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Abstract

The aim of this paper is to rigorously show the existence of a Dirac dispersion cone in a bubbly honeycomb phononic crystal comprised of bubbles of arbitrary shape. The main result is an asymptotic formula for the quasi-periodic Minnaert resonance frequencies close to the symmetry points K in the Brilloun zone. This shows the linear dispersion relation of a Dirac cone. Our findings in this paper are illustrated in the case of circular bubbles, where the multipole expansion method provides an efficient technique for computing the band structure.

Mathematics Subject Classification (MSC2000). 35R30, 35C20.

Keywords. Honeycomb lattice, Dirac cone, bubble, Minnaert resonance, subwavelength bandgap.

1 Introduction

Recently there have been many discoveries involving materials that exhibit intriguing wave propagation properties due to the presence of a Dirac cone in their band structures [10, 12, 13, 15, 16, 19, 26, 28, 29]. A Dirac cone is a linear intersection of two curves in the dispersion diagram, and is a consequence of non-trivial symmetry of the lattice. Dirac cones have typically been studied in the context of electron bands in graphene, where peculiar effects such as Klein tunneling and Zitterbewegung have been observed. Moreover, Dirac cones have been demonstrated in acoustic analogues of graphene.

A single gas bubble inside a liquid possesses an acoustic resonance frequency known as the Minnaert resonance frequency [1, 2, 24]. Due to the high density contrast in the material parameters, this frequency is of subwavelengh scale, *i.e.*, it corresponds to a wavelength much larger than the size of the bubble. This makes the Minnaert bubble an

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ideal component for constructing subwavelength scale metamaterials, see, for instance, [5, 6, 9, 17, 18, 20, 21, 22, 23, 30].

In [4], it was proved that a bubbly crystal features a bandgap opening close to the Minnaert resonance frequency, *i.e.*, in the deep subwavelength regime. This suggests the possibility of creating materials with Dirac cones at subwavelength scales, resulting in small scale metamaterials with Dirac singularities. Such materials have been experimentally and numerically studied in [28, 29]. The goal of this paper is to rigorously prove the existence of a Dirac cone in the deep subwavelength scale in a bubbly honeycomb crystal.

Apart from the very recent work [14], previous mathematical analyses of honeycomb lattice structures have typically been based on the tight-binding model [25, 27], assuming interaction only between the nearest neighbouring particles in the lattice. In this work, using layer potential theory and Gohberg-Sigal theory, we will demonstrate a method for analysing honeycomb structures without this assumption, starting only from the differential operator. The method is general with respect to the shape of the scatterer, and generalizes previous works on circular scatterers [26]. Moreover, the slopes of the Dirac cones can be asymptotically computed in terms of the density contrast.

The paper is organised as follows. In Section 2, we define the geometry of the bubbly honeycomb crystal and formulate the spectral resonance problem. We also introduce some well-known results regarding the quasi-periodic Green's function on the honeycomb lattice. In Section 3, we derive an asymptotic formula for the quasi-periodic Minnaert resonance in terms of the density contrast between the bubble and the surrounding medium. In Section 4, we rigorously show the existence of a Dirac dispersion cone in the bubbly honeycomb crystal. Our main result is stated in Theorem 4.1 which gives an asymptotic expansion of the quasi-periodic Minnaert resonance frequency for quasi-periodicities close to a Dirac point. In Section 5, we numerically compute the Dirac cones in the case of circular bubbles using the multipole expansion method. The paper ends with some concluding remarks in Section 6.

2 Problem statement and preliminaries

2.1 Problem formulation

In order to describe the problem under consideration, we start by describing the bubbly honeycomb crystal depicted in Figure 1. We consider a two-dimensional infinite honeycomb crystal in two dimensions. Define a hexagonal lattice Λ with lattice vectors:

$$l_1 = a\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad l_2 = a\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

Denote by Y a fundamental domain of the given lattice. Here, we take

$$Y := \{ s l_1 + t l_2 \mid 0 \le s, t \le 1 \}.$$

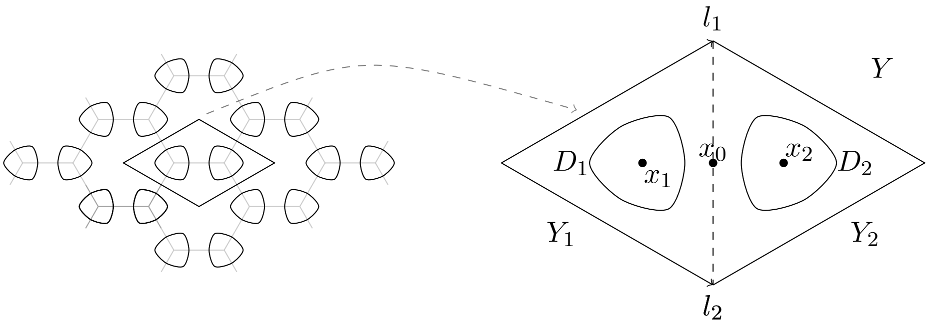


Figure 1: Illustration of the bubbly honeycomb crystal and quantities in the fundamental domain Y.

We decompose the fundamental domain Y into two parts:

$$Y = Y_1 \cup Y_2,$$

where

$$Y_1 = \{sl_1 + tl_2 \mid 0 \leq s, t, \text{ and } t + s \leq 1\}, \; Y_2 = Y \setminus Y_1.$$

Define the three points x_0, x_1, and x_2 as the centers of Y, Y_1, and Y_2, respectively, *i.e.*,

$$x_0 = \frac{l_1 + l_2}{2}, \quad x_1 = \frac{l_1 + l_2}{3}, \quad x_2 = \frac{2(l_1 + l_2)}{3}.$$

We will assume that each bubble in the crystal has a three-fold rotational symmetry and that each pair of adjacent bubbles has a two-fold rotational symmetry. More precisely, let R_1 and R_2 be the rotations by $-\frac{2\pi}{3}$ around x_1 and x_2, respectively, and let R_0 be the rotation around x_0 by π. These rotations can be written as

$$R_1 x = Rx + l_1, \quad R_2 x = Rx + 2l_1, \quad R_0 x = 2x_0 - x,$$

where R is the rotation by $-\frac{2\pi}{3}$ around the origin. Assume that the unit cell contains two bubbles D_j, $j = 1, 2$, each in the fundamental domain Y_j such that

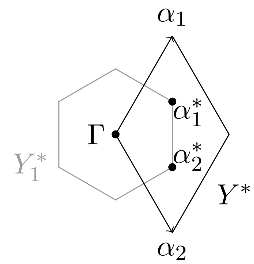
$$R_1 D_1 = D_1, \quad R_2 D_2 = D_2, \quad R_0 D_1 = D_2.$$

Denote the bubble dimer by $D = D_1 \cup D_2$.

