Compact Equivalent Inverse of the Electric Field Integral Operator on Screens

R. Hiptmair and C. Urzua-Torres

Research Report No. 2018-46
November 2018

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding: ETHIRA grant ETH-04 13-2
Submitted

Compact Equivalent Inverse of the Electric Field Integral Operator on Screens

R. Hiptmair and C. Urzúa-Torres

Abstract. We study explicit inverses of the variational electric field boundary integral operator on orientable topologically simple Lipschitz screens. We describe them as solution operators of variational problems set in low-regularity standard trace spaces. On flat disks these variational problems do not involve the inversion of any non-local operators and supply an inverse up to a compact perturbation. This result lays the foundation for operator preconditioning for the discretized electric field integral equation.

Mathematics Subject Classification (2010). 45P05, 31A10, 78A25.

Keywords. Boundary integral operators; Electric field integral equation; Screens; Hodge decomposition.

1. Introduction

1.1. "Simple" Lipschitz Screens

A simple Liptschitz screen in the sense of this article is a compact orientable two-dimensional Lipschitz manifold $\Gamma \subset \mathbb{R}^3$ with boundary $\partial\Gamma$, which is the image of the unit disk

$$\mathbb{D} := \{\mathbf{x} \in \mathbb{R}^3 \;\; x_3 = 0 \text{ and } \|\mathbf{x}\| < 1\}$$

under a bi-Lipschitz mapping. In particular, Γ need not be smooth; shapes with corners and kinks are admitted. Nevertheless, Γ has a tangent plane and an unit normal vector $\mathbf{n}$ almost everywhere. We point out that simple Lipschitz screens are a special case of the Lipschitz screens considered in [7], and, of course, of the even more general class of screens introduced in [14].

1.2. Electric Field Integral Equation on Screens

For a simple Lipschitz screen Γ the Electric Field Integral Equation (EFIE) in variational form reads: For fixed wave number $\mathsf{k} > 0$ and given $\mathbf{g} \in (\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$ seek $\boldsymbol{\xi} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that [7, Sect. 2.2]

$$\mathsf{a}_\mathsf{k}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \langle \mathbf{V}_\mathsf{k}\, \boldsymbol{\xi}\, ,\, \boldsymbol{\eta} \rangle_\Gamma - \frac{1}{\mathsf{k}^2} \langle \mathsf{V}_\mathsf{k} \,\mathrm{div}_\Gamma\, \boldsymbol{\xi}\, ,\, \mathrm{div}_\Gamma\, \boldsymbol{\eta} \rangle_\Gamma = \langle \mathbf{g}\, ,\, \boldsymbol{\eta} \rangle_\Gamma\,, \tag{1.1}$$

for all $\boldsymbol{\eta} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Here $\langle \cdot\, ,\, \cdot \rangle_\Gamma$ denotes the duality pairing extending the $L^2(\Gamma)$ inner product, $\mathsf{V}_\mathsf{k} : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ the weakly singular boundary integral operator for the Helmholtz operator $\Delta + \mathsf{k}^2$, and $\mathbf{V}_\mathsf{k} : \widetilde{\mathbf{H}}^{-1/2}(\Gamma) \to \mathbf{H}^{1/2}(\Gamma)$ its extension to surface vector fields. Notations for and properties of the trace spaces will be explained later in Section 2.

We are interested in the EFIE because it models frequency-domain electromagnetic scattering at perfectly electrically conducting objects, see [7, Sect. 3.1]. It is a mathematical foundation for the widely used boundary element method (BEM) in computational electromagnetics.

1.3. Motivation and Objectives

In this paper, we pursue the *construction* of bounded linear operators

$$\mathsf{N}_\mathsf{k} : (\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))' \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

which provide *compact-equivalent inverses* of the EFIE operator on simple screens in the sense that

$$\mathsf{N}_\mathsf{k}\, \mathsf{A}_\mathsf{k} = \mathsf{Id} + \mathsf{C}_\mathsf{k} \quad \text{in} \quad \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \tag{1.2}$$

where $\mathsf{A_k} : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to (\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$ is the EFIE operator induced by the bilinear form $\mathsf{a_k}$, and $\mathsf{C_k} : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is a *compact* operator.

In addition we demand that the evaluation of $\mathsf{N_k}\,\mathbf{g}$ for any $\mathbf{g} \in (\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$

(A) does not entail solving any integral equation, but merely the evaluation of integral operators on Γ, and

(B) entirely relies on solving variational equations in *low-regularity trace spaces*.

Remark 1.1. Recall from [12, Sect. 6] that the so-called Calderón identities on *closed surfaces* $\Gamma = \partial\Omega$, with $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain, imply

$$\mathsf{R}\,\mathsf{A_k}\,\mathsf{R}\,\mathsf{A_k} = \mathsf{Id} + \mathsf{M_k}, \tag{1.3}$$

where $\mathsf{R} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to (\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$ is the $\frac{\pi}{2}$-rotation operator and $\mathsf{M_k}$ the Magnetic Field Integral operator, which is compact on (closed) C^2-surfaces. Thus, for closed C^2 surfaces we can choose $\mathsf{N_k} := \mathsf{R}\,\mathsf{A_k}\,\mathsf{R}$.

Remark 1.2. The rationale behind **(A)** and **(B)** above is the use of (1.2) as basis for operator preconditioning of the linear systems of equations arising from low-order boundary element discretization of (1.1). This approach, harnessing the Calderón identity (1.3), has been successfully applied on closed surfaces [3] and scalar boundary integral equations on screens, and yields methods that are robust with respect to mesh refinement.

1.4. Related Work, Novelty and Outline

Our main new contribution is the explicit construction of a suitable operator $\mathsf{N_k}$ complying with (1.2) and **(A)** and **(B)** under the assumption that (compact-equivalent) inverses of the single-layer and hypersingular boundary integral operators (BIOs) on Γ for the Laplacian $-\Delta$ are available in the form of concrete BIOs. In [17] we verified this assumption for the disk $\mathbb{D}$. Thus, for this particular simple Lipschitz screen we have fully achieved the goals advertised above, but we hope that such inverses will be discovered for more general shapes in the future.

Therefore, we have decided to elaborate the construction of $\mathsf{N_k}$ in Section 3 for general simple Lipschitz screens. The key tool is the Hodge decomposition of the trace space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, which we recall in Section 2.2. The proper realization of $\mathsf{N_k}$ through variational equations is presented in Section 4.

Another important feature of the operator $\mathsf{N_k}$ is uniform stability in the low-frequency limit $\mathsf{k} \to 0$, as will be shown in Section 3.

The idea to tackle the EFIE by means of Hodge decompositions is well established, see [12, Sect. 6] also for screen problems [4, Sect. 3]. In these works, it was used as an analysis tool. In other works, most prominently [10] and [16], the Hodge decomposition served to convert the EFIE into boundary equations for scalar traces. Our policy for constructing $\mathsf{N_k}$ also draws on this trick. Similar ideas, though in a BEM setting, have recently been proposed for the construction of preconditioners in [1].

