

Scattering Coefficients of Inhomogeneous Objects and Their Application in Target Classification in Wave Imaging

L. Baldassari

Research Report No. 2018-23

June 2018

Latest revision: June 2018

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Scattering Coefficients of Inhomogeneous Objects and Their Application in Target Classification in Wave Imaging*

Lorenzo Baldassari[†]

Abstract

The aim of this paper is to provide and numerically test in the presence of measurement noise a procedure for target classification in wave imaging based on comparing frequency-dependent distribution descriptors with precomputed ones in a dictionary of learned distributions. Distribution descriptors for inhomogeneous objects are obtained from the scattering coefficients. First, we extract the scattering coefficients of the (inhomogeneous) target from the perturbation of the echoes. Then, for a collection of inhomogeneous targets, we build a frequency-dependent dictionary of distribution descriptors and use a matching algorithm in order to identify a target from the dictionary up to some translation, rotation and scaling.

Keywords: Helmholtz equation, Scattering coefficients, Inhomogeneous objects, Asymptotic expansion, Neumann-to-Dirichlet map, Target classification

Mathematics Subject Classification (2010): 35B20, 35B30, 35C20, 35R30

1 Introduction

There are several geometric and physical quantities associated with target classification such as eigenvalues and capacities [16]. The concept of the scattering coefficients is one of them. The notion appears naturally when we describe the perturbation of sounds emitted by animals such bats and dolphins due to the presence of targets whose material parameters (permeability and permittivity) are different from the ones of the background [12, 17, 18].

To mathematically introduce the concept of the scattering coefficients, we consider the Helmholtz problem in \mathbb{R}^2 for a given fixed frequency $\omega > 0$:

$$\begin{cases} \nabla \cdot (\chi(\mathbb{R}^2 \setminus \bar{B}) + \frac{1}{\sigma}\chi(B))\nabla u + \omega^2(\chi(\mathbb{R}^2 \setminus \bar{B}) + \mu\chi(B))u = 0 & \text{in } \mathbb{R}^2, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty. \end{cases} \quad (1.1)$$

Here, K is a positive constant, B is the target embedded in \mathbb{R}^2 with Lipschitz boundary, $\chi(B)$ (resp. $\chi(\mathbb{R}^2 \setminus \bar{B})$) is the characteristic function of B (resp. $\mathbb{R}^2 \setminus \bar{B}$), the positive constants σ and μ are the magnetic permeability and electric permittivity of the target, which are supposed to be different from the background permittivity and permeability (the constant function 1), U is the background solution,

*This work was supported by the SNF grant 200021-172483.

[†]Department of Mathematics, ETH Zürich, Rämistrasse 101, CH-8093 Zürich, Switzerland (lorenzo.baldassari@sam.math.ethz.ch).

i.e., a given solution to $(\Delta + \omega^2)U = 0$, and the solution u to (1.1) represents the perturbed wave. The perturbation $u - U$ due to the presence of the permittivity and permeability target B admits the following asymptotic expansion as $|x| \rightarrow \infty$, see [12]:

$$u(x) - U(x) = -\frac{i}{4} \sum_{n \in \mathbb{Z}} H_n^{(1)}(\omega|x|) e^{in\theta_x} \sum_{m \in \mathbb{Z}} W_{n,m}[B, \sigma, \mu, \omega] a_m(U), \quad (1.2)$$

where $H_n^{(1)}$ are the Hankel functions of the first kind of order n and $a_m(U)$ are constants such that $U(x) = \sum_{m \in \mathbb{Z}} a_m(U) J_m(\omega|x|) e^{im\theta_x}$ with J_m being the Bessel function of order m . The building blocks $W_{n,m}[B, \sigma, \mu, \omega]$ for the asymptotic expansion (1.2) are called the scattering coefficients. Note that the scattering coefficients $W_{n,m}[B, \sigma, \mu, \omega]$ can be reconstructed from the far-field measurements of u by a least-squared method. A stability analysis of the reconstruction is provided in [12].

This paper extends the results of [12] to targets with inhomogeneous permittivities and permeabilities. The concept of inhomogeneous scattering coefficients were first introduced in [4] and used later in [5] to prove resolution enhancement in high-contrast media. It is the purpose of this paper to extend the notion of scattering coefficients to objects with inhomogeneous permittivities and permeabilities and show their application in target classification. First, we prove important properties of the scattering coefficients such as translation, rotation and scaling formulas. Then, we construct distribution descriptors for multiple frequencies based on scaling, rotation, and translation properties of the scattering coefficients. Finally, we use a target identification algorithm in order to identify an inhomogeneous target from a dictionary of precomputed frequency-dependent distribution descriptors up to some translation, rotation and scaling. For the sake of simplicity, throughout this paper, we focus on two-dimensional models. However, the results can be easily extended to three dimensions.

The paper is organized as follows. In Section 2 we introduce the scattering coefficients for inhomogeneous targets and prove that they are the building blocks of the far-field expansion of the wave perturbation. Section 3 is devoted to the derivation of integral representations of the inhomogeneous scattering coefficients. In Section 4 we prove important properties for the inhomogeneous scattering coefficients, such as the exponential decay of the scattering coefficients. We also show translation, rotation and scaling property for the scattering coefficients. In Section 5 we construct the translation- and rotation-invariant distribution descriptor. We also observe that the inhomogeneous scattering coefficients are nothing else but the Fourier coefficients of the far-field pattern. In Section 6 we present numerical results in order to demonstrate the theoretical framework presented in previous sections. In particular, we investigate the identification of a target by the reconstruction of scattering coefficients from the measurements of the multistatic response matrix. A few concluding remarks are given in Section 7. In Appendix A, we provide integral representations for the case of piecewise constant (inhomogeneous) material parameters. In Appendix B, we present results of target identification using a full-view setting with no noise ($\sigma = 0\%$).

2 Scattering coefficients and asymptotic expansions

Let $\frac{1}{\sigma}$ be a bounded measurable function in \mathbb{R}^2 such that $\frac{1}{\sigma} - 1$ is compactly supported and

$$0 < \lambda_1 \leq \frac{1}{\sigma} \leq \lambda_2,$$

where λ_1, λ_2 are constants. Let μ be a bounded measurable function in \mathbb{R}^2 such that $\mu - 1$ is compactly supported and

$$0 < \lambda_3 \leq \mu \leq \lambda_4,$$

where λ_3, λ_4 are constants. For a given fixed frequency $\omega > 0$, we consider the following Helmholtz problem:

$$\begin{cases} \nabla \cdot \frac{1}{\sigma} \nabla u + \omega^2 \mu u = 0 & \text{in } \mathbb{R}^2, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty, \end{cases} \quad (2.1)$$

where U is a given solution to

$$\Delta U + \omega^2 U = 0, \quad (2.2)$$

and $|\cdot|$ denotes the Euclidean norm of \mathbb{R}^2 . In this section, we derive a full far-field expansion of $(u - U)(x)$ as $|x| \rightarrow \infty$. In the course of doing so, the notion of inhomogeneous scattering coefficients appears naturally.

Let B a bounded domain in \mathbb{R}^2 with Lipschitz boundary. We assume that B is such that

$$\text{supp} \left(\frac{1}{\sigma} - 1 \right) \subset B,$$

$$\text{supp}(\mu - 1) \subset B.$$

Suppose that B contains the origin. Note that (2.1) is equivalent to

$$\begin{cases} \nabla \cdot \frac{1}{\sigma} \nabla u + \omega^2 \mu u = 0 & \text{in } B, \\ \Delta u + \omega^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{B}, \\ u|_+ - u|_- = 0 & \text{on } \partial B, \\ \nu \cdot \nabla u|_+ - \nu \cdot \frac{1}{\sigma} \nabla u|_- = 0 & \text{on } \partial B, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty, \end{cases} \quad (2.3)$$

where ν is the outward normal vector at some $x \in \partial B$ and the subscripts \pm indicate the limits from outside and inside B , respectively.

In two dimensions, the fundamental solution $\Gamma_\omega(x)$ to the the Helmholtz equation

$$(\Delta + \omega^2)\Gamma_\omega(x) = \delta_0(x)$$

subject to the Sommerfeld outgoing radiation condition is given by

$$\Gamma_\omega(x) = -\frac{i}{4} H_0^{(1)}(\omega|x|).$$

Assume that ω^2 is not a Neumann eigenvalue of $-\frac{1}{\mu(x)} \nabla \cdot \frac{1}{\sigma(x)} \nabla$ on B . Let $N_{\sigma,\mu}$ be the Neumann function of problem

$$\begin{cases} \nabla \cdot \frac{1}{\sigma} \nabla u + \omega^2 \mu u = 0 & x \in B, \\ \frac{1}{\sigma} \frac{\partial u}{\partial \nu} = g & x \in \partial B, \end{cases} \quad (2.4)$$

that is, for each fixed $z \in B$, $N_{\sigma,\mu}(z, \cdot)$ is solution to

$$\begin{cases} \nabla_y \cdot \frac{1}{\sigma(y)} \nabla_y N_{\sigma,\mu}(z, y) + \omega^2 \mu(y) N_{\sigma,\mu}(z, y) = -\delta_z(y) & y \in B, \\ \frac{1}{\sigma(y)} \nabla_y N_{\sigma,\mu}(z, y) \nu_y = 0 & y \in \partial B. \end{cases} \quad (2.5)$$

We can prove the following result.

Proposition 2.1. *The function defined by*

$$\mathcal{N}_{B,\sigma,\mu}[g](x) := \int_{\partial B} N_{\sigma,\mu}(x,y)g(y) dS_y, \quad x \in B, \quad (2.6)$$

is the solution to (2.4). Moreover, the Neumann-to-Dirichlet (NtD) map $\Lambda_{\sigma,\mu} : H^{-\frac{1}{2}}(\partial B) \rightarrow H^{\frac{1}{2}}(\partial B)$ is well-defined, invertible and

$$\Lambda_{\sigma,\mu}[g](x) = \mathcal{N}_{B,\sigma,\mu}[g]|_{\partial B}(x) = u|_{\partial B}(x) \text{ for } x \in \partial B.$$

Proof. By (2.4), for each fixed $x \in B$, we have

$$\left(\nabla_y \cdot \frac{1}{\sigma(y)} \nabla_y u(y) + \omega^2 \mu(y) u(y) \right) N_{\sigma,\mu}(x,y) = 0.$$

By integrating over B and applying Green's formula, we have

$$\begin{aligned} & \int_{\partial B} g(y) N_{\sigma,\mu}(x,y) dS_y - \int_{\partial B} u(y) \frac{1}{\sigma(y)} \nabla_y N_{\sigma,\mu}(x,y) \cdot \nu_y dS_y \\ & + \int_B u(y) \left(\nabla_y \cdot \frac{1}{\sigma(y)} \nabla_y N_{\sigma,\mu}(x,y) + \omega^2 \mu(y) N_{\sigma,\mu}(x,y) \right) dy = 0. \end{aligned}$$

From (2.5), it follows that

$$u(x) = \int_{\partial B} N_{\sigma,\mu}(x,y)g(y) dS_y.$$

The second part of the proposition follows from the fact that ω^2 is not a Neumann eigenvalue of $-\Delta$ in B . \square

We can prove the following proposition.

Proposition 2.2. *For each $x \in \mathbb{R}^2 \setminus \bar{B}$, if u is the solution to (2.1) and U is such that (2.2) holds, then*

$$(u - U)(x) = \int_{\partial B} g(y) \Gamma_\omega(x-y) dS_y - \int_{\partial B} \Lambda_{\sigma,\mu}[g](y) \frac{\partial \Gamma_\omega}{\partial \nu_y} \Big|_+ (x-y) dS_y. \quad (2.7)$$

Proof. Observe that

$$\Delta(u - U)(y) + \omega^2(u - U)(y) = 0 \text{ for } y \in \mathbb{R}^2 \setminus \bar{B}.$$

Hence, for each fixed $x \in \mathbb{R}^2 \setminus \bar{B}$,

$$\int_{\mathbb{R}^2 \setminus \bar{B}} [\Delta(u - U)(y) + \omega^2(u - U)(y)] \Gamma_\omega(x-y) dy = 0.$$

Let $R > 0$ be such that $B \subset B_R$. By Green's formula, we have

$$\begin{aligned}
& - \int_{B_R \setminus \bar{B}} \nabla_y(u-U)(y) \nabla_y \Gamma_\omega(x-y) dy + \int_{\partial(B_R \setminus \bar{B})} \frac{\partial(u-U)}{\partial \nu_y}(y) \Gamma_\omega(x-y) dS_y \\
& + \int_{B_R \setminus \bar{B}} \omega^2(u-U)(y) \Gamma_\omega(x-y) dy = \\
& = \int_{\partial(B_R \setminus \bar{B})} \frac{\partial(u-U)}{\partial \nu_y}(y) \Gamma_\omega(x-y) dS_y - \int_{\partial(B_R \setminus \bar{B})} (u-U)(y) \frac{\partial \Gamma_\omega}{\partial \nu_y}(x-y) dS_y \\
& + \int_{B_R \setminus \bar{B}} (u-U)(y) [(\Delta + \omega^2) \Gamma_\omega(x-y)] dy = \\
& = 0.
\end{aligned} \tag{2.8}$$

Given $x \in \mathbb{R}^2 \setminus \bar{B}$, for R large enough,

$$\begin{aligned}
\int_{B_R \setminus \bar{B}} (u-U)(y) [(\Delta + \omega^2) \Gamma_\omega(x-y)] dy &= \int_{B_R \setminus \bar{B}} (u-U)(y) \delta_0(x-y) dy = \\
&= (u-U)(x).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
(u-U)(x) &= - \int_{\partial B} (u-U)(y) \frac{\partial \Gamma_\omega}{\partial \nu_y} \Big|_+ (x-y) dS_y + \int_{\partial B} \frac{\partial(u-U)}{\partial \nu_y}(y) \Gamma_\omega(x-y) dS_y = \\
&= - \int_{\partial B} u(y) \frac{\partial \Gamma_\omega}{\partial \nu_y} \Big|_+ (x-y) dS_y + \int_{\partial B} \frac{\partial u}{\partial \nu_y}(y) \Gamma_\omega(x-y) dS_y,
\end{aligned} \tag{2.9}$$

where the second equality holds from Green's formula and $\Delta U = -\omega^2 U$:

$$\begin{aligned}
& \int_{\partial B} U(y) \frac{\partial \Gamma_\omega}{\partial \nu_y} \Big|_+ (x-y) dS_y - \int_{\partial B} \frac{\partial U}{\partial \nu_y}(y) \Gamma_\omega(x-y) dS_y = \\
&= \int_B U(y) \Delta \Gamma_\omega(x-y) - \Delta U(y) \Gamma_\omega(x-y) dy = \\
&= \int_B U(y) (\delta_0(x-y) - \omega^2 \Gamma_\omega(x-y)) + \omega^2 U(y) \Gamma_\omega(x-y) dy = \\
&= \int_B U(y) \delta_0(x-y) dy = 0.
\end{aligned}$$

Thus, from the transmission conditions, it follows that (2.9) can be rewritten as

$$(u-U)(x) = - \int_{\partial B} \Lambda_{\sigma, \mu}[g](y) \frac{\partial \Gamma_\omega}{\partial \nu_y} \Big|_+ (x-y) dS_y + \int_{\partial B} g(y) \Gamma_\omega(x-y) dS_y.$$

□

For $x \in \mathbb{R}^2 \setminus \bar{B}$, we have

$$\Lambda_{1,1} \left(\frac{\partial \Gamma_\omega}{\partial \nu_y}(x-\cdot) \right) = \Gamma_\omega(x-\cdot) \text{ on } \partial B,$$

and hence

$$\int_{\partial B} \frac{\partial \Gamma_\omega}{\partial \nu_y} \Big|_+ (x-y) \Lambda_{\sigma,\mu}[g](y) dS_y = \int_{\partial B} \Gamma_\omega(x-y) \Lambda_{1,1}^{-1} \Lambda_{\sigma,\mu}[g](y) dS_y, \quad (2.10)$$

which is a consequence of the fact that $\Lambda_{1,1}$ is invertible and self-adjoint. The following result holds.

