \mathbb{R} \to \mathbb{R}$ is constructed such that G^{-1} is Lipschitz continuous and the stochastic process $Z = G \circ X$ is the unique strong solution of an SDE with Lipschitz continuous coefficients. It then follows by standard error estimates for the Euler-Maruyama scheme that there exist $c_1, c_2 \in (0, \infty)$ such that for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\big(\mathbb{E}\big[\|X - \widehat{X}_n\|_\infty\|^p\big]\big)^{1/p} &\leq c_1 \cdot \big(\mathbb{E}[\|Z - G \circ \widehat{X}_n\|_\infty^p]\big)^{1/p} \\
&\leq c_2/\sqrt{n} + c_1 \cdot \big(\mathbb{E}\big[\|\widehat{Z}_n - G \circ \widehat{X}_n\|_\infty^p\big]\big)^{1/p},
\end{aligned}
\tag{7}
$$

where $\widehat{Z}_n$ is the time-continous Euler-Maruyama scheme with step-size $1/n$ associated to the SDE for the stochastic process Z. Using further regularity properties of the function G it is shown in [16] that there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$(8) \qquad \big(\mathbb{E}\big[\|\widehat{Z}_n - G\circ \widehat{X}_n\|_\infty^p\big]\big)^{1/p} \le c/\sqrt{n} + c\cdot \Big(\mathbb{E}\Big[\Big|\int_0^1 1_B(\widehat{X}_{n,t}, \widehat{X}_{n,\underline{t}_n})\, dt\Big|^p\Big]\Big)^{1/p},$$

where

$$B = \Big(\bigcup_{i=1}^{k+1} (\xi_{i-1}, \xi_i)^2\Big)^c$$

is the set of pairs (x,y) in $\mathbb{R}^2$, which do not allow for a joint Lipschitz estimate of $|\mu(x)-\mu(y)|$ if μ has at least one discontinuity. Finally, using a large deviation argument it is shown in [16] that for every arbitrary small $\delta \in (0,1)$ there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$(9) \qquad \Big(\mathbb{E}\Big[\Big|\int_0^1 1_B(\widehat{X}_{n,t}, \widehat{X}_{n,\underline{t}_n})\, dt\Big|^p\Big]\Big)^{1/p} \le c\cdot n^{-(1-\delta)/(2p)}.$$

Combining(7) to (9) yields the rate of convergence $1/(2p)-$ for the p-th root of the p-th mean of the maximum error of the time-continuous Euler-Maruyama scheme.

We add that in [16] it is assumed that the coefficients μ and σ are bounded and the analysis is carried out only for $p = 2$. However, it is straightforward to adapt the proof technique to the case of a general $p \in [1,\infty)$, and in [23, Remark 4.2] it is stated that the proof techniques in [16] could be modified to cover the case of unbounded coefficients μ and σ as well.

Our proof of Theorem 1 follows the steps (7) and (8) but provides a much better estimate of the p-th mean occupation time of the set B than (9), namely

$$(10) \qquad \Big(\mathbb{E}\Big[\Big|\int_0^1 1_B(\widehat{X}_{n,t}, \widehat{X}_{n,\underline{t}_n})\, dt\Big|^p\Big]\Big)^{1/p} \le c/\sqrt{n},$$

which jointly with (7) and (8) yields the statement of Theorem 1. The estimate (10) is, essentially, obtained by employing the Markov property of the time-continuous Euler-Maruyama scheme $\widehat{X}_n$ relative to the corresponding grid points $1/n, 2/n, \dots, 1$, by using appropriate estimates of the expected occupation time of a neighborhood of a non-zero $\xi \in \mathbb{R}$ of σ by $\widehat{X}_n$ and by carrying out a detailed analysis of the probability of a sign change of $\widehat{X}_{n,t} - \xi$ relative to the sign of $\widehat{X}_{n,\underline{t}_n} - \xi$.

We briefly describe the structure of this section. In Subsection 3.1 we provide L_p-estimates of the solution X and the time-continuous Euler-Maruyama scheme $\widehat{X}_n$. Subsection 3.2 provides the Markov property of $\widehat{X}_n$ and occupation time estimates for $\widehat{X}_n$, which finally lead to the proof of the estimate (10), see Proposition 1. Subsection 3.3 contains the construction of the transformation G and provides the properties of G needed to carry out steps (7) and (8). The material presented in subsection 3.3 is essentially known from [15]. The proof of Theorem 1 is carried out in Subsection 3.4. Subsection 3.5 contains the proof of Theorem 2.

Throughout the following we make use of the fact that the functions μ and σ satisfy a linear growth condition, i.e. there exists $K \in (0,\infty)$ such that for all $x \in \mathbb{R}$,

$$(11) \qquad |\mu(x)| + |\sigma(x)| \le K\cdot (1+|x|).$$

This property is an immediate consequence of the assumptions (A1) and (A2).

3.1. **L_p-estimates of the solution and the time-continuous Euler-Maruyama scheme.** We have the following L_p-estimates for X, which follow from the linear growth property (11) of μ and σ by using standard arguments as in [18, Sec.2.4].

Lemma 1. *Let $p \in [1,\infty)$. Then there exists $c \in (0,\infty)$ such that for all $\delta \in [0,1]$ and all $t \in [0,1-\delta]$,*

$$\big(\mathbb{E}\big[\sup_{s\in[t,t+\delta]} |X(s)-X(t)|^p\big]\big)^{1/p} \leq c\cdot\sqrt{\delta}.$$

In particular,

$$\mathbb{E}\big[\|X\|_\infty^p\big] < \infty.$$

For technical reasons we have to provide L_p-estimates and some further properties of the time-continuous Euler-Maruyama scheme for the SDE (4) dependent on the initial value x_0. To be formally precise, for every $x \in \mathbb{R}$ we let X^x denote the unique strong solution of the SDE

$$\begin{aligned} dX_t^x &= \mu(X_t^x)\,dt + \sigma(X_t^x)\,dW_t, \quad t\in[0,1],\\ X_0^x &= x, \end{aligned} \tag{12}$$

and for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we use $\widehat{X}_n^x = (\widehat{X}_{n,t}^x)_{t\in[0,1]}$ to denote the time-continuous Euler-Maruyama scheme with step-size $1/n$ associated to the SDE (12), i.e. $\widehat{X}_{n,0}^x = x$ and

$$\widehat{X}_{n,t}^x = \widehat{X}_{n,\underline{t}_n}^x + \mu(\widehat{X}_{n,\underline{t}_n}^x)\cdot(t-\underline{t}_n) + \sigma(\widehat{X}_{n,\underline{t}_n}^x)\cdot(W_t - W_{\underline{t}_n})$$

for $t\in[0,1]$. In particular, $X = X^{x_0}$ and $\widehat{X}_n = \widehat{X}_n^{x_0}$ for every $n\in\mathbb{N}$. Furthermore, the integral representation

$$\widehat{X}_{n,t}^x = x + \int_0^t \mu(\widehat{X}_{n,\underline{s}_n}^x)\,ds + \int_0^t \sigma(\widehat{X}_{n,\underline{s}_n}^x)\,dW_s \tag{13}$$

holds for every $n\in\mathbb{N}$ and $t\in[0,1]$.

We have the following uniform L_p-estimates for $\widehat{X}_n^x$, $n\in\mathbb{N}$, which follow from (13) and the linear growth property (11) of μ and σ by using standard arguments.

Lemma 2. *Let $p\in[1,\infty)$. Then there exists $c\in(0,\infty)$ such that for all $x\in\mathbb{R}$, all $n\in\mathbb{N}$, all $\delta\in[0,1]$ and all $t\in[0,1-\delta]$,*

$$\big(\mathbb{E}\big[\sup_{s\in[t,t+\delta]} |\widehat{X}_{n,s}^x - \widehat{X}_{n,t}^x|^p\big]\big)^{1/p} \leq c\cdot(1+|x|)\cdot\sqrt{\delta}.$$

In particular,

$$\sup_{n\in\mathbb{N}}\big(\mathbb{E}\big[\|\widehat{X}_n^x\|_\infty^p\big]\big)^{1/p} \leq c\cdot(1+|x|).$$

3.2. **A Markov property and occupation time estimates for the time-continuous Euler-Maruyama scheme.** The following lemma provides a Markov property of the time-continuous Euler-Maruyama scheme $\widehat{X}_n^x$ relative to the gridpoints $1/n, 2/n, \dots, 1$.