2. Function Space Framework

2.1. Trace Operators and Trace Spaces

From [19, Ch. 3] we adopt standard notations and definitions for Sobolev spaces $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$, $-1 \le s \le 1$, on the simple Lipschitz screen Γ. Bold font will mark corresponding Sobolev spaces $\mathbf{H}^s(\Gamma)$ and $\widetilde{\mathbf{H}}^s(\Gamma)$ of vector fields on Γ. We point out that in the case of screens the vector Sobolev spaces satisfy duality relations analogous to the scalar case, i.e.

$$\widetilde{\mathbf{H}}^{-1/2}(\Gamma) \equiv \left(\mathbf{H}^{1/2}(\Gamma)\right)' \qquad \text{and} \qquad \mathbf{H}^{-1/2}(\Gamma) \equiv \left(\widetilde{\mathbf{H}}^{1/2}(\Gamma)\right)', \tag{2.1}$$

with $\mathbf{L}^2(\Gamma)$ as pivot space.

The variational EFIE (1.1) is set in a jump trace space for $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$. Theoretical investigations of these traces spaces started with [8] and [9] and were further developed in [11] and, for screens, in [7, Sect. 2] and [14]. For a very brief review, let us introduce the space of tangential square-integrable vector fields on the simple Lipschitz screen Γ

$$\mathbf{L}^2_t(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma) \,|\, \mathbf{u}\cdot\mathbf{n} = 0 \text{ a.e. on } \Gamma\}, \tag{2.2}$$

endowed with the $\mathbf{L}^2$-inner product. We define the *tangential trace* γ_t as the operator that suitably extends

$$\gamma_t(\mathbf{U}) = \mathbf{n} \times (\mathbf{U}_{|\Gamma} \times \mathbf{n}), \qquad \mathbf{U} \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^3. \tag{2.3}$$

We will make use of the following tangential trace space

$$\mathbf{H}_t^{1/2}(\Gamma) := \gamma_t(\mathbf{H}^1(\mathbb{R}^3)), \tag{2.4}$$

together with its dual space (relying on $\mathbf{L}_t^2(\Gamma)$ as pivot space)

$$\widetilde{\mathbf{H}}_t^{-1/2}(\Gamma) := (\mathbf{H}_t^{1/2}(\Gamma))'.$$

Next, we recall the space of div_Γ-conforming tangential surface vector fields with vanishing in-Γ normal component on $\partial\Gamma$defined in [7, Sect. 2,Def. 1] (there denoted as X)

$$\begin{aligned}\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \Big\{ \boldsymbol{\eta} \in \widetilde{\mathbf{H}}_t^{-1/2}(\Gamma) \mid \mathrm{div}_\Gamma \, \boldsymbol{\eta} \in \widetilde{H}^{-1/2}(\Gamma) \text{ and} \\ \langle \boldsymbol{\eta} \, , \, \mathbf{grad}_\Gamma \, v \rangle_\Gamma + \langle \mathrm{div}_\Gamma \, \boldsymbol{\eta} \, , \, v \rangle_\Gamma = 0 \, \forall v \in C_0^\infty(\mathbb{R}^3)_{|\Gamma} \Big\} \, ,\end{aligned} \tag{2.5}$$

and its dual space (with respect to $\mathbf{L}_t^2(\Gamma)$)

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = (\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'.$$

In addition, we define the spaces

$$H_*^{1/2}(\Gamma) := \{ g \in H^{1/2}(\Gamma) \mid \langle g \, , \, 1 \rangle_\Gamma = 0 \}, \tag{2.6}$$

$$\widetilde{H}_*^{-1/2}(\Gamma) := \{ \varphi \in \widetilde{H}^{-1/2}(\Gamma) \mid \langle \varphi \, , \, 1 \rangle_\Gamma = 0 \}, \tag{2.7}$$

that are dual to each other.

Proposition 2.1. *The following duality relation holds*

$$(H_*^{1/2}(\Gamma))' = \widetilde{H}_*^{-1/2}(\Gamma) \tag{2.8}$$

with $L^2(\Gamma)$ as pivot space.

Proof. We can rewrite $\widetilde{H}_*^{-1/2}(\Gamma)$ as quotient space

$$\widetilde{H}_*^{-1/2}(\Gamma) = \widetilde{H}^{-1/2}(\Gamma) / \left(v \mapsto \int_\Gamma v dS \right), \tag{2.9}$$

from where it is clear that

$$\widetilde{H}_*^{-1/2}(\Gamma) = \{ v \mapsto \varphi(v - \int_\Gamma v dS \cdot 1) \, : \, \varphi \in \widetilde{H}^{-1/2}(\Gamma) \}. \tag{2.10}$$

□

As homeomorphic image of the disk $\mathbb{D}$ the screen Γ is connected and has trivial co-homology; it has no holes. As a consequence we have the following result about surface differential operators and related spaces.

Theorem 2.2. *The surface differential operators* $\mathbf{curl}_\Gamma$ *and* div_Γ *generate the following deRham exact sequence of Hilbert spaces:*

$$\{0\} \to \widetilde{H}^{1/2}(\Gamma) \xrightarrow{\mathbf{curl}_\Gamma} \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \xrightarrow{\mathrm{div}_\Gamma} \widetilde{H}_*^{-1/2}(\Gamma) \to \{0\}. \tag{2.11}$$

Proof. This theorem is the essence of results from [9, Sect. 6], in particular [9, Proposition 4.7] and [9, Theorem 6.1]. Alternatively, one can pull back everything to the unit disk $\mathbb{D}$ and there use the smoothed Poincaré lifting invented in [15]. □

The exact sequence property implies the existence of surface scalar potentials

$$\ker(\mathrm{div}_\Gamma(\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))) = \mathrm{Im} \, (\mathbf{curl}_\Gamma(\widetilde{H}^{1/2}(\Gamma))). \tag{2.12}$$

In addition, we learn that the surface divergence operator

$$\mathrm{div}_\Gamma : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \widetilde{H}_*^{-1/2}(\Gamma) \tag{2.13}$$

is continuous and surjective.

2.2. Hodge Decomposition

Following the developments of [7, Sect. 2.4] we consider the Laplace-Beltrami operator with Neumann boundary conditions Δ_Γ^N in the variational sense. Setting

$$H^1_*(\Gamma) := \{v \in H^1(\Gamma) \,:\, \langle v\,,\,1\rangle_\Gamma = 0\}, \tag{2.14}$$

we can define $-\Delta_\Gamma^N : H^1_*(\Gamma) \to \widetilde{H}^{-1}(\Gamma)$ variationally as the operator induced by the bilinear form $(w, v) \mapsto \int_\Gamma \mathbf{grad}_\Gamma\, w\cdot\mathbf{grad}_\Gamma\, v d\Gamma$, $w, v \in H^1_*(\Gamma)$. This means that for $\psi \in \widetilde{H}^{-1}(\Gamma)$ the function $\left(-\Delta_\Gamma^N\right)^{-1}\psi \in H^1_*(\Gamma)$ is the (unique) solution of the following variational problem: seek $w \in H^1_*(\Gamma)$ such that

$$\int_\Gamma \mathbf{grad}_\Gamma\, w \cdot \mathbf{grad}_\Gamma\, v d\Gamma = \int_\Gamma \psi v d\Gamma \quad \forall v \in H^1_*(\Gamma). \tag{2.15}$$

Based on Δ_Γ^N we define the space

$$\mathcal{H}(\Gamma) := \{v \in H^1_*(\Gamma) \,:\, \Delta_\Gamma^N v \in \widetilde{H}^{-1/2}(\Gamma)\}, \tag{2.16}$$

and endow it with the graph norm. It is an ingredient in the definition of the Hodge decomposition.

