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Abstract

The Alekseev-Gröbner formula is a well known tool in numerical analysis for describing the effect that a perturbation of an ordinary differential equation (ODE) has on its solution. In this article we provide an extension of the Alekseev-Gröbner formula for Banach space valued ODEs under, loosely speaking, mild conditions on the perturbation of the considered ODEs.

Contents

1 Introduction

The Alekseev-Gröbner formula (see, e.g., Alekseev [1], Gröbner [6], and Hairer et al. [7, Theorem 14.5 in Chapter I]) is a well known tool in deterministic numerical analysis for describing the effect that a perturbation of an ordinary differential equation (ODE) has on its solution. Considering numerical methods for ODEs as appropriate perturbations of the underlying equations makes the Alekseev-Gröbner formula applicable for estimating errors of numerical methods (see, e.g., Hairer et al. [7, Theorem 7.9 in Chapter II], Iserles [8, Theorem 3.7], Iserles [9, Theorem 1], and Niesen [11, Theorem 1]). It is the main contribution of this work to provide an extension of the Alekseev-Gröbner formula for Banach space valued ODEs under, loosely speaking, mild conditions on the perturbation of the considered ODEs (see Corollary 5.2 in Section 5 and Theorem 1.1 below). As a consequence, our main result is well suited for the analysis of pathwise approximation errors between exact solutions of stochastic partial differential equations (SPDEs) of evolutionary type and their numerical approximations. In particular, it can be used as a key ingredient for establishing strong convergence rates for numerical approximations of SPDEs with a non-globally Lipschitz continuous, non-globally monotone nonlinearity, and additive trace-class noise. The precise result will be the subject of a future research article. In this introductory section we now present our main result. Theorem 1.1 is proven as Corollary 5.2 in Section 5 below.

Theorem 1.1. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0,\infty)$, let $f\colon [0,T]\times V \to V$ be a continuous function, assume for all $t \in [0,T]$ that $V \ni x \mapsto f(t,x) \in V$ is Fréchet differentiable, assume that $[0,T] \times V \ni (t,x) \mapsto (\frac{\partial}{\partial x} f)(t,x) \in L(V)$ is continuous, for every $x \in V$, $s \in [0,T]$ let $X^x_{s,(\cdot)}\colon [s,T] \to V$ be a continuous function which satisfies for all $t \in [s,T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau$, and let $Y, E\colon [0,T] \to V$ be strongly measurable functions which satisfy for all $t \in [0,T]$ that $\int_0^T [\|f(\tau, Y_\tau)\|_V + \|E_\tau\|_V]\, d\tau < \infty$ and $Y_t = Y_0 + \int_0^t [f(\tau, Y_\tau) + E_\tau]\, d\tau$. Then*

(i) it holds for all $s \in [0,T]$, $t \in [s,T]$ that $V \ni x \mapsto X^x_{s,t} \in V$ is Fréchet differentiable,

(ii) it holds for all $t \in [0,T]$ that $[0,t] \ni \tau \mapsto (\frac{\partial}{\partial x} X^{Y_\tau}_{\tau,t}) E_\tau \in V$ is strongly measurable,

(iii) it holds for all $t \in [0,T]$ that $\int_0^t \|(\frac{\partial}{\partial x} X^{Y_\tau}_{\tau,t}) E_\tau\|_V\, d\tau < \infty$, and

(iv) it holds for all $s \in [0,T]$, $t \in [s,T]$ that

$$Y_t = X^{Y_s}_{s,t} + \int_s^t (\tfrac{\partial}{\partial x} X^{Y_\tau}_{\tau,t}) E_\tau\, d\tau. \tag{1}$$

The rest of this article is structured as follows. In Section 2 we recall some elementary and well known properties for Banach space valued functions (see Lemmas 2.1–2.6, Corollary 2.7, and Lemma 2.8). Thereafter we combine these elementary results to prove an abstract version of the Alekseev-Gröbner formula for Banach space valued ODEs under, roughly speaking, restrictive conditions on the solution as well as on the perturbation of the considered ODE; see Proposition 2.9 in Section 2 below for details. Sections 3 and 4 are devoted to presenting in detail some partially well known results on continuous differentiability of solutions to a class of Banach space valued ODEs with respect to initial value, initial time, and current time (see Lemma 4.8 in Section 4 below). Finally, we combine Proposition 2.9, Lemma 3.7 (the flow property of solutions to ODEs), and Lemma 4.8 to establish in Corollary 5.2 in Section 5 below the main result of this article.

2 Extended chain rule property for Banach space valued functions

In this section we prove an abstract version of the Alekseev-Gröbner formula for Banach space valued ODEs under, loosely speaking, restrictive conditions in Proposition 2.9. This will be used in Section 5 to prove in Corollary 5.2 an extension of the Alekseev-Gröbner formula (cf., e.g., Hairer et al. [7, Theorem 14.5 in Chapter I]) for Banach space valued functions. In order to prove Proposition 2.9 we first recall some elementary auxiliary lemmas; see Lemmas 2.1–2.6, Corollary 2.7, and Lemma 2.8. In particular, we recall the fundamental theorem of calculus for Banach space valued functions in Lemmas 2.2–2.4 (cf., e.g., Prévôt & Röckner [12, Proposition A.2.3]) and we derive suitable extensions thereof in Lemma 2.6 (cf., e.g., [10, Lemma 2.1]) and Corollary 2.7. Thereafter, we combine Lemma 2.6, Corollary 2.7, and Lemma 2.8 (cf., e.g., Rudin [13, Theorem 7.17 and the remark thereafter]) to establish Proposition 2.9.

Lemma 2.1. *Let $(X, \mathcal{X})$ be a separable topological space, let $(Y, \mathcal{Y})$ be a topological space, and let $f \in \mathcal{C}(X, Y)$. Then f is strongly measurable.*

Proof of Lemma 2.1. Throughout this proof let $A \subseteq X$ be a countable dense subset of X. Note that the assumption that f is continuous ensures that for all $V \in \mathcal{Y}$ with $f(X) \cap V \neq \emptyset$ it holds that

$$\emptyset \neq \{x \in X : f(x) \in V\} \in \mathcal{X}. \tag{2}$$

This and the fact that $A \subseteq X$ is dense imply that for every $V \in \mathcal{Y}$ with $f(X) \cap V \neq \emptyset$ there exists $a \in A$ such that $f(a) \in V$. The fact that $A \subseteq X$ is countable therefore implies that $f(A) \subseteq f(X)$ is a countable dense subset of $f(X)$. This and the fact that f is measurable complete the proof of Lemma 2.1. ☐

Lemma 2.2. *Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a, \infty)$, and let $f \in \mathcal{C}([a, b], V)$. Then it holds for every $t \in [a, b]$ that $\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \|\frac{1}{h}\int_t^{t+h} f(s)\,ds - f(t)\|_V = 0$.*

Proof of Lemma 2.2. Throughout this proof let $(c_n)_{n\in\mathbb{N}} \subseteq \mathrm{im}(f)$ be a dense subset of $\mathrm{im}(f)$. Note that the fundamental theorem of calculus assures that for all $t \in [a, b]$, $n \in \mathbb{N}$ it holds that

$$\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \left| \frac{1}{h} \int_t^{t+h} \|f(s) - c_n\|_V \, ds - \|f(t) - c_n\|_V \right| = 0. \tag{3}$$

This implies for all $t \in [a, b]$, $n \in \mathbb{N}$ that

$$\begin{aligned}
&\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{1}{|h|} \int_{\min\{t,t+h\}}^{\max\{t,t+h\}} \|f(s) - f(t)\|_V \, ds \\
&\leq \limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{1}{|h|} \int_{\min\{t,t+h\}}^{\max\{t,t+h\}} \Big(\|f(s) - c_n\|_V + \|c_n - f(t)\|_V \Big) \, ds \\
&= \limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{1}{|h|} \int_{\min\{t,t+h\}}^{\max\{t,t+h\}} \|f(s) - c_n\|_V \, ds \\
&\quad + \limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{1}{|h|} \int_{\min\{t,t+h\}}^{\max\{t,t+h\}} \|c_n - f(t)\|_V \, ds \\
&= \|f(t) - c_n\|_V + \limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{|h|}{|h|} \|c_n - f(t)\|_V = 2\|f(t) - c_n\|_V.
\end{aligned} \tag{4}$$

The fact that $(c_n)_{n\in\mathbb{N}} \subseteq \operatorname{im}(f)$ is dense hence ensures for all $t \in [a,b]$ that

$$\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|_V \, ds = 0. \tag{5}$$

Therefore, we obtain for every $t \in [a,b]$ that

$$\begin{split} &\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \left\| \tfrac{1}{h} \int_t^{t+h} f(s)\, ds - f(t) \right\|_V \\ &\leq \limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \tfrac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|_V \, ds = 0. \end{split} \tag{6}$$

This completes the proof of Lemma 2.2. ☐

Lemma 2.3. *Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space and let $a \in \mathbb{R}$, $b \in (a,\infty)$, $f \in \mathcal{C}([a,b],V)$, $F\colon [a,b] \to V$ satisfy for all $t \in [a,b]$ that $F(t) = F(a) + \int_a^t f(s)\, ds$. Then it holds for every $t \in [a,b]$ that $\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \|\frac{1}{h}(F(t+h) - F(t) - f(t)h)\|_V = 0$.*

Proof of Lemma 2.3. Note that for all $t \in [a,b]$, $h \in [a-t, b-t]\backslash\{0\}$ it holds that

$$\tfrac{F(t+h)-F(t)}{h} = \tfrac{1}{h} \int_t^{t+h} f(s)\, ds. \tag{7}$$

Combining this with Lemma 2.2 (with $V = V$, $a = a$, $b = b$, $f = f$ in the notation of Lemma 2.2) implies for all $t \in [a,b]$ that

$$\limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \left\| \tfrac{F(t+h)-F(t)}{h} - f(t) \right\|_V = \limsup_{([a-t,b-t]\backslash\{0\})\ni h\to 0} \left\| \tfrac{1}{h} \int_t^{t+h} f(s)\, ds - f(t) \right\|_V = 0. \tag{8}$$

The proof of Lemma 2.3 is thus completed. ☐

Lemma 2.4. *Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space and let $a \in \mathbb{R}$, $b \in (a,\infty)$, $F \in \mathcal{C}^1([a,b],V)$. Then it holds for all $t \in [a,b]$ that*

$$F(t) - F(a) = \int_a^t F'(s)\, ds. \tag{9}$$

Proof of Lemma 2.4. Throughout this proof let $G\colon [a,b] \to V$ be the function which satisfies for all $t \in [a,b]$ that $G(t) - F(a) = \int_a^t F'(s)\, ds$ and for every $L \in L(V,\mathbb{R})$ let $p_L\colon [a,b] \to \mathbb{R}$ be the function which satisfies for all $t \in (a,b)$ that $p_L(t) = L(F(t) - G(t))$. Observe that Lemma 2.3 (with $V = V$, $a = a$, $b = b$, $f = F'$, $F = G$ in the notation of Lemma 2.3) shows that for all $t \in [a,b]$ it holds that

$$G'(t) = F'(t). \tag{10}$$

This and the fact that $F' \in \mathcal{C}([a,b],V)$ ensure that for all $L \in L(V,\mathbb{R})$, $t \in (a,b)$ it holds that $p_L \in \mathcal{C}^1([a,b],\mathbb{R})$ and $p_L'(t) = 0$. This proves that for every $L \in L(V,\mathbb{R})$ the function p_L is constant. The fact that the space $L(V,\mathbb{R})$ separates points (see, e.g., Brezis [2, Corollary 1.3 in Chaper 1]) hence shows that $[a,b] \ni t \mapsto F(t) - G(t)$ is a constant function. This demonstrates that for all $t \in [a,b]$ it holds that

$$F(t) - F(a) = \int_a^t F'(s)\, ds. \tag{11}$$

The proof of Lemma 2.4 is thus completed. ☐

Lemma 2.5. *Let (V, d_V) and (W, d_W) be metric spaces and let $a \in \mathbb{R}$, $b \in (a, \infty)$, $g \in \mathcal{C}([a,b] \times V, W)$. Then it holds that the function*

$$V \ni v \mapsto ([a,b] \ni t \mapsto g(t,v) \in W) \in \mathcal{C}([a,b], W) \tag{12}$$

is continuous.

Proof of Lemma 2.5. Throughout this proof let $\varepsilon \in (0, \infty)$. Observe that the assumption that $g \in \mathcal{C}([a,b] \times V, W)$ implies that there exists a function $\delta \colon [a,b] \times V \to (0, \infty)$ such that for all $x, x_0 \in V$, $s, t \in [a,b]$ with $\max\{|s-t|, d_V(x, x_0)\} < \delta_{t,x_0}$ it holds that

$$d_W(g(s,x), g(t,x_0)) < \tfrac{\varepsilon}{2}. \tag{13}$$

Moreover, note that the fact that $[a,b]$ is compact ensures that for every $x_0 \in V$ there exist $n \in \mathbb{N}$, $t_1, \ldots, t_n \in [a,b]$ such that $a = t_1 < \ldots < t_n = b$ and

$$[a,b] = \cup_{i=1}^n \{r \in [a,b] \colon |r - t_i| < \delta_{t_i, x_0}\}. \tag{14}$$

This and (13) demonstrate that for all $x_0 \in V$ there exist $n \in \mathbb{N}$, $t_1, \ldots, t_n \in [a,b]$ such that for all $x \in V$, $t \in [a,b]$ with $d_V(x, x_0) < \min\{\delta_{t_1,x_0}, \ldots, \delta_{t_n,x_0}\}$ it holds that

$$d_W(g(t,x), g(t,x_0)) \le \min_{i \in [0,n] \cap \mathbb{N}} \left| d_W(g(t,x), g(t_i, x_0)) + d_W(g(t_i, x_0), g(t, x_0)) \right| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \tag{15}$$

As $\varepsilon \in (0, \infty)$ was arbitrary, the proof of Lemma 2.5 is completed. □

Lemma 2.6. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a, \infty)$, $\phi \in \mathcal{C}^1(V, W)$, $F \colon [a,b] \to V$, and let $f \colon [a,b] \to V$ be a strongly measurable function which satisfies for all $t \in [a,b]$ that $\int_a^b \|f(s)\|_V \, ds < \infty$ and $F(t) - F(a) = \int_a^t f(s) \, ds$. Then*

(i) it holds that $[a,b] \ni s \mapsto \phi'(F(s)) f(s) \in W$ is strongly measurable,

(ii) it holds that $\int_a^b \|\phi'(F(s)) f(s)\|_W \, ds < \infty$, and

(iii) it holds for all $t \in [a,b]$ that $\phi(F(t)) - \phi(F(a)) = \int_a^t \phi'(F(s)) f(s) \, ds$.

Proof of Lemma 2.6. Throughout this proof let $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}([a,b], V)$ be functions which satisfy $\limsup_{n \to \infty} \int_a^b \|f_n(s) - f(s)\|_V \, ds = 0$ and let $F_n \colon [a,b] \to V$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$, $t \in [a,b]$ that $F_n(t) = F(a) + \int_a^t f_n(s) \, ds$. Observe that the fact that f is strongly measurable, the fact that the function $[a,b] \ni s \mapsto \phi'(F(s)) \in L(V, W)$ is continuous, and the fact that the function $V \times L(V, W) \ni (v, A) \mapsto Av \in W$ is continuous implies that $[a,b] \ni s \mapsto \phi'(F(s)) f(s) \in W$ is strongly measurable. The fact that the function $[a,b] \ni s \mapsto \phi'(F(s)) \in L(V, W)$ is continuous and the assumption that f is strongly measurable and integrable hence establish items (i) and (ii). Next observe that Lemma 2.3 (with $V = V$, $a = a$, $b = b$, $f = f_n$, $F = F_n$ for $n \in \mathbb{N}$ in the notation of Lemma 2.3) ensures that for every $n \in \mathbb{N}$ it holds that $F_n \in \mathcal{C}^1([a,b], V)$ and $F_n' = f_n$. Lemma 2.4 (with $V = W$, $a = a$, $b = b$, $F = \phi \circ F_n$ for $n \in \mathbb{N}$ in the notation of Lemma 2.4) and the chain rule for Fréchet derivatives therefore prove that for all $n \in \mathbb{N}$, $t \in [a,b]$ it holds that

$$\phi(F_n(t)) - \phi(F_n(a)) = \int_a^t \phi'(F_n(s)) f_n(s) \, ds. \tag{16}$$

Moreover, note that for all $n \in \mathbb{N}$, $t \in [a,b]$ it holds that

$$\|F_n(t) - F(t)\|_V = \left\| \int_a^t [f_n(s) - f(s)] \, ds \right\|_V \leq \int_a^b \|f_n(s) - f(s)\|_V \, ds. \tag{17}$$

This assures that

$$\limsup_{n\to\infty} (\sup_{s\in[a,b]} \|F_n(s) - F(s)\|_V) = 0. \tag{18}$$

The fact that $\phi \in \mathcal{C}^1(V,W)$ hence shows that for every $t \in [a,b]$ it holds that

$$\limsup_{n\to\infty} \|\phi(F_n(t)) - \phi(F(t))\|_W = 0 \tag{19}$$

and

$$\limsup_{n\to\infty} \|\phi'(F_n(t)) - \phi'(F(t))\|_{L(V,W)} = 0. \tag{20}$$

Next observe that for all $n \in \mathbb{N}$, $t \in [a,b]$ it holds that

$$\begin{aligned}
&\int_a^t \|\phi'(F_n(s))f_n(s) - \phi'(F(s))f(s)\|_W \, ds \\
&\leq \int_a^t \|\phi'(F_n(s))\|_{L(V,W)} \|f_n(s) - f(s)\|_V \, ds + \int_a^t \|\phi'(F_n(s)) - \phi'(F(s))\|_{L(V,W)} \|f(s)\|_V \, ds \\
&\leq \sup_{r\in[a,b]} \|\phi'(F(r) + (F_n(r) - F(r)))\|_{L(V,W)} \int_a^b \|f_n(s) - f(s)\|_V \, ds \\
&\quad + \sup_{r\in[a,b]} \|\phi'(F(r) + (F_n(r) - F(r))) - \phi'(F(r))\|_{L(V,W)} \int_a^b \|f(s)\|_V \, ds.
\end{aligned} \tag{21}$$

Lemma 2.5 (with $V = V$, $d_V = (V \times V \ni (v_1, v_2) \mapsto \|v_1 - v_2\|_V \in \mathbb{R})$, $W = L(V,W)$, $d_W = (W \times W \ni (w_1, w_2) \mapsto \|w_1 - w_2\|_W \in \mathbb{R})$, $a = a$, $b = b$, $g = ([a,b] \times V \ni (t,x) \mapsto \phi'(F(t)+x) \in L(V,W))$ in the notation of Lemma 2.5) and (18) prove that for every $\varepsilon \in (0,\infty)$ there exists $N \in \mathbb{N}$ such that for every $n \in [N,\infty) \cap \mathbb{N}$ it holds that

$$\sup_{r\in[a,b]} \|\phi'(F(r) + (F_n(r) - F(r))) - \phi'(F(r))\|_{L(V,W)} < \varepsilon. \tag{22}$$

In particular, this and the fact that $\phi' \circ F \in \mathcal{C}([a,b], L(V,W))$ imply that

$$\sup_{n\in\mathbb{N}} \sup_{r\in[a,b]} \|\phi'(F(r) + (F_n(r) - F(r)))\|_{L(V,W)} < \infty. \tag{23}$$

Combining (21)–(23) and the fact that $\limsup_{n\to\infty} \int_a^b \|f_n(s) - f(s)\|_V \, ds = 0$ ensures that for every $t \in [a,b]$ it holds that

$$\limsup_{n\to\infty} \left\| \int_a^t \phi'(F_n(s))f_n(s) \, ds - \int_a^t \phi'(F(s))f(s) \, ds \right\|_W = 0. \tag{24}$$

Moreover, observe that (19) shows that for all $t \in [a,b]$ it holds that

$$\limsup_{n\to\infty} \|\phi(F_n(t)) - \phi(F_n(a)) - [\phi(F(t)) - \phi(F(a))]\|_W = 0. \tag{25}$$

This, (16), and (24) establish item (iii). The proof of Lemma 2.6 is thus completed. ☐

Corollary 2.7. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a,\infty)$, $F\colon [a,b] \to V$, $\phi \in \mathcal{C}^1([a,b] \times V, W)$, $\phi_{1,0}\colon [a,b] \times V \to W$, $\phi_{0,1}\colon [a,b] \times V \to L(V,W)$ satisfy for all $t \in [a,b]$, $x \in V$ that $\phi_{1,0}(t,x) = (\frac{\partial}{\partial t}\phi)(t,x)$, $\phi_{0,1}(t,x) = (\frac{\partial}{\partial x}\phi)(t,x)$, and let $f\colon [a,b] \to V$ be a strongly measurable function which satisfies for all $t \in [a,b]$ that $\int_a^b \|f(s)\|_V \, ds < \infty$ and $F(t) - F(a) = \int_a^t f(s)\, ds$. Then*

