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First-Kind Galerkin Boundary Element Methods for the Hodge-Laplacian in Three Dimensions

Xavier Claeys and Ralf Hiptmair

Abstract. Boundary value problems for the Euclidean Hodge-Laplacian in three dimensions $-\boldsymbol{\Delta}_{\mathrm{HL}} := \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\,\mathrm{div}$ lead to variational formulations set in subspaces of $\boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain. Via a representation formula and Calderón identities we derive corresponding first-kind boundary integral equations set in trace spaces of $H^1(\Omega)$, $\boldsymbol{H}(\mathbf{curl}, \Omega)$, and $\boldsymbol{H}(\mathrm{div}, \Omega)$. They give rise to saddle-point variational formulations and feature kernels whose dimensions are linked to fundamental topological invariants of Ω.

Kernels of the same dimensions also arise for the linear systems generated by low-order conforming Galerkin boundary element (BE) discretization. On their complements, we can prove stability of the discretized problems, nevertheless. We prove that discretization does not affect the dimensions of the kernels and also illustrate this fact by numerical tests.

Keywords. Hodge-Laplacian, representation formula, Calderón indentities, first-kind boundary integral equations, harmonic vector fields, saddle-point problems, boundary element method (BEM).

AMS classification. 31A10, 45A05, 65N38.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain, connected, but not necessarily bounded, whose boundary $\Gamma := \partial\Omega$ is a compact Lipschitz polyhedron with exterior unit normal vector field $\boldsymbol{n}$. We are concerned with boundary integral equation techniques for numerically solving boundary value problems for the Hodge-Laplacian

$$-\boldsymbol{\Delta}_{\mathrm{HL}} := \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\,\mathrm{div}\ , \tag{1.1}$$

a formally self-adjoint, second-order, linear partial differential operator acting on vector fields.

This article builds upon the work of Kress [18, 19, 20] and our theoretical investigations in [10, Section 6] into boundary value problems for the Hodge-Helmholtz operator $-\boldsymbol{\Delta}_{\mathrm{HL}} - \kappa^2\mathsf{Id}$, $\kappa \geq 0$, and associated *first-kind* boundary integral equations. To establish the setting we will review and extend the results of [10, Sections 3 & 4] in Section 2 and Section 3. As a new aspect, the boundary integral equations in weak form will be studied based on the theory of variational saddle point problems. This perspective will also be adopted in the core of the paper, Section 4, where we identify boundary element spaces that give rise to "h-uniformly" stable Galerkin discretizations of the variational boundary integral equations. As a theoretical tool we use the Babuska-Brezzi theory of discrete saddle point problems. Section 5 supplements the theory by numerical explorations of discrete kernels.

Remark 1.1 (Hodge-Laplacian in exterior calculus). Differential geometry provides us with a much more general notion of Hodge-Laplacian as the operator

$$\mathbf{\Delta}_{\mathrm{HL}} := *\,\mathbf{d}*\mathbf{d} + \mathbf{d}*\mathbf{d}\,* \tag{1.2}$$

acting on ℓ-forms on a manifold (with boundary). Here $*$ is the so-called Hodge operator and $\mathbf{d}$ stands for the exterior derivative. Our concrete Hodge-Laplacian from (1.1) is an incarnation of (1.2) for $\ell = 1$ and on Euclidean space $\mathbb{R}^3$. This will be our sole focus and we refer to the monograph [23] for a comprehensive treatment of general Hodge-Laplacians. △

Remark 1.2 (Hodge-Laplacian and Maxwell's Equations). When enforcing vanishing divergence of solutions of $\mathbf{\Delta}_{\mathrm{HL}}\mathbf{U} = 0$ by choosing suitable boundary conditions, the Hodge-Laplacian offers a way to express the *zero-frequency limit* of Maxwell's equations in frequency domain, *cf.* the Introduction of [10] or [14]. Integral equations for the time-harmonic Maxwell equations are well established and usually comprise the so-called electric-field and magnetic-field boundary integral operators [9, Sect. 5]. Naturally, those are related to the integral operators investigated in this work and we point out that results about their null spaces in the static limit [11] will be mirrored in Section 3 of this work. △

2 Boundary Value Problems for the Hodge Laplacian

Straightforward integration by parts, elaborated for instance in [10, Section 3,(19)] and [18, Lemma 3.2], reveals that the fundamental symmetric bilinear form induced by $-\mathbf{\Delta}_{\mathrm{HL}}$ is

$$\mathsf{a}_{\mathrm{HL}}(\boldsymbol{U},\boldsymbol{V}) := (\mathbf{curl}\,\boldsymbol{U},\mathbf{curl}\,\boldsymbol{V})_{L^2(\Omega)} + (\operatorname{div}\boldsymbol{U},\operatorname{div}\boldsymbol{V})_{L^2(\Omega)} \ . \tag{2.1}$$

It is a more subtle matter to decide about meaningful choices for function spaces, on which to pose variational problems for a_{HL}. This will define boundary conditions and point to relevant trace operators. Since we rely on both to state pertinent boundary integral equations, we summarize [10, Section 3] in this section.

2.1 Spaces and Traces

Since Ω may be an unbounded exterior domain, we rely on the weighted space $\boldsymbol{\mathcal{X}}(\Omega)$ defined as the closure of $\boldsymbol{\mathcal{C}}^\infty_{\mathrm{comp}}(\overline{\Omega}) := \{\boldsymbol{V}_{|\Omega}, \boldsymbol{V} \in (\mathcal{C}^\infty(\overline{\Omega}))^3, \mathrm{supp}(\boldsymbol{V}) \text{ bounded}\}$

under the norm

$$\|\boldsymbol{V}\|^2_{\boldsymbol{\mathcal{X}}(\Omega)} := \|\mathbf{curl}\,\boldsymbol{V}\|^2_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{V}\|^2_{L^2(\Omega)} + \int_\Omega \frac{|\boldsymbol{V}(\mathbf{x})|^2}{1+|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}\ . \tag{2.2}$$

Consult [26, Sect. 1.4] or [24, Sect. 2.2] for further explanations on the significance of the weight functions. The maximal domain space of $\boldsymbol{\Delta}_{\mathrm{HL}}$ is

$$\begin{aligned}\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega) &:= \{\boldsymbol{V}\in\boldsymbol{\mathcal{X}}(\Omega):\ \boldsymbol{\Delta}_{\mathrm{HL}}\boldsymbol{V}\in\boldsymbol{L}^2(\Omega)\}\\ &= \{\boldsymbol{V}\in\boldsymbol{\mathcal{X}}(\Omega):\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{V}\in\boldsymbol{L}^2(\Omega),\ \mathbf{grad}\,\mathrm{div}\,\boldsymbol{V}\in\boldsymbol{L}^2(\Omega)\}\ .\end{aligned}$$

The space $\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega)$ supports the following *continuous and surjective* trace operators

$$\begin{aligned}
&\gamma_{\mathrm{T}}:\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega)\to\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)\ , && \gamma_{\mathrm{T}}\,\boldsymbol{U}(\mathbf{x}) := \boldsymbol{n}(\mathbf{x})\times(\boldsymbol{U}(\mathbf{x})\times\boldsymbol{n}(\mathbf{x}))\ ,\\
&\gamma_{\mathrm{N}}:\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega)\to H^{-\frac12}(\Gamma)\ , && \gamma_{\mathrm{N}}\,\boldsymbol{U}(\mathbf{x}) := \boldsymbol{U}(\mathbf{x})\cdot\boldsymbol{n}(\mathbf{x})\ ,\\
&\gamma_{\mathrm{R}}:\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega)\to\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\ , && \gamma_{\mathrm{R}}\,\boldsymbol{U}(\mathbf{x}) := \boldsymbol{n}\times\mathbf{curl}\,\boldsymbol{U}(\mathbf{x})\ ,\\
&\gamma_{\mathrm{D}}:\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega)\to H^{\frac12}(\Gamma)\ , && \gamma_{\mathrm{D}}\,\boldsymbol{U}(\mathbf{x}) := \mathrm{div}\,\boldsymbol{U}(\mathbf{x})\ ,
\end{aligned}$$

their pointwise definitions first considered for almost all $\mathbf{x}\in\Gamma$ and $\boldsymbol{U}\in\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega})$ and then extended to the function spaces by continuity. The trace spaces $H^{\frac12}(\Gamma)$ and $H^{-\frac12}(\Gamma)$ occurring above are classical Sobolev spaces, see [22, Ch. 3] for a comprehensive discussion. Less standard are the tangential trace spaces

$$\begin{aligned}\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) &:= \{\boldsymbol{v}\in\boldsymbol{H}_\perp^{-\frac12}(\Gamma):\ \mathrm{curl}_\Gamma\,\boldsymbol{v}\in H^{-\frac12}(\Gamma)\}\ ,\\ \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) &:= \{\boldsymbol{v}\in\boldsymbol{H}_\times^{-\frac12}(\Gamma):\ \mathrm{div}_\Gamma\boldsymbol{v}\in H^{-\frac12}(\Gamma)\}\ .\end{aligned} \tag{2.3}$$

where $\boldsymbol{H}_\perp^{-\frac12}(\Gamma)$ and $\boldsymbol{H}_\times^{-\frac12}(\Gamma)$ are the duals of the tangential and rotated tangential traces spaces of $\boldsymbol{H}^1(\Omega)$ onto Γ. These spaces along with the surface differential operators curl_Γ and div_Γ have been introduced in [5, 6] for piecewise smooth boundaries, in [8] for Lipschitz boundaries and summaries can be found in [4], [9, Sect. 2.2]. The spaces (2.3) are equipped with the natural graph norms.

By [8, Lemma 5.6] the spaces $H^{\frac12}(\Gamma)$ and $H^{-\frac12}(\Gamma)$, and $\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ are in duality with pivot spaces $L^2(\Gamma)$ and $\boldsymbol{L}^2_{\boldsymbol{t}}(\Gamma)$, respectively, where $\boldsymbol{L}^2_{\boldsymbol{t}}(\Gamma)$ is the space of tangential vectorfields in $(L^2(\Gamma))^3$. For the duality pairings we indiscriminately write $\langle\cdot,\cdot\rangle_\Gamma$. They also enter the general Green's formula [10, formula (19)] for $-\boldsymbol{\Delta}_{\mathrm{HL}}$ given by

$$\begin{aligned}-(\boldsymbol{U},\boldsymbol{\Delta}_{\mathrm{HL}}\boldsymbol{V})_{L^2(\Omega)} &= \mathsf{a}_{\mathrm{HL}}(\boldsymbol{U},\boldsymbol{V}) - \langle\gamma_{\mathrm{N}}(\boldsymbol{U}),\gamma_{\mathrm{D}}(\boldsymbol{V})\rangle_\Gamma + \langle\gamma_{\mathrm{T}}(\boldsymbol{U}),\gamma_{\mathrm{R}}(\boldsymbol{V})\rangle_\Gamma\\ &= -(\boldsymbol{\Delta}_{\mathrm{HL}}\boldsymbol{U},\boldsymbol{V})_{L^2(\Omega)} - \langle\gamma_{\mathrm{N}}(\boldsymbol{U}),\gamma_{\mathrm{D}}(\boldsymbol{V})\rangle_\Gamma + \langle\gamma_{\mathrm{D}}(\boldsymbol{U}),\gamma_{\mathrm{N}}(\boldsymbol{V})\rangle_\Gamma\\ &\quad - \langle\gamma_{\mathrm{R}}(\boldsymbol{U}),\gamma_{\mathrm{T}}(\boldsymbol{V})\rangle_\Gamma + \langle\gamma_{\mathrm{T}}(\boldsymbol{U}),\gamma_{\mathrm{R}}(\boldsymbol{V})\rangle_\Gamma\end{aligned} \tag{2.4}$$

which underscores the relevance of the traces introduced above.

Notation. Functions on Ω are designated with capital letters, those on Γ with small letters. We use different classes of symbols for functions in the four relevant trace spaces, plain roman characters for functions in $H^{\frac{1}{2}}(\Gamma)$, bold roman font for elements of $\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, greek and bold greek symbols for functions in $H^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, respectively. As a rule, bold typeface marks vector valued functions.

Kernels of tangential and normal traces define closed subspaces of $\boldsymbol{\mathcal{X}}(\Omega)$ that we denote as[1]:

$$\boldsymbol{\mathcal{X}}_T(\Omega) := \{\boldsymbol{V} \in \boldsymbol{\mathcal{X}}(\Omega) : \gamma_{\mathrm{T}}\, \boldsymbol{V} = 0 \text{ on } \Gamma\} \;,$$
$$\boldsymbol{\mathcal{X}}_N(\Omega) := \{\boldsymbol{V} \in \boldsymbol{\mathcal{X}}(\Omega) : \gamma_{\mathrm{N}}\, \boldsymbol{V} = 0 \text{ on } \Gamma\} \;.$$

For bounded Ω we have $\boldsymbol{\mathcal{X}}_T(\Omega) = \boldsymbol{\mathcal{X}}(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{\mathcal{X}}_N(\Omega) = \boldsymbol{\mathcal{X}}(\Omega) \cap \boldsymbol{H}_0(\mathrm{div}, \Omega)$. In this case an important compact embedding result holds [29], [1, Thm. 2.8]:

Lemma 2.1 (Compact embedding in $\boldsymbol{L}^2(\Omega)$). *If $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, $\boldsymbol{\mathcal{X}}_N(\Omega)$ and $\boldsymbol{\mathcal{X}}_T(\Omega)$ are compactly embedded in $\boldsymbol{L}^2(\Omega)$.*

By contrast, the embedding $\boldsymbol{\mathcal{X}}(\Omega) \subset \boldsymbol{L}^2(\Omega)$ itself is *not* compact [1, Prop. 2.7].

2.2 Boundary Conditions

Thanks to Lemma 2.1, restricting a_{HL} to $\boldsymbol{\mathcal{X}}_T(\Omega)$ or $\boldsymbol{\mathcal{X}}_N(\Omega)$ spawns operators $\boldsymbol{\Delta}_{\mathrm{T}} : \boldsymbol{\mathcal{X}}_T(\Omega) \to \boldsymbol{\mathcal{X}}_T(\Omega)'$ and $\boldsymbol{\Delta}_{\mathrm{N}} : \boldsymbol{\mathcal{X}}_N(\Omega) \to \boldsymbol{\mathcal{X}}_N(\Omega)'$, respectively, with compact resolvent, if Ω is bounded. This is one reason why the weak formulations of meaningful boundary value problems for $\boldsymbol{\Delta}_{\mathrm{HL}}$ should rely on either $\boldsymbol{\mathcal{X}}_T(\Omega)$ or $\boldsymbol{\mathcal{X}}_N(\Omega)$ as test spaces [10, Sect. 3].