Definition 2.3 (Hodge decomposition, [7, Sect. 2.4]**).** We call Hodge decomposition the following direct decomposition of the trace space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$:

$$\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_z(\Gamma) \bigoplus X_\perp(\Gamma), \tag{2.17}$$

with closed subspaces

$$X_z(\Gamma) :\,= \{\mathbf{v} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \,:\, \mathrm{div}_\Gamma\, \mathbf{v} = 0\} \tag{2.18}$$

and

$$X_\perp(\Gamma) = \mathbf{grad}_\Gamma\, \mathcal{H}(\Gamma). \tag{2.19}$$

Thanks to the trivial topology of Γ, the exact sequence of Theorem 2.2 guarantees the existence of scalar potentials

$$X_z(\Gamma) := \{\mathbf{v} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \,:\, \mathrm{div}_\Gamma\, \mathbf{v} = 0\} = \mathbf{curl}_\Gamma(\widetilde{H}^{1/2}(\Gamma)). \tag{2.20}$$

Therefore, we can rewrite (2.17) as [9, Theorem 6.4]

$$\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma(\widetilde{H}^{1/2}(\Gamma)) \bigoplus \mathbf{grad}_\Gamma\, \mathcal{H}(\Gamma). \tag{2.21}$$

Since the mapping $\mathbf{curl}_\Gamma : \widetilde{H}^{1/2}(\Gamma) \to X_z(\Gamma)$ is bijective, we can view this as a parameterization of $X_z(\Gamma)$ over $\widetilde{H}^{1/2}(\Gamma)$. In order to find a parameterization of $X_\perp(\Gamma)$, based on Theorem 2.2 let us introduce a *divergence lifting* $\mathsf{L} : \widetilde{H}^{-1/2}_*(\Gamma) \to X_\perp(\Gamma)$ as a right inverse of div_Γ in the sense that $\mathrm{div}_\Gamma \circ \mathsf{L} = \mathsf{Id}$, through

$$\mathsf{L} = -\,\mathbf{grad}_\Gamma \circ \left(-\Delta_\Gamma^N\right)^{-1}, \tag{2.22}$$

where $\left(-\Delta_\Gamma^N\right)^{-1} : \widetilde{H}^{-1/2}(\Gamma) \to \mathcal{H}(\Gamma)$ is to be understood in variational sense, *cf.* (2.15). More concretely, one computes $\mathsf{L}\,\psi$ for $\psi \in \widetilde{H}^{-1/2}_*(\Gamma)$ first by solving the variational problem (2.15), and then by applying $-\,\mathbf{grad}_\Gamma$.

By means of the lifting operator L, we find the following representation

$$\boxed{X_\perp(\Gamma) = -\,\mathbf{grad}_\Gamma\, \mathcal{H}(\Gamma) = \mathsf{L}\left(\widetilde{H}^{-1/2}_*(\Gamma)\right) = -\,\mathbf{grad}_\Gamma \circ \left(-\Delta_\Gamma^N\right)^{-1} \widetilde{H}^{-1/2}_*(\Gamma)}\,. \tag{2.23}$$

From this representation we can draw an important conclusion. We immediately see that $X_\perp(\Gamma)$ is continuously embedded in $\mathbf{L}^2_t(\Gamma)$, which, in turns, is *compactly embedded* in $\widetilde{\mathbf{H}}^{-1/2}_t(\Gamma)$ by Rellich's theorem [18, Theorem 4.1.6] and duality.

Lemma 2.4. *The space* $X_\perp(\Gamma)$ *as defined in* (2.19) *and endowed with the norm of* $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *is compactly embedded in* $\widetilde{\mathbf{H}}^{-1/2}_t(\Gamma)$.

3. Compact-Equivalent Inverses

As explained in Section 1.3, we aim to find an operator $\mathsf{N_k}$ such that

$$\mathsf{N_k}\,\mathsf{A_k} = \mathsf{Id} + \mathsf{C_k}\ : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma), \tag{3.1}$$

with a compact operator $\mathsf{C_k}$ that may also depend on the wave number k.

We begin by considering the scaled Hodge decompositions $\boldsymbol{\xi} = \boldsymbol{\xi}_z + \mathsf{k}\,\boldsymbol{\xi}_\perp$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_z + \mathsf{k}\,\boldsymbol{\eta}_\perp$ with $(\boldsymbol{\xi}_z,\boldsymbol{\xi}_\perp)$, $(\boldsymbol{\eta}_z,\boldsymbol{\eta}_\perp) \in X_z(\Gamma)\times X_\perp(\Gamma)$, and plug it into the EFIE variational problem

$$\mathsf{a_k}(\boldsymbol{\xi}_z + \mathsf{k}\,\boldsymbol{\xi}_\perp, \boldsymbol{\eta}_z + \mathsf{k}\,\boldsymbol{\eta}_\perp) = \langle \mathbf{g}\,,\, \boldsymbol{\eta}_z + \mathsf{k}\,\boldsymbol{\eta}_\perp\rangle_\Gamma\ , \quad \forall(\boldsymbol{\eta}_z,\boldsymbol{\eta}_\perp)\in X_z(\Gamma)\times X_\perp(\Gamma).$$

We split the terms and get

$$\begin{aligned}&\langle \mathbf{V}_\mathsf{k}\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_z\rangle_\Gamma + \mathsf{k}\,\{\langle \mathbf{V}_\mathsf{k}\,\boldsymbol{\xi}_\perp\,,\,\boldsymbol{\eta}_z\rangle_\Gamma + \langle \mathbf{V}_\mathsf{k}\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma \\ &\quad + \mathsf{k}\,\langle \mathbf{V}_\mathsf{k}\,\boldsymbol{\xi}_\perp\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma\} - \langle \mathsf{V_k}\,\mathrm{div}_\Gamma\,\boldsymbol{\xi}_\perp\,,\,\mathrm{div}_\Gamma\,\boldsymbol{\eta}_\perp\rangle_\Gamma = \langle \mathbf{g}\,,\,\boldsymbol{\eta}_z\rangle_\Gamma + \mathsf{k}\,\langle \mathbf{g}\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma\,,\end{aligned} \tag{3.2}$$

where the three "cross-terms" in braces are compact due to Lemma 2.4 and behave like $O(\mathsf{k})$ when $\mathsf{k}\to 0$.