The dual lattice of Λ, denoted Λ^*, is generated by α_1 and α_2 satisfying $\alpha_i \cdot l_j = 2\pi\delta_{ij}$ for $i, j = 1, 2$. Then

$$\alpha_1 = \frac{2\pi}{a}\left(\frac{1}{\sqrt{3}}, 1\right), \quad \alpha_2 = \frac{2\pi}{a}\left(\frac{1}{\sqrt{3}}, -1\right).$$

The Brillouin zone Y^* is defined as the torus $Y^* := \mathbb{R}^2 / \Lambda^*$ and can be represented either as



$$Y^* \simeq \{s\alpha_1 + t\alpha_2 \mid 0 \le s, t \le 1\},$$

or as the first Brillouin zone Y_1^*. The points

$$\alpha_1^* = \frac{2\alpha_1 + \alpha_2}{3}, \quad \alpha_2^* = \frac{\alpha_1 + 2\alpha_2}{3},$$

in the Brillouin zone are called *Dirac points*. Observe that in the torus Y^* we have

$$R\alpha_1^* = \frac{-\alpha_1 + \alpha_2}{3} = \frac{2\alpha_1 + \alpha_2}{3} = \alpha_1^*,$$

and similarly, $R\alpha_2^* = \alpha_2^*$.

Having defined the geometry, we now define the wave scattering problem in the bubbly honeycomb crystal. We will study the α-quasi-periodic Floquet component u of the total wave field for $\alpha \in Y^*$. We therefore consider the following α-quasi-periodic acoustic wave problem in Y:

$$\begin{cases} \nabla \cdot \dfrac{1}{\rho} \nabla u + \dfrac{\omega^2}{\kappa} u = 0 & \text{in } Y \backslash D, \\ \nabla \cdot \dfrac{1}{\rho_b} \nabla u + \dfrac{\omega^2}{\kappa_b} u = 0 & \text{in } D, \\ u_+ - u_- = 0 & \text{on } \partial D, \\ \dfrac{1}{\rho} \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{1}{\rho_b} \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D, \\ u(x + l) = e^{i\alpha \cdot l} u(x) & \text{for all } l \in \Lambda. \end{cases} \tag{2.1}$$

Here, $\frac{\partial}{\partial \nu}$ denotes the normal derivative on ∂D, and the subscripts $+$ and $-$ indicate the limits from outside and inside D, respectively. Let

$$v := \sqrt{\frac{\kappa}{\rho}}, \quad v_b := \sqrt{\frac{\kappa_b}{\rho_b}}, \quad k = \frac{\omega}{v}, \quad k_b = \frac{\omega}{v_b}. \tag{2.2}$$

Introduce the density contrast parameter δ as

$$\delta := \frac{\rho_b}{\rho}. \tag{2.3}$$

We assume that there is a high contrast in the density while the wave speeds are comparable, *i.e.*,

$$\delta \ll 1 \quad \text{and} \quad v, v_b = O(1).$$

2.2 Quasi-periodic Green's function for the honeycomb lattice

Define the $\alpha-$quasi-periodic Green's function $G^{\alpha,k}$ to satisfy

$$\Delta G^{\alpha,k} + k^2 G^{\alpha,k} = \sum_{n\in\Lambda} \delta(x-n)e^{i\alpha\cdot n}.$$

Then it can be shown that $G^{\alpha,k}$ is given by [3, 8]

$$G^{\alpha,k}(x) = \frac{1}{|Y|}\sum_{q\in\Lambda^*} \frac{e^{i(\alpha+q)\cdot x}}{k^2-|\alpha+q|^2}. \tag{2.4}$$

From now on, we assume that $|Y|=1$.

For a given bounded domain D in Y, with Lipschitz boundary ∂D, the single layer potential of the density function $\varphi \in L^2(\partial D)$ is defined by

$$\mathcal{S}_D^{\alpha,k}[\varphi](x) := \int_{\partial D} G^{\alpha,k}(x-y)\varphi(y)d\sigma(y), \quad x\in\mathbb{R}^2.$$

The following jump relations are well-known [3, 8]:

$$\frac{\partial}{\partial\nu}\mathcal{S}_D^{\alpha,k}[\varphi]\Big|_{\pm}(x) = \left(\pm\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,k})^*\right)[\varphi](x), \quad x\in\partial D, \tag{2.5}$$

where the Neumann-Poincaré operator $(\mathcal{K}_D^{-\alpha,k})^*$ is defined by

$$(\mathcal{K}_D^{-\alpha,k})^*[\varphi](x) = \text{p.v.}\int_{\partial D} \frac{\partial G^{\alpha,k}(x-y)}{\partial\nu_x}\varphi(y)d\sigma(y), \quad x\in\partial D.$$

It is known that $\mathcal{S}_D^{\alpha,0} : L^2(\partial D)\to H^1(\partial D)$ is invertible when $\alpha\neq 0$ [3, 8].

We can expand $G^{\alpha,k}$ with respect to k as follows:

$$G^{\alpha,k} = G^{\alpha,0} + \sum_{j=1}^{\infty} k^{2j} G_j^{\alpha,0},$$

where

$$G_j^{\alpha,0}(x) := -\sum_{q\in\Lambda^*}\frac{e^{i(\alpha+q)\cdot x}}{|\alpha+q|^{2j}}.$$

Using this expansion, we can expand the single layer potentials and the Neumann-Poincaré operators as

$$\mathcal{S}_D^{\alpha,k} = \mathcal{S}_D^{\alpha,0} + \sum_{j=1}^{\infty} k^{2j}\mathcal{S}_{D,j}^{\alpha,0} \quad \text{with} \quad \mathcal{S}_{D,j}^{\alpha,0}[\phi](x) := \int_{\partial D} G_j^{\alpha,0}(x-y)\phi(y)d\sigma(y),$$

and

$$(\mathcal{K}_D^{-\alpha,k})^* = (\mathcal{K}_D^{-\alpha,0})^* + \sum_{j=1}^{\infty} k^{2j}\mathcal{K}_{D,j}^{\alpha,0} \quad \text{with} \quad \mathcal{K}_{D,j}^{\alpha,0}[\phi](x) := \int_{\partial D}\frac{\partial}{\partial\nu_x}G_j^{\alpha,0}(x-y)\phi(y)d\sigma(y).$$

The following identity will be used in the sequel:

$$\int_{\partial D_j} \mathcal{K}_{D,1}^{\alpha,0}[\phi] = -\int_{D_j} \mathcal{S}_D^{\alpha,0}[\phi]. \tag{2.6}$$