In the former case, for given $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $f \in H^{\frac{1}{2}}(\Gamma)$ we consider:

Find $\boldsymbol{U} \in \boldsymbol{\mathcal{X}}(\Omega)$ with $\gamma_{\mathrm{T}}\, \boldsymbol{U} = \boldsymbol{g}$ such that

$$\mathsf{a}_{\mathrm{HL}}(\boldsymbol{U}, \boldsymbol{V}) = \langle \gamma_{\mathrm{N}}\, \boldsymbol{V}, f \rangle_\Gamma \quad \forall \boldsymbol{V} \in \boldsymbol{\mathcal{X}}_T(\Omega) \;. \tag{T}$$

Integrating by parts using (2.4) reveals the associated boundary value problem [2]:

$$\boldsymbol{\Delta}_{\mathrm{HL}} \boldsymbol{U} = 0 \quad \text{in } \Omega \quad , \quad \gamma_{\mathrm{T}}\, \boldsymbol{U} = \boldsymbol{g} \;, \quad \gamma_{\mathrm{D}}\, \boldsymbol{U} = f \quad \text{on } \Gamma \;. \tag{2.5}$$

In the latter case, the data are $\boldsymbol{\eta} \in \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, and we want to solve the variational problem:

[1] The subscripts N, T are reversed with respect to the notation in [1, Def. 2.4]

[2] In [18, Sect. 1] the boundary conditions in (2.5) are called "electric"

Find $\boldsymbol{U} \in \boldsymbol{\mathcal{X}}(\Omega)$ with $\gamma_{\mathrm{N}} \boldsymbol{U} = \varphi$ and

$$\mathsf{a}_{\mathrm{HL}}(\boldsymbol{U}, \boldsymbol{V}) = -\langle \boldsymbol{\eta}, \gamma_{\mathrm{T}} \boldsymbol{V} \rangle_\Gamma \quad \forall \boldsymbol{V} \in \boldsymbol{\mathcal{X}}_N(\Omega) . \tag{N}$$

This is the weak form of the boundary value problem [3]

$$\boldsymbol{\Delta}_{\mathrm{HL}} \boldsymbol{U} = 0 \quad \text{in } \Omega \quad , \quad \gamma_{\mathrm{R}} \boldsymbol{U} = \boldsymbol{\eta} \, , \quad \gamma_{\mathrm{N}} \boldsymbol{U} = \varphi \quad \text{on } \Gamma . \tag{2.6}$$

Taking the cue from the boundary conditions in (2.5) and (2.6), as in [10, Sect. 3.1] we introduce the compound trace spaces and operators

$$\mathsf{T}_{\mathrm{T}} := \begin{pmatrix} \gamma_{\mathrm{T}} \\ \gamma_{\mathrm{D}} \end{pmatrix} : \boldsymbol{\mathcal{X}}(\Omega) \to \mathcal{H}_T(\Gamma) := \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times H^{\frac{1}{2}}(\Gamma) \, ,$$

$$\mathsf{T}_{\mathrm{N}} := \begin{pmatrix} \gamma_{\mathrm{R}} \\ \gamma_{\mathrm{N}} \end{pmatrix} : \boldsymbol{\mathcal{X}}(\Omega) \to \mathcal{H}_N(\Gamma) := \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-\frac{1}{2}}(\Gamma) \, .$$

They are continous, surjective by [10, Lemma 3.2], and $\mathcal{H}_T(\Gamma)$ and $\mathcal{H}_N(\Gamma)$ are dual to each other with respect to the $L^2(\Gamma)$-type pairing

$$\left\langle\!\!\left\langle \begin{pmatrix} \boldsymbol{\mu} \\ \psi \end{pmatrix}, \begin{pmatrix} \boldsymbol{v} \\ w \end{pmatrix} \right\rangle\!\!\right\rangle_\Gamma := \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_\Gamma + \langle \psi, w \rangle_\Gamma \, , \quad \begin{pmatrix} \boldsymbol{\mu} \\ \psi \end{pmatrix} \in \mathcal{H}_N(\Gamma) \, , \; \begin{pmatrix} \boldsymbol{v} \\ w \end{pmatrix} \in \mathcal{H}_T(\Gamma) \, . \tag{2.7}$$

Remark 2.1 (Divergence-free solutions). If $f = 0$ in (2.5), then [23, Thm. 1.1(6)] asserts that $\mathrm{div}\, \boldsymbol{U} = 0$ for any solution $\boldsymbol{U}$. Thus, imposing $\gamma_{\mathrm{D}} \boldsymbol{U} = 0$ converts (T) to a variational formulation for the magnetostatic boundary value problem in terms of a Coulomb-gauged vector potential:

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{U} = 0 \, , \quad \mathrm{div}\, \boldsymbol{U} = 0 \quad \text{in } \Omega \quad , \quad \gamma_{\mathrm{T}} \boldsymbol{U} = \boldsymbol{g} \quad \text{on } \partial\Omega . \tag{2.8}$$

△

2.3 Kernels

From [26, Sects. 2.4 & 2.5] and [23, Sect. 1.1] we learn that both $\boldsymbol{\Delta}_{\mathrm{T}}$ and $\boldsymbol{\Delta}_{\mathrm{N}}$ may have finite-dimensional kernels implying non-uniqueness of solutions of (T) and (N). Firstly, we find

$$\mathrm{Ker}(\boldsymbol{\Delta}_{\mathrm{T}}) = \boldsymbol{\mathcal{Z}}_T(\Omega) := \{ \boldsymbol{V} \in \boldsymbol{\mathcal{X}}_T(\Omega) : \; \mathbf{curl}\, \boldsymbol{V} = 0 \text{ and } \mathrm{div}\, \boldsymbol{V} = 0 \} \, ,$$

[3] In the parlance of [18] the boundary conditions of (2.6) are “magnetic”

which is the space of *Dirichlet harmonic vector fields*. Its dimension agrees with the second Betti number $\beta_2(\Omega)$ of Ω, the number of holes in Ω. Hence, solutions of (T) can be unique only up to adding elements of $\mathcal{Z}_T(\Omega)$ and will exist, if and only if

$$\langle \gamma_{\mathrm{N}}\, \boldsymbol{Z}, f\rangle_\Gamma = 0 \quad \forall \boldsymbol{Z} \in \mathcal{Z}_T(\Omega)\ . \tag{2.9}$$

Similar considerations apply to (N), for which

$$\mathrm{Ker}(\boldsymbol{\Delta}_{\mathrm{N}}) = \mathcal{Z}_N(\Omega) := \{\boldsymbol{V} \in \mathcal{X}_N(\Omega) : \mathbf{curl}\, \boldsymbol{V} = 0 \text{ and } \operatorname{div} \boldsymbol{V} = 0\}\ ,$$

known as space of *Neumann harmonic vector fields*, with $\dim \mathcal{Z}_N(\Omega) = \beta_1(\Omega)$, the first Betti number of Ω, which can be read as the number of handles. Uniqueness of solutions of (N) can hold only modulo contributions from $\mathcal{Z}_N(\Omega)$ and existence requires the constraint on the data

$$\langle \boldsymbol{\eta}, \gamma_{\mathrm{T}}\, \boldsymbol{Z}\rangle_\Gamma = 0 \quad \forall \boldsymbol{Z} \in \mathcal{Z}_N(\Omega)\ . \tag{2.10}$$

3 First-Kind Boundary Integral Equations

3.1 Representation Formula and Calderón Identities

Solutions of both (T)/(2.5) and (N)/(2.6) allow a representation through boundary potentials. This has been the main result of [10, Sect. 4.2], see also [23, Ch. 3], and we give a special version below. The involved potentials rely on the fundamental solution for the scalar Laplacian in 3D, $\mathcal{G}(\mathbf{x}) := \frac{1}{4\pi\|\mathbf{x}\|}$, $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$, and are

- the *single layer potential*

$$\mathcal{SL}\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix} := -\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\boldsymbol{\mu}(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) + \mathbf{grad}_{\mathbf{x}} \int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\alpha(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\ , \tag{3.1}$$

- and the *double layer potential*

$$\mathcal{DL}\begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix} := \mathbf{curl}_{\mathbf{x}} \int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})(\boldsymbol{q}\times\boldsymbol{n})(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) + \int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\boldsymbol{n}(\mathbf{y})v(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\ , \tag{3.2}$$

for $\binom{\boldsymbol{\mu}}{\alpha} \in \mathcal{H}_N(\Gamma)$, $\binom{\boldsymbol{q}}{v} \in \mathcal{H}_T(\Gamma)$. Both potentials provide continuous linear operators mapping from compound trace spaces into $\mathcal{X}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega) \times \mathcal{X}(\boldsymbol{\Delta}_{\mathrm{HL}};\Omega') \cong \mathcal{X}(\boldsymbol{\Delta}_{\mathrm{HL}};\mathbb{R}^3\setminus\Gamma)$, $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$, see [10, Sect. 5].

A little extra notation is needed for a concise statement of the general representation formula: When tagged with a '+' superscript, trace operators applied to functions in $\boldsymbol{\mathcal{X}}(\boldsymbol{\Delta}_{\mathrm{HL}};\mathbb{R}^3 \setminus \Gamma)$ are to be taken from the exterior of Ω. Then, for every generic trace operator T, we can introduce the jump $[\mathsf{T}]_\Gamma = \mathsf{T} - \mathsf{T}^+$ and the average trace $\{\mathsf{T}\}_\Gamma := \frac{1}{2}(\mathsf{T} + \mathsf{T}^+)$. The next result was established in [10, Sect.4.2, in particular (38)], see also [18, Thm. 3.3].

Theorem 3.1 (Representation formula for solutions of $\boldsymbol{\Delta}_{\mathrm{HL}}\boldsymbol{U} = 0$). *If $\boldsymbol{U} \in \boldsymbol{\mathcal{X}}(\mathbb{R}^3 \setminus \Gamma)$ satisfies $\boldsymbol{\Delta}_{\mathrm{HL}}\boldsymbol{U} = 0$ in both Ω and Ω', then*

$$\boxed{\boldsymbol{U} = \boldsymbol{\mathcal{SL}}([\mathsf{T}_{\mathrm{N}}]_\Gamma \boldsymbol{U}) + \boldsymbol{\mathcal{DL}}([\mathsf{T}_{\mathrm{T}}]_\Gamma \boldsymbol{U}) \quad \text{in } \mathbb{R}^3 \setminus \Gamma} \,. \tag{3.3}$$

The jumps of the two potentials in terms of T_{T} and T_{N} are well-defined and satisfy fundamental *jump relations* given in [10, Thm. 5.1]:

$$\begin{aligned} [\mathsf{T}_{\mathrm{T}}]_\Gamma \circ \boldsymbol{\mathcal{DL}} = \mathsf{Id} \quad &, \quad [\mathsf{T}_{\mathrm{N}}]_\Gamma \circ \boldsymbol{\mathcal{DL}} = 0 \quad \text{in } \mathcal{H}_T(\Gamma)\,, \\ [\mathsf{T}_{\mathrm{T}}]_\Gamma \circ \boldsymbol{\mathcal{SL}} = 0 \quad &, \quad [\mathsf{T}_{\mathrm{N}}]_\Gamma \circ \boldsymbol{\mathcal{SL}} = \mathsf{Id} \quad \text{in } \mathcal{H}_N(\Gamma)\,. \end{aligned} \tag{3.4}$$

By convention *boundary integral operators* (BIOs) arise from applying averaged traces $\{\mathsf{T}_{\mathrm{T}}\}_\Gamma$ and $\{\mathsf{T}_{\mathrm{N}}\}_\Gamma$ to the potentials $\boldsymbol{\mathcal{SL}}$ and $\boldsymbol{\mathcal{DL}}$. We recover four different BIOs which map continuously between the appropriate compound trace spaces, e.g., $\{\mathsf{T}_{\mathrm{T}}\}_\Gamma \circ \boldsymbol{\mathcal{SL}} : \mathcal{H}_N(\Gamma) \to \mathcal{H}_T(\Gamma)$.