Now, let us recall the following result from literature

Lemma 3.1. $\mathsf{V_k} - \mathsf{V_0}\ :\ \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is compact and admits the asymptotic expansion* $\mathsf{V_k} - \mathsf{V_0} = O(\mathsf{k})$ *as* $\mathsf{k}\to 0$.

Proof. The first assertion follows from [20, Lemma 3.9.8] [13, Lemma 2.1]. The second is a slight generalization of what has been shown in [2, Appendix A]. □

Then, we exploit the fact that $\mathsf{V_k} - \mathsf{V_0}$ is compact and rewrite (3.2) as

$$\langle \mathbf{V}_0\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_z\rangle_\Gamma + \left\langle \widetilde{\mathsf{C}}_\mathsf{k}(\boldsymbol{\xi}_z + \boldsymbol{\xi}_\perp)\,,\,\boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\right\rangle_\Gamma - \langle \mathsf{V_0}\,\mathrm{div}_\Gamma\,\boldsymbol{\xi}_\perp\,,\,\mathrm{div}_\Gamma\,\boldsymbol{\eta}_\perp\rangle_\Gamma = \langle \mathbf{g}\,,\,\boldsymbol{\eta}_z\rangle_\Gamma + \mathsf{k}\,\langle \mathbf{g}\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma\,,$$

where the operator $\widetilde{\mathsf{C}}_\mathsf{k}$ contains all the compact terms from (3.2) plus some containing $\mathsf{V_k} - \mathsf{V_0}$ and $\mathbf{V}_\mathsf{k} - \mathbf{V}_0$.

We see that the final expression involves only two terms that are not compact and that they only act in either $X_z(\Gamma)$ or $X_\perp(\Gamma)$. This motivates that we define the operators $\mathsf{S}_z : X_z(\Gamma) \to (X_z(\Gamma))'$ and $\mathsf{S}_\perp : X_\perp(\Gamma) \to (X_\perp(\Gamma))'$ induced by them,

$$\langle \mathsf{S}_z\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_z\rangle_\Gamma := \langle \mathbf{V}_0\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_z\rangle_\Gamma \qquad \text{for } \boldsymbol{\xi}_z,\boldsymbol{\eta}_z \in X_z(\Gamma), \tag{3.3}$$

$$\langle \mathsf{S}_\perp\,\boldsymbol{\xi}_\perp\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma := \left\langle \mathsf{V_0}\,\widetilde{\mathrm{div}}_\Gamma\,\boldsymbol{\xi}_\perp\,,\,\widetilde{\mathrm{div}}_\Gamma\,\boldsymbol{\eta}_\perp\right\rangle_\Gamma \quad \text{for } \boldsymbol{\xi}_\perp,\boldsymbol{\eta}_\perp \in X_\perp(\Gamma), \tag{3.4}$$

and consider the following variational problem: For $\mathbf{g}\in(\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma))'$, find $\boldsymbol{\xi}_z \in X_z(\Gamma)$ and $\boldsymbol{\xi}_\perp \in X_\perp(\Gamma)$ such that

$$\langle \mathsf{S}_z\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_z\rangle_\Gamma = \langle \mathbf{g}\,,\,\boldsymbol{\eta}_z\rangle_\Gamma\,, \quad \forall \boldsymbol{\eta}_z \in X_z(\Gamma), \tag{3.5}$$

$$\langle \mathsf{S}_\perp\,\boldsymbol{\xi}_\perp\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma = \langle \mathbf{g}\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma\,, \quad \forall \boldsymbol{\eta}_\perp \in X_\perp(\Gamma). \tag{3.6}$$

As we want $\mathsf{N_k}$ to be a compact-equivalent inverse of $\mathsf{A_k}$, we point out that it suffices to solve (3.5) and (3.6). Let us denote the associated inverses by $\mathsf{N}_z := \mathsf{S}_z^{-1}$ and $\mathsf{N}_\perp := \mathsf{S}_\perp^{-1}$. Then, we define

$$\boxed{\mathsf{N_k} := \mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp = \mathsf{S}_z^{-1} - \mathsf{k}^2\mathsf{S}_\perp^{-1}}. \tag{3.7}$$

In other words, given $\mathbf{g}\in\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$, we can compute $\boldsymbol{\xi} = \mathsf{N_k}\,\mathbf{g} = (\mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp)\mathbf{g}$ as follows:

(I) To compute $\mathsf{N}_z\,\mathbf{g}$ we find $\boldsymbol{\xi}_z \in X_z(\Gamma)$ such that

$$\langle \mathbf{V}_0\,\boldsymbol{\xi}_z\,,\,\boldsymbol{\eta}_z\rangle_\Gamma = \langle \mathbf{g}\,,\,\boldsymbol{\eta}_z\rangle_\Gamma \quad \forall \boldsymbol{\eta}_z \in X_z(\Gamma). \tag{3.8}$$

Note that unique solvability of (3.8) is ensured by the $\widetilde{H}^{-1/2}(\Gamma)$-ellipticity of $\mathsf{V_0}$ [20, Theorem 3.5.9].

Equivalently, we can use the scalar potential representation (2.20) of $X_z(\Gamma)$ and solve: Find $u\in\widetilde{H}^{1/2}(\Gamma)$ such that

$$\langle \mathbf{V}_0\,\mathbf{curl}_\Gamma\,u\,,\,\mathbf{curl}_\Gamma\,v\rangle_\Gamma = \langle \mathbf{g}\,,\,\mathbf{curl}_\Gamma\,v\rangle_\Gamma\,, \quad \forall v\in\widetilde{H}^{1/2}(\Gamma), \tag{3.9}$$

which is the weak form of a hypersingular boundary integral equation for the Laplacian [20, Corollary 3.3.24]. Therefore, if we denote the corresponding hypersingular integral operator by $\mathsf{W_0}$, we can use $\mathsf{W}_0^{-1} : H^{-1/2}(\Gamma) \to \widetilde{H}^{1/2}(\Gamma)$ and write

$$u = \mathsf{W}_0^{-1}\circ\mathbf{curl}_\Gamma^*\,\mathbf{g}, \tag{3.10}$$

where $\mathbf{curl}_\Gamma^* : (\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)' \to (\widetilde{H}^{1/2}(\Gamma))' = H^{-1/2}(\Gamma)$. Finally, we conclude that

$$\mathsf{N}_z = \mathbf{curl}_\Gamma \circ \mathsf{W}_0^{-1} \circ (\mathbf{curl}_\Gamma)^* \tag{3.11}$$

(II) The evaluation of $\mathsf{N}_\perp$ boils down to solving: Find $\boldsymbol{\xi}_\perp \in X_\perp(\Gamma)$ such that

$$\langle \mathsf{V}_0 \,\mathrm{div}_\Gamma \,\boldsymbol{\xi}_\perp \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}_\perp \rangle_\Gamma = \langle \mathbf{g} \,,\, \boldsymbol{\eta}_\perp \rangle_\Gamma \,, \quad \forall \boldsymbol{\eta}_\perp \in X_\perp(\Gamma). \tag{3.12}$$

We again point out that existence and uniqueness of solutions of (3.12) follow by the $\widetilde{H}^{-1/2}(\Gamma)$-ellipticity of V_0 and the bijectivity of $\mathrm{div}_\Gamma : X_\perp(\Gamma) \to \widetilde{H}_*^{-1/2}(\Gamma)$ from Theorem 2.2.