Proposition 2.3. *For each $u, v \in L^2(\partial B)$ solutions of (2.1),*

$$\left\langle \Lambda_{\sigma,\mu} \left[\frac{1}{\sigma} \frac{\partial u}{\partial \nu} \right], \frac{1}{\sigma} \frac{\partial v}{\partial \nu} \right\rangle_{L^2(\partial B)} = \left\langle \frac{1}{\sigma} \frac{\partial u}{\partial \nu}, \Lambda_{\sigma,\mu} \left[\frac{1}{\sigma} \frac{\partial v}{\partial \nu} \right] \right\rangle_{L^2(\partial B)}.$$

Proof. For the sake of simplicity, let us prove the case in which $\sigma = \mu = 1$. By subtracting $\Delta u \bar{v} + \omega^2 u \bar{v} = 0$ to $\Delta \bar{v} u + \omega^2 \bar{v} u = 0$, we get

$$\Delta u \bar{v} - \Delta \bar{v} u = 0.$$

By Green's formula,

$$\begin{aligned} \left\langle \Lambda_{1,1} \left[\frac{\partial u}{\partial \nu} \right], \frac{\partial v}{\partial \nu} \right\rangle_{L^2(\partial B)} &= \int_{\partial B} u \frac{\partial \bar{v}}{\partial \nu} = \\ &= \int_B \Delta \bar{v} u - \Delta u \bar{v} + \int_{\partial B} \bar{v} \frac{\partial u}{\partial \nu} = \\ &= \left\langle \frac{\partial u}{\partial \nu}, \Lambda_{1,1} \left[\frac{\partial v}{\partial \nu} \right] \right\rangle_{L^2(\partial B)}. \end{aligned}$$

□

By (2.10), for $x \in \mathbb{R}^2 \setminus \bar{B}$, formula (2.7) becomes

$$(u - U)(x) = \int_{\partial B} \Gamma_\omega(x-y) \Lambda_{1,1}^{-1} (\Lambda_{1,1} - \Lambda_{\sigma,\mu})[g](y) dS_y. \quad (2.11)$$

For $|x| > |y|$, by Graf's addition formula [1],

$$\Gamma_\omega(x-y) = -\frac{i}{4} \sum_{n \in \mathbb{Z}} H_n^{(1)}(\omega|x|) e^{in\theta_x} J_n(\omega|y|) e^{-in\theta_y},$$

where $x = (|x|, \theta_x)$, $y = (|y|, \theta_y)$, $H_n^{(1)}$ is the Hankel function of the first kind of order n and J_n is the Bessel function of the first kind of order n . In the following, we use \mathcal{C}_m to denote the cylindrical wave of index $m \in \mathbb{Z}$ and of wave number ω , which is defined by

$$\mathcal{C}_m(x) = \mathcal{C}_{m,\omega}(y) := J_m(\omega|y|) e^{im\theta_y}. \quad (2.12)$$

Hence, (2.11) becomes:

$$(u - U)(x) = -\frac{i}{4} \sum_{n \in \mathbb{Z}} H_n^{(1)}(\omega|x|) e^{in\theta_x} \int_{\partial B} \overline{\mathcal{C}_n(y)} \Lambda_{1,1}^{-1} (\Lambda_{1,1} - \Lambda_{\sigma,\mu})[g](y) dS_y. \quad (2.13)$$

For each $m \in \mathbb{Z}$, let u_m be the solution to (2.1) when \mathcal{C}_m is the source term. For $x \in \partial B$, let us define

$$g_m(x) := \frac{1}{\sigma} \frac{\partial u_m}{\partial \nu} \Big|_- (x).$$

Since the family of cylindrical waves $(\mathcal{C}_m(x))_m$ is complete [8], U admits the following expansion:

$$U(x) = \sum_{m \in \mathbb{Z}} a_m(U) \mathcal{C}_m(x),$$

where $a_m(U)$ are constants. By linearity of (2.1), we get

$$g(x) = \frac{1}{\sigma} \frac{\partial u}{\partial \nu} \Big|_{-} (x) = \sum_{m \in \mathbb{Z}} a_m(U) g_m(x).$$

Then, for $|x| \rightarrow \infty$,

$$(u - U)(x) = -\frac{i}{4} \sum_{n, m \in \mathbb{Z}} H_n^{(1)}(\omega|x|) e^{in\theta_x} a_m(U) \int_{\partial B} \overline{\mathcal{C}_n(y)} \Lambda_{1,1}^{-1}(\Lambda_{1,1} - \Lambda_{\sigma, \mu}) [g_m](y) dS_y. \quad (2.14)$$

Now we can define the scattering coefficients associated with σ and μ .

Definition 2.4. We define the scattering coefficients associated with the inhomogeneous permittivity $\mu(x)$ and the permeability $\sigma(x)$ for a given fixed frequency $\omega > 0$ as follows:

$$W_{n,m}[B, \sigma, \mu, \omega] = \int_{\partial B} \overline{\mathcal{C}_n(y)} \Lambda_{1,1}^{-1}(\Lambda_{1,1} - \Lambda_{\sigma, \mu}) [g_m](y) dS_y.$$

Then, for $x \rightarrow \infty$, (2.14) becomes

$$(u - U)(x) = -\frac{i}{4} \sum_{n, m \in \mathbb{Z}} H_n^{(1)}(\omega|x|) e^{in\theta_x} W_{n,m}[B, \sigma, \mu, \omega] a_m(U). \quad (2.15)$$

From (2.15), we get the following theorem.

Theorem 2.5. *Let u be the solution to (2.1). If U admits the following expansion:*

$$U(x) = \sum_{m \in \mathbb{Z}} a_m(U) \mathcal{C}_m(x),$$

then we have

$$(u - U)(x) = -\frac{i}{4} \sum_{n, m \in \mathbb{Z}} H_n^{(1)}(\omega|x|) e^{in\theta_x} W_{n,m}[B, \sigma, \mu, \omega] a_m(U). \quad (2.16)$$

which holds uniformly as $|x| \rightarrow \infty$.

3 Integral representation of the scattering coefficients

In this section, we provide another definition of scattering coefficients which is based on integral formulations. In the following, we suppose that ω^2 is not a Dirichlet eigenvalue of $-\Delta$, unless stated otherwise.

Formula (2.11) suggests that the solution u to (2.1) for a given fixed frequency $\omega > 0$ may be represented as

$$u(x) = \begin{cases} U(x) + S_B^\omega[\phi](x) & x \in \mathbb{R}^2 \setminus \bar{B}, \\ \mathcal{N}_{B, \sigma, \mu}[\psi](x) & x \in B, \end{cases} \quad (3.1)$$

where the pair of densities $(\phi, \psi) \in L^2(\partial B) \times L^2(\partial B)$ satisfy the transmission conditions

$$\begin{cases} U(x) + S_B^\omega[\phi](x) = \mathcal{N}_{B,\sigma,\mu}[\psi](x) = \Lambda_{\sigma,\mu}[\psi](x), \\ \psi(x) = (\frac{1}{2}I + (K_B^\omega)^*)[\phi](x) + \frac{\partial U}{\partial \nu}(x), \end{cases} \quad x \in \partial B. \quad (3.2)$$

Here, the single-layer potential and the trace operator are given by

$$S_B^\omega[\phi](x) = \int_{\partial B} \Gamma_\omega(x-y)\phi(y) dS_y,$$

and

$$(K_B^\omega)^*[\phi](x) = \int_{\partial B} \frac{\partial \Gamma_\omega}{\partial \nu_x}(x-y)\phi(y) dS_y,$$

and $\mathcal{N}_{B,\sigma,\mu}$ is defined by (2.6). We now prove that the integral equation (3.1) is uniquely solvable.

Lemma 3.1. *The operator $\mathcal{A} : L^2(\partial B) \times L^2(\partial B) \rightarrow H^1(\partial B) \times L^2(\partial B)$ defined by*

$$\mathcal{A} = \begin{bmatrix} -S_B^\omega & \Lambda_{\sigma,\mu} \\ -(\frac{1}{2}I + (K_B^\omega)^*) & I \end{bmatrix}$$

is invertible.

As a consequence of Lemma 3.1, we get the following theorem.

Theorem 3.2. *The solution u to (2.1) can be represented in the form (3.1), where the pair $(\phi, \psi) \in L^2(\partial B) \times L^2(\partial B)$ is the solution to*

$$\mathcal{A} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} U \\ \frac{\partial U}{\partial \nu} \end{bmatrix}. \quad (3.3)$$

Proof of Lemma 3.1. Let $(F, G) \in H^1(\partial B) \times L^2(\partial B)$. Proving that

$$\mathcal{A} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \quad (3.4)$$

is uniquely solvable is equivalent to prove existence and uniqueness of a solution in $H_{\text{loc}}^1(\mathbb{R}^2)$ to the problem

$$\begin{cases} \nabla \cdot \frac{1}{\sigma} \nabla u + \omega^2 \mu u = 0 & \text{in } B, \\ \Delta u + \omega^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{B}, \\ u|_+ - u|_- = F & \text{in } \partial B, \\ \frac{\partial u}{\partial \nu}|_+ - \frac{1}{\sigma} \frac{\partial u}{\partial \nu}|_- = G & \text{in } \partial B, \\ \left| \frac{\partial u}{\partial |x|} - i\omega u \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty. \end{cases} \quad (3.5)$$

To prove the injectivity of \mathcal{A} , let us suppose that $F = G = 0$. Using the fact that

$$\int_{\partial B} \frac{\partial u}{\partial \nu} \Big|_+ \bar{u} = \int_{\partial B} \frac{1}{\sigma} \frac{\partial u}{\partial \nu} \Big|_- \bar{u} = \int_B \frac{1}{\sigma} |\nabla u|^2 - \omega^2 \mu |u|^2,$$

we find that the solution u satisfies

$$\Im \int_{\partial B} \frac{\partial u}{\partial \nu} \Big|_+ \bar{u} = 0.$$

By applying Lemma 11.3 of [10] and the unique continuation property to (3.5), we readily get $u = 0$ in \mathbb{R}^2 . Since

$$u(x) = \begin{cases} S_B^\omega[\phi](x) & x \in \mathbb{R}^2 \setminus \bar{B}, \\ \mathcal{N}_{B,\sigma,\mu}[\psi](x) & x \in B, \end{cases}$$

we get

$$\begin{aligned} S_B^\omega[\phi](x) &= 0 \quad x \in \mathbb{R}^2 \setminus \bar{B}, \\ \mathcal{N}_{B,\sigma,\mu}[\psi](x) &= 0 \quad x \in B. \end{aligned}$$

In particular, $\mathcal{N}_{B,\sigma,\mu}[\psi](x) = \Lambda_{\sigma,\mu}[\psi](x) = 0$ on ∂B . Since $\Lambda_{\sigma,\mu}$ is invertible, $\psi = 0$ on ∂B . On the other hand, $S_B^\omega[\phi](x) = 0$ on ∂B . Suppose that ω^2 is not a Dirichlet eigenvalue for $-\Delta$ on B . Since $(\Delta + \omega^2)S_B^\omega[\phi](x) = 0$ in B , we have $S_B^\omega[\phi](x) = 0$ in B , and hence in \mathbb{R}^2 . It then follows from [10] that

$$\phi = \frac{\partial S_B^\omega[\phi]}{\partial \nu} \Big|_+ - \frac{\partial S_B^\omega[\phi]}{\partial \nu} \Big|_- = 0 \quad \text{on } \partial D.$$

This finishes the proof of the injectivity of \mathcal{A} . Since u is solution to $\Delta u + \omega^2 u = 0$ in $\mathbb{R}^2 \setminus \bar{B}$ and $\left| \frac{\partial u}{\partial |x|} - i\omega u \right| \leq \frac{K}{|x|^{\frac{3}{2}}}$ as $|x| \rightarrow \infty$, then there exists $\phi \in L^2(\partial B)$ such that

$$u(x) = S_B^\omega[\phi](x), \quad x \in \mathbb{R}^2 \setminus \bar{B}. \quad (3.6)$$

If we set

$$\psi = \frac{1}{\sigma} \frac{\partial u}{\partial \nu} \Big|_-, \quad (3.7)$$

then

$$\Lambda_{\sigma,\mu}[\psi] = u|_-.$$

By (3.6),

$$\frac{\partial u}{\partial \nu} \Big|_+ = \left(\frac{1}{2}I + (K_B^\omega)^* \right) [\phi],$$

and hence

$$\psi = \frac{1}{\sigma} \frac{\partial u}{\partial \nu} \Big|_- = \frac{\partial u}{\partial \nu} \Big|_+ + G = \left(\frac{1}{2}I + (K_B^\omega)^* \right) [\phi] + G.$$

Thus, for $x \in \partial B$,

$$G(x) = - \left(\frac{1}{2}I + (K_B^\omega)^* \right) [\phi](x) + \psi(x),$$

and from the transmission condition

$$F(x) = \Lambda_{\sigma,\mu}[\psi](x) - S_B^\omega[\phi](x),$$

which shows that ϕ and $\psi = \frac{1}{\sigma} \frac{\partial u}{\partial \nu} \Big|_-$ solve (3.4). □

We can now define the scattering coefficients associated with μ and σ using the operator \mathcal{A} .

Definition 3.3. For $m \in \mathbb{Z}$, let $(\phi_m, \psi_m) \in L^2(\partial B) \times L^2(\partial B)$ be the solution to

$$\mathcal{A} \begin{bmatrix} \phi_m \\ \psi_m \end{bmatrix} = \begin{bmatrix} \mathcal{C}_m \\ \frac{\partial \mathcal{C}_m}{\partial \nu} \end{bmatrix} \text{ on } \partial B, \quad (3.8)$$

where \mathcal{C}_m is the cylindrical wave. For $n \in \mathbb{Z}$, we define the scattering coefficients associated with the permittivity distribution $\mu(x)$ and permeability $\sigma(x)$ for a given fixed frequency $\omega > 0$ as follows:

$$W_{n,m} = W_{n,m}[B, \sigma, \mu, \omega] = \int_{\partial B} \overline{\mathcal{C}_n(y)} \phi_m(y) dS_y. \quad (3.9)$$

4 Properties of the scattering coefficients

In this section, we prove important properties for the scattering coefficients.

4.1 Decay of the Scattering Coefficients

Like the homogeneous case, the coefficient $W_{n,m}$ decays exponentially as the orders m, n increase. We can prove the following Proposition.