It is immediate from the jump relations (3.4) and the representation formula (3.3) that compound traces of any $\boldsymbol{U} \in \boldsymbol{\mathcal{X}}(\Omega)$ (extended by zero to $\mathbb{R}^3$) satisfying $\boldsymbol{\Delta}_{\mathrm{HL}}\boldsymbol{U} = 0$ in Ω solve the following boundary integral equations, here written in compact block-operator form

$$\begin{pmatrix} \{\mathsf{T}_{\mathrm{T}}\}_\Gamma \circ \boldsymbol{\mathcal{DL}} + \frac{1}{2}\mathsf{Id} & \{\mathsf{T}_{\mathrm{T}}\}_\Gamma \circ \boldsymbol{\mathcal{SL}} \\ \{\mathsf{T}_{\mathrm{N}}\}_\Gamma \circ \boldsymbol{\mathcal{DL}} & \{\mathsf{T}_{\mathrm{N}}\}_\Gamma \circ \boldsymbol{\mathcal{SL}} + \frac{1}{2}\mathsf{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathrm{T}}\boldsymbol{U} \\ \mathsf{T}_{\mathrm{N}}\boldsymbol{U} \end{pmatrix} = \begin{pmatrix} \mathsf{T}_{\mathrm{T}}\boldsymbol{U} \\ \mathsf{T}_{\mathrm{N}}\boldsymbol{U} \end{pmatrix} \,. \tag{3.5}$$

This operator acting as a projection in $\mathcal{H}_T(\Gamma) \times \mathcal{H}_N(\Gamma)$ is called the *(interior) Calderón projector* [25, Sect. 3.6]. The same considerations can be applied to $\boldsymbol{U} \in \boldsymbol{\mathcal{X}}(\Omega')$, $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$, extended by zero to Ω. Observing sign flips, the action of boundary integral operators on exterior traces of $\boldsymbol{U}$ yields a formula for the *exterior Calderón projector*

$$\begin{pmatrix} -\{\mathsf{T}_{\mathrm{T}}\}_\Gamma \circ \boldsymbol{\mathcal{DL}} + \frac{1}{2}\mathsf{Id} & -\{\mathsf{T}_{\mathrm{T}}\}_\Gamma \circ \boldsymbol{\mathcal{SL}} \\ -\{\mathsf{T}_{\mathrm{N}}\}_\Gamma \circ \boldsymbol{\mathcal{DL}} & -\{\mathsf{T}_{\mathrm{N}}\}_\Gamma \circ \boldsymbol{\mathcal{SL}} + \frac{1}{2}\mathsf{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{\mathrm{T}}^+\boldsymbol{U} \\ \mathsf{T}_{\mathrm{N}}^+\boldsymbol{U} \end{pmatrix} = \begin{pmatrix} \mathsf{T}_{\mathrm{T}}^+\boldsymbol{U} \\ \mathsf{T}_{\mathrm{N}}^+\boldsymbol{U} \end{pmatrix} \,. \tag{3.6}$$

3.2 First-Kind BIE for (T)

In the case of (T)/(2.5) we know $\mathsf{T}_{\mathrm{T}}\boldsymbol{U} = \binom{g}{f} \in \mathcal{H}_T(\Gamma)$ on Γ and have to determine $\mathsf{T}_{\mathrm{N}}\boldsymbol{U} = \binom{\gamma_{\mathrm{R}}\boldsymbol{U}}{\gamma_{\mathrm{N}}\boldsymbol{U}} \in \mathcal{H}_N(\Gamma)$ in order to recover $\boldsymbol{U}$ in Ω through the representation formula. From (3.5) we can immediately extract the boundary integral equation of the first kind

$$(\{\mathsf{T}_{\mathrm{T}}\} \circ \boldsymbol{\mathcal{SL}})\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix} = \left(\tfrac{1}{2}\mathsf{Id} - \{\mathsf{T}_{\mathrm{T}}\} \circ \boldsymbol{\mathcal{DL}}\right)\begin{pmatrix}\boldsymbol{g}\\ f\end{pmatrix} \quad \text{in } \mathcal{H}_T(\Gamma)\ . \tag{3.7}$$

for the unknown compound trace $\binom{\boldsymbol{\mu}}{\alpha} \in \mathcal{H}_N(\Gamma)$. The duality of $\mathcal{H}_T(\Gamma)$ and $\mathcal{H}_N(\Gamma)$ supplies an equivalent variational equation:

$$\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix} \in \mathcal{H}_N(\Gamma) : \quad \left\langle\!\!\left\langle (\{\mathsf{T}_{\mathrm{T}}\} \circ \boldsymbol{\mathcal{SL}})\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma = \ell_T\begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix} \quad \forall \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix} \in \mathcal{H}_N(\Gamma)\ . \tag{3.8}$$

with the right-hand-side functional

$$\ell_T\begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix} := \left\langle\!\!\left\langle \left(\tfrac{1}{2}\mathsf{Id} - \{\mathsf{T}_{\mathrm{T}}\} \circ \boldsymbol{\mathcal{DL}}\right)\begin{pmatrix}\boldsymbol{g}\\ f\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma\ . \tag{3.9}$$

We denote the bilinear form on the left-hand side of (3.8) by $\mathsf{b}_T : \mathcal{H}_N(\Gamma) \times \mathcal{H}_N(\Gamma) \to \mathbb{R}$. Based on (3.1) we arrive at an integral representation [10, Formula (62)]

$$\begin{aligned} &\mathsf{b}_T\left(\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix}\right) = \left\langle\!\!\left\langle \{\mathsf{T}_{\mathrm{T}}\} \circ \boldsymbol{\mathcal{SL}}\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \\ &= -\int_\Gamma\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\left(\boldsymbol{\mu}(\mathbf{y})\cdot\boldsymbol{\eta}(\mathbf{x}) + \alpha(\mathbf{y})\mathrm{div}_\Gamma\boldsymbol{\eta}(\mathbf{x}) + \beta(\mathbf{x})\mathrm{div}_\Gamma\boldsymbol{\mu}(\mathbf{y})\right)\mathrm{d}S(\mathbf{y},\mathbf{x})\ . \end{aligned} \tag{3.10}$$

To recast it in a more concise, structure-revealing way, we introduce

$$(\varphi,\psi)_{-1/2} := \int_\Gamma\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\psi(\mathbf{y})\,\mathrm{d}S(\mathbf{y},\mathbf{x})\ , \quad \varphi,\psi \in H^{-\frac{1}{2}}(\Gamma)\ , \tag{3.11}$$

and analogous bilinear forms on $\boldsymbol{H}_\perp^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}_\times^{-\frac{1}{2}}(\Gamma)$, for which we retain the same notation. All three bilinear forms can serve as (equivalent) inner products in the respective trace spaces [25, Thm. 3.5.3], [7, Prop. 2]. Using these inner products (3.10) reads

$$\mathsf{b}_T\left(\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix}\right) = -\left(\boldsymbol{\mu},\boldsymbol{\eta}\right)_{-1/2} - \left(\alpha,\mathrm{div}_\Gamma\boldsymbol{\eta}\right)_{-1/2} - \left(\beta,\mathrm{div}_\Gamma\boldsymbol{\mu}\right)_{-1/2}\ . \tag{3.12}$$

Obviously, the variational problem (3.7) has a *saddle-point structure*. Writing $L_1(\boldsymbol{\eta}) + L_2(\beta)$ instead of the right-hand side linear functional ℓ_T in (3.7), it is equivalent to: Seek $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $\alpha \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\boxed{\begin{array}{lcll} (\boldsymbol{\mu},\boldsymbol{\eta})_{-1/2} + (\alpha,\mathrm{div}_\Gamma\boldsymbol{\eta})_{-1/2} & = & -L_1(\boldsymbol{\eta}) & \forall\boldsymbol{\eta} \in \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\ , \\ (\beta,\mathrm{div}_\Gamma\boldsymbol{\mu})_{-1/2} & = & -L_2(\beta) & \forall\beta \in H^{-\frac{1}{2}}(\Gamma)\ . \end{array}} \tag{3.13}$$

To start the discussion about existence and uniqueness of solutions of (3.7)/(3.13), we summarize the findings of [10, Sect. 7.1] in the following lemma, which is a straightforward consequence of the structure of (3.13).

Lemma 3.2 (Kernel of b_T). *The kernel of the bilinear form* b_T *from (3.12) is given by*

$$\mathrm{Ker}(\mathsf{b}_T) = \{0\} \times \mathcal{Z}_2(\Gamma) \;,$$

with

$$\mathcal{Z}_2(\Gamma) := \{\alpha \in H^{-\frac{1}{2}}(\Gamma) : \; (\alpha, \mathrm{div}_\Gamma \boldsymbol{\eta})_{-1/2} = 0 \; \forall \boldsymbol{\eta} \in \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)\} \;,$$

and its dimension agrees with the second Betti number $\beta_2(\Gamma)$ *of* Γ.

Another representation of $\mathrm{Ker}(\mathsf{b}_T)$ can be derived by means of the Calderón projectors. Harmonic vectorfields certainly satisfy the assumptions of Theorem 3.1. Thus, consider (3.5) for $\boldsymbol{U} \in \boldsymbol{\mathcal{Z}}_T(\Omega)$ and (3.6) for $\boldsymbol{U} \in \boldsymbol{\mathcal{Z}}_T(\Omega')$. Since $\mathsf{T}_\mathrm{T} \boldsymbol{U} = 0$ and $\mathsf{T}_\mathrm{T}^+ \boldsymbol{U} = 0$ for interior/exterior Dirichlet harmonic vector fields, we infer from (3.5) and (3.6) that, on one hand, $\mathsf{T}_\mathrm{N}\, \boldsymbol{\mathcal{Z}}_T(\Omega) + \mathsf{T}_\mathrm{N}^+\, \boldsymbol{\mathcal{Z}}_T(\Omega') \subset \mathrm{Ker}(\mathsf{b}_T)$. On the other hand, if $\binom{\zeta}{\nu} \in \mathrm{Ker}(\mathsf{b}_T) = \mathrm{Ker}(\{\mathsf{T}_\mathrm{T}\}_\Gamma \circ \boldsymbol{\mathcal{SL}})$, then $\boldsymbol{V} := \boldsymbol{\mathcal{SL}}\binom{\zeta}{\nu} \in \boldsymbol{\mathcal{X}}_T(\mathbb{R}^3 \setminus \Gamma)$ satisfies $\mathsf{T}_\mathrm{T}\, \boldsymbol{V} = 0$. It is a solution of (T) for vanishing data on both Ω and Ω' and, therefore, $\boldsymbol{V}|_\Omega$ will belong to $\mathrm{Ker}(\boldsymbol{\Delta}_\mathrm{T}) = \boldsymbol{\mathcal{Z}}_T(\Omega)$, and $\boldsymbol{V}|_{\Omega'}$ to $\boldsymbol{\mathcal{Z}}_T(\Omega')$. From the jump relations (3.4) we get $\binom{\zeta}{\nu} = [\mathsf{T}_\mathrm{N}\, \boldsymbol{V}]_\Gamma = \mathsf{T}_\mathrm{N}\, \boldsymbol{V} - \mathsf{T}_\mathrm{N}^+\, \boldsymbol{V}$. Altogether, we conclude the following alternative representation.

Lemma 3.3 (Kernel of b_T (II)). *The kernel of the bilinear form* b_T *is generated by traces of interior and exterior Dirichlet harmonic vector fields*

$$\mathrm{Ker}(\mathsf{b}_T) = \mathrm{Ker}(\{\mathsf{T}_\mathrm{T}\}_\Gamma \circ \boldsymbol{\mathcal{SL}}) = \mathsf{T}_\mathrm{N}\, \boldsymbol{\mathcal{Z}}_T(\Omega) + \mathsf{T}_\mathrm{N}^+\, \boldsymbol{\mathcal{Z}}_T(\Omega') \;. \tag{3.14}$$

This also matches the formula $\beta_2(\Gamma) = \beta_2(\Omega) + \beta_2(\Omega')$ [28, Ch. 11].

Remark 3.1. If Ω is a sphere, then $\beta_2(\Omega) = 0$, $\beta_2(\Omega') = 1$, and even in this case b_T will have a kernel, though $\mathrm{Ker}(\boldsymbol{\Delta}_\mathrm{T}) = \{0\}$. More generally, elements of $\mathrm{Ker}(\mathsf{b}_T)$ due to $\mathsf{T}_\mathrm{N}^+\, \boldsymbol{\mathcal{Z}}_T(\Omega')$ will be "spurious" in the sense that they are not induced by functions in $\mathrm{Ker}(\boldsymbol{\Delta}_\mathrm{T})$. However, when we plug the traces of $\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}_T(\Omega')$ into the representation formula (3.3) it returns a function that vanishes on Ω ("null field property"), because $\boldsymbol{Z}$, extended by zero into Ω, satisfies the assumptions of Theorem 3.1. △

Lemma 3.4 (Consistency of the right-hand side of (3.8)). *If the boundary data* $f \in H^{\frac{1}{2}}(\Gamma)$ *satisfy (2.9), then the right-hand side functional* $\ell_T \in \mathcal{H}_N(\Gamma)'$ *is consistent in the sense that*

$$\ell_T \begin{pmatrix} \boldsymbol{\eta} \\ \beta \end{pmatrix} = 0 \quad \forall \begin{pmatrix} \boldsymbol{\eta} \\ \beta \end{pmatrix} \in \mathrm{Ker}(\mathsf{b}_T) \;. \tag{3.15}$$

Proof. Using that $\boldsymbol{\eta} = 0$ for $\binom{\boldsymbol{\eta}}{\beta} \in \mathrm{Ker}(\mathsf{b}_T)$ and the integral representation of $\boldsymbol{\mathcal{DL}}$, we arrive at

$$\begin{aligned}
\ell_T\begin{pmatrix}0\\ \beta\end{pmatrix} &= \left\langle\!\!\left\langle \left(\tfrac{1}{2}\mathsf{Id} - \{\mathsf{T}_\mathrm{T}\}\circ\boldsymbol{\mathcal{DL}}\right)\begin{pmatrix}\boldsymbol{g}\\ f\end{pmatrix}, \begin{pmatrix}0\\ \beta\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \\
&= \tfrac{1}{2}\langle f,\beta\rangle_\Gamma - \left\langle (\{\gamma_\mathrm{D}\}_\Gamma\circ\boldsymbol{\mathcal{DL}})\begin{pmatrix}\boldsymbol{g}\\ f\end{pmatrix}, \beta\right\rangle_\Gamma \\
&= \tfrac{1}{2}\langle f,\beta\rangle_\Gamma - \left\langle \left\{\mathrm{div}\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\boldsymbol{n}(\mathbf{y})f(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\right\}_\Gamma, \beta\right\rangle_\Gamma \\
&= \tfrac{1}{2}\langle f,\beta\rangle_\Gamma - \left\langle \left\{\int_\Gamma (\mathbf{grad}\,\mathcal{G})(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{n}(\mathbf{y})f(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\right\}_\Gamma, \beta\right\rangle_\Gamma \\
&= \tfrac{1}{2}\langle f,\beta\rangle_\Gamma + \langle \mathsf{K}f,\beta\rangle_\Gamma\ ,
\end{aligned}$$

where $\mathsf{K} : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is the double layer boundary integral operator on Γ belonging to the scalar Laplacian, see [25, (3.6) and Sect. 3.3.3].

Next, from Lemma 3.3 we learn that for $\binom{\boldsymbol{\eta}}{\beta} \in \mathrm{Ker}(\mathsf{b}_T)$ we have $\beta \in \gamma_\mathrm{N}\,\boldsymbol{\mathcal{Z}}_T(\Omega) + \gamma_\mathrm{N}^+\,\boldsymbol{\mathcal{Z}}_T(\Omega')$. Moreover, harmonic vectorfields fit the assumptions of the representation formula of Theorem 3.1. Therefore, for $\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}_T(\Omega)$, since $\mathsf{T}_\mathrm{T}\,\boldsymbol{Z} = 0$ and $\gamma_\mathrm{R}\,\boldsymbol{Z} = 0$, we can write

$$\boldsymbol{Z} = \boldsymbol{\mathcal{SL}}(\mathsf{T}_\mathrm{N}\,\boldsymbol{Z}) = \boldsymbol{\mathcal{SL}}\begin{pmatrix}0\\ \gamma_\mathrm{N}\,\boldsymbol{Z}\end{pmatrix} = \mathbf{grad}_\mathbf{x}\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})(\gamma_\mathrm{N}\,\boldsymbol{Z})(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\ .$$

We recognize $\gamma_\mathrm{N}\,\boldsymbol{Z}$ as interior co-normal trace of the single layer potential for the scalar Laplacian. Using appropriate jump relations [25, Thm. 3.3.1], we find

$$\gamma_\mathrm{N}\,\boldsymbol{Z} = (\tfrac{1}{2}\mathsf{Id} + \mathsf{K}')\,\gamma_\mathrm{N}\,\boldsymbol{Z}\ , \tag{3.16}$$

with the adjoint double layer boundary integral operator $\mathsf{K}' : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ for the scalar Laplacian.