Next, we consider $G^{\alpha,k}$ near a Dirac point $\alpha = \alpha^*$. Since

$$G_1^{\alpha,0} - G_1^{\alpha^*,0} = O(|\alpha - \alpha^*|),$$

we have

$$G^{\alpha,k} = G^{\alpha,0} + G^{\alpha^*,k} - G^{\alpha^*,0} + O(k^2|\alpha - \alpha^*|).$$

Furthermore, the following asymptotic expansion of $G^{\alpha,k}$ holds for α near a Dirac point α^*:

$$G^{\alpha,k}(x) = G^{\alpha^*,k}(x) + \sum_{q \in \Lambda^*} \frac{e^{i(\alpha^*+q)\cdot x}}{k^2 - |\alpha^* + q|^2} \left(i(\alpha - \alpha^*) \cdot x + 2\frac{(\alpha^* + q)\cdot(\alpha - \alpha^*)}{k^2 - |\alpha^* + q|^2} \right) + O(|\alpha - \alpha^*|^2). \tag{2.7}$$

We define G_1^k by

$$G_1^k(x) := \sum_{q \in \Lambda^*} \frac{e^{i(\alpha^*+q)\cdot x}}{k^2 - |\alpha^* + q|^2} \left(ix + \frac{2(\alpha^* + q)}{k^2 - |\alpha^* + q|^2} \right).$$

Then, we have

$$G^{\alpha,k}(x) = G^{\alpha^*,k}(x) + G_1^k(x) \cdot (\alpha - \alpha^*) + O(|\alpha - \alpha^*|^2). \tag{2.8}$$

This motivates the definition of the integral operators $\mathcal{S}_{D,1}^k$, $\mathcal{K}_{D,1}^k$ as

$$\mathcal{S}_{D,1}^k[\phi](x) := \int_{\partial D} G_1^k(x - y)\phi(y) d\sigma(y),$$
$$\mathcal{K}_{D,1}^k[\phi](x) := \int_{\partial D} \frac{\partial}{\partial \nu_x} G_1^k(x - y)\phi(y),$$

for $\phi \in L^2(\partial D)$.

2.3 Quasi-periodic capacitance matrix

Let V_j^α be the solution to

$$\begin{cases} \Delta V_j^\alpha = 0 & \text{in } Y \setminus D, \\ V_j^\alpha = \delta_{ij} & \text{on } \partial D_i, \\ V_j^\alpha(x + l) = e^{i\alpha\cdot l} V_j^\alpha(x) & \forall l \in \Lambda. \end{cases} \tag{2.9}$$

We define the capacitance coefficients matrix $C^\alpha = (C_{ij}^\alpha)$ by

$$C_{ij}^\alpha := \int_{Y \setminus D} \nabla V_i^\alpha \cdot \overline{\nabla V_j^\alpha}, \quad i, j = 1, 2.$$

Let $\psi_j \in L^2(\partial D)$ be given by

$$\mathcal{S}_D^{\alpha,0}[\psi_j] = \delta_{ij} \quad \text{on } \partial D_i, \quad i = 1, 2. \tag{2.10}$$

By Green's formula, we see that the capacitance coefficients matrix $C^\alpha = (C_{ij}^\alpha)$ is also given by

$$C_{ij}^\alpha := -\int_{\partial D_j} \psi_i, \quad i, j = 1, 2.$$

We define two transformations T_1^α and T_2 of functions in Y by

$$(T_1^\alpha f)(x) := \begin{cases} e^{-i\alpha\cdot l_1} f(R_1 x), & x \in Y_1, \\ e^{-2i\alpha\cdot l_1} f(R_2 x), & x \in Y_2, \end{cases} \qquad (T_2 f)(x) := \overline{f(2x_0 - x)}. \tag{2.11}$$

Observe that for any α, the definitions of $T_1^\alpha f$ coincide on $\partial Y_1 \cap \partial Y_2$. Moreover, $T_1^{\alpha^*} f$ is α^*-quasi-periodic if f is α^*-quasi-periodic. We will denote by $T_1 := T_1^{\alpha^*}$.

We remark that the matrix C^α is Hermitian for any α. At Dirac points, the following result holds.

Lemma 2.1. *For Dirac points, the capacitance matrix is a constant multiple of the identity matrix.*

Proof. We prove the above statement for $\alpha^* = \frac{2\alpha_1+\alpha_2}{3}$. Define $\tau := e^{2\pi i/3}$. Since $T_2 V_1^{\alpha^*} = V_2^{\alpha^*}$, we have

$$C_{11}^{\alpha^*} = \int_{Y\setminus D} \nabla V_1^\alpha \cdot \overline{\nabla V_1^\alpha} = C_{22}^{\alpha^*}.$$

We can also check that

$$T_1 V_1^{\alpha^*} = \tau V_1^{\alpha^*}, \ T_1 V_2^{\alpha^*} = \tau^2 V_2^{\alpha^*}.$$

Then, it follows that

$$C_{12}^{\alpha^*} = \int_{Y\setminus D} \nabla V_1^{\alpha^*} \cdot \overline{\nabla V_2^{\alpha^*}} = \int_{Y\setminus D} \nabla T_1 V_1^{\alpha^*} \cdot \overline{\nabla T_1 V_2^{\alpha^*}} = \tau^2 C_{12}^{\alpha^*}.$$

Therefore, we have $C_{12}^{\alpha^*} = 0$ and so $C_{21}^{\alpha^*} = 0$. □

3 High-contrast subwavelength bands

In this section, we investigate the asymptotic behaviour of the band structure in the case of small δ. The main results are given in Theorem 3.1 and in equation (3.13). Throughout this section, we fix $\alpha^* = \frac{2\alpha_1+\alpha_2}{3}$ and consider $\alpha \neq 0$.

From, for instance, [7], we know that the solution to (2.1) can be represented using the single layer potentials $\mathcal{S}_D^{\alpha,k_b}$ and $\mathcal{S}_D^{\alpha,k}$ as follows:

$$u(x) = \begin{cases} \mathcal{S}_D^{\alpha,k}[\psi](x), & x \in Y \setminus \bar{D}, \\ \mathcal{S}_D^{\alpha,k_b}[\phi](x), & x \in D, \end{cases} \tag{3.1}$$

where the pair $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ satisfies

$$\begin{cases} \mathcal{S}_D^{\alpha,k_b}[\phi] - \mathcal{S}_D^{\alpha,k}[\psi] = 0 \\ \dfrac{1}{\rho_b}\left(-\dfrac{1}{2}I + (\mathcal{K}_D^{-\alpha,k_b})^*\right)[\phi] - \dfrac{1}{\rho}\left(\dfrac{1}{2}I + (\mathcal{K}_D^{-\alpha,k})^*\right)[\psi] = 0 \end{cases} \quad \text{on } \partial D. \tag{3.2}$$

We denote by

$$\mathcal{A}_\delta^{\alpha,\omega} := \begin{bmatrix} \mathcal{S}_D^{\alpha,k_b} & -\mathcal{S}_D^{\alpha,k} \\ -\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,k_b})^* & -\delta\left(\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,k})^*\right) \end{bmatrix}. \tag{3.3}$$

Then, (3.2) can be written as

$$\mathcal{A}_\delta^{\alpha,\omega}\begin{pmatrix}\phi \\ \psi\end{pmatrix} = 0. \tag{3.4}$$

It is well-known that the above integral equation has a non-trivial solution for some discrete frequencies ω. These can be viewed as the characteristic values of the operator-valued analytic function $\mathcal{A}_\delta^{\alpha,\omega}$ (with respect to ω); see [3, 8].