Lemma 3. *For all $x\in\mathbb{R}$, all $n\in\mathbb{N}$, all $j\in\{0,\dots,n-1\}$ and $\mathbb{P}^{\widehat{X}_{n,j/n}^x}$-almost all $y\in\mathbb{R}$ we have*

$$\mathbb{P}^{(\widehat{X}_{n,t}^x)_{t\in[j/n,1]}|\mathcal{F}_{j/n}} = \mathbb{P}^{(\widehat{X}_{n,t}^x)_{t\in[j/n,1]}|\widehat{X}_{n,j/n}^x}$$

as well as

$$\mathbb{P}^{(\widehat{X}^x_{n,t})_{t\in[j/n,1]}|\widehat{X}^x_{n,j/n}=y} = \mathbb{P}^{(\widehat{X}^y_{n,t})_{t\in[0,1-j/n]}}.$$

Proof. The lemma is an immediate consequence of the fact that, by definition of $\widehat{X}^x_n$, for every $\ell \in \{1,\dots,n\}$ there exists a mapping $\psi\colon \mathbb{R}\times C([0,\ell/n]) \to C([0,\ell/n])$ such that for all $x\in\mathbb{R}$ and all $i\in\{0,1,\dots,n-\ell\}$,

$$(\widehat{X}^x_{n,t+i/n})_{t\in[0,\ell/n]} = \psi\big(\widehat{X}^x_{n,i/n}, (W_{t+i/n}-W_{i/n})_{t\in[0,\ell/n]}\big). \qquad \square$$

Next, we provide an estimate for the expected occupation time of a neighborhood of a non-zero of σ by the time-continuous Euler-Maruyama scheme $\widehat{X}^x_n$.

Lemma 4. *Let $\xi\in\mathbb{R}$ satisfy $\sigma(\xi)\neq 0$. Then there exists $c\in(0,\infty)$ such that for all $x\in\mathbb{R}$, all $n\in\mathbb{N}$ and all $\varepsilon\in(0,\infty)$,*

$$\int_0^1 \mathbb{P}(\{|\widehat{X}^x_{n,t}-\xi|\le\varepsilon\})\,dt \le c\cdot(1+x^2)\cdot\Big(\varepsilon+\frac{1}{\sqrt{n}}\Big). \tag{14}$$

Proof. Let $x\in\mathbb{R}$ and $n\in\mathbb{N}$. By (13), (11) and Lemma 2 we see that $\widehat{X}^x_n$ is a continuous semi-martingale with quadratic variation

$$\langle\widehat{X}^x_n\rangle_t = x^2+\int_0^t \sigma^2\big(\widehat{X}^x_{n,\underline{s}_n}\big)\,ds,\quad t\in[0,1]. \tag{15}$$

For $a\in\mathbb{R}$ let $L^a(\widehat{X}^x_n)=(L^a_t(\widehat{X}^x_n))_{t\in[0,1]}$ denote the local time of $\widehat{X}^x_n$ at the point a. Thus, for all $a\in\mathbb{R}$ and all $t\in[0,1]$,

$$|\widehat{X}^x_{n,t}-a| = |x-a| + \int_0^t \operatorname{sgn}(\widehat{X}^x_{n,s}-a)\cdot\mu(\widehat{X}^x_{n,s})\,ds + \int_0^t \operatorname{sgn}(\widehat{X}^x_{n,s}-a)\cdot\sigma(\widehat{X}^x_{n,s})\,dW_s + L^a_t(\widehat{X}^x_n),$$

where $\operatorname{sgn}(z)=1_{(0,\infty)}(z)-1_{(-\infty,0]}(z)$ for $z\in\mathbb{R}$, see, e.g. [27, Chap. VI]. Hence, for all $a\in\mathbb{R}$ and all $t\in[0,1]$,

$$L^a_t(\widehat{X}^x_n) \le |\widehat{X}^x_{n,t}-x| + \int_0^t |\mu(\widehat{X}^x_{n,s})|\,ds + \Big|\int_0^t \operatorname{sgn}(\widehat{X}^x_{n,s}-a)\cdot\sigma(\widehat{X}^x_{n,s})\,dW_s\Big|.$$

Using the Hölder inequality, the Burkholder-Davis-Gundy inequality, (11) and the second estimate in Lemma 2 we conclude that there exists $c\in(0,\infty)$ such that for all $x\in\mathbb{R}$, all $n\in\mathbb{N}$, all $a\in\mathbb{R}$ and all $t\in[0,1]$,

$$\mathbb{E}\big[L^a_t(\widehat{X}^x_n)\big] \le c\cdot(1+|x|). \tag{16}$$

Let $\varepsilon\in(0,\infty)$. Using (15) and (16) we obtain by the occupation time formula that there exists $c\in(0,\infty)$ such that for all $x\in\mathbb{R}$, all $n\in\mathbb{N}$ and all $\varepsilon\in(0,\infty)$,

$$\mathbb{E}\Big[\int_0^1 1_{[\xi-\varepsilon,\xi+\varepsilon]}(\widehat{X}^x_{n,t})\cdot\sigma^2(\widehat{X}^x_{n,\underline{t}_n})\,dt\Big] = \int_{\mathbb{R}} 1_{[\xi-\varepsilon,\xi+\varepsilon]}(a)\,\mathbb{E}\big[L^a_t(\widehat{X}^x_n)\big]\,da \le c\cdot(1+|x|)\cdot\varepsilon. \tag{17}$$

Using the Lipschitz continuity of σ as well as (11) and Lemma 2 we obtain that there exist $c_1, c_2 \in (0,\infty)$ such that for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\Big[\int_0^1 \big|\sigma^2(\widehat{X}^x_{n,t}) - \sigma^2(\widehat{X}^x_{n,\underline{t}_n})\big|\, dt\Big] &\le c_1 \cdot \int_0^1 \mathbb{E}\big[|\widehat{X}^x_{n,t} - \widehat{X}^x_{n,\underline{t}_n}| \cdot (1 + \|\widehat{X}^x_n\|_\infty)\big]\, dt \\ &\le c_2 \cdot (1+x^2) \cdot \frac{1}{\sqrt{n}}. \end{aligned} \tag{18}$$

Since σ is continuous and $\sigma(\xi) \neq 0$ there exist $\kappa, \varepsilon_0 \in (0,\infty)$ such that

$$\inf_{|z-\xi|<\varepsilon_0} \sigma^2(z) \ge \kappa.$$

Observing (17) and (18) we conclude that there exists $c \in (0,\infty)$ such that for all $x \in \mathbb{R}$, all $n \in \mathbb{N}$ and all $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned}\int_0^1 \mathbb{P}(\{|\widehat{X}^x_{n,t} - \xi| \le \varepsilon\})\, dt &= \frac{1}{\kappa} \cdot \mathbb{E}\Big[\int_0^1 \kappa \cdot 1_{[\xi-\varepsilon,\xi+\varepsilon]}(\widehat{X}^x_{n,t})\, dt\Big] \\ &\le \frac{1}{\kappa} \cdot \mathbb{E}\Big[\int_0^1 1_{[\xi-\varepsilon,\xi+\varepsilon]}(\widehat{X}^x_{n,t}) \cdot \sigma^2(\widehat{X}^x_{n,t})\, dt\Big] \\ &\le \frac{1}{\kappa} \cdot \mathbb{E}\Big[\int_0^1 \big(1_{[\xi-\varepsilon,\xi+\varepsilon]}(\widehat{X}^x_{n,t}) \cdot \sigma^2(\widehat{X}^x_{n,\underline{t}_n}) + \big|\sigma^2(\widehat{X}^x_{n,t}) - \sigma^2(\widehat{X}^x_{n,\underline{t}_n})\big|\big)\, dt\Big] \\ &\le \frac{c}{\kappa} \cdot (1 + |x| + x^2) \cdot \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big), \end{aligned}$$

which completes the proof of the lemma. □

The following result shows how to transfer the condition of a sign change of $\widehat{X}_n - \xi$ at time t relative to its sign at the grid point $\underline{t}_n$ to a condition on the distance of $\widehat{X}_n$ and ξ at the time $\underline{t}_n - (t - \underline{t}_n)$.