Similarly, for $\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}_T(\Omega')$ we end up with the representation

$$\boldsymbol{Z} = -\boldsymbol{\mathcal{SL}}(\mathsf{T}_\mathrm{N}^+\,\boldsymbol{Z}) = -\,\mathbf{grad}\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})(\gamma_\mathrm{N}^+\,\boldsymbol{Z})(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\ . \tag{3.17}$$

Hence, we see that $\gamma_\mathrm{N}^+\,\boldsymbol{Z}$ is the exterior co-normal trace of a scalar single layer potential, which, by the jump relation, delivers

$$\gamma_\mathrm{N}^+\,\boldsymbol{Z} = (\tfrac{1}{2}\mathsf{Id} - \mathsf{K}')(\gamma_\mathrm{N}^+\,\boldsymbol{Z})\ . \tag{3.18}$$

Finally, we use the duality of K and K', the expression for ℓ_T derived above and the assumptions on f. This gives us

$$\ell_T\begin{pmatrix}0\\ \beta\end{pmatrix} = \langle f, (\tfrac{1}{2}\mathsf{Id}+\mathsf{K}')\beta\rangle_\Gamma = \begin{cases}\langle f,\beta\rangle_\Gamma = 0 & \text{for } \beta\in\gamma_{\mathrm{N}}\,\boldsymbol{\mathcal{Z}}_T(\Omega)\,,\\ 0 & \text{for } \beta\in\gamma_{\mathrm{N}}^{+}\,\boldsymbol{\mathcal{Z}}_T(\Omega')\,.\end{cases}$$

□

Lemma 3.5 (Solvability of restricted saddle point problem (3.13)). *The bilinear form of the saddle point variational problem (3.13) satisfies an inf-sup condition on the orthogonal complement* $\mathrm{Ker}(\mathsf{b}_T)^\perp$ *of* $\mathrm{Ker}(\mathsf{b}_T)$ *in* $\mathcal{H}_N(\Gamma)$.

Proof. Obviously the orthogonal complement of

$$\{\alpha\in H^{-\frac{1}{2}}(\Gamma):\ (\alpha,\mathrm{div}_\Gamma\boldsymbol{\eta})_{-1/2}=0\quad\forall\boldsymbol{\eta}\in\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\}$$

in $H^{-\frac{1}{2}}(\Gamma)$ is $\mathrm{div}_\Gamma\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, since $\mathrm{div}_\Gamma\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\subset H^{-\frac{1}{2}}(\Gamma)$ is a closed subspace. Hence, the set of $\boldsymbol{\mu}\in\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ satisfying the second line of (3.13) for $L_2=0$ agrees with $\mathrm{Ker}(\mathrm{div}_\Gamma)$ and $(\boldsymbol{\mu},\boldsymbol{\eta})\mapsto(\boldsymbol{\mu},\boldsymbol{\eta})_{-1/2}$ is clearly $\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$-elliptic on that space. Also an inf-sup condition for $(\alpha,\boldsymbol{\eta})\mapsto(\alpha,\mathrm{div}_\Gamma\boldsymbol{\eta})_{-1/2}$ is straightforward. Appealing to the abstract variational saddle point theory from [2, Sect. 4.2.3] finishes the proof. □

Corollary 3.6 (Existence and uniqueness of solutions of (3.8)).

(i) For any data $\binom{\boldsymbol{g}}{f}\in\mathcal{H}_T(\Gamma)$ *fulfilling the consistency condition (2.9), the variational problem (3.8) has a solution in* $\mathcal{H}_N(\Gamma)$*, unique up to contributions from the finite-dimensional kernel* $\mathrm{Ker}(\mathsf{b}_T)$.

(ii) Any solution $\binom{\boldsymbol{\mu}}{\alpha}\in\mathcal{H}_N(\Gamma)$ *of (3.20) will spawn a solution* $\boldsymbol{U}$ *of (T) via*

$$\boldsymbol{U}=\boldsymbol{\mathcal{SL}}\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix}+\boldsymbol{\mathcal{DL}}\begin{pmatrix}\boldsymbol{g}\\ f\end{pmatrix}.$$

Of course, for a solution $\boldsymbol{U}$ of (T), $\mathsf{T}_{\mathrm{N}}\boldsymbol{U}$ will supply a solution of (3.8).

3.3 First-Kind BIE for (N)

Now we consider the variational problem/boundary value problem (N)/(2.6) with $\mathsf{T}_{\mathrm{N}}\boldsymbol{U}=\binom{\boldsymbol{\eta}}{\varphi}\in\mathcal{H}_N(\Gamma)$ given and $\mathsf{T}_{\mathrm{T}}\boldsymbol{U}=\binom{\boldsymbol{u}}{w}\in\mathcal{H}_T(\Gamma)$ sought. By the second line of (3.5) the traces of a solution $\boldsymbol{U}$ of (N) are linked by the following first-kind boundary integral equation:

$$(\{\mathsf{T}_{\mathrm{N}}\}_\Gamma\circ\boldsymbol{\mathcal{DL}})\begin{pmatrix}\boldsymbol{u}\\ w\end{pmatrix}=\left(\tfrac{1}{2}\mathsf{Id}-\{\mathsf{T}_{\mathrm{N}}\}_\Gamma\circ\boldsymbol{\mathcal{SL}}\right)\begin{pmatrix}\boldsymbol{\eta}\\ \varphi\end{pmatrix}\quad\text{in }\mathcal{H}_N(\Gamma)\,.\tag{3.19}$$

Again, the duality between $\mathcal{H}_N(\Gamma)$ and $\mathcal{H}_T(\Gamma)$ furnishes an equivalent variational formulation:

$$\begin{aligned}\begin{pmatrix}\boldsymbol{u}\\ w\end{pmatrix} \in \mathcal{H}_T(\Gamma): \quad & \left\langle\!\!\left\langle (\{\mathsf{T_N}\}_\Gamma \circ \boldsymbol{\mathcal{DL}})\begin{pmatrix}\boldsymbol{u}\\ w\end{pmatrix}, \begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma = \ell_N\begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix} := \\ & \left\langle\!\!\left\langle \left(\tfrac{1}{2}\mathsf{Id} - \{\mathsf{T_N}\}_\Gamma \circ \boldsymbol{\mathcal{SL}}\right)\begin{pmatrix}\boldsymbol{\eta}\\ \varphi\end{pmatrix}, \begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \quad \forall \begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix} \in \mathcal{H}_T(\Gamma)\,. \end{aligned} \tag{3.20}$$

We write $\mathsf{b}_N : \mathcal{H}_T(\Gamma) \times \mathcal{H}_T(\Gamma) \to \mathbb{R}$ for the underlying continuous bilinear form and from (3.2) we find the integral representation [10, Sect.7.2, (80)]

$$\begin{aligned}\mathsf{b}_N\left(\begin{pmatrix}\boldsymbol{u}\\ w\end{pmatrix}, \begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix}\right) &:= \left\langle\!\!\left\langle (\{\mathsf{T_N}\}_\Gamma \circ \boldsymbol{\mathcal{DL}})\begin{pmatrix}\boldsymbol{u}\\ w\end{pmatrix}, \begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \\ &= -\int_\Gamma\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\,\mathrm{curl}_\Gamma\,\boldsymbol{u}(\mathbf{y})\ \mathrm{curl}_\Gamma\,\boldsymbol{q}(\mathbf{x})\,\mathrm{d}S(\mathbf{y},\mathbf{x}) \\ &\quad -\int_\Gamma\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\Big[(\boldsymbol{n}\times\boldsymbol{q})(\mathbf{x})\cdot\mathbf{curl}_\Gamma w(\mathbf{y}) \\ &\qquad\qquad + (\boldsymbol{n}\times\boldsymbol{u})(\mathbf{y})\cdot\mathbf{curl}_\Gamma v(\mathbf{x})\Big]\,\mathrm{d}S(\mathbf{y})\mathrm{d}S(\mathbf{x}) \\ &\quad +\int_\Gamma\int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})(w\boldsymbol{n})(\mathbf{y})(v\boldsymbol{n})(\mathbf{x})\,\mathrm{d}S(\mathbf{y},\mathbf{x})\,. \\ &= -\left(\mathrm{curl}_\Gamma\,\boldsymbol{u}, \mathrm{curl}_\Gamma\,\boldsymbol{q}\right)_{-1/2} + (w\boldsymbol{n}, v\boldsymbol{n})_{-1/2} \\ &\quad - (\boldsymbol{n}\times\boldsymbol{q}, \mathbf{curl}_\Gamma w)_{-1/2} - (\boldsymbol{n}\times\boldsymbol{u}, \mathbf{curl}_\Gamma v)_{-1/2}\,. \end{aligned} \tag{3.21}$$

Again, in (3.20) we face a variational saddle-point problem: Seek $\boldsymbol{u} \in \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $w \in H^{\frac{1}{2}}(\Gamma)$ such that

$$\boxed{\begin{array}{lclcl} (\mathrm{curl}_\Gamma\,\boldsymbol{u}, \mathrm{curl}_\Gamma\,\boldsymbol{q})_{-1/2} & + & (\boldsymbol{n}\times\boldsymbol{q}, \mathbf{curl}_\Gamma w)_{-1/2} & = & -F_1(\boldsymbol{q})\,, \\ (\boldsymbol{n}\times\boldsymbol{u}, \mathbf{curl}_\Gamma v)_{-1/2} & - & (w\boldsymbol{n}, v\boldsymbol{n})_{-1/2} & = & -F_2(v)\,. \end{array}} \tag{3.22}$$

for all $\boldsymbol{q} \in \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $v \in H^{\frac{1}{2}}(\Gamma)$. Here, $\binom{\boldsymbol{q}}{v} \mapsto F_1(\boldsymbol{q}) + F_2(v)$ abbreviates the right hand side of (3.20).

We recall the results of [10, Sect. 7.2] where we determined the kernel of b_N.

Lemma 3.7 (Kernel of b_N). *We have*

$$\mathrm{Ker}(\mathsf{b}_N) = \boldsymbol{\mathcal{Z}}_1(\Gamma) \times \{0\}\,,$$

with

$$\mathcal{Z}_1(\Gamma) := \left\{ \boldsymbol{u} \in \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) : \begin{array}{l} \mathrm{curl}_\Gamma\, \boldsymbol{u} = \mathrm{div}_\Gamma(\boldsymbol{n} \times \boldsymbol{u}) = 0 \;, \\ (\boldsymbol{n} \times \boldsymbol{u}, \mathbf{curl}_\Gamma v)_{-1/2} = 0 \;\forall v \in H^{\frac{1}{2}}(\Gamma) \end{array} \right\} ,$$

which is finite-dimensional with $\dim \mathrm{Ker}(\mathsf{b}_N) = \beta_1(\Gamma)$*, the first Betti number of the boundary* Γ.

Proof. Let $\binom{\boldsymbol{u}}{w}$ solve (3.22) for $F_1 = F_2 = 0$. Then, the choice $\boldsymbol{q} := \mathbf{grad}_\Gamma\, w$ reveals $\mathbf{curl}_\Gamma w = 0$. This can be used to deduce

(i) $\mathrm{curl}_\Gamma\, \boldsymbol{u} = 0$ from testing the first equation with $\boldsymbol{q} := \boldsymbol{u}$,

(ii) and $w = 0$ by testing the second equation with $v := w$.

Then, the orthogonality constraint in the definition of $\mathcal{Z}_1(\Gamma)$ is immediate. □

The identities (3.5) and (3.6) again pave the way for an alternative representation of the kernel. Applying (3.5) for $\boldsymbol{U} \in \mathcal{Z}_N(\Omega)$ and (3.6) for $\boldsymbol{U} \in \mathcal{Z}_N(\Omega')$, we find, since either $\mathsf{T}_\mathrm{N}\, \boldsymbol{U} = 0$ or $\mathsf{T}_\mathrm{N}^+ \boldsymbol{U} = 0$, that $\mathsf{T}_\mathrm{T}\, \mathcal{Z}_N(\Omega) + \mathsf{T}_\mathrm{T}^+\, \mathcal{Z}_N(\Omega') \subset \mathrm{Ker}(\mathsf{b}_N)$. Similar arguments as above establish the other inclusion. If $\binom{\boldsymbol{z}}{v} \in \mathrm{Ker}(\mathsf{b}_N) = \mathrm{Ker}(\{\mathsf{T}_\mathrm{N}\}_\Gamma \circ \mathcal{DL})$, then $\mathsf{T}_\mathrm{N}\, \mathbf{V} = 0$ for $\mathbf{V} = \mathcal{DL}\binom{\boldsymbol{z}}{v}$. This vector field is a solution of (N) for zero data and, thus, belongs to $\mathcal{Z}_N(\Omega)$ in Ω and $\mathcal{Z}_N(\Omega')$ outside. Then the jump relations give the desired inclusion and we have shown the following representation.

Lemma 3.8 (Kernel of b_N (II)). *Traces of interior and exterior Neumann harmonic vector fields span the kernel of* b_N*:*

$$\mathrm{Ker}(\mathsf{b}_N) = \mathrm{Ker}(\{\mathsf{T}_\mathrm{N}\}_\Gamma \circ \mathcal{DL}) = \mathsf{T}_\mathrm{T}\, \mathcal{Z}_N(\Omega) + \mathsf{T}_\mathrm{T}^+\, \mathcal{Z}_N(\Omega') \;. \tag{3.23}$$

This meshes well with the dimension formula $\beta_1(\Gamma) = \beta_1(\Omega) + \beta_1(\Omega')$, a consequence of Alexander duality in singular co-homology [28, Ch. 11].

