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Lower and upper bounds for strong approximation errors for numerical approximations of stochastic heat equations

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Abstract

Optimal upper and lower error estimates for strong full-discrete numerical approximations of the stochastic heat equation driven by space-time white noise are obtained. In particular, we establish the optimality of strong convergence rates for full-discrete approximations of stochastic Allen-Cahn equations with space-time white noise which have recently been obtained in [Becker, S., Gess, B., Jentzen, A., and Kloeden, P. E., Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations. *arXiv:1711.02423* (2017)].

1 Introduction

In this work we consider space-time discrete numerical methods for linear stochastic heat partial differential equations of the type

$$dX_t(x) = \Delta X_t(x)\, dt + dW_t(x) \tag{1}$$

with zero Dirichlet boundary conditions $X_t(0) = X_t(1) = 0$ for $t \in [0, T]$, $x \in (0, 1)$ where $T \in (0, \infty)$ is the time horizon under consideration and where $\frac{dW}{dt}$ is a space-time white noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, we analyse strong rates of convergence of a full-discrete exponential Euler method, proving optimal upper and lower estimates on the strong rate of convergence. The next result, Theorem 1.1 below, summarizes the main findings of this article.

Theorem 1.1. *Let $T \in (0, \infty)$, $p \in [2, \infty)$, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2((0, 1); \mathbb{R}), \langle\cdot,\cdot\rangle_{L^2((0,1);\mathbb{R})}, \|\cdot\|_{L^2((0,1);\mathbb{R})})$, $(P_n)_{n\in\mathbb{N}} \subseteq L(H)$, let $(e_n)_{n\in\mathbb{N}} \subseteq H$ be an orthonormal basis of H, let $A\colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical Wiener process, let $X\colon [0, T] \times \Omega \to H$ and $\mathcal{X}^{M,N}\colon [0, T] \times \Omega \to H$, $M, N \in \mathbb{N}$, be stochastic processes which satisfy that for all $t \in [0, T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that $X_t = \int_0^t e^{(t-s)A}\, dW_s$ and $\mathcal{X}_t^{M,N} = \int_0^t P_N e^{(t-\max(\{0,T/M,2T/M,\ldots\}\cap[0,s]))A}\, dW_s$, and assume for all $n \in \mathbb{N}$, $v \in H$ that $Ae_n = -\pi^2 n^2 e_n$ and $P_n(v) = \sum_{k=1}^n \langle e_k, v\rangle_H\, e_k$. Then there exist $c, C \in (0, \infty)$ such that*

(i) we have for all $M \in \mathbb{N}$ that

$$c\, M^{-1/4} \leq \lim_{n\to\infty} \sup_{t\in[0,T]} \Big(\mathbb{E}\big[\|X_t - \mathcal{X}_t^{M,n}\|_H^p\big]\Big)^{1/p} \leq C M^{-1/4} \tag{2}$$

and

(ii) we have for all $N \in \mathbb{N}$ that

$$c\, N^{-1/2} \leq \lim_{m\to\infty} \sup_{t\in[0,T]} \Big(\mathbb{E}\big[\|X_t - \mathcal{X}_t^{m,N}\|_H^p\big]\Big)^{1/p} \leq C N^{-1/2}. \tag{3}$$

Theorem 1.1 above is an immediate consequence of Corollary 2.4 below, Lemma 2.6 below, and Da Prato & Zabczyk [6, Lemma 7.7]. The recent article [1] establishes strong convergence rates for suitable space-time discrete approximation methods for stochastic Allen-Cahn equations of the type

$$dX_t(x) = \Delta X_t(x)\, dt + \big[\, a\, X_t(x) - b\, \big(X_t(x)\big)^3\, \big]\, dt + dW_t(x) \tag{4}$$

with zero Dirichlet boundary conditions $X_t(0) = X_t(1) = 0$ for $t \in [0,T]$, $x \in (0,1)$ where $a, b \in [0,\infty)$ are real numbers. Roughly speaking, in [1, Theorem 1.1] a spatial convergence rate of the order $1/2 - \varepsilon$ and a temporal convergence rate of the order $1/4 - \varepsilon$ have been established. More precisely, [1, Theorem 1.1] shows that for every $p, \varepsilon \in (0,\infty)$ there exists $C \in \mathbb{R}$ such that for all $M, N \in \mathbb{N}$ we have

$$\sup_{t\in[0,T]} \left(\mathbb{E}\big[\|X_t - \mathcal{X}_t^{M,N}\|^p_{L^2((0,1);\mathbb{R})}\big]\right)^{1/p} \leq C\left(M^{(\varepsilon-\frac{1}{4})} + N^{(\varepsilon-\frac{1}{2})}\right) \tag{5}$$

where $(\mathcal{X}_t^{M,N})_{t\in[0,T]}$ denotes the nonlinearity-truncated approximation scheme in [1] applied to (4). The results of this article, that is, inequalities (2) and (3), prove that these rates are essentially (up to an arbitrarily small polynomial order of convergence) optimal. We also refer, e.g., to [9, 25, 10, 8, 23, 20, 21, 7, 13, 14, 11, 2, 22, 15, 4, 3, 24, 19] for further research articles on explicit approximation schemes for stochastic differential equations with superlinearly growing non-linearities. Furthermore, related lower bounds for approximation errors in the linear case (i.e., in the case $a = b = 0$ in (4)) can, e.g., be found in Müller-Gronbach, Ritter, & Wagner [17, Theorem 1], Müller-Gronbach & Ritter [16, Theorem 1], Müller-Gronbach, Ritter, & Wagner [18, Theorem 4.2], Conus, Jentzen, & Kurniawan [5, Lemma 6.2], and Jentzen & Kurniawan [12, Corollary 9.4].

