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THE POINT-INTERACTION APPROXIMATION FOR THE FIELDS GENERATED BY CONTRASTED BUBBLES AT ARBITRARY FIXED FREQUENCIES

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Abstract. We deal with the linearized model of the acoustic wave propagation generated by small bubbles in the harmonic regime. We estimate the waves generated by a cluster of M small bubbles, distributed in a bounded domain Ω, with relative densities having contrasts of the order $a^\beta, \beta > 0$, where a models their relative maximum radius, $a \ll 1$. We provide useful and natural conditions on the number M, the minimum distance and the contrasts parameter β of the small bubbles under which the point interaction approximation (called also the Foldy-Lax approximation) is valid.

With the regimes allowed by our conditions, we can deal with a general class of such materials. Applications of these expansions in material sciences and imaging are immediate. For instance, they are enough to derive and justify the effective media of the cluster of the bubbles for a class of gases with densities having contrasts of the order a^β, $\beta \in (\frac{3}{2}, 2)$ and in this case we can handle any fixed frequency. In the particular and important case $\beta = 2$, we can handle any fixed frequency far or close (but distinct) from the corresponding Minnaert resonance.

1. Introduction

Diffusion by highly contrasted small particles is of fundamental importance in several branches of applied sciences as material sciences and imaging. We are interested in the models where these small particles have sizes at the micro scales as in the models related to gas bubbles. To describe properly the mathematical model we are dealing with in this work, let us denote by $\{D_s\}_{s=1}^M$ a finite collection of small particles in $\mathbb{R}^3$ of the form $D_s := \delta B_s + z_s$, where B_s are open, bounded (with Lipschitz boundaries), simply connected sets in $\mathbb{R}^3$ containing the origin, and z_s specify the locations of the particle. The parameter $\delta > 0$ characterizes the smallness assumption on the particles. We shall further assume that the Lipschitz constants of the open sets B_s are uniformly bounded. Let us consider piecewise constant densities of the form

$$\rho_\delta(x) = \begin{cases} \rho_0, \ x \in \mathbb{R}^3 \diagdown \overline{\cup_{l=1}^M D_l}, \\ \rho_s, \ x \in D_s, \ s = 1, ..., M, \end{cases} \tag{1.1}$$

and piecewise constant bulk modulus in the analogous form

$$k_\delta(x) = \begin{cases} k_0, \ x \in \mathbb{R}^3 \diagdown \overline{\cup_{l=1}^M D_l}, \\ k_s, \ x \in D_s, \ s = 1, ..., M, \end{cases} \tag{1.2}$$

where ρ_0, ρ_s, k_0, k_s are positive constants. Thus ρ_0 and k_0 denote the density and bulk modulus of the background medium and ρ_s and k_s denote the density and bulk modulus of the bubbles respectively.

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We are interested in the following problem describing the acoustic scattering by the collection of small bubbles $D_s, s = 1, ..., M$:

$$\begin{cases} \nabla.(\frac{1}{\rho_0}\nabla u) + \omega^2 \frac{1}{k_0} u = 0 \text{ in } \mathbb{R}^3 \diagdown \overline{\cup_{l=1}^M D_l}, \\ \nabla.(\frac{1}{\rho_s}\nabla u) + \omega^2 \frac{1}{k_s} u = 0 \text{ in } D_s,\ s = 1, ..., M, \\ u|_{-} - u|_{+} = 0, \text{ on } \partial D_s,\ s = 1, ..., M, \\ \frac{1}{\rho_s} \frac{\partial u}{\partial \nu^s}\Big|_{-} - \frac{1}{\rho_0} \frac{\partial u}{\partial \nu^s}\Big|_{+} = 0 \text{ on } \partial D_s,\ s = 1, ..., M, \end{cases} \tag{1.3}$$

where $\omega > 0$ is a given frequency. Here the total field $u := u^I + u^s$, where u^I denotes the incident field (we restrict to plane incident waves) and u^s denotes the scattered waves. The above set of equations have to be supplemented with the *Sommerfeld radiation condition* on u^s which we shall henceforth refer to as $(S.R.C)$.
Keeping in mind the positivity of the bulk modulus, the above problem can be equivalently formulated as

$$\begin{cases} \Delta u + \kappa_0^2 u = 0 \text{ in } \mathbb{R}^3 \diagdown \overline{\cup_{l=1}^M D_l}, \\ \Delta u + \kappa_s^2 u = 0 \text{ in } D_s,\ s = 1, ..., M, \\ u|_{-} - u|_{+} = 0, \text{ on } \partial D_s,\ s = 1, ..., M, \\ \frac{1}{\rho_s} \frac{\partial u}{\partial \nu^s}\Big|_{-} - \frac{1}{\rho_0} \frac{\partial u}{\partial \nu^s}\Big|_{+} = 0 \text{ on } \partial D_s,\ s = 1, ..., M, \\ \frac{\partial u^s}{\partial |x|} - i\kappa_0 u^s = o(\frac{1}{|x|}),\ |x| \to \infty\ (S.R.C), \end{cases} \tag{1.4}$$

where $\kappa_0^2 = \omega^2 \frac{\rho_0}{k_0}$ and $\kappa_s^2 = \omega^2 \frac{\rho_s}{k_s}$. As in the previous case, the total field $u := u^I + u^s$, with u^I denoting the acoustic incident field and u^s denoting the acoustic scattered field.

Definition 1.1. To describe the collection of small bubbles, we use the following parameters:

(1) $a := \max\limits_{1\leq m\leq M} diam(D_m) \ \big[= \delta \max\limits_{1\leq m\leq M} diam(B_m)\big]$,
(2) $d := \min\limits_{\substack{m\neq j \\ 1\leq m,j\leq M}} d_{mj}$, where $d_{mj} := dist(D_m, D_j)$,
(3) $\omega_{\max}$ as the upper bound of the used wave numbers, i.e. $\omega \in (0, \omega_{\max}]$,
(4) there exist constants $\zeta_m \in (0, 1]$ such that

$$B^3_{\zeta_m \frac{a}{2}}(z_m) \subset D_m \subset B^3_{\frac{a}{2}}(z_m),$$

where ζ_m are assumed to be uniformly bounded below by a positive constant.

The distribution of the small bubbles is modeled as follows:

(5) the number $M := M(a) := O(a^{-s}) \leq M_{max} a^{-s}$ with a given positive constant M_{max},
(6) the minimum distance $d := d(a) \approx a^t$, i.e. $d_{min} a^t \leq d(a) \leq d_{max} a^t$, with given positive constants d_{min} and d_{max},
(7) the coefficients k_m, ρ_m satisfy the conditions:

$$\frac{\rho_m}{\rho_0} = C_\rho a^\beta,\ \beta > 0, (\text{ i.e. } \frac{\rho_m}{\rho_0} \ll 1), \tag{1.5}$$

keeping the relative speed of propagation uniformly bounded, i.e.

$$\frac{\kappa_m^2}{\kappa_0^2} := \frac{\rho_m k_0}{k_m \rho_0} = \frac{\rho_m}{\rho_0} \frac{k_0}{k_m} = O(1),\ \text{as } a \ll 1. \tag{1.6}$$

Here the real numbers s, t and β are assumed to be non negative.

We call the upper bounds of the Lipschitz character of B_m's, $M_{max}, d_{min}, d_{max}$ and ω_{max} the set of the a priori bounds.

The scattering problem described above models the acoustic wave diffracted in the presence of small bubbles. In this case, the parameter β fixes the kind of medium we are considering, see [7,11,12,21]. To state our results, let us first denote $\hat{A}_l := \frac{1}{|\partial D_l|}\int_{\partial D_l}\int_{\partial D_l}\frac{(s-s')}{|s-s'|}\cdot\nu_{s'}\,ds'\,ds$ and define

$$\omega_M^2 := \frac{8\pi\; k_l}{(\rho_l-\rho_0)\hat{A}_l}.$$

Note that $\hat{A}_l$ is negative, as $\hat{A}_l = -\frac{2}{|\partial D_l|}\int_{\partial D_l}\int_{D_l}\frac{1}{|s-y|}dy\;d\sigma_l(s)$ by the divergence theorem, and since ρ_l satisfy (1.5) and $a \ll 1$, it follows that ω_M^2 is positive. In the case $\beta = 2$, to simplify the exposition of the results, we assume that the constant ω_M^2 is the same for all $l = 1, \ldots, M$. For example, this can hold if all the bubbles are identical in shape, and have the same density and bulk modulus. The main results of this work are stated in the following theorem.

Theorem 1.2. *Under the conditions $0 \leq t < \frac{1}{2}$, $0 \leq s \leq \frac{3}{2}$, $\beta = 1+\gamma$, with $0 \leq \gamma \leq 1$ and $s+\gamma \leq 2$ we have the following expansions.*

(1) *Assume that $\gamma < 1$ or $\gamma = 1$ with ω being away from ω_M, i.e. $|1-\frac{\omega_M^2}{\omega^2}| \geq l_0$ with a positive constant l_0 independent of $a, a \ll 1$. Then*

$$u^\infty(\hat{x},\theta) = \sum_{m=1}^{M}\Phi^\infty(\hat{x},z_m)Q_m + O(a^{2-s}+a^{1-2t}) \tag{1.7}$$

under the additional condition on t: $t \geq \frac{s}{3}$.

(2) *Assume that $\gamma = 1$ and the frequency ω is near ω_M, i.e. $1-\frac{\omega_M^2}{\omega^2} = l_M a^{h_1}$, $h_1 > 0$. Then*

$$u^\infty(\hat{x},\theta) = \sum_{m=1}^{M}\Phi^\infty(\hat{x},z_m)Q_m + O(a^{2-s-2h_1}+a^{3-2t-2s-2h_1}) \tag{1.8}$$

under the additional conditions on t and h_1 given by

- $t \geq \frac{s}{3}$ *and* $s+h_1 \leq 1$ *if* $l_M < 0$.
- $t \geq \frac{s}{3}$, $t \leq 1-h_1$ *and* $s+h_1 < \frac{3}{2}$ *if* $l_M > 0$.

The vector $(Q_m)_{m=1}^M$ is the solution of the following algebraic system

$${\mathbf{C_m}}^{-1}Q_m + \sum_{l\neq m}\Phi(z_l,z_m)Q_l = -u^I(z_m),\; m = 1,\ldots,M, \tag{1.9}$$

with

$$\mathbf{C_m} := \frac{\kappa_m^2|D_m|}{\frac{\rho_m}{\rho_m-\rho_0}-\frac{1}{8\pi}\kappa_m^2\hat{A}_m} \text{ and } \hat{A}_m := \frac{1}{|\partial D_m|}\int_{\partial D_m}\int_{\partial D_m}\frac{(s-s')}{|s-s'|}\cdot\nu_{s'}\,ds'\,ds. \tag{1.10}$$

The algebraic system (1.9) is invertible under one of the following conditions:

(1) *The coefficients $\mathbf{C_m}$ are negative and $\max|\mathbf{C_m}| = O(a^s)$, as $a \ll 1$. This condition holds if*

   (a) *$\gamma < 1$ or $\gamma = 1$ with ω being away from ω_M and we have the relations $0 \leq \gamma \leq 1$, $\gamma+s \leq 2$ and $\frac{s}{3} \leq t \leq 1$.*

   (b) *$\gamma = 1$ and the frequency ω approaches ω_M from below ($l_M < 0$), i.e. $\omega < \omega_M$, and we have the relations $\frac{s}{3} \leq t \leq 1$ and $1-h_1-s \geq 0$.*

(2) *The coefficients $\mathbf{C_m}$ are positive and one of the following conditions is fulfilled*

(a) $\max|\mathbf{C_m}| = O(a^t)$, *as* $a \ll 1$, *and* $\tau := \min_{1\leq j,m\leq M,\ j\neq m} cos(\kappa_0|z_m - z_j|) > 0$. *The first conditions holds if* $\gamma = 1$, *the frequency* ω *approaches* ω_M *from above* ($l_M > 0$), *i.e.* $\omega > \omega_M$, *and we have the relations* $0 \leq t \leq 1 - h_1$ *and* $s \leq 1$.

(b) $\max|\mathbf{C_m}| = O(a^s)$, *as* $a \ll 1$. *This condition holds if* $\gamma = 1$ *and the frequency* ω *approaches* ω_M *from above* ($l_M > 0$), *i.e.* $\omega > \omega_M$, *and we have the relations* $\frac{s}{3} \leq t \leq 1$ *and* $1-h_1-s \geq 0$.

The constants appearing in the error terms of (1.7) and (1.8) depend only on the a priori bounds mentioned above. In the case $\gamma = 1$ *(i.e.* $\beta = 2$*), we assume that the constant* C_ρ*, defined in (1.5) is larger then a certain constant depending only on those a priori bounds.*

Related to the model (1.4), the asymptotic expansion is derived formally in [11, 12] and justified mathematically in [7] in the case $\beta = 2$ when the frequency ω is close to the particular frequency ω_M. The particular frequency ω_M is shown to be an approximation, as $a \ll 1$, of a resonance known as the Minneart resonance, see [6].

The approximations provided in Theorem 1.2 are valid for a large class of bubbles. In particular, they can be used to derive the equivalent effective media at least in the following class:

(1) If $\gamma \in (\frac{1}{2}, 1)$, which means that $\beta \in (\frac{3}{2}, 2)$, and for any frequency ω.

(2) If $\gamma = 1$, i.e. $\beta = 2$, for any frequency ω away or very close (but distinct) from ω_M.

The error in the approximation in (1.7) is going to zero since $t < \frac{1}{2}$ and $s \leq \frac{3}{2}$.

The error term in (1.8) goes to zero provided s, h_1 and t satisfy the condition

$$s < \min\{2 - 2h_1,\ \frac{3 - 2t - 2h_1}{2}\}. \tag{1.11}$$

These conditions are fulfilled, in particular, if

$$s + h_1 \leq 1,\ h_1 < 1 \ \text{ and } t < \frac{1}{2}. \tag{1.12}$$

These last conditions are the regimes in which one can derive the effective media. Actually, in the case $l_M > 0$, one can also allow $s + h_1 > 1$ (but $s + h_1 < \frac{3}{2}$) and hence generate potential walls which are completely reflecting any incident wave sent from outside of its support. This phenomenon is already observed and justified in the framework of acoustic waves in the presence of very large number of holes, see [15].

We also observe that, in our approximations, we can handle the case $s = 0$ and $\frac{\rho_m}{\rho_m - \rho_0} - \frac{1}{8\pi}\kappa_m^2 \hat{A}_m = \zeta_m a^3$ with some constant ζ_m, by choosing $h_1 = 1$. In this case, the error term goes also to zero as soon as $t < \frac{1}{2}$. Hence, the condition $\frac{\rho_m}{\rho_m - \rho_0} - \frac{1}{8\pi}\kappa_0^2 \hat{A}_m = \zeta_m\, a^3$ implies that, as $a \to 0$, and choosing $t = 0$ for instance, $u^\infty(\hat{x}, \theta) = \sum_{m=1}^M \Phi^\infty(x, z_m) Q_m$ where ${\mathbf{C_m}}^{-1} := \zeta_m {\kappa_0}^{-2} |B_m|^{-1}$ and $(Q_m)_{m=1}^M$ is the solution of the algebraic system (1.9). This limiting field describes the Foldy-Lax field generated by the interaction of the point-like scatterers, given by the centers of our small bubbles, with scattering strengths modeled by $\mathbf{C_m}$'s, compare to [16,18] and see also [19]. This limiting field is also modeled by the Dirac-like potentials supported on the centers of the small bubbles, see [1]. What we have shown then is that if we inject to the background small bubbles characterized by $\frac{\rho_m}{\rho_0} \approx a^2$ and the frequency ω near to the Minnaert resonance as follows $\frac{\omega^2}{\omega_M^2} = 1 + l\ a$ the generated fields behave as the ones created by Dirac-like potentials (or point-like scatteres). This shows a way how to generate fields generated by singular potentials by injective regular small bubbles. This singular potentials are supported on points. In a forthcoming work, we show how to generate fields due to potentials supported on $1D$ curves (i.e. metawires), 2D surfaces (metasurfaces or metascreens) or 3D domains (metamaterials). Thanks to the kind of approximations we provide in Theorem 1.2, these different settings of the metamaterials will be treated in a unified way, namely as Dirac type potentials supported on curves, surfaces or domains.

The analysis of the wave propagation in the presence of highly heterogeneous media is an object of extensive study, see [20]. An important situation is when the heterogeneity of the medium is modeled by the high relative contrasts as the relative densities or the relative bulk modulus described above. In the recent years there was an increase of interest in describing the effective macroscopic models generated by periodically distributed microscopic structures characterized by high contrasted media, see [17] and the references therein, based on homogenization techniques. Compared to these works, we need to study the expansions of the fields generated by contrasted small bubbles that are not necessarily periodically distributed. Besides, we focus on the precise description of the dominant parts of the fields as the cluster of the bubbles becomes dense. For this, we propose to derive the point-interaction approximation of those fields using integral equation methods coupled with asymptotic expansions tools. This point-interaction approach has its roots back to the Foldy approximation method known in several branches of physics and engineering, see [1, 19]. The advantage of this approach is that we can characterize clearly the dominant fields generated by the interaction of the small bubbles between each other and also with the background medium. This approach was already tested and justified in our previous works [3, 14] when the contrast is modeled by high surface impedances and in [7] where the case $\gamma = 1$ (i.e. $\beta = 2$) and the frequency ω is close to the resonance ω_M. The purpose of our work here is to extend those last results to more general values of β and the whole range of frequencies ω.

Related to this bubbles model, other results were derived very recently. In particular, in the case $\beta = 2$ and ω close to the resonance ω_M, we find in [8] a mathematical framework for modeling metasurfaces with bubbles, in [9] a justification of the superfocusing of acoustic waves in the presence of gas bubbles and in [10] a justification of the bandgap opening due to periodically distributed bubbles.

As compared to the results in [7, 9], using our derived estimates, we can handle not only frequencies near the Minnaert resonance but arbitrary other fixed frequencies[1] and any gas modelled by densities having contrasts of the order a^β with $\beta \in (\frac{3}{2}, 2]$. In the particular case when $\beta = 2$ and ω is close to the resonance ω_M, we retrieve the results in [7]. Namely taking $s + h_1 = 1$ in (1.8), we see that we can get the effective media with an additive coefficient changing sign depending if ω is lower or higher than ω_M. The equivalent medium is absorbing if $\omega < \omega_M$ and reflecting if $\omega > \omega_M$. In addition, and as discussed above, if ω is close to ω_M and $\omega > \omega_M$, i.e. with $l_M > 0$ in (1.8), we can also take $1 < s + h_1 < \frac{3}{2}$. In these cases, the equivalent media behave as extremely reflecting media allowing no incident wave to penetrate inside it, see [15] for a different but related setting using holes. The quantification and justification of these results will be reported in a forthcoming work.

The rest of the paper is divided into the following sections. In section 2, we give the core proof of the result while in section 3, we provide the detailed proofs of the main tools used in section 2.

2. Proof of Theorem 1.2

2.1. **Representation of the solutions.** We recall that the single layer potential, double layer potential and the adjoint of the double layer potential are defined as

$$S^\kappa_{D_m}\phi(x) := \int_{\partial D_m} \Phi_\kappa(x,y)\phi(y)d\sigma_m(y),$$

$$K^\kappa_{D_m}\phi(x) := \int_{\partial D_m} \partial_{\nu_y}\Phi_\kappa(x,y)\phi(y)d\sigma_m(y),$$

$$(K^\kappa_{D_m})^*\phi(x) := \int_{\partial D_m} \partial_{\nu_y}\Phi_\kappa(x,y)\phi(y)d\sigma_m(y),$$

[1]This, of course, means that we are in the Rayleigh regime, i.e. $\omega a << 1$ as $a << 1$.

where

$$\Phi_\kappa(x,y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|},\ \text{for } x,y \in \mathbb{R}^3,$$

denotes the fundamental solution of the Helmholtz equation in three dimensions with a fixed wave number κ.

The problem (1.4) is well-posed, see [4,5] for instance. For the purpose of our analysis later on, we show that the total field can be represented as

$$u(x) = \begin{cases} u^I + \sum_{l=1}^M S_{D_l}^{\kappa_0}\phi_l(x),\ x \in \mathbb{R}^3 \diagdown \overline{\cup_{l=1}^M D_l}, \\ S_{D_s}^{\kappa_s}\psi_s(x),\ x \in D_s,\ s = 1, ..., M, \end{cases} \tag{2.1}$$

where ϕ_l, ψ_l are appropriate densities. Observe that we represent the field inside each $D_s,\ s = 1, \dots, M$, using a single density. This simplifies the presentation of the computations.

Using (2.1), the jump conditions across the boundary of the obstacles in (1.4) can be reformulated as follows. From the third identity in (1.4), we derive that on ∂D_s,

$$0 = u\Big|_+(x) - u\Big|_-(x) = u^I|_{\partial D_s}(x) + \sum_{l=1}^M S_{D_l}^{\kappa_0}\phi_l\Big|_+(x) - S_{D_s}^{\kappa_s}\psi_s\Big|_-(x).$$

Since the single-layer potentials are continuous, this implies that

$$S_{D_s}^{\kappa_s}\psi_s\Big|_{\partial D_s}(x) - \sum_{l=1}^M S_{D_l}^{\kappa_0}\phi_l\Big|_{\partial D_s}(x) = u^I\Big|_{\partial D_s}(x). \tag{2.2}$$

Again

$$\frac{1}{\rho_0}\frac{\partial u}{\partial \nu^s}\Big|_+(x) = \frac{1}{\rho_0}\frac{\partial u^I}{\partial \nu^s}\Big|_{\partial D_s}(x) + \sum_{l=1}^M \frac{1}{\rho_0}\frac{\partial (S_{D_l}^{\kappa_0}\phi_l)}{\partial \nu^s}\Big|_+(x), \quad \frac{1}{\rho_s}\frac{\partial u}{\partial \nu^s}\Big|_-(x) = \frac{1}{\rho_s}\frac{\partial (S_{D_s}^{\kappa_s}\psi_s)}{\partial \nu^s}\Big|_-(x),$$

and therefore from the fourth identity in (1.4), we derive that

$$\begin{aligned} 0 = \frac{1}{\rho_0}\ \frac{\partial u}{\partial \nu^s}\Big|_+(x) - \frac{1}{\rho_s}\ \frac{\partial u}{\partial \nu^s}\Big|_-(x) &= \frac{1}{\rho_0}\ \frac{\partial u^I}{\partial \nu^s}\Big|_{\partial D_s}(x) + \sum_{\substack{l=1 \\ l\neq s}}^M \frac{1}{\rho_0}\frac{\partial (S_{D_l}^{\kappa_0}\phi_l)}{\partial \nu^s}\Bigg|_{\partial D_s}(x) \\ &- \frac{1}{2\rho_0}\phi_s(x) + \frac{1}{\rho_0}(K_{D_s}^{\kappa_0})^*\phi_s(x) - \frac{1}{2\rho_s}\psi_s(x) - \frac{1}{\rho_s}(K_{D_s}^{\kappa_s})^*\psi_s(x) \end{aligned}$$

and hence

$$\frac{1}{\rho_0}\ \frac{\partial u^I}{\partial \nu^s}\Big|_{\partial D_s}(x) = -\frac{1}{\rho_0}\Big[-\frac{1}{2}Id + (K_{D_s}^{\kappa_0})^*\Big]\phi_s(x) + \frac{1}{\rho_s}\Big[\frac{1}{2}Id + (K_{D_s}^{\kappa_s})^*\Big]\psi_s(x) - \sum_{\substack{l=1 \\ l\neq s}}^M \frac{1}{\rho_0}\frac{\partial (S_{D_l}^{\kappa_0}\phi_l)}{\partial \nu^s}\Bigg|_{\partial D_s}(x). \tag{2.3}$$

The following proposition guarantees the existence of densities ϕ_l, ψ_l satisfying (2.2) and (2.3).

Proposition 2.1. *For each*

$$\prod_{i=1}^M (F_i, G_i) \in Y := \prod_{i=1}^M [H^1(\partial D_i) \times L^2(\partial D_i)],$$

there exists a unique solution

$$\prod_{i=1}^M (\phi_i, \psi_i) \in X := \prod_{i=1}^M [L^2(\partial D_i) \times L^2(\partial D_i)]$$

to the system of integral equations

$$\left(S_{D_i}^{\kappa_i}\psi_i - \sum_{l=1}^{M} S_{D_l}^{\kappa_0}\phi_l\right)\Bigg|_{\partial D_i} = F_i,$$

$$\frac{\rho_0}{\rho_i}\Big[\frac{1}{2}Id + (K_{D_i}^{\kappa_i})^*\Big]\psi_i(x) - \Big[-\frac{1}{2}Id + (K_{D_i}^{\kappa_0})^*\Big]\phi_i(x) - \sum_{\substack{l=1\\ l\neq i}}^{M} \frac{\partial (S_{D_l}^{\kappa_0}\phi_l)}{\partial \nu^i}\Bigg|_{\partial D_i} = G_i.$$

Proof. We outline a proof of this proposition in section 3.1. □

Note that in order to prove the representation (2.1), we apply the above proposition with $F_i := u^I|_{\partial D_i}$ and $G_i := \frac{\partial u^I}{\partial \nu^i}\Big|_{\partial D_i}$ respectively, where $i = 1, \dots, M$.

2.2. **Integral identities.** Let us recall that the capacitance Cap_l is defined as

$$Cap_l := \int_{\partial D_l} \left(S_{D_l}^{0}\right)^{-1}(1)(t)\; d\sigma_l(t). \tag{2.4}$$

It is known that $Cap_m \approx \delta$ and hence $Cap_m \approx a$.
We note that the system of integral equations (2.2)-(2.3) can be rewritten as follows. For $l = 1 \dots, M$,

$$S_{D_l}^{\kappa_0}(\psi_l - \phi_l) - \sum_{\substack{m=1\\ m\neq l}}^{M} S_{D_m}^{\kappa_0}\phi_m = u^I + [S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}]\psi_l, \quad \text{on } \partial D_l, \tag{2.5}$$

(2.6)

$$\frac{\rho_0}{\rho_l}[\frac{1}{2}Id + (K_{D_l}^{\kappa_0})^*]\psi_l - [-\frac{1}{2}Id + (K_{D_l}^{\kappa_0})^*]\phi_l - \sum_{\substack{m=1\\ m\neq l}}^{M} \frac{\partial (S_{D_m}^{\kappa_0}\phi_m)}{\partial \nu^l} = \frac{\partial u^I}{\partial \nu^l} + \frac{\rho_0}{\rho_l}[(K_{D_l}^{\kappa_0})^* - (K_{D_l}^{\kappa_l})^*]\psi_l, \quad \text{on } \partial D_l.$$

Our first step is to convert the system of double integral equations (2.5)-(2.6) for $(\phi_l, \psi_l)_{l=1}^M$ into a system of single integral equation for $(\phi_l)_{l=1}^M$. For this, we need to estimate first the source terms $[S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}]\psi_l$ and $[(K_{D_l}^{\kappa_0})^* - (K_{D_l}^{\kappa_l})^*]\psi_l$.
In the following two lemmas, we collect some approximations that we shall use.