Lemma 5. *Let $\xi \in \mathbb{R}$. Then there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$, all $0 \le s \le t \le 1$ with $\underline{t}_n - s \ge 1/n$ and all $A \in \mathcal{F}_s$,*

$$\begin{aligned}&\mathbb{P}\big(A \cap \{(\widehat{X}_{n,t} - \xi) \cdot (\widehat{X}_{n,\underline{t}_n} - \xi) \le 0\}\big) \\ &\qquad \le \frac{c}{n} \cdot \mathbb{P}(A) + c \cdot \int_{\mathbb{R}} \mathbb{P}\big(A \cap \big\{|\widehat{X}_{n,\underline{t}_n - (t-\underline{t}_n)} - \xi| \le \tfrac{c}{\sqrt{n}}(1+|z|)\big\}\big) \cdot e^{-\frac{z^2}{2}}\, dz. \end{aligned} \tag{19}$$

Proof. Choose $K \in (0,\infty)$ according to (11) and choose $n_0 \in \mathbb{N} \setminus \{1\}$ such that for all $n \ge n_0$,

$$12K \cdot (1 + |\xi|) \cdot \frac{1 + \sqrt{2\ln(n)}}{\sqrt{n}} \le \frac{1}{2}.$$

Without loss of generality we may assume that $n \ge n_0$. Let $0 \le s \le t \le 1$ with $\underline{t}_n - s \ge 1/n$ and let $A \in \mathcal{F}_s$. If $t = \underline{t}_n$ then for all $c \in (0,\infty)$ and all $z \in \mathbb{R}$ we have

$$\{(\widehat{X}_{n,t} - \xi) \cdot (\widehat{X}_{n,\underline{t}_n} - \xi) \le 0\} = \{\widehat{X}_{n,\underline{t}_n} - \xi = 0\} \subset \big\{|\widehat{X}_{n,\underline{t}_n - (t-\underline{t}_n)} - \xi| \le \tfrac{c}{\sqrt{n}}(1+|z|)\big\},$$

which implies that in this case (19) holds for all $c \ge 1/\sqrt{2\pi}$.

Now assume that $t > \underline{t}_n$ and put

$$Z_1 = \frac{W_t - W_{\underline{t}_n}}{\sqrt{t-\underline{t}_n}}, \quad Z_2 = \frac{W_{\underline{t}_n} - W_{\underline{t}_n-(t-\underline{t}_n)}}{\sqrt{t-\underline{t}_n}}, \quad Z_3 = \frac{W_{\underline{t}_n-(t-\underline{t}_n)} - W_{\underline{t}_n-1/n}}{\sqrt{1/n-(t-\underline{t}_n)}}.$$

Below we show that

$$\begin{aligned}(20)\qquad &\big\{(\widehat{X}_{n,t}-\xi)\cdot(\widehat{X}_{n,\underline{t}_n}-\xi)\le 0\big\}\cap\big\{\max_{i\in\{1,2,3\}}|Z_i|\le\sqrt{2\ln(n)}\big\}\\ &\qquad\subset\big\{|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}-\xi|\le 12K\cdot(1+|\xi|)\cdot(1+|Z_1|+|Z_2|)/\sqrt{n}\big\}.\end{aligned}$$

Note that Z_1, Z_2, Z_3 are independent and identically distributed standard normal random variables. Moreover, (Z_1, Z_2, Z_3) is independent of $\mathcal{F}_s$ since $s \le \underline{t}_n - 1/n$, (Z_1, Z_2) is independent of $\mathcal{F}_{\underline{t}_n-(t-\underline{t}_n)}$ and $\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}$ is $\mathcal{F}_{\underline{t}_n-(t-\underline{t}_n)}$-measurable. Using the latter facts jointly with (20) and a standard estimate of standard normal tail probabilities we obtain that

$$\begin{aligned}&\mathbb{P}\big(A\cap\{(\widehat{X}_{n,t}-\xi)\cdot(\widehat{X}_{n,\underline{t}_n}-\xi)\le 0\}\big)\\ &\quad\le\mathbb{P}\big(A\cap\{|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}-\xi|\le 12K\cdot(1+|\xi|)\cdot(1+|Z_1|+|Z_2|)/\sqrt{n}\}\big)\\ &\qquad\qquad+\mathbb{P}\big(A\cap\{\max_{i\in\{1,2,3\}}|Z_i|>\sqrt{2\ln(n)}\}\big)\\ &\quad\le\frac{2}{\pi}\int_{[0,\infty)^2}\mathbb{P}\big(A\cap\big\{|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}-\xi|\le 12K\cdot(1+|\xi|)\cdot\tfrac{1+z_1+z_2}{\sqrt{n}}\big\}\big)\cdot e^{-\frac{z_1^2+z_2^2}{2}}\,d(z_1,z_2)\\ &\qquad\qquad+6\mathbb{P}(A)\cdot\mathbb{P}\big(\{Z_1>\sqrt{2\ln(n)}\}\big)\\ &\quad\le\frac{2}{\pi}\int_{\mathbb{R}^2}\mathbb{P}\Big(A\cap\Big\{|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}-\xi|\le 12\sqrt{2}K\cdot(1+|\xi|)\cdot\frac{1+|\frac{z_1+z_2}{\sqrt{2}}|}{\sqrt{n}}\Big\}\Big)\cdot e^{-\frac{z_1^2+z_2^2}{2}}\,d(z_1,z_2)\\ &\qquad\qquad+\frac{6\mathbb{P}(A)}{\sqrt{2\pi\cdot 2\ln(n)}\cdot n}\\ &\quad=\frac{4}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathbb{P}\big(A\cap\big\{|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}-\xi|\le 12\sqrt{2}K\cdot(1+|\xi|)\cdot\tfrac{1+|z|}{\sqrt{n}}\big\}\big)\cdot e^{-\frac{z^2}{2}}\,dz+\frac{3\mathbb{P}(A)}{\sqrt{\pi\ln(n)}\cdot n},\end{aligned}$$

which yields (19).

It remains to prove the inclusion (20). To this end let $\omega \in \Omega$ and assume that

$$(21)\qquad (\widehat{X}_{n,t}(\omega)-\xi)\cdot(\widehat{X}_{n,\underline{t}_n}(\omega)-\xi)\le 0 \quad\text{and}\quad \max_{i\in\{1,2,3\}}|Z_i(\omega)|\le\sqrt{2\ln(n)}.$$

Using (11) and the fact that for all $a, b \in \mathbb{R}$,

$$(22)\qquad 1+|a|\le(1+|a-b|)\cdot(1+|b|),$$

we obtain

$$
\begin{aligned}
|\widehat{X}_{n,\underline{t}_n}(\omega)-\xi| &\le |(\widehat{X}_{n,\underline{t}_n}(\omega)-\xi)-(\widehat{X}_{n,t}(\omega)-\xi)|\\
&= |\mu(\widehat{X}_{n,\underline{t}_n}(\omega))\cdot(t-\underline{t}_n)+\sigma(\widehat{X}_{n,\underline{t}_n}(\omega))\cdot\sqrt{t-\underline{t}_n}\cdot Z_1(\omega)|\\
&\le K\cdot(1+|\widehat{X}_{n,\underline{t}_n}(\omega)|)\cdot\Big(\frac{1}{n}+\frac{1}{\sqrt{n}}\cdot|Z_1(\omega)|\Big)\\
&\le (1+|\widehat{X}_{n,\underline{t}_n}(\omega)-\xi|)\cdot K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|).
\end{aligned}
\tag{23}
$$

Since $n\ge n_0$ we have

$$
K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|)\le K\cdot(1+|\xi|)\cdot\frac{1+\sqrt{2\ln(n)}}{\sqrt{n}}\le\frac{1}{2},
$$

and therefore,

$$
|\widehat{X}_{n,\underline{t}_n}(\omega)-\xi|\le\frac{K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|)}{1-K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|)}\le 2K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|).
\tag{24}
$$

Similarly to (23), we obtain by (11) and (22) that

$$
|\widehat{X}_{n,\underline{t}_n}(\omega)-\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)|\le(1+|\widehat{X}_{n,\underline{t}_n-1/n}(\omega)-\xi|)\cdot K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_2(\omega)|)
\tag{25}
$$

and

$$
|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)-\widehat{X}_{n,\underline{t}_n-1/n}(\omega)|\le(1+|\widehat{X}_{n,\underline{t}_n-1/n}(\omega)-\xi|)\cdot K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_3(\omega)|).
\tag{26}
$$