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2 Lower and upper bounds for strong approximation errors of numerical approximations of linear stochastic heat equations

2.1 Setting

Let $\lfloor\cdot\rfloor_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, the functions which satisfy for all $h \in (0,\infty)$, $t \in \mathbb{R}$ that $\lfloor t\rfloor_h = \max(\{0, h, -h, 2h, -2h, \ldots\} \cap (-\infty, t])$, for every measure space $(\Omega, \mathcal{F}, \nu)$, every measurable space $(S, \mathcal{S})$, every set R, and every function $f \colon \Omega \to R$ let $[f]_{\nu,\mathcal{S}}$ the set given by $[f]_{\nu,\mathcal{S}} = \{g \colon \Omega \to S \colon [(\exists\, A \in \mathcal{F} \colon \nu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A) \cap (\forall\, A \in \mathcal{S} \colon g^{-1}(A) \in \mathcal{F})]\}$, let $T, \nu \in (0,\infty)$, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$

$= (L^2(\lambda_{(0,1)}; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})})$, $(e_n)_{n\in\mathbb{N}} \subseteq H$, $(P_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq L(H)$ satisfy for all $m \in \mathbb{N}$, $n \in \mathbb{N}\cup\{\infty\}$, $v \in H$ that $e_m = [(\sqrt{2}\sin(m\pi x))_{x\in(0,1)}]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}$ and $P_n(v) = \sum_{k=1}^n \langle e_k, v\rangle_H \, e_k$, let $A\colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H times the real number ν, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical Wiener process, and let $O\colon [0,T]\times\Omega \to H$ and $\mathcal{O}^{M,N}\colon [0,T] \times \Omega \to H$, $M, N \in \mathbb{N}$, be stochastic processes which satisfy for all $t \in [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$ that $[O_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A} \, dW_s$ and $[\mathcal{O}_t^{M,N}]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s\rfloor_{T/M})A} \, dW_s$.

2.2 Lower and upper bounds for Hilbert-Schmidt norms of Hilbert-Schmidt operators

Lemma 2.1. *Assume the setting in Section 2.1 and let $N \in \mathbb{N} \cup \{\infty\}$, $s_1, s_2, t \in [0, \infty)$ with $s_1 \leq s_2$. Then*

(i) we have that

$$\left(\sum_{n=1}^{\infty} \|P_N e^{s_1 A}(\mathrm{Id}_H - e^{tA}) \, e_n\|_H^2\right)^{1/2} \geq \left(\sum_{n=1}^{\infty} \|P_N e^{s_2 A}(\mathrm{Id}_H - e^{tA}) \, e_n\|_H^2\right)^{1/2} \tag{6}$$

and

(ii) we have that

$$\left(\sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_1 A}) \, e_n\|_H^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_2 A}) \, e_n\|_H^2\right)^{1/2}. \tag{7}$$

Proof of Lemma 2.1. Throughout this proof let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that $\mu_n = \nu\pi^2 n^2$. Next observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \|P_N e^{s_1 A}(\mathrm{Id}_H - e^{tA}) \, e_n\|_H^2 \\
&= \sum_{n=1}^{N} \|e^{s_1 A}(\mathrm{Id}_H - e^{tA}) \, e_n\|_H^2 = \sum_{n=1}^{N} \|e^{-\mu_n s_1}(1 - e^{-\mu_n t}) \, e_n\|_H^2 \\
&= \sum_{n=1}^{N} |e^{-\mu_n s_1}(1 - e^{-\mu_n t})|^2 \geq \sum_{n=1}^{N} |e^{-\mu_n s_2}(1 - e^{-\mu_n t})|^2 \\
&= \sum_{n=1}^{N} \|e^{s_2 A}(\mathrm{Id}_H - e^{tA}) \, e_n\|_H^2 = \sum_{n=1}^{\infty} \|P_N e^{s_2 A}(\mathrm{Id}_H - e^{tA}) \, e_n\|_H^2.
\end{aligned} \tag{8}$$

This establishes (i). Moreover, note that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_1 A})\, e_n\|_H^2 \\
&= \sum_{n=1}^{N} \|e^{tA}(\mathrm{Id}_H - e^{s_1 A})\, e_n\|_H^2 = \sum_{n=1}^{N} \|e^{-\mu_n t}(1 - e^{-\mu_n s_1})\, e_n\|_H^2 \\
&= \sum_{n=1}^{N} |e^{-\mu_n t}(1 - e^{-\mu_n s_1})|^2 \leq \sum_{n=1}^{N} |e^{-\mu_n t}(1 - e^{-\mu_n s_2})|^2 \\
&= \sum_{n=1}^{N} \|e^{tA}(\mathrm{Id}_H - e^{s_2 A})\, e_n\|_H^2 = \sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_2 A})\, e_n\|_H^2.
\end{aligned} \tag{9}
$$