Lemma 2.2. *The functions* $\left(S_{D_l}^0\right)^{-1}\left(\int_{\partial D_l} |\cdot - t|^n \phi_l(t)\; d\sigma_l(t)\right)$ *and* $\left(S_{D_l}^0\right)^{-1}\left((\cdot - z_l)^n\right)$ *exhibit the following approximate behavior.*

$$\left\|\left(S_{D_l}^0\right)^{-1}\left(\int_{\partial D_l} |\cdot - t|^n \phi_l(t) d\sigma_l(t)\right)\right\|_{L^2(\partial D_l)} = O\Big(a^{n+1}\|\phi_l\|_{L^2(\partial D_l)}\Big), \tag{2.7}$$

$$\left\|\left(S_{D_l}^0\right)^{-1}\left((\cdot - z_l)^n\right)\right\|_{L^2(\partial D_l)} = O(a^n). \tag{2.8}$$

Proof. We refer to section 3.2 for a proof of this lemma. □

Lemma 2.3. *The following approximations hold true.*

(2.9)

$$\left[S_{D_m}^{\kappa_0} - S_{D_m}^{\kappa_m}\right]\psi_m(s) = \frac{i}{4\pi}(\kappa_0 - \kappa_m)\int_{\partial D_m} \psi_m(s)\, d\sigma_m(s) + \underbrace{\sum_{n=2}^{\infty} \frac{i^n(\kappa_0^n - \kappa_m^n)}{4\pi n!}\int_{\partial D_m} |x - s|^{n-1}\psi_m(s)\, d\sigma_m(s)}_{=:Err1_m = O(a^2\|\psi_m\|)},$$

(2.10)

$$\int_{\partial D_m} \left[(K^{\kappa_0}_{D_m})^* - (K^{\kappa_m}_{D_m})^*\right] \psi_m(s) d\sigma_m(s) = \frac{1}{8\pi}(\kappa_0^2 - \kappa_m^2) \int_{\partial D_m} \psi_m(s) \left[\int_{\partial D_m} \frac{(s-t)}{|s-t|} \cdot \nu_t \, dt\right] d\sigma_m(s) - \frac{i}{4\pi}(\kappa_0^3 - \kappa_m^3)|D_m| \int_{\partial D_m} \psi_m(s) d\sigma_m(s) + \underbrace{Err2_m}_{[=O(a^5\|\psi_m\|)]},$$

We shall also sometimes use (2.10) *with* $\kappa_m = 0$*. In this case, we shall refer to* $Err2_m$ *as* $Err3_m$*.*

(2.11)

$$\int_{\partial D_l} \frac{\partial (S^{\kappa_0}_{D_m}\phi_m)}{\partial \nu^l} = - \kappa_0^2 |D_l| \left[\Phi_{\kappa_0}(z_l, z_m) \int_{\partial D_m} \phi_m(s) \, d\sigma_m(s) + \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot \int_{\partial D_m} (s - z_m)\phi_m(s) \, d\sigma_m(s)\right] - \kappa_0^2 \nabla_x \Phi_{\kappa_0}(z_l, z_m) \cdot \Big[\int_{D_l} (x - z_l) dx\Big] \int_{\partial D_m} \phi_m(s) \, d\sigma_m(s) - \underbrace{Err4_m}_{=O\left(\frac{a^6}{d^3_{ml}}\|\phi_m\|\right)}, \ m \neq l,$$

$$\int_{\partial D_l} \frac{\partial u^I}{\partial \nu^l} = - \kappa_0^2 \, |D_l| \, u^I(z_l) + \underbrace{Err5_l}_{=O(a^4)}, \tag{2.12}$$

$$\begin{aligned}
\psi_l - \phi_l = {} & \left(S^0_{D_l}\right)^{-1} \left(\frac{i(\kappa_0 - \kappa_l)}{4\pi} Q_l\right) \\
& + \left(S^0_{D_l}\right)^{-1} \left(\sum_{m \neq l} \left[\left(1 - \frac{i\kappa_l}{4\pi} Cap_l\right) \Phi_{\kappa_0}(z_l, z_m) + (s - z_l) \cdot \nabla_s \Phi_{\kappa_0}(z_l, z_m)\right] Q_m\right) \\
& + \left(S^0_{D_l}\right)^{-1} \left(\sum_{m \neq l} \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot V_m\right) + \left(S^0_{D_l}\right)^{-1} u^I(z_l) \\
& + Err6_l \Big[:= O\left(a + a^2\|\phi_l\| + \sum_{m \neq l} \frac{a^3}{d^3}\|\phi_m\|\right)\Big], \ in \ L^2,
\end{aligned} \tag{2.13}$$

(2.14)

$$\begin{aligned}
\int_{\partial D_l} \psi_l - \int_{\partial D_l} \phi_l = {} & Cap_l \left(\frac{i(\kappa_0 - \kappa_l)}{4\pi} Q_l\right) \\
& + \left(\sum_{m \neq l} \left[Cap_l \left(1 - \frac{i\kappa_l}{4\pi} Cap_l\right) \Phi_{\kappa_0}(z_l, z_m) + \nabla_s \Phi_{\kappa_0}(z_l, z_m) \cdot \int_{\partial D_l} \left(S^0_{D_l}\right)^{-1} (\cdot - z_l)(s) d\sigma_l(s)\right] Q_m\right) \\
& + Cap_l \left(\sum_{m \neq l} \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot V_m\right) + Cap_l \ u^I(z_l) \\
& + Err7_l \Big[:= O\left(a^2 + a^3\|\phi_l\| + \sum_{m \neq l} \frac{a^4}{d^3}\|\phi_m\|\right)\Big],
\end{aligned}$$

Proof. We defer the proof of the lemma to section 3.3. □

2.3. **A way of counting the number of bubbles.** In the following sections, we will repeatedly need to estimate sums of the form $\sum_{i=1,i\neq j}^{M} f(z_i, z_j)$ with functions f involving inverse power of distances, i.e. $|z_i - z_j|^{-k}$, $k \in \mathbb{R}_+$. Here z_j is the center of the small bubble D_j. Following [14], we describe here a way how to handle these sums by a proper counting. To do it, for any $m = 1, \dots, M$ fixed, we distinguish between the points z_j, $j \neq m$ by keeping them into different layers based on their distance from D_m. Let Ω_m, $1 \leq m \leq M$ be the cubes of center z_m such that each side is of size $(\frac{a}{2} + d^\alpha)$ with $0 \leq \alpha \leq 1$ and it contains only D_m. Let us arrange these cubes in a cuboid, for example unit rubiks cube in different layers. Hence, the total cubes upto the n^{th} layer consists $(2n+1)^3$ cubes for $n = 0, \dots, [d^{-\alpha}]$, and Ω_m is located on the center. It is clear then that the number of bubbles located in the n^{th}, $n \neq 0$ layer will be $[(2n+1)^3 - (2n-1)^3]$ and their distance from D_j is more than nd^α. With this way of counting, we deduce that for j fixed

$$\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} \leq \sum_{z_m\in\Omega_j, z_m\neq z_j} |z_l - z_j|^{-k} + \sum_{l=1}^{d^{-\alpha}} \sum_{z_i\in\Omega_l} |z_l - z_j|^{-k} \tag{2.15}$$

where

$$\sum_{z_m\in\Omega_j, z_m\neq z_j} |z_l - z_j|^{-k} = O(d^{-k})$$

and

$$\sum_{l=1}^{d^{-\alpha}} \sum_{z_i\in\Omega_l} |z_l - z_j|^{-k} \leq \sum_{l=1}^{d^{-\alpha}} ((2l+1)^3 - (2l-1)^3)(ld^\alpha)^{-k} = O(\sum_{l=1}^{d^{-\alpha}} l^2 (ld^\alpha)^{-k}) = O(d^{-\alpha\, k} \sum_{l=1}^{d^{-\alpha}} l^{2-k}).$$

Hence

$$\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(d^{-k}) + O(d^{-\alpha\, k} \sum_{l=1}^{d^{-\alpha}} l^{2-k}). \tag{2.16}$$

Observe that for $k = 0$, it is obvious that $\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = M - 1$. By the previous formulas we have $\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(1) + O(d^{-3\alpha})$. Recalling the way we counted, $d^{3\alpha}$ is the volume of the Ω_j'. Hence $d^{-3\alpha}$ is of the order of the number of the bubbles, i.e. $d^{-3\alpha} = O(M)$. As we set $M = O(a^{-s})$, we will be using the formula $3\alpha t = s$, and then $t \geq \frac{s}{3}$, as $\alpha \leq 1$.

For $k > 0$, we obtain the following formulas

(1) If $k \leq 2$, then

$$\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(d^{-k}) + O(d^{-\alpha\, k} \sum_{l=1}^{d^{-\alpha}} l^{2-k}) = O(d^{-k}) + O(d^{-3\alpha}). \tag{2.17}$$

(2) If $2 < k < 3$, then

$$\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(d^{-k}) + O(d^{-\alpha\, k} |\ln(d)|). \tag{2.18}$$

(3) If $k > 3$, then

$$\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(d^{-k}) + O(d^{-\alpha\, k} \sum_{l=1}^{d^{-\alpha}} l^{2-k}) = O(d^{-k}) + O(d^{(-2k+3)\alpha}). \tag{2.19}$$

If we do not count properly, then we would have $\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(M\ d^{-k}) = a^{-s-k\,t}$. Let us take $k = 1$ as an example. We have shown above that $\sum_{i=1,i\neq j}^{M} |z_i - z_j|^{-k} = O(a^{-t}) + O(d^{-3\alpha}) = O(a^{-t}) + O(a^{-3\alpha t}) = O(a^{-t}) + O(a^{-s})$ instead of $O(a^{-s-t})$. Hence, in this case, we gain an order of a^{-t} as soon as $s \geq t$. This kind of reasoning will be used in the next sections.

2.4. **The a priori approximation of the system of integral equations.** Our first step is to approximate the source term $[S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}]\psi_l$ in (2.5), that is, $S^{\kappa_0}_{D_l}(\psi_l - \phi_l) - \sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m = u^I + [S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}]\psi_l$ by single layers of the form $S^{\kappa_0}_{D_l} g_l$, where g_l is the unique solution of the problem

$$S^{\kappa_0}_{D_l} g_l(s) = \frac{i}{4\pi}(\kappa_0 - \kappa_l)\int_{\partial D_l}\psi_l(t)d\sigma_l(t), \text{ on } \partial D_l. \tag{2.20}$$

Hence

$$[S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}]\psi_l = S^{\kappa_0}_{D_l} g_l + Err1_l.$$

Let us define

$$H1 := S^{\kappa_0}_{D_l}(\psi_l - \phi_l) - \sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m - u^I - S^{\kappa_0}_{D_l} g_l,$$

and $\tilde{g}_l$ such that

$$S^{\kappa_0}_{D_l}\tilde{g}_l = Err1_l. \tag{2.21}$$

Using (2.20) and (2.21) in (2.9), it follows that $H1$ satisfies

$$\begin{cases} (\Delta + \kappa_0^2)(H1 - S^{\kappa_0}_{D_l}\tilde{g}_l) = 0 \text{ in } D_l,\\ H1 = S^{\kappa_0}_{D_l}\tilde{g}_l \text{ on } \partial D_l. \end{cases}$$

By the maximum principle, we can immediately conclude that $H1 = S^{\kappa_0}_{D_l}\tilde{g}_l$ in D_l. Taking the normal derivative from inside D_l, we have

$$\frac{\partial[S^{\kappa_0}_{D_l}(\psi_l - \phi_l)]}{\partial\nu^l}\Big|_{\partial D_l} - \sum_{\substack{m=1\\m\neq l}}^{M}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l} = \frac{\partial u^I}{\partial\nu^l}\Big|_{\partial D_l} + \frac{\partial(S^{\kappa_0}_{D_l} g_l)}{\partial\nu^l}\Big|_{\partial D_l} + Er_l,$$

where $Er_l := \frac{\partial[S^{\kappa_0}_{D_l}\tilde{g}_l]}{\partial\nu^l}\Big|_{\partial D_l}$. This gives us

$$\frac{\rho_0}{\rho_l}\Big[\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*\Big](\psi_l - \phi_l) - \sum_{\substack{m=1\\m\neq l}}^{M}\frac{\rho_0}{\rho_l}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l} = \frac{\rho_0}{\rho_l}\frac{\partial u^I}{\partial\nu^l}\Big|_{\partial D_l} + \frac{\rho_0}{\rho_l}\Big[\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*\Big]g_l + Er_{2,l},$$

where $Er_{2,l} := \frac{\rho_0}{\rho_l}Er_l$. Next we use this in the second identity (2.6) to derive

$$\begin{aligned} &-\frac{1}{2}\Big[\frac{\rho_0}{\rho_l} + 1\Big]\phi_l - \Big[\frac{\rho_0}{\rho_l} - 1\Big](K^{\kappa_0}_{D_l})^*\phi_l - \sum_{\substack{m=1\\m\neq l}}^{M}\Big(\frac{\rho_0}{\rho_l} - 1\Big)\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l}\\ &\qquad = \Big(\frac{\rho_0}{\rho_l} - 1\Big)\frac{\partial u^I}{\partial\nu^l}\Big|_{\partial D_l} + \frac{\rho_0}{\rho_l}\Big[\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*\Big]g_l - \frac{\rho_0}{\rho_l}[(K^{\kappa_0}_{D_l})^* - (K^{\kappa_l}_{D_l})^*]\psi_l + Er_{2,l}, \end{aligned}$$

which, in turn, implies that

$$\begin{aligned} &-\frac{1}{2}\Big[\frac{\rho_0 + \rho_l}{\rho_0 - \rho_l}\Big]\phi_l - (K^{\kappa_0}_{D_l})^*\phi_l - \sum_{\substack{m=1\\m\neq l}}^{M}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l}\\ &\quad = \frac{\partial u^I}{\partial\nu^l} + \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}\Big([(K^{\kappa_l}_{D_l})^* - (K^{\kappa_0}_{D_l})^*]\psi_l + [\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*]g_l\Big) + Er_{3,l}, \text{ on } \partial D_l, \end{aligned}$$

where $Er_{3,l} := \left(\frac{\rho_0}{\rho_l} - 1\right)^{-1} Er_{2,l}$. Hence, we have

$$
\begin{aligned}
-\frac{1}{2}\Big[\frac{\rho_0+\rho_l}{\rho_0-\rho_l}\Big]\phi_l - (K^0_{D_l})^*\phi_l = \frac{\partial u^I}{\partial \nu^l} + \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\Bigg([(K^{\kappa_l}_{D_l})^* - (K^{\kappa_0}_{D_l})^*]\psi_l + [\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*]g_l\Bigg) \\
+ \sum_{\substack{m=1\\m\neq l}}^{M} \frac{\partial (S^{\kappa_0}_{D_m}\phi_m)}{\partial \nu^l}\Big|_{\partial D_l} + [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*]\phi_l + Er_{3,l}, \quad \text{on } \partial D_l.
\end{aligned} \tag{2.22}
$$

Our aim henceforth would be to derive a suitable approximate system for $\int_{\partial D_l}\phi_l\, d\sigma_l$ from (2.22) above, leading us to the final algebraic system. To do so, we start by integrating (2.22) on ∂D_l, and since $K^0_{D_l}(1) = -\frac{1}{2}$, it follows that

$$
\begin{aligned}
-\frac{1}{2}\Big[\frac{\rho_0+\rho_l}{\rho_0-\rho_l} - 1\Big]\int_{\partial D_l}\phi_l\, d\sigma_l = \int_{\partial D_l}\frac{\partial u^I}{\partial \nu^l}\, d\sigma_l + \int_{\partial D_l}\Big[(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*\Big]\phi_l\, d\sigma_l + \sum_{\substack{m=1\\m\neq l}}^{M}\int_{\partial D_l}\frac{\partial (S^{\kappa_0}_{D_m}\phi_m)}{\partial \nu^l}\, d\sigma_l \\
+ \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\int_{\partial D_l}\Bigg([(K^{\kappa_l}_{D_l})^* - (K^{\kappa_0}_{D_l})^*]\psi_l + [\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*]g_l\Bigg)\, d\sigma_l + Er_{4,l},
\end{aligned} \tag{2.23}
$$

where $Er_{4,l} := \int_{\partial D_l} Er_{3,l}$. For $s \in \partial D_l$, let us define

$$
A_l(s) := \int_{\partial D_l}\frac{(s-t)}{|s-t|}\cdot \nu_t\, d\sigma_l(t).
$$

From the definition, it easily follows that $\|A_l\|_{L^2(\partial D_l)} = O(a^3)$. Next we observe that,

$$
\begin{aligned}
Er_{4,l} &= \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\int_{\partial D_l} Er_l = \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\int_{\partial D_l}[\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*]\tilde{g}_l \\
&= \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\int_{\partial D_l}[-(K^0_{D_l})^* + (K^{\kappa_0}_{D_l})^*]\tilde{g}_l \\
&= \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\frac{1}{8\pi}\kappa_0^2\int_{\partial D_l}\tilde{g}_l(s)A_l(s)d\sigma_l(s) + O(a^5\|\tilde{g}_l\|),
\end{aligned}
$$

where the last step follows from (2.10) applied to $\tilde{g}_l$.
In order to check the behavior of the $Er_{4,l}$, let us consider its dominating term $\int_{\partial D_l}\tilde{g}_l(s)A_l(s)d\sigma_l(s)$. From the definition of $\tilde{g}_l$, it can be observed that the dominating term of $\tilde{g}_l$ is $\left(S^{\kappa_0}_{D_l}\right)^{-1}\left(\int_{\partial D_l}|\cdot - t|\psi_l(t)d\sigma_l(t)\right)$ and hence it's sufficient to consider $\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|\cdot - t|\psi_l(t)d\sigma_l(t)\right)(s)\, A_l(s)d\sigma_l(s)$. Using (2.7), we can deduce that

$$
\begin{aligned}
&\Big|\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|\cdot - t|\psi_l(t)d\sigma_l(t)\right)(s)\, A_l(s)\; d\sigma_l(s)\Big| \\
&\le \Big\|\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|\cdot - t|\psi_l(t)d\sigma_l(t)\right)\Big\|_{L^2(\partial D_l)}\|A_l\|_{L^2(\partial D_l)} = O\left(a^5\|\psi_l\|\right).
\end{aligned} \tag{2.24}
$$

In addition, $\|\tilde{g}_l\| \le \|(S^{\kappa_0}_{D_l})^{-1}\|\|Err1_l\|_{L^2(\partial D_l)} = O(a^2\|\psi_l\|)$. Therefore $Er_{4,l} = O\left(a^5\|\psi_l\|\right)$.
In the following lemma, we note down some estimates for g_l that we shall use.

Lemma 2.4. *Similar to* (2.10), *g_l satisfies the estimate*

$$
\int_{\partial D_l}\left[(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*\right] g_l = \frac{1}{8\pi}\kappa_0^2\int_{\partial D_l} g_l(s)\left[\int_{\partial D_l}\frac{(s-t)}{|s-t|}\cdot \nu_t\, dt\right] d\sigma_l(s) \tag{2.25}
$$

$$- \frac{i}{4\pi}\kappa_0^3|D_l|\int_{\partial D_l} g_l(s)d\sigma_l(s) + \underbrace{Err8_l}_{=O(a^5\|g_l\|)}.$$

Also

$$g_l(s) = \frac{i}{4\pi}(\kappa_0 - \kappa_l)\left[Cap_l\ u^I(z_l) + \int_{\partial D_l}\phi_l + \sum_{m\neq l}\Phi_{\kappa_0}(z_l, z_m)\, Q_m Cap_l\right](S^0_{D_l})^{-1}(1)(s) + Err9_l\Big[:= O\left(a^2 + a^2\|\phi_l\| + \sum_{m\neq l}\frac{a^3}{d^2_{ml}}\|\phi_m\|\right)\Big],\ in\ L^2. \tag{2.26}$$

Proof. We defer the proof of this result to section 3.4. □

Now using lemma 2.3, (2.23) can be rewritten as

$$\begin{aligned}
&\frac{\rho_l}{\rho_l - \rho_0}\int_{\partial D_l}\phi_l\, d\sigma_l(s) + \sum_{\substack{m=1\\ m\neq l}}^{M}\Big(\kappa_0^2|D_l|\Big[\Phi_{\kappa_0}(z_l, z_m)\int_{\partial D_m}\phi_m(s)\, d\sigma_m(s) \\
&\quad + \nabla_t\Phi_{\kappa_0}(z_l, z_m)\cdot\int_{\partial D_m}(s - z_m)\phi_m(s)\, d\sigma_m(s)\Big] \\
&\quad + \kappa_0^2\nabla_x\Phi_{\kappa_0}(z_l, z_m)\cdot\Big[\int_{D_l}(x - z_l)dx\Big]\int_{\partial D_m}\phi_m(s)\, d\sigma_m(s) + Err4_m\Big) \\
&= -\kappa_0^2\,|D_l|\, u^I(z_l) + Err5_l + \frac{1}{8\pi}\kappa_0^2\int_{\partial D_l}\phi_l(s)\left[\int_{\partial D_l}\frac{(s-t)}{|s-t|}\cdot\nu_t\, d\sigma_l(t)\right]d\sigma_l(s) \\
&\quad - \frac{i}{4\pi}\kappa_0^3|D_l|\int_{\partial D_l}\phi_l(s)d\sigma_l(s) + Err3_l \\
&\quad + \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}\frac{1}{8\pi}(\kappa_l^2 - \kappa_0^2)\int_{\partial D_l}\psi_l(s)\left[\int_{\partial D_l}\frac{(s-t)}{|s-t|}\cdot\nu_t\, d\sigma_l(t)\right]d\sigma_l(s) \\
&\quad - \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}\frac{i}{4\pi}(\kappa_l^3 - \kappa_0^3)|D_l|\int_{\partial D_l}\psi_l(s)d\sigma_l(s) - \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}Err2_l \\
&\quad + \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}\frac{1}{8\pi}\kappa_0^2\int_{\partial D_l}g_l(s)\left[\int_{\partial D_l}\frac{(s-t)}{|s-t|}\cdot\nu_t\, d\sigma_l(t)\right]d\sigma_l(s) \\
&\quad - \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}\frac{i}{4\pi}\kappa_0^3|D_l|\int_{\partial D_l}g_l(s)d\sigma_l(s) + \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}Err8_l + Er_{4,l}.
\end{aligned} \tag{2.27}$$

Dividing by $|D_l|$ and making use of (2.13),(2.14) and (2.26) in (2.27), we obtain

(2.28)

$$\begin{aligned}
&\frac{\rho_l}{\rho_l - \rho_0}|D_l|^{-1}\int_{\partial D_l}\phi_l(s)\, d\sigma_l(s) \\
&\quad + \kappa_0^2\sum_{\substack{m=1\\ m\neq l}}^{M}\Big[\Phi_{\kappa_0}(z_l, z_m)\int_{\partial D_m}\phi_m(s)\, d\sigma_m(s) + \nabla_t\Phi_{\kappa_0}(z_l, z_m)\cdot\int_{\partial D_m}(s - z_m)\phi_m(s)\, d\sigma_m(s)\Big] \\
&\quad + \kappa_0^2|D_l|^{-1}\sum_{\substack{m=1\\ m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l, z_m)\cdot\Big[\int_{D_l}(x - z_l)dx\Big]\int_{\partial D_m}\phi_m(s)\, d\sigma_m(s) \\
&= -\kappa_0^2\, u^I(z_l) + \frac{1}{8\pi}\kappa_0^2|D_l|^{-1}\int_{\partial D_l}\phi_l(s)A_l(s)d\sigma_l(s) - \frac{i}{4\pi}\kappa_0^3\int_{\partial D_l}\phi_l(s)d\sigma_l(s)
\end{aligned}$$

$$
\begin{aligned}
&+\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{1}{8\pi}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\int_{\partial D_l}\phi_l(s)A_l(s)d\sigma_l(s)\\
&+\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{1}{8\pi}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\Bigg(\frac{i(\kappa_0-\kappa_l)}{4\pi}Q_l\\
&\qquad+\sum_{m\neq l}\left[\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\Phi_{\kappa_0}(z_l,z_m)+(\cdot-z_l)\cdot\nabla_s\Phi_{\kappa_0}(z_l,z_m)\right]Q_m\\
&\qquad+\sum_{m\neq l}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\,V_m\Bigg)(s)A_l(s)d\sigma_l(s)\\
&+\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{1}{8\pi}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\int_{\partial D_l}\left[\left(S^0_{D_l}\right)^{-1}u^I(z_l)+Err6_l\right]A_l(s)d\sigma_l(s)\\
&-\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{i}{4\pi}(\kappa_l^3-\kappa_0^3)\int_{\partial D_l}\phi_l(s)d\sigma_l(s)\\
&-\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{i}{4\pi}(\kappa_l^3-\kappa_0^3)\Bigg[\frac{i(\kappa_0-\kappa_l)}{4\pi}Cap_lQ_l+\sum_{m\neq l}\Big[Cap_l\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\Phi_{\kappa_0}(z_l,z_m)\\
&\qquad\qquad+\nabla_s\Phi_{\kappa_0}(z_l,z_m)\cdot\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(\cdot-z_l)(s)d\sigma_l(s)\Big]Q_m+Cap_l\sum_{m\neq l}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\,V_m\Bigg]\\
&-\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{i}{4\pi}(\kappa_l^3-\kappa_0^3)\left[Cap_lu^I(z_l)+Err7_l\right]\\
&+\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{1}{8\pi}\kappa_0^2|D_l|^{-1}\int_{\partial D_l}\frac{i}{4\pi}(\kappa_0-\kappa_l)\Big[Cap_lu^I(z_l)\\
&\qquad\qquad+Q_l+\sum_{m\neq l}\Phi_{\kappa_0}(z_l,z_m)\,Q_mCap_l\Big](S^0_{D_l})^{-1}(1)\,(s)A_l(s)d\sigma_l(s)\\
&+\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}\frac{1}{8\pi}\kappa_0^2|D_l|^{-1}\int_{\partial D_l}Err9_l\cdot A_l(s)\,d\sigma_l(s)+Er_{5,l},
\end{aligned}
$$

where

(2.29)

$$
\begin{aligned}
Er_{5,l}&=|D_l|^{-1}Er_{4,l}+|D_l|^{-1}\left[Err5_l-\sum_{\substack{m=1\\m\neq l}}^{M}Err4_m+\left(Err3_l+\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}(-Err2_l+Err8_l)\right)\right]\\
&=O(a^2\|\psi_l\|)+O\left(a+\sum_{m\neq l}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^2\|\psi_l\|+a^2\|\phi_l\|+a^2\|g_l\|\right)\\
&=O(a^2\|\psi_l\|)+O\left(a+\sum_{m\neq l}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^2\|\psi_l\|+a^2\|\phi_l\|+a^3\|\psi_l\|\right)\\
&=O\left(a+\sum_{m\neq l}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^2\|\phi_l\|+a^2\|\psi_l\|\right).
\end{aligned}
$$

Collecting the error terms together, we can rewrite (2.28) further as

(2.30)

$$
\begin{aligned}
&|D_l|^{-1}\int_{\partial D_l}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]A_l(s)\right)\phi_l(s)d\sigma_l(s)\\
&+\frac{i}{4\pi}\kappa_0^3\,Q_l+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Q_l\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{1}{16\pi^2}(\kappa_l^3-\kappa_0^3)(\kappa_0-\kappa_l)Cap_lQ_l\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{32\pi^2}(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)|D_l|^{-1}Q_l\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\left(1\right)(s)A_l(s)d\sigma_l(s)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\kappa_0^2|D_l|^{-1}\left(\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(1)\,(s)A_l(s)d\sigma_l(s)\right)\,Q_l\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\sum_{m\neq\, l}\left[Cap_l\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\Phi_{\kappa_0}(z_l,z_m)\right.\\
&\qquad\qquad\left.+\nabla_s\Phi_{\kappa_0}(z_l,z_m)\cdot\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(\cdot-z_l)(s)d\sigma_l(s)\right]Q_m\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{1}{8\pi}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\\
&\quad\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\left(\sum_{m\neq\, l}Q_m\left[\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\Phi_{\kappa_0}(z_l,z_m)+(\cdot-z_l)\cdot\nabla_s\Phi_{\kappa_0}(z_l,z_m)\right]\right)(s)A_l(s)d\sigma_l(s)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{32\pi^2}\kappa_0^2(\kappa_0-\kappa_l)|D_l|^{-1}\left[\sum_{m\neq\, l}\Phi_{\kappa_0}(z_l,z_m)\,Q_mCap_l\right]\int_{\partial D_l}(S^0_{D_l})^{-1}\,(1)\,(s)A_l(s)d\sigma_l(s)\\
&+\sum_{\substack{m=1\\ m\neq l}}^{M}\left[\kappa_0^2\Phi_{\kappa_0}(z_l,z_m)\right]Q_m\\
&+\kappa_0^2|D_l|^{-1}\sum_{\substack{m=1\\ m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)\cdot\Big[\int_{D_l}(x-z_l)dx\Big]Q_m\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{1}{8\pi}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\left(\sum_{m\neq\, l}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\,V_m\right)\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\left(1\right)(s)A_l(s)d\sigma_l(s)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\sum_{m\neq\, l}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\,V_m\\
&+\sum_{\substack{m=1\\ m\neq l}}^{M}\left[\kappa_0^2\nabla_t\Phi_{\kappa_0}(z_l,z_m)\right]\cdot\,V_m\\
=&-\kappa_0^2\,u^I(z_l)+\frac{\rho_0}{\rho_l-\rho_0}\frac{1}{8\pi}(\kappa_0^2-\kappa_l^2)|D_l|^{-1}u^I(z_l)\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}\left(1\right)(s)A_l(s)d\sigma_l(s)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_l^3-\kappa_0^3)Cap_lu^I(z_l)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{32\pi^2}\kappa_0^2(\kappa_l-\kappa_0)|D_l|^{-1}Cap_lu^I(z_l)\int_{\partial D_l}(S^0_{D_l})^{-1}\,(1)\,(s)A_l(s)d\sigma_l(s)+Er_{6,l},
\end{aligned}
$$

where

$$Er_{6,l} = Er_{5,l} + \left(1 - \frac{\rho_l}{\rho_0}\right)^{-1} \frac{1}{8\pi}(\kappa_l^2 - \kappa_0^2)|D_l|^{-1} \int_{\partial D_l} Err6_l \cdot A_l(s) d\sigma_l(s)$$
$$+ \left(1 - \frac{\rho_l}{\rho_0}\right)^{-1} \frac{1}{8\pi}\kappa_0^2|D_l|^{-1} \int_{\partial D_l} Err9_l \cdot A_l(s) d\sigma_l(s) - \left(1 - \frac{\rho_l}{\rho_0}\right)^{-1} \frac{i}{4\pi}(\kappa_l^3 - \kappa_0^3) Err7_l$$
$$= O\left(a + \sum_{m \neq l} \frac{a^3}{d_{ml}^3}\|\phi_m\| + a^2\|\phi_l\| + a^2\|\psi_l\|\right) + O\left(a + a^2\|\phi_l\| + \sum_{m \neq l} \frac{a^3}{d_{ml}^3}\|\phi_m\|\right)$$
$$+ O\left(a^2 + a^2\|\phi_l\| + \sum_{m \neq l} \frac{a^3}{d_{ml}^2}\|\phi_m\|\right) + O\left(a^2 + a^3\|\phi_l\| + \sum_{m \neq l} \frac{a^4}{d_{ml}^3}\|\phi_m\|\right)$$
$$= O\left(a + \sum_{m \neq l} \frac{a^3}{d_{ml}^3}\|\phi_m\| + a^2\|\phi_l\| + a^2\|\psi_l\|\right). \tag{2.31}$$

In the following lemma, we state some identities concerning the integrals with A_l in the integrand.