Proposition 4.1. *For a given fixed frequency $\omega > 0$, there is a constant K (depending on σ, μ and ω) such that*

$$|W_{n,m}[B, \sigma, \mu, \omega]| \leq \frac{K^{|n|+|m|}}{|n|^{|n|}|m|^{|m|}} \quad \forall n, m \in \mathbb{Z}. \quad (4.1)$$

Proof. Recall that

$$\mathcal{C}_m(x) = J_m(\omega|x|)e^{im\theta_x}.$$

Since

$$J_m(t) \sim \frac{1}{\sqrt{2\pi|m|}} \left(\frac{et}{2|m|} \right)^{|m|}$$

as $m \rightarrow \infty$, we have

$$\|\mathcal{C}_m\|_{L^2(\partial B)} + \|\nabla \mathcal{C}_m\|_{L^2(\partial B)} \leq \frac{K^{|m|}}{|m|^{|m|}}.$$

Then, with the same arguments as those of [7], there exists another constant K such that

$$\begin{aligned} \|\phi_m\|_{L^2(\partial B)} + \|\psi_m\|_{L^2(\partial B)} &\leq K \left(\|\mathcal{C}_m\|_{L^2(\partial B)} + \left\| \frac{\partial \mathcal{C}_m}{\partial \nu} \right\|_{L^2(\partial B)} \right) \leq \\ &\leq K (\|\mathcal{C}_m\|_{L^2(\partial B)} + \|\nabla \mathcal{C}_m\|_{L^2(\partial B)}). \end{aligned}$$

Hence, for another constant K ,

$$\|\phi_m\|_{L^2(\partial B)} \leq \frac{K^{|m|}}{|m|^{|m|}}.$$

Since

$$\left\| \overline{\mathcal{C}_n(y)} \right\|_{L^2(\partial B)} \leq \frac{K^{|n|}}{|n|^{|n|}},$$

by the definition of the scattering coefficients $W_{n,m}$, we have (4.1). □

4.2 Transformation formulas

We introduce the notation for translation, scaling, and rotation of a shape B as

$$B^z := B + z, \quad B^s := sB, \quad B^\theta = e^{i\theta}B,$$

and those of the material parameter σ (resp. μ) as

$$\sigma^z := \sigma(\cdot - z), \quad \sigma^s := \sigma(s^{-1}\cdot), \quad \sigma^\theta := \sigma(e^{-i\theta}\cdot).$$

We denote by $\phi_{U,B} = \phi_{U,B,\omega}$ the solution to (3.3) given the domain B , the source term U , and the frequency ω . We can prove that there exist explicit relations between the inhomogeneous scattering coefficients of B and B^z , B^s , B^θ . We prove the following Propositions.

Proposition 4.2 (Translation formula). *For any $z \in \mathbb{R}^2$, the following relation holds*

$$W_{n,m}[B^z, \sigma^z, \mu^z, \omega] = \sum_{a,b} \overline{\mathcal{C}_a(z)} \mathcal{C}_b(z) W_{n-a,m-b}[B, \sigma, \mu, \omega]. \quad (4.2)$$

Proof. Let $\tilde{\phi}_m = \phi_{\mathcal{C}_m, B^z}$ and $\tilde{y} = y + z$ with $y \in B$. By the definition of the scattering coefficients:

$$\begin{aligned} W_{n,m}[B^z, \sigma^z, \mu^z, \omega] &= \int_{\partial B^z} \overline{\mathcal{C}_n(\tilde{y})} \tilde{\phi}_m(\tilde{y}) \, dS_{\tilde{y}} = \\ &= \int_{\partial B} \overline{\mathcal{C}_n(y+z)} \tilde{\phi}_m(y+z) \, dS_y. \end{aligned} \quad (4.3)$$

From the identity [8],

$$\mathcal{C}_n(y-z) = \sum_{l \in \mathbb{Z}} \mathcal{C}_{l+n}(y) \overline{\mathcal{C}_l(z)},$$

we have

$$\overline{\mathcal{C}_n(y+z)} = \sum_{a \in \mathbb{Z}} \overline{\mathcal{C}_{a+n}(y)} \mathcal{C}_a(-z),$$

and

$$\mathcal{C}_m(\tilde{x}) = \mathcal{C}_m(x+z) = \sum_{b \in \mathbb{Z}} \mathcal{C}_{b+m}(x) \overline{\mathcal{C}_b(-z)},$$

where $\tilde{x} = x + z$ with $x \in B$. To find $\tilde{\phi}_m$, let us consider

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_m \\ \tilde{\psi}_m \end{bmatrix} (\tilde{x}) = \begin{bmatrix} -S_{B^z}^\omega [\tilde{\phi}_m](\tilde{x}) & \Lambda_{B^z, \sigma^z, \mu^z} [\tilde{\psi}_m](\tilde{x}) \\ -(\frac{1}{2}I + (K_{B^z}^\omega)^*) [\tilde{\phi}_m](\tilde{x}) & I [\tilde{\psi}_m](\tilde{x}) \end{bmatrix}.$$

Recall that $(\tilde{\phi}_m, \tilde{\psi}_m) \in L^2(\partial B^z) \times L^2(\partial B^z)$ is the solution to

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_m \\ \tilde{\psi}_m \end{bmatrix} (\tilde{x}) = \begin{bmatrix} \mathcal{C}_m(\tilde{x}) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_{\tilde{x}}}(\tilde{x}) \end{bmatrix}. \quad (4.4)$$

Let $\tilde{w} = w + z$ for $w \in B$. Let us prove that

$$\begin{bmatrix} \tilde{\phi}_m(\tilde{w}) \\ \tilde{\psi}_m(\tilde{w}) \end{bmatrix} = \sum_{b \in \mathbb{Z}} \overline{\mathcal{C}_b(-z)} \begin{bmatrix} \phi_{m+b}(w) \\ \psi_{m+b}(w) \end{bmatrix} \quad (4.5)$$

is the solution. Since

$$\begin{bmatrix} \mathcal{C}_m(\tilde{x}) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_{\tilde{x}}}(\tilde{x}) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_m(x+z) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_{x+z}}(x+z) \end{bmatrix} = \sum_{b \in \mathbb{Z}} \overline{\mathcal{C}_b(-z)} \begin{bmatrix} \mathcal{C}_{m+b}(x) \\ \frac{\partial \mathcal{C}_{m+b}}{\partial \nu_x}(x) \end{bmatrix},$$

if we prove that

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_m \\ \tilde{\psi}_m \end{bmatrix}(\tilde{x}) = \begin{bmatrix} -S_B^\omega[\tilde{\phi}_m(\cdot+z)](x) & \Lambda_{B,\sigma,\mu}[\tilde{\psi}_m(\cdot+z)](x) \\ -\left(\frac{1}{2}I + (K_B^\omega)^*\right)[\tilde{\phi}_m(\cdot+z)](x) & I[\tilde{\psi}_m(\cdot+z)](x) \end{bmatrix}, \quad (4.6)$$

then by the linearity of operator \mathcal{A} and the existence and uniqueness of a solution to system (4.4), we have (4.5). Let us prove (4.6). We write

$$\begin{aligned} S_{B^z}^\omega[\tilde{\phi}_m](\tilde{x}) &= \int_{B^z} \Gamma_\omega(\tilde{x} - \tilde{y}) \tilde{\phi}_m(\tilde{y}) dS_{\tilde{y}} = \\ &= \int_B \Gamma_\omega(x+z - (y+z)) \tilde{\phi}_m(y+z) dS_y = \\ &= S_B^\omega[\tilde{\phi}_m(\cdot+z)](x), \\ -\left(\frac{1}{2}I + (K_{B^z}^\omega)^*\right)[\tilde{\phi}_m](\tilde{x}) &= -\left(\frac{1}{2}I + (K_B^\omega)^*\right)[\tilde{\phi}_m(\cdot+z)](x), \\ \mathcal{N}_{B^z,\sigma^z,\mu^z}[\tilde{\psi}_m](\tilde{x}) &= \int_{\partial B^z} \tilde{N}_{\sigma^z,\mu^z}(\tilde{x}, \tilde{y}) \tilde{\psi}_m(\tilde{y}) dS_{\tilde{y}} = \\ &= \int_{\partial B} N_{\sigma,\mu}(x, y) \tilde{\psi}_m(y+z) dS_y = \\ &= \mathcal{N}_{B,\sigma,\mu}[\tilde{\psi}_m(\cdot+z)](x), \end{aligned}$$

where $\tilde{N}_{\sigma^z,\mu^z}(\tilde{x}, \tilde{y}) = N_{\sigma,\mu}(x, y)$ follows from the existence and uniqueness of the Neumann function result. In fact, for $\tilde{w} \in B^z$ and by a change of variables, the Neumann problem

$$\begin{cases} \nabla_{\tilde{y}} \cdot \frac{1}{\sigma^z(\tilde{y})} \nabla_{\tilde{y}} \tilde{N}_{\sigma^z,\mu^z}(\tilde{w}, \tilde{y}) + \omega^2 \mu^z(\tilde{y}) \tilde{N}_{\sigma^z,\mu^z}(\tilde{w}, \tilde{y}) = -\delta_{\tilde{w}}(\tilde{y}) & \tilde{y} \in B^z, \\ \frac{1}{\sigma^z(\tilde{y})} \frac{\partial \tilde{N}_{\sigma^z,\mu^z}}{\partial \nu_{\tilde{y}}}(\tilde{w}, \tilde{y}) = 0 & \tilde{y} \in \partial B^z, \end{cases} \quad (4.7)$$

can be rewritten as

$$\begin{cases} \nabla_y \cdot \frac{1}{\sigma(y)} \nabla_y \tilde{N}_{\sigma^z,\mu^z}(w+z, y+z) + \omega^2 \mu(y) \tilde{N}_{\sigma^z,\mu^z}(w+z, y+z) = -\delta_w(y) & y \in B, \\ \frac{1}{\sigma(y)} \frac{\partial \tilde{N}_{\sigma^z,\mu^z}}{\partial \nu_y}(w+z, y+z) = 0 & y \in \partial B. \end{cases} \quad (4.8)$$

From the uniqueness of the Neumann function, $\tilde{N}_{\sigma^z,\mu^z}(w+z, y+z) = N_{\sigma,\mu}(x, y)$ is the solution to (4.8).

Then, (4.3) yields

$$W_{n,m}[B^z, \sigma^z, \mu^z, \omega] = \sum_{a,b} \mathcal{C}_a(-z) \overline{\mathcal{C}_b(-z)} W_{n+a,m+b}[B, \sigma, \mu, \omega].$$

Since $\overline{\mathcal{C}_a(-z)} = \mathcal{C}_{-a}(z)$, (4.2) holds. □

Proposition 4.3 (Scaling formula). *For any $s > 0$, the following relation holds*

$$W_{n,m}[B^s, \sigma^s, \mu^s, \omega] = W_{n,m}[B, \sigma, \mu, s\omega]. \quad (4.9)$$

Proof. Let $\tilde{\phi}_{m,\omega} = \phi_{\mathcal{C}_{m,B^s,\omega}}$ and $\tilde{y} = sy$ with $y \in B$. Since $(|\tilde{y}|, \theta_{\tilde{y}}) = (s|y|, \theta_y)$, by the definition of the scattering coefficients:

$$\begin{aligned} W_{n,m}[B^s, \sigma^s, \mu^s, \omega] &= \int_{\partial B^s} \overline{\mathcal{C}_{n,\omega}(\tilde{y})} \tilde{\phi}_{m,\omega}(\tilde{y}) dS_{\tilde{y}} = \\ &= s \int_{\partial B} \overline{\mathcal{C}_{n,s\omega}(y)} \tilde{\phi}_{m,\omega}(sy) dS_y. \end{aligned} \quad (4.10)$$

We have

$$\mathcal{C}_{m,\omega}(\tilde{x}) = \mathcal{C}_{m,\omega}(sx) = \mathcal{C}_{m,s\omega}(x),$$

where $\tilde{x} = sx$ with $x \in B$. To find $\tilde{\phi}_{m,\omega}$, let us consider

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_{m,\omega} \\ \tilde{\psi}_{m,\omega} \end{bmatrix} (\tilde{x}) = \begin{bmatrix} -S_{B^s}^\omega [\tilde{\phi}_{m,\omega}](\tilde{x}) & \Lambda_{B^s, \sigma^s, \mu^s, \omega} [\tilde{\psi}_{m,\omega}](\tilde{x}) \\ -\left(\frac{1}{2}I + (K_{B^s}^\omega)^*\right) [\tilde{\phi}_{m,\omega}](\tilde{x}) & I[\tilde{\psi}_{m,\omega}](\tilde{x}) \end{bmatrix}.$$

Recall that $(\tilde{\phi}_{m,\omega}, \tilde{\psi}_{m,\omega}) \in L^2(\partial B^s) \times L^2(\partial B^s)$ is the solution to

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_{m,\omega} \\ \tilde{\psi}_{m,\omega} \end{bmatrix} (\tilde{x}) = \begin{bmatrix} \mathcal{C}_{m,\omega}(\tilde{x}) \\ \frac{\partial \mathcal{C}_{m,\omega}}{\partial \nu_{\tilde{x}}}(\tilde{x}) \end{bmatrix}. \quad (4.11)$$