The proof of Lemma 2.1 is thus completed. □

Lemma 2.2. *Assume the setting in Section 2.1 and let* $N \in \mathbb{N} \cup \{\infty\}$, $t \in (0, T]$. *Then*

$$
\begin{aligned}
&\left[\int_0^{\max\{0, t(N+1)^2 - (1+\sqrt{t})^2\}} \frac{(1 - e^{-\nu\pi^2 \min\{1, tN^2\}})^2}{2\nu\pi^2 (x + [1 + \sqrt{T}]^2)^{3/2}}\, dx \right]^{1/2} \\
&\qquad \leq \|P_N(-\sqrt{t}A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|_{HS(H)} \leq \left[\tfrac{1}{\pi\sqrt{\nu}} + \tfrac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right]^{1/2}.
\end{aligned} \tag{10}
$$

Proof of Lemma 2.2. Observe that

$$
\begin{aligned}
&\tfrac{1}{\sqrt{t}} \|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|_{HS(H)}^2 \\
&= \tfrac{1}{\sqrt{t}} \sum_{k=1}^{N} \|(-A)^{-1/2}(\mathrm{Id}_H - e^{tA}) e_k\|_H^2 = \tfrac{1}{\sqrt{t}} \sum_{k=1}^{N} \|(\nu\pi^2 k^2)^{-1/2}(1 - e^{-\nu\pi^2 k^2 t}) e_k\|_H^2 \\
&= \sum_{k=1}^{N} \frac{(1 - e^{-\nu\pi^2 k^2 t})^2}{\nu\pi^2 k^2 \sqrt{t}} = \sum_{k=1}^{N} \int_k^{k+1} \frac{(1 - e^{-\nu\pi^2 k^2 t})^2}{\nu\pi^2 k^2 \sqrt{t}}\, dx \\
&\geq \sum_{k=1}^{N} \int_k^{k+1} \frac{(1 - e^{-\nu\pi^2 (x-1)^2 t})^2}{\nu\pi^2 x^2 \sqrt{t}}\, dx \\
&= \int_1^{N+1} \frac{(1 - e^{-\nu\pi^2 (x-1)^2 t})^2}{\nu\pi^2 x^2 \sqrt{t}}\, dx \geq \int_{1+\min\{1/\sqrt{t}, N\}}^{N+1} \frac{(1 - e^{-\nu\pi^2 (x-1)^2 t})^2}{\nu\pi^2 x^2 \sqrt{t}}\, dx.
\end{aligned} \tag{11}
$$

This and the integral transformation theorem imply that

$$
\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|^2_{HS(H)} \\
&\geq \int_{1+\min\{1/\sqrt{t},N\}}^{N+1} \frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{\nu\pi^2 x^2\sqrt{t}}\,dx \\
&= \int_{(1+\min\{1/\sqrt{t},N\})^2}^{(N+1)^2} \frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2 x\sqrt{xt}}\,dx \\
&= \int_{t(1+\min\{1/\sqrt{t},N\})^2}^{t(N+1)^2} \frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2 x\sqrt{x}}\,dx \\
&= \int_{\min\{(1+\sqrt{t})^2,t(N+1)^2\}}^{t(N+1)^2} \frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2 x\sqrt{x}}\,dx \\
&= \int_0^{t(N+1)^2-\min\{(1+\sqrt{t})^2,t(N+1)^2\}} \frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2(x+\min\{(1+\sqrt{t})^2,t(N+1)^2\})^{3/2}}\,dx \\
&\geq \int_0^{\max\{0,t(N+1)^2-(1+\sqrt{t})^2\}} \frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2(x+[1+\sqrt{T}]^2)^{3/2}}\,dx.
\end{aligned}
\tag{12}
$$

Moreover, note that

$$
\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|^2_{HS(H)} \\
&= \tfrac{1}{\sqrt{t}}\sum_{k=1}^N \|(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})e_k\|^2_H = \tfrac{1}{\sqrt{t}}\sum_{k=1}^N \|(\nu\pi^2k^2)^{-1/2}(1-e^{-\nu\pi^2k^2t})e_k\|^2_H \\
&= \sum_{k=1}^N \frac{(1-e^{-\nu\pi^2k^2t})^2}{\nu\pi^2k^2\sqrt{t}} = \frac{(1-e^{-\nu\pi^2t})^2}{\nu\pi^2\sqrt{t}} + \sum_{k=2}^N\int_{k-1}^k \frac{(1-e^{-\nu\pi^2k^2t})^2}{\nu\pi^2k^2\sqrt{t}}\,dx.
\end{aligned}
\tag{13}
$$