Lemma 2.5. *The following identities hold true.*

$$\int_{\partial D_l} [\left(S_{D_l}^0\right)^{-1}(1)](s) A_l(s)\, d\sigma_l(s) = -8\pi|D_l|, \tag{2.32}$$

$$\int_{\partial D_l} [\left(S_{D_l}^0\right)^{-1}(\cdot - z_l)](s) A_l(s)\, d\sigma_l(s) = -8\pi \int_{D_l} (x - z_l)\, dx. \tag{2.33}$$

Proof. Observe that,

$$\int_{\partial D_l} [\left(S_{D_l}^0\right)^{-1}(1)](s) A_l(s)\, d\sigma_l(s) = -2\int_{D_l}\int_{\partial D_l} [\left(S_{D_l}^0\right)^{-1}(1)](s) \frac{1}{|s-y|}\, d\sigma_l(s)\, dy$$
$$= -8\pi \int_{D_l}\int_{\partial D_l} [\left(S_{D_l}^0\right)^{-1}(1)](s)\Phi_0(s,y)\, d\sigma_l(s)\, dy$$
$$= -8\pi \int_{D_l} S_{D_l}^0 \left(\left(S_{D_l}^0\right)^{-1}(1)\right)(y)\, dy.$$

Now, by denoting $f := S_{D_l}^0\left(\left(S_{D_l}^0\right)^{-1}(1)\right)$, we can observe that $\Delta f = 0$ in D_l and $f = 1$ on ∂D_l. Since the boundary integral equation has a unique solution $f = 1$ in $\bar{D}$, we can conclude that

$$\int_{\partial D_l} [\left(S_{D_l}^0\right)^{-1}(1)](s) A_l(s)\, d\sigma_l(s) = -8\pi|D_l|.$$

Similarly, we can prove (2.33). □

Let us denote the average of A_l as

$$\hat{A}_l := \frac{1}{|\partial D_l|}\int_{\partial D_l} A_l(s)\ d\sigma_l(s).$$

It is easy to see from the definitions that both A_l and $\hat{A}_l$ scale as $O(a^3)$ in L^2 norm and therefore

$$\|A_l - \hat{A}_l\|_{L^2(\partial D_l)} = O(a^3). \tag{2.34}$$

Remark 2.6. While dealing with the difference of A_l and $\hat{A}_l$, one is tempted to believe that the difference behaves better than A_l and such a result might be possible to derive using Poincaré type inequalities. But such a result is not true in this case. Indeed from the definition of A_l, we note that

$$A_l(s) = -2\int_{D_l} \frac{1}{|s-y|} dy. \tag{2.35}$$

Now, using the Poincaré type inequality mentioned in [2, Proposition 3.2], we can conclude that there exists a positive constant C_A independent of a such that

$$\begin{aligned}\|A-\hat{A}_l\|^2_{L^2(\partial D_l)} &\leq C_A\, a\int_{D_l}|\nabla A_l(x)|^2dx \leq 4C_A\, a\int_{D_l}\left|\int_{D_l}\frac{1}{|x-y|^2}dy\right|^2dx \\ &\leq 4C_A\, a^6\int_{B_l}\left|\int_{B_l}\frac{1}{|\xi-\eta|^2}d\xi\right|^2d\eta = \tilde{C}_A a^6,\end{aligned} \tag{2.36}$$

where

$$\tilde{C}_A := 4C_A\int_{B_l}\left|\int_{B_l}\frac{1}{|\xi-\eta|^2}d\xi\right|^2d\eta.$$

□

Now by making use of (2.32) and splitting A_l as $\hat{A}_l + (A_l - \hat{A}_l)$, we can rewrite (2.30) as

(2.37)

$$\begin{aligned}
&\Bigg[|D_l|^{-1}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]\hat{A}_l\right)+\frac{i}{4\pi}\kappa_0^3+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{1}{16\pi^2}(\kappa_l^3-\kappa_0^3)(\kappa_0-\kappa_l)Cap_l-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0-\kappa_l)\kappa_0^2\Bigg]Q_l\\
&+\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\int_{\partial D_l}\left(A_l(s)-\hat{A}_l(s)\right)\phi_l(s)d\sigma_l(s)\\
&+\Bigg[\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\\
&-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\kappa_0^2(\kappa_0-\kappa_l)Cap_l+\kappa_0^2\Bigg]\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m\\
&+\Bigg[\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(\cdot-z_l)(s)d\sigma_l(s)-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\\
&+\kappa_0^2|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\Bigg]\cdot\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m\\
&+\left[\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\right]\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m\\
=&\left[-\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_0^2-\kappa_l^2)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_l^3-\kappa_0^3)Cap_l-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\kappa_0^2(\kappa_l-\kappa_0)Cap_l\right]u^I(z_l)+Er_{6,l}.
\end{aligned}$$

To proceed further, we shall use the following result describing the structure of V_m. Let $\overline{z}_l := \frac{1}{|\partial D_l|}\int_{\partial D_l} s\, d\sigma_l(s)$.

Proposition 2.7. *The term V_m can be written as*

$$V_m := V_m^{dom} + V_m^{rem},$$

where

(2.38)

$$V_m^{dom} := \sum_{\substack{n=1\\n\neq m}}^{M}(\overline{z}_m - z_m)\Big(\lambda_m - \frac{1}{2}\Big)^{-1}\kappa_0^2|D_m|\Phi_{\kappa_0}(z_m,z_n)Q_n - \frac{1}{8\pi}\kappa_0^2(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m Q_m$$

$$- \frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m Q_m$$

$$- \frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m Cap_m \sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n$$

$$- \frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\int_{\partial D_m}(A_m(s) - \hat{A}_m)\Big(\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\Big)(S_{D_m}^0)^{-1}(s)d\sigma_m(s),$$

and V_m^{rem} *satisfies the estimate*

$$|V_m^{rem}| = O(a^{3-\gamma}) + O\Big(\Big(a^{4-\gamma} + \frac{a^{5-\gamma}}{d^2} + \frac{a^{5-\gamma}}{d^{3\alpha}} + \frac{a^4}{d^2} + \frac{a^4}{d^{3\alpha}} + \frac{a^{6-\gamma}}{d^3} + \frac{a^{6-\gamma}}{d^{3\alpha+1}}\Big)\|\phi\|\Big). \tag{2.39}$$

Proof. We refer to section 3.6 for a proof of this result. □

Remark 2.8. We shall also sometimes, for the sake of the analysis, split the term V_m^{dom} into $V_{m,1}^{dom}$ and $V_{m,2}^{dom}$, where

(2.40)

$$V_{m,1}^{dom} := -\frac{1}{8\pi}\kappa_0^2(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m Q_m - \frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m Q_m,$$

and

(2.41)

$$V_{m,2}^{dom} := \sum_{\substack{n=1\\n\neq m}}^{M}(\overline{z}_m - z_m)\Big(\lambda_m - \frac{1}{2}\Big)^{-1}\kappa_0^2|D_m|\Phi_{\kappa_0}(z_m,z_n)Q_n$$

$$- \frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m Cap_m \sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n$$

$$- \frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\int_{\partial D_m}(A_m(s) - \hat{A}_m)\Big(\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\Big)(S_{D_m}^0)^{-1}(s)d\sigma_m(s).$$

Note that unlike the terms in $V_{m,1}^{dom}$, the terms in $V_{m,2}^{dom}$ contain a summation and this makes a difference in the manner we deal with these terms and therefore we chose to distinguish between them. □

Next we need to deal with the quantities V_m and link them to Q_m to derive a closed system involving only Q_m. For this, we use the splitting of V_m into its dominant and remainder parts, V_m^{dom} and V_m^{rem} respectively, to obtain

(2.42)

$$\Bigg[|D_l|^{-1}\left(\frac{\rho_l}{\rho_l - \rho_0} + \frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l - \rho_0}(\kappa_l^2 - \kappa_0^2) - \kappa_0^2\right]\hat{A}_l\right) + \frac{i}{4\pi}\kappa_0^3 + \frac{\rho_0}{\rho_l - \rho_0}\frac{i}{4\pi}(\kappa_0^3 - \kappa_l^3)$$

$$+ \frac{\rho_0}{\rho_l - \rho_0}\frac{1}{16\pi^2}(\kappa_l^3 - \kappa_0^3)(\kappa_0 - \kappa_l)Cap_l - \frac{\rho_0}{\rho_l - \rho_0}\frac{i}{4\pi}(\kappa_l^2 - \kappa_0^2)(\kappa_0 - \kappa_l) - \frac{\rho_0}{\rho_l - \rho_0}\frac{i}{4\pi}(\kappa_0 - \kappa_l)\kappa_0^2\Bigg]\,Q_l$$

$$
\begin{aligned}
&+\Bigg[\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\\
&-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\kappa_0^2(\kappa_0-\kappa_l)Cap_l+\kappa_0^2\Bigg]\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m\\
&+\Bigg[\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(\cdot-z_l)(s)d\sigma_l(s)-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\\
&+\kappa_0^2|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\Bigg]\cdot\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m\\
&+\left[\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\right]\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m^{dom}\\
=&\left[-\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_0^2-\kappa_l^2)\right]u^I(z_l)-\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\int_{\partial D_l}\left(A_l(s)-\hat{A}_l(s)\right)\phi_l(s)d\sigma_l(s)\\
&+O\left((\kappa_0-\kappa_l)a\right)+O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{1}{d_{ml}^2}|V_m^{rem}|\right)+Er_{6,l}.
\end{aligned}
$$

We next have to deal with the term $\int_{\partial D_l}\left(A_l(s)-\hat{A}_l(s)\right)\phi_l(s)d\sigma_l(s)$ in the above identity. The following lemma provides an approximation of this term in terms of Q_l.

Proposition 2.9. *We have the following estimate*

(2.43)

$$
\begin{aligned}
\int_{\partial D_l}\left(A_l(s)-\hat{A}_l\right)\phi_l\ d\sigma_l(s)=&-R_l\cdot\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_s\Phi_{\kappa_0}(z_l,z_m)Q_m\\
&-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0-\kappa_l)\big[Q_l+\sum_{m\neq l}Cap_l\Phi_{\kappa_0}(z_l,z_m)Q_m\big]\left(8\pi|D_l|+\hat{A}_lCap_l\right)\\
&+O\left(a^4+a^3\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^5\|\phi_l\|+a^5\|\psi_l\|\right),
\end{aligned}
$$

where $R_l:=\int_{\partial D_l}\Big[\left(\lambda_l Id+K^0_{D_l}\right)^{-1}(A_l(\cdot)-\hat{A}_l)\Big](s)\nu_l(s)\,d\sigma_l(s)$ *and* $\lambda_l:=\frac{1}{2}\frac{\rho_0+\rho_l}{\rho_0-\rho_l}$.

Proof. We defer the proof of this result to section 3.5. □

Making use of Proposition 2.9 we can simplify (2.42) as below.

(2.44)

$$
\begin{aligned}
&\Bigg[|D_l|^{-1}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]\hat{A}_l\right)+\frac{i}{4\pi}\kappa_0^3+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\\
&+\frac{\rho_0}{\rho_l-\rho_0}\frac{1}{16\pi^2}(\kappa_l^3-\kappa_0^3)(\kappa_0-\kappa_l)Cap_l-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0-\kappa_l)\kappa_0^2\Bigg]Q_l
\end{aligned}
$$

$$+\left[\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)\right.$$
$$\left.-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\kappa_0^2(\kappa_0-\kappa_l)Cap_l+\kappa_0^2\right]\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m$$
$$+\left[\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(\cdot-z_l)(s)d\sigma_l(s)-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\right.$$
$$\left.+\kappa_0^2|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\right]\cdot\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m$$
$$+\left[\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\right]\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m^{dom}$$
$$=\left[-\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_0^2-\kappa_l^2)\right]u^I(z_l)+O\left((\kappa_0-\kappa_l)a\right)+O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{1}{d_{ml}^2}|V_m^{rem}|\right)+Er_{6,l}$$
$$-\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\left[-R_l\cdot\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_s\Phi_{\kappa_0}(z_l,z_m)Q_m\right.$$
$$-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0-\kappa_l)\big[Q_l+\sum_{m\neq l}Cap_l\Phi_{\kappa_0}(z_l,z_m)Q_m\big]\Big(8\pi|D_l|+\hat{A}_lCap_l\Big)$$
$$\left.+O\left(a^4+a^3\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^5\|\phi_l\|+a^5\|\psi_l\|\right)\right].$$

We can rewrite it further as, by making use of (2.8) and the expression for V_m^{dom},

(2.45)

$$\left[|D_l|^{-1}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]\hat{A}_l\right)+\frac{i}{4\pi}\kappa_0^3+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\right.$$
$$-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\Big[(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)+(\kappa_0-\kappa_l)\kappa_0^2\Big]$$
$$\left.-\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\frac{\rho_0}{\rho_l-\rho_0}\Big(8\pi|D_l|+\hat{A}_lCap_l\Big)\right]Q_l$$
$$+\left[\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\kappa_0^2(\kappa_0-\kappa_l)Cap_l\right.$$
$$\left.-\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\frac{\rho_0}{\rho_l-\rho_0}\Big(8\pi|D_l|+\hat{A}_lCap_l\Big)Cap_l\right]\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m$$
$$+\left[-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]+\kappa_0^2|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\right.$$

$$
\begin{aligned}
&-\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}R_l\right]\cdot\sum_{\substack{m=1\\m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m\\
&+\left[\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\right]\kappa_0^2\sum_{\substack{m=1\\m\neq l}}^{M}(\overline{z}_m-z_m)(\lambda_m-\frac{1}{2})^{-1}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\\
&\Big[-\frac{1}{8\pi}\hat{A}_mQ_m-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\hat{A}_mQ_m+|D_m|\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\\
&-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\hat{A}_mCap_m\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\\
&-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\Big(\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{k_0}(z_m,z_n)Q_n\Big)\int_{\partial D_m}(A_m(s)-\hat{A}_m)(S^0_{D_m})^{-1}(1)(s)\,d\sigma_m(s)\Big]\\
=&\left[-\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_0^2-\kappa_l^2)\right]u^I(z_l)\\
&+O\left((\kappa_0-\kappa_l)a\right)+O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{1}{d_{ml}^2}|V_m^{rem}|\right)+Er_{6,l}+O\left(a+\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^2\|\phi_l\|+a^2\|\psi_l\|\right)\\
=&-\kappa_l^2u^I(z_l)+O\left((\kappa_0-\kappa_l)a\right)+O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{1}{d_{ml}^2}|V_m^{rem}|\right)\\
&+Er_{6,l}+O\left(a+\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\|+a^2\|\phi_l\|+a^2\|\psi_l\|\right)+O(\rho_l(k_0-k_l)),
\end{aligned}
$$

where to get the last identity, we use the fact that

$$
\frac{\rho_0}{\rho_l-\rho_0}(\kappa_0^2-\kappa_l^2)\;=\;\left(1-\frac{\rho_l}{\rho_0}\right)^{-1}(\kappa_l^2-\kappa_0^2)\;=\;\kappa_l^2-\kappa_0^2+O\left(\rho_l(\kappa_0-\kappa_l)\right). \tag{2.46}
$$

Let us write

(2.47)

$$
\begin{aligned}
J_l:=&\left[\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)\left(1-\frac{i\kappa_l}{4\pi}Cap_l\right)+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\kappa_0^2(\kappa_0-\kappa_l)Cap_l\right.\\
&\left.-\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\frac{\rho_0}{\rho_l-\rho_0}\left(8\pi|D_l|+\hat{A}_lCap_l\right)Cap_l\right]\\
=&\ \kappa_l^2+O\left(\rho_l(\kappa_0-\kappa_l)\right)+iO\left((\kappa_0-\kappa_l)a\right),\ \text{(using (2.46))},
\end{aligned}
$$

where using (2.46), it follows that the real part J_l^r of J_l satisfies

$$
J_l^r=\kappa_0^2-\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)=\kappa_l^2+\frac{\rho_l}{\rho_0}(k_l^2-k_0^2)+O\left(\rho_l^2(\kappa_0-\kappa_l)\right).
$$

Let

(2.48)
$$\begin{aligned}
I_l :=& \Bigg[|D_l|^{-1}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]\hat{A}_l\right)+\frac{i}{4\pi}\kappa_0^3+\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\\
&-\frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}\Big[(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)+(\kappa_0-\kappa_l)\kappa_0^2\Big]\\
&-\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\frac{\rho_0}{\rho_l-\rho_0}\Big(8\pi|D_l|+\hat{A}_lCap_l\Big)\Bigg]\\
=& |D_l|^{-1}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[-\sum_{n=0}^{\infty}\left(\frac{\rho_l}{\rho_0}\right)^n(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]\hat{A}_l\right)+\frac{i}{4\pi}\kappa_0^3\\
&-\left[1+\sum_{n=1}^{\infty}\left(\frac{\rho_l}{\rho_0}\right)^n\right]\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)+\left[1+\sum_{n=1}^{\infty}\left(\frac{\rho_l}{\rho_0}\right)^n\right]\frac{i}{4\pi}\Big[(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)+(\kappa_0-\kappa_l)\kappa_0^2\Big]\\
&-\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\left[\left[1+\sum_{n=1}^{\infty}\left(\frac{\rho_l}{\rho_0}\right)^n\right](\kappa_l^2-\kappa_0^2)+\kappa_0^2\right]|D_l|^{-1}\left[1+\sum_{n=1}^{\infty}\left(\frac{\rho_l}{\rho_0}\right)^n\right]\Big(8\pi|D_l|+\hat{A}_lCap_l\Big)\\
=& I_l'+O\left(a^{-1}(\kappa_0-\kappa_l)\rho_l^2\right)+iO\left((\kappa_0-\kappa_l)\rho_l\right),
\end{aligned}$$

where

$$\begin{aligned}
I_l' :=& |D_l|^{-1}\left(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[-\kappa_l^2-\frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)\right]\hat{A}_l\right)+\frac{i}{4\pi}\kappa_0^3-\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)\\
&+\frac{i}{4\pi}\Big[(\kappa_l^2-\kappa_0^2)(\kappa_0-\kappa_l)+(\kappa_0-\kappa_l)\kappa_0^2\Big]-\frac{i}{32\pi^2}(\kappa_0-\kappa_l)\kappa_l^2|D_l|^{-1}(8\pi|D_l|+\hat{A}_lCap_l)\\
=& |D_l|^{-1}\Big(\frac{\rho_l}{\rho_l-\rho_0}+\frac{1}{8\pi}\left[-\kappa_l^2-\frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)\right]\hat{A}_l\Big)+i\Big[\frac{\kappa_l^3}{4\pi}-\frac{1}{32\pi^2}(\kappa_0-\kappa_l)\kappa_l^2|D_l|^{-1}\hat{A}_lCap_l\Big],
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
F_l :=& -\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]+\kappa_0^2|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]\\
&-\frac{1}{8\pi}\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}R_l\\
=& F_l'+O((\kappa_0-\kappa_l)\rho_la) \qquad \text{(using (2.46))},
\end{aligned} \tag{2.50}$$

where

$$F_l' := \kappa_l^2|D_l|^{-1}\Big[\int_{D_l}(x-z_l)dx\Big]+\frac{\kappa_l^2}{8\pi}|D_l|^{-1}R_l. \tag{2.51}$$

We can observe that J_l is not scaling. Also, let us denote the dominating term of I_l' by

$$I_l'^{d} := |D_l|^{-1}\Big(\frac{\rho_l}{\rho_l-\rho_0}-\frac{1}{8\pi}\kappa_l^2\hat{A}_l\Big),$$

and the remaining terms (of $O(1)$) by

$$I_l'^{s} := -\frac{1}{8\pi}\frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)\hat{A}_l|D_l|^{-1}+i\Big[\frac{\kappa_l^3}{4\pi}-\frac{1}{32\pi^2}(\kappa_0-\kappa_l)\kappa_l^2|D_l|^{-1}\hat{A}_lCap_l\Big].$$

Then

$$\begin{aligned}
\frac{1}{J_l} &= \left[J_l^r + iJ_l^i\right]^{-1} = \frac{1}{(J_l^r)^2}\left[J_l^r - iJ_l^i\right]\sum_{n=0}^{\infty}(-1)^n\left(\frac{J_l^i}{J_l^r}\right)^{2n} \\
&= \frac{1}{J_l^r}\left[1 - i\frac{J_l^i}{J_l^r}\right]\sum_{n=0}^{\infty}(-1)^n\left(\frac{J_l^i}{J_l^r}\right)^{2n} \\
&= \frac{1}{J_l^r} + O((\kappa_0-\kappa_l)a) \qquad [\text{since } J_l^i = O((\kappa_0-\kappa_l)a)] \\
&= \frac{1}{\kappa_l^2 + O((\kappa_0-\kappa_l)\rho_l)} + O((\kappa_0-\kappa_l)a) = \frac{1}{\kappa_l^2} + O((\kappa_0-\kappa_l)\rho_l) + O((\kappa_0-\kappa_l)a) \\
&= \frac{1}{\kappa_l^2} + O((\kappa_0-\kappa_l)a),
\end{aligned} \tag{2.52}$$

and hence

$$\frac{F_l'}{J_l} = \frac{F_l'}{\kappa_l^2} + O(a^2). \tag{2.53}$$

Also note that using (2.46), we have

$$\begin{aligned}
J_l^i &= \frac{\rho_0}{\rho_l-\rho_0}\Bigg[\frac{1}{4\pi}\kappa_l Cap_l(\kappa_l^2-\kappa_0^2) + \frac{1}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l - \frac{1}{4\pi}\kappa_0^2(\kappa_0-\kappa_l)Cap_l \\
&\qquad - \frac{1}{32\pi^2}(\kappa_0-\kappa_l)\left[\frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2)-\kappa_0^2\right]|D_l|^{-1}\left(8\pi|D_l| + \hat{A}_l Cap_l\right)Cap_l\Bigg] \\
&= \frac{\rho_0}{\rho_l-\rho_0}\Big[\frac{1}{4\pi}(\kappa_0-\kappa_l)\kappa_l^2 Cap_l + \frac{1}{32\pi^2}(\kappa_0-\kappa_l)\kappa_l^2\hat{A}_l|D_l|^{-1}Cap_l^2\Big] + O(a\rho_l(\kappa_0-\kappa_l)) \\
&= J_l^{i'} + O((\kappa_0-\kappa_l)a\rho_l),
\end{aligned} \tag{2.54}$$

where

$$J_l^{i'} := \frac{\rho_0}{\rho_l-\rho_0}\frac{1}{4\pi}(\kappa_0-\kappa_l)\kappa_l^2\Big[1 + \frac{1}{8\pi}\hat{A}_l|D_l|^{-1}Cap_l\Big]Cap_l. \tag{2.55}$$

Therefore, since $J_l^i = O((\kappa_0-\kappa_l)a))$, we can write

$$\begin{aligned}
\frac{I_l'}{J_l} &= I_l'\left[J_l^r + iJ_l^i\right]^{-1} = \frac{I_l'}{(J_l^r)^2}\left[J_l^r - iJ_l^i\right]\sum_{n=0}^{\infty}(-1)^n\left(\frac{J_l^i}{J_l^r}\right)^{2n} \\
&= \frac{I_l'^d + I_l'^s}{J_l^r}\left[1 - i\frac{J_l^i}{J_l^r}\right]\sum_{n=0}^{\infty}(-1)^n\left(\frac{J_l^i}{J_l^r}\right)^{2n} \\
&= \frac{1}{[\kappa_l^2 + \frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)]}\Big[I_l'^d - iI_l'^d\frac{J_l^{i'}}{[\kappa_l^2 + \frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)]}\Big] + \frac{1}{\kappa_l^2}\Big[I_l'^s - I_l'^d\frac{(J_l^{i'})^2}{\kappa_l^4}\Big] + O((\kappa_0-\kappa_l)a),
\end{aligned} \tag{2.56}$$

Using (2.46-2.50), we can rewrite (2.45) as

(2.57)

$$\frac{I'_l}{J_l}Q_l + \sum_{\substack{m=1\\ m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m + \frac{F'_l}{J_l}\cdot\sum_{\substack{m=1\\ m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m$$
$$+\frac{1}{J_l}\left[\kappa_0^2 - \frac{\rho_0}{\rho_l-\rho_0}(\kappa_l^2-\kappa_0^2) + \frac{\rho_0}{\rho_l-\rho_0}\frac{i}{4\pi}(\kappa_0^3-\kappa_l^3)Cap_l\right]\kappa_0^2\sum_{\substack{m=1\\ m\neq l}}^{M}(\overline{z}_m - z_m)(\lambda_m-\frac{1}{2})^{-1}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot$$
$$\Big[-\frac{1}{8\pi}\hat{A}_mQ_m - \frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\hat{A}_mQ_m + |D_m|\sum_{\substack{n=1\\ n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n$$
$$-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\hat{A}_mCap_m\sum_{\substack{n=1\\ n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n$$
$$-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\Big(\sum_{\substack{n=1\\ n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\Big)\int_{\partial D_m}(A_m(s)-\hat{A}_m)(S^0_{D_m})^{-1}(1)(s)\,d\sigma_m(s)\Big]$$
$$= -\frac{\kappa_l^2}{J_l}u^I(z_l) + \frac{1}{J_l}\Big[Er_6 + O(a) + O(a^2\|\phi_l\|) + \sum_{\substack{m=1\\ m\neq l}}^{M}O\left(\frac{a^3}{d_{ml}^3}\|\phi_m\|\right) + O(a^2\|\psi_l\|) + O\left(\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{1}{d_{ml}^2}|V_m^{rem}|\right)$$
$$+O((\kappa_0-\kappa_l)\rho_l) + O((\kappa_0-\kappa_l)\rho_l^2\|\phi_l\|) + O((\kappa_0-\kappa_l)a\rho_l\|\phi_l\|) + O((\kappa_0-\kappa_l)a^2\rho_l(\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{1}{d_{ml}^2}\|\phi_m\|))\Big].$$

Now let us assume that $\rho_l \simeq a^{1+\gamma}$, with $\gamma \geq 0$. Then from (2.39), we can deduce that

(2.58)
$$\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{1}{d_{ml}^2}|V_m^{rem}| = O\Big(\frac{a^{3-\gamma}}{d^2} + \frac{a^{3-\gamma}}{d^{3\alpha}}\Big)$$
$$+O\Big(\Big(\frac{a^{4-\gamma}}{d^2} + \frac{a^{4-\gamma}}{d^{3\alpha}} + \frac{a^4}{d^4} + \frac{a^4}{d^{2+3\alpha}} + \frac{a^4}{d^{6\alpha}} + \frac{a^{5-\gamma}}{d^4} + \frac{a^{5-\gamma}}{d^{2+3\alpha}} + \frac{a^{5-\gamma}}{d^{6\alpha}}$$
$$+\frac{a^{6-\gamma}}{d^5} + \frac{a^{6-\gamma}}{d^{3+3\alpha}} + \frac{a^{6-\gamma}}{d^{6\alpha+1}}\Big)\|\phi\|\Big).$$

Let us define $\mathbf{C}_l^{-1} := \frac{1}{[\kappa_l^2+\frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)]}\Big[{I'_l}^d - i{I'_l}^d\frac{{J_l^i}'}{[\kappa_l^2+\frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)]}\Big] + \frac{1}{\kappa_l^2}\Big[{I'_l}^s - {I'_l}^d\frac{({J_l^i}')^2}{\kappa_l^4}\Big]$. Then, using (2.46),(2.52) and (2.58), we can rewrite (2.57) as

(2.59)

$$\mathbf{C}_l^{-1}Q_l + \sum_{\substack{m=1\\ m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m + \frac{F'_l}{\kappa_l^2}\cdot\sum_{\substack{m=1\\ m\neq l}}^{M}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m$$
$$+\kappa_0^2\sum_{\substack{m=1\\ m\neq l}}^{M}(\overline{z}_m - z_m)(\lambda_m-\frac{1}{2})^{-1}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\Big[-\frac{1}{4\pi}\hat{A}_mQ_m + |D_m|\sum_{\substack{n=1\\ n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n$$
$$-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\Big(\sum_{\substack{n=1\\ n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\Big)\int_{\partial D_m}A_m(s)(S^0_{D_m})^{-1}(1)(s)\,d\sigma_m(s)\Big]$$

$$= -u^I(z_l) + O\Big(a + \frac{a^{3-\gamma}}{d^2} + \frac{a^{3-\gamma}}{d^{3\alpha}} + \Big(a^2 + \frac{a^3}{d^3} + \frac{a^3}{d^{3\alpha+1}} + \frac{a^{4-\gamma}}{d^2} + \frac{a^{4-\gamma}}{d^{3\alpha}} + \frac{a^4}{d^4} + \frac{a^4}{d^{2+3\alpha}} + \frac{a^4}{d^{6\alpha}} + \frac{a^{5-\gamma}}{d^4} + \frac{a^{5-\gamma}}{d^{2+3\alpha}} + \frac{a^{5-\gamma}}{d^{6\alpha}} + \frac{a^{6-\gamma}}{d^5} + \frac{a^{6-\gamma}}{d^{3+3\alpha}} + \frac{a^{6-\gamma}}{d^{6\alpha+1}}\Big)\|\phi\|\Big).$$