It can be shown that $\omega = 0$ is a characteristic value of $\mathcal{A}_0^{\alpha,\omega}$ because

$$\begin{bmatrix} \mathcal{S}_D^{\alpha,0} & -\mathcal{S}_D^{\alpha,0} \\ -\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,0})^* & 0 \end{bmatrix}$$

has a nontrivial kernel of two dimensions, which is generated by

$$\Psi_1 = \begin{pmatrix}\psi_1 \\ \psi_1\end{pmatrix} \quad \text{and} \quad \Psi_2 = \begin{pmatrix}\psi_2 \\ \psi_2\end{pmatrix},$$

where ψ_1 and ψ_2 are as in (2.10). Then Gohberg-Sigal theory [3, 8] tells us that there exists characteristic values $\omega_j^\alpha = \omega_j^\alpha(\delta),\ j = 1,2$ of $\mathcal{A}_\delta^{\alpha,\omega}$ such that $\omega_j^\alpha(0) = 0$ and ω_j^α depends on δ continuously.

Theorem 3.1. *The characteristic values $\omega_j^\alpha = \omega_j^\alpha(\delta),\ j = 1,2$ of $\mathcal{A}_\delta^{\alpha,\omega}$ can be approximated as*

$$\omega_j^\alpha = \sqrt{\frac{\delta\lambda_j^\alpha}{|D|}}v_b + O(\delta),$$

where $\lambda_j^\alpha,\ j = 1,2$ are eigenvalues of the quasi-periodic capacitance matrix

$$\begin{pmatrix} C_{11}^\alpha & C_{12}^\alpha \\ C_{21}^\alpha & C_{22}^\alpha \end{pmatrix}.$$

Proof. The integral equation (3.4) can be approximated by

$$\mathcal{S}_D^{\alpha,0}[\phi] - \mathcal{S}_D^{\alpha,0}[\psi] = O(\omega^2), \tag{3.5}$$

$$\left(-\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,0})^* + k_b^2\mathcal{K}_{D,1}^{\alpha,0}\right)[\phi] - \delta\left(\frac{1}{2}I + (\mathcal{K}_D^{\alpha,0})^*\right)[\psi] = O(\omega^4 + \delta^2). \tag{3.6}$$

Then, we get

$$\psi = \phi + O(\omega^2), \tag{3.7}$$

and inserting the above approximation into (3.6), we obtain that

$$\left(-\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,0})^* + k_b^2 \mathcal{K}_{D,1}^{\alpha,0}\right)[\phi] - \delta\left(\frac{1}{2}I + (\mathcal{K}_D^{\alpha,0})^*\right)[\phi] = O(\omega^4 + \delta^2). \tag{3.8}$$

Since $\ker\left(-\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,0})^*\right)$ is generated by ψ_1 and ψ_2, we may write ϕ as

$$\phi = a\psi_1 + b\psi_2 + O(\omega^2 + \delta), \tag{3.9}$$

where $|a| + |b| = 1$. Now, we integrate (3.8) on ∂D. Then, using (2.6), we get

$$-\frac{\omega^2|D|}{v_b^2}a + \delta(aC_{11}^\alpha + bC_{12}^\alpha) = O(\omega^4 + \delta^2), \tag{3.10}$$

$$-\frac{\omega^2|D|}{v_b^2}b + \delta(aC_{12}^\alpha + bC_{22}^\alpha) = O(\omega^4 + \delta^2). \tag{3.11}$$

Therefore, $\frac{\omega^2|D|}{\delta v_b^2}$ approximates the eigenvalues of the α-quasi-periodic capacitance matrix. This completes the proof. □

Since the quasi-periodic capacitance matrix has a double eigenvalue at Dirac points, we have that $\omega_1^{\alpha^*} = \omega_2^{\alpha^*} + O(\delta)$. The following proposition shows that in fact $\omega_1^{\alpha^*} = \omega_2^{\alpha^*}$ and that this is a double characteristic value.

Proposition 3.2. *For the Dirac point $\alpha = \alpha^*$, the first Bloch eigenfrequency $\omega^* := \omega(\alpha^*)$ is of multiplicity 2, i.e., $\mathcal{A}_\delta^{\alpha^*,\omega^*}$ has a two dimensional kernel.*

Proof. From the asymptotic expansion of the operator, the multiplicity of the characteristic value ω^* of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$ is at most 2. Suppose that ω^* is of multiplicity 1. Suppose also that

$$\mathcal{A}_\delta^{\alpha^*,\omega^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0$$

for a non-trivial pair (ϕ, ψ). Define $k^* = \omega^*/v$ and $k_b^* = \omega^*/v_b$ and let

$$u(x) = \begin{cases} \mathcal{S}_D^{\alpha^*,k^*}[\psi](x), & x \in Y \setminus \bar{D}, \\ \mathcal{S}_D^{\alpha^*,k_b^*}[\phi](x), & x \in D. \end{cases} \tag{3.12}$$

We can easily check that $T_1 u$ and $T_2 u$ also satisfy (2.1). Then u, $T_1 u$, and $T_2 u$ are linearly dependent and so,

$$T_1 u = c_1 u, \ T_2 u = c_2 u,$$

for some nonzero constants c_1 and c_2. Here, we observe that

$$T_1^3 = I, \ T_2^2 = I, \ T_1 T_2 f = T_2 T_1 f.$$

Then it follows that $c_1 = 1, \tau, \tau^2$, $c_2 = \pm 1$, and

$$c_1 c_2 u = T_1 T_2 u = T_2 T_1 u = \bar{c}_1 c_2 u.$$

Therefore, we get $c_1 = 1$, *i.e.*, $T_1 u = u$. This leads to a contradiction because we know from (3.9) that u is constant up to $O(\delta)$ on ∂D_j, $j = 1, 2$. This completes the proof. $\square$