Let $\tilde{w} = sw$ for $w \in B$. Let us prove that

$$\begin{bmatrix} \tilde{\phi}_{m,\omega}(\tilde{w}) \\ \tilde{\psi}_{m,\omega}(\tilde{w}) \end{bmatrix} = s^{-1} \begin{bmatrix} \phi_{m,s\omega}(w) \\ \psi_{m,s\omega}(w) \end{bmatrix} \quad (4.12)$$

is the solution. Since

$$\begin{bmatrix} \mathcal{C}_{m,\omega}(\tilde{x}) \\ \frac{\partial \mathcal{C}_{m,\omega}}{\partial \nu_{\tilde{x}}}(\tilde{x}) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{m,\omega}(sx) \\ \frac{\partial \mathcal{C}_{m,\omega}}{\partial \nu_{sx}}(sx) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{m,s\omega}(x) \\ s^{-1} \frac{\partial \mathcal{C}_{m,s\omega}}{\partial \nu_x}(x) \end{bmatrix},$$

if we prove that

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_{m,\omega} \\ \tilde{\psi}_{m,\omega} \end{bmatrix} (\tilde{x}) = \begin{bmatrix} -sS_B^{s\omega} [\tilde{\phi}_{m,\omega}(s \cdot)](x) & s\Lambda_{B,\sigma,\mu,s\omega} [\tilde{\psi}_{m,\omega}(s \cdot)](x) \\ -\left(\frac{1}{2}I + (K_B^{s\omega})^*\right) [\tilde{\phi}_{m,\omega}(s \cdot)](x) & I[\tilde{\psi}_{m,\omega}(s \cdot)](x) \end{bmatrix}, \quad (4.13)$$

then by the linearity of operator $\tilde{\mathcal{A}}$ and the existence and uniqueness of a solution to system (4.11), we have (4.12). Let us prove (4.13). We write

$$\begin{aligned} S_{B^s}^\omega [\tilde{\phi}_{m,\omega}](\tilde{x}) &= \int_{B^s} \Gamma_\omega(\tilde{x} - \tilde{y}) \tilde{\phi}_{m,\omega}(\tilde{y}) dS_{\tilde{y}} = \\ &= \int_B \Gamma_\omega(sx - sy) \tilde{\phi}_{m,\omega}(sy) s dS_y = \\ &= sS_B^{s\omega} [\tilde{\phi}_{m,\omega}(s \cdot)](x), \\ -\left(\frac{1}{2}I + (K_{B^s}^\omega)^*\right) [\tilde{\phi}_{m,\omega}](\tilde{x}) &= -\left(\frac{1}{2}I + (K_B^{s\omega})^*\right) [\tilde{\phi}_{m,\omega}(s \cdot)](x), \end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{B^s, \sigma^s, \mu^s, \omega}[\tilde{\psi}_{m, \omega}](\tilde{x}) &= \int_{\partial B^s} \tilde{N}_{\sigma^s, \mu^s, \omega}(\tilde{x}, \tilde{y}) \tilde{\psi}_{m, \omega}(\tilde{y}) dS_{\tilde{y}} = \\
&= s \int_{\partial B} N_{\sigma, \mu, s\omega}(x, y) \tilde{\psi}_{m, \omega}(s \cdot) dS_y = \\
&= s \mathcal{N}_{B, \sigma, \mu, s\omega}[\tilde{\psi}_{m, \omega}(s \cdot)](x),
\end{aligned}$$

where $\tilde{N}_{\sigma^s, \mu^s, \omega}(\tilde{x}, \tilde{y}) = N_{\sigma, \mu, s\omega}(x, y)$ follows from existence and uniqueness of the Neumann function. In fact, for $\tilde{w} \in B^s$ and by a change of variables, the Neumann problem

$$\begin{cases} \nabla_{\tilde{y}} \cdot \frac{1}{\sigma^s(\tilde{y})} \nabla_{\tilde{y}} \tilde{N}_{\sigma^s, \mu^s, \omega}(\tilde{w}, \tilde{y}) + \omega^2 \mu^s(\tilde{y}) \tilde{N}_{\sigma^s, \mu^s, \omega}(\tilde{w}, \tilde{y}) = -\delta_{\tilde{w}}(\tilde{y}) & \tilde{y} \in B^s, \\ \frac{1}{\sigma^s(\tilde{y})} \frac{\partial \tilde{N}_{\sigma^s, \mu^s, \omega}}{\partial \nu_{\tilde{y}}}(\tilde{w}, \tilde{y}) = 0 & \tilde{y} \in \partial B^s, \end{cases} \quad (4.14)$$

can be rewritten as

$$\begin{cases} \nabla_y \cdot \frac{1}{\sigma(y)} \nabla_y \tilde{N}_{\sigma^s, \mu^s, \omega}(sw, sy) + s^2 \omega^2 \mu(y) \tilde{N}_{\sigma^s, \mu^s, \omega}(sw, sy) = -\delta_w(y) & y \in B, \\ \frac{1}{\sigma(y)} \frac{\partial \tilde{N}_{\sigma^s, \mu^s, \omega}}{\partial \nu_y}(sw, sy) = 0 & y \in \partial B. \end{cases} \quad (4.15)$$

from the uniqueness of the Neumann function, $\tilde{N}_{\sigma^s, \mu^s, \omega}(sx, sy) = N_{\sigma, \mu, s\omega}(x, y)$ is the solution to (4.15).

Then, (4.10) yields

$$W_{n, m}[B^s, \sigma^s, \mu^s, \omega] = W_{n, m}[B, \sigma, \mu, s\omega].$$

□

Proposition 4.4 (Rotation formula). *For any θ , the following relation holds*

$$W_{n, m}[B^\theta, \sigma^\theta, \mu^\theta, \omega] = e^{i(m-n)\theta} W_{n, m}[B, \sigma, \mu, \omega]. \quad (4.16)$$

Proof. Let $\tilde{\phi}_m = \phi_{U_m, B^\theta}$ and $\tilde{y} = e^{i\theta} y$ with $y \in B$. Since $(|\tilde{y}|, \theta_{\tilde{y}}) = (|y|, \theta_y + \theta)$, by the definition of the scattering coefficients:

$$\begin{aligned}
W_{n, m}[B^\theta, \sigma, \mu, \omega] &= \int_{\partial B^\theta} \overline{\mathcal{C}_n(\tilde{y})} \tilde{\phi}_m(\tilde{y}) dS_{\tilde{y}} = \\
&= \int_{\partial B} \overline{\mathcal{C}_n(y)} e^{-in\theta} \tilde{\phi}_m(e^{i\theta} y) dS_y.
\end{aligned} \quad (4.17)$$

We have

$$\mathcal{C}_m(\tilde{x}) = \mathcal{C}_m(e^{i\theta} x) = \mathcal{C}_m(x) e^{im\theta},$$

where $\tilde{x} = e^{i\theta} x$ with $x \in B$. To find $\tilde{\phi}_m$, let us consider

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_m \\ \tilde{\psi}_m \end{bmatrix}(\tilde{x}) = \begin{bmatrix} -S_{B^\theta}^\omega[\tilde{\phi}_m](\tilde{x}) & \Lambda_{B^\theta, \sigma^\theta, \mu^\theta}[\tilde{\psi}_m](\tilde{x}) \\ -(\frac{1}{2}I + (K_{B^\theta}^\omega)^*)[\tilde{\phi}_m](\tilde{x}) & I[\tilde{\psi}_m](\tilde{x}) \end{bmatrix}.$$

Recall that $(\tilde{\phi}_m, \tilde{\psi}_m) \in L^2(\partial B^\theta) \times L^2(\partial B^\theta)$ is the solution to

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_m \\ \tilde{\psi}_m \end{bmatrix}(\tilde{x}) = \begin{bmatrix} \mathcal{C}_m(\tilde{x}) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_{\tilde{x}}}(\tilde{x}) \end{bmatrix}. \quad (4.18)$$

Let $\tilde{w} = e^{i\theta}w$ for $w \in B$. Let us prove that

$$\begin{bmatrix} \tilde{\phi}_m(\tilde{w}) \\ \tilde{\psi}_m(\tilde{w}) \end{bmatrix} = e^{im\theta} \begin{bmatrix} \phi_m(w) \\ \psi_m(w) \end{bmatrix} \quad (4.19)$$

is the solution. Since

$$\begin{bmatrix} \mathcal{C}_m(\tilde{y}) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_{\tilde{y}}}(\tilde{y}) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_m(e^{i\theta}y) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_{e^{i\theta}y}}(e^{i\theta}y) \end{bmatrix} = e^{im\theta} \begin{bmatrix} \mathcal{C}_m(y) \\ \frac{\partial \mathcal{C}_m}{\partial \nu_y}(y) \end{bmatrix},$$

if we prove that

$$\tilde{\mathcal{A}} \begin{bmatrix} \tilde{\phi}_m \\ \tilde{\psi}_m \end{bmatrix}(\tilde{x}) = \begin{bmatrix} -S_B^\omega[\tilde{\phi}_m(e^{i\theta} \cdot)](x) & \Lambda_{B,\sigma,\mu}[\tilde{\psi}_m(e^{i\theta} \cdot)](x) \\ -(\frac{1}{2}I + (K_B^\omega)^*)[\tilde{\phi}_m(e^{i\theta} \cdot)](x) & I[\tilde{\psi}_m(e^{i\theta} \cdot)](x) \end{bmatrix}, \quad (4.20)$$

then by the linearity of operator \mathcal{A} and the existence and uniqueness of a solution system (4.18), we have (4.19). Let us prove identity (4.20). We write

$$\begin{aligned} S_{B^\theta}^\omega[\tilde{\phi}_m](\tilde{x}) &= \int_{B^\theta} \Gamma_\omega(\tilde{x} - \tilde{y})\tilde{\phi}_m(\tilde{y}) dS_{\tilde{y}} = \\ &= \int_B \Gamma_\omega(x - y)\tilde{\phi}_m(e^{i\theta}y) s dS_y = \\ &= S_B^\omega[\tilde{\phi}_m(e^{i\theta} \cdot)](x), \\ -\left(\frac{1}{2}I + (K_{B^\theta}^\omega)^*\right)[\tilde{\phi}_m](\tilde{x}) &= -\left(\frac{1}{2}I + (K_B^\omega)^*\right)[\tilde{\phi}_m(e^{i\theta} \cdot)](x), \\ \mathcal{N}_{B^\theta, \tilde{\sigma}, \tilde{\mu}}[\tilde{\psi}_m](\tilde{x}) &= \int_{\partial B^\theta} \tilde{N}_{\tilde{\sigma}, \tilde{\mu}}(\tilde{x}, \tilde{y})\tilde{\psi}_m(\tilde{y}) dS_{\tilde{y}} = \\ &= \int_{\partial B} N_{\sigma,\mu}(x, y)\tilde{\psi}_m(e^{i\theta} \cdot) dS_y = \\ &= \mathcal{N}_{B,\sigma,\mu}[\tilde{\psi}_m(e^{i\theta} \cdot)](x), \end{aligned}$$

where $\tilde{N}_{\sigma^\theta, \mu^\theta}(\tilde{x}, \tilde{y}) = N_{\sigma,\mu}(x, y)$ follows from the existence and uniqueness of the Neumann function. In fact, for $\tilde{w} \in B^\theta$ and by the change of variables, the Neumann problem

$$\begin{cases} \nabla_{\tilde{y}} \cdot \frac{1}{\sigma^\theta(\tilde{y})} \nabla_{\tilde{y}} \tilde{N}_{\sigma^\theta, \mu^\theta}(\tilde{w}, \tilde{y}) + \omega^2 \mu^\theta(\tilde{y}) \tilde{N}_{\sigma^\theta, \mu^\theta}(\tilde{w}, \tilde{y}) = -\delta_{\tilde{w}}(\tilde{y}) & \tilde{y} \in B^\theta, \\ \frac{1}{\sigma^\theta(\tilde{y})} \frac{\partial \tilde{N}_{\sigma^\theta, \mu^\theta}}{\partial \nu_{\tilde{y}}}(\tilde{w}, \tilde{y}) = 0 & \tilde{y} \in \partial B^\theta, \end{cases} \quad (4.21)$$

can be rewritten as

$$\begin{cases} \nabla_y \cdot \frac{1}{\sigma(y)} \nabla_y \tilde{N}_{\sigma^\theta, \mu^\theta}(e^{i\theta}w, e^{i\theta}y) + \omega^2 \mu(y) \tilde{N}_{\sigma^\theta, \mu^\theta}(e^{i\theta}w, e^{i\theta}y) = -\delta_w(y) & y \in B, \\ \frac{1}{\sigma(y)} \frac{\partial \tilde{N}_{\sigma^\theta, \mu^\theta}}{\partial \nu_y}(e^{i\theta}w, e^{i\theta}y) = 0 & y \in \partial B. \end{cases} \quad (4.22)$$

By the uniqueness of the Neumann function, $\tilde{N}_{\sigma^\theta, \mu^\theta}(e^{i\theta}x, e^{i\theta}y) = N_{\sigma,\mu}(x, y)$ is the solution to (4.15).

Then, (4.17) yields

$$W_{n,m}[B^\theta, \sigma^\theta, \mu^\theta, \omega] = e^{i(m-n)\theta} W_{n,m}[B, \sigma, \mu, \omega].$$

□

5 Distribution descriptors and identification in a dictionary

In this section, we construct the distribution descriptors which are invariant to rigid transformations. In the following, we proceed as in [12]. We denote by B a reference shape of size 1 centered at the origin, so that the unknown target D is generated from B by a rotation with angle θ , a scaling $s > 0$, and a translation $z \in \mathbb{R}^2$ as

$$D = z + sR_\theta B.$$

5.1 Far-field pattern

To construct the distribution descriptors which are invariant to rigid transformations, we derive the far-field pattern of the scattering field in terms of the inhomogeneous scattering coefficients. The result is similar to the homogeneous case [11].

If U is given by a plane wave $e^{i\omega\xi \cdot x}$ such that $\omega\xi \cdot \omega\xi = \omega^2$, then by the Jacobi-Anger decomposition, we have

$$U(x) = \sum_{m \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} J_m(\omega|x|) e^{im\theta_x} = \sum_{m \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} \mathcal{C}_m(x).$$

By the linearity of operator \mathcal{A} and existence and uniqueness of a solution to system (4.11), we obtain

$$\phi = \sum_{m \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} \phi_m.$$

Using (2.11),

$$u - e^{i\omega\xi \cdot x} = (u - U)(x) = -\frac{i}{4} \sum_{m \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} \int_{\partial B} H_0^{(1)}(\omega|x - y|) \phi_m(y) dS_y. \quad (5.1)$$

Recall that [11]

$$H_0^{(1)}(\omega|x - y|) = \sqrt{\frac{2}{\pi\omega|x - y|}} e^{i(\omega|x - y| - \frac{\pi}{4})} + O(|x|^{-1}) \text{ as } x \rightarrow \infty.$$

Since $|x - y| = |x| - |y| \cos(\theta_x - \theta_y) + O(|x|^{-1})$, (5.1) becomes

$$u - e^{i\omega\xi \cdot x} = -ie^{-i\frac{\pi}{4}} \frac{e^{i\omega|x|}}{\sqrt{8\pi\omega|x|}} \sum_{m \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} \int_{\partial B} e^{-i\omega|y| \cos(\theta_x - \theta_y)} \phi_m(y) dS_y + O(|x|^{-1}). \quad (5.2)$$

By Jacobi-Anger identity,

$$e^{-i\omega|y| \cos(\theta_x - \theta_y)} = e^{i\omega|y| \cos(\theta_x - \theta_y + \pi)} = \sum_{n \in \mathbb{Z}} i^n J_n(\omega|y|) e^{in(\theta_x - \theta_y + \pi)},$$

(5.2) becomes

$$u - e^{i\omega\xi \cdot x} = -ie^{-i\frac{\pi}{4}} \frac{e^{i\omega|x|}}{\sqrt{8\pi\omega|x|}} \sum_{m, n \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} e^{in(\theta_x - \frac{\pi}{2})} \int_{\partial B} i^n J_n(\omega|y|) e^{-in(\theta_y - \frac{3}{2}\pi)} \phi_m(y) dS_y + O(|x|^{-1}). \quad (5.3)$$

Since $i^n e^{in\frac{3\pi}{2}} = i^n (-i)^n = 1$, we obtain

$$u - e^{i\omega\xi\cdot x} = -ie^{-i\frac{\pi}{4}} \frac{e^{i\omega|x|}}{\sqrt{8\pi\omega|x|}} \sum_{m,n\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)} e^{in(\theta_x-\frac{\pi}{2})} W_{n,m} + O(|x|^{-1}), \quad (5.4)$$

where $W_{n,m}$ are the inhomogeneous scattering coefficients.

We define the far-field pattern (the scattering amplitude) when the incident field is the plane wave $U(x) = e^{i\omega\xi\cdot x}$, $|\xi| = 1$, as the two-dimensional 2π -periodic function

$$A_B^\infty((\theta_\xi, \theta_x)^T; \sigma, \mu, \omega) = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)} \int_{\partial B} e^{-i\omega|y|\cos(\theta_x-\theta_y)} \phi_m(y) dS_y.$$

From (5.4), we get the following proposition.