Remark 2.10. We note that the dominant part of $\mathbf{C}_l^{-1}$ is given by $\frac{{I'}_l^d}{\kappa_l^2} = \frac{|D_l|^{-1}}{\kappa_l^2}\left[\frac{\rho_l}{\rho_l-\rho_0} - \frac{1}{8\pi}\kappa_l^2\hat{A}_l\right]$. When $\gamma < 1$ or the frequency is away from the resonance, the sign of the real part $(\mathbf{C}_l^{-1})^r$ of $\mathbf{C}_l^{-1}$ is given by the sign of the term $\frac{|D_l|^{-1}}{\kappa_l^2}\cdot\frac{\rho_l}{\rho_l-\rho_0}$ which is negative. Therefore in this case, $(\mathbf{C}_l^{-1})^r < 0,\ \forall\, l = 1,\dots,M$. When the frequency ω is near the resonance, we can write $1 - \left(\frac{\omega_M}{\omega}\right)^2 = l_M a^{h_1}$. In this case, we can write

$$\frac{{I'}_l^d}{\kappa_l^2} = \frac{\hat{A}_l|D_l|^{-1}\frac{\rho_l}{k_l}}{8\pi}\left[\frac{\omega_M^2}{\omega^2} - 1\right] = \frac{\hat{A}_l|D_l|^{-1}\frac{\rho_l}{k_l}}{8\pi}\Big[-l_M a^{h_1}\Big].$$

Therefore if $l_M > 0$, $(\mathbf{C}_l^{-1})^r > 0,\ \forall\, l = 1,\dots,M$ and if $l_M < 0$, $(\mathbf{C}_l^{-1})^r < 0,\ \forall\, l = 1,\dots,M$. □

Lemma 2.11. *For $m = 1,\cdots,M$, let us define*

$$Y_m := -u^I(z_m) + O\Big(a + \frac{a^{3-\gamma}}{d^2} + \frac{a^{3-\gamma}}{d^{3\alpha}} + \Big(a^2 + \frac{a^3}{d^3} + \frac{a^3}{d^{3\alpha+1}} + \frac{a^{4-\gamma}}{d^2} + \frac{a^{4-\gamma}}{d^{3\alpha}} + \frac{a^4}{d^4} + \frac{a^4}{d^{2+3\alpha}} + \frac{a^4}{d^{6\alpha}} + \frac{a^{5-\gamma}}{d^4} + \frac{a^{5-\gamma}}{d^{2+3\alpha}} + \frac{a^{5-\gamma}}{d^{6\alpha}} + \frac{a^{6-\gamma}}{d^5} + \frac{a^{6-\gamma}}{d^{3+3\alpha}} + \frac{a^{6-\gamma}}{d^{6\alpha+1}}\Big)\|\phi\|\Big).$$

Then the algebraic system (2.59) is invertible provided

- *in the case when $(\mathbf{C}_l^{-1})^r > 0,\ \forall\, l = 1,\dots,M$,*

$$\frac{\min\limits_{1\le m\le M}\mathbf{C}_m^r}{(\max\limits_{1\le m\le M}|\mathbf{C}_m|)^2} \ge \frac{3\tau}{5\pi\, d} + \max_{1\le m\le M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12} + Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac12}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12}, \tag{2.60}$$

- *in the case when $(\mathbf{C}_l^{-1})^r < 0,\ \forall\, l = 1,\dots,M$,*

$$\frac{\min\limits_{1\le m\le M}|\mathbf{C}_m^r|}{(\max\limits_{1\le m\le M}|\mathbf{C}_m|)^2} \ge C\sqrt{MM_{max}}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac12} + \max_{1\le m\le M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12} + Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac12}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12}, \tag{2.61}$$

and the solution vector $Q_m,\ m = 1,\dots,M$, satisfies either the estimate

$$\sum_{m=1}^M |Q_m|^2 \le 4\,(\max_{1\le m\le M}|\mathbf{C}_m|)^2 \sum_{m=1}^M |Y_m|^2 \Big(\frac{\min\limits_{1\le m\le M}|\mathbf{C}_m^r|}{\max\limits_{1\le m\le M}|\mathbf{C}_m|} - \Big(\frac{3\tau}{5\pi\, d} + \max_{1\le m\le M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12} + Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac12}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac12}\Big)\max_{1\le m\le M}|\mathbf{C}_m|\Big)^{-2}, \tag{2.62}$$

where $\tau := \min_{l\neq m}\cos(\kappa|z_m - z_l|) \geq 0$, *or the estimate*

(2.63)

$$\sum_{m=1}^{M}|Q_m|^2 \leq 4\,(\max_{1\leq m\leq M}|\mathbf{C}_m|)^2\sum_{m=1}^{M}|Y_m|^2$$
$$\Big(\frac{\min\limits_{1\leq m\leq M}|\mathbf{C}^r_m|}{\max\limits_{1\leq m\leq M}|\mathbf{C}_m|} - \Big(C\sqrt{MM_{max}}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}} + \max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}$$
$$+ Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\Big)\max_{1\leq m\leq M}|\mathbf{C}_m|\Big)^{-2},$$

where C *is a constant (depending on the Lipschitz characters of the obstacles) uniformly bounded with respect to* a.

Proof. We refer to section 3.7 for a proof of this lemma. □

Remark 2.12. From (2.62), we conclude that

(2.64)

$$\sum_{m=1}^{M}|Q_m| \leq 2M\,(\max_{1\leq m\leq M}|\mathbf{C}_m|)\max_{1\leq m\leq M}|Y_m|$$
$$\Big(\frac{\min\limits_{1\leq m\leq M}|\mathbf{C}^r_m|}{\max\limits_{1\leq m\leq M}|\mathbf{C}_m|} - \Big(\frac{3\tau}{5\pi\, d} + \max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}$$
$$+ Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\Big)\max_{1\leq m\leq M}|\mathbf{C}_m|\Big)^{-1},$$

while (2.63) yields

(2.65)

$$\sum_{m=1}^{M}|Q_m| \leq 2M\,(\max_{1\leq m\leq M}|\mathbf{C}_m|)\max_{1\leq m\leq M}|Y_m|$$
$$\Big(\frac{\min\limits_{1\leq m\leq M}|\mathbf{C}^r_m|}{\max\limits_{1\leq m\leq M}|\mathbf{C}_m|} - \Big(C\sqrt{MM_{max}}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}} + \max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}$$
$$+ Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\Big)\max_{1\leq m\leq M}|\mathbf{C}_m|\Big)^{-1}.$$

□

Remark 2.13. We note that if

$$(\max_{1\leq m\leq M}|\mathbf{C}_m|)\Big[\frac{3\tau}{5\pi d} + \max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}$$
$$+ Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}} + Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\Big] = O(1),$$

then the condition (2.60) holds. Since in this case the frequency is near the resonance ω_M (and $l_M > 0$), this is equivalent to the condition

$$a^{1-h_1}\cdot O(a^{-t} + a^{1-s-t} + a^{2-\gamma-s-t} + a^{3-\gamma-2s-t}) = O(1). \tag{2.66}$$

This leads to the additional conditions $2 - h_1 - s - t \geq 0$ and $1 - t - h_1 \geq 0$ provided $\gamma + s \leq 2$. As in this case $\gamma = 1$ and hence $s \leq 1$, the conditions reduce to $1 - t - h_1 \geq 0$ and $s \leq 1$. Note that (2.66) also

gives rise to the condition $2(a)$ in Theorem 1.2.
Similarly if

$$(\max_{1\leq m\leq M}|\mathbf{C}_m|)\Big[C\sqrt{MM_{max}}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}+\max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right|C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}$$
$$+Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}+Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\Big]=O(1),$$

then the condition (2.61) holds true. When the frequency is away from the resonance ω_M, and as we take $3\alpha t = s$ with $\alpha \in (0,1]$, this is equivalent to the condition

$$a^{2-\gamma}\cdot O(a^{-s}+a^{1-s-t}+a^{2-\gamma-s-t}+a^{3-\gamma-2s-t})=O(1),$$

which holds if $\gamma+s\leq 2$, $\frac{s}{3}\leq t\leq 1$ and $0\leq\gamma\leq 1$. This also gives rise to the condition $1(a)$ in Theorem 1.2.
If the frequency is near the resonance ω_M (and $l_M<0$), and as we take $3\alpha t = s$ with $\alpha\in(0,1]$, then the condition is equivalent to

$$a^{1-h_1}\cdot O(a^{-s}+a^{1-s-t}+a^{2-\gamma-s-t}+a^{3-\gamma-2s-t})=O(1), \tag{2.67}$$

which holds provided $1-h_1-s\geq 0$, $\gamma+s\leq 2, 0\leq t\leq 1, 0\leq\gamma\leq 1$ and $\frac{s}{3}\leq t$. Again as here also $\gamma=1$ and hence $s\leq 1$, these conditions reduce to $1-h_1-s\geq 0$ and $\frac{s}{3}\leq t\leq 1$. Also note that (2.67) gives rise to the condition $1(b)$ in Theorem 1.2.
The condition $2(b)$ follows from the fact that a condition similar to (2.67) can also be derived when the coefficients $\mathbf{C_m}$ are positive. □

From (2.64) or (2.65) and the definition of Y_m, we can derive the a priori estimate

$$\begin{aligned}\sum_{m=1}^{M}|Q_m| &= O\Big(M\max|\mathbf{C}_m|\Big[1+a+\frac{a^{3-\gamma}}{d^2}+\frac{a^{3-\gamma}}{d^{3\alpha}}+\Big(a^2+\frac{a^3}{d^3}+\frac{a^3}{d^{3\alpha+1}}+\frac{a^{4-\gamma}}{d^2}+\frac{a^{4-\gamma}}{d^{3\alpha}}+\frac{a^4}{d^4}+\frac{a^4}{d^{2+3\alpha}}+\frac{a^4}{d^{6\alpha}}\\
&\quad+\frac{a^{5-\gamma}}{d^4}+\frac{a^{5-\gamma}}{d^{2+3\alpha}}+\frac{a^{5-\gamma}}{d^{6\alpha}}+\frac{a^{6-\gamma}}{d^5}+\frac{a^{6-\gamma}}{d^{3+3\alpha}}+\frac{a^{6-\gamma}}{d^{6\alpha+1}}\Big)\|\phi\|\Big]\Big)\\
&= O\Big(M\max|\mathbf{C}_m|\Big[1+a+a^{3-\gamma-2t}+a^{3-\gamma-s}+\Big(a^2+a^{3-3t}+a^{3-s-t}\\
&\quad+a^{4-\gamma-2t}+a^{4-\gamma-s}+a^{4-4t}+a^{4-s-2t}+a^{4-2s}\\
&\quad+a^{5-\gamma-4t}+a^{5-\gamma-s-2t}+a^{5-\gamma-2s}+a^{6-\gamma-5t}+a^{6-\gamma-s-3t}+a^{6-\gamma-2s-t}\Big)\|\phi\|\Big]\Big)\\
&= O\Big(M\max|\mathbf{C}_m|\Big[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\Big]\Big),\end{aligned} \tag{2.68}$$

assuming that $0\leq t<\frac{1}{2}$, $0\leq s\leq\frac{3}{2}$, $0\leq\gamma\leq 1$ and $s+\gamma\leq 2$.
Using (2.68), we also note that

$$\begin{aligned}\frac{F'_l}{\kappa_l^2}\cdot\sum_{m\neq l}\nabla_x\Phi_{\kappa_0}(z_l,z_m)Q_m &= O\Big(\frac{a}{d^2}\sum_{m=1}^{M}|Q_m|\Big)\\
&= O\left(a^{1-2t}M\max|\mathbf{C}_m|\Big[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\Big]\right),\end{aligned} \tag{2.69}$$

$$-\frac{1}{4\pi}\kappa_0^2\sum_{\substack{m=1\\ m\neq l}}^{M}(\overline{z}_m-z_m)(\lambda_m-\frac{1}{2})^{-1}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\hat{A}_mQ_m \tag{2.70}$$

$$= \Big(\frac{a^{2-\gamma}}{d^2}\sum_{m=1}^{M}|Q_m|\Big) = O\left(a^{2-\gamma-2t}M\max|\mathbf{C}_m|\Big[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\Big]\right),$$

and

$$\kappa_0^2\sum_{\substack{m=1\\m\neq l}}^{M}(\overline{z}_m-z_m)(\lambda_m-\frac{1}{2})^{-1}\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot\Big[|D_m|\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n$$
$$-\frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\Big(\sum_{\substack{n=1\\n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\Big)\int_{\partial D_m}A_m(s)(S^0_{D_m})^{-1}(1)(s)\,d\sigma_m(s)\Big]$$
$$= O\left(a^{3-\gamma}\left(\sum_{m\neq l}\frac{1}{d^2_{ml}}\right)\frac{1}{d}\sum_{n=1}^{M}|Q_n|\right) = O\left(a^{3-\gamma}\left(\frac{1}{d^3}+\frac{1}{d^{3\alpha+1}}\right)\sum_{n=1}^{M}|Q_n|\right)$$
$$= O\Big(a^{3-\gamma-s-t}M\max|\mathbf{C}_m|\Big[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\Big]\Big). \tag{2.71}$$

Therefore we can rewrite (2.59) as

$$\mathbf{C}_l^{-1}Q_l+\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m = -u^I(z_l)+O\left(a+a^{3-\gamma-s}+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\right)$$
$$+O\Big((a^{1-2t}+a^{2-\gamma-2t}+a^{3-\gamma-s-t})M\max|\mathbf{C}_m|\Big[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\Big]\Big). \tag{2.72}$$

2.5. **The final approximations.**

Proposition 2.14. *Let the parameters t, s, γ satisfy the conditions $0 \leq t < \frac{1}{2},\ 0 \leq s \leq \frac{3}{2},\ 0 \leq \gamma \leq 1,\ s+\gamma \leq 2,\ \frac{s}{3} \leq t$ and let $M\max|\mathbf{C}_m| = O(a^{-h}),\ h < \frac{1}{2}$. Then for every l, we have*

$$\|\phi_l\| = O(a^{-\gamma}) + O(a^{-\gamma-h}).$$

Proof. We refer to section 3.8 for a proof of this result. □

Remark 2.15. In case $\gamma < 1$ or when the frequency is away from the resonance, we have $\max|\mathbf{C}_m| = O(a^{2-\gamma})$ whence it follows using $s+\gamma \leq 2$ that $M\max|\mathbf{C}_m| = O(a^{2-\gamma-s}) = O(1)$. Therefore in this case, we have $h \leq 0$ and then $\|\phi_l\| = O(a^{-\gamma})$.
In contrast with this case, near the resonance $\max_l|\mathbf{C}_l| = O(a^{1-h_1})$ where $h_1 \geq 0$. Therefore $M\max_l|\mathbf{C}_l| = O(a^{1-s-h_1}) = O(a^{-h})$. Now if $l_M < 0$, we need the condition $1-h_1-s \geq 0$, see Remark 2.13, which implies that $h \leq 0$ and then $\|\phi_l\| = O(a^{-\gamma})$. But if $l_M > 0$, we have only the condition $1-h_1 \geq t$ and $s \leq 1$ and it is possible to have $-1+s+h_1 > 0$ and hence $h > 0$. □

Combining (2.72) and Proposition 2.14, we deduce that under the conditions $0 \leq t < \frac{1}{2},\ 0 \leq s \leq \frac{3}{2},\ 0 \leq \gamma \leq 1, s+\gamma \leq 2$ and $1-2t-h > 0$ with $h < \frac{1}{2}$, [2] the vectors $(Q_l)_{l=1}^{M}$ satisfy

[2]Here, we do the computations when $h \geq 0$ in which case $\|\phi_l\| = O(a^{-\gamma-h})$. For the case $h < 0$, we have $\|\phi_l\| = O(a^{-\gamma})$. In this case, we use the rough estimate for $M\max|\mathbf{C}_m| = O(1)$ (instead of $M\max|\mathbf{C}_m| = O(a^{-h}) \to 0$ as $a \to 0$). Under these considerations, we derive the approximation, for the case $h < 0$, by taking $h = 0$ in all the computations we develop for the case $h > 0$. This is enough for our final approximation as the dominant terms are not affected.

(2.73)

$$
\begin{aligned}
&\mathbf{C}_l^{-1}Q_l + \sum_{\substack{m=1\\ m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m = -u^I(z_l) + O\left(a + a^{2-\gamma-h} + a^{3-\gamma-3t-h} + a^{3-\gamma-s-t-h} + a^{4-\gamma-2s-h}\right)\\
&+ O\Big((a^{1-2t} + a^{2-\gamma-2t} + a^{3-\gamma-s-t})a^{-h}\Big[1 + a + a^{2-\gamma-h} + a^{3-\gamma-h-3t} + a^{3-\gamma-h-s-t} + a^{4-\gamma-h-2s}\Big]\Big)\\
&= -u^I(z_l) + O\Big(a + a^{2-\gamma-h} + a^{3-\gamma-h-3t} + a^{3-\gamma-h-s-t} + a^{4-\gamma-h-2s} + a^{1-2t-h}\Big)\\
&= -u^I(z_l) + O\Big(a^{4-\gamma-h-2s} + a^{1-2t-h}\Big),
\end{aligned}
$$

where

$$
\mathbf{C}_l^{-1} = \frac{1}{[\kappa_l^2 + \frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)]}\Big[I_l'^{d} - iI_l'^{d}\frac{J_l^{i'}}{[\kappa_l^2 + \frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)]}\Big] + \frac{1}{\kappa_l^2}\Big[I_l'^{s} - I_l'^{d}\frac{(J_l^{i'})^2}{\kappa_l^4}\Big] \tag{2.74}
$$

with

$$
I_l' := I_l'^{d} + I_l'^{s} := |D_l|^{-1}\Big(\frac{\rho_l}{\rho_l-\rho_0} + \frac{1}{8\pi}\left[-\kappa_l^2 - \frac{\rho_l}{\rho_0}(\kappa_l^2-\kappa_0^2)\right]\hat{A}_l\Big) + i\Big[\frac{\kappa_l^3}{4\pi} - \frac{1}{32\pi^2}(\kappa_0-\kappa_l)\kappa_l^2|D_l|^{-1}\hat{A}_l Cap_l\Big]
$$

and

$$
J_l^{i'} = \frac{\rho_0}{\rho_l-\rho_0}\frac{1}{4\pi}(\kappa_0-\kappa_l)\kappa_l^2\Big[1 + \frac{1}{8\pi}\hat{A}_l|D_l|^{-1}Cap_l\Big]Cap_l.
$$

We recall that we have the representation of the scattered field of the form $u^s(x,\theta) = \sum_{m=1}^{M} S_{D_m}^{\kappa_0}\phi_m$.

From this we can deduce that the far-field can be approximated as

$$
u^\infty(\hat{x},\theta) = \sum_{m=1}^{M}\Phi_{\kappa_0}^\infty(\hat{x},z_m)Q_m + \sum_{m=1}^{M}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)\cdot V_m + O(Ma^{3-\gamma}).
$$

Now under the assumptions on t, s, γ, h, using (2.39), we see that

$$
\begin{aligned}
\sum_{m=1}^{M}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)\cdot V_m^{rem} &= O(M|V_m^{rem}|)\\
&= O\Big(a^{3-\gamma-s} + a^{4-2\gamma-s-h} + a^{5-2\gamma-s-h-2t} + a^{5-2\gamma-h-2s}\\
&\qquad + a^{4-\gamma-s-h-2t} + a^{4-\gamma-h-2s} + a^{6-2\gamma-s-h-3t} + a^{6-2\gamma-h-2s-t}\Big)\\
&= O\left(a^{3-\gamma-s} + a^{4-2\gamma-s-h} + a^{1-h} + a^{2-s-h}\right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\sum_{m=1}^{M}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)\cdot V_{m,1}^{dom} &= -\frac{\kappa_0^2}{8\pi}\sum_{m=1}^{M}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)\left(1 + \left(1-\frac{\rho_m}{\rho_0}\right)^{-1}\right)(\overline{z}_m - z_m)\left(\lambda_m - \frac{1}{2}\right)^{-1}\hat{A}_m Q_m\\
&= O\left(a^{2-\gamma}\sum_{m=1}^{M}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)Q_m\right),
\end{aligned}
$$

$$
\sum_{m=1}^{M}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)\cdot V_{m,2}^{dom} = \kappa_0^2\sum_{\substack{m=1\\ m\neq l}}^{M}(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\nabla\Phi_{\kappa_0}^\infty(\hat{x},z_m)\cdot
$$

$$
\Big[|D_m| - \frac{1}{8\pi}\Big(1-\frac{\rho_m}{\rho_0}\Big)^{-1}\int_{\partial D_m}A_m(s)(S_{D_m}^0)^{-1}(1)(s)\,d\sigma_m(s)\Big]\left(\sum_{\substack{n=1\\ n\neq m}}^{M}\Phi_{\kappa_0}(z_m,z_n)Q_n\right)
$$

$$= \kappa_0^2 \sum_{\substack{m=1\\ m\neq l}}^{M} (\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1} \cdot \nabla\Phi^\infty_{\kappa_0}(\hat{x}, z_m)\Big[|D_m| - \frac{1}{8\pi}\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}\int_{\partial D_m} A_m(s)(S^0_{D_m})^{-1}(1)(s)\, d\sigma_m(s)\Big]$$
$$\Big(- \mathbf{C}_m^{-1}Q_m - u^I(z_m) + O\left(a^{4-\gamma-h-2s} + a^{1-2t-h}\right)\Big)$$
$$= \underbrace{-\kappa_0^2 \sum_{\substack{m=1\\ m\neq l}}^{M} (\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1} \cdot \nabla\Phi^\infty_{\kappa_0}(\hat{x}, z_m)\Big[|D_m| - \frac{1}{8\pi}\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}\int_{\partial D_m} A_m(s)(S^0_{D_m})^{-1}(1)(s)\, d\sigma_m(s)\Big]\mathbf{C}_m^{-1}Q_m}_{=O(a\sum_{m=1}^M \nabla\Phi^\infty_{\kappa_0}(\hat{x},z_m)Q_m)}$$
$$+ O(a^{3-\gamma-s}) + O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s} + a^{1-2t-h}\right)\Big).$$

Thus under the conditions $0 \leq t < \frac{1}{2},\ 0 \leq s \leq \frac{3}{2},\ 0 \leq \gamma \leq 1, s + \gamma \leq 2$ and $1 - 2t - h > 0$ with $h < \frac{1}{2}$, using (2.68) and proposition 2.14, we have the following estimate for the far-field:

(2.75)
$$u^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} \Phi^\infty_{\kappa_0}(\hat{x}, z_m)Q_m - \frac{\kappa_0^2}{8\pi}\sum_{m=1}^{M} \nabla\Phi^\infty_{\kappa_0}(\hat{x}, z_m)\left(1 + \left(1 - \frac{\rho_m}{\rho_0}\right)^{-1}\right)(\overline{z}_m - z_m)\left(\lambda_m - \frac{1}{2}\right)^{-1}\hat{A}_m Q_m$$
$$- \kappa_0^2 \sum_{\substack{m=1\\ m\neq l}}^{M} (\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1} \cdot \nabla\Phi^\infty_{\kappa_0}(\hat{x}, z_m)\mathbf{C}_m^{-1}Q_m$$
$$\Big[|D_m| - \frac{1}{8\pi}\Big(1 - \frac{\rho_m}{\rho_0}\Big)^{-1}\int_{\partial D_m} A_m(s)(S^0_{D_m})^{-1}(1)(s)\, d\sigma_m(s)\Big]$$
$$+ O(a^{3-\gamma-s}) + O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s} + a^{1-2t-h}\right)\Big) + O\left(a^{3-\gamma-s} + a^{4-2\gamma-s-h} + a^{1-h} + a^{2-s-h}\right)$$
$$= \sum_{m=1}^{M} \Phi^\infty_{\kappa_0}(\hat{x}, z_m)Q_m + O\left(a\sum_{m=1}^{M} \nabla\Phi^\infty_{\kappa_0}(\hat{x}, z_m)Q_m\right) + O(a^{3-\gamma-s})$$
$$+ O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s} + a^{1-2t-h}\right)\Big) + O\left(a^{3-\gamma-s} + a^{4-2\gamma-s-h} + a^{1-h} + a^{2-s-h}\right)$$
$$= \sum_{m=1}^{M} \Phi^\infty_{\kappa_0}(\hat{x}, z_m)Q_m + O(a^{1-h} + a^{5-2s-\gamma-2h}) + O(a^{3-\gamma-s}) + O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s} + a^{1-2t-h}\right)\Big)$$
$$+ O\left(a^{4-2\gamma-s-h} + a^{2-s-h}\right).$$

Next let us observe that for $l = 1, \ldots, M$, we can write $\mathbf{C}_l^{-1} = \frac{{I'_l}^d}{\kappa_l^2} + Rem_l$, where $|Rem_l| = O(a^{-1+\gamma})$. Let the vectors $(\tilde{Q}_l)_{l=1}^M$ satisfy the algebraic system

$$\frac{{I'_l}^d}{\kappa_l^2}\tilde{Q}_l + \sum_{\substack{m=1\\ m\neq l}}^{M} \Phi_{\kappa_0}(z_l, z_m)\tilde{Q}_m = -u^I(z_l). \tag{2.76}$$

Note that the dominant terms in the systems satisfied by $(Q_l)_{l=1}^M$ and $(\tilde{Q}_l)_{l=1}^M$ are same and therefore the algebraic system (2.76) is invertible under the same conditions.
From (2.73) and (2.76), we observe that the vector $(\tilde{Q}_l - Q_l)_{l=1}^M$ satisfies the algebraic system

$$\frac{{I'_l}^d}{\kappa_l^2}(\tilde{Q}_l - Q_l) + \sum_{\substack{m=1\\ m\neq l}}^{M} \Phi_{\kappa_0}(z_l, z_m)(\tilde{Q}_m - Q_m) = Rem_l Q_l + O(a^{4-\gamma-2s-h} + a^{1-2t-h}).$$

Therefore

$$\sum_{l=1}^{M}|\tilde{Q}_l-Q_l|^2 \le C\left(\max_l\left|\frac{\kappa_l^2}{I_l^{\prime d}}\right|\right)^2\left[\sum_{l=1}^{M}|Rem_l|^2|Q_l|^2+\sum_{l=1}^{M}|O(a^{4-\gamma-2s-h}+a^{1-2t-h})|^2\right]$$
$$\le C(\max_l|\mathbf{C}_l|)^2\left[\max_l|Rem_l|^2\sum_{l=1}^{M}|Q_l|^2+\sum_{l=1}^{M}|O(a^{4-\gamma-2s-h}+a^{1-2t-h})|^2\right]$$
$$\le CM(\max_l|\mathbf{C}_l|)^4(\max_l|Rem_l|)^2\left[O(1+a^{3-s-t-\gamma-h}+a^{4-\gamma-2s-h})\right]^2$$
$$+CM(\max_l|\mathbf{C}_l|)^2\left[O(a^{4-\gamma-2s-h}+a^{1-2t-h})\right]^2,$$

where C is some generic constant. Hence %beginequation

$$\left(\sum_{l=1}^{M}|\tilde{Q}_l-Q_l|\right)^2 \le M\sum_{l=1}^{M}|\tilde{Q}_l-Q_l|^2 \le CM^2(\max_l|\mathbf{C}_l|)^4(\max_l|Rem_l|)^2\left[O(1+a^{3-s-t-\gamma-h}+a^{4-\gamma-2s-h})\right]^2$$
$$+CM^2(\max_l|\mathbf{C}_l|)^2\left[O(a^{4-\gamma-2s-h}+a^{1-2t-h})\right]^2,$$

which gives

(2.77)
$$\sum_{l=1}^{M}|\tilde{Q}_l-Q_l| \le CM(\max_l|\mathbf{C}_l|)^2(\max_l|Rem_l|)\left[O(1+a^{3-s-t-\gamma-h}+a^{4-\gamma-2s-h})\right]$$
$$+CM(\max_l|\mathbf{C}_l|)\left[O(a^{4-\gamma-2s-h}+a^{1-2t-h})\right]$$
$$\le CM(\max_l|\mathbf{C}_l|)^2a^{-1+\gamma}\left[O(1+a^{3-s-t-\gamma-h}+a^{4-\gamma-2s-h})\right]+O(a^{4-\gamma-2s-2h}+a^{1-2t-2h}).$$

Using (2.77) in (2.75), we obtain

(2.78)
$$u^\infty(\hat{x},\theta)=\sum_{m=1}^{M}\Phi^\infty_{\kappa_0}(\hat{x},z_m)\tilde{Q}_m+\sum_{m=1}^{M}\Phi^\infty_{\kappa_0}(\hat{x},z_m)(Q_m-\tilde{Q}_m)+O(a^{1-h}+a^{5-2s-\gamma-2h})$$
$$+O(a^{3-\gamma-s})+O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s}+a^{1-2t-h}\right)\Big)+O\left(a^{4-2\gamma-s-h}+a^{2-s-h}\right)$$
$$=\sum_{m=1}^{M}\Phi^\infty_{\kappa_0}(\hat{x},z_m)\tilde{Q}_m+O(a^{1-h}+a^{5-2s-\gamma-2h})$$
$$+O(a^{3-\gamma-s})+O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s}+a^{1-2t-h}\right)\Big)+O\left(a^{4-2\gamma-s-h}+a^{2-s-h}\right)$$
$$+M(\max_l|\mathbf{C}_l|)^2a^{-1+\gamma}\left[O(1+a^{3-s-t-\gamma-h}+a^{4-\gamma-2s-h})\right]+O(a^{4-\gamma-2s-2h}+a^{1-2t-2h}).$$

Finally from (2.78), we can conclude the following:

- We recall that when $\gamma<1$ or we are away from the resonance, we have $M\max_l|\mathbf{C}_l|=O(1)$ and $h=0$. Also $\max_l|\mathbf{C}_l|=O(a^{2-\gamma})$. Therefore (2.78) reduces to

(2.79)
$$u^\infty(\hat{x},\theta)=\sum_{m=1}^{M}\Phi^\infty_{\kappa_0}(\hat{x},z_m)\tilde{Q}_m+O(a+a^{5-2s-\gamma})$$
$$+O(a^{3-\gamma-s})+O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-2s}+a^{1-2t}\right)\Big)+O\left(a^{4-2\gamma-s}+a^{2-s}\right)$$
$$+a^{3-\gamma-s}\ O(1+a^{3-s-t-\gamma}+a^{4-\gamma-2s})+O(a^{4-\gamma-2s}+a^{1-2t})$$

$$= \sum_{m=1}^{M} \Phi^{\infty}_{\kappa_0}(\hat{x}, z_m)\tilde{Q}_m + O(a^{2-s} + a^{4-\gamma-2s} + a^{1-2t})$$
$$= \sum_{m=1}^{M} \Phi^{\infty}_{\kappa_0}(\hat{x}, z_m)\tilde{Q}_m + O(a^{2-s} + a^{1-2t})$$

where in the last line we use the fact $s + \gamma \leq 2$.