Now, we can ascertain the approximate dependence of ω_j^α on α. Using the fact that $T_2 V_1^\alpha = V_2^\alpha$, we have

$$C_{11}^\alpha = C_{22}^\alpha.$$

From

$$V_1^{R^2\alpha} = \begin{cases} V_1^\alpha(R_1 x), & x \in Y_1, \\ e^{-i\alpha\cdot l_1} V_1^\alpha(R_2 x), & x \in Y_2, \end{cases}$$

$$V_2^{R^2\alpha} = \begin{cases} e^{i\alpha\cdot l_1} V_2^\alpha(R_1 x), & x \in Y_1, \\ V_2^\alpha(R_2 x), & x \in Y_2, \end{cases}$$

it follows that

$$C_{11}^\alpha = C_{11}^{R\alpha} = C_{11}^{R^2\alpha}, \quad C_{12}^{R^2\alpha} = e^{-i\alpha\cdot l_1} C_{12}^\alpha.$$

Differentiating these expressions, and applying $R\alpha^* = \alpha^*$, we arrive at

$$\nabla_\alpha C_{11}^\alpha \Big|_{\alpha=\alpha^*} = 0, \quad \nabla_\alpha C_{12}^\alpha \Big|_{\alpha=\alpha^*} = c \begin{pmatrix} 1 \\ i \end{pmatrix},$$

where $c = \frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha^*}$. Hence,

$$C_{11}^\alpha = C_{11}^{\alpha^*} + O(|\alpha-\alpha^*|^2), \quad |C_{12}^\alpha| = \left| \frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha^*} \right| |\alpha-\alpha^*| + O(|\alpha-\alpha^*|^2).$$

Because C^α is Hermitian with identical diagonal elements, the eigenvalues are given by $C_{11}^\alpha \pm |C_{12}^\alpha|$. For α close to α^*, we find the following asymptotic behaviour:

$$\begin{aligned} C_{11}^\alpha \pm |C_{12}^\alpha| &= C_{11}^{\alpha^*} \pm |\nabla_\alpha C_{12}^\alpha \cdot (\alpha-\alpha^*)| + O(|\alpha-\alpha^*|^2) \\ &= C_{11}^{\alpha^*} \pm \left| \frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha^*} \right| |\alpha-\alpha^*| + O(|\alpha-\alpha^*|^2). \end{aligned}$$

Now, we conclude that $\lambda_1^{\alpha^*} = \lambda_2^{\alpha^*} = C_{11}^{\alpha^*}$ and

$$\omega_j^\alpha(\delta) = \sqrt{\frac{\delta C_{11}^{\alpha^*}}{|D|}} v_b \left(1 \pm \frac{\left| \frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha^*} \right|}{C_{11}^{\alpha^*}} |\alpha-\alpha^*| + O(|\alpha-\alpha^*|^2) \right) + O(\delta). \tag{3.13}$$

Equation (3.13) gives the asymptotic band structure for small δ, and suggests that the system has a Dirac cone. However, we do not know the behaviour of the error term $O(\delta)$, so we can not conclude the existence of a Dirac cone from equation (3.13) alone. This will be addressed in the following section.

Remark 1. *Theorem 3.1 shows that ω_j^α scales like $O(\delta)$ for small δ. In [2], it was found that ω_M scales according to $\omega_M^2 \ln \omega_M = O(\delta)$ in two dimensions. Thus, ω_j^α has a different asymptotic behaviour than the Minnaert resonance ω_M of a single bubble in free-space. The difference is explained by the quasi-periodic single layer potential not exhibiting a* log-*singularity as $\omega \to 0$.*

4 Conical behaviour of subwavelength bands at Dirac points

In this section, we prove the main result of the paper. As before, let ω^* be the characteristic value at $\alpha^* = \frac{2\alpha_1+\alpha_2}{3}$, so that

$$\omega^* = \sqrt{\frac{\delta C_{11}^{\alpha^*}}{|D|}} v_b + O(\delta).$$

Theorem 4.1. *The first band and the second band form a Dirac cone at α^*, i.e.,*

$$\omega_1^\alpha(\delta) = \omega^* - \lambda|\alpha - \alpha^*|\big[1 + O(|\alpha - \alpha^*|)\big],$$
$$\omega_2^\alpha(\delta) = \omega^* + \lambda|\alpha - \alpha^*|\big[1 + O(|\alpha - \alpha^*|)\big],$$

for some constant $\lambda > 0$. Moreover,

$$\lambda = \sqrt{\frac{\delta}{|D|C_{11}^{\alpha^*}}} v_b \left| \frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha^*} \right| + O(\delta),$$

which is nonzero for sufficiently small δ.

The idea of the proof is to expand the operator $\mathcal{A}_\delta^{\alpha,k}$ for α close to α^* for a fixed, nonzero δ. For the proof, we will need Lemmas 4.2, 4.3, and 4.4. In the following, we will interchangeably use T_1, T_2 and T_1^α as operators on $L^2(\partial D)$ and as operators on $L^2(\partial D) \times L^2(\partial D)$ (the latter defined by applying the operator coordinate-wise). We begin with the first lemma.

Lemma 4.2. *For every ω and α,*

$$T_1^\alpha \mathcal{A}_\delta^{R^2\alpha,\omega} = \mathcal{A}_\delta^{\alpha,\omega} T_1^\alpha, \tag{4.1}$$

and

$$T_2 \mathcal{A}_\delta^{\alpha,\omega} = \mathcal{A}_\delta^{\alpha,\omega} T_2. \tag{4.2}$$

Proof. To prove (4.1), observe that $R\Lambda^* = \Lambda^*$. Therefore, it follows that

$$G^{R^2\alpha,k}(x-y) = \frac{1}{|Y|}\sum_{q\in\Lambda^*} \frac{e^{i(R^2\alpha+q)\cdot(x-y)}}{k^2 - |R^2\alpha+q|^2} = \frac{1}{|Y|}\sum_{q\in\Lambda^*} \frac{e^{iR(R^2\alpha+q)\cdot(Rx-Ry)}}{k^2 - |R\left(R^2\alpha+q\right)|^2} = G^{\alpha,k}(Rx-Ry).$$

Then, we can check that

$$\begin{aligned}
\mathcal{S}_{D_1}^{\alpha,k}[\psi(R_1 y)](x) &= \int_{\partial D_1} G^{\alpha,k}(x-y)\psi(R_1 y)d\sigma(y)\\
&= \int_{\partial D_1} G^{R^2\alpha,k}(R_1 x - R_1 y)\psi(R_1 y)d\sigma(y)\\
&= \mathcal{S}_{D_1}^{R^2\alpha,k}[\psi(y)](R_1 x)\\
&= \begin{cases} \mathcal{S}_{D_1}^{R^2\alpha,k}[\psi(y)](R_1 x), & x\in Y_1,\\ e^{-i\alpha\cdot l_1}\mathcal{S}_{D_1}^{R^2\alpha,k}[\psi(y)](R_2 x), & x\in Y_2.\end{cases}
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
\mathcal{S}_{D_2}^{\alpha,k}[\psi(R_2 y)](x) &= \mathcal{S}_{D_2}^{R^2\alpha,k}[\psi(y)](R_2 x)\\
&= \begin{cases} e^{i\alpha\cdot l_1}\mathcal{S}_{D_2}^{R^2\alpha,k}[\psi(y)](R_1 x), & x\in Y_1,\\ \mathcal{S}_{D_2}^{R^2\alpha,k}[\psi(y)](R_2 x), & x\in Y_2.\end{cases}
\end{aligned}$$