Remark 3.2. A spherical Ω has no handle, that is, $\beta_1(\Omega) = \beta_1(\Omega') = 0$, so that in this case $\mathrm{Ker}(\mathsf{b}_N) = \{0\}$. In the case of more complex topology of Ω, comparing results of Section 2.3 and Lemma 3.8 points to a "spurious kernel components" for (3.20): the subspace contributed to the kernel by $\mathsf{T}_\mathrm{T}^+\, \mathcal{Z}_N(\Omega')$ does not correspond to actual kernal functions for $\boldsymbol{\Delta}_\mathrm{N}$. However, arguing as in Remark 3.1, on this subspace the representation formula of Theorem 3.1 produces a null field inside Ω. △

Lemma 3.9 (Consistency of right-hand side of (3.20)). *If* $\boldsymbol{\eta}$ *complies with the consistency condition (2.10), then* $\ell_N\binom{\boldsymbol{q}}{v} = 0$ *for all* $\binom{\boldsymbol{q}}{v} \in \mathrm{Ker}(\mathsf{b}_N)$.

Proof. As a tool we introduce the "Maxwell double layer potential" [9, Eq. (28)], [10, Sect. 5.3],

$$(\boldsymbol{\mathcal{M}}\boldsymbol{q})(\mathbf{x}) := \mathbf{curl}_\mathbf{x} \int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})(\boldsymbol{q} \times \boldsymbol{n})(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \;, \quad \mathbf{x} \notin \Gamma \;,$$

recall the jump relation $[\gamma_{\mathrm{T}}]\,\boldsymbol{\mathcal{M}}(\boldsymbol{q}) = \boldsymbol{q}$, and consider the "Maxwell double layer boundary integral operator" $\mathsf{C} := \{\gamma_{\mathrm{T}}\}\,\boldsymbol{\mathcal{M}} : \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \to \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ [9, Sect. 5].

A kernel element $\binom{\boldsymbol{q}}{v} \in \mathrm{Ker}(\mathsf{b}_N)$ satisfies $v = 0$ by Lemma 3.7 so that $\ell_N\binom{\boldsymbol{q}}{v}$ simplifies to

$$\begin{aligned}
\ell_N\begin{pmatrix}\boldsymbol{q}\\ v\end{pmatrix} &= \left\langle\!\!\left\langle \left(\tfrac{1}{2}\mathsf{Id} - \{\mathsf{T}_\mathrm{N}\}_\Gamma \circ \boldsymbol{\mathcal{SL}}\right)\begin{pmatrix}\boldsymbol{\eta}\\ \varphi\end{pmatrix}, \begin{pmatrix}\boldsymbol{q}\\ 0\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \\
&= \left\langle \tfrac{1}{2}\boldsymbol{\eta}, \boldsymbol{q}\right\rangle_\Gamma - \left\langle \left\{ +\,\mathbf{curl}_{\mathbf{x}} \int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})\boldsymbol{\eta}(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\right\} \times \boldsymbol{n}, \boldsymbol{q}\right\rangle_\Gamma \\
&= \left\langle \tfrac{1}{2}\boldsymbol{\eta}, \boldsymbol{q}\right\rangle_\Gamma - \left\langle \left\{ \int_\Gamma (\mathbf{grad}\,\mathcal{G})(\mathbf{x}-\mathbf{y}) \times \boldsymbol{\eta}(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\right\} \times \boldsymbol{n}, \boldsymbol{q}\right\rangle_\Gamma \\
&= \left\langle \tfrac{1}{2}\boldsymbol{\eta}, \boldsymbol{q}\right\rangle_\Gamma - \left\langle \left\{ \int_\Gamma (\mathbf{grad}\,\mathcal{G})(\mathbf{x}-\mathbf{y}) \times (\boldsymbol{q}\times\boldsymbol{n})(\mathbf{x})\,\mathrm{d}S(\mathbf{x})\right\}, \boldsymbol{\eta}\right\rangle_\Gamma \\
&= \left\langle \tfrac{1}{2}\boldsymbol{\eta}, \boldsymbol{q}\right\rangle_\Gamma + \left\langle \left\{ \mathbf{curl}_{\mathbf{y}} \int_\Gamma \mathcal{G}(\mathbf{x}-\mathbf{y})(\boldsymbol{q}\times\boldsymbol{n})(\mathbf{x})\,\mathrm{d}S(\mathbf{x})\right\}, \boldsymbol{\eta}\right\rangle_\Gamma \\
&= \left\langle \tfrac{1}{2}\boldsymbol{\eta}, \boldsymbol{q}\right\rangle_\Gamma + \langle \boldsymbol{\eta}, \mathsf{C}\boldsymbol{q}\rangle_\Gamma = \left\langle \eta, (\tfrac{1}{2}\mathsf{Id} + \mathsf{C})\boldsymbol{q}\right\rangle_\Gamma .
\end{aligned}$$

By Lemma 3.8, a kernel element $\binom{\boldsymbol{q}}{0} \in \mathrm{Ker}(\mathsf{b}_N)$ also satisfies $\boldsymbol{q} \in \gamma_{\mathrm{T}}\,\boldsymbol{\mathcal{Z}}_N(\Omega) + \gamma_{\mathrm{T}}^+\,\boldsymbol{\mathcal{Z}}_N(\Omega')$, that is, it is a tangential trace of Neumann harmonic vector fields. Those meet the assumptions of Theorem 3.1 and, therefore, allow an integral representation according to (3.3). For $\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}_N(\Omega)$ holds $\mathsf{T}_\mathrm{N}\,\boldsymbol{Z} = 0$ and we find

$$\boldsymbol{Z} = \boldsymbol{\mathcal{DL}}(\mathsf{T}_\mathrm{T}\,\boldsymbol{Z}) = \boldsymbol{\mathcal{DL}}\begin{pmatrix}\gamma_\mathrm{T}\,\boldsymbol{Z}\\ 0\end{pmatrix} = \boldsymbol{\mathcal{M}}(\gamma_\mathrm{T}\,\boldsymbol{Z}) \quad \text{in } \Omega\ .$$

Taking into account the jump relations, we obtain for the tangential trace

$$\gamma_\mathrm{T}\,\boldsymbol{Z} = (\mathsf{C} + \tfrac{1}{2}\mathsf{Id})(\gamma_\mathrm{T}\,\boldsymbol{Z})\ . \tag{3.24}$$

In the same vein, $\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}_N(\Omega')$ satisfies

$$\boldsymbol{Z} = -\boldsymbol{\mathcal{DL}}(\mathsf{T}_\mathrm{T}^+\,\boldsymbol{Z}) = -\boldsymbol{\mathcal{DL}}\begin{pmatrix}\gamma_\mathrm{T}^+\,\boldsymbol{Z}\\ 0\end{pmatrix} = -\boldsymbol{\mathcal{M}}(\gamma_\mathrm{T}^+\,\boldsymbol{Z}) \quad \text{in } \Omega'\ ,$$

from which we infer

$$\gamma_\mathrm{T}^+\,\boldsymbol{Z} = (\tfrac{1}{2}\mathsf{Id} - \mathsf{C})(\gamma_\mathrm{T}^+\,\boldsymbol{Z})\ . \tag{3.25}$$

Finally, from (3.24) and (3.25) we conclude

$$\ell_N\begin{pmatrix}\boldsymbol{q}\\0\end{pmatrix} = \left\langle \eta, (\tfrac{1}{2}\mathsf{Id}+\mathsf{C})\boldsymbol{q}\right\rangle_\Gamma = \begin{cases}\langle \eta, \boldsymbol{q}\rangle_\Gamma \overset{(2.10)}{=} 0 & \text{for } \boldsymbol{q}\in\gamma_{\mathrm{T}}\,\boldsymbol{\mathcal{Z}}_N(\Omega)\ ,\\ 0 & \text{for } \boldsymbol{q}\in\gamma_{\mathrm{T}}^{+}\,\boldsymbol{\mathcal{Z}}_N(\Omega')\ .\end{cases}$$

□

Lemma 3.10 (Restricted saddle point problem (3.22)). *The bilinear form associated with the variational problem (3.22) satisfies an inf-sup condition on the orthogonal complement* $\mathrm{Ker}(\mathsf{b}_N)^\perp$ *of* $\mathrm{Ker}(\mathsf{b}_N)$ *in* $\mathcal{H}_T(\Gamma)$.

Proof. On the space $\mathrm{Ker}(\mathsf{b}_N)^\perp$ we examine the "principal part" variational saddle point problem: Seek $\binom{\boldsymbol{u}}{w}\in\mathrm{Ker}(\mathsf{b}_N)^\perp$ such that

$$\begin{aligned}(\mathrm{curl}_\Gamma\,\boldsymbol{u},\mathrm{curl}_\Gamma\,\boldsymbol{q})_{-1/2} \quad + \quad (\boldsymbol{n}\times\boldsymbol{q},\mathbf{curl}_\Gamma w)_{-1/2} &= -F_1(\boldsymbol{q})\ ,\\ (\boldsymbol{n}\times\boldsymbol{u},\mathbf{curl}_\Gamma v)_{-1/2} \qquad\qquad\qquad\qquad &= -F_2(v)\ .\end{aligned} \tag{3.26}$$

for all $\binom{\boldsymbol{q}}{v}\in\mathrm{Ker}(\mathsf{b}_N)^\perp$. Owing to the formula $\mathrm{curl}_\Gamma\,\boldsymbol{w} = \mathrm{div}_\Gamma(\boldsymbol{w}\times\boldsymbol{n})$, the kernel of curl_Γ is

$$\boldsymbol{n}\times\mathrm{Ker}(\mathrm{curl}_\Gamma) = \mathbf{curl}_\Gamma H^{\frac{1}{2}}(\Gamma)\oplus(\boldsymbol{n}\times\boldsymbol{\mathcal{Z}}_1(\Gamma))\ , \tag{3.27}$$

where $\oplus$ indicates a $\boldsymbol{H}_\times^{-\frac{1}{2}}(\Gamma)$-orthogonal direct sum. Hence the orthogonality constraint on $\binom{\boldsymbol{u}}{w}$ plus the second equation of (3.26) (for $F_2=0$) completely suppress the kernel of the upper-left bilinear form $(\boldsymbol{u},\boldsymbol{q})\mapsto(\mathrm{curl}_\Gamma\,\boldsymbol{u},\mathrm{curl}_\Gamma\,\boldsymbol{q})_{-1/2}$. Since the range of curl_Γ is closed we conclude that the "$\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$-ellipticity on the kernel" holds for the saddle point problem (3.26).

The inf-sup condition for $(\boldsymbol{q},w)\mapsto(\boldsymbol{n}\times\boldsymbol{q},\mathbf{curl}_\Gamma w)_{-1/2}$ is clear from $\mathbf{grad}_\Gamma\,H^{\frac{1}{2}}(\Gamma)\subset\mathrm{Ker}(\mathsf{b}_N)^\perp$. By the theory of [2, Sect. 4.2.3], the bilinear form of (3.26) induces an isomorphism $\mathrm{Ker}(\mathsf{b}_N)^\perp\to(\mathrm{Ker}(\mathsf{b}_N))'$.

Observe that by Rellich's embedding theorem the bilinear form $(w,v)\mapsto(w\boldsymbol{n},v\boldsymbol{n})_{-1/2}$ is compact on $H^{\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$. Then apply a Fredholm alternative argument to finish the proof, because the bilinear form b_N is trivially injective on $\mathrm{Ker}(\mathsf{b}_N)^\perp$. □

Corollary 3.11 (Existence and uniqueness of solutions of (3.20)). *Under the constraint (2.10) on the data* $\binom{\eta}{\varphi}\in\mathcal{H}_N(\Gamma)$ *the variational problem (3.20) always has solutions, which are unique up to contributions from* $\mathrm{Ker}(\mathsf{b}_N)$.

Further, if $\binom{\boldsymbol{u}}{w}\in\mathcal{H}_T(\Gamma)$ *solves (3.20), then* $\boldsymbol{U} := \boldsymbol{\mathcal{SL}}\binom{\eta}{\varphi}+\boldsymbol{\mathcal{DL}}\binom{\boldsymbol{u}}{w}$ *solves (N).*

It goes without saying that for a solution $\boldsymbol{U}$ of (N), its trace $\mathsf{T}_{\mathrm{T}}\,\boldsymbol{U}$ solves (3.20).

4 Boundary Element Galerkin Discretization

For the remainder of this manuscript we assume, for the sake of simplicity, that Ω is a Lipschitz polyhedron. Then Γ can be equipped with a triangular surface mesh Γ_h consisting of flat triangles, see [25, Sect. 4.1.2] for details. We also refer to this reference for the notion of shape regularity of Γ_h.

4.1 Boundary Element Spaces

On Γ_h we introduce the following piecewise polynomial boundary element spaces:

- The space $\mathcal{S}^{1,0}(\Gamma_h)$ of Γ_h-piecewise linear, continuous scalar functions [25, Sect 4.1.7],
- the space $\mathcal{S}^{0,-1}(\Gamma_h)$ of Γ_h-piecewise constant scalar functions [25, Sect 4.1.3],
- and the spaces $\mathcal{E}^0(\Gamma_h)$ and $\mathcal{E}^0_\times(\Gamma_h)$ of piecewise linear tangential surface vector fields with continuous tangential and normal components, respectively, across interelement edges [9, Sect. 8].

The boundary element spaces are conforming, that is, subspaces of trace spaces

$$\mathcal{S}^{1,0}(\Gamma_h) \subset H^{\frac{1}{2}}(\Gamma) \quad , \qquad \mathcal{S}^{0,-1}(\Gamma_h) \subset H^{-\frac{1}{2}}(\Gamma) \; ,$$
$$\mathcal{E}^0(\Gamma_h) \subset \boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \quad , \qquad \mathcal{E}^0_\times(\Gamma_h) \subset \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \; .$$

As such, surface differential operators are well defined on the boundary element spaces. In fact, they generate discrete DeRham complexes, the horizontal sequences in

$$\begin{array}{ccccc}
\mathcal{S}^{1,0}(\Gamma_h) & \xrightarrow{\mathbf{grad}_\Gamma} & \mathcal{E}^0(\Gamma_h) & \xrightarrow{\mathrm{curl}_\Gamma} & \mathcal{S}^{0,-1}(\Gamma_h) \; , \\
\mathsf{Id}\big\downarrow & & \cdot\times\boldsymbol{n}\big\downarrow & & \mathsf{Id}\big\downarrow \\
\mathcal{S}^{1,0}(\Gamma_h) & \xrightarrow{\mathbf{curl}_\Gamma} & \mathcal{E}^0_\times(\Gamma_h) & \xrightarrow{\mathrm{div}_\Gamma} & \mathcal{S}^{0,-1}(\Gamma_h) \; .
\end{array} \tag{4.1}$$

Read $\cdot \times \boldsymbol{n}$ as the rotation of a tangential vector field counterclockwise around the normal $\boldsymbol{n}$ by $\pi/2$, which is an isometry $\mathcal{E}^0(\Gamma_h) \to \mathcal{E}^0_\times(\Gamma_h)$. Then this diagram will commute.