The fact that

$$\forall\, x\in(0,\infty), r\in[0,1]\colon\, x^{-r}(1-e^{-x})\leq 1, \tag{14}$$

the fact that

$$\forall\, x\in[1,\infty)\colon\, (x+1)^2\leq 4x^2, \tag{15}$$

and the integral transformation theorem hence yield that

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|^2_{HS(H)} \\
&\leq \frac{(1-e^{-\nu\pi^2 t})^{3/2}}{\pi\sqrt{\nu}} + \sum_{k=2}^{N}\int_{k-1}^{k} \frac{(1-e^{-\nu\pi^2(x+1)^2 t})^2}{\nu\pi^2 x^2\sqrt{t}}\,dx \\
&\leq \frac{1}{\pi\sqrt{\nu}} + \int_1^N \frac{(1-e^{-4\nu\pi^2 x^2 t})^2}{\nu\pi^2 x^2\sqrt{t}}\,dx \\
&= \frac{1}{\pi\sqrt{\nu}} + \int_1^{N^2} \frac{(1-e^{-4\nu\pi^2 x t})^2}{2\nu\pi^2 x\sqrt{xt}}\,dx = \frac{1}{\pi\sqrt{\nu}} + \int_t^{tN^2} \frac{(1-e^{-4\nu\pi^2 x})^2}{2\nu\pi^2 x\sqrt{x}}\,dx.
\end{aligned} \tag{16}$$

Again the fact that

$$\forall\, x \in (0,\infty), r \in [0,1]\colon x^{-r}(1-e^{-x}) \leq 1 \tag{17}$$

therefore ensures that

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|^2_{HS(H)} \\
&\leq \frac{1}{\pi\sqrt{\nu}} + \int_0^\infty \frac{(1-e^{-4\nu\pi^2 x})^2}{2\nu\pi^2 x\sqrt{x}}\,dx \\
&\leq \frac{1}{\pi\sqrt{\nu}} + 2\int_0^1 \frac{(1-e^{-4\nu\pi^2 x})}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{2\nu\pi^2 x\sqrt{x}}\,dx \\
&\leq \frac{1}{\pi\sqrt{\nu}} + 4\pi\sqrt{\nu}\int_0^1 \sqrt{1-e^{-4\nu\pi^2 x}}\,dx + \left[\frac{-1}{\nu\pi^2\sqrt{x}}\right]_{x=1}^{x=\infty} \\
&\leq \frac{1}{\pi\sqrt{\nu}} + 4\pi\sqrt{\nu} + \frac{1}{\nu\pi^2}.
\end{aligned} \tag{18}$$

Combining this and (12) completes the proof of Lemma 2.2. □

2.3 Lower and upper bounds for strong approximation errors of temporal discretizations of linear stochastic heat equations

Lemma 2.3. *Assume the setting in Section 2.1 and let $M \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$. Then*

$$
\begin{aligned}
&\frac{1}{M^{1/4}} \left[\int_0^{\max\left\{0, \frac{T(N+1)^2}{2M} - \left[1 + \frac{\sqrt{T}}{\sqrt{2M}}\right]^2\right\}} \frac{\sqrt{T}\left[1 - e^{-\nu\pi^2 T}\right]\left[1 - \exp(-\nu\pi^2 \min\{1, \frac{TN^2}{2M}\})\right]^2}{8\nu\pi^2\sqrt{2}(x + [1+\sqrt{T}]^2)^{3/2}} \, dx \right]^{1/2} \\
&\leq \|P_N O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \sup_{t \in [0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sup_{t \in [0,T]} \left[\int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s - \lfloor s \rfloor_{T/M})A})\|^2_{HS(H)} \, ds \right]^{1/2} \\
&\leq \frac{1}{M^{1/4}} \left[\frac{\sqrt{T}}{2} \left(\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right) \right]^{1/2}.
\end{aligned}
\tag{19}
$$

Proof of Lemma 2.3. Throughout this proof let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that $\mu_n = \nu\pi^2 n^2$ and let $\lceil\cdot\rceil_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be the functions which satisfy for all $h \in (0,\infty)$, $t \in \mathbb{R}$ that $\lceil t \rceil_h = \min(\{0, h, -h, 2h, -2h, \ldots\} \cap [t, \infty))$. Observe that Lemma 2.1 (i) ensures for all $t \in [0,T)$ that

$$
\begin{aligned}
&2 \int_{\lfloor t \rfloor_{T/M}}^{\lfloor t \rfloor_{T/M} + \frac{T}{M}} \mathbb{1}^{\mathbb{R}}_{[\lfloor s \rfloor_{T/M}, \lfloor s \rfloor_{T/M} + \frac{T}{2M}]}(s) \, \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \, ds \\
&= 2 \int_{\lfloor t \rfloor_{T/M}}^{\lfloor t \rfloor_{T/M} + \frac{T}{2M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \, ds \\
&\geq \int_{\lfloor t \rfloor_{T/M}}^{\lfloor t \rfloor_{T/M} + \frac{T}{2M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \, ds \\
&\quad + \int_{\lfloor t \rfloor_{T/M} + \frac{T}{2M}}^{\lfloor t \rfloor_{T/M} + \frac{T}{M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \, ds \\
&= \int_{\lfloor t \rfloor_{T/M}}^{\lfloor t \rfloor_{T/M} + \frac{T}{M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \, ds.
\end{aligned}
\tag{20}
$$

Therefore, we obtain that

$$
\begin{aligned}
&2\int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s\rfloor_{T/M},\lfloor s\rfloor_{T/M}+\frac{T}{2M}]}(s)\,\|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\,ds \\
&\geq \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\,ds.
\end{aligned}
\tag{21}
$$