(i) it holds that $[a,b] \ni s \mapsto [\phi_{1,0}(s,F(s)) + \phi_{0,1}(s,F(s))f(s)] \in W$ is strongly measurable,

(ii) it holds that $\int_a^b \|\phi_{1,0}(s,F(s)) + \phi_{0,1}(s,F(s))f(s)\|_W \, ds < \infty$, and

(iii) it holds for all $t \in [a,b]$ that

$$\phi(t,F(t)) - \phi(a,F(a)) = \int_a^t \Big[\phi_{1,0}(s,F(s)) + \phi_{0,1}(s,F(s))f(s)\Big]\, ds. \tag{26}$$

Proof of Corollary 2.7. Throughout this proof let $\Phi \in \mathcal{C}^1(\mathbb{R} \times V, W)$ be a function which satisfies for all $t \in [a,b]$, $x \in V$ that $\Phi(t,x) = \phi(t,x)$. Note that Lemma 2.6 (with $V = \mathbb{R} \times V$, $W = W$, $a = a$, $b = b$, $\phi = \Phi$, $F = ([a,b] \ni s \mapsto (s, F(s)) \in \mathbb{R} \times V)$, $f = ([a,b] \ni s \mapsto (1, f(s)) \in \mathbb{R} \times V)$ in the notation of Lemma 2.6) establishes items (i)–(iii). The proof of Corollary 2.7 is thus completed. ☐

Lemma 2.8. *Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a,\infty)$, $t_0 \in [a,b]$, $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^1([a,b],V)$, and assume that $(f_n(t_0))_{n\in\mathbb{N}} \subseteq V$ and $((f_n)')_{n\in\mathbb{N}} \subseteq \mathcal{C}([a,b],V)$ are convergent. Then there exists $F \in \mathcal{C}^1([a,b],V)$ such that for every $t \in [a,b]$ it holds that*

$$\limsup_{n\to\infty} \Big(\|f_n(t) - F(t)\|_V + \|(f_n)'(t) - F'(t)\|_V\Big) = 0. \tag{27}$$

Proof of Lemma 2.8. Throughout this proof let $F, g\colon [a,b] \to V$ be the functions which satisfy for all $t \in [a,b]$ that $\limsup_{n\to\infty} \sup_{s\in[0,T]} \|(f_n)'(s) - g(s)\|_V = 0$, $F(t_0) = \lim_{n\to\infty} f_n(t_0)$, and

$$F(t) = F(t_0) + \int_{t_0}^t g(s)\, ds. \tag{28}$$

Observe that Lemma 2.4 (with $V = V$, $a = a$, $b = b$, $F = f_n$ for $n \in \mathbb{N}$ in the notation of Lemma 2.4) shows that for all $n \in \mathbb{N}$, $t \in [a,b]$ it holds that

$$f_n(t) = f_n(t_0) + \int_{t_0}^t (f_n)'(s)\, ds. \tag{29}$$

The assumption that $((f_n)')_{n\in\mathbb{N}} \subseteq \mathcal{C}([a,b],V)$ converges ensures that

$$\sup\nolimits_{n\in\mathbb{N}} \sup\nolimits_{s\in[a,b]} \|(f_n)'(s)\|_V < \infty. \tag{30}$$

The dominated convergence theorem therefore proves that for every $t \in [a,b]$ it holds that

$$\limsup_{n\to\infty} \left\| \int_{t_0}^t g(s)\, ds - \int_{t_0}^t (f_n)'(s)\, ds \right\|_V = 0. \tag{31}$$

This and (29) imply that for all $t \in [a,b]$ it holds that

$$\lim_{n\to\infty} f_n(t) = F(t_0) + \int_{t_0}^t g(s)\, ds. \tag{32}$$

Equation (28) hence assures for all $t \in [a, b]$ that

$$F(t) = \lim_{n\to\infty} f_n(t). \tag{33}$$

The fact that $g \in \mathcal{C}([a, b], V)$, (28), and Lemma 2.3 (with $V = V$, $a = a$, $b = b$, $f = g$, $F = F$ in the notation of Lemma 2.3) establish that for every $t \in [a, b]$ it holds that $F \in \mathcal{C}^1([a, b], V)$ and $F'(t) = g(t)$. The proof of Lemma 2.8 is thus completed. $\square$

Proposition 2.9. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $t_0 \in \mathbb{R}$, $t \in (t_0, \infty)$, $\phi \in \mathcal{C}^1(V, V)$, $F\colon [t_0, t] \to V$, $\Phi\colon \{(u, r) \in [t_0, t]^2 \colon u \le r\} \times V \to V$, $\dot{\Phi}\colon \{(u, r) \in [t_0, t]^2 \colon u \le r\} \times V \to V$, $\Phi^\star\colon \{(u, r) \in [t_0, t]^2 \colon u \le r\} \times V \to L(V)$, let $f\colon [t_0, t] \to V$ be a strongly measurable function, assume that $([t_0, t] \times V \ni (u, x) \mapsto \Phi_{u,t}(x) \in V) \in \mathcal{C}^1([t_0, t] \times V, V)$, assume for all $x \in V$, $t_1 \in (t_0, t)$ that $([t_0, t_1] \ni u \mapsto \Phi_{u,t_1}(x) \in V) \in \mathcal{C}^1([t_0, t_1], V)$, $([t_0, t_1] \ni u \mapsto \dot{\Phi}_{u,t_1}(x) \in V) \in \mathcal{C}([t_0, t_1], V)$, $(\{(u, r) \in (t_0, t)^2 \colon u < r\} \ni (s, \tau) \mapsto \Phi_{s,\tau}(x) \in V) \in \mathcal{C}^1(\{(u, r) \in (t_0, t)^2 \colon u < r\}, V)$, assume for all $t_1 \in [t_0, t]$, $t_2 \in [t_1, t]$, $t_3 \in [t_2, t]$, $x \in V$ that $\int_{t_0}^t \|f(u)\|_V \, du < \infty$, $F(t_1) = F(t_0) + \int_{t_0}^{t_1} f(u) \, du$, $\Phi_{t_1,t_1}(x) = x$, $\Phi_{t_1,t_3}(x) = \Phi_{t_2,t_3}(\Phi_{t_1,t_2}(x))$, assume for all $t_1 \in (t_0, t)$, $t_2 \in (t_1, t)$, $x \in V$ that $\dot{\Phi}_{t_1,t_2}(x) = \frac{\partial}{\partial t_2}(\Phi_{t_1,t_2}(x))$, and assume for all $t_1 \in [t_0, t]$, $x \in V$ that $\Phi^\star_{t_1,t}(x) = \frac{\partial}{\partial x}(\Phi_{t_1,t}(x))$. Then*

(i) it holds that $[t_0, t] \ni s \mapsto \phi'(\Phi_{s,t}(F(s)))\Phi^\star_{s,t}(F(s))[\dot{\Phi}_{s,s}(F(s)) - f(s)] \in V$ is strongly measurable,

(ii) it holds that $\int_{t_0}^t \|\phi'(\Phi_{s,t}(F(s)))\Phi^\star_{s,t}(F(s))[\dot{\Phi}_{s,s}(F(s)) - f(s)]\|_V \, ds < \infty$, and

(iii) it holds that

$$\phi(\Phi_{t_0,t}(F(t_0))) - \phi(F(t)) = \int_{t_0}^t \phi'(\Phi_{s,t}(F(s)))\Phi^\star_{s,t}(F(s))\Big[\dot{\Phi}_{s,s}(F(s)) - f(s)\Big] \, ds. \tag{34}$$

Proof of Proposition 2.9. Throughout this proof let

$$\mathring{\Phi} = (\mathring{\Phi}_{t_1,t_2}(x))_{(t_1,t_2,x)\in\{(u,r)\in[t_0,t]^2 \colon u\le r\}\times V} \colon \{(u, r) \in [t_0, t]^2 \colon u \le r\} \times V \to V \tag{35}$$

be a function which satisfies for all $t_2 \in (t_0, t]$, $t_1 \in [t_0, t_2]$, $x \in V$ that $\mathring{\Phi}_{t_1,t_2}(x) = \frac{\partial}{\partial t_1}\Phi_{t_1,t_2}(x)$ and let $\varphi\colon [t_0, t] \times V \to V$ be the function which satisfies for all $s \in [t_0, t]$, $v \in V$ that $\varphi(s, v) = \Phi_{s,t}(v)$. Note that the assumption that $([t_0, t] \times V \ni (s, x) \mapsto \Phi_{s,t}(x) \in V) \in \mathcal{C}^1([t_0, t] \times V, V)$ shows that for every $\tau \in [t_0, t)$ it holds that $\varphi|_{[\tau,t]\times V} \in \mathcal{C}^1([\tau, t] \times V, V)$. Corollary 2.7 (with $V = V$, $W = V$, $a = \tau$, $b = t$, $\phi = \varphi|_{[\tau,t]\times V}$, $F = F|_{[\tau,t]}$, $f = f|_{[\tau,t]}$ for $\tau \in [t_0, t)$ in the notation of Corollary 2.7) therefore implies

(a) that $[t_0, t] \ni s \mapsto \Big[\Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s))\Big] \in V$ is strongly measurable,

(b) that $\int_{t_0}^t \|\Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s))\|_V \, ds < \infty$, and

(c) that for every $\tau \in [t_0, t]$ it holds that

$$\Phi_{t,t}(F(t)) - \Phi_{\tau,t}(F(\tau)) = F(t) - \Phi_{\tau,t}(F(\tau)) = \int_\tau^t \Big[\Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s))\Big] \, ds. \tag{36}$$

Lemma 2.6 (with $V = V$, $W = V$, $a = t_0$, $b = t$, $\phi = \phi$, $F = ([t_0, t] \ni s \mapsto \Phi_{s,t}(F(s)) \in V)$, $f = ([t_0, t] \ni s \mapsto \Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s)) \in V)$ in the notation of Lemma 2.6) hence shows

(A) that $[t_0,t] \ni s \mapsto \phi'(\Phi_{s,t}(F(s)))\Big[\Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s))\Big] \in V$ is strongly measurable,

(B) that $\int_{t_0}^t \Big\|\phi'(\Phi_{s,t}(F(s)))\Big[\Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s))\Big]\Big\|_V \, ds < \infty$, and

(C) that

$$\begin{aligned} &\phi(F(t)) - \phi(\Phi_{t_0,t}(F(t_0))) = \phi(\Phi_{t,t}(F(t))) - \phi(\Phi_{t_0,t}(F(t_0))) \\ &= \int_{t_0}^t \phi'(\Phi_{s,t}(F(s)))\Big[\Phi^\star_{s,t}(F(s))f(s) + \mathring{\Phi}_{s,t}(F(s))\Big] \, ds. \end{aligned} \tag{37}$$

Next observe that the assumption that $([t_0,t] \times V \ni (s,x) \mapsto \Phi_{s,t}(x) \in V) \in \mathcal{C}^1([t_0,t] \times V, V)$ and the chain rule ensure that for all $u \in (t_0,t)$, $s \in (t_0,u]$, $x \in V$ it holds that

$$\mathring{\Phi}_{s,t}(x) = \tfrac{\partial}{\partial s}\Phi_{s,t}(x) = \tfrac{\partial}{\partial s}(\Phi_{u,t}(\Phi_{s,u}(x))) = \Phi^\star_{u,t}(\Phi_{s,u}(x))\mathring{\Phi}_{s,u}(x). \tag{38}$$

Moreover, note that the assumption that for all $x \in V$ it holds that $\{(u,r) \in (t_0,t)^2 : u < r\} \ni (s,\tau) \mapsto \Phi_{s,\tau}(x) \in V$ is continuously differentiable implies that for all $s \in (t_0,t)$, $n \in [2,\infty) \cap \mathbb{N}$, $x \in V$ it holds that

$$\tfrac{\partial}{\partial s}\Phi_{s-(s-t_0)/n,s}(x) = (1 - {}^1\!/_n)\mathring{\Phi}_{s-(s-t_0)/n,s}(x) + \dot{\Phi}_{s-(s-t_0)/n,s}(x). \tag{39}$$

Combining the fact that for all $\varepsilon \in (0, {}^{(t-t_0)}\!/_6)$, $n \in [2,\infty) \cap \mathbb{N}$, $x \in V$ it holds that $[t_0+\varepsilon, t-\varepsilon] \ni s \mapsto \Phi_{s-(s-t_0)/n,s}(x) \in V$ is continuously differentiable, Lemma 2.8 (with $V = V$, $a = t_0 + \varepsilon$, $b = t - \varepsilon$, $t_0 = {}^{(t_0+t)}\!/_2$, $f_n = ([t_0+\varepsilon, t-\varepsilon] \ni s \mapsto \Phi_{s-(s-t_0)/n,s}(x) \in V)$ for $\varepsilon \in (0, {}^{(t-t_0)}\!/_6)$, $n \in [2,\infty) \cap \mathbb{N}$, $x \in V$ in the notation of Lemma 2.8), and the assumptions that $\forall\, x \in V, t_1 \in (t_0,t)\colon ([t_0,t_1] \ni u \mapsto \Phi^x_{u,t_1} \in V) \in \mathcal{C}^1([t_0,t_1],V)$ and $([t_0,t_1] \ni u \mapsto \dot{\Phi}^x_{u,t_1} \in V) \in \mathcal{C}([t_0,t_1],V)$ therefore proves that for all $s \in (t_0,t)$, $x \in V$ it holds that

$$\tfrac{\partial}{\partial s}\Phi_{s,s}(x) = \mathring{\Phi}_{s,s}(x) + \dot{\Phi}_{s,s}(x). \tag{40}$$

Hence, we obtain that for all $s \in (t_0,t)$, $x \in V$ it holds that

$$\mathring{\Phi}_{s,s}(x) + \dot{\Phi}_{s,s}(x) = 0. \tag{41}$$

This and (38) imply that for all $s \in (t_0,t)$, $x \in V$ it holds that

$$\mathring{\Phi}_{s,t}(x) = \Phi^\star_{s,t}(\Phi_{s,s}(x))\mathring{\Phi}_{s,s}(x) = \Phi^\star_{s,t}(x)\mathring{\Phi}_{s,s}(x) = -\Phi^\star_{s,t}(x)\dot{\Phi}_{s,s}(x). \tag{42}$$

Combining this with items (A)–(C) completes the proof of Proposition 2.9. ☐

3 Continuity of solutions to initial value problems

In this section we prove in Corollary 3.8 joint continuity of the solution to a Banach space valued ODE with respect to initial value, initial time, and current time. More precisely, we first apply Lemma 3.1 to prove a local existence and uniqueness result for initial value problems in Lemma 3.2. Then we combine Lemma 3.2, Lemma 3.3, Corollary 3.4, and Lemmas 3.5–3.7 to establish Corollary 3.8.

Lemma 3.1. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space and let $a \in \mathbb{R}$, $b \in [a, \infty)$, $s \in [a, b]$, $f \in \mathcal{C}([a, b] \times V, V)$, $X, Y \in \mathcal{C}([a, b], V)$ satisfy for all $t \in [a, b]$, $x \in V$ that $X_t - \int_s^t f(\tau, X_\tau)\, d\tau = Y_t - \int_s^t f(\tau, Y_\tau)\, d\tau$ and $\inf_{r \in (0,\infty)} \sup_{\tau \in [a,b]} \sup_{y \in V, \|x-y\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|x-z\|_V \leq r} \frac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} < \infty$. Then it holds for all $t \in [a, b]$ that $X_t = Y_t$.*

Proof of Lemma 3.1. Throughout this proof let $L_{x,r} \in [0, \infty]$, $x \in V$, $r \in (0, \infty)$, be the extended real numbers which satisfy for all $r \in (0, \infty)$, $x \in V$ that

$$L_{x,r} = \sup_{\tau \in [a,b]} \sup_{y \in V, \|x-y\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|x-z\|_V \leq r} \frac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V}, \tag{43}$$

let $\alpha = \sup(\{a\} \cup \{u \in [a, s] \colon X_u \neq Y_u\})$, and let $\beta = \inf(\{b\} \cup \{u \in [s, b] \colon X_u \neq Y_u\})$. Observe that the hypothesis that $X \colon [a, b] \to V$ and $Y \colon [a, b] \to V$ are continuous functions ensures that there exists a function $\delta \colon [a, b] \times (0, \infty) \to (0, \infty)$ such that for all $u \in [a, b]$, $\varepsilon \in (0, \infty)$, $t \in [u - \delta_{u,\varepsilon}, u + \delta_{u,\varepsilon}] \cap [a, b]$ it holds that

$$\|X_t - X_u\|_V < \varepsilon \qquad \text{and} \qquad \|Y_t - Y_u\|_V < \varepsilon. \tag{44}$$

This implies that for all $u \in [a, b]$ with $X_u = Y_u$ there exists $\varepsilon \in (0, \infty)$ with $L_{X_u,\varepsilon} < \infty$ such that for all $t \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a, b]$ it holds that

$$\begin{aligned}
&\|X_t - Y_t\|_V = \|(X_t - Y_t) - (X_u - Y_u)\|_V \\
&= \left\| \left[\int_s^t f(\tau, X_\tau)\, d\tau - \int_s^t f(\tau, Y_\tau)\, d\tau \right] - \left[\int_s^u f(\tau, X_\tau)\, d\tau - \int_s^u f(\tau, Y_\tau)\, d\tau \right] \right\|_V \\
&= \left\| \int_u^t f(\tau, X_\tau)\, d\tau - \int_u^t f(\tau, Y_\tau)\, d\tau \right\|_V \\
&= \left\| \int_{\min\{u,t\}}^{\max\{u,t\}} f(\tau, X_\tau)\, d\tau - \int_{\min\{u,t\}}^{\max\{u,t\}} f(\tau, Y_\tau)\, d\tau \right\|_V \\
&\leq \int_{\min\{u,t\}}^{\max\{u,t\}} \|f(\tau, X_\tau) - f(\tau, Y_\tau)\|_V\, d\tau \leq L_{X_u,\varepsilon} \int_{\min\{u,t\}}^{\max\{u,t\}} \|X_\tau - Y_\tau\|_V\, d\tau \\
&\leq L_{X_u,\varepsilon}\, |t - u| \left[\sup_{\tau \in [\min\{u,t\}, \max\{u,t\}]} \|X_\tau - Y_\tau\|_V \right] \\
&\leq L_{X_u,\varepsilon}\, |t - u| \left[\sup_{\tau \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a,b]} \|X_\tau - Y_\tau\|_V \right] \\
&\leq \frac{L_{X_u,\varepsilon}}{1 + 2L_{X_u,\varepsilon}} \left[\sup_{\tau \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a,b]} \|X_\tau - Y_\tau\|_V \right] \\
&\leq \frac{1}{2} \left[\sup_{\tau \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a,b]} \|X_\tau - Y_\tau\|_V \right].
\end{aligned} \tag{45}$$

Hence, we obtain that for all $u \in [a, b]$ with $X_u = Y_u$ there exists $\varepsilon \in (0, \infty)$ with $L_{X_u,\varepsilon} < \infty$ such that

$$\begin{aligned}
&\left[\sup_{\tau \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a,b]} \|X_\tau - Y_\tau\|_V \right] \\
&\leq \frac{1}{2} \left[\sup_{\tau \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a,b]} \|X_\tau - Y_\tau\|_V \right].
\end{aligned} \tag{46}$$

This shows that for all $u \in [a,b]$ with $X_u = Y_u$ there exists $\varepsilon \in (0,\infty)$ with $L_{X_u,\varepsilon} < \infty$ such that

$$\left[\sup_{\tau \in [u-\min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}, u+\min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u,\varepsilon})\}] \cap [a,b]} \|X_\tau - Y_\tau\|_V\right] = 0. \tag{47}$$

Therefore, we obtain for all $u \in [0,T]$ with $X_u = Y_u$ that there exists $\Delta \in (0,\infty)$ such that for all $t \in [u-\Delta, u+\Delta] \cap [a,b]$ it holds that

$$X_t = Y_t. \tag{48}$$

Moreover, observe that the fact that X and Y are continuous ensures that $X_\alpha = Y_\alpha$ and $X_\beta = Y_\beta$. Combining this with (48) demonstrates that $\alpha = a$ and $\beta = b$. The proof of Lemma 3.1 is thus completed. ☐