By the complex property we have $(\cdot,\cdot)_{-1/2}$-orthogonal decompositions

$$\{v_h \in \mathcal{S}^{1,0}(\Gamma_h) : \mathbf{curl}_\Gamma v_h = 0\} = \{0\} \oplus \mathcal{Z}_0(\Gamma_h) \; , \tag{4.2}$$
$$\{\boldsymbol{\eta}_h \in \mathcal{E}^0_\times(\Gamma_h) : \mathrm{div}_\Gamma \boldsymbol{\eta}_h = 0\} = \mathbf{curl}_\Gamma \mathcal{S}^{1,0}(\Gamma_h) \oplus (\boldsymbol{\mathcal{Z}}_1(\Gamma_h) \times \boldsymbol{n}) \; , \tag{4.3}$$
$$\mathcal{S}^{0,-1}(\Gamma_h) = \mathrm{curl}_\Gamma \, \mathcal{E}^0(\Gamma_h) \oplus \mathcal{Z}_2(\Gamma_h) \; . \tag{4.4}$$

This implicitly defines the so-called discrete co-homology spaces $\mathcal{Z}_0(\Gamma_h) \subset \mathcal{S}^{1,0}(\Gamma_h)$, $\boldsymbol{\mathcal{Z}}_1(\Gamma_h) \subset \mathcal{E}^0(\Gamma_h)$, and $\mathcal{Z}_2(\Gamma_h) \subset \mathcal{S}^{0,-1}(\Gamma_h)$. A deep result of algebraic topology [27, Ch. 4] ensures that the Betti numbers of Γ tell the dimensions of the co-homology spaces of the discrete DeRham complex [21, Sect. IV.1]:

$$\dim \mathcal{Z}_0(\Gamma_h) = \beta_2(\Gamma) \; , \quad \dim \boldsymbol{\mathcal{Z}}_1(\Gamma_h) = \beta_1(\Gamma) \; , \quad \dim \mathcal{Z}_2(\Gamma_h) = \beta_2(\Gamma) \; . \tag{4.5}$$

4.2 Stable BEM for (T)

The boundary element Galerkin discretization of the variational boundary integral equation (3.8) associated with (T) leads to: Seek $\binom{\boldsymbol{\mu}_h}{\alpha_h} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h) \subset \mathcal{H}_N(\Gamma)$ such that

$$\mathsf{b}_T\left(\begin{pmatrix}\boldsymbol{\mu}_h\\ \alpha_h\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix}\right) = \ell_T\begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix} \quad \forall\begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h)\,. \tag{4.6}$$

Lemma 4.1 (Discrete kernel of b_T). *The kernel of* b_T *on* $\mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h)$ *is*

$$\begin{aligned}\mathrm{Ker}_h(\mathsf{b}_T) &:= \left\{\begin{array}{l}\binom{\boldsymbol{\mu}_h}{\alpha_h} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h) : \\ \mathsf{b}_T\left(\binom{\boldsymbol{\mu}_h}{\alpha_h}, \binom{\boldsymbol{\eta}_h}{\beta_h}\right) = 0\,\forall\binom{\boldsymbol{\eta}_h}{\beta_h} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h)\end{array}\right\} \\ &= \{0\} \times \mathcal{Z}_2(\Gamma_h)\,.\end{aligned} \tag{4.7}$$

Proof. If $\binom{\boldsymbol{\mu}_h}{\alpha_h} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h)$ satisfies $\mathsf{b}_T\left(\binom{\boldsymbol{\mu}_h}{\alpha_h}, \binom{\boldsymbol{\eta}_h}{\beta_h}\right) = 0$ for all $\binom{\boldsymbol{\eta}_h}{\beta_h} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h)$, we first conclude from (4.4) and by testing with $\binom{0}{\beta_h}$ that $\mathrm{div}_\Gamma \boldsymbol{\mu}_h = 0$. Next, test with $\binom{\boldsymbol{\eta}_h}{0}$, $\mathrm{div}_\Gamma \boldsymbol{\eta}_h = 0$, which yields $\boldsymbol{\mu}_h = 0$. Testing with general $\binom{\boldsymbol{\eta}_h}{0}$ finishes the proof. □

The dimensions of $\mathrm{Ker}_h(\mathsf{b}_T)$ and $\mathrm{Ker}(\mathsf{b}_T)$ agree, but by no means we can expect $\mathrm{Ker}_h(\mathsf{b}_T) = \mathrm{Ker}(\mathsf{b}_T)$, because the kernels have been introduced as orthogonal complements in different spaces.

After having identified the kernel a key question is the stability of the discrete variational problem on its complement. To tackle this we rely on mapping properties of div_Γ.

Lemma 4.2 (Discrete closed range of div_Γ). *There is a constant $C > 0$ depending only on Γ and the shape-regularity of Γ_h such that*

$$\forall \varphi_h \in \mathrm{div}_\Gamma \mathcal{E}^0_\times(\Gamma_h) : \quad \exists \boldsymbol{\eta}_h \in \mathcal{E}^0_\times(\Gamma_h) : \quad \begin{array}{l}\mathrm{div}_\Gamma \boldsymbol{\eta}_h = \varphi_h\,, \\ \|\boldsymbol{\eta}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \le C\,\|\varphi_h\|_{H^{-\frac{1}{2}}(\Gamma)}\end{array}.$$

Proof. [9, Lemma 2] gives a bounded projection $\mathsf{R} : \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \boldsymbol{H}^{\frac{1}{2}}_\times(\Gamma)$ with $\mathrm{div}_\Gamma \mathsf{R} = \mathrm{div}_\Gamma$ and

$$\|\mathsf{R}\boldsymbol{\eta}\|_{H^{\frac{1}{2}}(\Gamma)} \le C\,\|\mathrm{div}_\Gamma \boldsymbol{\eta}\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{\eta} \in \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)\,. \tag{4.8}$$

Let Π_h stand for the canonical local edge projection into $\mathcal{E}^0_\times(\Gamma_h)$, for which [9, Lemma 16] gives the remarkable interpolation error estimate

$$\|\boldsymbol{\mu} - \Pi_h \boldsymbol{\mu}\|_{L^2(\Gamma)} \le C h^{1/2}\,\|\boldsymbol{\mu}\|_{H^{\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{\mu} \in \boldsymbol{H}^{\frac{1}{2}}_\times(\Gamma),\ \mathrm{div}_\Gamma \boldsymbol{\mu} \in \mathcal{S}^{0,-1}(\Gamma_h)\,. \tag{4.9}$$

for surface tangential vector fields with piecewise constant surface divergence. Both in (4.8) and (4.9) the constants merely depend on Γ and shape-regularity. Thus, on one hand

$$\|\Pi_h \mathsf{R}\boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|\Pi_h \mathsf{R}\boldsymbol{\mu}_h\|_{L^2(\Gamma)} \leq \|\mathsf{R}\boldsymbol{\mu}_h\|_{L^2(\Gamma)} + Ch^{1/2}\|\mathsf{R}\boldsymbol{\mu}_h\|_{H^{\frac{1}{2}}(\Gamma)}$$
$$\leq C\,\|\mathrm{div}_\Gamma \boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{\mu}_h \in \mathcal{E}^0_\times(\Gamma_h)\ ,$$

and, on the other hand, by the commuting diagram property [9, Eq. (57)] of Π_h,

$$\mathrm{div}_\Gamma \Pi_h \mathsf{R}\boldsymbol{\mu}_h = \mathsf{Q}_h \mathrm{div}_\Gamma \mathsf{R}\boldsymbol{\mu}_h = \mathsf{Q}_h \mathrm{div}_\Gamma \boldsymbol{\mu}_h = \mathrm{div}_\Gamma \boldsymbol{\mu}_h \quad \forall \boldsymbol{\mu}_h \in \mathcal{E}^0_\times(\Gamma_h)\ .$$

Here, Q_h designates the $L^2(\Gamma)$-orthogonal projection onto $\mathcal{S}^{0,-1}(\Gamma_h)$. Setting $\varphi_h := \mathrm{div}_\Gamma \boldsymbol{\mu}_h$, $\boldsymbol{\eta}_h := \Pi_h \mathsf{R}\boldsymbol{\mu}_h$ finishes the proof. □

Lemma 4.3 (Stability of discrete variational problem). *On the $\mathcal{H}_N(\Gamma)$-orthogonal complement of* $\mathrm{Ker}_h(\mathsf{b}_T)$ *in* $\mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h)$,

$$\mathrm{Ker}_h(\mathsf{b}_T)^\perp = \mathcal{E}^0_\times(\Gamma_h) \times \mathrm{div}_\Gamma \mathcal{E}^0_\times(\Gamma_h)\ ,$$

the bilinear form b_T *satisfies an inf-sup condition with a constant depending only on* Γ *and the shape-regularity of* Γ_h.

Proof. Recall that the variational problem (4.6) can be written in saddle point form analogously to (3.13). Thus, we can invoke the abstract saddle point theory of [2, Thm. 5.2.5]. "Ellipticity on the kernel" [2, Eq. 5.2.34] is straightfoward from (4.1) and the definitions of the norm on $\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. The other inf-sup condition [2, Eq. (5.2.33)] is immediate from Lemma 4.2 and will hold "h-uniformly". □

In addition, since

$$\mathrm{Ker}_h(\mathsf{b}_T)^\perp \subset \mathrm{Ker}(\mathsf{b}_T)^\perp = \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times \mathrm{div}_\Gamma \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)\ ,$$

and $\mathrm{Ker}_h(\mathsf{b}_T)^\perp = \{(\boldsymbol{\mu}_h, \rho_h(\alpha_h)), \boldsymbol{\mu}_h \in \mathcal{E}^0_\times(\Gamma_h), \alpha_h \in \mathcal{S}^{0,-1}(\Gamma_h)\}$, boundary element Galerkin solutions in $\mathrm{Ker}_h(()\mathsf{b}_T)^\perp$ enjoy quasi-optimality.

Corollary 4.4 (A priori BE error estimate). *Let* $(\boldsymbol{\mu}, \alpha)$ *and* $(\boldsymbol{\mu}_h, \alpha_h)$ *be the unique solutions of the variational problem (3.8) restricted to the* $\mathcal{H}_N(\Gamma)$*-orthogonal complements* $\mathrm{Ker}(\mathsf{b}_T)^\perp$ *and* $\mathrm{Ker}_h(\mathsf{b}_T)^\perp$, *respectively. Then*

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}_h\|_{\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} + \|\alpha - \alpha_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq$$
$$C\left(\inf_{\boldsymbol{\eta}_h \in \mathcal{E}^0_\times(\Gamma_h)} \|\boldsymbol{\mu} - \boldsymbol{\eta}_h\|_{\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} + \inf_{\boldsymbol{\nu}_h \in \mathcal{E}^0_\times(\Gamma_h)} \|\alpha - \mathrm{div}_\Gamma \boldsymbol{\nu}_h\|_{H^{-\frac{1}{2}}(\Gamma)}\right)\ .$$

We emphasize that the consistency of the right hand side of (3.8) established in Lemma 3.4 does not extend to (4.6), because $\mathrm{Ker}(\mathsf{b}_T) \neq \mathrm{Ker}_h(\mathsf{b}_T)$. Since $\beta_2(\Gamma) > 0$, which implies $\mathrm{Ker}_h(\mathsf{b}_T) \neq \{0\}$, solutions of (4.6) will not exist in general!

Remark 4.1. Let us describe a possible remedy to this inconsistency. Denote by $\Gamma_h^q, q = 1 \dots Q$, the connected components of Γ_h, and observe that $\mathrm{div}_\Gamma \mathcal{E}^0_\times(\Gamma_h)$ consists in those $\alpha_h \in \mathcal{S}^{0,-1}(\Gamma_h)$ satisfying $\int_{\Gamma_h^q} \alpha_h(\mathbf{y}) dS(\mathbf{y}) = 0$ for each $q = 1 \dots Q$ which rewrites $\alpha_h = \rho_h(\alpha_h)$ where

$$\rho_h(\alpha)(\mathbf{x}) := \alpha_h(\mathbf{x}) - \sum_{q=1}^{Q} 1_{\Gamma_h^q}(\mathbf{x}) \int_{\Gamma_h^q} \alpha_h(\mathbf{y}) dS(\mathbf{y}) / |\Gamma_h^q|.$$

The operator ρ_h is a projection of $\mathcal{S}^{0,-1}(\Gamma_h)$ onto $\mathrm{div}_\Gamma \mathcal{E}^0_\times(\Gamma_h)$ whose kernel consists of those functions that are constant on each Γ_h^q. Applying this projector reduces to updating a vector after the calculation of Q scalar products, which is computationally cheap. Now, in order to solve (3.8), the following discrete variational formulation can be used.

$$\begin{aligned}
&\text{Seek } (\boldsymbol{\mu}_h, \alpha_h) \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h) \text{ such that} \\
&\widetilde{\mathsf{b}_T}\left(\begin{pmatrix}\boldsymbol{\mu}_h\\ \alpha_h\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix}\right) = \widetilde{\ell}_T\begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix} \quad \forall \begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix} \in \mathcal{E}^0_\times(\Gamma_h) \times \mathcal{S}^{0,-1}(\Gamma_h) \\
&\text{where} \quad \widetilde{\mathsf{b}_T}\left(\begin{pmatrix}\boldsymbol{\mu}_h\\ \alpha_h\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}_h\\ \beta_h\end{pmatrix}\right) := \mathsf{b}_T\left(\begin{pmatrix}\boldsymbol{\mu}_h\\ \rho_h(\alpha_h)\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}_h\\ \rho_h(\beta_h)\end{pmatrix}\right) \\
&\qquad \widetilde{\ell}_T\begin{pmatrix}\boldsymbol{\mu}_h\\ \alpha_h\end{pmatrix} := \ell_T\begin{pmatrix}\boldsymbol{\mu}_h\\ \rho_h(\alpha_h)\end{pmatrix}
\end{aligned} \tag{4.10}$$

This discrete formulation very much looks like (4.6), except that $\mathrm{Ker}_h(\widetilde{\mathsf{b}_T}) = \{0\} \times \mathrm{Kern}(\rho_h)$. Besides, the right-hand side $\widetilde{\ell}_T$ always satisfies the compatibility condition $\widetilde{\ell}_T(\boldsymbol{\mu}_h, \alpha_h) = 0$ for all $(\boldsymbol{\mu}_h, \alpha_h) \in \mathrm{Ker}_h(\widetilde{\mathsf{b}_T})$. As a consequence, the conjugate gradient method applied to (4.10) will converge toward a solution despite $\mathrm{Ker}_h(\widetilde{\mathsf{b}_T})$ being non-trivial, see e.g [13, Thm.4.2] and [15, 16]. △

4.3 Stable BEM for (N)

The Galerkin discretization of (3.20) employs the conforming boundary element space $\mathcal{E}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h) \subset \mathcal{H}_T(\Gamma)$. We remind of the definition of $\boldsymbol{\mathcal{Z}}_1(\Gamma_h)$ in (4.3) and start with the discrete counterpart of Lemma 3.7.