Since $n\ge n_0$ we have $K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_3(\omega)|)\le 1/2$, and therefore we conclude from (26) that

$$
\begin{aligned}
1+|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)-\xi| &\ge 1+|\widehat{X}_{n,\underline{t}_n-1/n}(\omega)-\xi|-|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)-\widehat{X}_{n,\underline{t}_n-1/n}(\omega)|\\
&\ge(1+|\widehat{X}_{n,\underline{t}_n-1/n}(\omega)-\xi|)/2.
\end{aligned}
\tag{27}
$$

Using (24), (25) and (27) we obtain

$$
\begin{aligned}
&|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)-\xi|\\
&\quad\le|\widehat{X}_{n,\underline{t}_n}(\omega)-\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)|+|\widehat{X}_{n,\underline{t}_n}(\omega)-\xi|\\
&\quad\le(1+|\widehat{X}_{n,\underline{t}_n-1/n}(\omega)-\xi|)\cdot 3K\cdot(1+|\xi|)\cdot\tfrac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|+|Z_2(\omega)|)\\
&\quad\le(1+|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)-\xi|)\cdot 6K\cdot(1+|\xi|)\cdot\tfrac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|+|Z_2(\omega)|).
\end{aligned}
\tag{28}
$$

Since $n\ge n_0$ we have $6K\cdot(1+|\xi|)\cdot\frac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|+|Z_2(\omega)|)\le 1/2$, which jointly with (28) yields

$$
|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)}(\omega)-\xi|\le 12K\cdot(1+|\xi|)\cdot\tfrac{1}{\sqrt{n}}\cdot(1+|Z_1(\omega)|+|Z_2(\omega)|).
$$

This finishes the proof of (20). □

Using Lemmas 3, 4 and 5 we can now establish the following two estimates on the probability of sign changes of $\widehat{X}_n-\xi$ relative to its sign at the gridpoints $0,1/n,\dots,1$.

Lemma 6. *Let $\xi \in \mathbb{R}$ satisfy $\sigma(\xi) \neq 0$ and let*

$$A_{n,t} = \{(\widehat{X}_{n,t} - \xi) \cdot (\widehat{X}_{n,\underline{t}_n} - \xi) \leq 0\}$$

for all $n \in \mathbb{N}$ and $t \in [0,1]$. Then the following two statements hold.

(i) *There exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$, all $s \in [0,1)$ and all $A \in \mathcal{F}_s$,*

$$\int_s^1 \mathbb{P}(A \cap A_{n,t})\, dt \leq \frac{c}{\sqrt{n}} \cdot \big(\mathbb{P}(A) + \mathbb{E}\big[1_A \cdot (\widehat{X}_{n,\underline{s}_n+1/n} - \xi)^2\big]\big).$$

(ii) *There exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$, all $s \in [0,1)$ and all $A \in \mathcal{F}_s$,*

$$\int_s^1 \mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{t}_n+1/n} - \xi)^2\big]\, dt \leq \frac{c}{n} \cdot \big(\mathbb{P}(A) + \mathbb{E}\big[1_A \cdot (\widehat{X}_{n,\underline{s}_n+1/n} - \xi)^2\big]\big).$$

Proof. Let $n \in \mathbb{N}$, $s \in [0,1)$ and $A \in \mathcal{F}_s$. In the following we use $c_1, c_2, \dots \in (0,\infty)$ to denote unspecified positive constants, which neither depend on n nor on s nor on A.

We first prove part (i) of the lemma. Clearly we may assume that $s < 1 - 1/n$. Then $\underline{s}_n \leq 1 - 2/n$ and we have

$$\int_s^1 \mathbb{P}(A \cap A_{n,t})\, dt \leq \frac{2}{n} \cdot \mathbb{P}(A) + \int_{\underline{s}_n+2/n}^1 \mathbb{P}(A \cap A_{n,t})\, dt. \tag{29}$$

If $t \in [\underline{s}_n + 2/n, 1]$ then $\underline{t}_n \geq \underline{s}_n + 2/n$, which implies $\underline{t}_n - 1/n \geq \underline{s}_n + 1/n \geq s$. We may thus apply Lemma 5 to conclude that there exists $c_1 \in (0,\infty)$ such that

$$\begin{aligned} &\int_s^1 \mathbb{P}(A \cap A_{n,t})\, dt \\ &\quad \leq \frac{c_1}{n} \cdot \mathbb{P}(A) + c_1 \cdot \int_{\mathbb{R}} \int_{\underline{s}_n+2/n}^1 \mathbb{P}\big(A \cap \big\{|\widehat{X}_{n,\underline{t}_n-(t-\underline{t}_n)} - \xi| \leq \tfrac{c_1}{\sqrt{n}}(1+|z|)\big\}\big) \cdot e^{-\frac{z^2}{2}}\, dz\, dt \\ &\quad = \frac{c_1}{n} \cdot \mathbb{P}(A) + c_1 \cdot \int_{\mathbb{R}} \int_{\underline{s}_n+1/n}^{1-1/n} \mathbb{P}\big(A \cap \big\{|\widehat{X}_{n,t} - \xi| \leq \tfrac{c_1}{\sqrt{n}}(1+|z|)\big\}\big) \cdot e^{-\frac{z^2}{2}}\, dz\, dt. \end{aligned} \tag{30}$$

By the fact that $A \in \mathcal{F}_{\underline{s}_n+1/n}$ and by the first part of Lemma 3 we obtain that for all $z \in \mathbb{R}$,

$$\begin{aligned} &\int_{\underline{s}_n+1/n}^{1-1/n} \mathbb{P}\big(A \cap \big\{|\widehat{X}_{n,t} - \xi| \leq \tfrac{c_1}{\sqrt{n}}(1+|z|)\big\}\big)\, dt \\ &\qquad = \mathbb{E}\Big[1_A \cdot \mathbb{E}\Big[\int_{\underline{s}_n+1/n}^{1-1/n} 1_{\{|\widehat{X}_{n,t}-\xi| \leq \frac{c_1}{\sqrt{n}}(1+|z|)\}}\, dt \Big| \widehat{X}_{n,\underline{s}_n+1/n}\Big]\Big]. \end{aligned} \tag{31}$$

Moreover, by the second part of Lemma 3 and by Lemma 4 we obtain that there exists $c_2 \in (0,\infty)$ such that for all $z, x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}\Big[\int_{\underline{s}_n+1/n}^{1-1/n} 1_{\{|\widehat{X}_{n,t}-\xi| \leq \frac{c_1}{\sqrt{n}}(1+|z|)\}}\, dt \Big| \widehat{X}_{n,\underline{s}_n+1/n} = x\Big] \\ &\quad = \mathbb{E}\Big[\int_0^{1-2/n-\underline{s}_n} 1_{\{|\widehat{X}^x_{n,t}-\xi| \leq \frac{c_1}{\sqrt{n}}(1+|z|)\}}\, dt\Big] \leq c_2 \cdot (1+x^2) \cdot \Big(\frac{c_1}{\sqrt{n}} \cdot (1+|z|) + \frac{1}{\sqrt{n}}\Big). \end{aligned} \tag{32}$$

Combining (31) and (32) and using the fact that for all $a, b \in \mathbb{R}$,

$$1 + a^2 \leq 2\,(1 + (a-b)^2) \cdot (1 + b^2),$$

we conclude that for all $z \in \mathbb{R}$,

$$\begin{aligned} &\int_{\underline{s}_n+1/n}^{1-1/n} \mathbb{P}\big(A \cap \big\{|\widehat{X}_{n,t} - \xi| \leq \tfrac{c_1}{\sqrt{n}}(1+|z|)\big\}\big)\, dt \\ &\qquad \leq \tfrac{c_2(c_1+1)}{\sqrt{n}} \cdot (1+|z|) \cdot \mathbb{E}\big[1_A \cdot (1 + \widehat{X}^2_{n,\underline{s}_n+1/n})\big] \\ &\qquad \leq \tfrac{2c_2(c_1+1)}{\sqrt{n}} \cdot (1+\xi^2) \cdot (1+|z|) \cdot \big(\mathbb{P}(A) + \mathbb{E}\big[1_A \cdot (\widehat{X}_{n,\underline{s}_n+1/n} - \xi)^2\big]\big). \end{aligned} \tag{33}$$

Inserting (33) into (30) and observing that $\int_{\mathbb{R}} (1+|z|) \cdot e^{-z^2/2}\, dz < \infty$ completes the proof of part (i) of the lemma.