Unfortunately, the space $X_\perp(\Gamma)$ is not a low-regularity trace space and thus (3.12) violates **(B)**. Nevertheless, (2.23) permits us to write $\boldsymbol{\xi}_\perp = \mathsf{L}\,\psi$, $\psi \in \widetilde{H}_*^{-1/2}(\Gamma)$ and recast (3.12) as

$$\langle \mathsf{V}_0 \,\mathrm{div}_\Gamma \,\mathsf{L}\,\psi \,,\, \mathrm{div}_\Gamma \,\mathsf{L}\,\phi \rangle_\Gamma = \langle \mathbf{g} \,,\, \mathsf{L}\,\phi \rangle_\Gamma \quad \forall \phi \in \widetilde{H}_*^{-1/2}(\Gamma), \tag{3.13}$$

which reduces to

$$\langle \mathsf{V}_0 \,\psi \,,\, \phi \rangle_\Gamma = \langle \mathsf{L}^* \,\mathbf{g} \,,\, \phi \rangle_\Gamma \quad \forall \phi \in \widetilde{H}_*^{-1/2}(\Gamma), \tag{3.14}$$

when using $\mathrm{div}_\Gamma \circ \mathsf{L} = \mathsf{Id}$ and the adjoint operator $\mathsf{L}^* : \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \to H_*^{1/2}(\Gamma)$ of L.

Rewriting the above with $\mathsf{V}_0^{-1} : H^{1/2}(\Gamma) \to \widetilde{H}^{-1/2}(\Gamma)$, we have

$$\mathsf{N}_\perp = \mathsf{L} \circ \mathsf{V}_0^{-1} \circ \mathsf{L}^* \,. \tag{3.15}$$

Theorem 3.2. *For any* $\mathsf{k} > 0$*, the continuous operators*

$$\mathsf{N}_\mathsf{k} := \mathsf{N}_z - \mathsf{k}^2 \,\mathsf{N}_\perp \;:\; \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma).$$

satisfy

$$\mathsf{N}_\mathsf{k}\,\mathsf{A}_\mathsf{k} = \mathsf{Id} + \mathsf{C}_\mathsf{k} \;:\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma), \tag{3.16}$$

with compact operators C_k *that are uniformly bounded as* $\mathsf{k} \to 0$.

Proof. For $\mathbf{g} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $\boldsymbol{\eta} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ we have

$$\begin{aligned}\mathsf{a}_\mathsf{k}(\mathsf{N}_\mathsf{k}\,\mathbf{g},\boldsymbol{\eta}) &= \langle \mathbf{V}_0 \,\mathsf{N}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma - \frac{1}{\mathsf{k}^2}\langle \mathsf{V}_0 \,\mathrm{div}_\Gamma \,\mathsf{N}_\mathsf{k}\,\mathbf{g} \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}\rangle_\Gamma \\ &\quad + \langle (\mathbf{V}_\mathsf{k} - \mathbf{V}_0)\,\mathsf{N}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma - \frac{1}{\mathsf{k}^2}\langle (\mathsf{V}_\mathsf{k} - \mathsf{V}_0)\,\mathrm{div}_\Gamma \,\mathsf{N}_\mathsf{k}\,\mathbf{g} \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}\rangle_\Gamma \,,\end{aligned}$$

where the last two terms are compact due to Lemma 3.1. For short, we gather these terms and write

$$\langle \mathsf{T}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma = \langle (\mathbf{V}_\mathsf{k} - \mathbf{V}_0)\,\mathsf{N}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma - \frac{1}{\mathsf{k}^2}\langle (\mathsf{V}_\mathsf{k} - \mathsf{V}_0)\,\mathrm{div}_\Gamma \,\mathsf{N}_\mathsf{k}\,\mathbf{g} \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}\rangle_\Gamma \,.$$

Now, let us plug in $\mathsf{N}_k = \mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp$ and obtain

$$\begin{aligned}\mathsf{a}_\mathsf{k}((\mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp)\mathbf{g},\boldsymbol{\eta}) &= \langle \mathbf{V}_0 \,\mathsf{N}_z\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma - \mathsf{k}^2\,\langle \mathbf{V}_0 \,\mathsf{N}_\perp\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma \\ &\quad + \langle \mathsf{V}_0 \,\mathrm{div}_\Gamma \,\mathsf{N}_\perp\,\mathbf{g} \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}\rangle_\Gamma + \langle \mathsf{T}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}\rangle_\Gamma \,,\end{aligned}$$

where we have already used the fact that N_z maps to $X_z(\Gamma)$ and that $X_z(\Gamma) = \ker \mathrm{div}_\Gamma$ in $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$.

Then, by also plugging in the Hodge decomposition $\boldsymbol{\eta} = \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp$, we arrive to

$$\begin{aligned}\mathsf{a}_\mathsf{k}((\mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp)\mathbf{g},\boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp) &= \langle \mathbf{V}_0 \,\mathsf{N}_z\,\mathbf{g} \,,\, \boldsymbol{\eta}_z\rangle_\Gamma + \langle \mathbf{V}_0 \,\mathsf{N}_z\,\mathbf{g} \,,\, \boldsymbol{\eta}_\perp\rangle_\Gamma \\ &\quad - \mathsf{k}^2\,\langle \mathbf{V}_0 \,\mathsf{N}_\perp\,\mathbf{g} \,,\, \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\rangle_\Gamma \\ &\quad + \langle \mathsf{V}_0 \,\mathrm{div}_\Gamma \,\mathsf{N}_\perp\,\mathbf{g} \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}_\perp\rangle_\Gamma + \langle \mathsf{T}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\rangle_\Gamma \,,\end{aligned}$$

where we have again employed $X_z(\Gamma) = \ker \mathrm{div}_\Gamma$.

Finally, let us re-order the right hand side and plug in the definition of $\mathsf{N}_\perp$

$$\begin{aligned}\mathsf{a}_\mathsf{k}((\mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp)\mathbf{g},\boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp) &= \langle \mathbf{V}_0 \,\mathsf{N}_z\,\mathbf{g} \,,\, \boldsymbol{\eta}_z\rangle_\Gamma + \langle \mathsf{V}_0 \,\mathrm{div}_\Gamma \,\mathsf{L}\,\mathsf{V}_0\,\mathsf{L}^*\,\mathbf{g} \,,\, \mathrm{div}_\Gamma \,\boldsymbol{\eta}_\perp\rangle_\Gamma \\ &\quad + \langle \mathsf{T}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\rangle_\Gamma \\ &\quad + \langle \mathbf{V}_0 \,\mathsf{N}_z\,\mathbf{g} \,,\, \boldsymbol{\eta}_\perp\rangle_\Gamma - \mathsf{k}^2\,\langle \mathbf{V}_0 \,\mathsf{N}_\perp\,\mathbf{g} \,,\, \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\rangle_\Gamma \,.\end{aligned}$$

With this it becomes clear that the first line gives us the identity due to the definitions of N_z and $\mathsf{N}_\perp$. On the other hand, we have that the expressions on the last line are compact as a consequence of Lemma 2.4. Hence, collecting all compact terms as C_k, we find

$$\mathsf{a}_\mathsf{k}((\mathsf{N}_z - \mathsf{k}^2\,\mathsf{N}_\perp)\mathbf{g},\boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp) = \langle \mathbf{g} \,,\, \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\rangle_\Gamma + \langle \mathsf{C}_\mathsf{k}\,\mathbf{g} \,,\, \boldsymbol{\eta}_z + \boldsymbol{\eta}_\perp\rangle_\Gamma \,,$$

and therefore the desired identity plus compact.