Lemma 4.5 (Discrete kernel of b_N). *The kernel of* b_N *from (3.21) restricted to* $(\mathcal{E}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)) \times (\mathcal{E}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h))$ *is*

$$\mathrm{Ker}_h(\mathsf{b}_N) = \boldsymbol{\mathcal{Z}}_1(\Gamma_h) \times \{0\} .$$

Proof. We study solutions $\binom{\boldsymbol{u}_h}{w_h} \in \boldsymbol{\mathcal{E}}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)$ of the variational saddle point problem (3.22) for $F_1 = F_2 = 0$ and $\boldsymbol{\mathcal{E}}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)$ as test space. Hardly surprising, the proof runs parallel to that of Lemma 3.7.

As a consequence of (4.1), testing the first equation of (3.22) with $\boldsymbol{q} := \mathbf{grad}_\Gamma w_h$ yields $\mathbf{curl}_\Gamma w_h = 0$. Then, from testing with $\boldsymbol{q} := \boldsymbol{u}_h$, we learn $\mathbf{curl}_\Gamma \boldsymbol{u}_h = 0$. We can choose $v := w_h$ in the second equation and conclude $\|\boldsymbol{n} w_h\|_{H^{-\frac{1}{2}}(\Gamma)} = 0$, which implies $w_h = 0$. Next, the second equation of (3.22) reveals that $(\boldsymbol{n} \times \boldsymbol{u}_h, \mathbf{curl}_\Gamma v_h)_{-1/2} = 0$ for any $v_h \in \mathcal{S}^{1,0}(\Gamma_h)$. Finally, recall the definition (4.3) of $\boldsymbol{\mathcal{Z}}_1(\Gamma)$. □

Again, the discrete variational problem is "h-uniformly" stable on the complement of its kernel.

Lemma 4.6 (Stability on complement of kernel). *The bilinear form* b_N *satisfies an inf-sup condition on the* $\mathcal{H}_T(\Gamma)$*-orthogonal complement* $\mathrm{Ker}_h(\mathsf{b}_N)^\perp$ *of* $\mathrm{Ker}_h(\mathsf{b}_N)$ *in* $\boldsymbol{\mathcal{E}}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)$*, with constant depending only on* Γ *and the shape regularity of* Γ_h*.*

Proof. To begin with, we consider the variational saddle point problem (3.22) on the $(\cdot,\cdot)_{-1/2}$-orthogonal complement of $\boldsymbol{\mathcal{Z}}_1(\Gamma_h) \times \mathcal{Z}_0(\Gamma_h)$ in $\boldsymbol{\mathcal{E}}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)$. We aim to apply the abstract result of [3, Thm. III.4.11] for augmented saddle point problems.

A rotated version of Lemma 4.2 gives the existence of a constant $C > 0$ depending only on Γ and the shape regularity of Γ_h such that

$$\begin{aligned} &\forall \varphi_h \in \mathrm{curl}_\Gamma \boldsymbol{\mathcal{E}}^0(\Gamma_h): \quad \exists \boldsymbol{q}_h \in \boldsymbol{\mathcal{E}}^0(\Gamma_h): \\ &\mathrm{curl}_\Gamma \boldsymbol{q}_h = \varphi_h \quad \text{and} \quad \|\boldsymbol{q}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \|\varphi_h\|_{H^{-\frac{1}{2}}(\Gamma)} . \end{aligned}$$

In combination with (4.3) we conclude

$$\|\boldsymbol{q}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \|\mathrm{curl}_\Gamma \boldsymbol{q}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{q}_h \in \mathrm{Ker}_h(\mathrm{curl}_\Gamma)^\perp . \tag{4.11}$$

Here and below $\perp$ designates orthogonal complements in the respective boundary element spaces with respect to the inner products $(\cdot,\cdot)_{-1/2}$. In analogy to (3.27) we have

$$\mathrm{Ker}_h(\mathrm{curl}_\Gamma) \times \boldsymbol{n} = \mathbf{curl}_\Gamma H^{\frac{1}{2}}(\Gamma) \oplus (\boldsymbol{n} \times \boldsymbol{\mathcal{Z}}_1(\Gamma_h)) . \tag{4.12}$$

Combined with (4.11) and the constraint $\boldsymbol{u} \perp \boldsymbol{\mathcal{Z}}_1(\Gamma_h)$ this confirms "ellipticity on the kernel" in the sense of [3, Thm III.4.3 (i)]. The inf-sup condition of [3, Thm. III.4.3 (ii)] follows from (4.1), because $\mathbf{curl}_\Gamma \mathcal{S}^{1,0}(\Gamma_h) \subset \boldsymbol{n} \times \boldsymbol{\mathcal{E}}^0(\Gamma_h)$.

We continue with the crucial observation that $\mathcal{Z}_0(\Gamma)$ is the space of functions that are piecewise constant on the connected components of Γ. As a consequence, $\mathcal{Z}_0(\Gamma_h) \subset \mathcal{S}^{1,0}(\Gamma_h)$ for any surface triangulation Γ_h and we trivially have the "h-uniform" norm equivalence $\|v_h \boldsymbol{n}\|_{H^{-\frac{1}{2}}(\Gamma)} \approx \|v_h\|_{H^{\frac{1}{2}}(\Gamma)}$ on $\mathcal{Z}_0(\Gamma_h)$. Appealing to [3, Thm. III.4.11] finishes the proof. □

Parallel to Section 4.2 we encounter the situation that, in general $\mathrm{Ker}(\mathsf{b}_N) \neq \mathrm{Ker}_h(\mathsf{b}_N)$, so that the consistency of the right hand side of (3.20) according to Lemma 3.9 does not carry over to the discrete setting. This means that, in general, (3.20) restricted to $\mathcal{E}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)$ will fail to possess a solution. At least, existence and uniqueness of Galerkin solutions in $\mathrm{Ker}_h(\mathsf{b}_N)^\perp$ is guaranteed by Lemma 4.6.

Recall that the *gap* between two non-trivial subspaces $V, W \subset X$ of a normed space X is defined as [17, Chap. 4, §2]

$$\delta(V,W) = \sup_{v \in V \setminus \{0\}} \frac{1}{\|v\|_X} \inf_{w \in W} \|v - w\|_X \ . \tag{4.13}$$

Then, in the current concrete setting Lemma A.1 with $H := \mathcal{H}_T(\Gamma)$, $V := \mathrm{Ker}(\mathsf{b}_N)^\perp$, $V_h := \mathrm{Ker}_h(\mathsf{b}_N)^\perp$, together with Lemmata A.2 and A.3, implies the following a priori error estimate. Note that Lemmata 3.10 and 4.6 ensure that the required inf-sup conditions hold uniformly in the discretization parameters.

Theorem 4.7 (A priori Galerkin error estimate). *Write* $(\boldsymbol{u}, w)$ *and* $(\boldsymbol{u}_h, w_h)$ *for the unique solution of the variational problem (3.19) restricted to* $\mathrm{Ker}(\mathsf{b}_N)^\perp$ *and* $\mathrm{Ker}_h(\mathsf{b}_N)^\perp$, *respectively.*

Then there is a constant $C > 0$ *depending only on* Γ *and the shape-regularity of* Γ_h *such that*

$$\left\| \begin{pmatrix} \boldsymbol{u} - \boldsymbol{u}_h \\ w - w_h \end{pmatrix} \right\|_{\mathcal{H}_T(\Gamma)} \leq C \Big\{ \inf_{(\boldsymbol{v}_h, v_h) \in \mathcal{E}^0(\Gamma_h) \times \mathcal{S}^{1,0}(\Gamma_h)} \left\| \begin{pmatrix} \boldsymbol{u} - \boldsymbol{v}_h \\ w - v_h \end{pmatrix} \right\|_{\mathcal{H}_T(\Gamma)} + \left\| \begin{pmatrix} \boldsymbol{u} \\ w \end{pmatrix} \right\|_{\mathcal{H}_T(\Gamma)} \delta(\boldsymbol{\mathcal{Z}}_1(\Gamma), \boldsymbol{\mathcal{Z}}_1(\Gamma_h)) \Big\} \ .$$

The gap term can be omitted, if $\boldsymbol{\mathcal{Z}}_1(\Gamma) = \boldsymbol{\mathcal{Z}}_1(\Gamma_h) = \{0\}$.

We have also used that in a Hilbert space setting, the two gaps between closed subspaces agree provided that neither of them is zero.

Remark 4.2 (Convergence of gap). The speed with which $\delta(\boldsymbol{\mathcal{Z}}_1(\Gamma), \boldsymbol{\mathcal{Z}}_1(\Gamma_h)) \to 0$ under mesh refinement, will depend on the smoothness of the fields in $\boldsymbol{\mathcal{Z}}_1(\Gamma)$ and appropriate local mesh refinement in the vicinity of edges and corners of Γ, where singularities of harmonic vector fields will usually be located. △

Remark 4.3. In terms of implementation, the stable discrete variational boundary integral equations for case (N) according to Lemma 4.6 are realized by using

- scalar potential representations (4.3) of discrete divergence-free surface vector fields,
- and constraints enforced by means of Lagrangian multipliers.

This is suggested by the representation as $H^{-\frac{1}{2}}(\Gamma)$-orthogonal sum

$$(\boldsymbol{\mathcal{Z}}_1(\Gamma_h)\times\boldsymbol{n})^{\perp} = \underbrace{\{\boldsymbol{\eta}_h\in\mathcal{E}^0_{\times}(\Gamma_h) : \operatorname{div}_\Gamma\boldsymbol{\eta}_h = 0\}}_{=:\mathcal{E}^0_{\times}(\Gamma_h,\operatorname{div}_\Gamma 0)}{}^{\perp} + \mathbf{curl}_\Gamma\mathcal{S}^{1,0}(\Gamma_h)\;, \tag{4.14}$$

which follows from (4.3). Hence, in the discretized version of (3.22) we set

$$\boldsymbol{n}\times\boldsymbol{u}_h = \boldsymbol{\mu}_h + \operatorname{curl}_\Gamma p_h\;,\quad \boldsymbol{\mu}_h\in\mathcal{E}^0_{\times}(\Gamma_h)\;,\quad p_h\in\mathcal{Z}_0(\Gamma_h)^{\perp}\;, \tag{4.15}$$

and add the constraint

$$(\boldsymbol{\mu}_h,\boldsymbol{\zeta}_h)_{-1/2} = 0 \quad \forall\boldsymbol{\zeta}_h\in\mathcal{E}^0_{\times}(\Gamma_h,\operatorname{div}_\Gamma 0)\;.$$

This leads to the extended vartiational saddle-point problem: seek $\boldsymbol{\mu}_h\in\mathcal{E}^0_{\times}(\Gamma_h)$, $p_h\in\mathcal{Z}_0(\Gamma_h)^{\perp}$, $\boldsymbol{\rho}_h\in\mathcal{E}^0_{\times}(\Gamma_h,\operatorname{div}_\Gamma 0)$, $w_h\in\mathcal{S}^{1,0}(\Gamma_h)$ such that

$$\begin{array}{lllll}
(\operatorname{div}_\Gamma\boldsymbol{\mu}_h,\operatorname{div}_\Gamma\boldsymbol{\nu}_h)_{-1/2} & + & (\boldsymbol{\nu}_h,\boldsymbol{\rho}_h)_{-1/2} & & = & -F_1(\boldsymbol{\nu}_h)\;,\\
(\boldsymbol{\mu}_h,\boldsymbol{\zeta}_h)_{-1/2} & & & & = & 0\;,\\
 & & & (\operatorname{curl}_\Gamma w_h,\operatorname{curl}_\Gamma q_h)_{-1/2} & = & -F_1(\operatorname{curl}_\Gamma q_h)\;,\\
(\operatorname{curl}_\Gamma p_h,\operatorname{curl}_\Gamma v_h)_{-1/2} & & & -(w_h\boldsymbol{n},v_h\boldsymbol{n})_{-1/2} & = & -F_2(v_h)\;,
\end{array}$$

for all $\boldsymbol{\nu}_h\in\mathcal{E}^0_{\times}(\Gamma_h)$, $q_h\in\mathcal{Z}_0(\Gamma_h)^{\perp}$, $\boldsymbol{\zeta}_h\in\mathcal{E}^0_{\times}(\Gamma_h,\operatorname{div}_\Gamma 0)$, and $v_h\in\mathcal{S}^{1,0}(\Gamma_h)$. Then $\mathbf{u}_h$ can be recovered from (4.15). Moreober, the constraint implicit in $\mathcal{E}^0_{\times}(\Gamma_h,\operatorname{div}_\Gamma 0)$ can be imposed through piecewise constant Lagrangian multipliers. △

5 Numerical Examples

For two test cases we conduct a numerical exploration of the kernels $\operatorname{Ker}_h(\mathsf{b}_T)$ and $\operatorname{Ker}_h(\mathsf{b}_N)$ to supplement the theoretical results obtained in Lemma 4.1 and 4.5. We focus on the dimensions of these spaces, for which we established the formulas

$$\dim\operatorname{Ker}_h(\mathsf{b}_T) = \beta_2(\Gamma)\;, \qquad \text{see Lemma 3.3,} \tag{5.1}$$

$$\dim\operatorname{Ker}_h(\mathsf{b}_N) = \beta_1(\Gamma)\;, \qquad \text{see Lemma 3.7.} \tag{5.2}$$