Proposition 5.1. *Let $(\theta_\xi, \theta_x)^T \in [0, 2\pi]^2$. Then, we have*

$$A_B^\infty((\theta_\xi, \theta_x)^T; \sigma, \mu, \omega) = \sum_{m,n\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)} e^{in(\theta_x-\frac{\pi}{2})} W_{n,m}[B, \sigma, \mu, \omega]. \quad (5.5)$$

5.2 Translation- and rotation-invariant distribution descriptors

A simple relation exists between far-field patterns of $D = z + sR_\theta B$ and B . The following result generalizes to the inhomogeneous case the result proved in [12] in the case of homogenous scattering coefficients.

Proposition 5.2. *Let $D = z + sR_\theta B$. We denote by θ_z the angle of z in polar coordinates, and define*

$$\phi_z((\theta_\xi, \theta_x)^T) := e^{i\omega|z|\cos(\theta_\xi-\theta_z)} e^{-i\omega|z|\cos(\theta_x-\theta_z)}.$$

Then, we have

$$A_D^\infty((\theta_\xi, \theta_x)^T; \sigma, \mu, \omega) = \phi_z((\theta_\xi, \theta_x)^T) A_B^\infty((\theta_\xi - \theta, \theta_x - \theta)^T, \sigma, \mu, s\omega).$$

Proof. By transformation formulas (4.2), (4.9), and (4.16), we have

$$\begin{aligned} W_{n,m}[D, \sigma, \mu, \omega] &= \sum_{a,b\in\mathbb{Z}} \overline{\mathcal{C}_a(z)} \mathcal{C}_b(z) W_{n-a,m-b}[sR_\theta B, \sigma, \mu, \omega] = \\ &= \sum_{a,b\in\mathbb{Z}} \overline{\mathcal{C}_a(z)} \mathcal{C}_b(z) W_{n-a,m-b}[R_\theta B, \sigma, \mu, s\omega] = \\ &= \sum_{a,b\in\mathbb{Z}} \overline{\mathcal{C}_a(z)} \mathcal{C}_b(z) e^{i(m-b)\theta} e^{-i(n-a)\theta} W_{n-a,m-b}[B, \sigma, \mu, s\omega]. \end{aligned} \quad (5.6)$$

Therefore, using the Jacobi-Anger identity,

$$\begin{aligned} e^{i\omega|z|\cos(\theta_\xi-\theta_z)} &= e^{i\omega|z|\cos(\theta_z-\theta_\xi)} = \sum_{a\in\mathbb{Z}} i^a J_a(\omega|z|) e^{ia(\theta_z-\theta_\xi)} = \sum_{a\in\mathbb{Z}} J_a(\omega|z|) e^{ia(\frac{\pi}{2}-\theta_\xi-\theta_z)} = \\ &= \sum_{a\in\mathbb{Z}} \mathcal{C}_a(z) e^{ia(\frac{\pi}{2}-\theta_\xi)}, \end{aligned}$$

we obtain

$$\begin{aligned}
A_D^\infty((\theta_\xi, \theta_x)^T; \sigma, \mu, \omega) &= \sum_{m, n \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} e^{in(\theta_x - \frac{\pi}{2})} W_{n, m}[D, \sigma, \mu, \omega] = \\
&= \sum_{m, n, a, b \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \theta_\xi)} e^{in(\theta_x - \frac{\pi}{2})} \overline{\mathcal{C}_a(z)} \mathcal{C}_b(z) e^{i(m-b)\theta} e^{-i(n-a)\theta} W_{n-a, m-b}[B, \sigma, \mu, s\omega] = \\
&= \sum_{m, n, a, b \in \mathbb{Z}} \overline{\mathcal{C}_a(z)} e^{-ia(\frac{\pi}{2} - \theta_x)} e^{ia(\frac{\pi}{2} - \theta_x)} \mathcal{C}_b(z) e^{im(\frac{\pi}{2} - \theta_\xi)} e^{in(\theta_x - \frac{\pi}{2})} e^{i(m-b)\theta} e^{-i(n-a)\theta} W_{n-a, m-b}[B, \sigma, \mu, s\omega] = \\
&= \sum_{m, n, a, b \in \mathbb{Z}} \overline{\mathcal{C}_a(z)} e^{-ia(\frac{\pi}{2} - \theta_x)} \mathcal{C}_b(z) e^{im(\frac{\pi}{2} - \theta_\xi)} e^{i(n-a)(\theta_x - \frac{\pi}{2})} e^{i(m-b)\theta} e^{-i(n-a)\theta} W_{n-a, m-b}[B, \sigma, \mu, s\omega] = \\
&= \sum_{m, n, a, b \in \mathbb{Z}} \overline{\mathcal{C}_a(z)} e^{-ia(\frac{\pi}{2} - \theta_x)} \mathcal{C}_b(z) e^{-ib(\theta_\xi - \frac{\pi}{2})} e^{i(m-b)(\frac{\pi}{2} - \theta_\xi)} e^{i(n-a)(\theta_x - \frac{\pi}{2})} e^{i(m-b)\theta} e^{-i(n-a)\theta} W_{n-a, m-b}[B, \sigma, \mu, s\omega] = \\
&= \sum_{m', n', a, b \in \mathbb{Z}} \overline{\mathcal{C}_a(z)} e^{-ia(\frac{\pi}{2} - \theta_x)} \mathcal{C}_b(z) e^{-ib(\theta_\xi - \frac{\pi}{2})} e^{im'(\frac{\pi}{2} - \theta_\xi)} e^{in'(\theta_x - \frac{\pi}{2})} e^{im'\theta} e^{-in'\theta} W_{n', m'}[B, \sigma, \mu, s\omega] = \\
&= \sum_{m', n' \in \mathbb{Z}} e^{i\omega|z| \cos(\theta_\xi - \theta_x)} e^{-i\omega|z| \cos(\theta_x - \theta_x)} e^{im'(\frac{\pi}{2} - \theta_\xi)} e^{in'(\theta_x - \frac{\pi}{2})} e^{i(m'-n')\theta} W_{n', m'}[B, \sigma, \mu, s\omega] = \\
&= \sum_{m', n' \in \mathbb{Z}} \phi_z((\theta_\xi, \theta_x)^T) e^{im'(\frac{\pi}{2} - \theta_\xi)} e^{in'(\theta_x - \frac{\pi}{2})} W_{n', m'}[B^\theta, \sigma, \mu, s\omega] = \\
&= \phi_z((\theta_\xi, \theta_x)^T) A_B^\infty((\theta_\xi - \theta, \theta_x - \theta)^T, \sigma, \mu, s\omega).
\end{aligned}$$

□

In the following, we introduce the descriptor construction based on the far-field pattern. We proceed as in [12]. Given $\eta = (\theta_\xi, \theta_x)^T$, we define the frequency-dependent *distribution descriptor* of an inhomogeneous object D as follows:

$$S_D(v; \sigma, \mu, \omega) := \int_{[0, 2\pi]^2} |A_D^\infty(\eta; \sigma, \mu, \omega) A_D^\infty(\eta - v; \sigma, \mu, \omega)| d\eta. \quad (5.7)$$

The distribution descriptor S_D is invariant to any translation and rotation. More precisely, we can prove the following identity.

Proposition 5.3. *Let $D = z + sR_\theta B$. We have*

$$S_D(v; \sigma, \mu, \omega) = S_B(v; \sigma, \mu, s\omega).$$

Proof. Given $\eta = (\eta_1, \eta_2)$, $\theta = (\theta, \theta)$, by $|\phi_z(\cdot)| = 1$ and

$$\begin{aligned}
\int_{[0, 2\pi]^2} A_B^\infty(\eta - \theta; \sigma, \mu, s\omega) d\eta &= \int_{[0, 2\pi]^2} \sum_{m, n \in \mathbb{Z}} e^{im(\frac{\pi}{2} - (\eta_1 - \theta))} e^{in((\eta_2 - \theta) - \frac{\pi}{2})} W_{n, m}[B, \sigma, \mu, s\omega] d\eta = \\
&= \int_{[-\theta, 2\pi - \theta]^2} \sum_{m, n \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \eta'_1)} e^{in(\eta'_2 - \frac{\pi}{2})} W_{n, m}[B, \sigma, \mu, s\omega] d\eta' = \\
&= \int_{[0, 2\pi]^2} \sum_{m, n \in \mathbb{Z}} e^{im(\frac{\pi}{2} - \eta'_1)} e^{in(\eta'_2 - \frac{\pi}{2})} W_{n, m}[B, \sigma, \mu, s\omega] d\eta' = \\
&= \int_{[0, 2\pi]^2} A_B^\infty(\eta'; \sigma, \mu, s\omega) d\eta',
\end{aligned}$$

we have

$$\begin{aligned}
S_D(v; \sigma, \mu, \omega) &= \int_{[0, 2\pi]^2} |A_D^\infty(\eta; \sigma, \mu, \omega) A_D^\infty(\eta - v; \sigma, \mu, \omega)| d\eta = \\
&= \int_{[0, 2\pi]^2} |\phi_z(\eta) A_B^\infty(\eta - (\theta, \theta), \sigma, \mu, s\omega) \phi_z(\eta - v) A_B^\infty(\eta - v - (\theta, \theta); \sigma, \mu, s\omega)| d\eta = \\
&= \int_{[0, 2\pi]^2} |A_B^\infty(\eta - (\theta, \theta), \sigma, \mu, s\omega) A_B^\infty(\eta - (\theta, \theta) - v; \sigma, \mu, s\omega)| d\eta = \\
&= \int_{[0, 2\pi]^2} |A_B^\infty(\eta', \sigma, \mu, s\omega) A_B^\infty(\eta' - v; \sigma, \mu, s\omega)| d\eta' = \\
&= S_B(v, \sigma, \mu, s\omega).
\end{aligned}$$

□

6 Numerical experiments

In this section, we present a variety of numerical results in order to demonstrate the applicability of the theoretical framework presented in the previous sections. In particular, we investigate the identification of a target by reconstructing inhomogeneous scattering coefficients from the measurements of the multistatic response (MSR) matrix. In the multistatic configuration, directions of incidence and observations are sampled. For each incident direction, the scattered wave is measured in all the observation directions [8]. The overall procedure is similar to the one of [12] for the homogeneous case. In the following, we consider the case of piecewise constant (inhomogeneous) material parameters. For a collection of (inhomogeneous) targets, based on the code developed in [19] for homogeneous targets, we build a frequency dependent dictionary of distribution descriptors and use a target identification algorithm like the one of [12] in order to identify an inhomogeneous target from the dictionary up to some translation, rotation and scaling. Our dictionary will include three kinds of objects:

- *Homogeneous targets*, i.e. a disk, a triangle, *etc.*
- *Inhomogeneous targets with one inclusion inside*, i.e. a circular inclusion inside a circular target, *etc.*
- *Inhomogeneous targets with two (distinct) inclusions inside*, i.e. a circular inclusion and a square inside a circular target, *etc.*

Note that these inclusions have different material parameters than the ones of target and the background. In the following, we use the results of Section 4 for a suitable integral representation of the solutions for the case of an inhomogeneous target with one inclusion inside and the case of an inhomogeneous object with two inclusions inside (see Appendix A). The case of a homogeneous target is taken into account in [12]. Finally, we perform numerical experiments in order to test the performance of the inhomogeneous scattering coefficients in inhomogeneous target identification.

Given a target D , we proceed as follows:

- *Data simulation.* The MSR matrix is simulated for a frequency range $[\omega_{\min}, \omega_{\max}]$ by evaluating the integral representation (3.1), where the densities are computed by solving (3.8). We adopt a circular acquisition system (a full-view acquisition).

- *Reconstruction of scattering coefficients.* For each frequency, we reconstruct the matrix $\mathbf{W} = (W_{mn}[D^{-z_0}])_{mn}$ of scattering coefficients directly by the formula (3.32) of [12].
- *Target identification.* We calculate the distribution descriptors and use a target identification algorithm, see [12].

6.1 Dictionary

The dictionary \mathcal{D} that we consider is composed by 14 targets with different material parameters:

- 6 elements of the dictionary are homogeneous targets: a disk, an ellipse, a triangle, a square, a rectangle, and the letter A (see Figure 1). All homogeneous targets share the same permittivity $\mu = 3$ and permeability $\sigma = 3$.
- 5 elements of the dictionary are inhomogeneous targets with a single inclusion inside: a disk with a circular inclusion inside, a disk with an ellipse inside, a disk with a triangular inclusion inside, a disk with a square inside, and a disk with a rectangular inclusion inside (see Figure 1). Note that these 6 targets share the same permittivity ($\mu_e = 3$) and permeability ($\sigma_e = 3$) for the exterior domain, while all inclusions have permittivity $\mu_i = 6$ and permeability $\sigma_i = 6$.
- 3 elements of the dictionary are (inhomogeneous) disks with two distinct inclusions inside: two circular inclusions for the first disk, a circle and an ellipse for the second disk, and two distinct ellipses for the third one (see Figure 1). These 3 targets share the same permittivity ($\mu_e = 3$) and permeability ($\sigma_e = 3$) for the exterior domain, while the two distinct inclusions have permittivity $\mu_i = 6$ and permeability $\sigma_i = 6$, $i = 1, 2$.

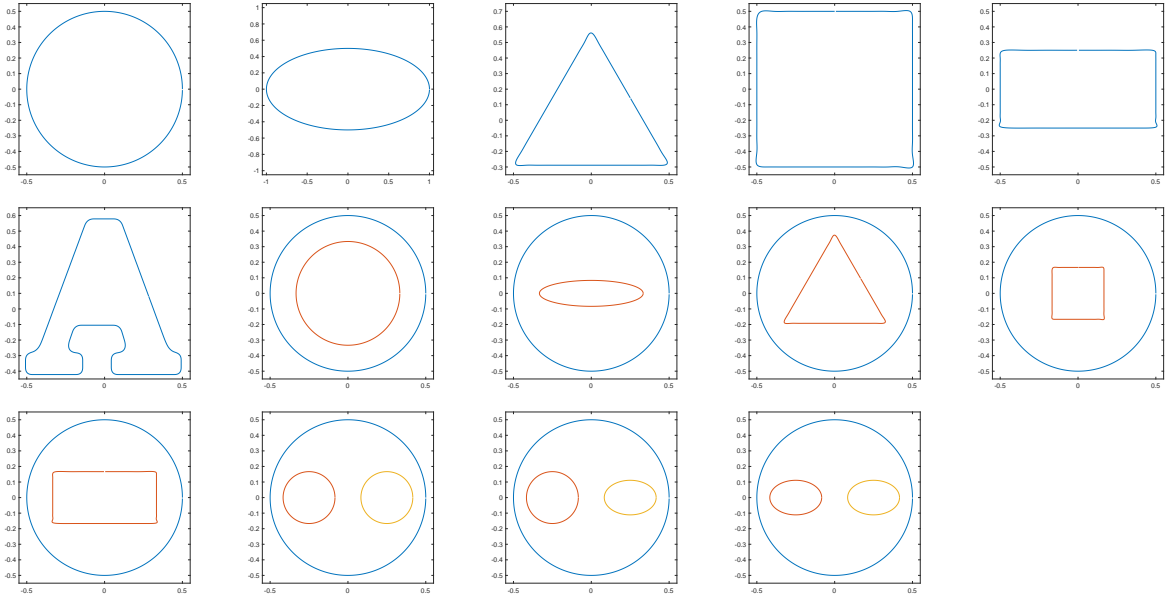


Figure 1: *Dictionary of targets*

6.2 Acquisition system

We generate a circular acquisition system using plane waves: the receivers x_r are uniformly distributed on a circle of radius R and centered at z_0 , and the sources are plane waves with equally distributed wave direction. More precisely, for the r th receiver we have $|x_r - z_0| = R$ and the angle $\theta_r := \theta_{x_r - z_0} = 2\pi r/N_r$, and the s th plane wave source is given by

$$U_s(x) = e^{ik_0 \xi_s \cdot x},$$

where the vector ξ_s is such that $|\xi_s| = 1$ and $\theta_s := 2\pi s/N_s$. We denote by N_s the total number of plane waves as sources, and by N_r the number of receivers. Note that z_0 can be obtained using some localization algorithm [8]. Here, we assume that z_0 is close to the center of the target D .