The compact terms in C_k are

$$\langle \mathsf{C}_\mathsf{k}\,\mathbf{g}\,,\,\boldsymbol{\eta}_z+\boldsymbol{\eta}_\perp\rangle_\Gamma = \langle \mathbf{V}_0\,\mathsf{N}_z\,\mathbf{g}\,,\,\boldsymbol{\eta}_\perp\rangle_\Gamma + \langle (\mathbf{V}_\mathsf{k}-\mathbf{V}_0)\,\mathsf{N}_\mathsf{k}\,\mathbf{g}\,,\,\boldsymbol{\eta}\rangle_\Gamma + \langle (\mathsf{V}_\mathsf{k}-\mathsf{V}_0)\,\mathrm{div}_\Gamma\,\mathsf{N}_\perp\,\mathbf{g}\,,\,\mathrm{div}_\Gamma\,\boldsymbol{\eta}_\perp\rangle_\Gamma$$
$$-\,\mathsf{k}^2\,\langle \mathbf{V}_0\,\mathsf{N}_\perp\,\mathbf{g}\,,\,\boldsymbol{\eta}_z+\boldsymbol{\eta}_\perp\rangle_\Gamma$$

from where it is clear that C_k remains bounded for $\mathsf{k}\to 0$. $\square$

Remark 3.3. If one uses compact-equivalent inverses of W_0 and V_0 in the construction of N_z and $\mathsf{N}_\perp$, then the resulting operator N_k would still be a compact-equivalent inverse of A_k.

4. Mixed Variational formulation for $\mathsf{N}_\perp$

This Section is devoted to derive a formulation of N_k that complies with **(B)**. Difficulties arise specifically from $\mathsf{N}_\perp$ and we start by briefly discussing why one cannot use straightforward variational formulations.

Remark 4.1. From (3.15), one is tempted to compute $\mathsf{N}_\perp\,\mathbf{g}$, with $\mathbf{g}\in(\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$, through the four following steps:

I. Seek $\mathsf{L}^*\,\mathbf{g}=u\in H^1_*(\Gamma)$ such that
$$\int_\Gamma \mathbf{grad}_\Gamma\,u\cdot\mathbf{grad}_\Gamma\,v dS=\int_\Gamma \mathbf{g}\cdot\mathbf{grad}_\Gamma\,v dS,\qquad \forall v\in H^1_*(\Gamma). \tag{4.1}$$
II. Take $\mu=\mathsf{V}_0^{-1}\,u\in\widetilde{H}_*^{-1/2}(\Gamma)$.

III. Find $w\in H^1_*(\Gamma)$ such that
$$\int_\Gamma \mathbf{grad}_\Gamma\,w\cdot\mathbf{grad}_\Gamma\,v dS=\int_\Gamma \mu v dS,\qquad \forall v\in H^1_*(\Gamma). \tag{4.2}$$
IV. Compute $\mathsf{N}_\perp\,\mathbf{g}=-\,\mathbf{grad}_\Gamma\,w$.

Nevertheless, this is not possible. Problematic is the right hand side of the variational problem (4.1). It is well-defined only if $\mathbf{grad}_\Gamma\,v\in\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, and thus we need $v\in\mathcal{H}(\Gamma)$, with $\mathcal{H}(\Gamma)$ as defined in (2.16). However, $\mathcal{H}(\Gamma)$ is not a low-regularity trace space and thus violates **(B)**.

As a remedy, we switch to a mixed variational formulation to compute $\mathsf{N}_\perp\,\mathbf{g}$. Recall
$$\mathsf{N}_\perp\,\mathbf{g}=\mathsf{L}\circ\mathsf{V}_0^{-1}\circ\mathsf{L}^*\,\mathbf{g},\qquad \mathsf{L}=-\,\mathbf{grad}_\Gamma\circ(-\Delta_\Gamma^N)^{-1},\qquad \text{and }\ \mathsf{L}^*=(-\Delta_\Gamma^N)^{-*}\circ\mathrm{div}_\Gamma\,. \tag{4.3}$$
Let us define
$$\widetilde{\mathbf{H}}^{0,-1/2}(\mathrm{div}_\Gamma,\Gamma):=\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\cap\mathbf{L}^2_t(\Gamma), \tag{4.4}$$
and note that due to the elliptic lifting of the Laplace-Beltrami operator [10, Sect. 5.2.1], we have
$$X_\perp(\Gamma)=\mathbf{grad}\,\mathcal{H}(\Gamma)\subset\mathbf{L}^2_t(\Gamma), \tag{4.5}$$
and therefore $X_\perp(\Gamma)\subset\widetilde{\mathbf{H}}^{0,-1/2}(\mathrm{div}_\Gamma,\Gamma)$.

In order to introduce the mixed formulation to compute (4.3), we split the evaluation of $\mathsf{N}_\perp\,\mathbf{g}$, $\mathbf{g}\in(\widetilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$, into two stages: First compute $u=\mathsf{L}^*\,\mathbf{g}$, and then $\mathsf{N}_\perp\,\mathbf{g}=\mathsf{L}\,\mathsf{V}_0^{-1}\,u$.

We now analyze each of these steps separately:

- $u:=\mathsf{L}^*\,\mathbf{g}\in H^{1/2}_*(\Gamma)$ solves
$$-\,\Delta_\Gamma^N u=\mathrm{div}_\Gamma\,\mathbf{g}, \tag{4.6}$$
which holds, if and only if, $\mathrm{div}_\Gamma(\mathbf{grad}_\Gamma\,u+\mathbf{g})=0$. This can be rewritten as a first-order system [6, Example 1.2,Chapter 2] with a flux variable $\boldsymbol{\mu}\in\widetilde{\mathbf{H}}^{0,-1/2}(\mathrm{div}_\Gamma,\Gamma)$ such that
$$\begin{aligned}\boldsymbol{\mu}&=\mathbf{grad}_\Gamma\,u+\mathbf{g}\\ \mathrm{div}_\Gamma\,\boldsymbol{\mu}&=0.\end{aligned} \tag{4.7}$$