Next note that Itô's isometry implies for all $t \in [0,T]$ that

$$
\begin{aligned}
&\|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|^2_H\Big] = \mathbb{E}\bigg[\Big\|\int_0^t P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s\rfloor_{T/M})A})\,dW_s\Big\|^2_H\bigg] \\
&= \int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s\rfloor_{T/M})A})\|^2_{HS(H)}\,ds.
\end{aligned}
\tag{22}
$$

This, the fact that $\forall\, s\in[0,T]\colon T-\lfloor T-s\rfloor_{T/M} = \lceil s\rceil_{T/M}$,and Lemma 2.1 (ii) ensure that

$$
\begin{aligned}
&\|P_N O_T - \mathcal{O}_T^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{(T-s-\lfloor T-s\rfloor_{T/M})A})\|^2_{HS(H)}\,ds \\
&= \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{(\lceil s\rceil_{T/M}-s)A})\|^2_{HS(H)}\,ds \\
&\geq \int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s\rfloor_{T/M},\lfloor s\rfloor_{T/M}+\frac{T}{2M}]}(s)\,\|P_N e^{sA}(\mathrm{Id}_H - e^{(\lceil s\rceil_{T/M}-s)A})\|^2_{HS(H)}\,ds \\
&\geq \int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s\rfloor_{T/M},\lfloor s\rfloor_{T/M}+\frac{T}{2M}]}(s)\,\|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\,ds.
\end{aligned}
\tag{23}
$$

Inequality (21) hence proves that

$$
\begin{aligned}
&\|P_N O_T - \mathcal{O}_T^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&\geq \frac{1}{2}\left[2\int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s\rfloor_{T/M},\lfloor s\rfloor_{T/M}+\frac{T}{2M}]}(s)\,\|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\,ds\right] \\
&\geq \frac{1}{2}\int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\,ds \\
&= \frac{1}{2}\int_0^T \sum_{k=1}^N \|e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})e_k\|^2_H\,ds \\
&= \frac{1}{2}\int_0^T \sum_{k=1}^N |e^{-\mu_k s}(1 - e^{-\mu_k\frac{T}{2M}})|^2\,ds \\
&= \frac{1}{2}\sum_{k=1}^N \frac{(1-e^{-2\mu_k T})}{2\mu_k}\,|1 - e^{-\mu_k\frac{T}{2M}}|^2.
\end{aligned}
\tag{24}
$$

Lemma 2.2 therefore implies that

$$
\begin{aligned}
&\|P_N O_T - \mathcal{O}_T^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&\geq \frac{1}{4}(1-e^{-\mu_1 T})\sum_{k=1}^N \left|\frac{(1-e^{-\mu_k\frac{T}{2M}})}{\sqrt{\mu_k}}\right|^2 \\
&= \frac{1}{4}(1-e^{-\mu_1 T})\sum_{k=1}^N \|(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{2M}A})e_k\|^2_H \\
&= \frac{1}{4}(1-e^{-\mu_1 T})\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \\
&\geq \frac{\sqrt{T}(1-e^{-\mu_1 T})}{4\sqrt{2M}} \\
&\quad\cdot\left[\int_0^{\max\left\{0,\frac{T(N+1)^2}{2M}-\left[1+\frac{\sqrt{T}}{\sqrt{2M}}\right]^2\right\}} \frac{\left[1-\exp(-\nu\pi^2\min\{1,\frac{TN^2}{2M}\})\right]^2}{2\nu\pi^2(x+[1+\sqrt{T}]^2)^{3/2}}\,dx\right].
\end{aligned}
\tag{25}
$$

In the next step observe that (22) and Lemma 2.1 (ii) assure that

$$\begin{aligned}
\sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} &\leq \sup_{t\in[0,T]} \int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)}\,ds \\
&= \sup_{t\in[0,T]} \int_0^t \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)}\,ds \\
&= \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)}\,ds \\
&= \int_0^T \sum_{k=1}^N \|e^{sA}(\mathrm{Id}_H - e^{\frac{T}{M}A})e_k\|^2_H\,ds.
\end{aligned} \tag{26}$$

Lemma 2.2 hence yields that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&\leq \int_0^T \sum_{k=1}^N |e^{-\mu_k s}(1 - e^{-\mu_k \frac{T}{M}})|^2\,ds = \sum_{k=1}^N \frac{(1 - e^{-2\mu_k T})}{2\mu_k} |1 - e^{-\mu_k \frac{T}{M}}|^2 \\
&\leq \frac{1}{2}\sum_{k=1}^N \left|\frac{(1 - e^{-\mu_k \frac{T}{M}})}{\sqrt{\mu_k}}\right|^2 = \frac{1}{2}\sum_{k=1}^N \|(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{M}A})e_k\|^2_H \\
&= \frac{1}{2}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)} \leq \frac{\sqrt{T}}{2\sqrt{M}}\left[\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu}\right].
\end{aligned} \tag{27}$$

Combining this with (22) and (25) completes the proof of Lemma 2.3. □

In the next result, Corollary 2.4, we specialize Lemma 2.3 to the case $N = \infty$ where no spatial discretization is applied to the stochastic process $O\colon [0,T]\times\Omega \to H$.