For this experiment, we adopt a circular acquisition system with $R = 3$, $N_s = 91$, and $N_r = 91$. For simplicity, we always choose the center $z_0 = [0, 0]^T$. Figure 2 illustrates this acquisition system for the three different elements of the dictionary \mathcal{D} .

6.3 Measurements

In each numerical experiment, the unknown target D is obtained from one of the elements of the dictionary by a rotation with angle $\theta = \pi/3$, a scaling $s = 1.2$ and a translation $z = [-0.5, 0.5]^T$.

Each element of the dictionary is approximated by 2^9 points. We reconstruct the matrix $\mathbf{W} = (W_{mn})_{mn}$ of scattering coefficients at order 25. Figure 3 plots the relative error of the (analytical) reconstruction $\|\mathbf{W}^{\text{est}} - \mathbf{W}\|_F / \|\mathbf{W}\|_F$ as a function of K for the three kinds of targets in the dictionary ($\|\cdot\|_F$ denotes the Frobenius norm of matrices). It can be seen that the reconstruction is robust: for example, in the case of a disk with a circular inclusion inside, with 20% of noise, the error is less than 10% for an order K up to 45.

6.4 Scale estimation

Given an unknown target $D_n = z + sR_\theta B_n$ and a dictionary of (inhomogeneous) objects $\mathcal{D} = (B_n)_n$, by measurements we reconstruct the distribution descriptor $S_{D_n}(v; \omega)$ and build a frequency dependent dictionary of distribution descriptors $(S_{B_n}(v; \omega))_n$.

Note that the distribution descriptor of the target S_{D_n} is frequency dependent. As we proved in the previous sections, since the frequency ω is coupled with the scaling factor s , which is unknown and arbitrary in $(0, \infty)$, to adapt the distribution descriptor S_{D_n} to target identification we assume that the physical operating frequency is limited, that is $0 < \omega_{\min} \leq \omega \leq \omega_{\max} < \infty$, and that $0 < s_{\min} \leq s \leq s_{\max} < \infty$, which means that the target we are interested in should not be too small or too large. Finally, s^{est} can be estimated as in [12] by solving

$$s^{\text{est}} = \arg \min_{s \in [s_{\min}, s_{\max}]} \left\{ \int_{\omega_{\min}}^{\omega_{\max}} \left(\int_{[0, 2\pi]^2} [S_{D_n}(v; \omega) - S_{B_n}(v; s\omega)] dv \right)^2 d\omega \right\}. \quad (6.1)$$

Note that a wide range of frequencies $[\omega_{\min}, \omega_{\max}]$ brings more information and therefore improves the estimation (6.1).

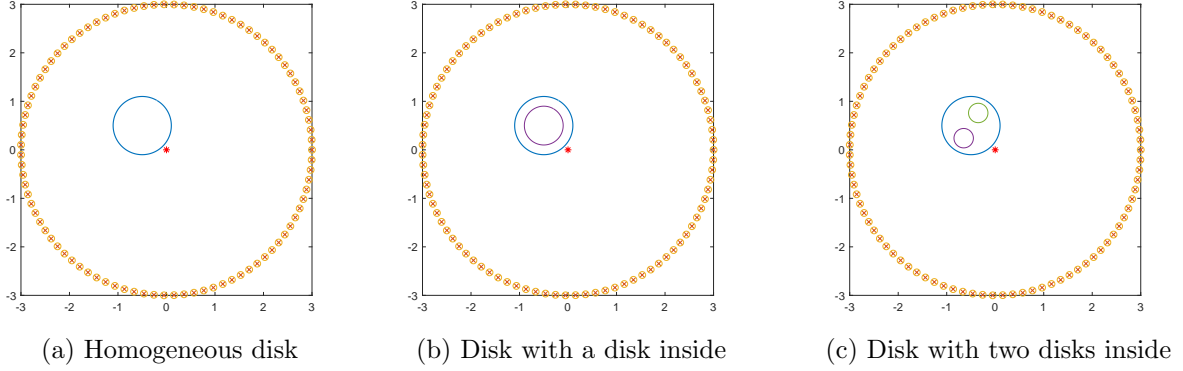


Figure 2: Circular acquisition systems for three kinds of targets. We adopt full aperture with $N_s = 91$ plane wave sources (angular position is marked by \circ) and $N_r = 91$ receivers (marked by \times). Measurement center is marked by $*$. Figure (a) shows a circular acquisition system for an homogeneous target (a disk); (b) shows a circular acquisition system for an inhomogeneous target (a disk with a circular inclusion inside); (c) shows a circular acquisition system for an inhomogeneous disk with two distinct inclusions inside.

6.5 Numerical implementation

We can solve (6.1) by sampling. The overall procedure is similar to the one of [12] for the homogeneous case. Let N_ω^{dic} , N_ω , N_v , and N_δ be positive integers. We define:

- $(\omega_l^{\text{dic}})_{l=0, \dots, N_\omega^{\text{dic}}}$ uniformly distributed points on $[\omega_{\min}^{\text{dic}}, \omega_{\max}^{\text{dic}}]$, with

$$\begin{aligned}\omega_{\min}^{\text{dic}} &:= \omega_{\min} s_{\min}, \\ \omega_{\max}^{\text{dic}} &:= \omega_{\max} s_{\max}.\end{aligned}$$

- $(\omega_k)_{k=0, \dots, N_\omega}$ uniformly distributed points on $[\omega_{\min}, \omega_{\max}]$.
- $((v_i^1, v_j^2))_{i,j=1, \dots, N_v}$ uniformly distributed points on $[0, 2\pi]^2$.
- $(s_t)_{t=0, \dots, N_\delta}$ uniformly distributed points on $[s_{\min}, s_{\max}]$.
- $I_k(s) := \{1 \leq l \leq N_\omega^{\text{dic}}, \text{ such that } \omega_{l-1}^{\text{dic}} \leq s\omega_k \leq \omega_l^{\text{dic}}\}$.

. The distribution descriptors S_{B_n} and S_{D_n} are sampled at discrete positions as follows:

$$S_{ijk}^{D_n} := S_{D_n}((v_i^1, v_j^2); \omega_k), \quad S_{ijl}^{B_n} := S_{B_n}((v_i^1, v_j^2); \omega_l).$$

Finally, we discretize the functional inside the argmin in (6.1):

$$J(t; D_n, B_n) = \sum_{k=0}^{N_\omega} \sum_{l \in I_k(s_t)} \left(\sum_{i,j=1}^{N_v} (S_{ijk}^{D_n} - S_{ijl}^{B_n}) \right)^2,$$

and the scaling factor s^{est} can be estimated by solving

$$\epsilon(D_n, B_n) = \min_{t=0, \dots, N_\delta} J(t; D_n, B_n).$$

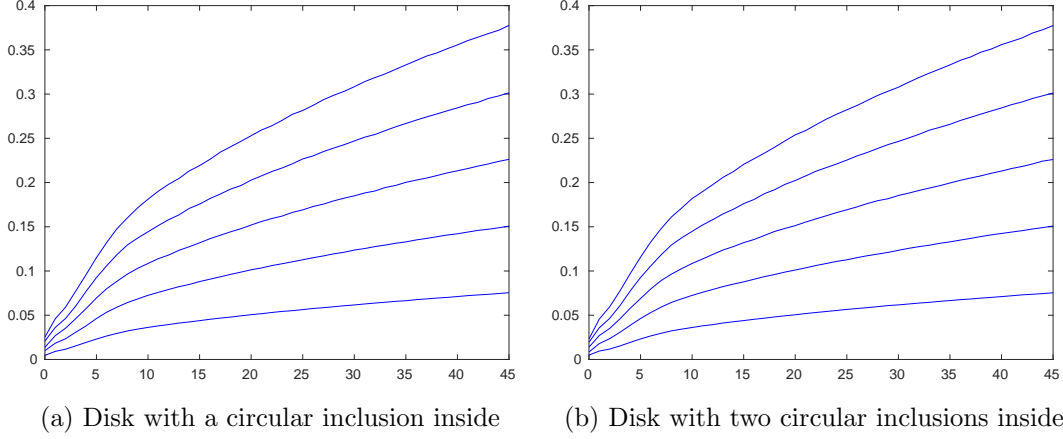


Figure 3: Relative error of the reconstruction $\|\mathbf{W}^{\text{est}} - \mathbf{W}\|_F / \|\mathbf{W}\|_F$ for the systems (b) and (c) in Figures 2 at a different order K and fixed frequency $\omega = 0.75\pi$. The curves from bottom to top correspond to percentage of noise $\sigma_0 = 20\%$, 40% , 60% , 80% , and 100% . The experiments have been repeated 100 times.

6.6 Frequency-dependent dictionary and matching algorithm

We construct the frequency-dependent dictionary of distribution descriptors as follows. For a collection of standard elements of the dictionary $(B_n)_n$, we precompute the discrete samples $(S_{ijl}^{B_n})_{ijl}$ of the distribution descriptor $S_{B_n}(v; \omega)$, for $v \in [0, 2\pi]^2$ and $\omega \in [\omega_{\min}^{\text{dic}}, \omega_{\max}^{\text{dic}}]$. The discrete samples $((S_{ijl}^{B_n})_{ijl})_n$ constitute our frequency-dependent dictionary.

Assume that our (inhomogeneous) target D is generated by an element of the dictionary $(B_n)_n$, up to some unknown translation, rotation, and scaling. Suppose that the scaling factor is such that $s_{\min} \leq s \leq s_{\max}$, where s_{\min} and s_{\max} are known. In order to detect the target D among the elements of the dictionary, we compute the discrete samples $(S_{ijk}^D)_{ijk}$ of the distribution descriptor $S_D(v; \omega)$, and calculate $\epsilon(D, B_n)$ for all elements of the above mentioned dictionary. The minimizer of $(\epsilon(D, B_n))_n$ is taken as the identified target and is expected to give the best estimation of s^{est} . This procedure is described in detail in Algorithm 1, which was first introduced by Ammari *et al.* [12].

Algorithm 1 Target identification algorithm

Input: $(S_{ijk}^D)_{ijk}$ of unknown target D ; $((S_{ijl}^{B_n})_{ijl})_n$ of the whole dictionary.
for B_n in the dictionary **do**
 $\epsilon_n \leftarrow \epsilon(D, B_n)$;
 $n \leftarrow n + 1$;
end for
Output: The true dictionary element $n^* \leftarrow \arg \min_n \epsilon_n$.

6.7 Parameter settings for identification and scaling estimation

For this experiment, the frequency-dependent dictionary of distribution descriptors $((S_{ijl}^{B_n})_{ijl})_n$ is computed for the range of frequency $[\omega_{\min}^{\text{dic}}, \omega_{\max}^{\text{dic}}] = [0.25\pi, 1.5\pi]$, with $N_{\omega}^{\text{dic}} = 78$ and $N_v = 512$. Data simulation is conducted for the range of operating frequency $[\omega_{\min}, \omega_{\max}] = [0.5\pi, \pi]$ with $N_{\omega} = 52$. The range of valid scaling factor is $[s_{\min}, s_{\max}] = [0.5, 1.5]$, with $N_{\delta} = 250$.

6.8 Results of target identification

Now, we present results of target identification obtained using the full-view setting of Figure 2:

- It can be seen that the identification succeeded for all targets with noise σ_0 up to 50%. In the case of $\sigma_0 = 0\%$ (see Appendix B), the error bars of each identified target have very different numerical value compared to those of the other elements of the dictionary. This means that recognition works well and a dictionary of large size can be used in practice.
- Figures 4 and 5 show the error bars for the dictionary of Figure 1 for all inhomogeneous targets with noise $\sigma_0 = 40\%$, 80%. The m th error bar in the n th group describes the error $\epsilon(D, B_m)$ of the matching experiment using the generating element of the dictionary B_n . The shortest bar in each group is the target identified by the matching procedure and is marked in green; the true target is marked in red where the identification fails. For $\sigma_0 = 40\%$, identification succeeded and s^{est} is also close to the true value $s = 1.2$, see Figure 6. For $\sigma_0 = 80\%$, identification failed for two inhomogeneous targets.

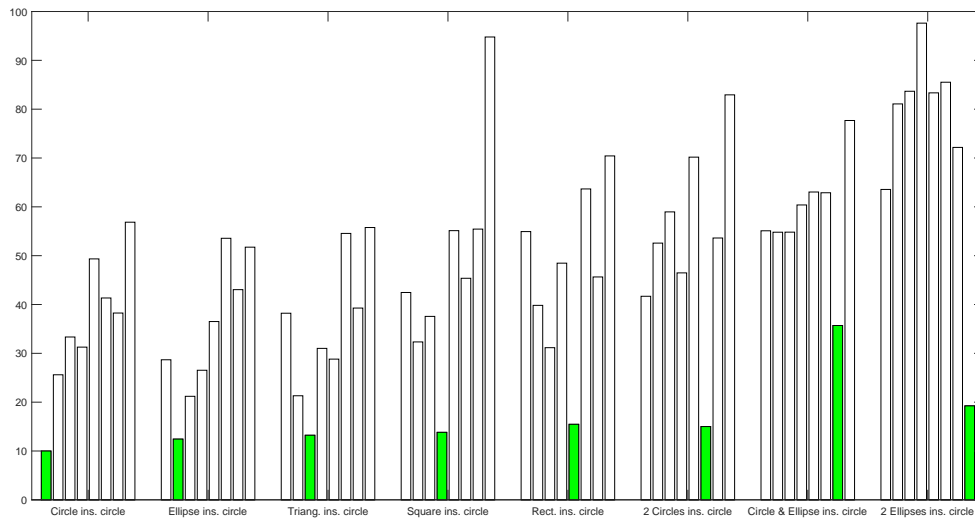


Figure 4: Results of identification for all inhomogeneous objects in the full-view setting and $\sigma_0 = 40\%$. Measurements have been repeated 1000 times.