We next prove part (ii). Clearly,

$$\begin{aligned} &\int_s^1 \mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{t}_n+1/n} - \xi)^2\big]\, dt \\ &\quad = \int_s^{\underline{s}_n+1/n} \mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{t}_n+1/n} - \xi)^2\big]\, dt + \int_{\underline{s}_n+1/n}^1 \mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{t}_n+1/n} - \xi)^2\big]\, dt. \end{aligned}$$

If $t \in [s, \underline{s}_n + 1/n)$ then $\underline{t}_n = \underline{s}_n$ and therefore

$$\begin{aligned} \int_s^{\underline{s}_n+1/n} \mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{t}_n+1/n} - \xi)^2\big]\, dt &= \int_s^{\underline{s}_n+1/n} \mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{s}_n+1/n} - \xi)^2\big]\, dt \\ &\leq \int_s^{\underline{s}_n+1/n} \mathbb{E}\big[1_A \cdot (\widehat{X}_{n,\underline{s}_n+1/n} - \xi)^2\big]\, dt \\ &\leq \frac{1}{n} \cdot \mathbb{E}\big[1_A \cdot (\widehat{X}_{n,\underline{s}_n+1/n} - \xi)^2\big]. \end{aligned} \tag{34}$$

Next, let $t \in [\underline{s}_n + 1/n, 1]$. Clearly, we have on A_t,

$$|\widehat{X}_{n,\underline{t}_n+1/n} - \xi| \leq |\widehat{X}_{n,\underline{t}_n+1/n} - \widehat{X}_{n,t}| + |\widehat{X}_{n,t} - \xi| \leq |\widehat{X}_{n,\underline{t}_n+1/n} - \widehat{X}_{n,t}| + |\widehat{X}_{n,t} - \widehat{X}_{n,\underline{t}_n}|.$$

Hence, by Lemma 3(i),

$$\begin{aligned} &\mathbb{E}\big[1_{A\cap A_{n,t}} \cdot (\widehat{X}_{n,\underline{t}_n+1/n} - \xi)^2\big] \\ &\qquad \leq \mathbb{E}\big[1_A \cdot (|\widehat{X}_{n,\underline{t}_n+1/n} - \widehat{X}_{n,t}| + |\widehat{X}_{n,t} - \widehat{X}_{n,\underline{t}_n}|)^2\big] \\ &\qquad = \mathbb{E}\big[1_A \cdot \mathbb{E}\big[(|\widehat{X}_{n,\underline{t}_n+1/n} - \widehat{X}_{n,t}| + |\widehat{X}_{n,t} - \widehat{X}_{n,\underline{t}_n}|)^2 \big| \widehat{X}_{n,\underline{s}_n+1/n}\big]\big]. \end{aligned} \tag{35}$$

If $t \geq \underline{s}_n + 1/n$ then $\underline{t}_n \geq \underline{s}_n + 1/n$. Hence, by Lemma 3(ii) and Lemma 2 we obtain that there exist $c_1, c_2 \in (0, \infty)$ such that for all $t \in [\underline{s}_n + 1/n, 1]$ and all $x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}\big[(|\widehat{X}_{n,\underline{t}_n+1/n} - \widehat{X}_{n,t}| + |\widehat{X}_{n,t} - \widehat{X}_{n,\underline{t}_n}|)^2 \big| \widehat{X}_{n,\underline{s}_n+1/n} = x\big] \\ &\qquad = \mathbb{E}[(|\widehat{X}^x_{n,\underline{t}_n-\underline{s}_n} - \widehat{X}^x_{n,t-\underline{s}_n-1/n}| + |\widehat{X}^x_{n,t-\underline{s}_n-1/n} - \widehat{X}^x_{\underline{t}_n-\underline{s}_n-1/n}|)^2] \\ &\qquad \leq c_1 \cdot (1+x^2) \cdot 1/n \leq c_2 \cdot (1 + (x-\xi)^2) \cdot 1/n. \end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$\begin{aligned}(37)\qquad &\int_{\underline{s}_n+1/n}^1 \mathbb{E}\big[1_{A\cap A_{n,t}}\cdot(\widehat{X}_{n,\underline{t}_n+1/n}-\xi)^2\big]\,dt\\ &\quad\le \frac{c_2}{n}\cdot\int_{\underline{s}_n+1/n}^1 \mathbb{E}\big[1_A\cdot(1+(\widehat{X}_{n,\underline{s}_n+1/n}-\xi)^2)\big]\,dt\\ &\quad\le \frac{c_2}{n}\cdot\big(\mathbb{P}(A)+\mathbb{E}\big[1_A\cdot(\widehat{X}_{n,\underline{s}_n+1/n}-\xi)^2)\big]\big).\end{aligned}$$

Combining (34) with (37) completes the proof of part (ii) of the lemma. □

We are ready to establish the main result in this section, which provides a p-th mean estimate of the Lebesgue measure of the set of times t of a sign change of $\widehat{X}_{n,t}-\xi$ relative to the sign of $\widehat{X}_{n,\underline{t}_n}-\xi$.

Proposition 1. *Let $\xi\in\mathbb{R}$ satisfy $\sigma(\xi)\neq 0$ and let $p\in[1,\infty)$. Then there exists $c\in(0,\infty)$ such that for all $n\in\mathbb{N}$,*

$$\mathbb{E}\Big[\Big|\int_0^1 1_{\{(\widehat{X}_{n,t}-\xi)\cdot(\widehat{X}_{n,\underline{t}_n}-\xi)\le 0\}}\,dt\Big|^p\Big]^{1/p}\le\frac{c}{\sqrt{n}}. \qquad (38)$$

Proof. Clearly, it suffices to consider only the case $p\in\mathbb{N}$. For $n\in\mathbb{N}$ and $t\in[0,1]$ put $A_{n,t}=\{(\widehat{X}_{n,t}-\xi)\cdot(\widehat{X}_{n,\underline{t}_n}-\xi)\le 0\}$ as in Lemma 6, and for $n,p\in\mathbb{N}$ let

$$a_{n,p}=\mathbb{E}\Big[\Big(\int_0^1 1_{A_{n,t}}\,dt\Big)^p\Big].$$

We prove by induction on p that for every $p\in\mathbb{N}$ there exists $c\in(0,\infty)$ such that for all $n\in\mathbb{N}$,

$$a_{n,p}\le c\cdot n^{-p/2}. \qquad (39)$$

First assume that $p=1$. Using Lemma 6(i) with $s=0$ and $A=\Omega$ we obtain that there exists $c\in(0,\infty)$ such that for all $n\in\mathbb{N}$,

$$a_{n,1}=\int_0^1\mathbb{P}(A_{n,t})\,dt\le\frac{c}{\sqrt{n}}\cdot(1+\mathbb{E}[(\widehat{X}_{n,1/n}-\xi)^2])\le\frac{c}{\sqrt{n}}\cdot(1+2\xi^2+2\sup_{j\in\mathbb{N}}\mathbb{E}[\|\widehat{X}_j\|_\infty^2]).$$

Observing Lemma 2 we thus see that (39) holds for $p=1$.