Lemma 3.2. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space and let $R, h, \varepsilon \in (0,\infty)$, $s_0 \in \mathbb{R}$, $L, M, \delta \in [0,\infty)$, $x_0 \in V$, $f \in \mathcal{C}(\mathbb{R} \times V, V)$ satisfy for all $x \in V$ that*

$$\inf\nolimits_{r\in(0,\infty)} \sup\nolimits_{\tau\in[s_0-h,s_0+h]} \sup\nolimits_{y\in V, \|x-y\|_V \le r} \sup\nolimits_{z\in V\setminus\{y\}, \|x-z\|_V \le r} \tfrac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} < \infty, \tag{49}$$

$$L = \sup\nolimits_{\tau\in[s_0-h,s_0+h]} \sup\nolimits_{y\in V, \|x_0-y\|_V \le R+\varepsilon} \sup\nolimits_{z\in V\setminus\{y\}, \|x_0-z\|_V \le R+\varepsilon} \tfrac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V}, \tag{50}$$

$$M = \sup\nolimits_{\tau\in[s_0-h,s_0+h]} \sup\nolimits_{y\in V, \|x_0-y\|_V \le R+\varepsilon} \|f(\tau,y)\|_V, \tag{51}$$

and $\delta = \min\{\varepsilon/(2M+1), 1/(4L+1), h\}$. Then

(i) it holds for every $s \in [s_0-\delta, s_0+\delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \le R\}$ that there exists a unique continuous function $X^x_{s,(\cdot)} = (X^x_{s,t})_{t\in[s_0-\delta,s_0+\delta]} \colon [s_0-\delta, s_0+\delta] \to V$ such that for all $t \in [s_0-\delta, s_0+\delta]$ it holds that

$$X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau \tag{52}$$

and

(ii) it holds that

$$\sup\nolimits_{s,t\in[s_0-\delta,s_0+\delta]} \sup\nolimits_{x\in V, \|x_0-x\|_V \le R} \|X^x_{s,t} - x_0\|_V \le R + \varepsilon. \tag{53}$$

Proof of Lemma 3.2. Throughout this proof let $\mathcal{A}$ be the set given by

$$\mathcal{A} = \Big\{\psi \in \mathcal{C}([s_0-\delta, s_0+\delta], V) \colon \sup\nolimits_{t\in[s_0-\delta,s_0+\delta]} \|\psi(t) - x_0\|_V \le R + \varepsilon\Big\}. \tag{54}$$

Note that for all $s, t \in [s_0-\delta, s_0+\delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \le R\}$, $\psi \in \mathcal{A}$ it holds that

$$\begin{aligned}\Big\|x + \int_s^t f(\tau, \psi(\tau))\, d\tau - x_0\Big\|_V &\le \|x - x_0\|_V + \int_{\min\{s,t\}}^{\max\{s,t\}} \|f(\tau, \psi(\tau))\|_V\, d\tau \\ &\le R + M|t-s| \le R + 2M\delta \le R + (2M+1)\delta \le R + \varepsilon.\end{aligned} \tag{55}$$

This ensures that there exist functions $B_{s,x} \colon \mathcal{A} \to \mathcal{A}$, $s \in [s_0-\delta, s_0+\delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \le R\}$, such that for all $s, t \in [s_0-\delta, s_0+\delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \le R\}$, $\psi \in \mathcal{A}$ it holds that

$$(B_{s,x}(\psi))(t) = x + \int_s^t f(\tau, \psi(\tau))\, d\tau. \tag{56}$$

Next observe that (50) and (54) demonstrate that for all $s, t \in [s_0 - \delta, s_0 + \delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$, $\psi_1, \psi_2 \in \mathcal{A}$ it holds that

$$\begin{aligned}
&\|(B_{s,x}(\psi_1))(t) - (B_{s,x}(\psi_2))(t)\|_V \leq \left\| \int_s^t [f(\tau, \psi_1(\tau)) - f(\tau, \psi_2(\tau))] \, d\tau \right\|_V \\
&\leq \int_{\min\{s,t\}}^{\max\{s,t\}} \|f(\tau, \psi_1(\tau)) - f(\tau, \psi_2(\tau))\|_V \, d\tau \leq L \int_{\min\{s,t\}}^{\max\{s,t\}} \|\psi_1(\tau) - \psi_2(\tau)\|_V \, d\tau \\
&\leq L|t-s| \left[\sup_{\tau \in [s_0 - \delta, s_0 + \delta]} \|\psi_1(\tau) - \psi_2(\tau)\|_V \right] \leq 2L\delta \left[\sup_{\tau \in [s_0 - \delta, s_0 + \delta]} \|\psi_1(\tau) - \psi_2(\tau)\|_V \right] \\
&\leq \frac{(4L+1)\delta}{2} \left[\sup_{\tau \in [s_0 - \delta, s_0 + \delta]} \|\psi_1(\tau) - \psi_2(\tau)\|_V \right] \leq \frac{1}{2} \left[\sup_{\tau \in [s_0 - \delta, s_0 + \delta]} \|\psi_1(\tau) - \psi_2(\tau)\|_V \right].
\end{aligned} \tag{57}$$

This shows that for all $s \in [s_0 - \delta, s_0 + \delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$, $\psi_1, \psi_2 \in \mathcal{A}$ it holds that

$$\left[\sup_{t \in [s_0 - \delta, s_0 + \delta]} \|(B_{s,x}(\psi_1))(t) - (B_{s,x}(\psi_2))(t)\|_V \right] \leq \frac{1}{2} \left[\sup_{t \in [s_0 - \delta, s_0 + \delta]} \|\psi_1(t) - \psi_2(t)\|_V \right]. \tag{58}$$

Banach's fixed point theorem hence proves that there exist continuous functions $X^x_{s,(\cdot)} = (X^x_{s,t})_{t \in [s_0 - \delta, s_0 + \delta]} \colon [s_0 - \delta, s_0 + \delta] \to V$, $s \in [s_0 - \delta, s_0 + \delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$, such that for all $s, t \in [s_0 - \delta, s_0 + \delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$ it holds that

$$X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau \tag{59}$$

and

$$\|X^x_{s,t} - x_0\|_V \leq R + \varepsilon. \tag{60}$$

Combining this and Lemma 3.1 (with $V = V$, $a = s_0 - \delta$, $b = s_0 + \delta$, $s = s$, $f = ([s_0 - \delta, s_0 + \delta] \times V \ni (\tau, y) \mapsto f(\tau, y) \in V)$, $X = ([s_0 - \delta, s_0 + \delta] \ni t \mapsto X^x_{s,t} \in V)$ for $s \in [s_0 - \delta, s_0 + \delta]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$ in the notation of Lemma 3.1) completes the proof of Lemma 3.2. ☐

Lemma 3.3. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space and let $a \in \mathbb{R}$, $b \in (a, \infty)$, $r \in (0, \infty)$, $x \in V$, $f \in \mathcal{C}([a,b] \times V, V)$ satisfy that*

$$\sup\nolimits_{\tau \in [a,b]} \sup\nolimits_{y \in V, \|x-y\|_V \leq r} \sup\nolimits_{z \in V \setminus \{y\}, \|x-z\|_V \leq r} \tfrac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} < \infty. \tag{61}$$

Then

$$\sup\nolimits_{\tau \in [a,b]} \sup\nolimits_{y \in V, \|x-y\|_V \leq r} \|f(\tau, y)\|_V < \infty. \tag{62}$$

Proof of Lemma 3.3. Note that (61) and the hypothesis that $f \colon [a,b] \times V \to V$ is a continuous function ensure that

$$\begin{aligned}
&\sup\nolimits_{\tau \in [a,b]} \sup\nolimits_{y \in V, \|x-y\|_V \leq r} \|f(\tau, y)\|_V \\
&\leq \left[\sup\nolimits_{\tau \in [a,b]} \sup\nolimits_{y \in V, \|x-y\|_V \leq r} \|f(\tau, y) - f(\tau, x)\|_V \right] + \left[\sup\nolimits_{\tau \in [a,b]} \|f(\tau, x)\|_V \right] \\
&= \left[\sup\nolimits_{\tau \in [a,b]} \sup\nolimits_{y \in V \setminus \{x\}, \|x-y\|_V \leq r} \|f(\tau, y) - f(\tau, x)\|_V \right] + \left[\sup\nolimits_{\tau \in [a,b]} \|f(\tau, x)\|_V \right] \\
&\leq r \left[\sup\nolimits_{\tau \in [a,b]} \sup\nolimits_{y \in V \setminus \{x\}, \|x-y\|_V \leq r} \left(\tfrac{\|f(\tau,y) - f(\tau,x)\|_V}{\|y-x\|_V} \right) \right] + \left[\sup\nolimits_{\tau \in [a,b]} \|f(\tau, x)\|_V \right] \\
&\leq r \left[\sup_{\tau \in [a,b]} \sup_{z \in V, \|x-z\|_V \leq r} \sup_{y \in V \setminus \{z\}, \|x-y\|_V \leq r} \left(\tfrac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} \right) \right] + \left[\sup\nolimits_{\tau \in [a,b]} \|f(\tau, x)\|_V \right] < \infty.
\end{aligned} \tag{63}$$

The proof of Lemma 3.3 is thus completed. ☐

Corollary 3.4. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T, R, \varepsilon \in (0, \infty)$, $s_0 \in [0, T]$, $x_0 \in V$, $f \in \mathcal{C}([0,T] \times V, V)$, for every $x \in V$, $s \in [0,T]$ let $X^x_{s,(\cdot)} = (X^x_{s,t})_{t \in [s,T]} \colon [s, T] \to V$ be a continuous function which satisfies for all $t \in [s,T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau$, and assume for all $x \in V$ that $\sup_{\tau \in [0,T]} \sup_{y \in V, \|x_0 - y\|_V \leq R+\varepsilon} \sup_{z \in V \setminus \{y\}, \|x_0 - z\|_V \leq R+\varepsilon} \frac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} < \infty$ and $\inf_{r \in (0,\infty)} \sup_{\tau \in [0,T]} \sup_{y \in V, \|x-y\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|x-z\|_V \leq r} \frac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} < \infty$. Then*

(i) there exists $\delta \in (0, \infty)$ such that for every $s \in [s_0 - \delta, s_0 + \delta] \cap [0, T]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$ there exists a unique continuous function $Y^x_{s,(\cdot)} = (Y^x_{s,t})_{t \in [s_0-\delta, s_0+\delta] \cap [0,T]} \colon [s_0 - \delta, s_0 + \delta] \cap [0,T] \to V$ such that for all $t \in [s_0 - \delta, s_0 + \delta] \cap [0, T]$ it holds that

$$Y^x_{s,t} = x + \int_s^t f(\tau, Y^x_{s,\tau})\, d\tau, \tag{64}$$

(ii) for all $s, t \in [s_0 - \delta, s_0 + \delta] \cap [0, T]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$ with $s \leq t$ it holds that $X^x_{s,t} = Y^x_{s,t}$, and

(iii) for all $s \in [s_0 - \delta, s_0 + \delta] \cap [0, T]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$ it holds that

$$\sup_{t \in [s_0 - \delta, s_0 + \delta]} \|Y^x_{s,t} - x_0\|_V \leq R + \varepsilon. \tag{65}$$

Proof of Corollary 3.4. Throughout this proof let $F \in \mathcal{C}(\mathbb{R} \times V, V)$ be the function which satisfies for all $t \in \mathbb{R}$, $x \in V$ that

$$F(t, x) = f(\min\{T, \max\{0, t\}\}, x), \tag{66}$$

let $L \in [0, \infty)$ be the real number given by

$$L = \sup_{\tau \in [0,T]} \sup_{y \in V, \|x_0 - y\|_V \leq R+\varepsilon} \sup_{z \in V \setminus \{y\}, \|x_0 - z\|_V \leq R+\varepsilon} \frac{\|F(\tau,y) - F(\tau,z)\|_V}{\|y-z\|_V}, \tag{67}$$

let $M \in [0, \infty]$ be the extended real number given by

$$M = \sup_{\tau \in [0,T]} \sup_{y \in V, \|x_0 - y\|_V \leq R+\varepsilon} \|F(\tau, y)\|_V, \tag{68}$$

and let $\delta \in [0, \infty)$ be the real number given by $\delta = \min\{\varepsilon/(2M+1), 1/(4L+1), 1\}$. Note that Lemma 3.3 (with $V = V$, $a = 0$, $b = T$, $r = R + \varepsilon$, $x = x_0$, $f = F$ in the notation of Lemma 3.3) proves that $M < \infty$. This ensures that $\delta \in (0, \infty)$. Combining this, the fact that $M < \infty$, and item (i) of Lemma 3.2 (with $V = V$, $R = R$, $h = 1$, $\varepsilon = \varepsilon$, $s_0 = s_0$, $L = L$, $M = M$, $\delta = \delta$, $x_0 = x_0$, $f = F$ in the notation of item (i) of Lemma 3.2) establishes item (i). The fact that $\forall\, s \in [s_0 - \delta, s_0 + \delta] \cap [0, T], x \in \{v \in V \colon \|x_0 - v\|_V \leq R\} \colon Y^x_{s,s} = X^x_{s,s}$ and Lemma 3.1 (with $V = V$, $a = s$, $b = \min\{s_0 + \delta, T\}$, $f = ([s, \min\{s_0 + \delta, T\}] \times V \ni (t, y) \mapsto F(t, y) \in V)$, $X = ([s, \min\{s_0 + \delta, T\}] \ni t \mapsto X^x_{s,t} \in V)$, $Y = ([s, \min\{s_0 + \delta, T\}] \ni t \mapsto Y^x_{s,t} \in V)$ for $s \in [s_0 - \delta, s_0 + \delta] \cap [0, T]$, $x \in \{v \in V \colon \|x_0 - v\|_V \leq R\}$ in the notation of Lemma 3.1) hence show that item (ii) holds. In addition, item (ii) of Lemma 3.2 (with $V = V$, $R = R$, $h = 1$, $\varepsilon = \varepsilon$, $s_0 = s_0$, $L = L$, $M = M$, $\delta = \delta$, $x_0 = x_0$, $f = F$ in the notation of item (ii) of Lemma 3.2) establishes item (iii). This completes the proof of Corollary 3.4. ☐

Lemma 3.5. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T, R \in (0, \infty)$, $s_0 \in [0, T]$, $x_0 \in V$, $f \in \mathcal{C}([0,T] \times V, V)$, for every $x \in V$, $s \in [0,T]$ let $X^x_{s,(\cdot)} = (X^x_{s,t})_{t \in [s,T]} \colon [s, T] \to V$ be a* *continuous function which satisfies for all $t \in [s,T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\,d\tau$, assume that* $\inf_{r\in(0,\infty)} \sup_{\tau\in[0,T]} \sup_{y\in V, \|x_0-y\|_V \le R+r} \sup_{z\in V\setminus\{y\}, \|x_0-z\|_V\le R+r} \frac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} < \infty$, *and assume for all $x \in V$ that* $\inf_{r\in(0,\infty)} \sup_{\tau\in[0,T]} \sup_{y\in V, \|x-y\|_V\le r} \sup_{z\in V\setminus\{y\}, \|x-z\|_V\le r} \frac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} < \infty$. *Then there exists $\delta \in (0,\infty)$ such that*

$$\{(u,v) \in ([s_0-\delta, s_0+\delta]\cap[0,T])^2 \colon u \le v\} \times \{v \in V \colon \|x_0 - v\|_V \le R\} \ni (s,t,x) \mapsto X^x_{s,t} \in V \quad (69)$$

is uniformly continuous.

Proof of Lemma 3.5. Throughout this proof let $\varepsilon, L \in \mathbb{R}$ be real numbers which satisfy that

$$L = \sup_{\tau\in[0,T]} \sup_{y\in V, \|x_0-y\|_V\le R+\varepsilon} \sup_{z\in V\setminus\{y\}, \|x_0-z\|_V\le R+\varepsilon} \frac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} \quad (70)$$

and let $M \in [0,\infty]$ be the extended real number given by

$$M = \sup_{\tau\in[0,T]} \sup_{y\in V, \|x_0-y\|_V\le R+\varepsilon} \|f(\tau,y)\|_V. \quad (71)$$

Note that Lemma 3.3 (with $V = V$, $a = 0$, $b = T$, $r = R+\varepsilon$, $x = x_0$, $f = f$ in the notation of Lemma 3.3) shows that $M < \infty$. Next observe that Corollary 3.4 (with $V = V$, $T = T$, $R = R$, $\varepsilon = \varepsilon$, $s_0 = s_0$, $x_0 = x_0$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $x \in V$, $(s,t) \in [0,T]$ with $s \le t$ in the notation of Corollary 3.4) ensures that there exists $\delta \in (0,\infty)$ such that for all $s \in [s_0-\delta, s_0+\delta] \cap [0,T]$, $t \in [s, \min\{s_0+\delta, T\}]$, $x \in \{v \in V \colon \|x_0 - v\|_V \le R\}$ it holds that

$$\|X^x_{s,t} - x_0\|_V \le R + \varepsilon. \quad (72)$$

Moreover, note that for all $s,t,u \in [\max\{s_0-\delta, 0\}, \min\{s_0+\delta, T\}]$, $x,y \in \{v \in V \colon \|x_0 - v\|_V \le R\}$ with $s,u \in [0,t]$ it holds that

$$\begin{aligned}
\|X^x_{s,t} - X^y_{u,t}\|_V &\le \|x-y\|_V + \left\| \int_s^t f(\tau, X^x_{s,\tau})\,d\tau - \int_u^t f(\tau, X^y_{u,\tau})\,d\tau \right\|_V \\
&\le \|x-y\|_V + \max\left\{ \left\| \int_s^{\max\{s,u\}} f(\tau, X^x_{s,\tau})\,d\tau \right\|_V, \left\| \int_u^{\max\{s,u\}} f(\tau, X^y_{u,\tau})\,d\tau \right\|_V \right\} \\
&\quad + \left\| \int_{\max\{s,u\}}^t f(\tau, X^x_{s,\tau}) - f(\tau, X^y_{u,\tau})\,d\tau \right\|_V \\
&\le \|x-y\|_V + \max\left\{ \left\| \int_s^{\max\{s,u\}} f(\tau, X^x_{s,\tau})\,d\tau \right\|_V, \left\| \int_u^{\max\{s,u\}} f(\tau, X^y_{u,\tau})\,d\tau \right\|_V \right\} \\
&\quad + \int_{\max\{s,u\}}^t \|f(\tau, X^x_{s,\tau}) - f(\tau, X^y_{u,\tau})\|_V\,d\tau.
\end{aligned} \quad (73)$$

Combining this with (72) proves that for all $s,t,u \in [\max\{s_0-\delta, 0\}, \min\{s_0+\delta, T\}]$, $x,y \in \{v \in V \colon \|x_0 - v\|_V \le R\}$ with $s,u \in [0,t]$ it holds that

$$\|X^x_{s,t} - X^y_{u,t}\|_V \le \|x-y\|_V + M|u-s| + L \int_u^t \|X^x_{s,\tau} - X^y_{u,\tau}\|_V\,d\tau. \quad (74)$$

The fact that $M < \infty$ and Gronwall's lemma therefore imply that for all $s,t,u \in [\max\{s_0 - \delta, 0\}, \min\{s_0+\delta, T\}]$, $x,y \in \{v \in V \colon \|x_0 - v\|_V \le R\}$ with $s,u \in [0,t]$ it holds that

$$\|X^x_{s,t} - X^y_{u,t}\|_V \le (\|x-y\|_V + M|u-s|)e^{L|t-u|}. \quad (75)$$

In addition, note that (72) shows that for all $s,t,\tau \in [\max\{s_0-\delta,0\},\min\{s_0+\delta,T\}]$, $x \in \{v \in V\colon \|x_0-v\|_V \le R\}$ with $s \le \min\{t,\tau\}$ it holds that

$$\|X^x_{s,t} - X^x_{s,\tau}\|_V \le \int_{\min\{\tau,t\}}^{\max\{\tau,t\}} \|f(r,X^x_{s,r})\|_V \, dr \le M|t-\tau|. \tag{76}$$

Combining this with (75) assures that for all $s,t,u,\tau \in [\max\{s_0-\delta,0\},\min\{s_0+\delta,T\}]$, $x,y \in \{v \in V\colon \|x_0-v\|_V \le R\}$ with $s \le t$, $u \le \tau \le t$ it holds that

$$\begin{aligned}\|X^x_{s,t} - X^y_{u,\tau}\|_V &\le \|X^x_{s,t} - X^y_{u,t}\|_V + \|X^y_{u,t} - X^y_{u,\tau}\|_V \\ &\le (\|x-y\|_V + M|u-s|)e^{LT} + M|t-\tau|.\end{aligned} \tag{77}$$

The fact that $M < \infty$ establishes (69). The proof of Lemma 3.5 is thus completed. ☐

Lemma 3.6. *Let $(V,\|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space and let $T \in (0,\infty)$, $x_0 \in V$, $f \in \mathcal{C}^{0,1}([0,T]\times V, V)$. Then*

$$\inf_{r\in(0,\infty)} \sup_{\tau\in[0,T]} \sup_{y\in V,\|x_0-y\|_V\le r} \sup_{z\in V\setminus\{y\},\|x_0-z\|_V\le r} \tfrac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} < \infty. \tag{78}$$