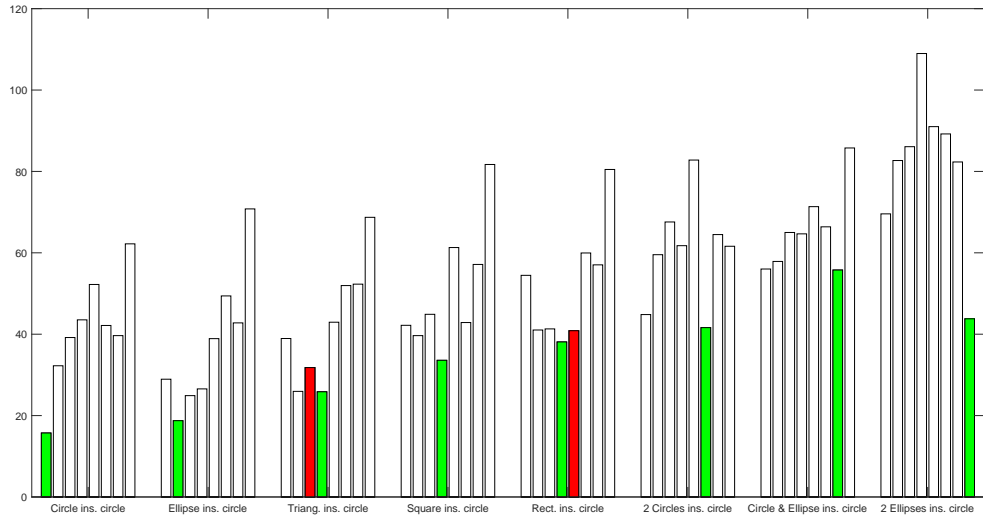


Figure 5: Results of identification for all inhomogeneous objects in the full-view setting and $\sigma_0 = 80\%$. Identification failed for two targets. Measurements have been repeated 1000 times.

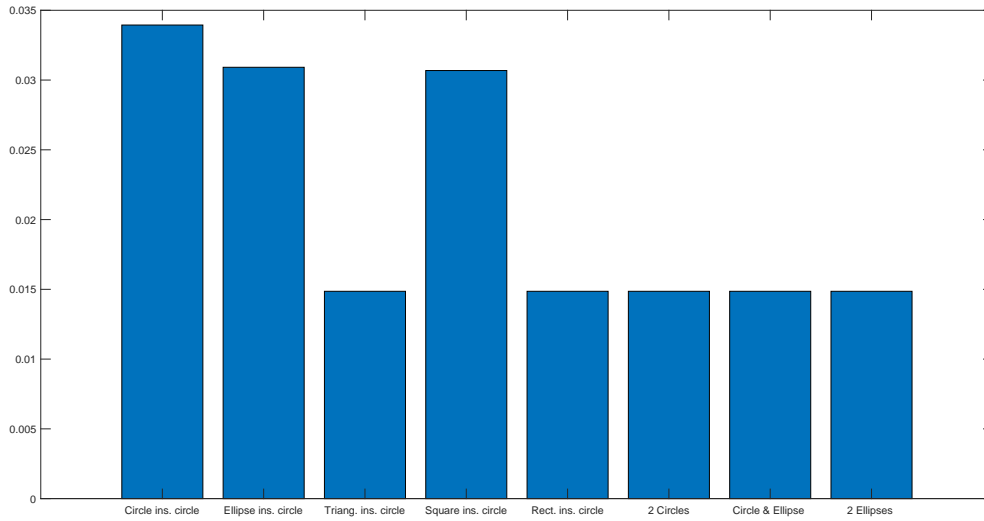


Figure 6: Difference between the estimated scaling factor and the true one ($s = 1.2$) at $\sigma_0 = 40\%$. Measurements have been repeated 1000 times.

- Figure 7 shows the probability of recognition for the inhomogeneous targets of the dictionary at different noise levels. Measurements have been repeated 1000 times.

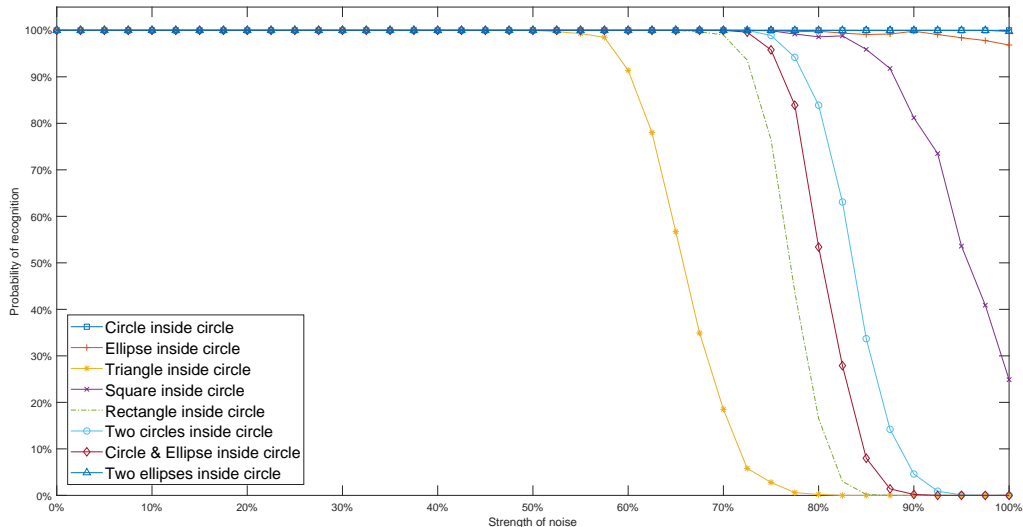


Figure 7: Probability of recognition for all inhomogeneous targets of the dictionary of Figure 1.

7 Concluding remarks

In this paper, we have presented a framework of target identification for inhomogeneous objects. We have provided and numerically tested in the presence of measurement noise a procedure for target classification in wave imaging based on matching on a dictionary of precomputed frequency-dependent distribution descriptors. The construction of such frequency-dependent distribution descriptors is based on the properties of the inhomogeneous scattering coefficients. For a collection of inhomogeneous targets, we first extracted the scattering coefficients from the reflected waves and then used a target identification algorithm in order to identify an inhomogeneous target from the dictionary up to some translation, rotation and scaling. It can be seen that the identification succeeded for all targets with noise σ_0 up to 50%.

A Piecewise constant distributions

In the appendix, we provide an integral representation of the solution to (2.1) for the special case of a domain B with piecewise constant electric permittivity μ and magnetic permeability σ . This can be seen as a particular case of (3.1).

A.1 The case of an inhomogeneous object with one inclusion inside

We consider the case of a domain B with one inclusion B_i inside. B is immersed in an homogeneous medium. B_i has different constant permeability and permittivity than the one of B and the background.

Let us consider the following Helmholtz problem

$$\begin{cases} \nabla \cdot \frac{1}{\sigma} \nabla u + \omega^2 \mu u = 0 & \text{in } \mathbb{R}^2, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty, \end{cases} \quad (\text{A.1})$$

where

$$\begin{cases} \frac{1}{\sigma}(x) = \frac{1}{\sigma_i} \chi_{B_i}(x) + \frac{1}{\sigma_e} \chi_{B_e \setminus \overline{B_i}}(x) + \frac{1}{\sigma_0} \chi_{\mathbb{R}^2 \setminus \overline{B_e}}(x), \\ \mu(x) = \mu_i \chi_{B_i}(x) + \mu_e \chi_{B_e \setminus \overline{B_i}}(x) + \mu_0 \chi_{\mathbb{R}^2 \setminus \overline{B_e}}(x), \end{cases}$$

with $B_i \subset B_e = B$. Let us define $k_0 = \omega$, $k_e = \omega \sqrt{\sigma_e \mu_e}$ and $k_i = \omega \sqrt{\sigma_i \mu_i}$. Solution to (A.1) should satisfy

$$\begin{cases} \Delta u + k_0^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B_e}, \\ \Delta u + k_e^2 u = 0 & \text{in } B_e \setminus \overline{B_i}, \\ \Delta u + k_i^2 u = 0 & \text{in } B_i, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty, \end{cases} \quad (\text{A.2})$$

with the following transmission conditions

$$\begin{cases} u|_+ = u|_- & \text{on } \partial B_e, \\ u|_+ = u|_- & \text{on } \partial B_i, \\ \frac{1}{\sigma_0} \frac{\partial u}{\partial \nu} \Big|_+ = \frac{1}{\sigma_e} \frac{\partial u}{\partial \nu} \Big|_- & \text{on } \partial B_e, \\ \frac{1}{\sigma_e} \frac{\partial u}{\partial \nu} \Big|_+ = \frac{1}{\sigma_i} \frac{\partial u}{\partial \nu} \Big|_- & \text{on } \partial B_i. \end{cases} \quad (\text{A.3})$$

Given the cylindrical wave \mathcal{C}_n of index $n \in \mathbb{Z}$ and of wave number k_0 , we look for a solution to (A.1) of the form

$$u_n(x) = \begin{cases} \mathcal{C}_n(x) + S_{B_e}^{k_0}[\phi](x) & \text{in } \mathbb{R}^2 \setminus \overline{B_e}, \\ S_{B_e}^{k_e}[\gamma](x) + S_{B_i}^{k_e}[\eta](x) & \text{in } B_e \setminus \overline{B_i}, \\ S_{B_i}^{k_i}[\psi](x) & \text{in } B_i, \end{cases} \quad (\text{A.4})$$

where the densities ψ_n, γ_n, η_n and ϕ_n are the solutions to

$$\begin{cases} S_{B_i}^{k_i}[\psi_n](x) = S_{B_e}^{k_e}[\gamma_n](x) + S_{B_i}^{k_e}[\eta_n](x) & \text{on } \partial B_i, \\ S_{B_e}^{k_e}[\gamma_n](x) + S_{B_i}^{k_e}[\eta_n](x) = \mathcal{C}_n(x) + S_{B_e}^{k_0}[\phi_n](x) & \text{on } \partial B_e, \\ \frac{1}{\sigma_i} \frac{\partial S_{B_i}^{k_i}[\psi_n]}{\partial \nu} \Big|_- = \frac{1}{\sigma_e} \frac{\partial S_{B_e}^{k_e}[\gamma_n]}{\partial \nu} \Big|_+ + \frac{1}{\sigma_e} \frac{\partial S_{B_i}^{k_e}[\eta_n]}{\partial \nu} \Big|_+ & \text{on } \partial B_i, \\ \frac{1}{\sigma_e} \frac{\partial S_{B_e}^{k_e}[\gamma_n]}{\partial \nu} \Big|_- + \frac{1}{\sigma_e} \frac{\partial S_{B_i}^{k_e}[\eta_n]}{\partial \nu} \Big|_- = \frac{1}{\sigma_0} \frac{\partial \mathcal{C}_n}{\partial \nu} \Big|_+ + \frac{1}{\sigma_0} \frac{\partial S_{B_e}^{k_0}[\phi_n]}{\partial \nu} \Big|_+ & \text{on } \partial B_e. \end{cases} \quad (\text{A.5})$$

As we have proved in (3.9), the scattering coefficient of order n, m associated to the target B with (inhomogeneous) piecewise constant permittivity and permeability is

$$W_{n,m}[B, \sigma, \mu, \omega] = \int_{\partial B_e} \overline{C_n(y)} \phi_m(y) dS_y.$$

A.2 The case of an inhomogeneous object with two (distinct) inclusions inside

Now we take into account the case of a domain B with two inclusions B_1 and B_2 inside. B is immersed in a homogeneous medium. B_1 and B_2 have different constant permeability and permittivity than the one of B and the background.

Let us consider the following Helmholtz problem

$$\begin{cases} \nabla \cdot \frac{1}{\sigma} \nabla u + \omega^2 \mu u = 0 & \text{in } \mathbb{R}^2, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty, \end{cases} \quad (\text{A.6})$$

where

$$\begin{aligned} \frac{1}{\sigma}(x) &= \frac{1}{\sigma_1} \chi_{B_1}(x) + \frac{1}{\sigma_2} \chi_{B_2}(x) + \frac{1}{\sigma_e} \chi_{B_e \setminus \overline{(B_1 \cup B_2)}}(x) + \frac{1}{\sigma_0} \chi_{\mathbb{R}^2 \setminus \overline{B_e}}(x), \\ \mu(x) &= \mu_1 \chi_{B_1}(x) + \mu_2 \chi_{B_2}(x) + \mu_e \chi_{B_e \setminus \overline{(B_1 \cup B_2)}}(x) + \mu_0 \chi_{\mathbb{R}^2 \setminus \overline{B_e}}(x), \end{aligned}$$

with $B_1, B_2 \subset B_e = B$. Let us define $k_0 = \omega$, $k_e = \omega \sqrt{\sigma_e \mu_e}$ and $k_i = \omega \sqrt{\sigma_i \mu_i}$ for $i = 1, 2$. The solution to (A.6) should satisfy

$$\begin{cases} \Delta u + k_0^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B_e}, \\ \Delta u + k_e^2 u = 0 & \text{in } B_e \setminus \overline{B_1 \cup B_2}, \\ \Delta u + k_2^2 u = 0 & \text{in } B_2, \\ \Delta u + k_1^2 u = 0 & \text{in } B_1, \\ \left| \frac{\partial(u-U)}{\partial|x|} - i\omega(u-U) \right| \leq \frac{K}{|x|^{\frac{3}{2}}} & \text{if } |x| \rightarrow \infty, \end{cases} \quad (\text{A.7})$$

with the following transmission conditions

$$\begin{cases} u|_+ = u|_- & \text{on } \partial B_e, \\ u|_+ = u|_- & \text{on } \partial B_2, \\ u|_+ = u|_- & \text{on } \partial B_1, \\ \frac{1}{\sigma_0} \frac{\partial u}{\partial \nu} \Big|_+ = \frac{1}{\sigma_e} \frac{\partial u}{\partial \nu} \Big|_- & \text{on } \partial B_e, \\ \frac{1}{\sigma_e} \frac{\partial u}{\partial \nu} \Big|_+ = \frac{1}{\sigma_2} \frac{\partial u}{\partial \nu} \Big|_- & \text{on } \partial B_2, \\ \frac{1}{\sigma_e} \frac{\partial u}{\partial \nu} \Big|_+ = \frac{1}{\sigma_1} \frac{\partial u}{\partial \nu} \Big|_- & \text{on } \partial B_1. \end{cases} \quad (\text{A.8})$$

As in the previous case, we look for a solution to (A.6) of the form

$$u_n(x) = \begin{cases} C_n(x) + S_{B_e}^{k_0}[\phi](x) & \text{in } \mathbb{R}^2 \setminus \overline{B_e}, \\ S_{B_e}^{k_e}[\gamma](x) + S_{B_2}^{k_e}[\eta](x) + S_{B_1}^{k_e}[\zeta](x) & \text{in } B_e \setminus \overline{B_1 \cup B_2}, \\ S_{B_2}^{k_2}[\psi](x) & \text{in } B_2, \\ S_{B_1}^{k_1}[\xi](x) & \text{in } B_1, \end{cases} \quad (\text{A.9})$$

where the densities $\phi_n, \gamma_n, \eta_n, \zeta_n, \psi_n$ and ξ_n are the solutions to