Next, let $q\in\mathbb{N}$ and assume that (39) holds for all $p\in\{1,\dots,q\}$. Clearly,

$$a_{n,q+1}=(q+1)!\cdot\int_0^1\int_{t_1}^1\dots\int_{t_q}^1\mathbb{P}(A_{n,t_1}\cap A_{n,t_2}\cap\ldots\cap A_{n,t_{q+1}})\,dt_{q+1}\,\ldots\,dt_2\,dt_1.$$

First applying Lemma 6(i) with $A=A_{n,t_1}\cap\ldots\cap A_{n,t_q}$ and $s=t_q$, then applying $(q-1)$-times Lemma 6(ii) with $A=A_{n,t_1}\cap\ldots\cap A_{n,t_j}$ and $s=t_j$ for $j=q-1,\dots,1$, and finally applying Lemma 6(ii) with $A=\Omega$ and $s=0$ we conclude that there exist constants $c_1,c_2,c_3\in(0,\infty)$

such that for all $n \in \mathbb{N}$,

$$\begin{aligned}
a_{n,q+1} &\leq \frac{c_1}{\sqrt{n}} \cdot \Big(a_{n,q} + \int_0^1 \dots \int_{t_{q-1}}^1 \mathbb{E}\big[1_{A_{n,t_1} \cap \dots \cap A_{n,t_q}} \cdot (\widehat{X}_{n,\underline{t_q}_n + 1/n} - \xi)^2]) \, dt_q \, \dots \, dt_1 \Big) \\
&\leq c_2 \cdot \Big(\frac{a_{n,q}}{\sqrt{n}} + \frac{a_{n,q-1}}{n^{3/2}} + \dots + \frac{a_{n,1}}{n^{q-1/2}} + \frac{1}{n^{q-1/2}} \cdot \int_0^1 \mathbb{E}\big[1_{A_{n,t_1}} \cdot (\widehat{X}_{n,\underline{t_1}_n + 1/n} - \xi)^2 \big] \, dt_1 \Big) \\
&\leq c_2 \cdot \Big(\frac{a_{n,q}}{\sqrt{n}} + \frac{a_{n,q-1}}{n^{3/2}} + \dots + \frac{a_{n,1}}{n^{q-1/2}} + \frac{c_3}{n^{q+1/2}} \cdot \big(1 + 2\xi^2 + 2 \sup_{j \in \mathbb{N}} \mathbb{E}[\|\widehat{X}_j\|_\infty^2]\big) \Big).
\end{aligned}$$

Employing Lemma 2 and the induction hypothesis yields the validity of (39) for $p = q+1$, which finishes the proof of the proposition. □

3.3. **The transformed equation.** We turn to the construction and the properties of the mapping $G\colon \mathbb{R} \to \mathbb{R}$ that is used to switch from the SDE (4) to an SDE with Lipschitz continuous coefficients. The material presented in this subsection is essentially known from [15].

Lemma 7. *There exists a function $G\colon \mathbb{R} \to \mathbb{R}$ with the following properties.*

(i) *G is differentiable with*

$$0 < \inf_{x \in \mathbb{R}} G'(x) \leq \sup_{x \in \mathbb{R}} G'(x) < \infty.$$

In particular, G is Lipschitz continuous and has an inverse $G^{-1}\colon \mathbb{R} \to \mathbb{R}$ that is Lipschitz continuous as well.

(ii) *The derivative G' of G is Lipschitz continuous hence absolutely continuous. Moreover, G' has a bounded Lebesgue-density $G''\colon \mathbb{R} \to \mathbb{R}$ that is Lipschitz continuous on each of the intervals $(\xi_0, \xi_1), \dots, (\xi_k, \xi_{k+1})$ and such that the functions*

$$\widetilde{\mu} = (G' \cdot \mu + \tfrac{1}{2} G'' \cdot \sigma^2) \circ G^{-1} \quad \textit{and} \quad \widetilde{\sigma} = (G' \cdot \sigma) \circ G^{-1}$$

are Lipschitz continuous.

Proof. We only provide a sketch of the proof. If $k = 0$ then μ and σ are Lipschitz continuous and we can take $G(x) = x$ for all $x \in \mathbb{R}$.

Now, assume that $k \in \mathbb{N}$. Since μ is Lipschitz continuous on each of the intervals $(\xi_0, \xi_1), \dots,$ (ξ_k, ξ_{k+1}) it is easy to see that the one-sided limits $\mu(\xi_i-)$ and $\mu(\xi_i+)$ exist for all $i \in \{1, \dots, k\}$. For $i \in \{1, \dots, k\}$ put

$$\alpha_i = \frac{\mu(\xi_i-) - \mu(\xi_i+)}{2\sigma^2(\xi_i)},$$

let $\rho \in (0, \infty]$ be given by

$$\rho = \begin{cases} \frac{1}{6|\alpha_1|}, & \text{if } k = 1, \\ \min\big(\big\{\frac{1}{6|\alpha_i|} \colon i \in \{1, \dots, k\}\big\} \cup \big\{\frac{\xi_i - \xi_{i-1}}{2} \colon i \in \{2, \dots, k\}\big\}\big), & \text{if } k \geq 2, \end{cases}$$

where we use the convention $1/0 = \infty$, let $\nu \in (0, \rho)$, let $\phi\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(x) = (1 - x^2)^3 \cdot 1_{[-1,1]}(x),$$

and define $G\colon \mathbb{R}\to\mathbb{R}$ by

$$G(x) = x + \sum_{i=1}^{k} \alpha_i \cdot (x-\xi_i)\cdot |x-\xi_i| \cdot \phi\Big(\frac{x-\xi_i}{\nu}\Big).$$

It is straightforward to check that G is differentiable with $\sup_{x\in\mathbb{R}} G'(x) < \infty$. For the proof of $\inf_{x\in\mathbb{R}} G'(x) > 0$ see Lemma 2.2 in [15].

Put $\Theta = \{\xi_1,\dots,\xi_k\}$. It is straightforward to check that G' is Lipschitz continuous and continuously differentiable on $\mathbb{R}\setminus\Theta$, $(G_{|\mathbb{R}\setminus\Theta})''$ is bounded, Lipschitz continuous on each of the intervals $(\xi_0,\xi_1),\dots,(\xi_k,\xi_{k+1})$ and has one-sided limits $(G_{|\mathbb{R}\setminus\Theta})''(\xi_i-)$ and $(G_{|\mathbb{R}\setminus\Theta})''(\xi_i+)$ for all $i\in\{1,\dots,k\}$. Moreover, one can show that for all $i\in\{1,\dots,k\}$,

$$(G'\cdot\mu + \tfrac{1}{2}G''\cdot\sigma^2)(\xi_i+) = (G'\cdot\mu + \tfrac{1}{2}G''\cdot\sigma^2)(\xi_i-). \tag{40}$$

By a slight abuse of notation we define an extension $G''\colon\mathbb{R}\to\mathbb{R}$ by taking

$$G''(\xi_i) = (G_{|\mathbb{R}\setminus\Theta})''(\xi_i+) + \frac{2G'(\xi_i)\cdot(\mu(\xi_i+)-\mu(\xi_i))}{\sigma^2(\xi_i)} \tag{41}$$

for $i\in\{1,\dots,k\}$. Clearly, G'' is then a bounded Lebesgue-density of G'. Furthermore, it is straightforward to check that $\widetilde{\mu}$ and $\widetilde{\sigma}$ are Lipschitz continuous, which completes the proof of the lemma. □

Next, choose G according to Lemma 7 and define a stochastic process $Z\colon[0,1]\times\Omega\to\mathbb{R}$ by

$$Z_t = G(X_t),\quad t\in[0,1]. \tag{42}$$

Lemma 8. *The process Z is the unique strong solution of the SDE*

$$\begin{aligned} dZ_t &= \widetilde{\mu}(Z_t)\,dt + \widetilde{\sigma}(Z_t)\,dW_t, \quad t\in[0,1],\\ Z_0 &= G(x_0)\end{aligned} \tag{43}$$

with $\widetilde{\mu}$ and $\widetilde{\sigma}$ according to Lemma 7(ii).

Proof. According to Lemma 7(ii), G' is absolutely continuous. We may therefore apply Itô's formula, see e.g. [13, Problem 3.7.3], to conclude that for every $t\in[0,1]$ we have $\mathbb{P}$-a.s.,

$$G(X_t) = G(x_0) + \int_0^t (G'(X_s)\cdot\mu(X_s) + \tfrac{1}{2}G''(X_s)\cdot\sigma^2(X_s))\,ds + \int_0^t G'(X_s)\cdot\sigma(X_s)\,dW_s,$$

which implies that Z is a strong solution of the SDE (43). Due to the Lipschitz continuity of $\widetilde{\mu}$ and $\widetilde{\sigma}$, see Lemma 7(ii), the strong solution of (43) is unique. □

For every $n\in\mathbb{N}$ we use $\widehat{Z}_n = (\widehat{Z}_{n,t})_{t\in[0,1]}$ to denote the time-continuous Euler-Maruyama scheme with step-size $1/n$ associated to the SDE (43), i.e. $\widehat{Z}_{n,0} = G(x_0)$ and

$$\widehat{Z}_{n,t} = \widehat{Z}_{n,i/n} + \widetilde{\mu}(\widehat{Z}_{n,i/n})\cdot(t-i/n) + \widetilde{\sigma}(\widehat{Z}_{n,i/n})\cdot(W_t - W_{i/n})$$

for $t\in(i/n,(i+1)/n]$ and $i\in\{0,\dots,n-1\}$. The following estimates are standard error bounds for the time-continuous Euler-Maruyama scheme associated to an SDE with Lipschitz continuous coefficients.