Thus, we get

$$\mathcal{S}_{D}^{\alpha,k}[T_1^\alpha\psi](x) = T_1^\alpha\mathcal{S}_{D}^{R^2\alpha,k}[\psi](x), \quad x\in Y.$$

This proves (4.1). Equation (4.2) follows easily from the fact that

$$\overline{G^{\alpha,k}(x-y)} = G^{\alpha,k}\big((2x_0-x)-(2x_0-y)\big).$$

This concludes the proof. □

Lemma 4.3. *There are two elements Φ_1,Φ_2 in the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$ which satisfy*

$$T_1\Phi_1=\tau\Phi_1,\ T_1\Phi_2=\tau^2\Phi_2,\ T_2\Phi_1=\Phi_2.$$

Proof. Let A be the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$, which, from Proposition 3.2, is of dimension two. By Lemma 4.2, T_1 and T_2 are operators from A onto itself. Since $T_1^3=I$ and $\ker(T_1-I)$ is trivial in A by the argument as in the proof of Proposition 3.2, there is an element $\Phi\in A$ such that either $T_1\Phi=\tau\Phi$ or $T_1\Phi=\tau^2\Phi$. In the case $T_1\Phi=\tau\Phi$, since

$$T_1(T_2\Phi_1)=T_2(T_1\Phi_1)=\tau^2T_2\Phi_1,$$

we let $\Phi_1:=\Phi$ and $\Phi_2:=T_2\Phi_1$ while in the second case, defining $\Phi_2:=\Phi$ and $\Phi_1:=T_2\Phi$ proves the claim.

□

Using the expansion (2.8), we decompose $\mathcal{A}_\delta^{\alpha,\omega}$ into

$$\begin{aligned}
\mathcal{A}_\delta^{\alpha,\omega} &= \mathcal{A}_\delta^{\alpha^*,\omega} + \begin{pmatrix}\mathcal{S}_1^{k_b} & -\mathcal{S}_1^{k}\\ \mathcal{K}_1^{k_b} & -\delta\mathcal{K}_1^{k}\end{pmatrix}\cdot(\alpha-\alpha^*)+O(|\alpha-\alpha^*|^2)\\
&:= \mathcal{A}_\delta^{\alpha^*,\omega}+\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)+O(|\alpha-\alpha^*|^2).
\end{aligned}\tag{4.3}$$

Here, $\cdot$ means the standard inner product taken component-wise. We observe that

$$(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} = \frac{T}{\omega - \omega^*}, + E^\omega$$

where T is an operator onto the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$ and E^ω is analytic in ω.

Next, we investigate some properties of T. It is easy to check that T vanishes on the range of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$. By Lemma 4.2, we have

$$TT_1 = T_1 T, \quad TT_2 = T_2 T.$$

We also have the following result.

Lemma 4.4. *For every α, on restriction to the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$, it holds that*

$$T(\mathcal{A}_{\delta,1}^{\omega^*} \cdot \alpha)T_1 = T_1 T(\mathcal{A}_{\delta,1}^{\omega^*} \cdot R^2\alpha).$$

Proof. By Lemma 4.2, we have

$$(\mathcal{A}_\delta^{\alpha,\omega})^{-1} T_1^\alpha = T_1^\alpha (\mathcal{A}_\delta^{R^2\alpha,\omega})^{-1}. \tag{4.4}$$

Using (4.3) and the Neumann series, we get

$$(\mathcal{A}_\delta^{\alpha,\omega})^{-1} = (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} + (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega} \cdot (\alpha - \alpha^*)(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} + O(|\alpha - \alpha^*|^2),$$
$$(\mathcal{A}_\delta^{R^2\alpha,\omega})^{-1} = (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} + (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega} \cdot (R^2\alpha - R^2\alpha^*)(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} + O(|\alpha - \alpha^*|^2).$$

We also expand T_1^α as

$$T_1^\alpha = T_1 + \hat{T}_1 + O(|\alpha - \alpha^*|^2),$$

where $\hat{T}_1$ is of order $O(|\alpha - \alpha^*|)$. Putting these asymptotic formulas into (4.4) and collecting terms of order $O(|\alpha - \alpha^*|)$, we get

$$\begin{aligned}&(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\hat{T}_1 + (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega} \cdot (\alpha - \alpha^*)(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} T_1 \\ &= \hat{T}_1 (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1} + T_1 (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega} \cdot (R^2\alpha - R^2\alpha^*)(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}.\end{aligned}$$

Applying $\mathcal{A}_\delta^{\alpha^*,\omega}$ on the above identity, we obtain

$$(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\hat{T}_1\mathcal{A}_\delta^{\alpha^*,\omega} + (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega} \cdot (\alpha - \alpha^*)T_1 = \hat{T}_1 + T_1(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega} \cdot (R^2\alpha - R^2\alpha^*).$$

Integrating over a small contour around ω^*, we have

$$T\hat{T}_1\mathcal{A}_\delta^{\alpha^*,\omega^*} + T\mathcal{A}_{\delta,1}^{\omega^*} \cdot (\alpha - \alpha^*)T_1 = T_1 T\mathcal{A}_{\delta,1}^{\omega^*} \cdot (R^2\alpha - R^2\alpha^*).$$

On the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$, the first term vanishes. Therefore, we get

$$T\mathcal{A}_{\delta,1}^{\omega^*} \cdot (\alpha - \alpha^*)T_1 = T_1 T\mathcal{A}_{\delta,1}^{\omega^*} \cdot (R^2\alpha - R^2\alpha^*).$$

on the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$. This completes the proof. $\square$

4.1 Proof of Theorem 4.1

Now, we are ready to prove our main result, Theorem 4.1. Let $\partial V(\omega^*)$ be a neighbourhood of ω^* containing only the two characteristic values $\omega_j^\alpha(\delta)$ for $j = 1, 2$. Then Gohberg-Sigal theory [3, 8] tells us that the characteristic values $\omega_1^\alpha(\delta)$ and $\omega_2^\alpha(\delta)$ of $\mathcal{A}_\delta^{\alpha,\omega}$ near α^* satisfy

$$f(\omega_1^\alpha(\delta)) + f(\omega_2^\alpha(\delta)) = \frac{\mathrm{tr}}{2\pi i}\int_{\partial V(\omega^*)} (\mathcal{A}_\delta^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}_\delta^{\alpha,\omega} f(\omega)\, d\omega,$$

for an analytic function $f(\omega)$. As in [3, 8], we have

$$f(\omega_1^\alpha(\delta)) + f(\omega_2^\alpha(\delta)) = -\frac{\mathrm{tr}}{2\pi i}\int_{\partial V(\omega^*)} \sum_{p=1}^{\infty} f(\omega)\frac{d}{d\omega}\left[(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}(\mathcal{A}_\delta^{\alpha^*,\omega} - \mathcal{A}_\delta^{\alpha,\omega})\right]^p d\omega.$$