Proof of Lemma 3.6. Throughout this proof let $f_{0,1}\colon [0,T]\times V \to L(V)$ be the function which satisfies for all $t \in [0,T]$, $x \in V$ that $f_{0,1}(t,x) = (\frac{\partial}{\partial x} f)(t,x)$. Note that the assumption that $f \in \mathcal{C}^{0,1}([0,T]\times V,V)$ implies that there exists a function $\delta\colon [0,T] \to (0,\infty)$ such that for all $x \in V$, $s,t \in [0,T]$ with $\max\{|s-t|, \|x-x_0\|_V\} < \delta_t$ it holds that

$$\|f_{0,1}(s,x) - f_{0,1}(t,x_0)\|_{L(V)} < \tfrac{1}{2}. \tag{79}$$

Moreover, observe that the fact that $[0,T]$ is compact ensures that there exist $n \in \mathbb{N}$, $t_1,\dots,t_n \in [0,T]$ such that

$$0 = t_1 < \ldots < t_n = T \qquad \text{and} \qquad [0,T] = \cup_{i=1}^n \{r \in [0,T]\colon |r-t_i| < \delta_{t_i}\}. \tag{80}$$

This and (79) demonstrate that there exist $n \in \mathbb{N}$, $t_1,\dots,t_n \in [a,b]$ such that for all $x \in V$, $t \in [0,T]$ with $\|x-x_0\|_V < \min\{\delta_{t_1},\dots,\delta_{t_n}\}$ it holds that

$$\begin{aligned}&\|f_{0,1}(t,x) - f_{0,1}(t,x_0)\|_V \\ &\le \min_{i\in[0,n]\cap\mathbb{N}} \Big|\|f_{0,1}(t,x) - f_{0,1}(t_i,x_0)\|_V + \|f_{0,1}(t_i,x_0) - f_{0,1}(t,x_0)\|_V\Big| < \tfrac{1}{2} + \tfrac{1}{2} = 1.\end{aligned} \tag{81}$$

Hence, we obtain that for all $t \in [0,T]$, $y,z \in V$ with $\max\{\|y-x_0\|_V, \|z-x_0\|_V\} < \min\{\delta_{t_1},\dots,\delta_{t_n}\}$ it holds that

$$\begin{aligned}&\|f(t,y) - f(t,z)\|_V = \left\| \int_0^1 f_{0,1}(t, z+s(y-z))(y-z)\, ds \right\|_V \\ &\le \int_0^1 \|f_{0,1}(t,z+s(y-z))\|_{L(V)} \|y-z\|_V \, ds \\ &\le \|f_{0,1}(t,x_0)\|_V \|y-z\|_V + \int_0^1 \|f_{0,1}(t,z+s(y-z)) - f_{0,1}(t,x_0)\|_{L(V)} \|y-z\|_V \, ds \\ &< (\|f_{0,1}(t,x_0)\|_V + 1)\|y-z\|_V \le \Big(\sup_{\tau\in[0,T]} \|f_{0,1}(\tau,x_0)\|_V + 1\Big)\|y-z\|_V < \infty.\end{aligned} \tag{82}$$

The proof of Lemma 3.6 is thus completed. ☐

Lemma 3.7. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0,\infty)$, $f \in \mathcal{C}^{0,1}([0,T] \times V, V)$, and for every $x \in V$, $s \in [0,T]$ let $X^x_{s,(\cdot)} = (X^x_{s,t})_{t\in[s,T]} \colon [s,T] \to V$ be a continuous function which satisfies for all $t \in [s,T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau$. Then it holds for all $x \in V$, $t_1 \in [0,T]$, $t_2 \in [t_1,T]$, $t_3 \in [t_2,T]$ that $X^{X^x_{t_1,t_2}}_{t_2,t_3} = X^x_{t_1,t_3}$.*

Proof of Lemma 3.7. Throughout this proof let $f_{0,1} \colon [0,T] \times V \to L(V)$ be the function which satisfies for all $t \in [0,T]$, $x \in V$ that $f_{0,1}(t,x) = (\frac{\partial}{\partial x} f)(t,x)$. Observe that for all $x \in V$, $t_1 \in [0,T]$, $t_2 \in [t_1,T]$, $t_3 \in [t_2,T]$ it holds that

$$X^{X^x_{t_1,t_2}}_{t_2,t_3} = X^x_{t_1,t_2} + \int_{t_2}^{t_3} f\Big(\tau, X^{X^x_{t_1,t_2}}_{t_2,\tau}\Big)\, d\tau. \tag{83}$$

This, the assumption that $\forall\, x \in V, s \in [0,T] \colon ([s,T] \ni t \mapsto X^x_{s,t} \in V) \in \mathcal{C}([s,T],V)$, and the assumption that $f \in \mathcal{C}^{0,1}([0,T] \times V, V)$ imply that for all $x \in V$, $t_1 \in [0,T]$, $t_2 \in [t_1,T]$, $t_3 \in [t_2,T]$ it holds that

$$\begin{aligned}
&\|X^{X^x_{t_1,t_2}}_{t_2,t_3} - X^x_{t_1,t_3}\|_V = \bigg\| \int_{t_2}^{t_3} f\Big(\tau, X^{X^x_{t_1,t_2}}_{t_2,\tau}\Big)\, d\tau + X^x_{t_1,t_2} - X^x_{t_1,t_3} \bigg\|_V \\
&= \bigg\| \int_{t_2}^{t_3} f\Big(\tau, X^{X^x_{t_1,t_2}}_{t_2,\tau}\Big)\, d\tau - \int_{t_2}^{t_3} f(\tau, X^x_{t_1,\tau})\, d\tau \bigg\|_V \le \int_{t_2}^{t_3} \Big\| f\Big(\tau, X^{X^x_{t_1,t_2}}_{t_2,\tau}\Big) - f(\tau, X^x_{t_1,\tau}) \Big\|_V\, d\tau \\
&= \int_{t_2}^{t_3} \bigg\| \int_0^1 f_{0,1}\Big(\tau, X^x_{t_1,\tau} + \Big(X^{X^x_{t_1,t_2}}_{t_2,\tau} - X^x_{t_1,\tau}\Big) r\Big) \Big(X^{X^x_{t_1,t_2}}_{t_2,\tau} - X^x_{t_1,\tau}\Big)\, dr \bigg\|_V\, d\tau \\
&\le \int_{t_2}^{t_3} \bigg(\int_0^1 \Big\| f_{0,1}\Big(\tau, X^x_{t_1,\tau} + \Big(X^{X^x_{t_1,t_2}}_{t_2,\tau} - X^x_{t_1,\tau}\Big) r\Big) \Big\|_{L(V)}\, dr \bigg) \Big\| X^{X^x_{t_1,t_2}}_{t_2,\tau} - X^x_{t_1,\tau} \Big\|_V\, d\tau \\
&\le \sup_{r\in[0,1], t\in[t_2,T]} \Big\| f_{0,1}\Big(t, X^x_{t_1,t} + \Big(X^{X^x_{t_1,t_2}}_{t_2,t} - X^x_{t_1,t}\Big) r\Big) \Big\|_{L(V)} \int_{t_2}^{t_3} \Big\| X^{X^x_{t_1,t_2}}_{t_2,\tau} - X^x_{t_1,\tau} \Big\|_V\, d\tau < \infty.
\end{aligned} \tag{84}$$

Gronwall's lemma hence shows for all $x \in V$, $t_1 \in [0,T]$, $t_2 \in [t_1,T]$, $t_3 \in [t_2,T]$ that $X^x_{t_1,t_3} = X^{X^x_{t_1,t_2}}_{t_2,t_3}$. The proof of Lemma 3.7 is thus completed. ☐

Corollary 3.8. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0,\infty)$, $f \in \mathcal{C}^{0,1}([0,T] \times V, V)$, and for every $x \in V$, $s \in [0,T]$ let $X^x_{s,(\cdot)} \colon [s,T] \to V$ be a continuous function which satisfies for all $t \in [s,T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau$. Then it holds that $\{(u,v) \in [0,T]^2 \colon u \le v\} \times V \ni (s,t,x) \mapsto X^x_{s,t} \in V$ is a continuous function.*

Proof of Corollary 3.8. Throughout this proof we denote by $\angle_T \subseteq [0,T]^2$ the set given by $\angle_T = \{(s,t) \in [0,T]^2 \colon s \le t\}$, let $(s_0,t_0,x_0) \in \angle_T \times V$, and let $\varepsilon \in (0,\infty)$. Note that Lemma 3.6 (with $V = V$, $T = T$, $x_0 = x$, $f = f$ for $x \in V$ in the notation of Lemma 3.6) shows that there exists a function $r \colon V \to (0,\infty)$ such that for every $x \in V$ it holds that

$$\sup\nolimits_{\tau\in[0,T]} \sup\nolimits_{y\in V, \|x-y\|_V \le 2r_x} \sup\nolimits_{z\in V\setminus\{y\}, \|x-z\|_V \le 2r_x} \tfrac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} < \infty. \tag{85}$$

Lemma 3.5 (with $V = V$, $T = T$, $R = r_x$, $s_0 = s$, $x_0 = x$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of Lemma 3.5) hence ensures that there exists a function $\delta \colon [0,T] \times V \to (0,\infty)$ such that for all $s \in [0,T]$, $x \in V$ it holds that

$$(\angle_T \cap [s - \delta_{s,x}, s + \delta_{s,x}]^2) \times \{v \in V \colon \|x - v\|_V \le r_x\} \ni (u,t,y) \mapsto X^y_{u,t} \in V \tag{86}$$

is a uniformly continuous function. Furthermore, note that the fact that $\forall\, x \in V, s \in [0,T]\colon ([s,T] \ni t \mapsto X^x_{s,t} \in V) \in \mathcal{C}([s,T],V)$ ensures that there exists a function $\Delta\colon [0,T] \to (0,\infty)$ such that for all $t, \tau \in [s_0,T]$ with $|t-\tau| < \Delta_t$ it holds that

$$\|X^{x_0}_{s_0,t} - X^{x_0}_{s_0,\tau}\|_V \leq \tfrac{1}{2} r_{X^{x_0}_{s_0,t}}. \tag{87}$$

This implies that there exist non-empty intervals $I_{s,x} \subseteq [0,T]$, $s \in [0,T]$, $x \in V$, such that for all $s \in (0,T)$, $x \in V$ it holds that $I_{s,x} = (\max\{s-\delta_{s,x}, s-\Delta_s, 0\}, \min\{s+\delta_{s,x}, s+\Delta_s, T\})$, $I_{0,x} = [0, \min\{\delta_{0,x}, \Delta_0\})$, and $I_{T,x} = (T - \min\{\delta_{T,x}, \Delta_T\}, T]$. In addition, observe that the fact that $\cup_{s\in[s_0,t_0]} I_{s,X^{x_0}_{s_0,s}} \supseteq [s_0,t_0]$ and the fact that $[s_0,t_0]$ is a compact set assure that there exist $n \in [2,\infty) \cap \mathbb{N}$, $s_1,\ldots,s_n \in (s_0,t_0]$ such that $s_1 < \ldots < s_n = t_0$ and

$$[s_0,t_0] \subseteq \cup_{j=0}^n I_{s_j,X^{x_0}_{s_0,s_j}}. \tag{88}$$

This demonstrates that there exist real numbers $\tau_j \in (s_0,t_0)$, $j \in [1,n] \cap \mathbb{N}$, such that for all $j \in [1,n] \cap \mathbb{N}$ it holds that

$$\tau_j \in \Big((I_{s_{j-1},X^{x_0}_{s_0,s_{j-1}}}) \cap (I_{s_j,X^{x_0}_{s_0,s_j}}) \Big). \tag{89}$$

Moreover, note that (87) (with $t = s_n$, $\tau = \tau_n$) ensures that $\|X^{x_0}_{s_0,s_n} - X^{x_0}_{s_0,\tau_n}\|_V \leq \frac{1}{2} r_{X^{x_0}_{s_0,s_n}}$. Combining this with (86) (with $s = s_n$, $x = X^{x_0}_{s_0,s_n}$) and Lemma 3.7 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of Lemma 3.7) proves that there exists $\varepsilon_n \in (0, \min\{1, \delta_{s_n,X^{x_0}_{s_0,s_n}}, \frac{1}{2} r_{X^{x_0}_{s_0,s_n}}\})$ such that for all $s \in [0,\tau_n]$, $t \in [\tau_n,T]$, $x \in V$ with $\max\{\|X^{x_0}_{s_0,\tau_n} - X^x_{s,\tau_n}\|_V, |t_0 - t|\} < \varepsilon_n$ it holds that $\|X^{x_0}_{s_0,s_n} - X^x_{s,\tau_n}\|_V \leq r_{X^{x_0}_{s_0,s_n}}$ and

$$\|X^{x_0}_{s_0,t_0} - X^x_{s,t}\|_V = \|X^{x_0}_{s_0,s_n} - X^x_{s,t}\|_V = \Big\| X^{X^{x_0}_{s_0,\tau_n}}_{\tau_n,t_0} - X^{X^x_{s,\tau_n}}_{\tau_n,t} \Big\|_V < \varepsilon. \tag{90}$$

Furthermore, observe that (87) (with $t = s_j$, $\tau = \tau_j$ for $j \in [1,n-1] \cap \mathbb{N}$) ensures that $\|X^{x_0}_{s_0,s_j} - X^{x_0}_{s_0,\tau_j}\|_V \leq \frac{1}{2} r_{X^{x_0}_{s_0,s_j}}$. This, (86) (with $s = s_j$, $x = X^{x_0}_{s_0,s_j}$ for $j \in [1,n-1] \cap \mathbb{N}$), and Lemma 3.7 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of Lemma 3.7) ensure that there exist $\varepsilon_j \in (0, \min\{1, \delta_{s_j,X^{x_0}_{s_0,s_j}}, \frac{1}{2} r_{X^{x_0}_{s_0,s_j}}\})$, $j \in [1,n-1] \cap \mathbb{N}$, such that for all $j \in [1,n-1] \cap \mathbb{N}$, $s \in [0,\tau_j]$, $x \in V$ with $\|X^{x_0}_{s_0,\tau_j} - X^x_{s,\tau_j}\|_V < \varepsilon_j$ it holds that $\|X^{x_0}_{s_0,s_j} - X^x_{s,\tau_j}\|_V \leq r_{X^{x_0}_{s_0,s_j}}$ and

$$\|X^{x_0}_{s_0,\tau_{j+1}} - X^x_{s,\tau_{j+1}}\|_V = \Big\| X^{X^{x_0}_{s_0,\tau_j}}_{\tau_j,\tau_{j+1}} - X^{X^x_{s,\tau_j}}_{\tau_j,\tau_{j+1}} \Big\|_V < \varepsilon_{j+1}. \tag{91}$$

In addition, note that (86) (with $s = s_0$, $x = x_0$) shows that there exists $\varepsilon_0 \in (0, \min\{1, \delta_{s_0,x_0}, r_{x_0}\})$ such that for all $s \in [0,\tau_1]$, $x \in V$ with $\max\{\|x_0 - x\|_V, |s_0 - s|\} < \varepsilon_0$ it holds that

$$\|X^{x_0}_{s_0,\tau_1} - X^x_{s,\tau_1}\|_V < \varepsilon_1. \tag{92}$$

Combining this with (90) and (91) implies that for all $s \in [0,\tau_1]$, $t \in [s,T]$, $x \in V$ with $\max\{\|x_0 - x\|_V, |s_0 - s|\} < \varepsilon_0$ and $|t_0 - t| < \min\{\varepsilon_n, |t_0 - \tau_n|\}$ it holds that

$$\|X^{x_0}_{s_0,t_0} - X^x_{s,t}\|_V < \varepsilon. \tag{93}$$

Hence, we obtain that there exists $\delta \in (0,\infty)$ such that for all $(s,t,x) \in \angle_T \times V$ with $\max\{|s - s_0|, |t - t_0|, \|x - x_0\|_V\} < \delta$ it holds that

$$\|X^{x_0}_{s_0,t_0} - X^x_{s,t}\|_V < \varepsilon. \tag{94}$$

The proof of Corollary 3.8 is thus completed. ◻

4 Continuous differentiability of solutions to initial value problems

In this section we prove in Lemma 4.8 (cf., e.g., Driver [5, Theorem 19.13]) differentiability properties of solutions to initial value problems. In order to do so, we recall a few elementary auxiliary results in Lemmas 4.1–4.3 (cf., e.g., Driver [5, Theorem 19.14]), Lemma 4.4 (cf., e.g., Driver [5, Theorem 19.7]), and Lemmas 4.5–4.6. Then we combine them to establish continuous differentiability of the solution to the considered initial value problem with respect to the initial data in Lemma 4.7. In addition, we establish in Lemma 4.8 continuous differentiability of the solution to the considered initial value problem with respect to the initial time as well as the current time.

Lemma 4.1. *Let $(U, \|\cdot\|_U)$, $(V, \|\cdot\|_V)$, and $(W, \|\cdot\|_W)$ be $\mathbb{R}$-Banach spaces and let $T \in (0,\infty)$, $\angle_T = \{(s,t) \in [0,T]^2 \colon s \leq t\}$, $f \in \mathcal{C}([0,T] \times U, L(V,W))$, $y \in \mathcal{C}(\angle_T, U)$, $h \in \mathcal{C}(\angle_T, V)$. Then $\angle_T \ni (s,t) \mapsto \int_s^t f(\tau, y(s,\tau))h(s,\tau)\,d\tau \in W$ is continuous.*

Proof of Lemma 4.1. Throughout this proof let $X \colon \angle_T \to W$ be the function which satisfies for all $(s,t) \in \angle_T$ that $X_{s,t} = \int_s^t f(\tau, y(s,\tau))h(s,\tau)\,d\tau$. Observe that for all $s \in [0,T]$, $t,u \in [s,T]$ with $t \leq u$ it holds that

$$\begin{aligned} \|X_{s,u} - X_{s,t}\|_W &= \Big\| \int_t^u f(\tau, y(s,\tau))h(s,\tau)\,d\tau \Big\|_W \\ &\leq |t-u| \Big[\sup_{(r,\tau)\in\angle_T} \|f(\tau, y(r,\tau))h(r,\tau)\|_W \Big]. \end{aligned} \tag{95}$$

Hence, we obtain that for all $s \in [0,T]$, $t,u \in [s,T]$ it holds that

$$\|X_{s,t} - X_{s,u}\|_W \leq |t-u| \Big[\sup_{(r,\tau)\in\angle_T} \|f(\tau, y(r,\tau))h(r,\tau)\|_W \Big]. \tag{96}$$

In addition, note that for all $s,u,t \in [0,T]$ with $s,u \in [0,t]$ it holds that

$$\begin{aligned} &\|X_{s,t} - X_{u,t}\|_W = \Big\| \int_s^t f(\tau, y(s,\tau))h(s,\tau)\,d\tau - \int_u^t f(\tau, y(u,\tau))h(u,\tau)\,d\tau \Big\|_W \\ &\leq \Big\| \int_{\max\{s,u\}}^t f(\tau, y(\max\{s,u\},\tau))h(\max\{s,u\},\tau)\,d\tau \\ &\quad - \int_{\min\{s,u\}}^t f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\,d\tau \Big\|_W \\ &= \Big\| \int_{\max\{s,u\}}^t f(\tau, y(\max\{s,u\},\tau))h(\max\{s,u\},\tau)\,d\tau \\ &\quad - \Big[\int_{\min\{s,u\}}^{\max\{s,u\}} f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\,d\tau \\ &\quad + \int_{\max\{s,u\}}^t f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\,d\tau \Big] \Big\|_W \\ &\leq \int_{\max\{s,u\}}^t \|f(\tau, y(\max\{s,u\},\tau))h(\max\{s,u\},\tau) - f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\|_W\,d\tau \\ &\quad + \int_{\min\{s,u\}}^{\max\{s,u\}} \|f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\|_W\,d\tau. \end{aligned} \tag{97}$$

This implies that for all $s, u, t \in [0, T]$ with $s, u \in [0, t]$ it holds that

$$\begin{aligned} \|X_{s,t} - X_{u,t}\|_W \leq \int_{\max\{s,u\}}^t \|f(\tau, y(s, \tau))h(s, \tau) - f(\tau, y(u, \tau))h(u, \tau)\|_W \, d\tau \\ + |s - u| \bigg[\sup_{(r,\tau) \in \angle_T} \|f(\tau, y(r, \tau))h(r, \tau)\|_W \bigg]. \end{aligned} \tag{98}$$

Combining (95) and (98) assures that for all $s, t, u, v \in [0, T]$ with $s \leq t$, $u \leq v$, and $u \leq t$ it holds that

$$\begin{aligned} &\|X_{s,t} - X_{u,v}\|_W \leq \|X_{s,t} - X_{u,t}\|_W + \|X_{u,t} - X_{u,v}\|_W \\ &\leq (|t - v| + |s - u|) \big[\sup_{(r,\tau) \in \angle_T} \|f(\tau, y(r, \tau))h(r, \tau)\|_V \big] \\ &\quad + \int_{\max\{s,u\}}^t \|f(\tau, y(s, \tau))h(s, \tau) - f(\tau, y(u, \tau))h(u, \tau)\|_V \, d\tau. \end{aligned} \tag{99}$$