Integrating by parts we deduce the following mixed variational problem: Find $\boldsymbol{\mu}\in\widetilde{\mathbf{H}}^{0,-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $u\in H^{1/2}_*(\Gamma)$ such that
$$\begin{array}{llll}\langle\boldsymbol{\mu}\,,\,\mathbf{j}\rangle_\Gamma & +\,\langle u\,,\,\mathrm{div}_\Gamma\,\mathbf{j}\rangle_\Gamma & =\langle\mathbf{g}\,,\,\mathbf{j}\rangle_\Gamma\,, & \forall\mathbf{j}\in\widetilde{\mathbf{H}}^{0,-1/2}(\mathrm{div}_\Gamma,\Gamma),\\ \langle\mathrm{div}_\Gamma\,\boldsymbol{\mu}\,,\,v\rangle_\Gamma & & =0, & \forall v\in H^{1/2}_*(\Gamma).\end{array} \tag{4.8}$$

- $w := \mathsf{L}\,\mathsf{V}_0^{-1}\,u \in \mathcal{H}(\Gamma)$ is obtained by solving

$$-\,\Delta_\Gamma^N w = \mathsf{V}_0^{-1}\,u \tag{4.9}$$

  and taking $-\,\mathbf{grad}_\Gamma\, w$. Its first-order formulation is given by:

$$\mathbf{grad}_\Gamma\, w = \boldsymbol{\eta},$$
$$\operatorname{div}_\Gamma \boldsymbol{\eta} = \mathsf{V}_0^{-1}\,u,$$

  with flux field $\boldsymbol{\eta} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma)$.

  From this we get the following *mixed variational problem*: Find $\boldsymbol{\eta} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma)$, and $w \in H_*^{1/2}(\Gamma)$ such that

$$\begin{aligned} &\langle \boldsymbol{\eta}\,,\,\mathbf{q}\rangle_\Gamma &&+\langle w\,,\,\operatorname{div}_\Gamma \mathbf{q}\rangle_\Gamma &&= 0, &&\forall \mathbf{q} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma),\\ &\langle \operatorname{div}_\Gamma \boldsymbol{\eta}\,,\,v\rangle_\Gamma && &&= \left\langle \mathsf{V}_0^{-1}\,u\,,\,v\right\rangle_\Gamma, &&\forall v \in H_*^{1/2}(\Gamma). \end{aligned} \tag{4.10}$$

Theorem 4.2. *The mixed problems* (4.8) *and* (4.10) *have unique solutions and are stable.*

Proof. This result follows from showing the two assumptions of the abstract theory of variational saddle-point theory [5, Theorem 4.3]. Therefore, we need to verify that the following two estimates hold:

(c1) $|\langle \mathbf{q}\,,\,\mathbf{q}\rangle_\Gamma| \geq C\,\|\mathbf{q}\|^2_{\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)}, \forall \mathbf{q} \in \mathcal{V}$ with $C > 0$ and

$$\mathcal{V} := \{\mathbf{j} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma) \,:\, \langle \operatorname{div}_\Gamma \mathbf{j}\,,\,u\rangle_\Gamma = 0\,\forall u \in H_*^{1/2}(\Gamma)\}.$$

(c2) The exists $c_b > 0$ such that

$$\sup_{\mathbf{j}\in\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)} \frac{|\langle \operatorname{div}_\Gamma \mathbf{j}\,,\,u\rangle_\Gamma|}{\|\mathbf{j}\|_{\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)}} \geq c_b\,\|u\|_{H^{1/2}(\Gamma)} \qquad \forall u \in H_*^{1/2}(\Gamma).$$

On the one hand, (c1) follows from the definition of $\mathcal{V}$ and the graph norm

$$\|\mathbf{j}\|^2_{\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)} = \|\mathbf{j}\|^2_{\mathbf{L}^2(\Gamma)} + \|\operatorname{div}_\Gamma\|^2_{\mathbf{L}^2(\Gamma)}.$$

On the other hand, (c2) is a consequence of the surjectivity of $\operatorname{div}_\Gamma : X_\perp(\Gamma) \to \widetilde{H}_*^{-1/2}(\Gamma)$ from Theorem 2.2, since it implies that $\mathsf{L} : \widetilde{H}_*^{-1/2}(\Gamma) \to X_\perp(\Gamma)$ is a continuous mapping from $\widetilde{H}_*^{-1/2}(\Gamma)$ to $\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma)$. Therefore, duality gives

$$\begin{aligned} \sup_{\mathbf{j}\in\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)} \frac{|\langle \operatorname{div}_\Gamma \mathbf{j}\,,\,u\rangle_\Gamma|}{\|\mathbf{j}\|_{\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)}} &= \sup_{\varphi\in\widetilde{H}_*^{-1/2}(\Gamma)} \frac{|\langle \operatorname{div}_\Gamma \mathsf{L}\,\varphi\,,\,u\rangle_\Gamma|}{\|\mathsf{L}\,\varphi\|_{\widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma,\Gamma)}}\\ &\geq C_{\mathsf{L}} \sup_{\varphi\in\widetilde{H}_*^{-1/2}(\Gamma)} \frac{|\langle \varphi\,,\,u\rangle_\Gamma|}{\|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}} = C_{\mathsf{L}}\,\|u\|_{H^{1/2}(\Gamma)} \qquad \forall u \in H_*^{1/2}(\Gamma). \end{aligned}$$

□

By the mixed variational formulations we have found a way to evaluate $\mathsf{N}_\mathsf{k}\,\mathbf{g}$, $\mathbf{g} \in \left(\widetilde{\mathbf{H}}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)\right)'$, meeting all requirements listed in Section 1.3, provided that we can realize V_0^{-1} and W_0^{-1} through simply applying a BIO. Summing up, we remark that we split the evaluation of N_k into the computation of its two components N_z and $\mathsf{N}_\perp$ as follows:

① $\mathsf{N}_z\,\mathbf{g}$ is obtained by finding $u \in \widetilde{H}^{1/2}(\Gamma)$ such that

$$\langle u\,,\,v\rangle_\Gamma = \left\langle \mathsf{W}_0^{-1}\,\mathbf{curl}_\Gamma^*\,\mathbf{g}\,,\,v\right\rangle_\Gamma \quad \forall v \in \widetilde{H}^{1/2}(\Gamma), \tag{4.11}$$

and applying $\mathbf{curl}_\Gamma$: $\mathsf{N}_z\,\mathbf{g} := \mathbf{curl}_\Gamma\,u$.