Using this formula twice, with $f(\omega) = \omega - \omega^*$ and $f(\omega) = (\omega - \omega^*)^2$, we have

$$\begin{aligned}
\omega_1^\alpha(\delta) - \omega^* + \omega_2^\alpha(\delta) - \omega^* &= -\frac{\mathrm{tr}}{2\pi i}\int_{\partial V(\omega^*)} (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)d\omega + O(|\alpha-\alpha^*|^2),\\
(\omega_1^\alpha(\delta) - \omega^*)^2 + (\omega_2^\alpha(\delta) - \omega^*)^2 &= \frac{\mathrm{tr}}{2\pi i}\int_{\partial V(\omega^*)} 2(\omega-\omega^*)\left[(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)\right]^2 d\omega\\
&\quad + O(|\alpha-\alpha^*|^3),
\end{aligned}$$

where we have used integration by parts in the second equation. To finish the proof, it suffices to show that

$$\begin{aligned}
\frac{\mathrm{tr}}{2\pi i}\int_{\partial V(\omega^*)} (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)d\omega &= 0,\\
\frac{\mathrm{tr}}{2\pi i}\int_{\partial V(\omega^*)} 2(\omega-\omega^*)\left[(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)\right]^2 d\omega &= C|\alpha-\alpha^*|^2,
\end{aligned}$$

for some constant C. Indeed, this would imply that $\omega_j^\alpha(\delta) = \omega^* \pm \lambda|\alpha-\alpha^*|[1+O(|\alpha-\alpha^*|)]$, and the expression for λ follows from (3.13).

We see that

$$\frac{1}{2\pi i}\int_{\partial V(\omega^*)} (\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)d\omega = T\mathcal{A}_{\delta,1}^{\omega^*}\cdot(\alpha-\alpha^*), \tag{4.5}$$

is an operator from the kernel of $\mathcal{A}_\delta^{\alpha^*,\omega^*}$ onto itself. Similarly, we get

$$\frac{1}{2\pi i}\int_{\partial V(\omega^*)} (\omega-\omega^*)\left[(\mathcal{A}_\delta^{\alpha^*,\omega})^{-1}\mathcal{A}_{\delta,1}^{\omega}\cdot(\alpha-\alpha^*)\right]^2 d\omega = [T\mathcal{A}_{\delta,1}^{\omega^*}\cdot(\alpha-\alpha^*)]^2. \tag{4.6}$$

For $\alpha = \begin{pmatrix}\alpha_{(1)}\\ \alpha_{(2)}\end{pmatrix}$, let

$$\mathcal{A}_{\delta,1}^{\omega^*}\cdot\alpha = \mathcal{A}_{\delta,11}^{\omega^*}\alpha_{(1)} + \mathcal{A}_{\delta,12}^{\omega^*}\alpha_{(2)}.$$

Suppose that

$$T\mathcal{A}^{\omega}_{\delta,11}[\Phi_1] = a\Phi_1 + b\Phi_2,$$
$$T\mathcal{A}^{\omega}_{\delta,12}[\Phi_1] = c\Phi_1 + d\Phi_2.$$

Since $R^2\alpha = \frac{1}{2}\begin{pmatrix} -\alpha_{(1)}-\sqrt{3}\alpha_{(2)} \\ \sqrt{3}\alpha_{(1)}-\alpha_{(2)} \end{pmatrix}$ and $T_1\Phi_1 = \tau\Phi_1$, we obtain

$$T(\mathcal{A}^{\omega^*}_{\delta,1}\cdot\alpha)T_1[\Phi_1] = \tau\big(\alpha_{(1)}(a\Phi_1+b\Phi_2)+\alpha_{(2)}(c\Phi_1+d\Phi_2)\big)$$
$$\begin{aligned} T_1T(\mathcal{A}^{\omega^*}_{\delta,1}\cdot R^2\alpha)[\Phi_1] &= T_1\left[\frac{-\alpha_{(1)}-\sqrt{3}\alpha_{(2)}}{2}(a\Phi_1+b\Phi_2)+\frac{\sqrt{3}\alpha_{(1)}-\alpha_{(2)}}{2}(c\Phi_1+d\Phi_2)\right] \\ &= \frac{-\alpha_{(1)}-\sqrt{3}\alpha_{(2)}}{2}(a\tau\Phi_1+b\tau^2\Phi_2)+\frac{\sqrt{3}\alpha_{(1)}-\alpha_{(2)}}{2}(c\tau\Phi_1+d\tau^2\Phi_2). \end{aligned}$$

By Lemma 4.4, it follows that

$$2a = -a+\sqrt{3}c,\ 2b = -\tau b+\sqrt{3}\tau d,\ 2c = -\sqrt{3}a - c,\ 2d = -\sqrt{3}\tau b - \tau d.$$

Solving these equations we obtain $a = c = 0,\ d = -\frac{\sqrt{3}\tau}{2+\tau}b = -ib$, which means

$$T\mathcal{A}^{\omega^*}_{\delta,11}[\Phi_1] = b\Phi_2,\ T\mathcal{A}^{\omega^*}_{\delta,12}[\Phi_1] = -ib\Phi_2.$$

Similarly, we have

$$T\mathcal{A}^{\omega^*}_{\delta,11}[\Phi_2] = \tilde{a}\Phi_1,\ T\mathcal{A}^{\omega^*}_{\delta,12}[\Phi_2] = i\tilde{a}\Phi_1,$$

for some constant $\tilde{a}$. Now, we arrive at

$$T\mathcal{A}^{\omega^*}_{\delta,1}\cdot(\alpha-\alpha^*)[\Phi_1] = b\left((\alpha_{(1)}-\alpha^*_{(1)}) - i(\alpha_{(2)}-\alpha^*_{(2)})\right)\Phi_2,$$
$$T\mathcal{A}^{\omega^*}_{\delta,1}\cdot(\alpha-\alpha^*)[\Phi_2] = \tilde{a}\left((\alpha_{(1)}-\alpha^*_{(1)}) + i(\alpha_{(2)}-\alpha^*_{(2)})\right)\Phi_1.$$