We also aim to confirm that these dimension counts are stable under quadrature and round-off error, which are inevitable in the computation of Galerkin boundary element matrices. In order to study the kernels, we consider the variational eigenvalue problems

$$\mathfrak{u}\in\mathcal{H}_N(\Gamma)\setminus\{0\},\ \lambda\in\mathbb{C} :\quad \mathsf{b}_T(\mathfrak{u},\mathfrak{v}) = \lambda(\mathfrak{u},\mathfrak{v})_{\mathcal{H}_N(\Gamma)}\quad \forall\mathfrak{v}\in\mathcal{H}_N(\Gamma)\;, \tag{5.3}$$

$$\mathfrak{p}\in\mathcal{H}_T(\Gamma)\setminus\{0\},\ \lambda\in\mathbb{C} :\quad \mathsf{b}_N(\mathfrak{p},\mathfrak{q}) = \lambda(\mathfrak{p},\mathfrak{q})_{\mathcal{H}_T(\Gamma)}\quad \forall\mathfrak{q}\in\mathcal{H}_T(\Gamma)\;, \tag{5.4}$$

and their Galerkin discretization relying on the lowest-order boundary element trial and test boundary element spaces introduced in Section 4. The concrete definitions of the scalar products are

$$\left(\begin{pmatrix}\boldsymbol{\mu}\\ \alpha\end{pmatrix}, \begin{pmatrix}\boldsymbol{\eta}\\ \beta\end{pmatrix}\right)_{\mathcal{H}_N(\Gamma)} := (\mathrm{div}_\Gamma \boldsymbol{\mu}, \mathrm{div}_\Gamma \boldsymbol{\eta})_{-1/2} + (\boldsymbol{\mu}, \boldsymbol{\eta})_{-1/2} + (\alpha, \beta)_{-1/2} \;, \tag{5.5}$$

on the one hand, and

$$\begin{aligned}\left(\begin{pmatrix}\mathbf{u}\\ w\end{pmatrix}, \begin{pmatrix}\mathbf{q}\\ v\end{pmatrix}\right)_{\mathcal{H}_T(\Gamma)} := \; & (\mathrm{curl}_\Gamma \mathbf{u}, \mathrm{curl}_\Gamma \mathbf{q})_{-1/2} + (\mathbf{u}, \mathbf{q})_{-1/2} + \\ & (\mathbf{grad}_\Gamma w, \mathbf{grad}_\Gamma v)_{-1/2} + (w, v)_{-1/2}\end{aligned} \tag{5.6}$$

on the other hand. For the assembly of Galerkin matrices we used the C++ boundary element library BEMTOOL[4] developed by one of the authors (X. Claeys) under Lesser GNU Public Licence. The double integrals defining the entries of the Galerkin matrices are evaluated approximately based on

(i) the regularizing transformations described in detail in [25, § 5.2] in case of singular integrands, combined with tensorised 1D Gauss-Legendre rules,

(ii) tensorised 3-point quadrature rules on triangles in the case of smooth integrands.

The resulting generalized symmetric matrix eigenvalue problems were solved using the routine EIGH from the package SCIPY under PYTHON 3.5.3. The computations were carried out for two model geometries, see Figure 1,

(i) the cube $\Omega :=]-1,1[^3$ with $\beta_2(\Gamma) = 1$, $\beta_1(\Gamma) = 0$,

(ii) the "cubistic doughnut" $\Omega = (]-2,2[^2 \times]-1,1[) \backslash]-1,1[^3$ topologically equivalent to a torus, with $\beta_2(\Gamma) = 1$, $\beta_1(\Gamma) = 2$.

Figure 1. The two model geometries rendered with the coarsest triangular surface meshes used in the computations

The triangulations were obtained using the GMSH[5] mesh generator [12]. Results for sequences of three meshes obtained by regular refinement are plotted in Figures 2

[4] `https://github.com/xclaeys/BemTool`
[5] `http://gmsh.info/`

and 3. Independently of the meshwidth we observe a stable number of near-zero eigenvalues in each case, which tells us the dimensions of the discrete kernels. Evidently, those agree with the theoretical predictions, see the figure captions for more explanations.

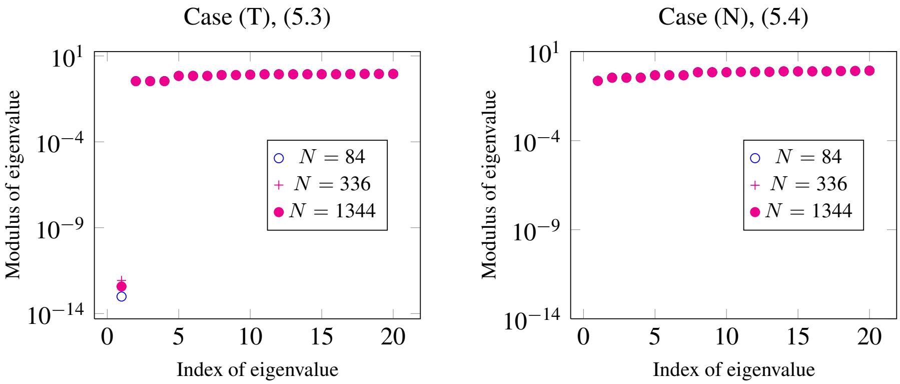

Figure 2. **Cube:** Plot of the 20 smallest approximate eigenvalues (in modulus) obtained by Galerkin boundary element discretization of (5.3) (left) and (5.4) (right) for different numbers N of flat triangles of the surface meshes. The dimensions of kernels are 1 (T) and 0 (N), respectively, in accordance with Lemma 4.1 and Lemma 4.5.

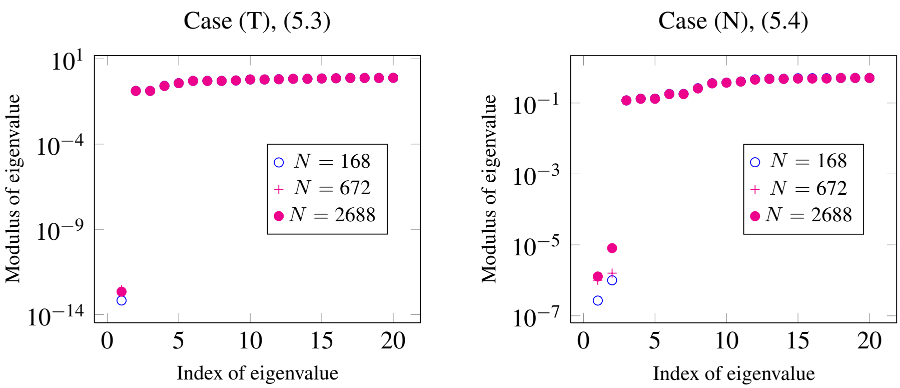

Figure 3. "**Cubistic doughnut**": Plot of the 20 smallest approximate eigenvalues (in modulus) obtained by Galerkin boundary element discretization of (5.3) (left) and (5.4) (right) for different numbers N of flat triangles of the surface meshes. From Lemma 4.1 and Lemma 4.5 we expect kernel dimensions of 1 (T) and 2 (N). Discretization errors affect the eigenvalues more severely in the case (N), where the discrete kernel $\mathrm{Ker}_h(\mathsf{b}_N)$ is not a subspace of the boundary element space and contains functions with strong edge singularities.

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A Gap-Based Abstract Auxiliary Estimates

Let a be a continuous bilinear form on the real Hilbert space H, whose norm will be denoted by $\|\cdot\|$. We write $V \subset H$ and $V_h \subset H$ for a pair of closed subspaces, unrelated a priori. On both the bilinear form a satisfies the *inf-sup conditions*

$$\sup_{v\in V} \frac{\mathsf{a}(w,v)}{\|v\|} \geq \gamma \|w\| \quad \forall w \in V \ , \qquad \sup_{w\in V} \frac{\mathsf{a}(w,v)}{\|v\|} > 0 \ , \tag{A.1}$$

$$\sup_{v_h\in V_h} \frac{\mathsf{a}(w_h,v_h)}{\|v_h\|} \geq \gamma_h \|w_h\| \quad \forall w_h \in V_h \ , \qquad \sup_{w_h\in V_h} \frac{\mathsf{a}(w_h,v_h)}{\|v_h\|} > 0 \ , \tag{A.2}$$

with constants $\gamma, \gamma_h > 0$. For every bounded linear functional $\ell \in H'$ they ensure existence and uniqueness of the solutions $u \in V$ and $u_h \in V_h$ of the following two variational problems.

$$u \in V: \quad \mathsf{a}(u, v) = \ell(v) \qquad \forall v \in V\,, \tag{A.3}$$

$$u_h \in V_h: \quad \mathsf{a}(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V_h\,. \tag{A.4}$$

Lemma A.1 (Variant of the second Strang lemma, *cf.* [3, III.1.2]). *Assuming (A.1) and (A.2) there is a constant $C > 0$ only depending on γ, γ_h and $\|\mathsf{a}\|$ such that*

$$\|u - u_h\| \leq C \left(\inf_{v_h \in V_h} \|u - v_h\| + \|\ell\|\, \delta(V_h, V) \right)\,,$$

where δ is the gap between two subspaces introduced in (4.13), and u, u_h are the solutions of the variational problems (A.3) and (A.4).

Proof. For any $v_h \in V_h$ and $v \in V$,

$$\begin{aligned} \|u - u_h\| &\leq \|u - v_h\| + \|u_h - v_h\| \\ &\leq \|u - v_h\| + \frac{1}{\gamma_h} \sup_{w_h \in V_h} \frac{a(u_h - u + u - v_h, w_h - v + v)}{\|w_h\|} \\ &\leq \big(1 + \frac{\|\mathsf{a}\|}{\gamma_h}\big) \|u - v_h\| + \frac{1}{\gamma_h} \sup_{w_h \in V_h} \frac{\ell(w_h) - \ell(v) - \mathsf{a}(u, w_h - v)}{\|w_h\|} \\ &\leq \big(1 + \frac{\|\mathsf{a}\|}{\gamma_h}\big) \|u - v_h\| + \frac{\|\ell\|}{\gamma_h}\big(1 + \frac{\|\mathsf{a}\|}{\gamma}\big) \sup_{w_h \in V_h} \frac{\|w_h - v\|}{\|w_h\|}\,. \end{aligned}$$

In the last step we use the estimate $\|u\| \leq \gamma^{-1} \|\ell\|$ implied by (A.1). Then fix $v \in V$ to be the best approximant of w_h in V to conclude the assertion. □

In the sequel let $H_h \subset H$ be a closed subspace, and so are $Z_h \subset H_h$ and $Z \subset H$. We define the two orthogonal complements

$$V := Z^\perp \quad \text{in} \quad H \quad , \quad V_h := Z_h^\perp \quad \text{in} \quad H_h\,.$$

Be aware that $V_h \subset H_h$.

Lemma A.2 (Gaps and complements). *The gap between V_h and V can be bounded according to*

$$\delta(V_h, V) \leq \delta(Z, Z_h)\,. \tag{A.5}$$

Proof. Writing P_V for the orthogonal projection on to V and parallel to Z, we seek to estimate the best approximation error $v_h - \mathsf{P}_V v_h$ for some $v_h \in V_h$. Since, $z := v_h - v \in Z$ for $v := \mathsf{P}_V v_h \in V$, by orthogonality $z \perp V$:

$$\begin{aligned}\|v_h - v\|^2 &= (v_h - v, z)_H = (v_h, z)_H = (v_h, z - z_h)_H \\ &\leq \|v_h\|\,\|z\| \sup_{w\in Z} \frac{\|w - \mathsf{P}_{Z_h} w\|}{\|w\|} \leq \|v_h\|\,\|z\|\,\delta(Z, Z_h)\ ,\end{aligned}$$

where we chose $z_h \in Z_h$ as the best approximant $\mathsf{P}_{Z_h} z$ of z in Z_h. Dividing by $\|v_h\|\,\|z\|$ and taking the supremum over all $v_h \in V_h$ we conclude the relationship (A.5) of gaps. □

Lemma A.3 (Gap-based best approximation estimate). *For any $v \in V$ we have the best approximation estimate*

$$\inf_{v_h \in V_h} \|v - v_h\| \leq 2 \inf_{w_h \in H_h} \|v - w_h\| + \delta(Z_h, Z)\,\|v\|\ .$$

Proof. Fix $v \in V$ and let p_h be its unique best approximant in H_h, which means $p_h = \mathsf{P}_{H_h} v$. Split orthogonally

$$p_h = v_h + z_h\ ,\quad v_h \in V_h\ ,\quad z_h \in Z_h\ .$$

By orthogonality $Z_h \perp V_h$ and $Z \perp V$, we find

$$\begin{aligned}\|z_h\|^2 &= (p_h - v_h, z_h)_H = (p_h, z_h)_H = (p_h - v, z_h)_H + (v, z_h - z)_H \\ &\leq \|p_h - v\|\,\|z_h\| + \|v\|\,\|z_h\| \cdot \frac{\|z_h - z\|}{\|z_h\|}\ ,\end{aligned}$$

with $z := \mathsf{P}_Z z_h$ the best approximant of z_h in Z. Thus we have found

$$\|z_h\| \leq \|v - p_h\| + \delta(Z_h, Z)\,\|v\|\ .$$

We finish the proof by invoking the triangle inequality:

$$\|v - v_h\| = \|v - p_h + p_h - v_h\| \leq \|v - p_h\| + \|z_h\| \leq 2\,\|v - p_h\| + \delta(Z_h, Z)\,\|v\|\ .$$

□

B Accompanying Software

The complete codes used to conduct the numerical tests of this article are freely available on the github repository `https://github.com/xclaeys/HodgeLaplaceBEM`. Generating the executable code requires only a standard C++ compiler, the Boost math library dedicated to special functions (see `https://www.boost.org`) as well as PYTHON3 and SCIPY.

After compilation of one of the files `Formulation-T.cpp` or `Formulation-N.cpp`, execution of the corresponding programs assembles the BEM matrices by means of the C++ library BEMTool. The matrices are then exported in Ascii format. The corresponding spectrum can then be computed by launching the PYTHON3 script `compute-eigs.py`. As an output, the full list of eigenvalues are written in the file `eigvals.txt`.

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