The dominated convergence theorem hence completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $\varepsilon, T \in (0, \infty)$, $\angle_T = \{(s, t) \in [0, T]^2 \colon s \leq t\}$, $y \in \mathcal{C}(\angle_T, V)$, $f \in \mathcal{C}^{0,1}([0, T] \times V, V)$, and let $f_{0,1} \colon [0, T] \times V \to L(V)$ be the function which satisfies for all $t \in [0, T]$, $x \in V$ that $f_{0,1}(t, x) = (\frac{\partial}{\partial x} f)(t, x)$. Then there exists $\delta \in (0, \infty)$ such that for all $h \in \mathcal{C}(\angle_T, V)$ with $\sup_{(s,t) \in \angle_T} \|h(s, t)\|_V < \delta$ it holds that*

$$\sup_{r \in [0,1]} \sup_{(s,t) \in \angle_T} \|f_{0,1}(t, y(s, t) + rh(s, t)) - f_{0,1}(t, y(s, t))\|_{L(V)} < \varepsilon. \tag{100}$$

Proof of Lemma 4.2. Throughout this proof let $C = y(\angle_T) = \{y(s, t) \in V \colon (s, t) \in \angle_T\}$. Note that the assumption that $f \in \mathcal{C}^{0,1}([0, T] \times V, V)$ ensures that there exists a function $\delta \colon [0, T] \times V \to (0, \infty)$ such that for all $(t, x), (\tau, \xi) \in [0, T] \times V$ with $|t - \tau| + \|x - \xi\|_V < \delta_{t,x}$ it holds that

$$\|f_{0,1}(\tau, \xi) - f_{0,1}(t, x)\|_{L(V)} < \tfrac{\varepsilon}{2}. \tag{101}$$

Moreover, observe that

$$[0, T] \times C \subseteq \cup_{(t,x) \in [0,T] \times C} \{(\tau, \xi) \in [0, T] \times V \colon |t - \tau| + \|x - \xi\|_V < \tfrac{1}{2} \delta_{t,x}\}. \tag{102}$$

The fact that $[0, T] \times C$ is a compact set therefore implies that there exist $n \in \mathbb{N}$, $(t_1, x_1), \ldots, (t_n, x_n) \in [0, T] \times C$ such that

$$[0, T] \times C \subseteq \cup_{i=1}^n \{(\tau, \xi) \in [0, T] \times V \colon |t_i - \tau| + \|x_i - \xi\|_V < \tfrac{1}{2} \delta_{t_i,x_i}\}. \tag{103}$$

This and (101) show that there exist $n \in \mathbb{N}$, $(t_1, x_1), \ldots, (t_n, x_n) \in [0, T] \times C$ such that for all $(s, t) \in \angle_T$, $r \in [0, 1]$, $h \in \mathcal{C}(\angle_T, V)$ with $\sup_{(u,\tau) \in \angle_T} \|h(u, \tau)\|_V < \min_{i \in [1,n] \cap \mathbb{N}} \delta_{t_i,x_i}/2$ it holds that

$$\begin{aligned} &\|f_{0,1}(t, y(s, t) + rh(s, t)) - f_{0,1}(t, y(s, t))\|_{L(V)} \\ &\leq \min_{i \in [1,n] \cap \mathbb{N}} \Big[\|f_{0,1}(t, y(s, t) + rh(s, t)) - f_{0,1}(t_i, x_i)\|_{L(V)} \\ &\quad + \|f_{0,1}(t_i, x_i) - f_{0,1}(t, y(s, t))\|_{L(V)} \Big] \\ &\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{104}$$

The proof of Lemma 4.2 is thus completed. □

Lemma 4.3. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0, \infty)$, $\angle_T = \{(s,t) \in [0,T]^2 \colon s \leq t\}$, $f = (f(t,x))_{(t,x)\in[0,T]\times V} \in \mathcal{C}^{0,1}([0,T] \times V, V)$, $F \colon \mathcal{C}(\angle_T, V) \to \mathcal{C}(\angle_T, V)$ satisfy for all $(s,t) \in \angle_T$, $y \in \mathcal{C}(\angle_T, V)$ that $(F(y))(s,t) = \int_s^t f(\tau, y(s,\tau))\, d\tau$, and let $f_{0,1} \colon [0,T] \times V \to L(V)$ be the function which satisfies for all $t \in [0,T]$, $x \in V$ that $f_{0,1}(t,x) = (\frac{\partial}{\partial x} f)(t,x)$. Then it holds for all $y, h \in \mathcal{C}(\angle_T, V)$, $(s,t) \in \angle_T$ that $F \in \mathcal{C}^1(\mathcal{C}(\angle_T, V), \mathcal{C}(\angle_T, V))$ and*

$$(F'(y)h)(s,t) = \int_s^t f_{0,1}(\tau, y(s,\tau)) h(s,\tau)\, d\tau. \tag{105}$$

Proof of Lemma 4.3. Note that for all $y, h \in \mathcal{C}(\angle_T, V)$, $(s,\tau) \in \angle_T$ it holds that

$$f(\tau, y(s,\tau) + h(s,\tau)) - f(\tau, y(s,\tau)) = \int_0^1 f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) h(s,\tau)\, dr. \tag{106}$$

This ensures that for all $y, h \in \mathcal{C}(\angle_T, V)$, $(s,t) \in \angle_T$ it holds that

$$\begin{aligned}
&(F(y+h))(s,t) - (F(y))(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau)) h(s,\tau)\, d\tau \\
&= \int_s^t \Big[f(\tau, y(s,\tau) + h(s,\tau)) - f(\tau, y(s,\tau)) - f_{0,1}(\tau, y(s,\tau)) h(s,\tau) \Big]\, d\tau \\
&= \int_s^t \left(\int_0^1 \Big[f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau)) \Big] h(s,\tau)\, dr \right) d\tau.
\end{aligned} \tag{107}$$

Hence, we obtain for all $y, h \in \mathcal{C}(\angle_T, V)$ that

$$\begin{aligned}
&\sup_{(s,t)\in\angle_T} \left\| (F(y+h))(s,t) - (F(y))(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau)) h(s,\tau)\, d\tau \right\|_V \leq \sup_{(s,t)\in\angle_T} \|h(s,t)\|_V \\
&\cdot \sup_{(s,t)\in\angle_T} \left[\int_s^t \left(\int_0^1 \left\| f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau)) \right\|_{L(V)} dr \right) d\tau \right] \\
&\leq T \left[\sup_{(s,t)\in\angle_T} \|h(s,t)\|_V \right] \left[\sup_{r\in[0,1]} \sup_{(s,\tau)\in\angle_T} \| f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau)) \|_{L(V)} \right].
\end{aligned} \tag{108}$$

Moreover, observe that Lemma 4.2 (with $V = V$, $\varepsilon = \varepsilon$, $T = T$, $y = y$, $f = f$ for $y \in \mathcal{C}(\angle_T, V)$, $\varepsilon \in (0,\infty)$ in the notation of Lemma 4.2) shows that for all $y \in \mathcal{C}(\angle_T, V)$, $\varepsilon \in (0,\infty)$ there exists $\delta \in (0,\infty)$ such that for all $h \in \mathcal{C}(\angle_T, V)$ with $\sup_{(u,\tau)\in\angle_T} \|h(u,\tau)\|_V < \delta$ it holds that

$$\sup\nolimits_{r\in[0,1]} \sup\nolimits_{(s,\tau)\in\angle_T} \| f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau)) \|_{L(V)} < \varepsilon. \tag{109}$$

Combining this with (108) implies that for all $y \in \mathcal{C}(\angle_T, V)$ it holds that

$$\limsup_{(\mathcal{C}(\angle_T, V)\setminus\{0\}) \ni h \to 0} \frac{\sup_{(s,t)\in\angle_T} \|(F(y+h))(s,t) - (F(y))(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau))\, h(s,\tau)\, d\tau\|_V}{\sup_{(s,t)\in\angle_T} \|h(s,t)\|_V} = 0. \tag{110}$$

Lemma 4.1 (with $U = V$, $V = V$, $W = V$, $T = T$, $f = ([0,T] \times V \ni (t,x) \mapsto f_{0,1}(t,x) \in L(V))$, $y = y$, $h = h$ in the notation of Lemma 4.1) therefore proves that F is Fréchet differentiable and that for all $y, h \in \mathcal{C}(\angle_T, V)$, $(s,t) \in \angle_T$ it holds that

$$(F'(y)h)(s,t) = \int_s^t f_{0,1}(\tau, y(s,\tau))\, h(s,\tau)\, d\tau. \tag{111}$$

This ensures that for all $y, g, h \in \mathcal{C}(\angle_T, V)$ it holds that

$$\begin{aligned}
&\sup_{(s,t)\in\angle_T} \|(F'(y+h)g)(s,t) - (F'(y)g)(s,t)\|_V \\
&\leq \sup_{(s,t)\in\angle_T} \int_s^t \|f_{0,1}(\tau, y(s,\tau) + h(s,\tau))\, g(s,\tau) - f_{0,1}(\tau, y(s,\tau))\, g(s,\tau)\|_V \, d\tau \\
&\leq T \sup_{(s,\tau)\in\angle_T} \|f_{0,1}(\tau, y(s,\tau) + h(s,\tau)) - f_{0,1}(\tau, y(s,\tau))\|_{L(V)} \sup_{(s,\tau)\in\angle_T} \|g(s,\tau)\|_V.
\end{aligned} \tag{112}$$

Combining this with (109) shows that for all $y \in \mathcal{C}(\angle_T, V)$ it holds that

$$\limsup_{\mathcal{C}(\angle_T,V)\ni h\to 0} \|F'(y+h) - F'(y)\|_{L(\mathcal{C}(\angle_T,V),\mathcal{C}(\angle_T,V))} = 0. \tag{113}$$

The proof of Lemma 4.3 is thus completed. ☐

Lemma 4.4. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be nontrivial $\mathbb{R}$-Banach spaces, let $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}^1(Y, Z)$, assume that $g \circ f \in \mathcal{C}^1(X, Z)$, and assume for all $x \in X$ that $g'(f(x))$ is bijective and that $[g'(f(x))]^{-1} \in L(Z, Y)$. Then*

(i) it holds that $f \in \mathcal{C}^1(X, Y)$ and

(ii) it holds for all $x \in X$ that $f'(x) = [g'(f(x))]^{-1}(g \circ f)'(x)$.

Proof of Lemma 4.4. Throughout this proof let $o_1, o_2\colon X \times X \to Z$ be the functions which satisfy for all $x, h \in X$ that

$$o_1(x,h) = g(f(x+h)) - g(f(x)) - g'(f(x))(f(x+h) - f(x)) \tag{114}$$

and

$$o_2(x,h) = g(f(x+h)) - g(f(x)) - (g \circ f)'(x)h. \tag{115}$$

Observe that (114) and (115) imply for all $x, h \in X$ that

$$\begin{aligned}
f(x+h) - f(x) &= [g'(f(x))]^{-1}\big((g \circ f)'(x)h + o_2(x,h) - o_1(x,h)\big) \\
&= [g'(f(x))]^{-1}(g \circ f)'(x)h + [g'(f(x))]^{-1}o_2(x,h) - [g'(f(x))]^{-1}o_1(x,h).
\end{aligned} \tag{116}$$

Moreover, note that the fact that f is continuous and the assumption that g is differentiable assure that for every $x \in X$ there exists a function $w\colon X \to [0,\infty)$ such that for every $h \in X$ it holds that $\limsup_{X\ni u\to 0} w(u) = 0$ and

$$\|o_1(x,h)\|_Z = w(h) \cdot \|f(x+h) - f(x)\|_Y. \tag{117}$$

This shows that for every $x \in X$ there exists $\delta \in (0,\infty)$ such that for all $h \in X$ with $\|h\|_X < \delta$ it holds that

$$\|[g'(f(x))]^{-1}\|_{L(Z,Y)} \|o_1(x,h)\|_Z \leq \tfrac{1}{2}\|f(x+h) - f(x)\|_Y. \tag{118}$$

Equation (116) hence proves that for every $x \in X$ there exists $\delta \in (0,\infty)$ such that for all $h \in X$ with $\|h\|_X < \delta$ it holds that

$$\begin{aligned}
\|f(x+h) - f(x)\|_Y &\leq \|[g'(f(x))]^{-1}(g \circ f)'(x)\|_{L(X,Y)} \|h\|_X \\
&\quad + \|[g'(f(x))]^{-1}\|_{L(Z,Y)} \|o_2(x,h)\|_Z + \tfrac{1}{2}\|f(x+h) - f(x)\|_Y.
\end{aligned} \tag{119}$$

Therefore, we establish that for every $x \in X$ there exists $\delta \in (0, \infty)$ such that for all $h \in X$ with $\|h\|_X < \delta$ it holds that

$$\tfrac{\|f(x+h)-f(x)\|_Y}{2} \leq \|[g'(f(x))]^{-1}(g \circ f)'(x)\|_{L(X,Y)}\|h\|_X + \|[g'(f(x))]^{-1}\|_{L(Z,Y)}\|o_2(x,h)\|_Z. \quad (120)$$

Furthermore, note that the fact that

$$\forall\, x \in X\colon \limsup_{(X\backslash\{0\})\ni h\to 0} \tfrac{\|o_2(x,h)\|_Z}{\|h\|_X} = 0 \quad (121)$$

ensures that for every $x \in X$ there exists $\delta \in (0, \infty)$ such that for all $h \in X$ with $0 < \|h\|_X < \delta$ it holds that

$$\tfrac{\|[g'(f(x))]^{-1}\|_{L(Z,Y)}\|o_2(x,h)\|_Z}{\|h\|_X} \leq \tfrac{1}{2}. \quad (122)$$

Equation (120) hence proves that for every $x \in X$ there exists $\delta \in (0, \infty)$ such that for all $h \in X$ with $0 < \|h\|_X < \delta$ it holds that

$$\begin{aligned} \tfrac{\|f(x+h)-f(x)\|_Y}{\|h\|_X} &\leq 2\|[g'(f(x))]^{-1}(g \circ f)'(x)\|_{L(X,Y)} + 2\tfrac{\|[g'(f(x))]^{-1}\|_{L(Z,Y)}\|o_2(x,h)\|_Z}{\|h\|_X} \\ &\leq 2\|[g'(f(x))]^{-1}(g \circ f)'(x)\|_{L(X,Y)} + 1. \end{aligned} \quad (123)$$

This shows that for every $x \in X$ there exists $\delta \in (0, \infty)$ such that for all $h \in \{y \in X\colon f(x+y) \neq f(x)\}$ with $0 < \|h\|_X < \delta$ it holds that

$$\tfrac{\|o_1(x,h)\|_Z}{\|f(x+h)-f(x)\|_Y} = \tfrac{\|o_1(x,h)\|_Z}{\|h\|_X} \cdot \tfrac{\|h\|_X}{\|f(x+h)-f(x)\|_Y} \geq \tfrac{\|o_1(x,h)\|_Z}{\|h\|_X} \cdot \tfrac{1}{2\|[g'(f(x))]^{-1}(g\circ f)'(x)\|_{L(X,Y)}+1}. \quad (124)$$

Combining this with (117) implies that for every $x \in X$ it holds that

$$\limsup_{(X\backslash\{0\})\ni h\to 0} \tfrac{\|o_1(x,h)\|_Z}{\|h\|_X} = 0. \quad (125)$$

Equation (116) therefore establishes that for every $x \in X$ there exists $\delta \in (0, \infty)$ such that for all $h \in X$ with $0 < \|h\|_X < \delta$ it holds that

$$\begin{aligned} &\tfrac{\|f(x+h)-f(x)-[g'(f(x))]^{-1}(g\circ f)'(x)h\|_Y}{\|h\|_X} = \tfrac{\|[g'(f(x))]^{-1}(o_2(x,h)-o_1(x,h))\|_Y}{\|h\|_X} \\ &\leq \|[g'(f(x))]^{-1}\|_{L(Z,Y)} \tfrac{\|o_2(x,h)-o_1(x,h)\|_Z}{\|h\|_X} \leq \|[g'(f(x))]^{-1}\|_{L(Z,Y)} \Big[\tfrac{\|o_2(x,h)\|_Z}{\|h\|_X} + \tfrac{\|o_1(x,h)\|_Z}{\|h\|_X}\Big]. \end{aligned} \quad (126)$$

This, (121), and (125) demonstrate that f is differentiable and that item (ii) holds. The fact that $\{B \in L(Y,Z)\colon B \text{ is invertible}\} \ni C \mapsto C^{-1} \in \{B \in L(Z,Y)\colon B \text{ is invertible}\}$ is continuous (cf., e.g., Deitmar & Echterhoff [4, Lemma 2.1.5]) and the fact that $X \ni x \mapsto g'(f(x)) \in L(Y,Z)$ is continuous assure that $X \ni x \mapsto [g'(f(x))]^{-1} \in L(Z,Y)$ is continuous. Item (ii) hence establishes item (i). The proof of Lemma 4.4 is thus completed. ☐

Lemma 4.5. *Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space and let $T \in (0,\infty)$, $\angle_T = \{(s,t) \in [0,T]^2\colon s \leq t\}$, $\phi \in \mathcal{C}(\angle_T, V)$, $A \in \mathcal{C}(\angle_T, L(V))$. Then there exists a unique function $y \in \mathcal{C}(\angle_T, V)$ such that for all $(s,t) \in \angle_T$ it holds that*

$$y(s,t) = \phi(s,t) + \int_s^t A(s,\tau)y(s,\tau)\, d\tau. \quad (127)$$

Proof of Lemma 4.5. Throughout this proof let $\mathcal{A}\colon \mathbb{R}\times[0,T]\to L(V)$ be a continuous function which satisfies for all $(s,\tau)\in\angle_T$ that $\mathcal{A}(s,\tau)=A(s,\tau)$, let $f_n\colon \mathcal{C}(\angle_T,V)\to\mathcal{C}(\angle_T,V)$, $n\in\mathbb{N}$, be the functions which satisfy for all $n\in\mathbb{N}$, $x\in\mathcal{C}(\angle_T,V)$, $(s,t)\in\angle_T$ that $(f_1(x))(s,t)=\phi(s,t)+\int_s^t A(s,\tau)x(s,\tau)\,d\tau$ (see Lemma 4.1 (with $U=\mathbb{R}$, $V=V$, $W=V$, $T=T$, $f=([0,T]\times\mathbb{R}\ni(\tau,s)\mapsto\mathcal{A}(s,\tau)\in L(V))$, $y=(\angle_T\ni(s,\tau)\mapsto s\in\mathbb{R})$, $h=x$ for $x\in\mathcal{C}(\angle_T,V)$ in the notation of Lemma 4.1)) and $(f_{n+1}(x))(s,t)=(f_n(f_1(x)))(s,t)$, and let $\mathcal{N}\subseteq\mathbb{N}$ satisfy that

$$\mathcal{N}=\left\{\begin{array}{r} n\in\mathbb{N}\colon \Big(\forall\, x,y\in\mathcal{C}(\angle_T,V),(s,t)\in\angle_T\colon \|(f_n(x))(s,t)-(f_n(y))(s,t)\|_V \\ \leq \frac{[\sup_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}(t-s)]^n}{n!}\sup_{(u,\tau)\in\angle_T}\|x(u,\tau)-y(u,\tau)\|_V\Big)\end{array}\right\}. \tag{128}$$

Note that for all $x,y\in\mathcal{C}(\angle_T,V)$, $(s,t)\in\angle_T$ it holds that

$$\begin{aligned}
&\|(f_1(x))(s,t)-(f_1(y))(s,t)\|_V=\left\|\int_s^t A(s,\tau)[x(s,\tau)-y(s,\tau)]\,d\tau\right\|_V\\
&\leq\int_s^t\|A(s,\tau)[x(s,\tau)-y(s,\tau)]\|_V\,d\tau\\
&\leq(t-s)\sup\nolimits_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}\sup\nolimits_{(u,\tau)\in\angle_T}\|x(u,\tau)-y(u,\tau)\|_V.
\end{aligned}\tag{129}$$

This implies that $1\in\mathcal{N}$ and that f_1 is continuous. Moreover, observe that for all $n\in\mathcal{N}$, $x,y\in\mathcal{C}(\angle_T,V)$, $(s,t)\in\angle_T$ it holds that

$$\begin{aligned}
&\|(f_{n+1}(x))(s,t)-(f_{n+1}(y))(s,t)\|_V\\
&=\left\|\int_s^t A(s,\tau)[(f_n(x))(s,\tau)-(f_n(y))(s,\tau)]\,d\tau\right\|_V\\
&\leq\int_s^t\|A(s,\tau)[(f_n(x))(s,\tau)-(f_n(y))(s,\tau)]\|_V\,d\tau\\
&\leq\sup_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}\int_s^t\|(f_n(x))(s,\tau)-(f_n(y))(s,\tau)\|_V\,d\tau\\
&\leq\sup_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}\sup_{(u,\tau)\in\angle_T}\|x(u,\tau)-y(u,\tau)\|_V\int_s^t\frac{[\sup_{(u,v)\in\angle_T}\|A(u,v)\|_{L(V)}(\tau-s)]^n}{n!}\,d\tau\\
&=\frac{[\sup_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}(t-s)]^{n+1}}{(n+1)!}\sup\nolimits_{(u,\tau)\in\angle_T}\|x(u,\tau)-y(u,\tau)\|_V.
\end{aligned}\tag{130}$$