- We recall that near the resonance $\max_l |\mathbf{C}_l| = O(a^{1-h_1})$ and therefore $O(a^{-h}) = M \max_l |\mathbf{C}_l| = O(a^{-s+1-h_1})$ leading to the condition $h = -1 + s + h_1$.
  Also since $\gamma = 1$, we obtain that $M(\max_l |\mathbf{C}_l|)^2 a^{-1+\gamma} = O(a^{s-2h})$. Then from (2.78), we conclude that

$$
\begin{aligned}
u^{\infty}(\hat{x}, \theta) &= \sum_{m=1}^{M} \Phi^{\infty}_{\kappa_0}(\hat{x}, z_m)\tilde{Q}_m + O(a^{1-h} + a^{5-2s-\gamma-2h}) \\
&\quad + O(a^{3-\gamma-s}) + O\Big(a^{3-\gamma-s}\left(a^{4-\gamma-h-2s} + a^{1-2t-h}\right)\Big) + O\left(a^{4-2\gamma-s-h} + a^{2-s-h}\right) \\
&\quad + a^{s-2h}\, O(1 + a^{3-s-t-\gamma-h} + a^{4-\gamma-2s-h}) + O(a^{4-\gamma-2s-2h} + a^{1-2t-2h}) \\
&= \sum_{m=1}^{M} \Phi^{\infty}_{\kappa_0}(\hat{x}, z_m)\tilde{Q}_m + O(a^{1-h} + a^{2-s-2h} + a^{1-2t-2h} + a^{s-2h}) \\
&= \sum_{m=1}^{M} \Phi^{\infty}_{\kappa_0}(\hat{x}, z_m)\tilde{Q}_m + O(a^{2-s-h_1} + a^{4-3s-2h_1} + a^{3-2t-2s-2h_1} + a^{2-s-2h_1}) \\
&= \sum_{m=1}^{M} \Phi^{\infty}_{\kappa_0}(\hat{x}, z_m)\tilde{Q}_m + O(a^{3-2t-2s-2h_1} + a^{2-s-2h_1})
\end{aligned}
\tag{2.80}
$$

where in the last line we use the fact that $s \leq 1$ (as we have $s + \gamma \leq 2$ and $\gamma = 1$).

(1) When $l_M < 0$, then we have seen that we need that $h \leq 0$, see Remark 2.15 and Remark 2.13). Hence in this case we need $s + h_1 \leq 1$.

(2) When $l_M > 0$, we need only $h < \frac{1}{2}$ which means that $s + h_1 < \frac{3}{2}$.

Note that the error term in (2.80) goes to zero provided s, h_1 and t satisfy the condition

$$s < \min\{2 - 2h_1,\ \frac{3 - 2t - 2h_1}{2}\}. \tag{2.81}$$

These conditions are fulfilled if

$$s + h_1 \leq 1,\ h_1 < 1 \ \text{ and } t < \frac{1}{2}. \tag{2.82}$$

These last conditions are the regimes in which one can derive the effective media. Actually, in the case $l_M > 0$, one can allow $s + h_1 > 1$ (but $s + h_1 < \frac{3}{2}$) and hence generate very large effective potentials.

3. Proofs of auxiliary results

3.1. **Proof of Proposition 2.1:** Let us define operators $T, T_0 : X \to Y$ by

$$\Big(\phi_1,\psi_1,...,\phi_M,\psi_M\Big) \mapsto \Bigg(\Big(S_{D_1}^{\kappa_1}\psi_1 - \sum_{l=1}^{M} S_{D_l}^{\kappa_0}\phi_l\Big)\Bigg|_{\partial D_1},$$

$$\frac{\rho_0}{\rho_1}\Big[\frac{1}{2}Id + (K_{D_1}^{\kappa_1})^*\Big]\psi_1 - \Big[-\frac{1}{2}Id + (K_{D_1}^{\kappa_0})^*\Big]\phi_1 - \sum_{\substack{l=1\\l\neq 1}}^{M} \frac{\partial(S_{D_l}^{\kappa_0}\phi_l)}{\partial\nu^1}\Bigg|_{\partial D_1}, ...,$$

$$\Big(S_{D_M}^{\kappa_M}\psi_M - \sum_{l=1}^{M} S_{D_l}^{\kappa_0}\phi_l\Big)\Bigg|_{\partial D_M},$$

$$\frac{\rho_0}{\rho_M}\Big[\frac{1}{2}Id + (K_{D_M}^{\kappa_M})^*\Big]\psi_M - \Big[-\frac{1}{2}Id + (K_{D_M}^{\kappa_0})^*\Big]\phi_M - \sum_{\substack{l=1\\l\neq M}}^{M} \frac{\partial(S_{D_l}^{\kappa_0}\phi_l)}{\partial\nu^M}\Bigg|_{\partial D_M} \Bigg)$$

and

$$\Big(\phi_1,\psi_1,...,\phi_M,\psi_M\Big) \mapsto \Big(\Big(S_{D_1}^{0}\psi_1 - S_{D_1}^{0}\phi_1\Big)\Big|_{\partial D_1},\ \frac{\rho_0}{\rho_1}\Big[\frac{1}{2}Id + (K_{D_1}^{0})^*\Big]\psi_1 - \Big[-\frac{1}{2}Id + (K_{D_1}^{0})^*\Big]\phi_1, ...,$$
$$\Big(S_{D_M}^{0}\psi_M - S_{D_M}^{0}\phi_M\Big)\Big|_{\partial D_M},\ \frac{\rho_0}{\rho_M}\Big[\frac{1}{2}Id + (K_{D_M}^{0})^*\Big]\psi_M - \Big[-\frac{1}{2}Id + (K_{D_M}^{0})^*\Big]\phi_M\Big).$$

We note that the operator $T - T_0 : X \to Y$ is a compact operator. To see this, first of all we observe that the typical terms in $(T - T_0)(\phi_1, \psi_1, ..., \phi_M, \psi_M)$ are of the form

$$-\sum_{\substack{l=1\\l\neq i}}^{M} S_{D_l}^{\kappa_0}\phi_l + (S_{D_i}^{\kappa_i} - S_{D_i}^{0})\psi_i - (S_{D_i}^{\kappa_0} - S_{D_i}^{0})\phi_i$$

and

$$-\sum_{\substack{l=1\\l\neq i}}^{M} \frac{\partial(S_{D_l}^{\kappa_0}\phi_l)}{\partial\nu^i} + \frac{\rho_0}{\rho_i}\Big[(K_{D_i}^{\kappa_i})^* - (K_{D_i}^{0})^*\Big]\psi_i - \Big[(K_{D_i}^{\kappa_0})^* - (K_{D_i}^{0})^*\Big]\phi_i.$$

The first term gives rise to a compact operator since $(S_{D_i}^{\kappa_i} - S_{D_i}^{0}), (S_{D_i}^{\kappa_0} - S_{D_i}^{0})$ are compact and also for $l \neq i$, $S_{D_l}^{\kappa_0}\phi_l$ is compact. A similar argument works for the second term as well, and therefore it follows that $T - T_0$ is a compact operator.
Also note that the operator T_0 is invertible. This follows directly as in the proof of Theorem 11.4, Page 189, of [4], observing that T_0 is diagonal with a 2×2 operator, at the diagonal, of the form

$$(\phi_l,\psi_l) \mapsto \Big(\Big(S_{D_l}^{0}\psi_l - S_{D_l}^{0}\phi_l\Big)\Big|_{\partial D_l},\ \frac{\rho_0}{\rho_l}\Big[\frac{1}{2}Id + (K_{D_l}^{0})^*\Big]\psi_l - \Big[-\frac{1}{2}Id + (K_{D_l}^{0})^*\Big]\phi_l\Big).$$

Therefore to prove that T is invertible, it is sufficient to prove that T is injective, as an application of the Fredholm alternative.
Let us therefore suppose that $T(\phi_1, \psi_1, ..., \phi_M, \psi_M) = 0$. Then

$$u(x) = \begin{cases} \sum_{l=1}^{M} S_{D_l}^{\kappa_0}\phi_l(x),\ x \in \mathbb{R}^3 \diagdown \overline{\cup_{l=1}^{M} D_l} \\ S_{D_s}^{\kappa_s}\psi_s(x),\ x \in D_s \end{cases}$$

is the unique solution to the problem under consideration.
Now

$$\int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_{+} \bar{u}\ d\sigma = \frac{\rho_0}{\rho_i}\int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_{-} \bar{u} d\sigma = \frac{\rho_0}{\rho_i}\int_{D_i} (|\nabla u|^2 - \kappa_i^2|u|^2)\ dx$$

which is real and therefore

$$\mathrm{Im} \int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_{+} \bar{u}\ d\sigma = 0,$$

which in turn implies that $u = 0$ in $\mathbb{R}^3 \diagdown \cup_{l=1}^M D_l$.
Again, u solves the equation

$$(\Delta + \kappa_i^2)u = 0 \text{ in } D_i,$$
$$u = \frac{\partial u}{\partial \nu^i} = 0 \text{ on } \partial D_i$$

and hence by unique continuation property, $u = 0$ in D_i. Arguing similarly for each obstacle D_i, we obtain that $u = 0$ in $\mathbb{R}^3$. In particular, this implies that

$$\sum_{l=1}^M S_{D_l}^{\kappa_0} \phi_l(x) = 0 \text{ on } \partial D_i.$$

Since $\Big(\Delta + \kappa_0^2\Big)(\sum_{l=1}^M S_{D_l}^{\kappa_0}\phi_l) = 0$ in D_i and κ_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on D_i (which holds since $a \ll 1$), it follows that

$$\sum_{l=1}^M S_{D_l}^{\kappa_0} \phi_l = 0 \text{ in } D_i$$

and therefore since this is true for each i, we have

$$\sum_{l=1}^M S_{D_l}^{\kappa_0} \phi_l = 0 \text{ in } \mathbb{R}^3. \tag{3.1}$$

Now for a fixed i, we know that whenever $l \neq i$, we have $\frac{\partial S_{D_l}^{\kappa_0}\phi_l}{\partial \nu^i}\Big|_+ = \frac{\partial S_{D_l}^{\kappa_0}\phi_l}{\partial \nu^i}\Big|_-$ and therefore we obtain using (3.1),

$$\phi_i = \frac{\partial S_{D_i}^{\kappa_0}\phi_i}{\partial \nu^i}\Big|_+ - \frac{\partial S_{D_i}^{\kappa_0}\phi_i}{\partial \nu^i}\Big|_- = 0 \text{ on } \partial D_i.$$

Next we prove that $\psi_i = 0$ on ∂D_i, for all i. Since we already know that $\phi_i = 0\ \forall\ i$, it follows from the form of the solution u and the zero jump condition that $S_{D_i}^{\kappa_i}\psi_i = 0$ on ∂D_i. Using this fact and the fact that $\Big(\Delta + \kappa_i^2\Big) S_{D_i}^{\kappa_i}\psi_i = 0$ in $\mathbb{R}^3 \diagdown \overline{D_i}$, we obtain that $S_{D_i}^{\kappa_i}\psi_i = 0$ in $\mathbb{R}^3 \diagdown \overline{D_i}$. Recalling the fact that we have already proved that $u = 0$ in D_i and hence $S_{D_i}^{\kappa_i}\psi_i = 0$ on ∂D_i, it therefore follows that $S_{D_i}^{\kappa_i}\psi_i = 0$ in $\mathbb{R}^3$. Now we can proceed just in the case of ϕ_l by using the Neumann jump of $S_{D_i}^{\kappa_i}\psi_i$ across D_i to obtain that $\psi_i = 0$.

3.2. **Proof of Lemma 2.2.** In order to prove (2.7), we first set $\big(S_{D_l}^0\big)^{-1}\Bigg(\int_{\partial D_l} |\cdot - t|^n \phi_l(t) d\sigma_l(t)\Bigg) =: \chi_l$. As

$$\delta^{n+2} \int_{\partial B_l} |\hat{s} - \hat{t}|^n \hat{\phi}_l(\hat{t})\ d\hat{\sigma}_l(\hat{t}) = \int_{\partial D_l} |s - t|^n \phi_l(t)\ d\sigma_l(t) = \Big[S_{D_l}^0 \chi_l\Big](s) = \delta \Big[S_{B_l}^0 \hat{\chi}_l\Big](\hat{s}), \tag{3.2}$$

then

$$\hat{\chi}_l = \delta^{n+1} (S_{B_l}^0)^{-1} \Big(\int_{\partial B_l} |\cdot - \hat{t}|^n \hat{\phi}_l(\hat{t})\ d\hat{\sigma}_l(\hat{t}) \Big),$$

where $[S_{B_l}^0 \hat{\chi}_l](\hat{s}) := \int_{\partial B_l} \frac{1}{4\pi|\hat{s} - \hat{t}|} \hat{\chi}_l(\hat{t})\ d\hat{\sigma}_l(\hat{t})$ and $d\hat{\sigma}_l$ denotes the surface measure on ∂B_l.
Therefore

$$\begin{aligned}\frac{1}{\delta}\|\chi_l\|_{L^2(\partial D_l)} = \|\hat{\chi}_l\|_{L^2(\partial B_l)} &= \delta^{n+1} \|(S_{B_l}^0)^{-1}\|_{\mathcal{L}(H^1(\partial B_l), L^2(\partial B_l))} \Big\| \int_{\partial B_l} |\cdot - \hat{t}|^n\ \hat{\phi}_l(\hat{t})\ d\hat{\sigma}_l(\hat{t}) \Big\|_{H^1(\partial B_l)} \\ &= O(\delta^{n+1} \|\hat{\phi}_l\|_{L^2(\partial B_l)}) = O(\delta^n \|\phi_l\|_{L^2(\partial D_l)}),\end{aligned}$$

whence it follows that

$$\left\|(S^0_{D_l})^{-1}\left(\int_{\partial D_l}|\cdot - t|^n\phi_l(t)d\sigma_l(t)\right)\right\|_{L^2(\partial D_l)} = \|\chi_l\|_{L^2(\partial D_l)} = O\Big(\delta^{n+1}\|\phi_l\|_{L^2(\partial D_l)}\Big)$$
$$= O\Big(a^{n+1}\|\phi_l\|_{L^2(\partial D_l)}\Big).$$

Similarly, to prove (2.8), we write $(S^0_{D_l})^{-1}\Big((\cdot - z_l)^n\Big) = \eta_l$. Then

$$\delta^n\hat{s}^n = (s-z_l)^n = \Big[S^0_{D_l}\eta_l\Big](s) = \delta\Big[S^0_{B_l}\hat{\eta}_l\Big](\hat{s}),$$

which implies

$$\hat{\eta}_l = \delta^{n-1}(S^0_{B_l})^{-1}(\cdot^n),$$

and hence

$$\frac{1}{\delta}\|\eta_l\|_{L^2(\partial D_l)} = \|\hat{\eta}_l\|_{L^2(\partial B_l)} = \delta^{n-1}\|(S^0_{B_l})^{-1}(\cdot^n)\|_{L^2(\partial B_l)}$$
$$\Rightarrow \|(S^0_{D_l})^{-1}\Big((\cdot - z_l)^n\Big)\|_{L^2(\partial D_l)} = \|\eta_l\|_{L^2(\partial D_l)} = \delta^n\|(S^0_{B_l})^{-1}(\cdot^n)\|_{L^2(\partial B_l)} = O(a^n).$$

3.3. **Proof of Lemma 2.3.** In order to prove (2.9), we proceed as follows.

(3.3)

$$\begin{aligned}
\big[S^{\kappa_0}_{D_m} - S^{\kappa_m}_{D_m}\big]\psi_m(x) &= \int_{\partial D_m}[\Phi_{\kappa_0} - \Phi_{\kappa_m}](x,s)\psi_m(s)\,d\sigma_m(s) \\
&= \frac{i}{4\pi}(\kappa_0 - \kappa_m)\int_{\partial D_m}\psi_m(s)\,d\sigma_m(s) + \underbrace{\sum_{n=2}^{\infty}\frac{i^n(\kappa_0^n - \kappa_m^n)}{4\pi n!}\int_{\partial D_m}|x-s|^{n-1}\psi_m(s)\,d\sigma_m(s)}_{=:Err1_m} \\
&= \frac{i}{4\pi}(\kappa_0 - \kappa_m)\int_{\partial D_m}\psi_m(s)\,d\sigma_m(s) + O\left(\sum_{n=2}^{\infty}\frac{(\kappa_0^n + \kappa_m^n)}{4\pi n!}a^{n-1}\int_{\partial D_m}|\psi_m(s)|\,d\sigma_m(s)\right) \\
&= \frac{i}{4\pi}(\kappa_0 - \kappa_m)\int_{\partial D_m}\psi_m(s)\,d\sigma_m(s) + O\left(\frac{a^2}{4\pi}\left[\frac{\kappa_0^2}{1-\kappa_0\,a} + \frac{\kappa_m^2}{1-\kappa_m\,a}\right]\|\psi_m\|\right) \\
&= \frac{i}{4\pi}(\kappa_0 - \kappa_m)\int_{\partial D_m}\psi_m(s)\,d\sigma_m(s) + O\left(a^2\|\psi_m\|\right),
\end{aligned}$$

whence (2.9) follows. Note that here we use the fact that $|k_0 a|, |k_m a| < \frac{1}{2}$, which holds since $a \ll 1$.

In order to prove (2.10), we first observe that

$$\int_{\partial D_l}(K^\kappa_D)^*\psi(s)d\sigma_l(s) = \int_{\partial D_l}\psi(s)(K^\kappa_D)(1)(s)d\sigma_l(s) = \int_{\partial D_l}\psi(s)\left[\int_{\partial D_l}\frac{\partial}{\partial\nu_t}\Phi_\kappa(s,t)d\sigma_l(t)\right]d\sigma_l(s)$$
$$= \int_{\partial D_l}\psi(s)\left[\int_{\partial D_l}\nabla_t\Phi_\kappa(s,t)\cdot\nu_t\,d\sigma_l(t)\right]d\sigma_l(s).$$

Then we can write

$$\int_{\partial D_l}\left[(K^{\kappa_0}_{D_l})^* - (K^{\kappa_l}_{D_l})^*\right]\psi_l(s)d\sigma_l(s) = \int_{\partial D_l}\psi_l(s)\left[\int_{\partial D_l}\nabla_t\Big(\Phi_{\kappa_0} - \Phi_{\kappa_l}\Big)(s,t)\cdot\nu_t\,d\sigma_l(t)\right]d\sigma_l(s)$$
$$= \int_{\partial D_l}\psi_l(s)\left[\int_{\partial D_l}\nabla_t\left(\sum_{n=1}^{\infty}\frac{i^n(\kappa_0^n - \kappa_l^n)}{4\pi\ n!}|s-t|^{n-1}\right)\cdot\nu_t\,d\sigma_l(t)\right]d\sigma_l(s)$$

$$
\begin{aligned}
&= \int_{\partial D_l} \psi_l(s) \Bigg[\int_{\partial D_l} \Bigg(\sum_{n=2}^{\infty} \frac{i^n(\kappa_l^n - \kappa_0^n)}{4\pi\ n!} (n-1)|s-t|^{n-2} \frac{(s-t)}{|s-t|} \Bigg) \cdot \nu_t d\sigma_l(t) \Bigg] d\sigma_l(s) \\
&= \frac{1}{8\pi}(\kappa_0^2 - \kappa_l^2) \int_{\partial D_l} \psi_l(s) \left[\int_{\partial D_l} \frac{(s-t)}{|s-t|} \cdot \nu_t\, d\sigma_l(t) \right] d\sigma_l(s) \\
&\quad + \sum_{n=3}^{\infty} \frac{i^n(\kappa_l^n - \kappa_0^n)}{4\pi\ n!} (n-1) \int_{\partial D_l} \psi_l(s) \Bigg[\int_{\partial D_l} \Bigg(|s-t|^{n-2} \frac{(s-t)}{|s-t|} \Bigg) \cdot \nu_t d\sigma_l(t) \Bigg] d\sigma_l(s) \\
&= \frac{1}{8\pi}(\kappa_0^2 - \kappa_l^2) \int_{\partial D_l} \psi_l(s) \left[\int_{\partial D_l} \frac{(s-t)}{|s-t|} \cdot \nu_t\, d\sigma_l(t) \right] d\sigma_l(s) - \frac{i}{4\pi}(\kappa_0^3 - \kappa_l^3)|D_l| \int_{\partial D_l} \psi_l(s) d\sigma_l(s) \\
&\quad + \underbrace{\sum_{n=4}^{\infty} \frac{i^n(\kappa_l^n - \kappa_0^n)}{4\pi\ n!} (n-1) \int_{\partial D_l} \psi_l(s) \Bigg[\int_{\partial D_l} \Bigg(|s-t|^{n-2} \frac{(s-t)}{|s-t|} \Bigg) \cdot \nu_t d\sigma_l(t) \Bigg] d\sigma_l(s)}_{=:Err2_l = O(a^5 \|\psi_l\|)},
\end{aligned}
$$

where the last step follows from the divergence theorem. Thus (2.10) follows.

Next we establish (2.11). For this, we first observe that

$$
\begin{aligned}
\int_{\partial D_l} \Big[\frac{\partial S_{D_m}^{\kappa_0} \phi_m}{\partial \nu^l} \Big](s)\ d\sigma_l(s) &= \int_{\partial D_l} \Big[\int_{\partial D_m} \frac{\partial \Phi_{\kappa_0}(s,t)}{\partial \nu^l(s)} \phi_m(t)\ d\sigma_m(t) \Big] d\sigma_l(s) \\
&= \int_{\partial D_m} \Big[\int_{\partial D_l} \frac{\partial \Phi_{\kappa_0}(s,t)}{\partial \nu^l(s)} d\sigma_l(s) \Big] \phi_m(t) d\sigma_m(t) \\
&= \int_{\partial D_m} \Big[\int_{D_l} \Delta \Phi_{\kappa_0}(x,t)\ dx \Big] \phi_m(t)\ d\sigma_m(t) \\
&= -\kappa_0^2 \int_{\partial D_m} \Big[\int_{D_l} \Phi_{\kappa_0}(x,t)\ dx \Big] \phi_m(t)\ d\sigma_m(t).
\end{aligned}
$$

We can write this as

$$
\begin{aligned}
\int_{\partial D_l} \Big[\frac{\partial S_{D_m}^{\kappa_0} \phi_m}{\partial \nu^l} \Big](s)\ d\sigma_l(s) &= -\kappa_0^2 \int_{\partial D_m} \Phi_{\kappa_0}(z_l,t)|D_l|\phi_m(t) d\sigma_m(t) \\
&\quad - \kappa_0^2 \int_{\partial D_m} \nabla_x \Phi_{\kappa_0}(z_l,t) \cdot \Big[\int_{D_l} (x - z_l) dx \Big] \phi_m(t) d\sigma_m(t) \\
&\quad \underbrace{- \kappa_0^2 \int_{\partial D_m} \Big[\int_{D_l} (\Phi_{\kappa_0}(x,t) - \Phi_{\kappa_0}(z_l,t) - (x-z_l) \cdot \nabla_x \Phi_{\kappa_0}(z_l,t)) dx \Big] \phi_m(t) d\sigma_m(t)}_{=:G_{1_{ml}}^{\kappa_0} = O(\frac{a^6}{d_{ml}^3} \|\phi_m\|)}
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\int_{\partial D_l} \Big[\frac{\partial S_{D_m}^{\kappa_0} \phi_m}{\partial \nu^l} \Big](s)\ d\sigma_l(s) &= -\kappa_0^2 \int_{\partial D_m} \Phi_{\kappa_0}(z_l,z_m)|D_l|\phi_m(t) d\sigma_m(t) \\
&\quad - \kappa_0^2 \int_{\partial D_m} |D_l| \nabla_y \Phi_{\kappa_0}(z_l,z_m) \cdot (t - z_m) \phi_m(t) d\sigma_m(t) \\
&\quad \underbrace{- \kappa_0^2 \int_{\partial D_m} |D_l| \big[\Phi_{\kappa_0}(z_l,t) - \Phi_{\kappa_0}(z_l,z_m) - \nabla_t \Phi_{\kappa_0}(z_l,z_m) \cdot (t - z_m) \big] \phi_m(t) d\sigma_m(t)}_{=:G_{2_{ml}}^{\kappa_0} = O(\frac{a^6}{d_{ml}^3} \|\phi_m\|)} - G_{1_{ml}}^{\kappa_0} \\
&\quad - \kappa_0^2 \int_{\partial D_m} \nabla_x \Phi_{\kappa_0}(z_l,z_m) \cdot \Big[\int_{D_l} (x - z_l) dx \Big] \phi_m(t) d\sigma_m(t)
\end{aligned}
$$

$$- \underbrace{\kappa_0^2 \int_{\partial D_m} [\nabla_x \Phi_{\kappa_0}(z_l, t) - \nabla_x \Phi_{\kappa_0}(z_l, z_m)] \cdot \Big[\int_{D_l} (x - z_l) dx \Big] \phi_m(t) d\sigma_m(t)}_{=: G^{\kappa_0}_{3_{ml}} = O(\frac{a^6}{d^3_{ml}} \|\phi_m\|)}$$

and therefore

(3.4)
$$\begin{aligned}
\int_{\partial D_l} \Big[\frac{\partial S^{\kappa_0}_{D_m} \phi_m}{\partial \nu^l} \Big](s)\ d\sigma_l(s) &= -\kappa_0^2 \Phi_{\kappa_0}(z_l, z_m) |D_l| \int_{\partial D_m} \phi_m(t) d\sigma_m(t) \\
&- \kappa_0^2 |D_l| \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot \int_{\partial D_m} (t - z_m) \phi_m(t) d\sigma_m(t) \\
&- \kappa_0^2 \nabla_x \Phi_{\kappa_0}(z_l, z_m) \cdot \Big[\int_{D_l} (x - z_l) dx \Big] \int_{\partial D_m} \phi_m(t) d\sigma_m(t) - \underbrace{[G^{\kappa_0}_{1_{ml}} + G^{\kappa_0}_{2_{ml}} + G^{\kappa_0}_{3_{ml}}]}_{=: Err4_m = O(\frac{a^6}{d^3_{ml}} \|\phi_m\|)},
\end{aligned}$$

whence (2.11) follows.