Therefore, we can conclude that

$$\operatorname{tr} T\mathcal{A}^{\omega^*}_{\delta,1}\cdot(\alpha-\alpha^*) = 0,$$
$$\operatorname{tr}\,[T\mathcal{A}^{\omega^*}_{\delta,1}\cdot(\alpha-\alpha^*)]^2 = 2\tilde{a}b|\alpha-\alpha^*|^2,$$

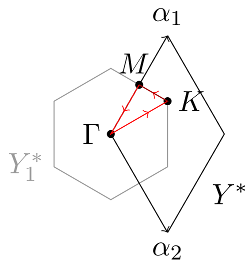
which together with (4.5) and (4.6) complete the proof. □

5 Numerical illustrations

We consider the bubbly honeycomb crystal as described in Section 2, and additionally assume that the bubbles are circular with radius R. The center-to-center distance between adjacent bubbles is assumed to be one and the material parameters are such that $v = v_b = 1$. The lattice basis vectors are given by

$$l_1 = \left(3, \sqrt{3}\right),\quad l_2 = \left(3, -\sqrt{3}\right),$$

and the reciprocal basis vectors are defined as



$$\alpha_1 = 2\pi \left(\frac{1}{6}, \frac{1}{2\sqrt{3}} \right), \quad \alpha_2 = 2\pi \left(\frac{1}{6}, -\frac{1}{2\sqrt{3}} \right).$$

Using the same notation as in Section 2, this corresponds to $a = 2\sqrt{3}$. We also define the symmetry points in the reciprocal space as follows:

$$\Gamma = (0,0), \quad K = \frac{2\alpha_1 + \alpha_2}{3} \quad M = \frac{\alpha_1}{2}.$$

We use the multipole expansion method to compute the band diagrams [4]. Because D has two connected components D_1 and D_2, we can identify $L^2(\partial D) = L^2(\partial D_1) \times L^2(\partial D_2)$. This gives the following matrix expressions of $\mathcal{S}_D^{\alpha,k}$ and $(\mathcal{K}_D^{-\alpha,k})^*$:

$$\mathcal{S}_D^{\alpha,k}[\phi] = \begin{pmatrix} \mathcal{S}_{D_1}^{\alpha,k} & \mathcal{S}_{D_2}^{\alpha,k} \\ \mathcal{S}_{D_1}^{\alpha,k} & \mathcal{S}_{D_2}^{\alpha,k} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (\mathcal{K}_D^{-\alpha,k})^*[\phi] = \begin{pmatrix} (\mathcal{K}_{D_1}^{-\alpha,k})^* & \frac{\partial}{\partial \nu}\mathcal{S}_{D_2}^{\alpha,k} \\ \frac{\partial}{\partial \nu}\mathcal{S}_{D_1}^{\alpha,k} & (\mathcal{K}_{D_2}^{-\alpha,k})^* \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Here, $\phi \in L^2(\partial D)$ is represented by $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in L^2(\partial D_1) \times L^2(\partial D_2)$. Using these expressions, the integral operator $\mathcal{A}_\delta^{\alpha,\omega}$ defined in equation (3.3) can be discretised with the multipole expansion method as described in [4, Appendix C]. We consider the band structure along the line $M\Gamma KM$ in the following numerical examples:

(i) (Dilute regime). We set $R = 1/50$ and $\delta = 1/9000$. The band structure is given in Figure 2. The left subfigure shows the first four bands. The right subfigure shows the first two bands, which correspond to subwavelength curves and which cross at K. Observe that the crossing is a linear dispersion which means that it signifies a Dirac point.

(ii) (Non-dilute regime) We set $R = 1/5$ and $\delta = 1/1000$. The band structure is given in Figure 3. In this non-dilute regime, there is still a Dirac cone at the point K.

6 Concluding remarks

In this paper, we have rigorously proven the existence of a Dirac cone in the subwavelength regime in a bubbly phononic crystal with a honeycomb lattice structure. We have illustrated our main results with different numerical experiments. In future works, we plan to further study topological phenomena in bubbly crystals. In particular, we will rigorously show the existence of localized edge states at the surface of a topologically non-trivial bubbly crystal. Similar to [9], we will also study a high-frequency homogenization of a bubbly honeycomb phononic crystal.

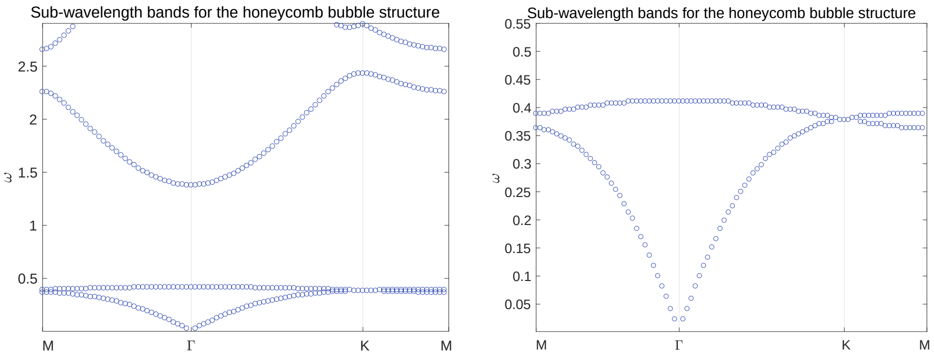


Figure 2: (left) The band structure of a bubbly honeycomb phononic crystal with $R = 1/50$ and $\delta = 1/9000$. The distance between the adjacent bubbles is one. (right) The band structure upon zooming in on the subwavelength region.

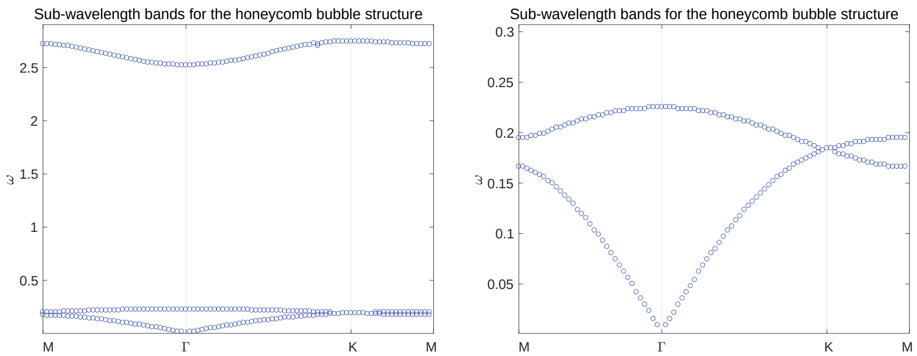


Figure 3: (left) The band structure of a bubbly honeycomb phononic crystal with $R = 1/5$ and $\delta = 1/1000$. The distance between the adjacent bubbles is one. (right) The band structure upon zooming in on the subwavelength region.

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