Corollary 2.4. *Assume the setting in Section 2.1 and let $M \in \mathbb{N}$. Then*

$$\begin{aligned}
&\frac{1}{M^{1/4}}\left[\int_0^\infty \frac{\sqrt{T}(1 - e^{-\nu\pi^2 T})(1 - e^{-\nu\pi^2})^2}{8\nu\pi^2\sqrt{2}(x + [1 + \sqrt{T}]^2)^{3/2}}\,dx\right]^{1/2} \\
&\leq \liminf_{N\to\infty} \|P_N O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \limsup_{N\to\infty} \|P_N O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \|O_T - \mathcal{O}_T^{M,\infty}\|_{\mathcal{L}^2(\mathbb{P};H)} = \liminf_{N\to\infty} \sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \limsup_{N\to\infty} \sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,\infty}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&\leq \frac{1}{M^{1/4}}\left[\frac{\sqrt{T}}{2}\left(\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu}\right)\right]^{1/2}.
\end{aligned} \tag{28}$$

2.4 Lower and upper bounds for strong approximation errors of spatial discretizations of linear stochastic heat equations

Lemma 2.5. *Assume the setting in Section 2.1. Then*

$$\limsup_{M\to\infty} \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left\| \int_0^t \left(P_N e^{(t-s)A} - P_N e^{(t-\lfloor s \rfloor_{T/M})A} \right) dW_s \right\|_{L^2(\mathbb{P};H)} = 0. \tag{29}$$

Proof of Lemma 2.5. Throughout this proof let $\alpha \in (0, 1/4)$ and let $\beta \in (1/4, 1/2 - \alpha)$. Note that the fact that $4\beta > 1$ shows that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \|P_N\|^2_{HS(H,H_{-\beta})} \\ &= \sup_{N\in\mathbb{N}} \left[\sum_{k=1}^N \|e_k\|^2_{H_{-\beta}} \right] = \sum_{k=1}^\infty \|(-A)^{-\beta} e_k\|^2_H \\ &= \sum_{k=1}^\infty |(\nu\pi^2 k^2)^{-\beta}|^2 = \sum_{k=1}^\infty \frac{1}{(\sqrt{\nu}\pi k)^{4\beta}} < \infty. \end{aligned} \tag{30}$$

Next observe that for all $M, N \in \mathbb{N}$, $t \in [0,T]$ we have that

$$\begin{aligned} &\int_0^t \|P_N e^{(t-s)A} (\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|^2_{HS(H)}\, ds \\ &\leq \|(-A)^{-\beta} P_N\|^2_{HS(H)} \int_0^t \|(-A)^{\beta} e^{(t-s)A} (\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|^2_{L(H)}\, ds \\ &= \|P_N\|^2_{HS(H,H_{-\beta})} \int_0^t \|(-A)^{(\alpha+\beta)} e^{(t-s)A} (-A)^{-\alpha} (\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|^2_{L(H)}\, ds \\ &\leq \|P_N\|^2_{HS(H,H_{-\beta})} \\ &\quad \cdot \int_0^t \|(-A)^{(\alpha+\beta)} e^{(t-s)A}\|^2_{L(H)} \|(-A)^{-\alpha} (\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|^2_{L(H)}\, ds. \end{aligned} \tag{31}$$

The fact that

$$\forall\, s \in [0,\infty), r \in [0,1] \colon \|(-sA)^r e^{sA}\|_{L(H)} \leq 1 \tag{32}$$

and the fact that

$$\forall\, s \in (0,\infty), r \in [0,1] \colon \|(-sA)^{-r} (\mathrm{Id}_H - e^{sA})\|_{L(H)} \leq 1 \tag{33}$$

hence prove for all $M, N \in \mathbb{N}$, $t \in [0,T]$ that

$$
\begin{aligned}
&\int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|_{HS(H)}^2 \, ds \\
&\leq \|P_N\|_{HS(H,H_{-\beta})}^2 \int_0^t (t-s)^{-2(\alpha+\beta)}(s - \lfloor s \rfloor_{T/M})^{2\alpha} \, ds \\
&\leq \frac{T^{2\alpha}}{M^{2\alpha}} \|P_N\|_{HS(H,H_{-\beta})}^2 \int_0^t (t-s)^{-2(\alpha+\beta)} \, ds \\
&= \frac{t^{(1-2\alpha-2\beta)} T^{2\alpha}}{(1-2\alpha-2\beta) M^{2\alpha}} \|P_N\|_{HS(H,H_{-\beta})}^2 .
\end{aligned}
\tag{34}
$$

Itô's isometry therefore ensures for all $M \in \mathbb{N}$ that

$$
\begin{aligned}
&\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \left\| \int_0^t \left(P_N e^{(t-s)A} - P_N e^{(t-\lfloor s \rfloor_{T/M})A} \right) dW_s \right\|_{L^2(\mathbb{P};H)}^2 \\
&= \sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \int_0^t \|P_N e^{(t-s)A} - P_N e^{(t-\lfloor s \rfloor_{T/M})A}\|_{HS(H)}^2 \, ds \\
&\leq \frac{T^{(1-2\beta)}}{(1-2\alpha-2\beta) M^{2\alpha}} \left[\sup_{N \in \mathbb{N}} \|P_N\|_{HS(H,H_{-\beta})}^2 \right].
\end{aligned}
\tag{35}
$$