The fact that $1\in\mathcal{N}$ hence shows that $\mathcal{N}=\mathbb{N}$. This proves that for all $n\in\mathbb{N}$, $x,y\in\mathcal{C}(\angle_T,V)$, $(s,t)\in\angle_T$ it holds that

$$\|(f_n(x))(s,t)-(f_n(y))(s,t)\|_V\leq\frac{[\sup_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}(t-s)]^n}{n!}\sup\nolimits_{(u,\tau)\in\angle_T}\|x(u,\tau)-y(u,\tau)\|_V. \tag{131}$$

In particular, for all $m,n\in\mathbb{N}$, $x\in\mathcal{C}(\angle_T,V)$, $(s,t)\in\angle_T$ it holds that

$$\begin{aligned}
&\|(f_n(x))(s,t)-(f_{n+1}(x))(s,t)\|_V\\
&\leq\frac{[\sup_{(u,\tau)\in\angle_T}\|A(u,\tau)\|_{L(V)}(t-s)]^n}{n!}\sup\nolimits_{(u,\tau)\in\angle_T}\|x(u,\tau)-(f_1(x))(u,\tau)\|_V.
\end{aligned}\tag{132}$$

Therefore, we establish that for all $m, n \in \mathbb{N}$, $x, y \in \mathcal{C}(\angle_T, V)$ it holds that

$$\begin{aligned}
&\sup\nolimits_{(s,t)\in\angle_T} \|(f_n(x))(s,t) - (f_{m+n}(y))(s,t)\|_V \\
&\le \sum_{k=0}^{m-1} \sup\nolimits_{(s,t)\in\angle_T} \|(f_{n+k}(x))(s,t) - (f_{n+k+1}(y))(s,t)\|_V \\
&\le \Big(\sup\nolimits_{(u,\tau)\in\angle_T} \|x(u,\tau) - (f_1(x))(u,\tau)\|_V\Big) \sum_{k=0}^{m-1} \tfrac{[\sup_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} T]^{n+k}}{(n+k)!} \\
&\le \tfrac{(\sup_{(u,\tau)\in\angle_T} \|x(u,\tau)-(f_1(x))(u,\tau)\|_V)[\sup_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} T]^n}{n!} \sum_{k=0}^{m-1} \tfrac{[\sup_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} T]^k}{k!} \\
&\le \tfrac{(\sup_{(u,\tau)\in\angle_T} \|x(u,\tau)-(f_1(x))(u,\tau)\|_V)[\sup_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} T]^n}{n!} \exp\Big(\sup\nolimits_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} T\Big).
\end{aligned} \tag{133}$$

This ensures that for every $x \in \mathcal{C}(\angle_T, V)$ the sequence $(f_n(x))_{n\in\mathbb{N}} \subseteq \mathcal{C}(\angle_T, V)$ is Cauchy. The fact that $\mathcal{C}(\angle_T, V)$ is a Banach space hence shows that for every $x \in \mathcal{C}(\angle_T, V)$ there exists $z \in \mathcal{C}(\angle_T, V)$ such that

$$\limsup\nolimits_{n\to\infty} \Big(\sup\nolimits_{(s,t)\in\angle_T} \|(f_n(x))(s,t) - z(s,t)\|_V\Big) = 0. \tag{134}$$

Moreover, note that for all $x, z \in \mathcal{C}(\angle_T, V)$ with $\limsup_{n\to\infty}(\sup_{(s,t)\in\angle_T} \|f_n(x)(s,t) - z(s,t)\|_V) = 0$ it holds that

$$\begin{aligned}
&\sup\nolimits_{(s,t)\in\angle_T} \|(f_1(z))(s,t) - z(s,t)\|_V \\
&\le \limsup\nolimits_{n\to\infty} \Big(\sup\nolimits_{(s,t)\in\angle_T} \|(f_1(z))(s,t) - (f_1(f_n(x)))(s,t)\|_V\Big) \\
&\quad + \limsup\nolimits_{n\to\infty} \Big(\sup\nolimits_{(s,t)\in\angle_T} \|(f_1(f_n(x)))(s,t) - z(s,t)\|_V\Big) \\
&= \limsup\nolimits_{n\to\infty} \Big(\sup\nolimits_{(s,t)\in\angle_T} \|(f_1(z))(s,t) - (f_1(f_n(x)))(s,t)\|_V\Big) \\
&\quad + \limsup\nolimits_{n\to\infty} \Big(\sup\nolimits_{(s,t)\in\angle_T} \|(f_{n+1}(x))(s,t) - z(s,t)\|_V\Big) \\
&= \limsup\nolimits_{n\to\infty} \Big(\sup\nolimits_{(s,t)\in\angle_T} \|(f_1(z))(s,t) - (f_1(f_n(x)))(s,t)\|_V\Big).
\end{aligned} \tag{135}$$

Combining this with (134) and the fact that $f_1 \colon \mathcal{C}(\angle_T, V) \to \mathcal{C}(\angle_T, V)$ is continuous demonstrates that there exists $z \in \mathcal{C}(\angle_T, V)$ such that

$$f_1(z) = z. \tag{136}$$

In addition, note that (131) implies that for all $N \in \mathbb{N}$, $x, y \in \mathcal{C}(\angle_T, V)$ with $f_1(x) = x$, $f_1(y) = y$ it holds that

$$\begin{aligned}
&\sum_{n=1}^{N} \sup_{(s,t)\in\angle_T} \|x(s,t) - y(s,t)\|_V = \sum_{n=1}^{N} \sup_{(s,t)\in\angle_T} \|(f_n(x))(s,t) - (f_n(y))(s,t)\|_V \\
&\le \sum_{n=1}^{N} \tfrac{[\sup_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} (t-s)]^n}{n!} \sup\nolimits_{(u,\tau)\in\angle_T} \|x(u,\tau) - y(u,\tau)\|_V \\
&\le \Big(\sup\nolimits_{(u,\tau)\in\angle_T} \|x(u,\tau) - y(u,\tau)\|_V\Big) \exp\Big(\sup\nolimits_{(u,\tau)\in\angle_T} \|A(u,\tau)\|_{L(V)} T\Big) < \infty.
\end{aligned} \tag{137}$$

This and (136) show that there exists a unique $z \in \mathcal{C}(\angle_T, V)$ such that $f_1(z) = z$. The proof of Lemma 4.5 is thus completed. ∎

Lemma 4.6. *Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space and let $T \in (0,\infty)$, $\angle_T = \{(s,t) \in [0,T]^2 \colon s \leq t\}$, $\phi \in \mathcal{C}(\angle_T, V)$, $A \in \mathcal{C}(\angle_T, L(V))$, $y \in \mathcal{C}(\angle_T, V)$ satisfy for all $(s,t) \in \angle_T$ that*

$$y(s,t) = \phi(s,t) + \int_s^t A(s,\tau) y(s,\tau) \, d\tau. \tag{138}$$

Then

$$\sup_{(s,t)\in\angle_T} \|y(s,t)\|_V \leq \Big(\sup_{(s,t)\in\angle_T} \|\phi(s,t)\|_V \Big) \exp\Big(T \sup_{(s,t)\in\angle_T} \|A(s,t)\|_{L(V)} \Big). \tag{139}$$

Proof of Lemma 4.6. Note that for all $(s,t) \in \angle_T$ it holds that

$$\begin{aligned}
\|y(s,t)\|_V &\leq \|\phi(s,t)\|_V + \int_s^t \|A(s,\tau)y(s,\tau)\|_V \, d\tau \\
&\leq \|\phi(s,t)\|_V + \sup_{(u,v)\in\angle_T} \|A(u,v)\|_{L(V)} \int_s^t \|y(s,\tau)\|_V \, d\tau \\
&\leq \sup_{(u,v)\in\angle_T} \|\phi(u,v)\|_V + \sup_{(u,v)\in\angle_T} \|A(u,v)\|_{L(V)} \int_s^t \|y(s,\tau)\|_V \, d\tau < \infty.
\end{aligned} \tag{140}$$

Gronwall's lemma hence shows that for all $(s,t) \in \angle_T$ it holds that

$$\|y(s,t)\|_V \leq \Big(\sup_{(u,v)\in\angle_T} \|\phi(u,v)\|_V \Big) \exp\Big(\sup_{(u,v)\in\angle_T} \|A(u,v)\|_{L(V)} (t-s) \Big). \tag{141}$$

The proof of Lemma 4.6 is thus completed. ◻

Lemma 4.7. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0,\infty)$, $\angle_T = \{(s,t) \in [0,T]^2 \colon s \leq t\}$, $f \in \mathcal{C}^{0,1}([0,T]\times V, V)$, let $f_{0,1} \colon [0,T]\times V \to L(V)$ be the function which satisfies for all $t \in [0,T]$, $x \in V$ that $f_{0,1}(t,x) = (\frac{\partial}{\partial x} f)(t,x)$, and for every $x \in V$ let $(X^x_{s,t})_{(s,t)\in\angle_T} \colon \angle_T \to V$ be a continuous function which satisfies for all $(s,t) \in \angle_T$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau$. Then*

(i) it holds that $(\angle_T \times V \ni (s,t,x) \mapsto X^x_{s,t} \in V) \in \mathcal{C}^{0,0,1}(\angle_T \times V, V)$ and

(ii) it holds for all $x, y \in V$, $(s,t) \in \angle_T$ that

$$\Big(\tfrac{\partial}{\partial x} X^x_{s,t}\Big) y = y + \int_s^t f_{0,1}(\tau, X^x_{s,\tau}) \Big(\tfrac{\partial}{\partial x} X^x_{s,\tau}\Big) y \, d\tau. \tag{142}$$

Proof of Lemma 4.7. Throughout this proof let $F \colon \mathcal{C}(\angle_T, V) \to \mathcal{C}(\angle_T, V)$ and $G, H \colon V \to \mathcal{C}(\angle_T, V)$ be the functions which satisfy for all $v \in V$, $(s,t) \in \angle_T$, $z \in \mathcal{C}(\angle_T, V)$ that $(F(z))(s,t) = z(s,t) - \int_s^t f(\tau, z(s,\tau)) \, d\tau$, $(G(v))(s,t) = X^v_{s,t}$, and $(H(v))(s,t) = v$. Note that Lemma 4.3 (with $V = V$, $T = T$, $f = f$, $F = (\mathcal{C}(\angle_T, V) \ni z \mapsto (\angle_T \ni (s,t) \mapsto \int_s^t f(\tau, z(s,\tau)) \, d\tau) \in \mathcal{C}(\angle_T, V))$, $f_{0,1} = f_{0,1}$ in the notation of Lemma 4.3) proves that for all $y, h \in \mathcal{C}(\angle_T, V)$, $(s,t) \in \angle_T$ it holds that $F \in \mathcal{C}^1(\mathcal{C}(\angle_T, V), \mathcal{C}(\angle_T, V))$ and

$$(F'(y)h)(s,t) = h(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau)) h(s,\tau) \, d\tau. \tag{143}$$

Therefore, we obtain that for all $v \in V$, $h \in \mathcal{C}(\angle_T, V)$, $(s,t) \in \angle_T$ it holds that

$$h(s,t) = (F'(G(v))h)(s,t) + \int_s^t f_{0,1}(\tau, G(v)(s,\tau)) h(s,\tau) \, d\tau. \tag{144}$$

Combining this with Lemma 4.5 (with $V = V$, $T = T$, $\phi = (\angle_T \ni (s, \tau) \mapsto (F'(G(v))h)(s, \tau) \in V)$, $A = \big(\angle_T \ni (s, \tau) \mapsto f_{0,1}(\tau, G(v)(s, \tau)) \in L(V)\big)$ for $h \in \mathcal{C}(\angle_T, V)$, $v \in V$ in the notation of Lemma 4.5) shows that for every $v \in V$ it holds that $F'(G(v)) \in L(\mathcal{C}(\angle_T, V), \mathcal{C}(\angle_T, V))$ is invertible. In addition, Lemma 4.6 (with $V = V$, $T = T$, $\phi = (\angle_T \ni (s, \tau) \mapsto (F'(G(v))h)(s, \tau) \in V)$, $A = \big(\angle_T \ni (s, \tau) \mapsto f_{0,1}(\tau, G(v)(s, \tau)) \in L(V)\big)$, $y = h$ for $h \in \mathcal{C}(\angle_T, V)$, $v \in V$ in the notation of Lemma 4.6) ensures that for every $v \in V$ it holds that

$$[F'(G(v))]^{-1} \in L(\mathcal{C}(\angle_T, V), \mathcal{C}(\angle_T, V)). \tag{145}$$

Moreover, observe that for all $v \in V$, $(s, t) \in \angle_T$ it holds that

$$\begin{aligned}(F(G(v)))(s, t) &= (G(v))(s, t) - \int_s^t f(\tau, (G(v))(s, \tau))\, d\tau \\ &= X^v_{s,t} - \int_s^t f(\tau, X^v_{s,\tau})\, d\tau = v = (H(v))(s, t).\end{aligned} \tag{146}$$

Next we intend to prove that

$$G \in \mathcal{C}(V, \mathcal{C}(\angle_T, V)). \tag{147}$$

For this note that Corollary 3.8 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s, t) \in \angle_T$, $x \in V$ in the notation of Corollary 3.8) shows that $\angle_T \times V \ni (s, t, x) \mapsto X^x_{s,t} \in V$ is continuous. This implies that there exists a function $\delta\colon (0, \infty) \times \angle_T \times V \to (0, \infty)$ such that for all $\varepsilon \in (0, \infty)$, $(s_0, t_0), (s, t) \in \angle_T$, $v_0, v \in V$ with $|s_0 - s| + |t_0 - t| + \|v_0 - v\|_V < \delta^\varepsilon_{s_0,t_0,v_0}$ it holds that

$$\|X^{v_0}_{s_0,t_0} - X^v_{s,t}\|_V < \varepsilon/2. \tag{148}$$

The fact that for all $v_0 \in V$, $\varepsilon \in (0, \infty)$ it holds that

$$\angle_T \subseteq \cup_{(s_0,t_0)\in\angle_T} \{(s, t) \in \angle_T \colon |s_0 - s| + |t_0 - t| < \delta^\varepsilon_{s_0,t_0,v_0}/2\} \tag{149}$$

and the compactness of $\angle_T$ prove that for all $v_0 \in V$, $\varepsilon \in (0, \infty)$ there exist $N \in \mathbb{N}$, $(s_1, t_1), \ldots, (s_N, t_N) \in \angle_T$ such that

$$\angle_T = \cup_{n=1}^N \{(s, t) \in \angle_T \colon |s_n - s| + |t_n - t| < \delta^\varepsilon_{s_n,t_n,v_0}/2\}. \tag{150}$$

Combining this with (148) demonstrates that for all $v_0 \in V$, $\varepsilon \in (0, \infty)$ there exist $N \in \mathbb{N}$, $(s_1, t_1), \ldots, (s_N, t_N) \in \angle_T$ such that for all $v \in V$, $(s, t) \in \angle_T$ with $\|v_0 - v\|_H < \frac{1}{2} \min_{n\in[1,N]\cap\mathbb{N}} \delta^\varepsilon_{s_n,t_n,v_0}$ it holds that

$$\|X^{v_0}_{s,t} - X^v_{s,t}\|_H \leq \min_{n\in[1,N]\cap\mathbb{N}} \Big(\|X^{v_0}_{s,t} - X^{v_0}_{s_n,t_n}\|_H + \|X^{v_0}_{s_n,t_n} - X^v_{s,t}\|_H\Big) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{151}$$

This ensures that (147) holds. Furthermore, note that H is continuously differentiable and that for all $v, w \in V$ it holds that

$$H'(v)w = (\angle_T \ni (s, \tau) \mapsto w \in V). \tag{152}$$

Equation (146) hence shows that $F \circ G$ is continuously differentiable. This, (145), (147), the facts that for every $v \in V$ it holds that F is continuously differentiable and $F'(G(v)) \in L(\mathcal{C}(\angle_T, V), \mathcal{C}(\angle_T, V))$ is invertible, and Lemma 4.4 (with $X = V$, $Y = \mathcal{C}(\angle_T, V)$, $Z =$

$\mathcal{C}(\angle_T, V)$, $f = G$, $g = F$ for $v \in V$ in the notation of Lemma 4.4) imply that G is continuously differentiable and that for every $v \in V$ it holds that

$$G'(v) = [F'(G(v))]^{-1} H'(v). \tag{153}$$

Hence, we obtain that

$$(V \ni x \mapsto X^x \in \mathcal{C}(\angle_T, V)) \in \mathcal{C}^1(V, \mathcal{C}(\angle_T, V)). \tag{154}$$

This implies that for every $x \in V$, $\varepsilon \in (0,\infty)$ there exists $\Delta \in (0,\infty)$ such that for all $s \in [0,T]$, $t, u \in [s,T]$, $\tau \in [u,T]$, $y \in V$ with $|s-u| + |t-\tau| + \|x-y\|_V < \Delta$ it holds that

$$\begin{aligned} \|X^x_{s,t} - X^y_{u,\tau}\|_V &\leq \|X^x_{s,t} - X^x_{s,\tau}\|_V + \|X^x_{s,\tau} - X^x_{u,\tau}\|_V + \|X^x_{u,\tau} - X^y_{u,\tau}\|_V \\ &\leq \|X^x_{s,t} - X^x_{s,\tau}\|_V + \|X^x_{s,\tau} - X^x_{u,\tau}\|_V + \sup\nolimits_{(v,r)\in\angle_T} \|X^x_{v,r} - X^y_{v,r}\|_V < \varepsilon. \end{aligned} \tag{155}$$

Moreover, observe that (154) proves that for every $x \in V$, $\varepsilon \in (0,\infty)$ there exists $\Delta \in (0,\infty)$ such that for all $s \in [0,T]$, $t, u \in [s,T]$, $\tau \in [u,T]$, $y \in V$ with $|s-u| + |t-\tau| + \|x-y\|_V < \Delta$ it holds that

$$\begin{aligned} &\left\| \tfrac{\partial}{\partial x} X^x_{s,t} - \tfrac{\partial}{\partial x} X^y_{u,\tau} \right\|_{L(V)} \\ &\leq \left\| \tfrac{\partial}{\partial x} X^x_{s,t} - \tfrac{\partial}{\partial x} X^x_{s,\tau} \right\|_{L(V)} + \left\| \tfrac{\partial}{\partial x} X^x_{s,\tau} - \tfrac{\partial}{\partial x} X^x_{u,\tau} \right\|_{L(V)} + \left\| \tfrac{\partial}{\partial x} X^x_{u,\tau} - \tfrac{\partial}{\partial x} X^y_{u,\tau} \right\|_{L(V)} \\ &\leq \left\| \tfrac{\partial}{\partial x} X^x_{s,t} - \tfrac{\partial}{\partial x} X^x_{s,\tau} \right\|_{L(V)} + \left\| \tfrac{\partial}{\partial x} X^x_{s,\tau} - \tfrac{\partial}{\partial x} X^x_{u,\tau} \right\|_{L(V)} + \sup_{(v,r)\in\angle_T} \left\| \tfrac{\partial}{\partial x} X^x_{v,r} - \tfrac{\partial}{\partial x} X^y_{v,r} \right\|_{L(V)} < \varepsilon. \end{aligned} \tag{156}$$

This and (155) establish item (i). In addition, note that (152) and (153) ensure that for all $v, w \in V$, $(s,t) \in \angle_T$ it holds that $(F'(G(v))G'(v)w)(s,t) = w$. Display (143) hence demonstrates that for all $(s,t) \in \angle_T$, $v, w \in V$ it holds that

$$\begin{aligned} w = (F'(G(v))G'(v)w)(s,t) &= (G'(v)w)(s,t) - \int_s^t f_{0,1}(\tau, G(v)(\tau))(G'(v)w)(s,\tau)\, d\tau \\ &= \tfrac{\partial}{\partial x} X^v_{s,t} w - \int_s^t f_{0,1}(\tau, X^v_{s,\tau}) \big(\tfrac{\partial}{\partial x} X^v_{s,\tau} w\big)\, d\tau. \end{aligned} \tag{157}$$

The proof of Lemma 4.7 is thus completed. ◻

Lemma 4.8. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0,\infty)$, $\angle_T = \{(s,t) \in [0,T]^2 \colon s \leq t\}$, $f \in \mathcal{C}^{0,1}([0,T] \times V, V)$, let $f_{0,1} \colon [0,T] \times V \to L(V)$ be the function which satisfies for all $t \in [0,T]$, $x \in V$ that $f_{0,1}(t,x) = (\frac{\partial}{\partial x} f)(t,x)$, and for every $x \in V$, $s \in [0,T]$ let $X^x_{s,(\cdot)} = (X^x_{s,t})_{t\in[s,T]} \colon [s,T] \to V$ be a continuous function which satisfies for all $t \in [s,T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau$. Then*

(i) it holds for all $x \in V$, $t \in (0,T]$ that $([0,t] \ni s \mapsto X^x_{s,t} \in V) \in \mathcal{C}^1([0,t], V)$,

(ii) it holds that $\{(r,u) \in [0,T] \times (0,T] \colon r \leq u\} \times V \ni (s,t,x) \mapsto \frac{\partial}{\partial s} X^x_{s,t} \in V$ is continuous,

(iii) it holds that there exists a unique continuous function $C \colon \{(r,u) \in [0,T]^2 \colon r \leq u\} \times V \to V$ which satisfies for all $t \in (0,T]$, $s \in [0,t]$, $x \in V$ that $C^x_{s,t} = \frac{\partial}{\partial s} X^x_{s,t}$,

(iv) it holds for all $x \in V$, $t \in (0,T]$, $s \in [0,t]$ that

$$\tfrac{\partial}{\partial s} X^x_{s,t} = -f(s,x) + \int_s^t f_{0,1}(\tau, X^x_{s,\tau}) (\tfrac{\partial}{\partial s} X^x_{s,\tau})\, d\tau, \tag{158}$$

(v) *it holds that* $(\angle_T \times V \ni (s,t,x) \mapsto X^x_{s,t} \in V) \in \mathcal{C}^{0,0,1}(\angle_T \times V, V)$,

(vi) *it holds for all* $x, y \in V$, $(s,t) \in \angle_T$ *that*

$$(\tfrac{\partial}{\partial x} X^x_{s,t})y = y + \int_s^t f_{0,1}(\tau, X^x_{s,\tau})(\tfrac{\partial}{\partial x} X^x_{s,\tau})y \, d\tau, \tag{159}$$

(vii) *it holds for all* $x \in V$, $t \in (0,T]$, $s \in [0,t]$ *that*

$$\tfrac{\partial}{\partial s} X^x_{s,t} = -\big(\tfrac{\partial}{\partial x} X^x_{s,t}\big) f(s,x), \tag{160}$$

(viii) *it holds for all* $x \in V$, $s \in [0,T)$ *that* $([s,T] \ni t \mapsto X^x_{s,t} \in V) \in \mathcal{C}^1([s,T], V)$,

(ix) *it holds that* $\{(r,u) \in [0,T) \times [0,T] \colon r \leq u\} \times V \ni (s,t,x) \mapsto \frac{\partial}{\partial t} X^x_{s,t} \in V$ *is continuous,*

(x) *it holds that there exists a unique continuous function* $D\colon \{(r,u) \in [0,T]^2 \colon r \leq u\} \times V \to V$ *which satisfies for all* $s \in [0,T)$, $t \in [s,T]$, $x \in V$ *that* $D^x_{s,t} = \frac{\partial}{\partial t} X^x_{s,t}$, *and*

(xi) *it holds for all* $x \in V$, $s \in [0,T)$, $t \in [s,T]$ *that* $\frac{\partial}{\partial t} X^x_{s,t} = f(t, X^x_{s,t})$.