The proof of (2.12) follows easily from the observation

$$\int_{\partial D_l} \frac{\partial u^I}{\partial \nu^l} = -\, \kappa_0^2 \, |D_l| \, u^I(z_l) \underbrace{- \kappa_0^2 \int_{D_l} (u^I(y) - u^I(z_l) \, dy}_{=: Err5_l = O(a^4)} . \tag{3.5}$$

We shall next derive the approximations (2.13) and (2.14). First of all, we note that

$$\begin{aligned}
S^{d_{\kappa_0}}_{D_l} \phi_l(s) &= \int_{\partial D_l} \frac{e^{i\kappa_0 |s-y|} - 1}{4\pi \, |s - y|} \phi_l(y) \, d\sigma_l(y) = \int_{\partial D_l} \sum_{n=1}^{\infty} \frac{1}{4\pi} \frac{(i\kappa_0)^n}{n!} |s - y|^{n-1} \phi_l(y) \, d\sigma_l(y) \\
&= \frac{i\kappa_0}{4\pi} \int_{\partial D_l} \phi_l(y) \, d\sigma_l(y) + \int_{\partial D_l} \sum_{n=2}^{\infty} \frac{1}{4\pi} \frac{(i\kappa_0)^n}{n!} |s - y|^{n-1} \phi_l(y) \, d\sigma_l(y) \\
&= \begin{cases} \frac{i\kappa_0}{4\pi} Q_l + O(a^2 \|\phi_l\|), \text{ in } L^\infty \text{ and } H^1, \\ \frac{i\kappa_0}{4\pi} Q_l + O(a^3 \|\phi_l\|), \text{ in } L^2. \end{cases}
\end{aligned} \tag{3.6}$$

From (2.5), we have that on ∂D_l,

$$\begin{aligned}
\psi_l(s) &= \big(S^{\kappa_l}_{D_l}\big)^{-1} \left[u^I + \sum_{m=1}^{M} S^{\kappa_0}_{D_m} \phi_m \right](s) \\
&= \big(S^{0}_{D_l}\big)^{-1} \left[u^I + S^{\kappa_0}_{D_l} \phi_l + \sum_{\substack{m=1 \\ m \neq l}}^{M} S^{\kappa_0}_{D_m} \phi_m \right](s) \\
&\quad + \big(S^{0}_{D_l}\big)^{-1} \left[\sum_{n=1}^{\infty} (-1)^n \left(S^{d_{\kappa_l}}_{D_l} \big(S^{0}_{D_l}\big)^{-1} \right)^n \left(u^I + S^{\kappa_0}_{D_l} \phi_l + \sum_{\substack{m=1 \\ m \neq l}}^{M} S^{\kappa_0}_{D_m} \phi_m \right) \right](s) \\
&= \big(S^{0}_{D_l}\big)^{-1} \left[u^I(z_l) + (s - z_l) \cdot \nabla u^I(z_l) + O(a^2) + S^{0}_{D_l} \phi_l + S^{d_{\kappa_0}}_{D_l} \phi_l + \sum_{\substack{m=1 \\ m \neq l}}^{M} S^{\kappa_0}_{D_m} \phi_m \right](s) \\
&\quad + \big(S^{0}_{D_l}\big)^{-1} \left[\sum_{n=1}^{\infty} (-1)^n \left(S^{d_{\kappa_l}}_{D_l} \big(S^{0}_{D_l}\big)^{-1} \right)^n \left(u^I + S^{0}_{D_l} \phi_l + S^{d_{\kappa_0}}_{D_l} \phi_l + \sum_{\substack{m=1 \\ m \neq l}}^{M} S^{\kappa_0}_{D_m} \phi_m \right) \right](s),
\end{aligned}$$

where the approximation above is in a point-wise sense and $S^{d_{k_l}}_{D_l}\phi(s) := \int_{\partial D_l} \frac{e^{ik_l|s-t|}-1}{4\pi|s-t|}\phi(t)\, d\sigma_l(t)$. Using (2.8) and (3.6), we can write this as

$$\psi_l(s) = \phi_l(s) + \left(S^0_{D_l}\right)^{-1}\left(u^I(z_l)\right) + \left(S^0_{D_l}\right)^{-1}\left(S^{d_{\kappa_0}}_{D_l} - S^{d_{\kappa_l}}_{D_l}\right)\phi_l(s) + O(a) + O(a^2\|\phi_l\|)$$
$$+ \left[\left(S^0_{D_l}\right)^{-1}\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right] + \left(S^0_{D_l}\right)^{-1}\left[\sum_{n=1}^{\infty}(-1)^n\left(S^{d_{\kappa_l}}_{D_l}\left(S^0_{D_l}\right)^{-1}\right)^n\left(\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right)\right]$$

with error estimates in L^2 sense. This further implies, using $(S^{d_{k_0}}_{D_l} - S^{d_{k_l}}_{D_l})\phi_l = (S^{k_0}_{D_l} - S^{k_l}_{D_l})\phi_l$ and (2.9), that

$$\psi_l(s) = \phi_l(s) + \left(S^0_{D_l}\right)^{-1}\left(u^I(z_l)\right) + \frac{i(\kappa_0-\kappa_l)}{4\pi}Q_l\left(S^0_{D_l}\right)^{-1}(1)(s)$$
$$+ \left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}\sum_{n=2}^{\infty}\frac{1}{4\pi}\frac{i^n(\kappa_0^n-\kappa_l^n)}{n!}|s-y|^{n-1}\phi_l(y)\, d\sigma_l(y)\right)(s)$$
$$+ \left[\left(S^0_{D_l}\right)^{-1}\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right] + \left(S^0_{D_l}\right)^{-1}\left[\sum_{n=1}^{\infty}(-1)^n\left(S^{d_{\kappa_l}}_{D_l}\left(S^0_{D_l}\right)^{-1}\right)^n\left(\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right)\right]$$
$$+ O(a) + O(a^2\|\phi_l\|),\ \text{in } L^2,$$

and hence, keeping only the first terms in the two infinite series,

$$\psi_l(s) = \phi_l(s) + \left(S^0_{D_l}\right)^{-1}\left(u^I(z_l)\right) + \frac{i(\kappa_0-\kappa_l)}{4\pi}Q_l\left(S^0_{D_l}\right)^{-1}(1)(s)$$
$$+ \frac{i^2(\kappa_0^2-\kappa_l^2)}{8\pi}\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|\cdot - y|\phi_l(y)\, d\sigma_l(y)\right)(s)$$
$$+ \left[\left(S^0_{D_l}\right)^{-1}\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right] - \left(S^0_{D_l}\right)^{-1}\left[S^{d_{\kappa_l}}_{D_l}\left(S^0_{D_l}\right)^{-1}\left(\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right)\right]$$
$$+ O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d^2_{lm}}\|\phi_m\|\right) + O(a) + O(a^2\|\phi_l\|).$$

Using (2.7), we can write this as

$$\psi_l(s) = \phi_l(s) + \left(S^0_{D_l}\right)^{-1}\left(u^I(z_l)\right) + \frac{i(\kappa_0-\kappa_l)}{4\pi}Q_l\left(S^0_{D_l}\right)^{-1}(1)(s) + \left[\left(S^0_{D_l}\right)^{-1}\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right]$$
$$- \left(S^0_{D_l}\right)^{-1}\left[S^{d_{\kappa_l}}_{D_l}\left(S^0_{D_l}\right)^{-1}\left(\sum_{\substack{m=1\\m\neq l}}^{M} S^{\kappa_0}_{D_m}\phi_m\right)\right] + O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d^2_{lm}}\|\phi_m\|\right) + O(a) + O(a^2\|\phi_l\|)\ \text{in } L^2. \tag{3.7}$$

For a better understanding of the dominating terms in (3.7), we next estimate the last two terms. Using Taylor series expansion, for $s \in \partial D_l$ and $t \in \partial D_m$, we can write

$$\Phi_{\kappa_0}(s,t) = \Phi_{\kappa_0}(z_l,t) + (s-z_l)\cdot\nabla_s\Phi_{\kappa_0}(z_l,t) + \frac{1}{2}(s-z_l)^2\cdot\nabla_s\nabla_s\Phi_{\kappa_0}(z_l,t) \tag{3.8}$$
$$+ \sum_{|\alpha|=3}\frac{|\alpha|}{\alpha!}(s-z_l)^\alpha\int_0^1(1-\beta)^{|\alpha|-1}D^\alpha_s\Phi_{\kappa_0}(z_l+\beta(s-z_l),t)d\beta$$

$$
\begin{aligned}
&= \Phi_{\kappa_0}(z_l, z_m) + (t - z_m) \cdot \nabla_t \Phi_{\kappa_0}(z_l, z_m) + (s - z_l) \cdot \nabla_s \Phi_{\kappa_0}(z_l, z_m) \\
&\quad + \frac{1}{2}(s - z_l)^2 \cdot \nabla_s \nabla_s \Phi_{\kappa_0}(z_l, t) \\
&\quad + \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!}(t - z_m)^\alpha \int_0^1 (1 - r)^{|\alpha|-1} D_t^\alpha \Phi_{\kappa_0}(z_l, z_m + r(t - z_m))\, dr \\
&\quad + (s - z_l) \cdot \sum_{|\alpha|=1} \frac{|\alpha|}{\alpha!}(t - z_m)^\alpha \int_0^1 (1 - r)^{|\alpha|-1} D_t^\alpha \nabla_s \Phi_{\kappa_0}(z_l, z_m + r(t - z_m))\, dr \\
&\quad + \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha!}(s - z_l)^\alpha \int_0^1 (1 - \beta)^{|\alpha|-1} D_s^\alpha \Phi_{\kappa_0}(z_l + \beta(s - z_l), t)\, d\beta,
\end{aligned}
$$

which gives us

$$
\begin{aligned}
(S_{D_m}^{\kappa_0} \phi_m)(s) &= \Phi_{\kappa_0}(z_l, z_m)\, Q_m + \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot V_m + (s - z_l) \cdot \nabla_s \Phi_{\kappa_0}(z_l, z_m)\, Q_m \\
&\quad + \int_{\partial D_m} \Bigg(\frac{1}{2}(s - z_l)^2 \cdot \nabla_s \nabla_s \Phi_{\kappa_0}(z_l, t) \\
&\quad + \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!}(t - z_m)^\alpha \int_0^1 (1 - r)^{|\alpha|-1} D_t^\alpha \Phi_{\kappa_0}(z_l, z_m + r(t - z_m))\, dr \\
&\quad + (s - z_l) \cdot \sum_{|\alpha|=1} \frac{|\alpha|}{\alpha!}(t - z_m)^\alpha \int_0^1 (1 - r)^{|\alpha|-1} D_t^\alpha \nabla_s \Phi_{\kappa_0}(z_l, z_m + r(t - z_m))\, dr \\
&\quad + \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha!}(s - z_l)^\alpha \int_0^1 (1 - \beta)^{|\alpha|-1} D_s^\alpha \Phi_{\kappa_0}(z_l + \beta(s - z_l), t)\, d\beta \Bigg) \phi_m(t) d\sigma_m(t).
\end{aligned} \tag{3.9}
$$

Using (2.8) and (3.9), we have

(3.10)

$$
\begin{aligned}
\left(S_{D_l}^0\right)^{-1} (S_{D_m}^{\kappa_0} \phi_m)(s) &= \Phi_{\kappa_0}(z_l, z_m)\, Q_m \left(S_{D_l}^0\right)^{-1}(1)(s) + \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot V_m \left(S_{D_l}^0\right)^{-1}(1)(s) \\
&\quad + \Big[\left(S_{D_l}^0\right)^{-1} (\cdot - z_l) \Big](s) \cdot \nabla_s \Phi_{\kappa_0}(z_l, z_m)\, Q_m \\
&\quad + \left(S_{D_l}^0\right)^{-1} \Bigg(\int_{\partial D_m} \Bigg[\frac{1}{2}(\cdot - z_l)^2 \cdot \nabla_s \nabla_s \Phi_{\kappa_0}(z_l, t) \\
&\quad + \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!}(t - z_m)^\alpha \int_0^1 (1 - r)^{|\alpha|-1} D_t^\alpha \Phi_{\kappa_0}(z_l, z_m + r(t - z_m))\, dr \\
&\quad + (s - z_l) \cdot \sum_{|\alpha|=1} \frac{|\alpha|}{\alpha!}(t - z_m)^\alpha \int_0^1 (1 - r)^{|\alpha|-1} D_t^\alpha \nabla_s \Phi_{\kappa_0}(z_l, z_m + r(t - z_m))\, dr \\
&\quad + \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha!}(\cdot - z_l)^\alpha \int_0^1 (1 - \beta)^{|\alpha|-1} D_s^\alpha \Phi_{\kappa_0}(z_l + \beta(\cdot - z_l), t)\, d\beta \Bigg] \phi_m(t) d\sigma_m(t) \Bigg)(s).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left(S_{D_l}^0\right)^{-1} (S_{D_m}^{\kappa_0} \phi_m)(s) &= \Phi_{\kappa_0}(z_l, z_m)\, Q_m \left(S_{D_l}^0\right)^{-1}(1)(s) + \nabla_t \Phi_{\kappa_0}(z_l, z_m) \cdot V_m \left(S_{D_l}^0\right)^{-1}(1)(s) \\
&\quad + \Big[\left(S_{D_l}^0\right)^{-1} (\cdot - z_l) \Big](s) \cdot \nabla_s \Phi_{\kappa_0}(z_l, z_m)\, Q_m + O\left(\frac{a^3}{d_{ml}^3} \|\phi_m\| \right) \text{ in } L^2,
\end{aligned} \tag{3.11}
$$

where the last term in (3.10), which is of order $O\left(\frac{a^4}{d_{ml}^4}\|\phi_m\|\right)$, is absorbed in $O\left(\frac{a^3}{d_{ml}^3}\|\phi_m\|\right)$ as $\frac{a}{d_{ml}}=O(1)$. To estimate the other term in (3.7), we note that

$$\begin{aligned}
\left(S^0_{D_l}\right)^{-1} S^{d_{\kappa_l}}_{D_l} \left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(s) &= \left(S^0_{D_l}\right)^{-1} \Bigg(\frac{i\kappa_l}{4\pi}\int_{\partial D_l} \left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t) \\
&\qquad + \int_{\partial D_l}\sum_{n=2}^{\infty}\frac{1}{4\pi}\frac{(i\kappa_l)^n}{n!}|s-t|^{n-1}\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t)\Bigg) \\
&= \left(S^0_{D_l}\right)^{-1} \Bigg(\frac{i\kappa_l}{4\pi}\int_{\partial D_l} \left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t) \\
&\qquad + \frac{(i\kappa_l)^2}{8\pi}\int_{\partial D_l}|s-t|\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t) \\
&\qquad + \int_{\partial D_l}\sum_{n=3}^{\infty}\frac{1}{4\pi}\frac{(i\kappa_l)^n}{n!}|s-t|^{n-1}\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t)\Bigg) \\
&= \left(S^0_{D_l}\right)^{-1} \Bigg(\frac{i\kappa_l}{4\pi}\int_{\partial D_l} \left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t) \\
&+ \frac{(i\kappa_l)^2}{8\pi}\int_{\partial D_l}|s-t|\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t)\Bigg) + O(\frac{a^3}{d_{ml}^2}\|\phi_m\|),\ \text{in } L^2 .
\end{aligned}$$

Using (3.11), we can further write this as

$$\begin{aligned}
&\left(S^0_{D_l}\right)^{-1} S^{d_{\kappa_l}}_{D_l} \left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(s) \\
&= \frac{i\kappa_l}{4\pi}\left(S^0_{D_l}\right)^{-1}\Bigg(\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l + \nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m Cap_l \\
&\quad + \nabla_s\Phi_{\kappa_0}(z_l,z_m)\,Q_m\cdot\int_{\partial D_l}\left[\left(S^0_{D_l}\right)^{-1}(\cdot - z_l)\right](t)d\sigma_l(t) + O\left(\frac{a^4}{d_{ml}^3}\|\phi_m\|\right)_{L^\infty}\Bigg) \\
&\quad + \frac{(i\kappa_l)^2}{8\pi}\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|s-t|\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t)\right) + O\Big(\frac{a^3}{d_{ml}^2}\|\phi_m\|\Big) \\
&= \frac{i\kappa_l}{4\pi}\Bigg(\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l + \nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m Cap_l \\
&\quad + \nabla_s\Phi_{\kappa_0}(z_l,z_m)\,Q_m\cdot\int_{\partial D_l}\left[\left(S^0_{D_l}\right)^{-1}(\cdot - z_l)\right](t)d\sigma_l(t) + O\left(\frac{a^4}{d_{ml}^3}\|\phi_m\|\right)_{L^\infty}\Bigg)\left(S^0_{D_l}\right)^{-1}(1) \\
&\quad + \frac{(i\kappa_l)^2}{8\pi}\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|s-t|\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t)\right) + O\Big(\frac{a^3}{d_{ml}^2}\|\phi_m\|\Big),\ \text{in } L^2 ,
\end{aligned}$$

where by $(\cdot)_{L^\infty}$ we mean the point-wise error estimate. Therefore, using (2.8) and the fact $V_m = O(a^2\|\phi_m\|)$, we get

(3.12)

$$\begin{aligned}
\left(S^0_{D_l}\right)^{-1} S^{d_{\kappa_l}}_{D_l} \left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(s) &= \frac{i\kappa_l}{4\pi}\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l\left(S^0_{D_l}\right)^{-1}(1)(s) \\
&+ \frac{(i\kappa_l)^2}{8\pi}\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|s-t|\left(S^0_{D_l}\right)^{-1} (S^{\kappa_0}_{D_m}\phi_m)(t)\, d\sigma_l(t)\right) + O(\frac{a^3}{d_{ml}^2}\|\phi_m\|)
\end{aligned}$$

$$= \frac{i\kappa_l}{4\pi}\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l\left(S^0_{D_l}\right)^{-1}(1)(s) + O(\frac{a^3}{d^2_{ml}}\|\phi_m\|),\ \text{in } L^2\ ,$$

where we have also used the estimate

$$\left\|\left(S^0_{D_l}\right)^{-1}\left(\int_{\partial D_l}|\cdot - t|\left(S^0_{D_l}\right)^{-1}(S^{\kappa_0}_{D_m}\phi_m)(t))d\sigma_l(t)\right)(s)\right\|_{L^2(\partial D_l)} = O\left(\frac{a^4}{d_{ml}}\|\phi_m\|\right),$$

the proof of which follows by a similar argument as in the proof of lemma 2.2.

Now, by making use of (3.11-3.12) in (3.7), we obtain

$$\begin{aligned}
\psi_l(s) =& \phi_l(s) + \left(S^0_{D_l}\right)^{-1}\left(u^I(z_l)\right) + \frac{i(\kappa_0-\kappa_l)}{4\pi}Q_l\left(S^0_{D_l}\right)^{-1}(1)(s) \\
&+ \sum_{\substack{m=1\\m\neq l}}^{M}\Bigg[\Phi_{\kappa_0}(z_l,z_m)\,Q_m\left(S^0_{D_l}\right)^{-1}(1)(s) + \nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m\left(S^0_{D_l}\right)^{-1}(1)(s) \\
&\quad + \left[\left(S^0_{D_l}\right)^{-1}(\cdot - z_l)\right](s)\cdot\nabla_s\Phi_{\kappa_0}(z_l,z_m)\,Q_m - \frac{i\kappa_l}{4\pi}\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l\left(S^0_{D_l}\right)^{-1}(1)\Bigg] \\
&+ O(a) + O(a^2\|\phi_l\|) + O\left(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d^3_{lm}}\|\phi_m\|\right),\ \text{in } L^2,
\end{aligned}\tag{3.13}$$

whence (2.13) follows. The approximation (2.14) can then be obtained by integrating (3.13) on ∂D_l as follows.

$$\begin{aligned}
\int_{\partial D_l}(\psi_l-\phi_l) =& Cap_l\ u^I(z_l) + \frac{i(\kappa_0-\kappa_l)}{4\pi}Q_l Cap_l \\
&+ \sum_{\substack{m=1\\m\neq l}}^{M}\Bigg[\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l + \nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m Cap_l \\
&\quad - \frac{i\kappa_l}{4\pi}\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l^2 + \nabla_s\Phi_{\kappa_0}(z_l,z_m)\,Q_m\cdot\int_{\partial D_l}\left(S^0_{D_l}\right)^{-1}(\cdot - z_l)(s)d\sigma_l(s)\Bigg] \\
&+ \underbrace{O\left(a^2 + a^3\|\phi_l\| + \sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^4}{d^3_{lm}}\|\phi_m\|\right)}_{=:Err7_l},
\end{aligned}\tag{3.14}$$

where $Err7_l$ is in the point-wise sense.

3.4. **Proof of lemma 2.4.** The proof of (2.25) follows as in the case of (2.10). To prove (2.26), we note that from the definition of g_l, we have

$$\begin{aligned}
g_l(s) =& \frac{i}{4\pi}(\kappa_0-\kappa_l)(S^{\kappa_0}_{D_l})^{-1}\left(\int_{\partial D_l}\psi_l\right)(s) \\
=& \frac{i}{4\pi}(\kappa_0-\kappa_l)\left[(S^0_{D_l})^{-1} + \sum_{n=1}^{\infty}(-1)^n(S^0_{D_l})^{-1}\left(S^{d_{\kappa_0}}_{D_l}(S^0_{D_l})^{-1}\right)^n\right]\left(\int_{\partial D_l}\psi_l\right)(s) \\
=& \frac{i}{4\pi}(\kappa_0-\kappa_l)(S^0_{D_l})^{-1}\left(\int_{\partial D_l}\psi_l\right)(s) + O(a^2\|\psi_l\|),\ \text{in } L^2,
\end{aligned}$$

where we use (2.7) applied to ψ_l with $n=0$ and the fact that for any $f \in L^2(\partial D_l)$, $\|S_{D_l}^{d_{\kappa_0}} f\|_{H^1(\partial D_l)} = O(a^2\|f\|_{L^2(\partial D_l)})$.
Using (3.13) and (3.14), we can write this as

$$
\begin{aligned}
g_l(s) =& \frac{i}{4\pi}(\kappa_0-\kappa_l)\left[\int_{\partial D_l}\psi_l\right](S_{D_l}^0)^{-1}(1)(s) + O\left(a^2 + a^2\|\phi_l\| + \sum_{m\neq l}\frac{a^3}{d_{ml}}\|\phi_m\|\right), \text{ in } L^2 \\
=& \frac{i}{4\pi}(\kappa_0-\kappa_l)\left[Cap_l\ u^I(z_l) + \int_{\partial D_l}\phi_l + \sum_{m\neq l}\Phi_{\kappa_0}(z_l,z_m)\,Q_m Cap_l\right](S_{D_l}^0)^{-1}(1)(s) \\
&+ O\left(a^2 + a^2\|\phi_l\| + \sum_{m\neq l}\frac{a^3}{d_{ml}^2}\|\phi_m\|\right), \text{ in } L^2,
\end{aligned} \tag{3.15}
$$

whence (2.26) follows.

3.5. **Proof of Proposition 2.9.** First of all, we note that using the definition of g_l and $\tilde{g}_l$ and (2.9), (2.22) can be rewritten as

$$
\begin{aligned}
\phi_l =& -[\lambda_l\ Id + (K_{D_l}^0)^*]^{-1}\Big[\frac{\partial u^I}{\partial \nu^l} + \sum_{\substack{m=1\\ m\neq l}}^{M}\frac{\partial(S_{D_m}^{\kappa_0}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l} + [(K_{D_l}^{\kappa_0})^* - (K_{D_l}^0)^*]\phi_l \\
&+ \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}[(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*]\psi_l \\
&+ \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}[\frac{1}{2}Id + (K_{D_l}^{\kappa_0})^*]\left(S_{D_l}^{\kappa_0}\right)^{-1}\left(S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}\right)\psi_l\Big], \text{ on } \partial D_l.
\end{aligned} \tag{3.16}
$$

Then we have

$$
\begin{aligned}
&\int_{\partial D_l}\Big(A_l(s)-\hat{A}_l\Big)\phi_l(s)\ d\sigma_l(s) \\
&= -\int_{\partial D_l}\left[\Big(\lambda_l\ Id + K_{D_l}^0\Big)^{-1}\Big(A_l(\cdot)-\hat{A}_l\Big)\right](s)\left[\frac{\partial u^I}{\partial\nu^l}(s) + \Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\left([(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*]\psi_l(s)\right.\right. \\
&\left.\left. +[\frac{1}{2}Id + (K_{D_l}^{\kappa_0})^*]\left(S_{D_l}^{\kappa_0}\right)^{-1}\left(S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}\right)\psi_l(s)\right) + \sum_{\substack{m=1\\ m\neq l}}^{M}\frac{\partial(S_{D_m}^{\kappa_0}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l}(s) + [(K_{D_l}^{\kappa_0})^* - (K_{D_l}^0)^*]\phi_l(s)\right]d\sigma_l(s).
\end{aligned} \tag{3.17}
$$

To estimate the terms in the right hand side of equation (3.17), we make use of (2.34) and the facts that $A_l - \hat{A}_l \in L_0^2$, and the operator $\Big(\lambda_l\ Id + K_{D_l}^0\Big)^{-1}$ maps L_0^2 to L_0^2 and is uniformly bounded with respect to a and proceed as follows. By Cauchy-Schwarz inequality and as $\|A_l - \hat{A}_l\|_{L^2(\partial D_l)} = O(a^3)$, we have

$$
\int_{\partial D_l}\left[\Big(\lambda_l Id + K_{D_l}^0\Big)^{-1}(A_l(\cdot)-\hat{A}_l)\right](s)\frac{\partial u^I}{\partial\nu^l}(s)d\sigma_l(s) = O(a^4), \tag{3.18}
$$

and as $\|(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*\|_{L^2(\partial D_l)} = O(a^2)$, we get

$$
\int_{\partial D_l}\left[\Big(\lambda_l Id + K_{D_l}^0\Big)^{-1}(A_l(\cdot)-\hat{A}_l)\right](s)\ [(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*]\psi_l(s)d\sigma_l(s) = O(a^3\cdot a^2\|\psi_l\|) = O(a^5\|\psi_l\|). \tag{3.19}
$$

Similarly, we have

$$
\int_{\partial D_l}\left[\Big(\lambda_l Id + K_{D_l}^0\Big)^{-1}(A_l(\cdot)-\hat{A}_l)\right](s)\ [(K_{D_l}^{\kappa_0})^* - (K_{D_l}^0)^*]\phi_l(s)d\sigma_l(s) = O(a^5\|\phi_l\|). \tag{3.20}
$$

Next for $s \in \partial D_l$, we use the expansion $\frac{\partial (S^{\kappa_0}_{D_m}\phi_m)}{\partial \nu^l}(s) = \nabla_x \Phi_{\kappa_0}(z_l, z_m) \cdot \nu_l(s) Q_m + O\left(\frac{a^3}{d^3_{ml}}\|\phi_m\|\right)$ with the error estimate in the L^2 sense, to deduce that

$$\int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}(A_l(\cdot) - \hat{A}_l)\right](s) \sum_{\substack{m=1\\m\neq l}}^{M} \frac{\partial (S^{\kappa_0}_{D_m}\phi_m)}{\partial \nu^l}(s) d\sigma_l(s)$$
$$= R_l \cdot \sum_{\substack{m=1\\m\neq l}}^{M} \nabla_s \Phi_{\kappa_0}(z_l, z_m) Q_m + O\left(a^3 \sum_{\substack{m=1\\m\neq l}}^{M} \frac{a^3}{d^3_{ml}} \|\phi_m\|\right), \tag{3.21}$$

where the term R_l behaves as $O(a^4)$ and is defined as

$$R_l := \int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}(A_l(\cdot) - \hat{A}_l)\right](s)\nu_l(s)\, d\sigma_l(s). \tag{3.22}$$

The remaining term can be dealt with in the following manner. First we write

$$\int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}\left(A_l(\cdot) - \hat{A}_l\right)\right](s)\ [\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*]\left(S^{\kappa_0}_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s)\ d\sigma_l(s)$$
$$= \left(\frac{1}{2} - \lambda_l\right)\int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}\left(A_l(\cdot) - \hat{A}_l\right)\right](s)\ \left(S^{\kappa_0}_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s)\ d\sigma_l(s)$$
$$+ \int_{\partial D_l}\left(A_l(s) - \hat{A}_l\right)\ \left(S^{\kappa_0}_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s)\ d\sigma_l(s)$$
$$+ \int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}\left(A_l(\cdot) - \hat{A}_l\right)\right](s)\ [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*]\left(S^{\kappa_0}_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s)\ d\sigma_l(s)$$
$$= \left(\frac{1}{2} - \lambda_l\right)\int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}\left(A_l(\cdot) - \hat{A}_l\right)\right](s)\ \sum_{n=0}^{\infty}(-1)^n\left(\left(S^0_{D_l}\right)^{-1} S^{d_{\kappa_0}}_{D_l}\right)^n\left(S^0_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s) d\sigma_l(s)$$
$$+ \int_{\partial D_l}\left(A_l(\cdot) - \hat{A}_l\right)(s)\ \left[Id + \sum_{n=1}^{\infty}(-1)^n\left(\left(S^0_{D_l}\right)^{-1} S^{d_{\kappa_0}}_{D_l}\right)^n\right]\left(S^0_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s) d\sigma_l(s)$$
$$+ \int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}\left(A_l(\cdot) - \hat{A}_l\right)\right](s)\ [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*]\left(S^{\kappa_0}_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s) d\sigma_l(s)$$
$$= \frac{\rho_l}{\rho_l - \rho_0} O(a^4\|\psi_l\|) + O(a^5\|\psi_l\|) + O(a^6\|\psi_l\|) + \int_{\partial D_l}\left(A_l(s) - \hat{A}_l\right)\ \left(S^0_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s) d\sigma_l(s)$$
$$= \frac{i}{4\pi}(\kappa_0 - \kappa_l)\left(\int_{\partial D_l}\psi_l(t)\ d\sigma_l(t)\right)\int_{\partial D_l}\left(A_l(s) - \hat{A}_l\right)\ \left(S^0_{D_l}\right)^{-1}(1)(s) d\sigma_l(s) + O(a^5\|\psi_l\|)$$
$$+ \int_{\partial D_l}\left(A_l(s) - \hat{A}_l\right)\ \left(S^0_{D_l}\right)^{-1}\left[\sum_{n=2}^{\infty}\frac{i^n(\kappa_0^n - \kappa_l^n)}{4\pi n!}\int_{\partial D_l}|t - s|^{n-1}\psi_l(t)\, d\sigma_l(t)\right] d\sigma_l(s),$$

where we use lemma 2.3 and the fact that $\rho_l \sim a^{1+\gamma}, \gamma \geq 0$.
Using (2.7) and (2.32), we can then deduce

(3.23)
$$\int_{\partial D_l} \left[\left(\lambda_l Id + K^0_{D_l}\right)^{-1}\left(A_l(\cdot) - \hat{A}_l\right)\right](s)\ [\frac{1}{2}Id + (K^{\kappa_0}_{D_l})^*]\left(S^{\kappa_0}_{D_l}\right)^{-1}\left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right)\psi_l(s)\ d\sigma_l(s)$$
$$= \frac{i}{4\pi}(\kappa_0 - \kappa_l)\left(\int_{\partial D_l}\psi_l(t) d\sigma_l(t)\right)\int_{\partial D_l}\left(A_l(s) - \hat{A}_l\right)\ \left(S^0_{D_l}\right)^{-1}(1)(s)\ d\sigma_l(s) + O(a^5\|\psi_l\|) + O(a^3 \cdot a^2\|\psi_l\|)$$
$$= -\frac{i}{4\pi}(\kappa_0 - \kappa_l)\left(8\pi|D_l| + \hat{A}_l Cap_l\right)\int_{\partial D_l}\psi_l(t)\ d\sigma_l(t) + O(a^5\|\psi_l\|).$$

Now by substituting (3.18-3.23) in (3.17) and using (2.14), we have

$$
\begin{aligned}
(3.24)\quad & \int_{\partial D_l} \Big(A_l(s) - \hat{A}_l\Big)\phi_l(s)d\sigma_l(s) \\
&= -R_l \cdot \sum_{\substack{m=1\\ m\neq l}}^{M} \nabla_s \Phi^{\kappa_0}(z_l, z_m) Q_m + \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1} \frac{i}{4\pi}(\kappa_0 - \kappa_l)\Big(8\pi|D_l| + \hat{A}_l Cap_l\Big) \int_{\partial D_l} \psi_l(s)\, d\sigma_l(s) \\
&\quad + O\left(a^4 + a^3 \sum_{\substack{m=1\\ m\neq l}}^{M} \frac{a^3}{d_{ml}^3}\|\phi_m\| + a^5\|\phi_l\| + a^5\|\psi_l\|\right) \\
&= -R_l \cdot \sum_{\substack{m=1\\ m\neq l}}^{M} \nabla_s \Phi^{\kappa_0}(z_l, z_m) Q_m \\
&\quad + \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1} \frac{i}{4\pi}(\kappa_0 - \kappa_l)\left[Q_l + \sum_{m\neq l} Cap_l \Phi_{\kappa_0}(z_l, z_m) Q_m\right]\Big(8\pi|D_l| + \hat{A}_l Cap_l\Big) \\
&\quad + O\left(a^4 + a^3 \sum_{\substack{m=1\\ m\neq l}}^{M} \frac{a^3}{d_{ml}^3}\|\phi_m\| + a^5\|\phi_l\| + a^5\|\psi_l\|\right),
\end{aligned}
$$

whence (2.43) follows.