Lemma 9. *Let $p\in[1,\infty)$. Then there exists $c\in(0,\infty)$ such that for all $n\in\mathbb{N}$,*

(i) $\mathbb{E}\big[\|\widehat{Z}_n\|_\infty^p\big] \le c$,

(ii) $\big(\mathbb{E}\big[\|Z - \widehat{Z}_n)\|_\infty^p\big]\big)^{1/p} \le c/\sqrt{n}$.

Finally, we provide an estimate for the transformed time-continuous Euler-Maruyama scheme $G \circ \widehat{X}_n = (G(\widehat{X}_{n,t}))_{t\in[0,1]}$.

Lemma 10. *Let $p \in [1,\infty)$. Then there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{E}\big[\|G \circ \widehat{X}_n\|_\infty^p\big] \le c.$$

Proof. According to Lemma 7(i), G is Lipschitz continous and hence satisfies a linear growth condition, i.e. there exists $c \in (0,\infty)$ such that $|G(x)| \le c \cdot (1 + |x|)$ for all $x \in \mathbb{R}$. Hence

$$\|G \circ \widehat{X}_n\|_\infty \le c \cdot (1 + \|\widehat{X}_n\|_\infty),$$

which jointly with Lemma 2 implies the statement of the lemma. □

3.4. **Proof of Theorem 1.** We choose G and a Lebesgue density G'' of G according to Lemma 7, define Z by (42), and for every $n \in \mathbb{N}$ we define a function $u_n \colon [0,1] \to [0,\infty)$ by

$$u_n(t) = \mathbb{E}\big[\sup_{s\in[0,t]} |G(\widehat{X}_{n,s}) - \widehat{Z}_{n,s}|^p\big].$$

Note that the functions u_n, $n \in \mathbb{N}$, are well-defined and bounded due to Lemma 9(i) and Lemma 10.

Below we show that there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and all $t \in [0,1]$,

$$u_n(t) \le c \cdot \Big(\frac{1}{n^{p/2}} + \sum_{i=1}^{k} \mathbb{E}\Big[\Big|\int_0^1 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}}\, ds\Big|^p\Big] + \int_0^t u_n(s)\, ds\Big). \tag{44}$$

Using Proposition 1 we conclude from (44) that there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and all $t \in [0,1]$,

$$u_n(t) \le c \cdot \Big(\frac{1}{n^{p/2}} + \int_0^t u_n(s)\, ds\Big).$$

By Gronwall's inequality it then follows that there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$u_n(1) \le \frac{c}{n^{p/2}}. \tag{45}$$

Using the fact that G^{-1} is Lipschitz continuous, see Lemma 7(i), as well as Lemma 9(ii) and (45) we conclude that there exist $c_1, c_2 \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\big[\|X - \widehat{X}_n\|_\infty^p\big] \le c_1 \cdot \mathbb{E}\big[\|Z - G \circ \widehat{X}_n\|_\infty^p\big] \le 2^p \cdot c_1 \cdot \big(\mathbb{E}\big[\|Z - \widehat{Z}_n\|_\infty^p\big] + u_n(1)\big) \le \frac{c_2}{n^{p/2}},$$

which yields the statement of Theorem 1.

It remains to prove (44). Let $n \in \mathbb{N}$. Clearly, for every $t \in [0,1]$,

$$\widehat{Z}_{n,t} = G(x_0) + \int_0^t \widetilde{\mu}(\widehat{Z}_{n,\underline{s}_n})\, ds + \int_0^t \widetilde{\sigma}(\widehat{Z}_{n,\underline{s}_n})\, dW_s.$$

Since G' is absolutely continuous, see Lemma 7(ii), we may apply Itô's formula, see e.g. [13, Problem 3.7.3], to obtain that $\mathbb{P}$-a.s. for all $t \in [0,1]$,

$$\begin{aligned} G(\widehat{X}_{n,t}) &= G(x_0) + \int_0^t (G'(\widehat{X}_{n,s}) \cdot \mu(\widehat{X}_{n,\underline{s}_n}) + \tfrac{1}{2} G''(\widehat{X}_{n,s}) \cdot \sigma^2(\widehat{X}_{n,\underline{s}_n}))\, ds \\ &\qquad + \int_0^t G'(\widehat{X}_{n,s}) \cdot \sigma(\widehat{X}_{n,\underline{s}_n})\, dW_s \\ &= G(x_0) + \int_0^t \widetilde{\mu}(G(\widehat{X}_{n,\underline{s}_n}))\, ds + \int_0^t \big(G'(\widehat{X}_{n,s}) - G'(\widehat{X}_{n,\underline{s}_n})\big) \cdot \mu(\widehat{X}_{n,\underline{s}_n})\, ds \\ &\qquad + \int_0^t \widetilde{\sigma}(G(\widehat{X}_{n,\underline{s}_n}))\, dW_s + \int_0^t \big(G'(\widehat{X}_{s,n}) - G'(\widehat{X}_{n,\underline{s}_n})\big) \cdot \sigma(\widehat{X}_{n,\underline{s}_n})\, dW_s \\ &\qquad + \frac{1}{2} \cdot \int_0^t \big(G''(\widehat{X}_{n,s}) - G''(\widehat{X}_{n,\underline{s}_n})\big) \cdot \sigma^2(\widehat{X}_{n,\underline{s}_n})\, ds. \end{aligned}$$

It follows that $\mathbb{P}$-a.s. for all $t \in [0,1]$,

$$G(\widehat{X}_{n,t}) - \widehat{Z}_{n,t} = \sum_{i=1}^{3} V_{n,i,t},$$

where

$$\begin{aligned} V_{n,1,t} &= \int_0^t (\widetilde{\mu}(G(\widehat{X}_{n,\underline{s}_n})) - \widetilde{\mu}(\widehat{Z}_{n,\underline{s}_n}))\, ds + \int_0^t (\widetilde{\sigma}(G(\widehat{X}_{n,\underline{s}_n})) - \widetilde{\sigma}(\widehat{Z}_{n,\underline{s}_n}))\, dW_s, \\ V_{n,2,t} &= \int_0^t \big(G'(\widehat{X}_{n,s}) - G'(\widehat{X}_{n,\underline{s}_n})\big) \cdot \mu(\widehat{X}_{n,\underline{s}_n})\, ds + \int_0^t \big(G'(\widehat{X}_{n,s}) - G'(\widehat{X}_{n,\underline{s}_n})\big) \cdot \sigma(\widehat{X}_{n,\underline{s}_n})\, dW_s, \\ V_{n,3,t} &= \frac{1}{2} \cdot \int_0^t \big(G''(\widehat{X}_{n,s}) - G''(\widehat{X}_{n,\underline{s}_n})\big) \cdot \sigma^2(\widehat{X}_{n,\underline{s}_n})\, ds. \end{aligned}$$

Hence, for all $t \in [0,1]$,

$$u_n(t) \le 3^p \cdot \sum_{i=1}^{3} \mathbb{E}\big[\sup_{s\in[0,t]} |V_{n,i,s}|^p\big]. \tag{46}$$

We next estimate the single summands on the right hand side of (46). Using the Hölder inequality, the Burkholder-Davis-Gundy inequality and the Lipschitz continuity of $\widetilde{\mu}$ and $\widetilde{\sigma}$, see Lemma 7(ii), we obtain that there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and all $t \in [0,1]$,

$$\mathbb{E}\big[\sup_{s\in[0,t]} |V_{n,1,s}|^p\big] \le c \cdot \int_0^t \mathbb{E}\big[|G(\widehat{X}_{n,\underline{s}_n})) - \widehat{Z}_{n,\underline{s}_n}|^p\big]\, ds \le c \cdot \int_0^t u_n(s)\, ds. \tag{47}$$