② The computation of $\mathsf{N}_\perp\,g$ boils down to the following two steps:

(i) Seek $\boldsymbol{\mu} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma)$, $u \in H_*^{1/2}(\Gamma)$ such that

$$\begin{aligned} &\langle \boldsymbol{\mu}\,,\,\mathbf{j}\rangle_\Gamma &&+\langle u\,,\,\operatorname{div}_\Gamma \mathbf{j}\rangle_\Gamma &&= \langle \mathbf{g}\,,\,\mathbf{j}\rangle_\Gamma &&\forall \mathbf{j} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma),\\ &\langle \operatorname{div}_\Gamma \boldsymbol{\mu}\,,\,v\rangle_\Gamma && &&= 0 &&\forall v \in H_*^{1/2}(\Gamma). \end{aligned} \tag{4.12}$$

(ii) Seek $\boldsymbol{\xi}_\perp \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma)$, $w \in H_*^{1/2}(\Gamma)$ such that:

$$\begin{aligned} &\langle \boldsymbol{\xi}_\perp\,,\,\mathbf{q}\rangle_\Gamma &&+\langle w\,,\,\operatorname{div}_\Gamma \mathbf{q}\rangle_\Gamma &&= 0 &&\forall \mathbf{q} \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma),\\ &\langle \operatorname{div}_\Gamma \boldsymbol{\xi}_\perp\,,\,v\rangle_\Gamma && &&= \left\langle \mathsf{V}_0^{-1}\,u\,,\,v\right\rangle_\Gamma &&\forall v \in H_*^{1/2}(\Gamma). \end{aligned} \tag{4.13}$$

Then $\mathsf{N}_\perp\,\mathbf{g} := \boldsymbol{\xi}_\perp \in \widetilde{\mathbf{H}}^{0,-1/2}(\operatorname{div}_\Gamma, \Gamma) \subset \widetilde{\mathbf{H}}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$

5. Conclusion: Compact-equivalent inverse on the disk $\mathbb{D}$

In light of (3.11) and (3.15), it becomes clear that the explicit computation of N_k relies on the availability of closed-form integral operator formulas for W_0^{-1} and V_0^{-1}. From [17, Eq. (3.1)–(3.4)], we have explicit formulas for these inverse operators on $\mathbb{D}$ and easily computable expressions for the associated symmetric bilinear forms:

$$\mathsf{a}_{\overline{\mathsf{V}}}(\upsilon, \phi) := \frac{2}{\pi^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \upsilon(\mathbf{y})\phi(\mathbf{x})\frac{S(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}d\mathbb{D}(\mathbf{y})d\mathbb{D}(\mathbf{x}), \quad \forall \upsilon, \phi \in H^{-1/2}(\mathbb{D}), \tag{5.1}$$

$$\begin{aligned}\mathsf{a}_{\overline{\mathsf{W}}}(u, v) := &\frac{2}{\pi^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{S(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathbf{curl}_{\mathbb{D},\mathbf{x}}\, u(\mathbf{x}) \cdot \mathbf{curl}_{\mathbb{D},\mathbf{y}}\, v(\mathbf{y}) d\mathbb{D}(\mathbf{x})d\mathbb{D}(\mathbf{y}) \\ &+ \frac{2}{\pi^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{u(\mathbf{x})v(\mathbf{y})}{\omega(\mathbf{x})\omega(\mathbf{y})}d\mathbb{D}(\mathbf{x})d\mathbb{D}(\mathbf{y}), \qquad \forall u, v \in H^{1/2}(\mathbb{D}),\end{aligned} \tag{5.2}$$

with $\omega(\mathbf{x}) := \sqrt{1-\|\mathbf{x}\|^2}$, for $\mathbf{x} \in \mathbb{D}$, and $S \in L^\infty(\mathbb{D}\times\mathbb{D})$ given by $S(\mathbf{x},\mathbf{y}) := \tan^{-1}\left(\frac{\omega(\mathbf{x})\omega(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\right)$, $\mathbf{x} \neq \mathbf{y}$.

Hence, on $\mathbb{D}$, solving the variational problems (4.11), (4.12) and (4.13) does not entail inverting a BIO after replacing W_0^{-1} by $\overline{\mathsf{V}}$ and V_0^{-1} by $\overline{\mathsf{W}}$.

References

[1] S. Adrian, F. Andriulli, and T. Eibert, *On a refinement-free Calderón multiplicative preconditioner for the electric field integral equation*, Preprint arXiv:1803.08333 [math.NA], arXiv, 2018.

[2] H. Ammari, P. Millien, M. Ruiz, and H. Zhang, *Mathematical analysis of plasmonic nanoparticles: the scalar case*, Arch. Ration. Mech. Anal., 224 (2017), pp. 597–658.

[3] F. Andriulli, K. Cools, H. Bagci, F. Olyslager, A. Buffa, S. Christiansen, and E. Michielssen, *A multiplicative Calderon preconditioner for the electric field integral equation*, IEEE Trans. Antennas and Propagation, 56 (2008), pp. 2398–2412.

[4] A. Bespalov, N. Heuer, and R. Hiptmair, *Convergence of the natural hp-BEM for the electric field integral equation on polyhedral surfaces*, SIAM J. Numer. Anal., 48 (2010), pp. 1518–1529.

[5] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, 2007.

[6] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[7] A. Buffa and S. H. Christiansen, *The electric field integral equation on Lipschitz screens: definitions and numerical approximation*, Numer. Math., 94 (2003), pp. 229–267.

[8] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[9] ——, *On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., 24 (2001), pp. 31–48.

[10] A. Buffa, M. Costabel, and C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Math., 92 (2002), pp. 679–710.

[11] A. Buffa, M. Costabel, and D. Sheen, *On traces for* $\mathbf{H}(\mathbf{curl}, \Omega)$ *in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[12] A. Buffa and R. Hiptmair, *Galerkin boundary element methods for electromagnetic scattering*, in Topics in Computational Wave Propagation. Direct and inverse Problems, M. Ainsworth, P. Davis, D. Duncan, P. Martin, and B. Rynne, eds., vol. 31 of Lecture Notes in Computational Science and Engineering, Springer, Berlin, 2003, pp. 83–124.

[13] A. Buffa and R. Hiptmair, *Regularized combined field integral equations*, Numer. Math., 100 (2005), pp. 1–19.

[14] X. Claeys and R. Hiptmair, *Integral equations for electromagnetic scattering at multi-screens*, Integral Equations Operator Theory, 84 (2016), pp. 33–68.

[15] M. Costabel and A. McIntosh, *On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains*, Math. Z., 265 (2010), pp. 297–320.

[16] C. Epstein and L. Greengard, *Debye sources and the numerical solution of the time-harmonic Maxwell equations*, Comm. Pure Appl. Math., 63 (2010), pp. 413–463.

[17] R. Hiptmair, C. Jerez-Hanckes, and C. Urzúa-Torres, *Closed-Form Inverses of the Weakly Singular and Hypersingular Operators on Disks*, Integral Equations Operator Theory, 90 (2018), p. 90:4.

[18] G. C. Hsiao and W. L. Wendland, *Boundary integral equations*, vol. 164 of Applied Mathematical Sciences, Springer-Verlag, Berlin, 2008.

[19] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, UK, 2000.

[20] S. Sauter and C. Schwab, *Boundary element methods*, vol. 39 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2010.

R. Hiptmair
Seminar for Applied Mathematics, ETH Zurich,
Raemistrasse 101,
8092 Zurich,
Switzerland
e-mail: `ralf.hiptmair@sam.math.ethz.ch`

C. Urzúa-Torres
Institut für Angewandte Mathematik, TU Graz,
Steyrergasse 30,
8010 Graz,
Austria
e-mail: `urzua@math.tugraz.at`