Proof of Lemma 4.8. Throughout this proof let $g\colon [-T,T]^3 \times V \to \mathbb{R}$ be a function which satisfies for all $s \in [0,T]$, $h \in [-s, T-s]\backslash\{0\}$, $\tau \in [\max\{s, s+h\}, T]$, $x \in V$ with $X^x_{s+h,\tau} - X^x_{s,\tau} \neq 0$ that $g(s,h,\tau,x) = \frac{\|f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau}) - f_{0,1}(\tau, X^x_{s,\tau})(X^x_{s+h,\tau} - X^x_{s,\tau})\|_V}{\|X^x_{s+h,\tau} - X^x_{s,\tau}\|_V}$, which satisfies for all $s \in [0,T]$, $h \in [-s, T-s]$, $\tau \in [\max\{s, s+h\}, T]$, $x \in V$ with $X^x_{s+h,\tau} - X^x_{s,\tau} = 0$ that $g(s,h,\tau,x) = 0$, and which satisfies for all $s, h \in [-T,T]$, $\tau \in [-T, \max\{s, s+h\})$, $x \in V$ that $g(s,h,\tau,x) = 0$. Observe that Corollary 3.8 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of Corollary 3.8) ensures that

$$(\angle_T \times V \ni (s,t,x) \mapsto X^x_{s,t} \in V) \in \mathcal{C}(\angle_T \times V, V). \tag{161}$$

This and the assumption that $f \in \mathcal{C}^{0,1}([0,T] \times V, V)$ show that for every $x \in V$ it holds that $(\angle_T \ni (s,t) \mapsto f_{0,1}(s, X^x_{s,t}) \in L(V)) \in \mathcal{C}(\angle_T, L(V))$. Lemma 4.5 (with $V = V$, $T = T$, $\phi = (\angle_T \ni (s,t) \mapsto -f(s,x) \in V)$, $A = (\angle_T \ni (s,t) \mapsto f_{0,1}(s, X^x_{s,t}) \in L(V))$ for $x \in V$ in the notation of Lemma 4.5) therefore proves that for every $x \in V$ there exists a unique function $Y^x \in \mathcal{C}(\angle_T, V)$ such that for all $(s,t) \in \angle_T$ it holds that

$$Y^x_{s,t} = -f(s,x) + \int_s^t f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau} \, d\tau. \tag{162}$$

This ensures that there exists a function $q\colon [-T,T]^3 \times V \to \mathbb{R}$ which satisfies for all $s \in [0,T]$, $h \in [-s, T-s]\backslash\{0\}$, $\tau \in [\max\{s, s+h\}, T]$, $x \in V$ that $q(s,h,\tau,x) = \|{(X^x_{s+h,\tau} - X^x_{s,\tau})}/{h} - Y^x_{s,\tau}\|_V$ and which satisfies for all $s \in [0,T]$, $h \in [-T,T]$, $\tau \in [-T, \max\{s, s+h\})$, $x \in V$ that $q(s,h,\tau,x) = 0$. In the next step we note that the triangle inequality implies that for all $x \in V$, $s \in [0,T]$, $h \in [-s, T-s]\backslash\{0\}$, $t \in [\max\{s+h, s\}, T]$ it holds that

$$\begin{aligned}
\Big\| \tfrac{X^x_{s+h,t} - X^x_{s,t}}{h} - Y^x_{s,t} \Big\|_V &= \Big\| -\tfrac{1}{|h|} \int_{\min\{s,s+h\}}^{\max\{s,s+h\}} f(\tau, X^x_{\min\{s,s+h\},\tau}) \, d\tau + f(s,x) \\
&\quad + \int_{\max\{s,s+h\}}^t \tfrac{f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau})}{h} \, d\tau - \int_s^t f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau} \, d\tau \Big\|_V \\
&\leq \Big\| -\int_{\min\{s,s+h\}}^{\max\{s,s+h\}} \tfrac{1}{|h|} f(\tau, X^x_{\min\{s,s+h\},\tau}) \, d\tau + f(s,x) \Big\|_V + \Big\| \int_s^{\max\{s,s+h\}} f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau} \, d\tau \Big\|_V \\
&\quad + \int_{\max\{s,s+h\}}^t \Big\| \tfrac{f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau})}{h} - f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau} \Big\|_V \, d\tau.
\end{aligned} \tag{163}$$

Next we intend to prove an upper bound for the l.h.s. of (163). To do so, we will estimate the terms on the r.h.s. of (163) separately. For this note that the fact that $\forall\, x \in V \colon (\angle_T \ni (s,t) \mapsto X^x_{s,t} \in V) \in \mathcal{C}(\angle_T, V)$ shows that for all $x \in V$, $s \in [0,T]$ it holds that

$$\sup_{h \in [-s,T-s]\backslash\{0\}} \left\| \int_{\min\{s,s+h\}}^{\max\{s,s+h\}} \tfrac{1}{|h|} f(\tau, X^x_{\min\{s,s+h\},\tau})\, d\tau - f(s,x) \right\|_V < \infty. \tag{164}$$

In addition, the assumption that $\forall\, x \in V, s \in [0,T] \colon ([s,T] \ni t \mapsto X^x_{s,t} \in V) \in \mathcal{C}([s,T],V)$ and the fact that $\forall\, x \in V \colon (\angle_T \ni (s,t) \mapsto Y^x_{s,t} \in V) \in \mathcal{C}(\angle_T, V)$ ensure for all $x \in V$, $s \in [0,T]$ that

$$\sup_{h \in [-s,T-s]\backslash\{0\}} \left\| \int_s^{\max\{s,s+h\}} f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau}\, d\tau \right\|_V < \infty. \tag{165}$$

Moreover, the triangle inequality proves that for all $x \in V$, $s \in [0,T]$, $h \in [-s, T-s]\backslash\{0\}$, $t \in [\max\{s,s+h\}, T]$ it holds that

$$\begin{aligned}
&\int_{\max\{s,s+h\}}^t \left\| \tfrac{f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau})}{h} - f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau} \right\|_V d\tau \\
&\leq \int_{\max\{s,s+h\}}^t \left\| \tfrac{f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau}) - f_{0,1}(\tau, X^x_{s,\tau})(X^x_{s+h,\tau} - X^x_{s,\tau})}{h} \right\|_V d\tau \\
&\quad + \int_{\max\{s,s+h\}}^t \left\| \tfrac{f_{0,1}(\tau, X^x_{s,\tau})(X^x_{s+h,\tau} - X^x_{s,\tau} - Y^x_{s,\tau} h)}{h} \right\|_V d\tau.
\end{aligned} \tag{166}$$

Furthermore, note that (161) assures that there exists $\delta \colon (0,\infty) \times V \to (0,\infty)$ such that for all $(s_1,t_1), (s_2,t_2) \in \angle_T$, $\varepsilon \in (0,\infty)$, $x \in V$ with $\max\{|s_1 - s_2|, |t_1 - t_2|\} < \delta^x_\varepsilon$ it holds that

$$\|X^x_{s_1,t_1} - X^x_{s_2,t_2}\|_V < \varepsilon. \tag{167}$$

This implies that for all $\varepsilon \in (0,\infty)$, $x \in V$, $s \in [0,T]$, $h \in [-s,T-s]\backslash\{0\}$, $t \in [\max\{s,s+h\},T]$ with $|h| < \delta^x_\varepsilon$ it holds that

$$\begin{aligned}
&\int_{\max\{s,s+h\}}^t \left\| \tfrac{f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau}) - f_{0,1}(\tau, X^x_{s,\tau})(X^x_{s+h,\tau} - X^x_{s,\tau})}{h} \right\|_V d\tau \\
&\leq \int_{\max\{s,s+h\}}^t \left(\tfrac{\|f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau})\|_V}{|h|} + \tfrac{\|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \|X^x_{s+h,\tau} - X^x_{s,\tau}\|_V}{|h|} \right) d\tau \\
&\leq \left(\sup_{\tau \in [0,T]} \sup_{y \in V, \|x-y\|_V \leq \varepsilon} \sup_{z \in V\backslash\{y\}, \|x-z\|_V \leq \varepsilon} \tfrac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} + \sup_{\tau \in [s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right) \\
&\quad \cdot \int_{\max\{s,s+h\}}^t \tfrac{\|X^x_{s+h,\tau} - X^x_{s,\tau}\|_V}{|h|}\, d\tau.
\end{aligned} \tag{168}$$

Therefore, it holds for all $\varepsilon \in (0,\infty)$, $x \in V$, $s \in [0,T]$, $h \in [-s,T-s]\backslash\{0\}$, $t \in [\max\{s,s+h\},T]$ with $|h| < \delta^x_\varepsilon$ that

$$\begin{aligned}
&\int_{\max\{s,s+h\}}^t \left\| \tfrac{f(\tau, X^x_{s+h,\tau}) - f(\tau, X^x_{s,\tau}) - f_{0,1}(\tau, X^x_{s,\tau})(X^x_{s+h,\tau} - X^x_{s,\tau})}{h} \right\|_V d\tau \\
&\leq \left(\sup_{\tau \in [0,T]} \sup_{y \in V, \|x-y\|_V \leq \varepsilon} \sup_{z \in V\backslash\{y\}, \|x-z\|_V \leq \varepsilon} \tfrac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} + \sup_{\tau \in [s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right) \\
&\quad \cdot \left(\int_{\max\{s,s+h\}}^t \tfrac{\|X^x_{s+h,\tau} - X^x_{s,\tau} - Y^x_{s,\tau} h\|_V}{|h|}\, d\tau + \int_{\max\{s,s+h\}}^t \|Y^x_{s,\tau}\|_V\, d\tau \right).
\end{aligned} \tag{169}$$

Furthermore, observe that for all $x \in V$, $s \in [0,T]$, $h \in [-s, T-s]\backslash\{0\}$, $t \in [\max\{s, s+h\}, T]$ it holds that

$$\begin{aligned}
&\int_{\max\{s,s+h\}}^{t} \left\| \tfrac{f_{0,1}(\tau, X^x_{s,\tau})(X^x_{s+h,\tau} - X^x_{s,\tau} - Y^x_{s,\tau} h)}{h} \right\|_V d\tau \\
&\leq \int_{\max\{s,s+h\}}^{t} \| f_{0,1}(\tau, X^x_{s,\tau}) \|_{L(V)} \tfrac{\| X^x_{s+h,\tau} - X^x_{s,\tau} - Y^x_{s,\tau} h \|_V}{|h|} \, d\tau \\
&\leq \sup\nolimits_{\tau \in [s,T]} \| f_{0,1}(\tau, X^x_{s,\tau}) \|_{L(V)} \int_{\max\{s,s+h\}}^{t} \tfrac{\| X^x_{s+h,\tau} - X^x_{s,\tau} - Y^x_{s,\tau} h \|_V}{|h|} \, d\tau.
\end{aligned} \tag{170}$$

Combining (163), (166), and (169) hence proves that for all $\varepsilon \in (0,\infty)$, $x \in V$, $s \in [0,T]$, $h \in [-s, T-s]\backslash\{0\}$, $t \in [\max\{s, s+h\}, T]$ with $|h| < \delta^x_\varepsilon$ it holds that

$$\begin{aligned}
&\left\| \tfrac{X^x_{s+h,t} - X^x_{s,t}}{h} - Y^x_{s,t} \right\|_V \\
&\leq \left\| \int_{\min\{s,s+h\}}^{\max\{s,s+h\}} \tfrac{1}{|h|} f(\tau, X^x_{\min\{s,s+h\},\tau}) \, d\tau - f(s,x) \right\|_V + \left\| \int_{s}^{\max\{s,s+h\}} f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau} \, d\tau \right\|_V \\
&\quad + \left(\sup_{\tau \in [0,T]} \sup_{y \in V, \|x-y\|_V \leq \varepsilon} \sup_{z \in V \backslash \{y\}, \|x-z\|_V \leq \varepsilon} \tfrac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} + 2 \sup_{\tau \in [s,T]} \| f_{0,1}(\tau, X^x_{s,\tau}) \|_{L(V)} \right) \\
&\quad \cdot \left(\int_{\max\{s,s+h\}}^{t} \left\| \tfrac{X^x_{s+h,\tau} - X^x_{s,\tau}}{h} - Y^x_{s,\tau} \right\|_V d\tau + \int_{\max\{s,s+h\}}^{t} \| Y^x_{s,\tau} \|_V \, d\tau \right).
\end{aligned} \tag{171}$$

The fact that $\forall\, x \in V \colon (\angle_T \ni (s,t) \mapsto Y^x_{s,t} \in V) \in \mathcal{C}(\angle_T, V)$, (164), (165), and Lemma 3.6 (with $V = V$, $T = T$, $x_0 = x_0$, $f = f$ in the notation of Lemma 3.6) therefore imply that for every $x \in V$, $s \in [0,T]$ there exist $\varepsilon, C \in (0,\infty)$ such that for all $h \in [-s, T-s]\backslash\{0\}$, $t \in [\max\{s, s+h\}, T]$ with $|h| < \delta^x_\varepsilon$ it holds that

$$\left\| \tfrac{X^x_{s+h,t} - X^x_{s,t}}{h} - Y^x_{s,t} \right\|_V \leq C + C \int_{\max\{s,s+h\}}^{t} \left\| \tfrac{X^x_{s+h,\tau} - X^x_{s,\tau}}{h} - Y^x_{s,\tau} \right\|_V d\tau < \infty. \tag{172}$$

Gronwall's lemma hence ensures that for all $x \in V$, $s \in [0,T]$ there exist $\varepsilon, C \in (0,\infty)$ such that for all $h \in [-s, T-s]\backslash\{0\}$, $t \in [\max\{s, s+h\}, T]$ with $|h| < \delta^x_\varepsilon$ it holds that

$$\left\| \tfrac{X^x_{s+h,t} - X^x_{s,t}}{h} - Y^x_{s,t} \right\|_V \leq C \exp(C(t - \max\{s, s+h\})) \leq C e^{Ct}. \tag{173}$$

In the next step we intend to establish that for all $x \in V$, $t \in (0,T]$, $s \in [0,t]$ it holds that

$$\limsup_{([-s,t-s]\backslash\{0\}) \ni h \to 0} \left\| \tfrac{X^x_{s+h,t} - X^x_{s,t}}{h} - Y^x_{s,t} \right\|_V = 0. \tag{174}$$

For this we will analyze the terms on the r.h.s. of (163) separately. First, note that Lemma 2.2 (with $V = V$, $a = s$, $b = T$, $f = ([s,T] \ni \tau \mapsto f(\tau, X^x_{s,\tau}) \in V)$ for $s \in [0,T)$, $x \in V$ in the notation of Lemma 2.2) shows that for all $x \in V$, $s \in [0,T)$ it holds that

$$\limsup_{(0,T-s] \ni h \to 0} \left\| \int_s^{s+h} \tfrac{1}{h} f(\tau, X^x_{s,\tau}) \, d\tau - f(s,x) \right\|_V = 0. \tag{175}$$

In addition, (161) implies for all $x \in V$, $s \in (0,T]$ that

$$\begin{aligned}
&\limsup_{[-s,0)\ni h\to 0} \left\| \int_{s+h}^{s} \tfrac{1}{h} f(\tau, X^x_{s+h,\tau})\, d\tau + f(s,x) \right\|_V \\
&= \limsup_{[-s,0)\ni h\to 0} \left\| \int_{s+h}^{s} \tfrac{1}{h} (f(\tau, X^x_{s+h,\tau}) - f(s,x))\, d\tau \right\|_V \\
&\leq \limsup_{[-s,0)\ni h\to 0} \left(\tfrac{1}{|h|} \int_{s+h}^{s} d\tau \cdot \sup_{\tau\in[s+h,s]} \|f(\tau, X^x_{s+h,\tau}) - f(s,x)\|_V \right) \\
&= \limsup_{[-s,0)\ni h\to 0} \sup_{\tau\in[s+h,s]} \|f(\tau, X^x_{s+h,\tau}) - f(s,x)\|_V = 0.
\end{aligned} \tag{176}$$

This and (175) prove for all $x \in V$, $s \in [0,T]$ that

$$\limsup_{([-s,T-s]\backslash\{0\})\ni h\to 0} \left\| \int_{\min\{s,s+h\}}^{\max\{s,s+h\}} \tfrac{1}{|h|} f(\tau, X^x_{\min\{s,s+h\},\tau})\, d\tau - f(s,x) \right\|_V = 0. \tag{177}$$

Furthermore, observe that the fact that $\forall\, x \in V \colon (\angle_T \ni (s,t) \mapsto Y^x_{s,t} \in V) \in \mathcal{C}(\angle_T, V)$ and (162) prove for all $x \in V$, $s \in [0,T]$ that

$$\begin{aligned}
&\limsup_{([-s,T-s]\backslash\{0\})\ni h\to 0} \left\| \int_{s}^{\max\{s,s+h\}} f_{0,1}(\tau, X^x_{s,\tau}) Y^x_{s,\tau}\, d\tau \right\|_V \\
&= \limsup_{([-s,T-s]\backslash\{0\})\ni h\to 0} \|Y^x_{s,\max\{s,s+h\}} + f(s,x)\|_V = 0.
\end{aligned} \tag{178}$$

Moreover, note that the triangle inequality, (167), and Lemma 3.6 (with $V = V$, $T = T$, $x_0 = x$, $f = f$ for $x \in V$ in the notation of Lemma 3.6) demonstrate that for all $s \in [0,T]$, $x \in V$ there exists $\varepsilon \in (0,\infty)$ such that for all $h \in [-s, T-s]$, $t \in [s,T]$ with $|h| < \delta^x_\varepsilon$ it holds that

$$|g(s,h,t,x)| \leq \left(\sup_{\tau\in[0,T]} \sup_{\substack{y\in V, \|x-y\|_V\leq\varepsilon,\\ z\in V\backslash\{y\}, \|x-z\|_V\leq\varepsilon}} \tfrac{\|f(\tau,y)-f(\tau,z)\|_V}{\|y-z\|_V} + \sup_{\tau\in[s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right) < \infty. \tag{179}$$