Furthermore, using the Hölder inequality, the Burkholder-Davis-Gundy inequality as well as the Lipschitz continuity of G', see Lemma 7(ii), and employing (11) as well as Lemma 2 we conclude that there exist $c_1, c_2, c_3 \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and all $t \in [0,1]$,

$$\begin{aligned} \mathbb{E}\big[\sup_{s\in[0,t]} |V_{n,2,s}|^p\big] &\le c_1 \cdot \int_0^t \mathbb{E}\big[|G'(\widehat{X}_{n,s})) - G'(\widehat{X}_{n,\underline{s}_n})|^p \cdot (|\mu(\widehat{X}_{n,\underline{s}_n})|^p + |\sigma(\widehat{X}_{n,\underline{s}_n})|^p)\big]\, ds \\ &\le c_2 \cdot \int_0^t \big(\mathbb{E}\big[|\widehat{X}_{n,s} - \widehat{X}_{n,\underline{s}_n}|^{2p}\big]\big)^{1/2} \cdot \big(1 + \mathbb{E}\big[|\widehat{X}_{n,\underline{s}_n}|^{2p}\big]\big)^{1/2}\, ds \le \frac{c_3}{n^{p/2}}. \end{aligned} \tag{48}$$

For estimating $\mathbb{E}\big[\sup_{s\in[0,t]} |V_{n,3,s}|^p\big]$ we put

$$B = \Big(\bigcup_{i=1}^{k+1} (\xi_{i-1}, \xi_i)^2\Big)^c$$

and we note that $B = \bigcup_{i=1}^k \{(x,y) \in \mathbb{R}^2 : (x-\xi_i)\cdot(y-\xi_i) \le 0\}$. Using Lemma 7(ii) and (11) we obtain that there exists $c \in (0,\infty)$ such that for all $x, y \in \mathbb{R}$,

$$|G''(x)\cdot\sigma^2(y) - G''(x)\cdot\sigma^2(y)| \le \begin{cases} c\cdot(1+y^2)\cdot|x-y|, & (x,y)\in B^c, \\ c\cdot(1+y^2), & (x,y)\in B. \end{cases}$$

Hence there exists $c \in (0,\infty)$ such that for all $t \in [0,1]$,

$$\begin{aligned} \sup_{s\in[0,t]} |V_{n,3,s}|^p \le c \cdot \Big(\Big|\int_0^t (1+\widehat{X}^2_{n,\underline{s}_n}) \cdot |\widehat{X}_{n,s} - \widehat{X}_{n,\underline{s}_n}|\, ds\Big|^p \\ + \Big|\int_0^t (1+\widehat{X}^2_{n,\underline{s}_n}) \cdot 1_{\{(\widehat{X}_{n,s}, \widehat{X}_{n,\underline{s}_n})\in B\}}\, ds\Big|^p\Big). \end{aligned} \tag{49}$$

Using Lemma 2 we obtain as in (48) that there exists $c \in (0,\infty)$ such that for all $t \in [0,1]$,

$$\mathbb{E}\Big[\Big|\int_0^t (1+\widehat{X}^2_{n,\underline{s}_n}) \cdot |\widehat{X}_{n,s} - \widehat{X}_{n,\underline{s}_n}|\, ds\Big|^p\Big] \le \frac{c}{n^{p/2}}. \tag{50}$$

Furthermore, for all $i \in \{1,\dots,k\}$ and all $s \in [0,1]$,

$$\begin{aligned} |\widehat{X}_{n,\underline{s}_n}| \cdot 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}} &\le (|\xi_i| + |\widehat{X}_{n,\underline{s}_n} - \xi_i|) \cdot 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}} \\ &\le (|\xi_i| + |\widehat{X}_{n,\underline{s}_n} - \widehat{X}_{n,s}|) \cdot 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}}, \end{aligned}$$

which yields that for all $s \in [0,1]$,

$$(1+\widehat{X}^2_{n,\underline{s}_n}) \cdot 1_{\{(\widehat{X}_{n,s}, \widehat{X}_{n,\underline{s}_n})\in B\}} \le (1 + 2\max_{i=1,\dots,k} \xi_i^2) \cdot \sum_{i=1}^k 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}} + 2(\widehat{X}_{n,\underline{s}_n} - \widehat{X}_{n,s})^2.$$

By the latter inequality and Lemma 2 we conclude that there exists $c \in (0,\infty)$ such that for all $t \in [0,1]$,

$$\begin{aligned} &\mathbb{E}\Big[\Big|\int_0^t (1+\widehat{X}^2_{n,\underline{s}_n}) \cdot 1_{\{(\widehat{X}_{n,s}, \widehat{X}_{n,\underline{s}_n})\in B\}}\, ds\Big|^p\Big] \\ &\qquad \le c \cdot \sum_{i=1}^k \mathbb{E}\Big[\Big|\int_0^t 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}}\, ds\Big|^p\Big] + \frac{c}{n^p}. \end{aligned} \tag{51}$$

Combining (49), (50) and (51) we see that there exists $c \in (0,\infty)$ such that for all $t \in [0,1]$,

$$\mathbb{E}\big[\sup_{s\in[0,t]} |V_{n,3,s}|^p\big] \le \frac{c}{n^{p/2}} + c\cdot \sum_{i=1}^k \mathbb{E}\Big[\Big|\int_0^t 1_{\{(\widehat{X}_{n,s}-\xi_i)\cdot(\widehat{X}_{n,\underline{s}_n}-\xi_i)\le 0\}}\, ds\Big|^p\Big],$$

which jointly with (46), (47) and (48) yields the estimate (44) and hereby completes the proof of Theorem 1.

3.5. **Proof of Theorem 2.** Clearly, for all $n \in \mathbb{N}$,

$$\big(\mathbb{E}\big[\|X - \overline{X}_n\|_q^p\big]\big)^{1/p} \le \big(\mathbb{E}\big[\|X - \widehat{X}_n\|_q^p\big]\big)^{1/p} + \big(\mathbb{E}\big[\|\widehat{X}_n - \overline{X}_n\|_q^p\big]\big)^{1/p}. \tag{52}$$

Moreover, by Theorem 1 there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E}\big[\|X - \widehat{X}_n\|_q^p\big]\big)^{1/p} \le \big(\mathbb{E}\big[\|X - \widehat{X}_n\|_\infty^p\big]\big)^{1/p} \le c/\sqrt{n}. \tag{53}$$

For $n \in \mathbb{N}$ define a stochastic process $\overline{W}_n = (\overline{W}_{n,t})_{t\in[0,1]}$ by

$$\overline{W}_{n,t} = (n\cdot t - i)\cdot W_{n,(i+1)/n} + (i+1-n\cdot t)\cdot W_{n,i/n}$$

for $t \in [i/n,(i+1)/n]$ and $i \in \{0,\dots,n-1\}$. Then for every $r \in [1,\infty)$ there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E}\big[\|W - \overline{W}_n\|_q^r\big]\big)^{1/r} \le \begin{cases} c/\sqrt{n}, & \text{if } q<\infty,\\ c\sqrt{\ln(n+1)}/\sqrt{n}, & \text{if } q=\infty, \end{cases} \tag{54}$$

see, e.g. [28] for the case $q \in [1,\infty)$ and [1] for the case $q = \infty$.

Note that for all $n \in \mathbb{N}$ and all $t \in [0,1]$,

$$\begin{aligned} |\widehat{X}_{n,t} - \overline{X}_{n,t}| &= \Big|\sum_{i=0}^{n-1} \sigma(\widehat{X}_{n,i/n})\cdot 1_{[i/n,(i+1)/n]}(t)\cdot (W_t - \overline{W}_{n,t})\Big| \\ &\le \sup_{s\in[0,1]} |\sigma(\widehat{X}_{n,s})|\cdot |W_t - \overline{W}_{n,t}|. \end{aligned}$$

Hence, by (11) and Lemma 2 there exist $c_1, c_2 \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} \big(\mathbb{E}\big[\|\widehat{X}_n - \overline{X}_n\|_q^p\big]\big)^{1/p} &\le c_1\cdot\big(1 + \big(\mathbb{E}\big[\|\widehat{X}_n\|_\infty^{2p}\big]\big)^{1/(2p)}\big)\cdot\big(\mathbb{E}\big[\|W - \overline{W}_n\|_q^{2p}\big]\big)^{1/(2p)} \\ &\le c_2\cdot\big(\mathbb{E}\big[\|W - \overline{W}_n\|_q^{2p}\big]\big)^{1/(2p)}, \end{aligned}$$

which jointly with (54), (53) and (52) completes the proof of the theorem.

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