3.6. **Proof of proposition 2.7.** We note that using (3.16), we can write

(3.25)

$$
\begin{aligned}
\int_{\partial D_l} (s - z_l)_p\, \phi_l\, d\sigma_l(s) &= \underbrace{-\int_{\partial D_l} (s - z_l)_p\, [\lambda_l Id + (K_{D_l}^0)^*]^{-1} \frac{\partial u^I}{\partial \nu^l}\, d\sigma_l(s)}_{:=E_l^p} \\
&\quad \underbrace{- \sum_{\substack{m=1\\ m\neq l}}^{M} \int_{\partial D_l} (s - z_l)_p\, [\lambda_l Id + (K_{D_l}^0)^*]^{-1} \frac{\partial (S_{D_m}^{\kappa_0}\phi_m)}{\partial \nu^l}\Big|_{\partial D_l}\, d\sigma_l(s)}_{:=B_{\phi l}^p} \\
&\quad \underbrace{- \int_{\partial D_l} (s - z_l)_p\, [\lambda_l Id + (K_{D_l}^0)^*]^{-1}[(K_{D_l}^{\kappa_0})^* - (K_{D_l}^0)^*]\phi_l\, d\sigma_l(s)}_{=:K_{\phi l}^p} \\
&\quad - \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1} \underbrace{\int_{\partial D_l} (s - z_l)_p\, [\lambda_l Id + (K_{D_l}^0)^*]^{-1}[(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*]\psi_l\, d\sigma_l(s)}_{=:K_{\psi l}^p} \\
&\quad - \Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1} \underbrace{\int_{\partial D_l} (s - z_l)_p\, [\lambda_l Id + (K_{D_l}^0)^*]^{-1}[\frac{1}{2} Id + (K_{D_l}^{\kappa_0})^*]\left(S_{D_l}^{\kappa_0}\right)^{-1}\left(S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}\right)\psi_l\, d\sigma_l(s)}_{=:K_{Ggl}^p}.
\end{aligned}
$$

Let us recall that $\overline{z}_l = \frac{1}{|\partial D_l|}\int_{\partial D_l} s\, d\sigma_l(s)$. By definition, we have $(s - \overline{z}_l) \in (L_0^2(\partial D_l))^3$. In the sequel, we will repeatedly use this property.

To estimate E_l^p, using $\overline{z}_l$ we write

(3.26)
$$\begin{aligned} E_l^p &:= \int_{\partial D_l} (s-z_l)_p \; [\lambda_l Id + (K^0_{D_l})^*]^{-1} \frac{\partial u^I}{\partial \nu^l} \; d\sigma_l(s) \\ &= \underbrace{\int_{\partial D_l} (s-\overline{z}_l)_p \; [\lambda_l Id + (K^0_{D_l})^*]^{-1} \frac{\partial u^I}{\partial \nu^l} \; d\sigma_l(s)}_{I} + \underbrace{\int_{\partial D_l} (\overline{z}_l - z_l)_p \; [\lambda_l Id + (K^0_{D_l})^*]^{-1} \frac{\partial u^I}{\partial \nu^l} \; d\sigma_l(s)}_{II}. \end{aligned}$$

To estimate the term II, we proceed as follows. We note that

(3.27)
$$II = \int_{\partial D_l} (\overline{z}_l - z_l)_p \; [\lambda_l Id + (K^0_{D_l})^*]^{-1} \frac{\partial u^I}{\partial \nu^l} \; d\sigma_l(s) = (\overline{z}_l - z_l)_p \int_{\partial D_l} \underbrace{[\lambda_l Id + (K^0_{D_l})^*]^{-1} \frac{\partial u^I}{\partial \nu^l}}_{f} \; d\sigma_l(s),$$

and we can write

$$[\lambda_l Id + (K^0_{D_l})^*] f = \frac{\partial u^I}{\partial \nu^l}.$$

Now integrating over ∂D_l, we find that

$$\begin{aligned} \int_{\partial D_l} [\lambda_l Id + (K^0_{D_l})^*] f(s) \; d\sigma_l(s) &= \int_{\partial D_l} \frac{\partial u^I}{\partial \nu^l}(s) \; d\sigma_l(s) = -\kappa_0^2 \int_{D_l} u^I(x) \; dx \\ \Rightarrow \Big(\lambda_l - \frac{1}{2}\Big) \int_{\partial D_l} f(s) \; d\sigma_l(s) &= \int_{\partial D_l} f(s) \; [\lambda_l Id + K^0_{D_l}](1) \; d\sigma_l(s) = -\kappa_0^2 \int_{D_l} u^I(x) \; dx \\ \Rightarrow \int_{\partial D_l} f(s) \; d\sigma_l(s) &= -\Big(\lambda_l - \frac{1}{2}\Big)^{-1} \kappa_0^2 \int_{D_l} u^I(x) \; dx. \end{aligned} \tag{3.28}$$

Using this in (3.27), it follows that

$$II = -(\overline{z}_l - z_l)_p \Big(\lambda_l - \frac{1}{2}\Big)^{-1} \kappa_0^2 \int_{D_l} u^I(x) \; dx. \tag{3.29}$$

In order to estimate I, we note that,

$$I = \int_{\partial D_l} (s-\overline{z}_l)_p \; [\lambda_l Id + (K^0_{D_l})^*]^{-1} \frac{\partial u^I}{\partial \nu^l} \; d\sigma_l(s) = \int_{\partial D_l} [\lambda_l Id + K^0_{D_l}]^{-1} (s-\overline{z}_l)_p \; \frac{\partial u^I}{\partial \nu^l} \; d\sigma_l(s).$$

We recall that $\big[\lambda Id + K^0_{D_l}\big]^{-1} : L^2_0(\partial D_l) \to L^2_0(\partial D_l)$ is uniformly bounded with respect to λ for $|\lambda| \in [\frac{1}{2}, +\infty)$. Using the fact that $(s - \overline{z}_l)$ is mean-free, we can now conclude that

(3.30)
$$|I| \leq \| [\lambda_l Id + K^0_{D_l}]^{-1} (s-\overline{z}_l)_p \|_{L^2(\partial D_l)} \Big\| \frac{\partial u^I}{\partial \nu^l} \Big\|_{L^2(\partial D_l)} \leq C \| (s-\overline{z}_l)_p \|_{L^2(\partial D_l)} \Big\| \frac{\partial u^I}{\partial \nu^l} \Big\|_{L^2(\partial D_l)} = O(a^3).$$

Combining (3.29) and (3.30), we can therefore write

$$\begin{aligned} -E_l^p &= (\overline{z}_l - z_l)_p \Big(\lambda_l - \frac{1}{2}\Big)^{-1} \kappa_0^2 \int_{D_l} u^I(x) \; dx + O(a^3) \\ &= (\overline{z}_l - z_l)_p \Big(\lambda_l - \frac{1}{2}\Big)^{-1} \kappa_0^2 u^I(z_l) |D_l| + O(a^3) + O(a^5 \rho_l^{-1}) = O(a^{3-\gamma}), \end{aligned} \tag{3.31}$$

as $\lambda_l - \frac{1}{2} \sim \rho_l$ and we assume that $\rho_l \simeq a^{1+\gamma}$, with $\gamma \geq 0$.
Next let us estimate the term K^p_{Ggl}.

$$K^p_{Ggl} = \int_{\partial D_l} (s-z_l)_p \; [\lambda_l Id + (K^0_{D_l})^*]^{-1} \Big[\frac{1}{2} Id + (K^{\kappa_0}_{D_l})^*\Big] \left(S^{\kappa_0}_{D_l}\right)^{-1} \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s) \; d\sigma_l(s)$$

$$= \int_{\partial D_l} [\lambda_l Id + K^0_{D_l}]^{-1}(s-z_l)_p \cdot \left[\frac{1}{2} Id + (K^{\kappa_0}_{D_l})^*\right] \left(S^{\kappa_0}_{D_l}\right)^{-1} \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s)$$
$$= \left(\frac{1}{2} - \lambda_l\right) \int_{\partial D_l} [\lambda_l Id + K^0_{D_l}]^{-1}(s-z_l)_p \cdot \left(S^{\kappa_0}_{D_l}\right)^{-1} \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s)$$
$$+ \int_{\partial D_l} (s-z_l)_p \ \left(S^{\kappa_0}_{D_l}\right)^{-1} \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s)$$
$$+ \int_{\partial D_l} [\lambda_l Id + K^0_{D_l}]^{-1}(s-z_l)_p \cdot [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \left(S^{\kappa_0}_{D_l}\right)^{-1} \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s)$$
$$= \frac{1}{2}\frac{\rho_l}{\rho_l - \rho_0} \int_{\partial D_l} (s-z_l)_p [\lambda_l Id + (K^0_{D_l})^*]^{-1} \left(S^0_{D_l}\right)^{-1} \sum_{n=0}^{\infty} \left(S^{d_{\kappa_0}}_{D_l} \left(S^0_{D_l}\right)^{-1}\right)^n \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s)$$
$$+ \int_{\partial D_l} \left(S^0_{D_l}\right)^{-1} (s-z_l)_p \cdot \sum_{n=0}^{\infty} \left(S^{d_{\kappa_0}}_{D_l} \left(S^0_{D_l}\right)^{-1}\right)^n \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s)$$
$$+ \int_{\partial D_l} (s-z_l)_p [\lambda_l Id + (K^0_{D_l})^*]^{-1} [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \left(S^{\kappa_0}_{D_l}\right)^{-1} \left(S^{\kappa_0}_{D_l} - S^{\kappa_l}_{D_l}\right) \psi_l(s)\, d\sigma_l(s).$$

Then using (2.8), we can deduce that

$$K^p_{Ggl} = O(\rho_l a^2 \rho_l^{-1} a^{-1} a^2 \|\psi_l\|) + O(a\ a^2 \|\psi_l\|) + O(a^2 \rho_l^{-1} a^2 a^{-1} a^2 \|\psi_l\|) \tag{3.32}$$
$$= O(a^3 \|\psi_l\|) + O(a^5 \rho_l^{-1} \|\psi_l\|).$$

Next, we shall estimate the term $K^p_{\phi_l}$.

$$K^p_{\phi_l} = \int_{\partial D_l} (s-z_l)_p\ [\lambda_l Id + (K^0_{D_l})^*]^{-1} [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \phi_l(s)\, d\sigma_l(s)$$
$$= \underbrace{\int_{\partial D_l} (s-\overline{z}_l)_p\ [\lambda_l Id + (K^0_{D_l})^*]^{-1} [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \phi_l(s)\, d\sigma_l(s)}_{K^{p,1}_{\phi_l}}$$
$$+ \underbrace{\int_{\partial D_l} (\overline{z}_l - z_l)_p\ [\lambda_l Id + (K^0_{D_l})^*]^{-1} [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \phi_l(s)\, d\sigma_l(s)}_{K^{p,2}_{\phi_l}}.$$

Now, the first term in the right hand side above can be estimated in the following manner using the fact that $(s-\overline{z}_l)_p \in L^2_0(\partial D_l)$.

$$K^{p,1}_{\phi_l} = \int_{\partial D_l} (s-\overline{z}_l)_p\ [\lambda_l Id + (K^0_{D_l})^*]^{-1} [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \phi_l(s)\, d\sigma_l(s)$$
$$= \int_{\partial D_l} [\lambda_l Id + K^0_{D_l}]^{-1} (s-\overline{z}_l)_p \cdot [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \phi_l(s)\, d\sigma_l(s)$$
$$= O(a^2\, a^2 \|\phi_l\|) = O(a^4 \|\phi_l\|).$$

To estimate the second term $K^{p,2}_{\phi_l}$, we observe that

$$K^{p,2}_{\phi_l} = (\overline{z}_l - z_l)_p \int_{\partial D_l} [\lambda_l Id + K^0_{D_l}]^{-1}(1) \cdot [(K^{\kappa_0}_{D_l})^* - (K^0_{D_l})^*] \phi_l(s)\, d\sigma_l(s)$$
$$= (\overline{z}_l - z_l)_p \int_{\partial D_l} [K^{\kappa_0}_{D_l} - K^0_{D_l}] [\lambda_l Id + K^0_{D_l}]^{-1}(1) \cdot \phi_l(s)\, d\sigma_l(s).$$

Let us denote $h := [\lambda_l Id + K^0_{D_l}]^{-1}(1)$. Then

$$\begin{aligned}
K^{p,2}_{\phi_l} &= (\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} (\nabla_t \Phi_{\kappa_0}(s,t) - \nabla_t \Phi_0(s,t)) \cdot \nu^l(t) h(t)\ d\sigma_l(t)\Big] d\sigma_l(s) \\
&= (\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} \Big(\sum_{n=2}^{\infty} -\frac{\kappa_0^n i^n}{4\pi n!}(n-1)|s-t|^{n-2}\frac{(s-t)}{|s-t|}\Big) \cdot \nu^l(t) h(t) d\sigma_l(t)\Big] d\sigma_l(s) \\
&= \frac{1}{8\pi}\kappa_0^2 (\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} \frac{(s-t)}{|s-t|} \cdot \nu^l(t) h(t)\ d\sigma_l(t)\Big] d\sigma_l(s) \\
&\quad + \frac{i\kappa_0^3}{12\pi}(\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} [(s-t)\cdot \nu^l(t)] h(t) d\sigma_l(t)\Big] d\sigma_l(s) \\
&\quad + (\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} \Big(\sum_{n=4}^{\infty} -\frac{\kappa_0^n i^n}{4\pi n!}(n-1)|s-t|^{n-2}\frac{(s-t)}{|s-t|}\Big) \cdot \nu^l(t) h(t) d\sigma_l(t)\Big] d\sigma_l(s).
\end{aligned}$$

Note that the third term in the right hand side of the above identity is of order $O(a^6 \rho_l^{-1} \|\phi_l\|)$. To understand the first two terms better, we now proceed as follows.
Let us write

$$\int_{\partial D_l} [(s-t)\cdot \nu^l(t)] h(t) d\sigma_l(t) = \int_{\partial D_l} \underbrace{[\lambda_l Id + (K^0_{D_l})^*]^{-1}[(s-t)\cdot \nu^l(t)]}_{f_1}\ d\sigma_l(t).$$

Now $[\lambda_l Id + (K^0_{D_l})^*] f_1 = (s-t)\cdot \nu^l(t)$ and hence

$$\begin{aligned}
&\int_{\partial D_l} [\lambda_l Id + (K^0_{D_l})^*] f_1(t)\ d\sigma_l(t) = \int_{\partial D_l} (s-t)\cdot \nu^l(t) d\sigma_l(t) \\
&\Longrightarrow \Big(\lambda_l - \frac{1}{2}\Big)\int_{\partial D_l} f_1(t)\ d\sigma_l(t) = \int_{\partial D_l} f_1(t)[\lambda_l Id + K^0_{D_l}](1)\ d\sigma_l(t) = \int_{\partial D_l} (s-t)\cdot \nu^l(t) d\sigma_l(t) = -3|D_l| \\
&\Longrightarrow \int_{\partial D_l} [(s-t)\cdot \nu^l(t)] h(t) d\sigma_l(t) = -3\Big(\lambda_l - \frac{1}{2}\Big)^{-1}|D_l|.
\end{aligned}$$

Therefore

$$\frac{i\kappa_0^3}{12\pi}(\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} [(s-t)\cdot \nu^l(t)] h(t) d\sigma_l(t)\Big] d\sigma_l(s) = -\frac{i\kappa_0^3}{4\pi}\Big(\lambda_l - \frac{1}{2}\Big)^{-1}|D_l|(\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s) d\sigma_l(s).$$

Next, we consider the term $\frac{1}{8\pi}\kappa_0^2 (\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} \frac{(s-t)}{|s-t|} \cdot \nu^l(t) h(t)\ d\sigma_l(t)\Big] d\sigma_l(s)$. Again we write

$$\int_{\partial D_l} \frac{(s-t)}{|s-t|} \cdot \nu^l(t) h(t)\ d\sigma_l(t) = \int_{\partial D_l} \underbrace{[\lambda_l Id + (K^0_{D_l})^*]^{-1}\Big(\frac{(s-t)}{|s-t|}\cdot \nu^l(t)\Big)}_{f_2}\ d\sigma_l(t),$$

then arguing as in the previous case, we obtain

$$\int_{\partial D_l} f_2(t)\ d\sigma_l(t) = \Big(\lambda_l - \frac{1}{2}\Big)^{-1} \int_{\partial D_l} \frac{s-t}{|s-t|}\cdot \nu^l(t)\ d\sigma_l(t) = \Big(\lambda_l - \frac{1}{2}\Big)^{-1} A_l(s)$$

and hence using (2.43), it follows that

$$\begin{aligned}
&\frac{1}{8\pi}\kappa_0^2 (\overline{z}_l - z_l)_p \int_{\partial D_l} \phi_l(s)\Big[\int_{\partial D_l} \frac{(s-t)}{|s-t|} \cdot \nu^l(t) h(t)\ d\sigma_l(t)\Big] d\sigma_l(s) \\
&= \frac{1}{8\pi}\kappa_0^2 (\overline{z}_l - z_l)_p (\lambda_l - \frac{1}{2})^{-1} \int_{\partial D_l} A_l(s)\phi_l(s)\ d\sigma_l(s) \\
&= \frac{1}{8\pi}\kappa_0^2 (\overline{z}_l - z_l)_p (\lambda_l - \frac{1}{2})^{-1} \hat{A}_l Q_l + \frac{1}{8\pi}\kappa_0^2 (\overline{z}_l - z_l)_p (\lambda_l - \frac{1}{2})^{-1} \int_{\partial D_l} (A_l - \hat{A}_l)(s)\phi_l(s)\ d\sigma_l(s)
\end{aligned}$$

$$= \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l + \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\Big[- R_l \cdot \sum_{\substack{m=1\\m\neq l}}^{M} \nabla_s \Phi_{\kappa_0}(z_l, z_m) Q_m$$
$$- \frac{\rho_0}{\rho_l - \rho_0}\frac{i}{4\pi}(\kappa_0 - \kappa_l)\big[Q_l + \sum_{m\neq l} Cap_l \Phi_{\kappa_0}(z_l, z_m)Q_m\big]\Big(8\pi|D_l| + \hat{A}_l Cap_l\Big)$$
$$+ O\left(a^4 + a^3\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\| + a^5\|\phi_l\| + a^5\|\psi_l\|\right)\Big].$$

Combining the estimates above, we finally have

(3.33)

$$-K^p_{\phi_l} = \frac{i\kappa_0^3}{4\pi}\Big(\lambda_l - \frac{1}{2}\Big)^{-1}|D_l|(\overline{z}_l - z_l)_p Q_l - \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l$$
$$+ \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\Big[R_l \cdot \sum_{\substack{m=1\\m\neq l}}^{M} \nabla_s \Phi_{\kappa_0}(z_l, z_m) Q_m$$
$$+ \frac{\rho_0}{\rho_l - \rho_0}\frac{i}{4\pi}(\kappa_0 - \kappa_l)\big[Q_l + \sum_{m\neq l} Cap_l \Phi_{\kappa_0}(z_l, z_m)Q_m\big]\Big(8\pi|D_l| + \hat{A}_l Cap_l\Big)$$
$$- O\left(a^4 + a^3\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\| + a^5\|\phi_l\| + a^5\|\psi_l\|\right)\Big] + O(a^4\|\phi_l\|) + O(a^6\rho_l^{-1}\|\phi_l\|)$$
$$= -\frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l + O\left(a^{4-\gamma} + a^{3-\gamma}\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\| + a^{5-\gamma}\|\phi_l\| + a^{5-\gamma}\|\psi_l\|\right)$$
$$+ O(a^{4-\gamma}\|\phi_l\|) + O(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^{5-\gamma}}{d_{ml}^2}\|\phi_m\|) + O(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^{5-\gamma}}{d_{ml}}\|\phi_m\|).$$

Proceeding as in the case of $K^p_{\phi_l}$, we can write

$$-K^p_{\psi_l} = \frac{i\kappa_0^3}{4\pi}\Big(\lambda_l - \frac{1}{2}\Big)^{-1}|D_l|(\overline{z}_l - z_l)_p \int_{\partial D_l}\psi_l(s)\ d\sigma_l(s) - \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l\int_{\partial D_l}\psi_l(s)\ d\sigma_l(s)$$
$$- \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\int_{\partial D_l}(A_l - \hat{A}_l)(s)\psi_l(s)\ d\sigma_l(s) + O(a^4\|\psi_l\|) + O(a^6\rho_l^{-1}\|\psi_l\|),$$

and therefore using (2.13),(2.14) and (2.43), it follows that

$$-K^p_{\psi_l} = O(a^6\rho_l^{-1}\|\psi_l\|) + O(a^4\|\psi_l\|)$$
$$+ \Big(\lambda_l - \frac{1}{2}\Big)^{-1}(\overline{z}_l - z_l)_p\Big[\frac{i\kappa_0^3}{4\pi}|D_l| - \frac{1}{8\pi}\kappa_0^2\hat{A}_l\Big]\Big[Q_l + Cap_l\ u^I(z_l) + \frac{i(\kappa_0 - \kappa_l)}{4\pi}Q_l Cap_l$$
$$+ \sum_{\substack{m=1\\m\neq l}}^{M}\Big(\Phi_{k_0}(z_l, z_m)Q_m Cap_l + \nabla_t\Phi_{k_0}(z_l, z_m)\cdot V_m Cap_l - \frac{i\kappa_l}{4\pi}\Phi_{\kappa_0}(z_l, z_m)Q_m C_l^2$$
$$+ \nabla_s\Phi_{\kappa_0}(z_l, z_m)Q_m\int_{\partial D_l}(S^0_{D_l})^{-1}(\cdot - z_l)(s)\ d\sigma_l(s)\Big) + O(a^2 + a^3\|\phi_l\| + \sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^4}{d_{ml}^3}\|\phi_m\|)\Big]$$

$$+\frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\Big[R_l \cdot \sum_{\substack{m=1\\m\neq l}}^{M} \nabla_s\Phi_{\kappa_0}(z_l, z_m)Q_m$$

$$+ \frac{\rho_0}{\rho_l - \rho_0}\frac{i}{4\pi}(\kappa_0 - \kappa_l)\big[Q_l + \sum_{m\neq l} Cap_l\Phi_{\kappa_0}(z_l, z_m)Q_m\big]\Big(8\pi|D_l| + \hat{A}_l Cap_l\Big)$$

$$- O\left(a^4 + a^3\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^3}{d_{ml}^3}\|\phi_m\| + a^5\|\phi_l\| + a^5\|\psi_l\|\right)\Big]$$

$$- \frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)_p(\lambda_l - \frac{1}{2})^{-1}\int_{\partial D_l}(A_l(s) - \hat{A}_l)\Big[\left(S^0_{D_l}\right)^{-1}\left(\frac{i(\kappa_0 - \kappa_l)}{4\pi}Q_l\right)$$

$$+ \left(S^0_{D_l}\right)^{-1}\left(\sum_{m\neq l}\left[\left(1 - \frac{i\kappa_l}{4\pi}Cap_l\right)\Phi_{\kappa_0}(z_l, z_m) + (s - z_l)\cdot\nabla_s\Phi_{\kappa_0}(z_l, z_m)\right]Q_m\right)$$

$$+ \left(S^0_{D_l}\right)^{-1}\left(\sum_{m\neq l}\nabla_t\Phi_{\kappa_0}(z_l, z_m)\cdot V_m\right) + \left(S^0_{D_l}\right)^{-1}u^I(z_l) + O\left(a + a^2\|\phi_l\| + \sum_{m\neq l}\frac{a^3}{d^3}\|\phi_m\|\right)\Big],$$

where the last error estimate mentioned above is in the sense of L^2, while the others are point-wise. Therefore we can write

(3.34)

$$-\left(1 - \frac{\rho_l}{\rho_0}\right)^{-1}K^p_{\psi_l} = -\frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l$$

$$-\frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Cap_l\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l, z_m)Q_m$$

$$-\frac{1}{8\pi}\kappa_0^2\Big(1 - \frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\int_{\partial D_l}(A_l(s) - \hat{A}_l)\Big(\sum_{\substack{m=1\\m\neq l}}^{M}\Phi_{\kappa_0}(z_l, z_m)Q_m\Big)(S^0_{D_l})^{-1}(s)d\sigma_l(s)$$

$$+ O(a^{5-\gamma}\|\psi_l\|) + O(a^4\|\psi_l\|) + O(a^{4-\gamma}\|\phi_l\|) + O(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^{5-\gamma}}{d_{ml}}\|\phi_m\|)$$

$$+ O(a^{3-\gamma}) + O(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^{5-\gamma}}{d_{ml}^2}\|\phi_m\|) + O(\sum_{\substack{m=1\\m\neq l}}^{M}\frac{a^{6-\gamma}}{d_{ml}^3}\|\phi_m\|).$$

Finally, we deal with the term $B^p_{\phi_l}$ in the following manner. As in the previous cases, using $\overline{z}_l$, we first write

$$B^p_{\phi_l} = \sum_{\substack{m=1\\m\neq l}}^{M}\int_{\partial D_l}(s - z_l)_p\,[\lambda_l Id + (K^0_{D_l})^*]^{-1}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}d\sigma_l(s) \tag{3.35}$$

$$= \underbrace{\sum_{\substack{m=1\\m\neq l}}^{M}\int_{\partial D_l}(s - \overline{z}_l)_p\,[\lambda_l Id + (K^0_{D_l})^*]^{-1}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}d\sigma_l(s)}_{B^{p,1}_{\phi_l}}$$

$$+\underbrace{\sum_{\substack{m=1\\m\neq l}}^{M}\int_{\partial D_l}(\overline{z}_l-z_l)_p\,[\lambda_l Id+(K^0_{D_l})^*]^{-1}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}d\sigma_l(s)}_{B^{p,2}_{\phi_l}}.$$

The first term can be dealt with as

$$\begin{aligned}B^{p,1}_{\phi_l}&=\sum_{\substack{m=1\\m\neq l}}^{M}\int_{\partial D_l}(s-\overline{z}_l)_p\,[\lambda_l Id+(K^0_{D_l})^*]^{-1}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}\,d\sigma_l(s)\\&=\sum_{\substack{m=1\\m\neq l}}^{M}\int_{\partial D_l}[\lambda_l Id+K^0_{D_l}]^{-1}(s-\overline{z}_l)_p\cdot\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}d\sigma_l(s)=\sum_{\substack{m=1\\m\neq l}}^{M}O\left(\frac{a^4}{d^2_{ml}}\|\phi_m\|\right),\end{aligned}$$

using the fact that $(s-\overline{z}_l)_p\in L^2_0$. To deal with the second term, we note that

$$\int_{\partial D_l}(\overline{z}_l-z_l)_p\,[\lambda_l Id+(K^0_{D_l})^*]^{-1}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}d\sigma_l(s)=(\overline{z}_l-z_l)_p\int_{\partial D_l}f_3(s)\,d\sigma_l(s),$$

where f_3 satisfies

$$[\lambda_l Id+(K^0_{D_l})^*]f_3=\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}.$$