Combining this with (30) completes the proof of Lemma 2.5. ☐

Lemma 2.6. *Assume the setting in Section 2.1 and let $N \in \mathbb{N}$. Then*

$$
\begin{aligned}
&\left[\frac{\sqrt{1-e^{-\nu T}}}{2\pi\sqrt{\nu}} \right] \frac{1}{\sqrt{N}} \leq \liminf_{M \to \infty} \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \limsup_{M \to \infty} \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \liminf_{M \to \infty} \sup_{t \in [0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \limsup_{M \to \infty} \sup_{t \in [0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \|O_T - P_N O_T\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sup_{t \in [0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P};H)} \leq \left[\frac{1}{\pi\sqrt{2\nu}} \right] \frac{1}{\sqrt{N}}.
\end{aligned}
\tag{36}
$$

Proof of Lemma 2.6. Throughout this proof let $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that

$$\mu_n = \nu \pi^2 n^2. \tag{37}$$

Note that Parseval's identity shows that for all $t \in [0,T]$ we have that

$$
\begin{aligned}
&\|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \mathbb{E}\big[\|O_t - P_N O_t\|_H^2\big] = \mathbb{E}\left[\sum_{k=N+1}^{\infty} |\langle e_k, O_t\rangle_H|^2\right] = \sum_{k=N+1}^{\infty} \mathbb{E}\big[|\langle e_k, O_t\rangle_H|^2\big] \\
&= \sum_{k=N+1}^{\infty} \mathbb{E}\left[\left|\int_0^t \langle e_k, e^{(t-s)A} dW_s\rangle_H\right|^2\right] = \sum_{k=N+1}^{\infty} \mathbb{E}\left[\left|\int_0^t \langle e^{(t-s)A} e_k, dW_s\rangle_H\right|^2\right].
\end{aligned}
\tag{38}
$$

Itô's isometry hence proves for all $t \in [0,T]$ that

$$
\begin{aligned}
&\|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sum_{k=N+1}^{\infty} \mathbb{E}\left[\left|\int_0^t e^{-\mu_k (t-s)} \, \langle e_k, dW_s\rangle_H\right|^2\right] \\
&= \sum_{k=N+1}^{\infty} \int_0^t e^{-2\mu_k (t-s)} \, ds = \sum_{k=N+1}^{\infty} \int_0^t e^{-2\mu_k s} \, ds = \sum_{k=N+1}^{\infty} \frac{(1 - e^{-2\mu_k t})}{2\mu_k}.
\end{aligned}
\tag{39}
$$

This shows that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} = \|O_T - P_N O_T\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sum_{k=N+1}^{\infty} \frac{\big(1 - e^{-2\mu_k T}\big)}{2\mu_k} = \sum_{k=N+1}^{\infty} \frac{\big(1 - e^{-2\nu\pi^2 k^2 T}\big)}{2\nu\pi^2 k^2} \\
&\geq \left[\frac{1 - e^{-\nu T}}{2\nu\pi^2}\right] \left[\sum_{k=N+1}^{\infty} \frac{1}{k^2}\right] \geq \left[\frac{1 - e^{-\nu T}}{2\nu\pi^2}\right] \left[\sum_{k=N+1}^{\infty} \int_k^{k+1} \frac{1}{x^2} \, dx\right] \\
&= \left[\frac{1 - e^{-\nu T}}{2\nu\pi^2}\right] \left[\int_{N+1}^{\infty} \frac{1}{x^2} \, dx\right] = \left[\frac{1 - e^{-\nu T}}{2\nu\pi^2}\right] \left[-\frac{1}{x}\right]_{x=N+1}^{x=\infty} \\
&= \left[\frac{1 - e^{-\nu T}}{2\nu\pi^2}\right] \frac{1}{(N+1)} \geq \left[\frac{1 - e^{-\nu T}}{2\nu\pi^2}\right] \frac{1}{(N+N)} = \left[\frac{1 - e^{-\nu T}}{4\nu\pi^2}\right] \frac{1}{N}.
\end{aligned}
\tag{40}
$$

This implies that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sum_{k=N+1}^{\infty} \frac{\left(1-e^{-2\nu\pi^2k^2T}\right)}{2\nu\pi^2k^2} \leq \left[\frac{1}{2\nu\pi^2}\right]\left[\sum_{k=N+1}^{\infty}\frac{1}{k^2}\right] \\
&\leq \left[\frac{1}{2\nu\pi^2}\right]\left[\sum_{k=N+1}^{\infty}\int_{k-1}^{k}\frac{1}{x^2}\,dx\right] = \left[\frac{1}{2\nu\pi^2}\right]\left[\int_{N}^{\infty}\frac{1}{x^2}\,dx\right] \\
&= \left[\frac{1}{2\nu\pi^2}\right]\left[-\frac{1}{x}\right]_{x=N}^{x=\infty} = \left[\frac{1}{2\nu\pi^2}\right]\frac{1}{N}.
\end{aligned}
\tag{41}
$$