Fatou's lemma and (173) therefore ensure that for all $x \in V$, $s \in [0,T]$ there exist $\varepsilon, C \in (0,\infty)$ such that for all $t \in [s,T]$ with $|t| + |s| > 0$ it holds that

$$\begin{aligned}
&\limsup_{([-\min\{s,\delta^x_\varepsilon\},\min\{t-s,\delta^x_\varepsilon\}]\backslash\{0\})\ni h\to 0} \int_{\max\{s,s+h\}}^{t} \left\| \tfrac{f(\tau,X^x_{s+h,\tau})-f(\tau,X^x_{s,\tau})-f_{0,1}(\tau,X^x_{s,\tau})(X^x_{s+h,\tau}-X^x_{s,\tau})}{h} \right\|_V d\tau \\
&\leq \limsup_{([-\min\{s,\delta^x_\varepsilon\},\min\{t-s,\delta^x_\varepsilon\}]\backslash\{0\})\ni h\to 0} \int_s^t g(s,h,\tau,x)\, d\tau \\
&\quad \cdot \left(\sup_{v\in[-\min\{s,\delta^x_\varepsilon\},\min\{t-s,\delta^x_\varepsilon\}]\backslash\{0\}, \tau\in[\max\{s,s+v\},T]} \tfrac{\|X^x_{s+v,\tau}-X^x_{s,\tau}\|_V}{|v|} \right) \\
&\leq \int_s^t \limsup_{([-s,t-s]\backslash\{0\})\ni h\to 0} g(s,h,\tau,x)\, d\tau \cdot \left(Ce^{CT} + \sup\nolimits_{\tau\in[s,T]} \|Y^x_{s,\tau}\|_V \right) = 0.
\end{aligned} \tag{180}$$

Combining (163), (166), (170), (173), (177), (178), and Fatou's lemma hence proves that for all $x \in V$, $s \in [0,T]$, $t \in [s,T]$ with $|t| + |s| > 0$ it holds that

$$\begin{aligned}
&\limsup_{([-s,t-s]\backslash\{0\})\ni h\to 0} q(s,h,t,x) \\
&\leq \sup_{\tau\in[s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \limsup_{([-s,t-s]\backslash\{0\})\ni h\to 0} \int_{\max\{s,s+h\}}^t \tfrac{\|X^x_{s+h,\tau}-X^x_{s,\tau}-Y^x_{s,\tau}h\|_V}{|h|}\,d\tau \\
&= \sup_{\tau\in[s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \limsup_{([-s,t-s]\backslash\{0\})\ni h\to 0} \int_s^t q(s,h,\tau,x)\,d\tau \\
&\leq \sup_{\tau\in[s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \int_s^t \limsup_{([-s,t-s]\backslash\{0\})\ni h\to 0} q(s,h,\tau,x)\,d\tau < \infty.
\end{aligned} \tag{181}$$

Gronwall's lemma hence establishes (174). In particular, we obtain that for all $x \in V$, $t \in (0,T]$, $s \in [0,t]$ it holds that $[0,t] \ni u \mapsto X^x_{u,t} \in V$ is differentiable and that

$$\tfrac{\partial}{\partial s} X^x_{s,t} = Y^x_{s,t}. \tag{182}$$

This and (162) establish items (i) and (iv). Moreover, note that (161) and Lemma 4.7 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$, $y = -f(s,x)$ for $x \in V$, $(s,t) \in \angle_T$ in the notation of Lemma 4.7) prove that items (v) and (vi) hold and that for all $x \in V$, $(s,t) \in \angle_T$ it holds that

$$-\Big(\tfrac{\partial}{\partial x} X^x_{s,t}\Big) f(s,x) = -f(s,x) + \int_s^t f_{0,1}(\tau, X^x_{s,\tau})\Big(-\Big(\tfrac{\partial}{\partial x} X^x_{s,\tau}\Big) f(s,x)\Big)\,d\tau. \tag{183}$$

Furthermore, observe that the fact that $(\angle_T \times V \ni (s,t,x) \mapsto \frac{\partial}{\partial x} X^x_{s,t} \in L(V)) \in \mathcal{C}(\angle_T \times V, L(V))$, the fact that $(\angle_T \times V \ni (s,t,x) \mapsto f(t,x) \in V) \in \mathcal{C}(\angle_T \times V, V)$, and the fact that $(L(V) \times V \ni (A,x) \mapsto Ax \in V) \in \mathcal{C}(L(V) \times V, V)$ ensure that

$$\Big(\angle_T \times V \ni (s,t,x) \mapsto -\Big(\tfrac{\partial}{\partial x} X^x_{s,t}\Big) f(s,x)\Big) \in \mathcal{C}(\angle_T \times V, V). \tag{184}$$

Combining this with (162) and (183) demonstrates that for all $x \in V$, $(s,t) \in \angle_T$ it holds that

$$Y^x_{s,t} = -\Big(\tfrac{\partial}{\partial x} X^x_{s,t}\Big) f(s,x). \tag{185}$$

Equations (182) and (184) therefore establish items (ii), (iii), and (vii). Next observe that Lemma 2.3 (with $V = V$, $a = s$, $b = T$, $f = ([s,T] \ni t \mapsto f(t, X^x_{s,t}) \in V)$, $F = ([s,T] \ni t \mapsto X^x_{s,t} \in V)$ for $s \in [0,T)$, $x \in V$ in the notation of Lemma 2.3) proves that for all $x \in V$, $s \in [0,T)$, $t \in [s,T]$ it holds that

$$\limsup_{([s-t,T-t]\backslash\{0\})\ni h\to 0} \tfrac{\|X^x_{s,t+h}-X^x_{s,t}-f(t,X^x_{s,t})h\|_V}{|h|} = 0. \tag{186}$$

This ensures that for all $x \in V$, $s \in [0,T)$ it holds that $[s,T] \ni t \mapsto X^x_{s,t} \in V$ is differentiable and that for all $x \in V$, $s \in [0,T)$, $t \in [s,T]$ it holds that

$$\tfrac{\partial}{\partial t} X^x_{s,t} = f(t, X^x_{s,t}). \tag{187}$$

Combining this with (161) establishes items (viii), (ix), (x), and (xi). The proof of Lemma 4.8 is thus completed. ☐

5 Alekseev-Gröbner formula

In this section we combine Proposition 2.9, Lemma 3.7, and Lemma 4.8 to establish in Corollary 5.2 an extension of the Alekseev-Gröbner formula (cf., e.g., Hairer et al. [7, Theorem 14.5 in Chapter I]) for Banach space valued functions.

Lemma 5.1. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space, and let $a \in \mathbb{R}$, $b \in (a, \infty)$, $\phi \in \mathcal{C}^{0,1}([a,b] \times V, W)$ satisfy for all $x \in V$ that $([a,b] \ni t \mapsto \phi(t,x) \in W) \in \mathcal{C}^1([a,b], W)$ and $([a,b] \times V \ni (t,y) \mapsto (\frac{\partial}{\partial t}\phi)(t,y) \in W) \in \mathcal{C}([a,b] \times V, W)$. Then there exists $\Phi \in \mathcal{C}^1(\mathbb{R} \times V, W)$ such that for all $t \in [a,b]$, $x \in V$ it holds that $\phi(t,x) = \Phi(t,x)$.*

Proof of Lemma 5.1. Throughout this proof let $\Phi\colon \mathbb{R} \times V \to W$ be the function which satisfies for all $t \in \mathbb{R}$, $x \in V$, $k \in \mathbb{N}$ that

$$\Phi(t,x) = \begin{cases} \phi(t,x) & : (t,x) \in [a,b] \times V \\ 2\phi(a,x) - \phi(2a-t,x) & : (t,x) \in [a-(b-a),a) \times V \\ 2\phi(b,x) - \phi(2b-t,x) & : (t,x) \in (b, b+(b-a)] \times V \end{cases} \tag{188}$$

and

$$\Phi(t,x) = \begin{cases} \phi(t,x) & : (t,x) \in [a,b] \times V \\ 2\Phi(b-k(b-a),x) - \Phi(2(b-k(b-a))-t,x) & : (t,x) \in [a-k(b-a), b-k(b-a)) \times V \\ 2\Phi(a+k(b-a),x) - \Phi(2(a+k(b-a))-t,x) & : (t,x) \in (a+k(b-a), b+k(b-a)] \times V \end{cases}, \tag{189}$$

let $\Phi_{1,0}\colon \mathbb{R} \times V \to W$ be the function which satisfies for all $t \in \mathbb{R}$, $x \in V$, $k \in \mathbb{N}$ that

$$\Phi_{1,0}(t,x) = \begin{cases} (\frac{\partial}{\partial t}\phi)(t,x) & : (t,x) \in [a,b] \times V \\ \Phi_{1,0}(2(b-k(b-a))-t,x) & : (t,x) \in [a-k(b-a), b-k(b-a)) \times V \\ \Phi_{1,0}(2(a+k(b-a))-t,x) & : (t,x) \in (a+k(b-a), b+k(b-a)] \times V \end{cases}, \tag{190}$$

and let $\Phi_{0,1}\colon \mathbb{R} \times V \to L(V,W)$ be the function which satisfies for all $t \in \mathbb{R}$, $x \in V$, $k \in \mathbb{N}$ that

$$\Phi_{0,1}(t,x) = \begin{cases} (\frac{\partial}{\partial x}\phi)(t,x) & : (t,x) \in [a,b] \times V \\ 2\Phi_{0,1}(b-k(b-a),x) - \Phi_{0,1}(2(b-k(b-a))-t,x) & : (t,x) \in [a-k(b-a), b-k(b-a)) \times V \\ 2\Phi_{0,1}(a+k(b-a),x) - \Phi_{0,1}(2(a+k(b-a))-t,x) & : (t,x) \in (a+k(b-a), b+k(b-a)] \times V \end{cases}. \tag{191}$$

Note that the fact that $[a,b] \times V \ni (t,x) \mapsto \phi(t,x) \in W$ is continuous ensures that

$$\Phi \in \mathcal{C}(\mathbb{R} \times V, W). \tag{192}$$

Next observe that for all $t \in \mathbb{R}$, $x \in V$ it holds that $\mathbb{R} \ni s \mapsto \Phi(s,x) \in W$ is differentiable and that

$$\Phi_{1,0}(t,x) = (\tfrac{\partial}{\partial t}\Phi)(t,x). \tag{193}$$

Furthermore, note that the fact that $([a,b] \times V \ni (t,x) \mapsto (\frac{\partial}{\partial t}\phi)(t,x) \in W) \in \mathcal{C}([a,b] \times V, W)$ assures that

$$\Phi_{1,0} \in \mathcal{C}(\mathbb{R} \times V, W). \tag{194}$$

Combining this and (193) proves that

$$(\mathbb{R} \times V \ni (t,x) \mapsto (\tfrac{\partial}{\partial t}\Phi)(t,x) \in W) \in \mathcal{C}(\mathbb{R} \times V, W). \tag{195}$$

In addition, note that for all $t \in \mathbb{R}$, $x \in V$ it holds that $V \ni v \mapsto \Phi(t, v) \in W$ is differentiable and that

$$\Phi_{0,1}(t, x) = (\tfrac{\partial}{\partial x}\Phi)(t, x). \tag{196}$$

Moreover, observe that the fact that $([a, b] \times V \ni (t, x) \mapsto (\frac{\partial}{\partial x}\phi)(t, x) \in L(V, W)) \in \mathcal{C}([a, b] \times V, L(V, W))$ implies that

$$\Phi_{0,1} \in \mathcal{C}(\mathbb{R} \times V, L(V, W)). \tag{197}$$

This and (196) show that

$$(\mathbb{R} \times V \ni (t, x) \mapsto (\tfrac{\partial}{\partial x}\Phi)(t, x) \in L(V, W)) \in \mathcal{C}(\mathbb{R} \times V, L(V, W)). \tag{198}$$

Combining (192), (195), (198), and, e.g., Coleman [3, Corollary 3.4] completes the proof of Lemma 5.1. ☐

Corollary 5.2. *Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0, \infty)$, $f \in \mathcal{C}^{0,1}([0, T] \times V, V)$, let $Y, E\colon [0, T] \to V$ be strongly measurable functions, for every $x \in V$, $s \in [0, T]$ let $X^x_{s,(\cdot)} = (X^x_{s,t})_{t\in[s,T]}\colon [s, T] \to V$ be a continuous function which satisfies for all $t \in [s, T]$ that $X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau})\, d\tau$, and assume for all $t \in [0, T]$ that $\int_0^T [\|f(\tau, Y_\tau)\|_V + \|E_\tau\|_V]\, d\tau < \infty$ and $Y_t = Y_0 + \int_0^t [f(\tau, Y_\tau) + E_\tau]\, d\tau$. Then*

(i) it holds that $(\{(u, r) \in [0, T]^2 : u \leq r\} \times V \ni (s, t, x) \mapsto X^x_{s,t} \in V) \in \mathcal{C}^{0,0,1}(\{(u, r) \in [0, T]^2 : u \leq r\} \times V, V)$,

(ii) it holds that $\{(u, r) \in (0, T)^2 : u < r\} \times V \ni (s, t, x) \mapsto X^x_{s,t} \in V$ is continuously differentiable,

(iii) it holds for all $t \in [0, T]$ that $[0, t] \ni \tau \mapsto (\frac{\partial}{\partial x}X^{Y_\tau}_{\tau,t})E_\tau \in V$ is strongly measurable,

(iv) it holds for all $t \in [0, T]$ that $\int_0^t \|(\frac{\partial}{\partial x}X^{Y_\tau}_{\tau,t})E_\tau\|_V\, d\tau < \infty$, and

(v) it holds for all $s \in [0, T]$, $t \in [s, T]$ that

$$Y_t = X^{Y_s}_{s,t} + \int_s^t (\tfrac{\partial}{\partial x}X^{Y_\tau}_{\tau,t})E_\tau\, d\tau. \tag{199}$$

Proof of Corollary 5.2. Throughout this proof let $\angle_T \subseteq [0, T]^2$ be the set given by $\angle_T = \{(s, t) \in [0, T]^2 \colon s \leq t\}$ and let $\Phi\colon \angle_T \times V \to V$ be the function which satisfies for all $(s, t) \in \angle_T$, $x \in V$ that $\Phi_{s,t}(x) = X^x_{s,t}$. Note that, e.g., Coleman [3, Corollary 3.4] and items (i), (ii), (v), (viii), and (ix) of Lemma 4.8 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s, t) \in \angle_T$, $x \in V$ in the notation of items (i), (ii), (v), (viii), and (ix) of Lemma 4.8) establish items (i) and (ii). It thus remains to prove items (iii)–(v). For this observe that item (x) of Lemma 4.8 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s, t) \in \angle_T$, $x \in V$ in the notation of item (x) of Lemma 4.8) ensures that there exists a unique continuous function $\dot{\Phi}\colon \angle_T \times V \to V$ which satisfies for all $s \in [0, T)$, $t \in [s, T]$, $x \in V$ that

$$\dot{\Phi}_{s,t}(x) = \tfrac{\partial}{\partial t}(\Phi_{s,t}(x)). \tag{200}$$

Next observe that item (i) ensures that there exists a function $\Phi^\star\colon \angle_T \times V \to L(V)$ which satisfies for all $(s, t) \in \angle_T$, $x \in V$ that

$$\Phi^\star_{s,t}(x) = \tfrac{\partial}{\partial x}(\Phi_{s,t}(x)). \tag{201}$$

Moreover, note that items (i), (ii), and (v) of Lemma 4.8 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of items (i), (ii), and (v) of Lemma 4.8) and Lemma 5.1 (with $V = V$, $W = V$, $a = s$, $b = t$, $\phi = ([s,t] \times V \ni (u,x) \mapsto X^x_{u,t} \in V)$ for $s \in [0,T)$, $t \in (s,T]$ in the notation of Lemma 5.1) ensure that for all $s \in [0,T)$, $t \in (s,T]$ it holds that

$$([s,t] \times V \ni (u,x) \mapsto X^x_{u,t} \in V) \in \mathcal{C}^1([s,t] \times V, V). \tag{202}$$

Combining item (ii), Lemma 3.7 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of Lemma 3.7), items (i) and (viii) of Lemma 4.8 (with $V = V$, $T = T$, $f = f$, $X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T$, $x \in V$ in the notation of items (i) and (viii) of Lemma 4.8), and Proposition 2.9 (with $V = V$, $t_0 = s$, $t = t$, $\phi = \operatorname{Id}_V$, $F = Y|_{[s,t]}$, $\Phi = \Phi|_{\{(u,r)\in[s,t]^2\colon u \leq r\}\times V}$, $\dot{\Phi}_{v,w} = \dot{\Phi}_{v,w}$, $\Phi^\star_{v,w} = \Phi^\star_{v,w}$, $f = ([s,t] \ni \tau \mapsto f(\tau, Y_\tau) + E_\tau \in V)$ for $s \in [0,T)$, $t \in (s,T]$, $v \in [s,t]$, $w \in [v,t]$ in the notation of Proposition 2.9) hence demonstrates that for all $(s,t) \in \angle_T$ it holds that

$$[s,t] \ni \tau \mapsto \Phi^\star_{\tau,t}(Y_\tau)\Big[\dot{\Phi}_{\tau,\tau}(Y_\tau) - f(\tau, Y_\tau) - E_\tau\Big] \in V \text{ is strongly measurable,} \tag{203}$$

$$\int_s^t \Big\|\Phi^\star_{\tau,t}(Y_\tau)\Big[\dot{\Phi}_{\tau,\tau}(Y_\tau) - f(\tau, Y_\tau) - E_\tau\Big]\Big\|_V \, d\tau < \infty, \text{ and} \tag{204}$$

$$X^{Y_s}_{s,t} - Y_t = \int_s^t \Phi^\star_{\tau,t}(Y_\tau)\Big[\dot{\Phi}_{\tau,\tau}(Y_\tau) - f(\tau, Y_\tau) - E_\tau\Big] \, d\tau. \tag{205}$$

Moreover, note that Lemma 2.3 (with $V = V$, $a = 0$, $b = t$, $f = ([0,t] \ni s \mapsto f(s, X^x_{s,t}) \in V)$, $F = ([0,t] \ni s \mapsto X^x_{s,t} \in V)$ for $t \in [0,T]$, $x \in V$ in the notation of Lemma 2.3) shows that for all $\tau \in [0,T)$, $t \in [\tau, T]$, $x \in V$ it holds that

$$\dot{\Phi}_{\tau,t}(x) = f(t, X^x_{\tau,t}). \tag{206}$$

Combining this, (203)–(205), and the fact that $\forall\, \tau \in [0,T)$, $t \in [\tau,T]$, $x \in V\colon \Phi^\star_{\tau,t}(x) = \frac{\partial}{\partial x} X^x_{\tau,t}$ establishes items (iii)–(v). The proof of Corollary 5.2 is thus completed. ☐

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References

[1] Alekseev, V. M. An estimate for the perturbations of the solutions of ordinary differential equations. *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, 2 (1961), 28–36.

[2] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[3] Coleman, R. *Calculus on normed vector spaces.* Universitext. Springer, New York, 2012.

[4] Deitmar, A., and Echterhoff, S. *Principles of harmonic analysis*, second ed. Universitext. Springer, Cham, 2014.

[5] Driver, B. *Analysis Tools with Applications.* Preprint, 2003. Available online at http://www.math.ucsd.edu/~bdriver/231-02-03/Lecture_Notes/PDE-Anal-Book/analpde1.pdf.

[6] Gröbner, W. *Die Lie-Reihen und ihre Anwendungen*, vol. 3 of *Mathematische Monographien.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1960.

[7] Hairer, E., Nørsett, S. P., and Wanner, G. *Solving ordinary differential equations. I*, second ed., vol. 8 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, 1993. Nonstiff problems.

[8] Iserles, A. *A First Course in the Numerical Analysis of Differential Equations*, 2nd ed. Cambridge University Press, New York, NY, USA, 2008.

[9] Iserles, A., and Söderlind, G. Global bounds on numerical error for ordinary differential equations. *J. Complexity 9*, 1 (1993), 97–112. Festschrift for Joseph F. Traub, Part I.

[10] Jentzen, A., Salimova, D., and Welti, T. Strong convergence for explicit space-time discrete numerical approximation methods for stochastic Burgers equations. *arXiv:1710.07123* (2017), 60 pages.

[11] Niesen, J. A priori estimates for the global error committed by Runge-Kutta methods for a nonlinear oscillator. *LMS J. Comput. Math. 6* (2003), 18–28.

[12] Prévôt, C., and Röckner, M. *A concise course on stochastic partial differential equations*, vol. 1905 of *Lecture Notes in Mathematics.* Springer, Berlin, 2007.

[13] Rudin, W. *Principles of mathematical analysis*, third ed. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. International Series in Pure and Applied Mathematics.