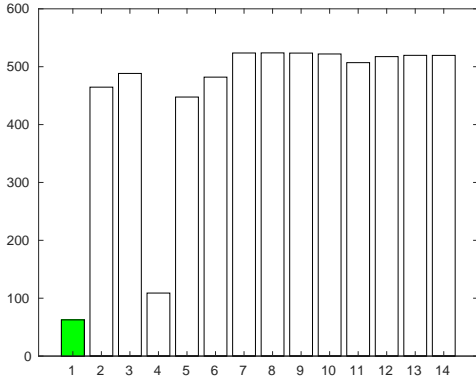
$$\begin{cases} S_{B_1}^{k_1}[\xi_n](x) = S_{B_e}^{k_e}[\gamma_n](x) + S_{B_2}^{k_e}[\eta_n](x) + S_{B_1}^{k_e}[\zeta_n](x) & \text{on } \partial B_1, \\ S_{B_2}^{k_2}[\psi_n](x) = S_{B_e}^{k_e}[\gamma_n](x) + S_{B_2}^{k_e}[\eta_n](x) + S_{B_1}^{k_e}[\zeta_n](x) & \text{on } \partial B_2, \\ S_{B_e}^{k_e}[\gamma_n](x) + S_{B_2}^{k_e}[\eta_n](x) + S_{B_1}^{k_e}[\zeta_n](x) = \mathcal{C}_n(x) + S_{B_e}^{k_0}[\phi_n](x) & \text{on } \partial B_e, \\ \left. \frac{1}{\sigma_1} \frac{\partial S_{B_1}^{k_1}[\psi_n]}{\partial \nu} \right|_- = \left. \frac{1}{\sigma_e} \frac{\partial S_{B_e}^{k_e}[\gamma_n]}{\partial \nu} \right|_- + \left. \frac{1}{\sigma_e} \frac{\partial S_{B_2}^{k_e}[\eta_n]}{\partial \nu} \right|_- + \left. \frac{1}{\sigma_e} \frac{\partial S_{B_1}^{k_e}[\zeta_n]}{\partial \nu} \right|_- & \text{on } \partial B_1, \\ \left. \frac{1}{\sigma_2} \frac{\partial S_{B_2}^{k_2}[\xi_n]}{\partial \nu} \right|_- = \left. \frac{1}{\sigma_e} \frac{\partial S_{B_e}^{k_e}[\gamma_n]}{\partial \nu} \right|_- + \left. \frac{1}{\sigma_e} \frac{\partial S_{B_2}^{k_e}[\eta_n]}{\partial \nu} \right|_- + \left. \frac{1}{\sigma_e} \frac{\partial S_{B_1}^{k_e}[\zeta_n]}{\partial \nu} \right|_- & \text{on } \partial B_2, \\ \left. \frac{1}{\sigma_e} \frac{\partial S_{B_e}^{k_e}[\gamma_n]}{\partial \nu} \right|_- + \left. \frac{1}{\sigma_e} \frac{\partial S_{B_2}^{k_e}[\eta_n]}{\partial \nu} \right|_- + \left. \frac{1}{\sigma_e} \frac{\partial S_{B_1}^{k_e}[\zeta_n]}{\partial \nu} \right|_- = \left. \frac{1}{\sigma_0} \frac{\partial \mathcal{C}_n}{\partial \nu} \right|_+ + \left. \frac{1}{\sigma_0} \frac{\partial S_{B_e}^{k_0}[\phi_n]}{\partial \nu} \right|_+ & \text{on } \partial B_e. \end{cases} \quad (\text{A.10})$$

Again, the scattering coefficient of order n, m associated to the target B with (inhomogeneous) piecewise constant permittivity and permeability is given by

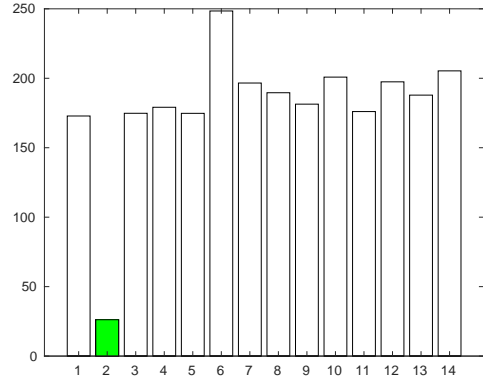
$$W_{n,m}[b, \sigma, \mu, \omega] = \int_{\partial B_e} \overline{C_n(y)} \phi_m(y) dS_y.$$

B Target identification with $\sigma_0 = 0\%$

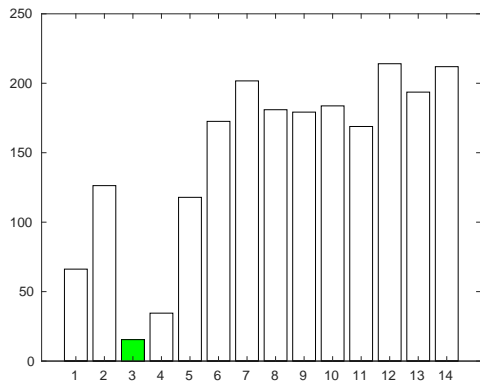
We present results of target identification obtained using the full-view setting of Figure 2 with no noise ($\sigma_0 = 0\%$). The computation of the error $\epsilon(D, B_n)$ is represented by error bars in Figure 8, where the m th error bar in the n th figure corresponds to the error $\epsilon(D, B_m)$ of the matching experiment using the generating element of the dictionary B_n . The shortest bar in each group is the identified target and is marked in green, while the true target is marked in red where the identification fails.



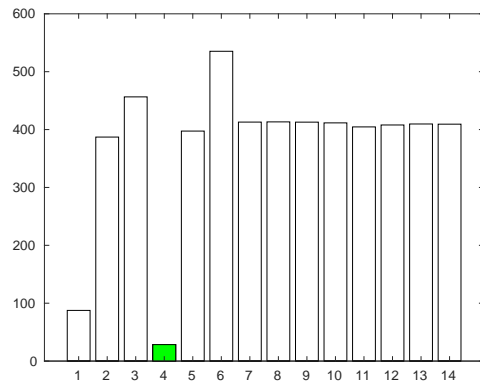
(1) Disk



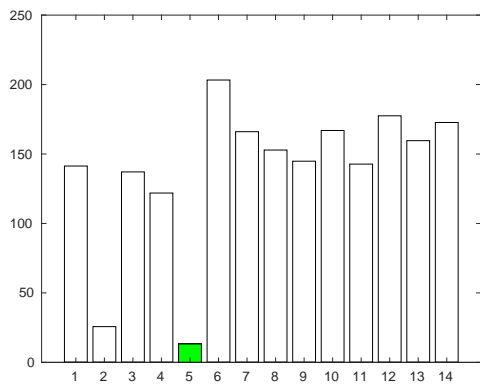
(2) Ellipse



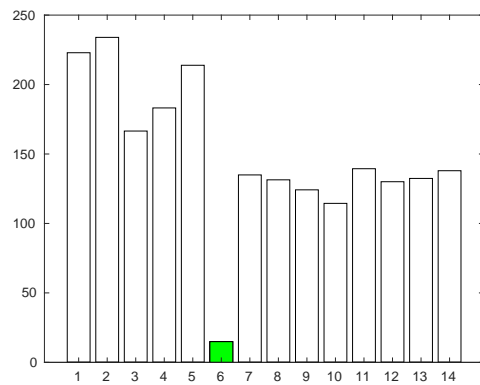
(3) Triangle



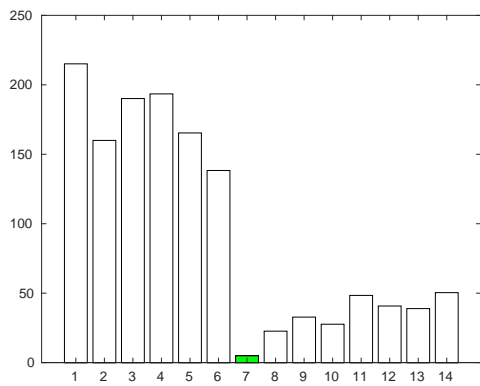
(4) Square



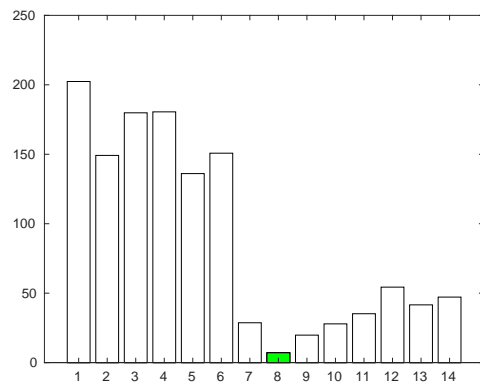
(5) Rectangle



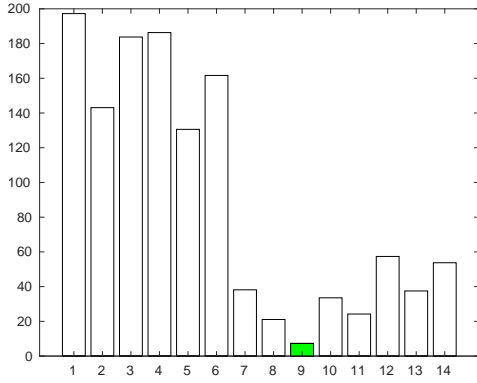
(6) Letter A



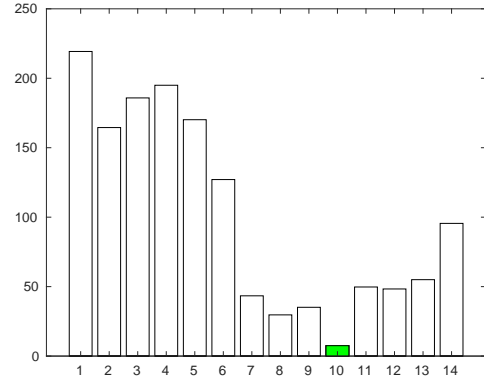
(7) Disk with a circular inclusion



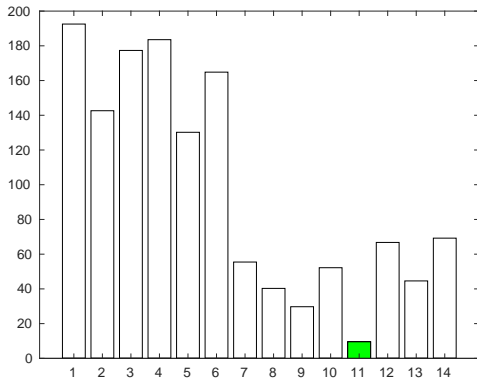
(8) Ellipse inside a disk



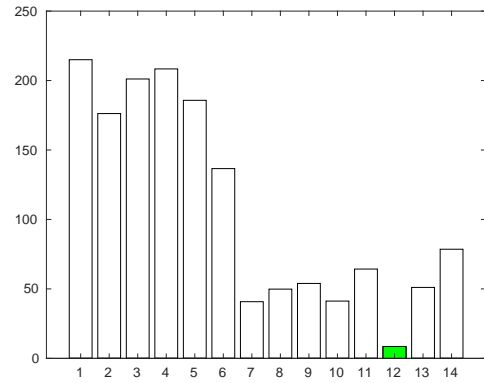
(9) Triangle inside a disk



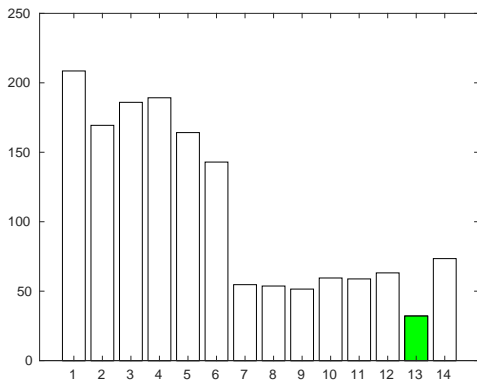
(10) Square inside a disk



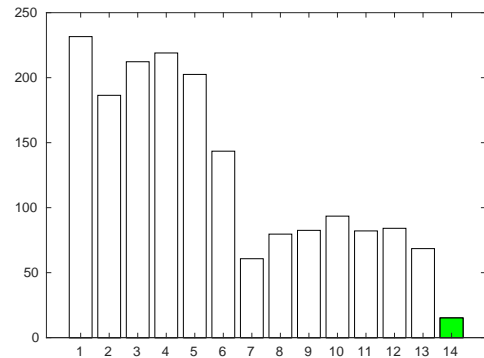
(11) Rectangle inside a disk



(12) Disk with two circular inclusions



(13) Disk with a disk and an ellipse inside



(14) Disk with two ellipses inside

Figure 8: Results of identification for all elements of the dictionary in the full-view and no noise.

Acknowledgments

The author gratefully acknowledges Prof. H. Ammari for his guidance. During the preparation of this work, the author was financially supported by a Swiss National Science Foundation grant (number 200021-172483).

References

- [1] M. Abramowitz and I. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series 55, Dover, New York, 1964.
- [2] H. Ammari, T. Boulier, J. Garnier, *Shape recognition and classification in electro-sensing*, Proceedings of the National Academy of Sciences USA, 111 (2014), 11652–11657.
- [3] H. Ammari, T. Boulier, J. Garnier, W. Jing, H. Kang, and H. Wang, *Target identification using dictionary matching of generalized polarization tensors*, Found. Comput. Math., 14 (2014), 27–62.
- [4] H. Ammari, Y. T. Chow, and J. Zou, *The concept of heterogeneous scattering coefficients and its application in inverse medium scattering*, SIAM J. Math. Anal., 46 (2014), 2905–2935.
- [5] H. Ammari, Y. T. Chow, and J. Zou, *Super-resolution in imaging high contrast targets from the perspective of scattering coefficients*, J. Math. Pures Appl., 111 (2018), 191–226.
- [6] H. Ammari, Y. Deng, H. Kang, and H. Lee, *Reconstruction of inhomogeneous conductivities via generalized polarization tensors*, Ann. IHP Anal. Non Lin., 31 (2014), 877–897.
- [7] H. Ammari and H. Kang, *Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities*, J. Math. Anal. Appl. 296 (2004), pp. 190-208.
- [8] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Sølna, and G. Wang, *Mathematical and Statistical Methods for Multistatic Imaging*, Springer, Cham, Switzerland, 2013.
- [9] H. Ammari, J. Garnier, and P. Millien, *Backpropagation imaging in nonlinear harmonic holography in the presence of measurement and medium noises*, SIAM J. Imaging Sci., 7 (2014), 239–276.
- [10] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Mathematics, Vol. 1846, Springer-Verlag, Berlin, 2004.
- [11] H. Ammari, H. Kang, H. Lee, and M. Lim, *Enhancement of Near-Cloaking. Part II: The Helmholtz Equation*, Comm. Math. Phys., 317 (2013), 485–502.
- [12] H. Ammari, M.P. Tran, and H. Wang, *Shape identification and classification in echolocation*, SIAM J. Imaging Sci., 7 (2014), 1883–1905.
- [13] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Math. Sciences 93, Springer- Verlag, New York, 1992.
- [14] L. Kleeman and R. Kuc, *Mobile robot sonar for target localization and classification*, Internat. J. Robotics Res., 14 (1995), pp. 295-318.

- [15] J.C. Nédélec, *Quelques propriétés des dérivées logarithmiques des fonctions de Hankel*, C. R. Acad. Sci. Paris, Série I, 314 (1992), 507-510.
- [16] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Ann. Math. Stud., vol. 27, Princeton University Press, Princeton, NJ, 1951.
- [17] J.A. Simmons, *Perception of echo phase information in bat sonar*, Science, 204 (1979), pp. 1336-1338.
- [18] J.A. Simmons, M.B. Fenton, and M.J. O'Farrell, *Echolocation and pursuit of prey by bats*, Science, 203 (1979), pp. 16-21.
- [19] H. Wang, Shape identification in electro-sensing, <https://github.com/yanncalec/SIES>.