In addition, note that the triangle inequality and Lemma 2.5 prove that

$$
\begin{aligned}
&\limsup_{M\to\infty}\sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \limsup_{M\to\infty}\sup_{t\in[0,T]}\left\|\int_0^t \left(e^{(t-s)A} - P_N e^{(t-\lfloor s\rfloor_{T/M})A}\right) dW_s\right\|_{L^2(\mathbb{P};H)} \\
&= \limsup_{M\to\infty}\sup_{t\in[0,T]}\left\|\int_0^t \left(e^{(t-s)A} - P_N e^{(t-s)A}\right) dW_s\right. \\
&\qquad \left. + \int_0^t \left(P_N e^{(t-s)A} - P_N e^{(t-\lfloor s\rfloor_{T/M})A}\right) dW_s\right\|_{L^2(\mathbb{P};H)} \\
&\leq \limsup_{M\to\infty}\sup_{t\in[0,T]}\left\|\int_0^t \left(e^{(t-s)A} - P_N e^{(t-s)A}\right) dW_s\right\|_{L^2(\mathbb{P};H)} \\
&\qquad + \limsup_{M\to\infty}\sup_{t\in[0,T]}\left\|\int_0^t \left(P_N e^{(t-s)A} - P_N e^{(t-\lfloor s\rfloor_{T/M})A}\right) dW_s\right\|_{L^2(\mathbb{P};H)} \\
&= \sup_{t\in[0,T]}\left\|\int_0^t \left(e^{(t-s)A} - P_N e^{(t-s)A}\right) dW_s\right\|_{L^2(\mathbb{P};H)} = \sup_{t\in[0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P};H)}.
\end{aligned}
\tag{42}
$$

Furthermore, observe that the triangle inequality, Lemma 2.5, and (40) ensure that

$$
\begin{aligned}
&\liminf_{M\to\infty} \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \liminf_{M\to\infty} \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-\lfloor s \rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&= \liminf_{M\to\infty} \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-s)A})\, dW_s \right. \\
&\quad \left. + \int_0^T (P_N e^{(T-s)A} - P_N e^{(T-\lfloor s \rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&\geq \liminf_{M\to\infty} \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-s)A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&\quad - \liminf_{M\to\infty} \left\| \int_0^T (P_N e^{(T-s)A} - P_N e^{(T-\lfloor s \rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&= \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-s)A})\, dW_s \right\|_{L^2(\mathbb{P};H)} = \|O_T - P_N O_T\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sup_{t\in[0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P};H)}.
\end{aligned}
\tag{43}
$$

Combining this with (40)–(42) completes the proof of Lemma 2.6. □

2.5 Lower and upper bounds for strong approximation errors of full discretizations of linear stochastic heat equations

Theorem 2.7. *Assume the setting in Section 2.1 and let $M, N \in \mathbb{N}$. Then*

$$\begin{aligned}
&\frac{1}{M^{1/4}} \left[\int_0^{\max\left\{0, \frac{T(N+1)^2}{2M} - \left[1 + \frac{\sqrt{T}}{\sqrt{2M}}\right]^2\right\}} \frac{\sqrt{T} \left[1 - e^{-\nu\pi^2 T}\right] \left[1 - \exp(-\nu\pi^2 \min\{1, \frac{TN^2}{2M}\})\right]^2}{32\nu\pi^2\sqrt{2}(x + [1+\sqrt{T}]^2)^{3/2}} \, dx \right]^{1/2} \\
&\quad + \frac{1}{N^{1/2}} \left[\frac{\sqrt{1 - e^{-\nu T}}}{4\pi\sqrt{\nu}} \right] \\
&\leq \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \leq \sup_{t \in [0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&\leq \frac{1}{M^{1/4}} \left[\frac{\sqrt{T}}{2} \left(\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right) \right]^{1/2} + \frac{1}{N^{1/2}} \left[\frac{1}{\pi\sqrt{2\nu}} \right].
\end{aligned} \tag{44}$$

Proof of Theorem 2.7. Observe that the fact that P_N is self-adjoint ensures for all $x \in H$, $y \in P_N(H)$ that

$$\begin{aligned}
&\langle x - P_N(x), P_N(x) - y \rangle_H \\
&= \langle x - P_N(x), P_N(x) - P_N(y) \rangle_H = \langle x - P_N(x), P_N(x - y) \rangle_H \\
&= \langle P_N(x - P_N(x)), x - y \rangle_H = \langle P_N(x) - P_N(x), x - y \rangle_H \\
&= \langle 0, x - y \rangle_H = 0.
\end{aligned} \tag{45}$$

This implies for all $t \in [0, T]$ that

$$\begin{aligned}
&\|O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \mathbb{E}\Big[\|O_t - \mathcal{O}_t^{M,N}\|_H^2\Big] = \mathbb{E}\Big[\|O_t - P_N O_t + P_N O_t - \mathcal{O}_t^{M,N}\|_H^2\Big] \\
&= \mathbb{E}\big[\|O_t - P_N O_t\|_H^2\big] + 2\,\mathbb{E}\Big[\langle O_t - P_N O_t, P_N O_t - \mathcal{O}_t^{M,N} \rangle_H\Big] \\
&\quad + \mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|_H^2\Big] \\
&= \|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} + \|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)}.
\end{aligned} \tag{46}$$

Combining this with Lemma 2.3, Lemma 2.6, and the fact that

$$\forall\, x, y \in [0, \infty)\colon \sqrt{x}/2 + \sqrt{y}/2 \leq \max\{\sqrt{x}, \sqrt{y}\} \leq \sqrt{x + y} \leq \sqrt{x} + \sqrt{y} \tag{47}$$

completes the proof of Theorem 2.7. ∎

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