From this we can deduce, as in the earlier cases and using (2.11), that

$$\begin{aligned}\int_{\partial D_l}f_3(s)\,d\sigma_l(s)&=\Big(\lambda_l-\frac{1}{2}\Big)^{-1}\int_{\partial D_l}\frac{\partial(S^{\kappa_0}_{D_m}\phi_m)}{\partial\nu^l}d\sigma_l(s)\\&=\Big(\lambda_l-\frac{1}{2}\Big)^{-1}\Big[-\kappa_0^2|D_l|\Big(\Phi_{\kappa_0}(z_l,z_m)Q_m+\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m\Big)\\&\qquad-\kappa_0^2\nabla_x\Phi_{\kappa_0}(z_l,z_m)\cdot\Big[\int_{D_l}(x-z_l)dx\Big]Q_m-Err4_m\Big].\end{aligned}$$

Therefore

(3.36)

$$\begin{aligned}-B^p_{\phi_l}&=-\sum_{\substack{m=1\\m\neq l}}^{M}(\overline{z}_l-z_l)_p\Big(\lambda_l-\frac{1}{2}\Big)^{-1}\Big[-\kappa_0^2|D_l|\Big(\Phi_{\kappa_0}(z_l,z_m)Q_m+\nabla_t\Phi_{\kappa_0}(z_l,z_m)\cdot V_m\Big)\\&\qquad-\kappa_0^2\nabla_x\Phi_{\kappa_0}(z_l,z_m)\cdot\Big[\int_{D_l}(x-z_l)dx\Big]Q_m-Err4_m\Big]+O\left(\frac{a^4}{d^2_{ml}}\|\phi_m\|\right)\\&=\sum_{\substack{m=1\\m\neq l}}^{M}(\overline{z}_l-z_l)\Big(\lambda_l-\frac{1}{2}\Big)^{-1}\kappa_0^2|D_l|\Phi_{\kappa_0}(z_l,z_m)Q_m\\&\quad+\sum_{\substack{m=1\\m\neq l}}^{M}\Big[O(\frac{a^{5-\gamma}}{d^2_{ml}}\|\phi_m\|)+O(\frac{a^{6-\gamma}}{d^3_{ml}}\|\phi_m\|)+O\left(\frac{a^4}{d^2_{ml}}\|\phi_m\|\right)\Big].\end{aligned}$$

Using (3.31),(3.32),(3.33), (3.34), (3.36) in (3.25), we can write

$$V_l=\int_{\partial D_l}(s-z_l)\phi_l\,d\sigma_l(s)=V^{dom}_l+V^{rem}_l, \tag{3.37}$$

where

$$V^{dom}_l:=\sum_{\substack{m=1\\m\neq l}}^{M}(\overline{z}_l-z_l)\Big(\lambda_l-\frac{1}{2}\Big)^{-1}\kappa_0^2|D_l|\Phi_{\kappa_0}(z_l,z_m)Q_m-\frac{1}{8\pi}\kappa_0^2(\overline{z}_l-z_l)(\lambda_l-\frac{1}{2})^{-1}\hat{A}_lQ_l$$

$$
\begin{aligned}
&- \frac{1}{8\pi}\kappa_0^2\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l \\
&- \frac{1}{8\pi}\kappa_0^2\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Cap_l \sum_{\substack{m=1\\ m\neq l}}^{M} \Phi_{\kappa_0}(z_l,z_m)Q_m \\
&- \frac{1}{8\pi}\kappa_0^2\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\int_{\partial D_l}(A_l(s)-\hat{A}_l)\Big(\sum_{\substack{m=1\\ m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m\Big)(S^0_{D_l})^{-1}(s)d\sigma_l(s) \\
&= V^{dom}_{l,1} + V^{dom}_{l,2},
\end{aligned}
$$

with

$$
\begin{aligned}
V^{dom}_{l,1} &:= -\frac{1}{8\pi}\kappa_0^2(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l - \frac{1}{8\pi}\kappa_0^2\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Q_l, \\
V^{dom}_{l,2} &:= \sum_{\substack{m=1\\ m\neq l}}^{M}(\overline{z}_l - z_l)\Big(\lambda_l - \frac{1}{2}\Big)^{-1}\kappa_0^2|D_l|\Phi_{\kappa_0}(z_l,z_m)Q_m \\
&\quad - \frac{1}{8\pi}\kappa_0^2\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\hat{A}_l Cap_l \sum_{\substack{m=1\\ m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m \\
&\quad - \frac{1}{8\pi}\kappa_0^2\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}(\overline{z}_l - z_l)(\lambda_l - \frac{1}{2})^{-1}\int_{\partial D_l}(A_l(s)-\hat{A}_l)\Big(\sum_{\substack{m=1\\ m\neq l}}^{M}\Phi_{\kappa_0}(z_l,z_m)Q_m\Big)(S^0_{D_l})^{-1}(s)d\sigma_l(s),
\end{aligned}
$$

and by V^{rem}_l, we denote the rest of the terms. The remainder V^{rem}_l satisfies the estimate

$$
\begin{aligned}
|V^{rem}_l| &= O(a^{3-\gamma}) + O(a^{4-\gamma}\|\phi_l\|) \\
&\quad + O\Big(\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{a^{5-\gamma}}{d^2_{ml}}\|\phi_m\|\Big) + O\Big(\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{a^{5-\gamma}}{d_{ml}}\|\phi_m\|\Big) + O\Big(\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{a^{6-\gamma}}{d^3_{ml}}\|\phi_m\|\Big) + O\Big(\sum_{\substack{m=1\\ m\neq l}}^{M}\frac{a^{4}}{d^2_{ml}}\|\phi_m\|\Big) \\
&= O(a^{3-\gamma}) + O\Big(\Big(a^{4-\gamma} + \frac{a^{5-\gamma}}{d^2} + \frac{a^{5-\gamma}}{d^{3\alpha}} + \frac{a^4}{d^2} + \frac{a^4}{d^{3\alpha}} + \frac{a^{6-\gamma}}{d^3} + \frac{a^{6-\gamma}}{d^{3\alpha+1}}\Big)\|\phi\|\Big),
\end{aligned}
$$

where we use the fact that $\rho_l \simeq a^{1+\gamma}$, with $\gamma \geq 0$ and $0 \leq \alpha \leq 1$. Also note that in the above estimate, we have majorised $\|\phi_i\|$ by $\|\phi\|$.

3.7. **Proof of the invertibility of the algebraic system.** We rewrite (2.59) in the following compact form;

$$
(\mathbf{C_I} + \mathbf{B} + \mathbf{B}' + \mathbf{R}_1 + \mathbf{R}_2)\mathbf{Q} = \mathbf{Y}, \tag{3.38}
$$

where $\mathbf{Q}, \mathbf{Y} \in \mathbb{C}^{M\times 1}$ and $\mathbf{C_I}, \mathbf{B}, \mathbf{B}', \mathbf{R}_1, \mathbf{R}_2 \in \mathbb{C}^{M\times M}$ are defined as

$$
\mathbf{B}(l,m) := \begin{cases} \Phi_{\kappa_0}(z_l,z_m), & \text{if} \quad l \neq m \\ 0, & \text{if} \quad l = m \end{cases}, \tag{3.39}
$$

$$
\mathbf{B}'(l,m) := \begin{cases} \frac{F'_l}{\kappa_l^2}\nabla_x\Phi_{\kappa_0}(z_l,z_m), & \text{if} \quad l \neq m \\ 0, & \text{if} \quad l = m \end{cases}, \tag{3.40}
$$

$$
\mathbf{C_I}(l,m) := \begin{cases} 0, & \text{if} \quad l \neq m \\ \mathbf{C}_l^{-1}, & \text{if} \quad l = m \end{cases}, \tag{3.41}
$$

$$
\mathbf{Q} := \begin{pmatrix} Q_1 & Q_2 & \dots & Q_M \end{pmatrix}^\top \text{ and } \mathrm{Y} := \begin{pmatrix} Y_1 & Y_2 & \dots & Y_M \end{pmatrix}^\top, \tag{3.42}
$$

and $\mathbf{R}_1 = \mathbf{P}\mathbf{P}_1, \mathbf{R}_2 = \mathbf{P}\mathbf{P}_2$ with the matrices $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2$ defined as

$$\mathbf{P}(l,m) := \begin{cases} \nabla_t \Phi_{\kappa_0}(z_l, z_m), & \text{if} \quad l \neq m \\ 0, & \text{if} \quad l = m \end{cases}, \tag{3.43}$$

$$\mathbf{P}_1(m,n) := \begin{cases} \kappa_0^2(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\Big[|D_m| & \\ -\frac{1}{8\pi}(1 - \frac{\rho_m}{\rho_0})^{-1} \int_{\partial D_m} A_m(s)(S^0_{D_m})^{-1}(s) d\sigma_m(s)\Big] \Phi_{\kappa_0}(z_m, z_n), & \text{if} \quad m \neq n \\ 0, & \text{if} \quad m = n \end{cases}, \tag{3.44}$$

$$\mathbf{P}_2(m,n) := \begin{cases} 0, & \text{if} \quad m \neq n \\ -\frac{1}{4\pi}\kappa_0^2(\overline{z}_m - z_m)(\lambda_m - \frac{1}{2})^{-1}\hat{A}_m, & \text{if} \quad m = n \end{cases}. \tag{3.45}$$

Our strategy here is to follow the methodology in [13]. In this direction, we multiply the system with $\mathbf{Q}^r$ and $\mathbf{Q}^i$, add the resulting identities and then use the fact that the matrices $\mathbf{C_I}, \mathbf{B}$ are self-adjoint to derive the inequality

$$\begin{aligned} &\langle \mathbf{C_I}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{B}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{B'}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{C_I}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle + \langle \mathbf{B}^i \mathbf{Q}^i, \mathbf{Q}^i \rangle + \langle \mathbf{B'}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle \\ &+ \langle \mathbf{B'}^i \mathbf{Q}^r, \mathbf{Q}^i \rangle - \langle \mathbf{B'}^i \mathbf{Q}^i, \mathbf{Q}^r \rangle + \langle \mathbf{R_1}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{R_1}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle + \langle \mathbf{R_1}^i \mathbf{Q}^r, \mathbf{Q}^i \rangle - \langle \mathbf{R_1}^i \mathbf{Q}^i, \mathbf{Q}^r \rangle \\ &+ \langle \mathbf{R_2}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{R_2}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle + \langle \mathbf{R_2}^i \mathbf{Q}^r, \mathbf{Q}^i \rangle - \langle \mathbf{R_2}^i \mathbf{Q}^i, \mathbf{Q}^r \rangle \\ &= \langle \mathbf{Y}^r, \mathbf{Q}^r \rangle + \langle \mathbf{Y}^i, \mathbf{Q}^i \rangle \leq 2 \left(\sum_{m=1}^{M} |Y_m|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{M} |Q_m|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.46}$$

Let us first consider the case when $(\mathbf{C}_l^{-1})^r > 0, \ \forall \ l = 1, \dots, M$. In this case, proceeding as in [13], we can obtain

$$\langle \mathbf{B}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{B}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle \geq -\frac{3\tau}{5\pi\, d} \sum_{m=1}^{M} |Q_m|^2, \tag{3.47}$$

where $\tau := \min_{1 \leq j,m \leq M,\ j \neq m} cos(\kappa_0 |z_m - z_j|)$ and is assumed to be non-negative.
We can also observe that

$$\langle \mathbf{C_I}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{C_I}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle \geq \min_m (\mathbf{C}_m^{-1})^r \sum_{m=1}^{M} |Q_m|^2 \geq \frac{\min\limits_{1 \leq m \leq M} (\mathbf{C}_m)^r}{(\max\limits_{1 \leq m \leq M} |\mathbf{C}_m|)^2} \sum_{m=1}^{M} |Q_m|^2. \tag{3.48}$$

We would like to note that $(\mathbf{C}_m^{-1})^r$ and $(\mathbf{C}_m)^r$ have the same sign.
Next using Cauchy-Schwarz inequality, we can write

$$\begin{aligned} \langle \mathbf{B'}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{B'}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle &\geq -\|\mathbf{B'}\|_2 \sum_{m=1}^{M} |Q_m|^2, \\ \langle \mathbf{B'}^i \mathbf{Q}^r, \mathbf{Q}^i \rangle \geq -\|\mathbf{B'}^i\|_2 \|Q^r\| \|Q^i\| &\geq -\|\mathbf{B'}\|_2 \sum_{m=1}^{M} |Q_m|^2, \\ -\langle \mathbf{B'}^i \mathbf{Q}^i, \mathbf{Q}^r \rangle \geq -\|\mathbf{B'}^i\|_2 \|Q^i\| \|Q^r\| &\geq -\|\mathbf{B'}\|_2 \sum_{m=1}^{M} |Q_m|^2, \end{aligned}$$

and therefore

$$\begin{aligned} &\langle \mathbf{B'}^r \mathbf{Q}^r, \mathbf{Q}^r \rangle + \langle \mathbf{B'}^r \mathbf{Q}^i, \mathbf{Q}^i \rangle + \langle \mathbf{B'}^i \mathbf{Q}^r, \mathbf{Q}^i \rangle - \langle \mathbf{B'}^i \mathbf{Q}^i, \mathbf{Q}^r \rangle \\ &\geq -3 \max_{1 \leq m \leq M} \left| \frac{F'_m}{\kappa_m^2} \right| \frac{\kappa_0 + 1}{4\pi} \left[\sum_{\substack{m,l=1 \\ m \neq l}}^{M} \frac{1}{d_{ml}^4} \right]^{\frac{1}{2}} \sum_{m=1}^{M} |Q_m|^2 \end{aligned} \tag{3.49}$$

$$\geq -3 \max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| \frac{\kappa_0+1}{4\pi} C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\sum_{n=1}^{[d^{-\alpha}]}[(2n+1)^3-(2n-1)^3]\frac{1}{n^4d^{4\alpha}}\right]^{\frac{1}{2}} \sum_{m=1}^{M}|Q_m|^2$$

$$\geq -\max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}} \sum_{m=1}^{M}|Q_m|^2 \quad ,$$

where C denotes a generic constant that is bounded in terms of a. Similarly we derive

(3.50)
$$\langle \mathbf{R_1}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle+\langle \mathbf{R_1}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle+\langle \mathbf{R_1}^i\mathbf{Q}^r,\mathbf{Q}^i\rangle-\langle \mathbf{R_1}^i\mathbf{Q}^i,\mathbf{Q}^r\rangle \geq -Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\sum_{m=1}^{M}|Q_m|^2$$

and

(3.51)
$$\langle \mathbf{R_2}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle+\langle \mathbf{R_2}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle+\langle \mathbf{R_2}^i\mathbf{Q}^r,\mathbf{Q}^i\rangle-\langle \mathbf{R_2}^i\mathbf{Q}^i,\mathbf{Q}^r\rangle \geq -Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\sum_{m=1}^{M}|Q_m|^2 \quad .$$

Making use of (3.47-3.51) in (3.46), the required estimate (2.62) follows.
Let us next consider the case when $(\mathbf{C}_l^{-1})^r<0,\ \forall\ l=1,\dots,M$.
In this case, we multiply the identity (3.46) with -1 and note that using Cauchy-Schwarz inequality, we can still write

$$-\langle \mathbf{Y}^r,\mathbf{Q}^r\rangle-\langle \mathbf{Y}^i,\mathbf{Q}^i\rangle \leq 2\left(\sum_{m=1}^{M}|Y_m|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{M}|Q_m|^2\right)^{\frac{1}{2}}.$$

As in the previous case, we derive the estimates

$$-\langle \mathbf{B'}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle-\langle \mathbf{B'}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle-\langle \mathbf{B'}^i\mathbf{Q}^r,\mathbf{Q}^i\rangle+\langle \mathbf{B'}^i\mathbf{Q}^i,\mathbf{Q}^r\rangle \geq -\max_{1\leq m\leq M}\left|\frac{F'_m}{\kappa_m^2}\right| C\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}} \sum_{m=1}^{M}|Q_m|^2,$$

$$-\langle \mathbf{R_1}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle-\langle \mathbf{R_1}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle-\langle \mathbf{R_1}^i\mathbf{Q}^r,\mathbf{Q}^i\rangle+\langle \mathbf{R_1}^i\mathbf{Q}^i,\mathbf{Q}^r\rangle \geq -Ca^{3-\gamma}MM_{max}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\sum_{m=1}^{M}|Q_m|^2$$

and

$$-\langle \mathbf{R_2}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle-\langle \mathbf{R_2}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle-\langle \mathbf{R_2}^i\mathbf{Q}^r,\mathbf{Q}^i\rangle+\langle \mathbf{R_2}^i\mathbf{Q}^i,\mathbf{Q}^r\rangle \geq -Ca^{2-\gamma}\sqrt{MM_{max}}\left[\frac{1}{d^4}+\frac{1}{d^{5\alpha}}\right]^{\frac{1}{2}}\sum_{m=1}^{M}|Q_m|^2 \quad .$$

To deal with the terms $-\langle \mathbf{B}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle$ and $-\langle \mathbf{B}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle$, in contrast to the earlier case, we use Cauchy-Schwarz inequality to derive

$$-\langle \mathbf{B}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle-\langle \mathbf{B}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle \geq -C\sqrt{MM_{max}}\left[\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right]^{\frac{1}{2}}\sum_{m=1}^{M}|Q_m|^2.$$

Also

$$-\langle \mathbf{C_I}^r\mathbf{Q}^r,\mathbf{Q}^r\rangle-\langle \mathbf{C_I}^r\mathbf{Q}^i,\mathbf{Q}^i\rangle \geq \min_{1\leq m\leq M}(-(\mathbf{C}_m^{-1})^r)\sum_{m=1}^{M}|Q_m|^2=\min_{1\leq m\leq M}\left(\frac{|\mathbf{C}_m^r|}{|\mathbf{C}_m|^2}\right)\sum_{m=1}^{M}|Q_m|^2$$

$$\geq \frac{\min\limits_{1\leq m\leq M}|\mathbf{C}_m^r|}{(\max\limits_{1\leq m\leq M}|\mathbf{C}_m|)^2}\sum_{m=1}^{M}|Q_m|^2.$$

Combining the above estimates, we can now conclude (2.63). We would like to remark that the argument in this case can be applied also to the case when $(\mathbf{C}_l^{-1})^r > 0,\ \forall\, l = 1,\dots,M$, but the estimate would be worse than that already achieved.
We note that in view of remark 2.10, these two are the only possible cases and hence the proof is complete.

3.8. **Proof of proposition 2.14.** First of all, we note that using (2.9),(2.20) and (2.21), we can rewrite (2.22) on ∂D_l as

(3.52)

$$\begin{aligned}
&\left[\lambda_l Id + (K_{D_l}^{\kappa_0})^*\right]\phi_l + \sum_{\substack{m=1\\ m\neq l}}^{M}\frac{\partial(S_{D_m}^{\kappa_0}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l} + \frac{\partial u^I}{\partial\nu^l}\\
&\qquad= -\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\left([(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*]\psi_l + [\frac{1}{2}Id + (K_{D_l}^{\kappa_0})^*]\left(S_{D_l}^{\kappa_0}\right)^{-1}\left(S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}\right)\psi_l\right)\\
&\qquad= -\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\left([(K_{D_l}^{\kappa_l})^* - (K_{D_l}^{\kappa_0})^*] + [(K_{D_l}^{\kappa_0})^* - (K_{D_l}^{0})^*]\left(S_{D_l}^{\kappa_0}\right)^{-1}\left(S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}\right)\right)\psi_l\\
&\qquad\quad -\Big(1-\frac{\rho_l}{\rho_0}\Big)^{-1}\left([\frac{1}{2}Id + (K_{D_l}^{0})^*]\left(S_{D_l}^{\kappa_0}\right)^{-1}\left(S_{D_l}^{\kappa_0} - S_{D_l}^{\kappa_l}\right)\psi_l\right)\\
&\qquad= O(a^2\|\psi_l\|) + O(a^2\cdot a^{-1}\cdot a^2\|\psi_l\|) + \underbrace{O(a^{-1}\cdot a^2\|\psi_l\|)}_{\in L_0^2} = O(a^2\|\psi_l\|) + \underbrace{O(a\|\psi_l\|)}_{\in L_0^2}\ \text{ in } L^2\\
&\qquad= O\left(a^2\left[1 + \|\phi_l\| + \sum_{m\neq l}\frac{a}{d_{ml}}\|\phi_m\| + \sum_{m\neq l}\frac{a^2}{d_{ml}^2}\|\phi_m\|\right]\right)\\
&\qquad\qquad + \underbrace{O\left(a\left[1 + \|\phi_l\| + \sum_{m\neq l}\frac{a}{d_{ml}}\|\phi_m\| + \sum_{m\neq l}\frac{a^2}{d_{ml}^2}\|\phi_m\|\right]\right)}_{\in L_0^2}\ \text{ in } L^2\\
&\qquad= E_{1,l} + E_{2,l}.
\end{aligned}$$

Now let us consider the system

$$[\lambda_l Id + (K_{D_l}^{\kappa_0})^*]\phi_l + \sum_{\substack{m=1\\ m\neq l}}^{M}\frac{\partial(S_{D_m}^{\kappa_0}\phi_m)}{\partial\nu^l}\Big|_{\partial D_l} = -\frac{\partial u^I}{\partial\nu^l}\Big|_{\partial D_l} + E_{1,l} + E_{2,l}. \tag{3.53}$$

We can further rewrite it as

$$(\boldsymbol{L} + \boldsymbol{K})\phi = -\partial_\nu u^{\boldsymbol{I}} + E_1 + E_2, \tag{3.54}$$

where $\boldsymbol{L} := (\boldsymbol{L}_{lm})_{l,m=1}^M$ and $\boldsymbol{K} := (\boldsymbol{K}_{lm})_{l,m=1}^M$, with

$$\boldsymbol{L}_{lm} = \begin{cases} [\lambda_l Id + (K_{D_l}^{\kappa_0})^*], & l = m\\ 0, & else\end{cases}, \qquad \boldsymbol{K}_{lm} = \begin{cases} \frac{\partial}{\partial\nu^l}S_{D_m}^{\kappa_0}, & l\neq m\\ 0, & else\end{cases}, \tag{3.55}$$

$$\partial_\nu u^I := \left(\frac{\partial u^I}{\partial\nu^1}\cdots\frac{\partial u^I}{\partial\nu^M}\right)^T, \tag{3.56}$$

$$\phi := (\phi_1\dots\phi_M)^T, \tag{3.57}$$

and $E_i := (E_{i,1}\ \dots\ E_{i,M})^T,\ i = 1, 2$. Let us also set

$$\boldsymbol{\Phi}^c_{\kappa_0}\phi := \left((\boldsymbol{\Phi}^c_{\kappa_0}\phi)_1 \dots (\boldsymbol{\Phi}^c_{\kappa_0}\phi)_M\right)$$

where

$$(\boldsymbol{\Phi}^c_{\kappa_0}\phi)_l(s) := \sum_{\substack{m=1\\m\neq l}}^{M} \frac{\partial}{\partial \nu^l}\Phi_{\kappa_0}(s, z_m)Q_m,$$

$$\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0}\phi := \left((\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_1 \dots (\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_M\right)$$

where

$$(\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_l(s) := \sum_{\substack{m=1\\m\neq l}}^{M} \nabla_s\nabla_t\Phi_{\kappa_0}(s, z_m)\cdot \nu^l(s)\cdot V_m,$$

$$\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0}\phi := \left((\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_1 \dots (\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_M\right)$$

where

$$(\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_l(s) := \sum_{\substack{m=1\\m\neq l}}^{M} (s-z_l)\cdot \frac{\partial}{\partial \nu^l}\nabla_t\nabla_s\Phi_{k_0}(s, z_m)\cdot V_m,$$

and

$$\acute{\boldsymbol{K}}\phi := \boldsymbol{K}\phi - \boldsymbol{\Phi}^c_0\phi - [\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\Phi}^c_0]\phi - \boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0\phi - [\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0]\phi_z - [\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_0]\phi,$$

with $(\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0\phi_z)_l := (\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0}\phi)_l(z_l) = \sum_{m\neq l}\nabla_s\nabla_t\Phi_{\kappa_0}(z_l, z_m)\cdot\nu^l(s)\cdot V_m$.
Using the above notations and the fact that the matrix L is invertible, we can write ϕ as

$$\begin{aligned}\phi = &- \boldsymbol{L}^{-1}\partial_\nu u^I - \boldsymbol{L}^{-1}\boldsymbol{\Phi}^c_0\phi - \boldsymbol{L}^{-1}[\boldsymbol{\Phi}^c_{k_0} - \boldsymbol{\Phi}^c_0]\phi - \boldsymbol{L}^{-1}\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0\phi - \boldsymbol{L}^{-1}[\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0]\phi_z\\ &- \boldsymbol{L}^{-1}[\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_0]\phi - \boldsymbol{L}^{-1}\acute{\boldsymbol{K}}\phi + \boldsymbol{L}^{-1}E_1 + \boldsymbol{L}^{-1}E_2.\end{aligned} \tag{3.58}$$

Now using the fact that $\boldsymbol{\Phi}^c_0\phi$, $\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0\phi$,$[\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0]\phi_z$ and E_2 are mean-free and $\boldsymbol{L}^{-1}$ doesn't scale while acting on mean-free vectors but in general $\|\boldsymbol{L}^{-1}\|_{\mathcal{L}(L^2,L^2)} = O\left(\rho_l^{-1}\right)$, we obtain

$$\|(\boldsymbol{L}^{-1}\partial_\nu u^I)_l\| = O\left(\frac{a}{|\rho_l|}\right), \quad \|(\boldsymbol{L}^{-1}\boldsymbol{\Phi}^c_0\phi + \boldsymbol{L}^{-1}[\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\Phi}^c_0]\phi)_l\| = O\left(\left[\frac{1}{d^2} + \frac{1}{|\rho_l|}\right] a\sum_{m\neq l}|Q_m|\right),$$

$$\|(\boldsymbol{L}^{-1}\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0\phi)_l\| = O\left(\sum_{m\neq l}\frac{1}{d^3_{ml}}a|V_m|\right), \quad \|(\boldsymbol{L}^{-1}[\boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^1}\boldsymbol{\Phi}^c_0]\phi_z)_l\| = O\left(\sum_{m\neq l}\frac{1}{d_{ml}}a|V_m|\right),$$

$$\|(\boldsymbol{L}^{-1}[\boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_{\kappa_0} - \boldsymbol{\nabla^2}\boldsymbol{\Phi}^c_0]\phi)_l\| = O\left(\sum_{m\neq l}\frac{1}{|\rho_l|}\frac{1}{d^2_{ml}}a^2|V_m|\right), \quad \|(\boldsymbol{L}^{-1}\acute{\boldsymbol{K}}\phi)_l\| = O\left(\frac{1}{|\rho_l|}\max_l\sum_{m\neq l}\frac{a^4}{d^4_{ml}}\|\phi\|\right),$$

$$\|(\boldsymbol{L}^{-1}E_1)_l\| = O\left(\frac{a^2}{|\rho_l|}\left[1 + \|\phi_l\| + \sum_{m\neq l}\frac{a}{d_{ml}}\|\phi_m\| + \sum_{m\neq l}\frac{a^2}{d^2_{ml}}\|\phi_m\|\right]\right),$$

and

$$\|(\boldsymbol{L}^{-1}E_2)_l\| = O\left(a\left[1 + \|\phi_l\| + \sum_{m\neq l}\frac{a}{d_{ml}}\|\phi_m\| + \sum_{m\neq l}\frac{a^2}{d^2_{ml}}\|\phi_m\|\right]\right).$$

Therefore we can write
(3.59)

$$\|\phi_l\| = O\left(\frac{a}{|\rho_l|}\right) + O\left(\left[\frac{1}{d^2} + \frac{1}{|\rho_l|}\right] a\sum_{m\neq l}|Q_m|\right) + O\left(\sum_{m\neq l}\left(\frac{1}{d^3_{ml}}a|V^{rem}_m| + \frac{1}{|\rho_l|}\frac{1}{d^2_{ml}}a^2|V^{rem}_m|\right)\right)$$

$$+O\left(\sum_{m\neq l}\left(\frac{1}{d_{ml}^3}a|V_{m,1}^{dom}|+\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_{m,1}^{dom}|\right)\right)+O\left(\sum_{m\neq l}\left(\frac{1}{d_{ml}^3}a|V_{m,2}^{dom}|+\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_{m,2}^{dom}|\right)\right)$$
$$+O\left(\frac{1}{|\rho_l|}\max_l\sum_{m\neq l}\frac{a^4}{d_{ml}^4}\|\phi\|\right)+O\left(\frac{a^2}{|\rho_l|}\left[1+\|\phi_l\|+\sum_{m\neq l}\frac{a}{d_{ml}}\|\phi_m\|+\sum_{m\neq l}\frac{a^2}{d_{ml}^2}\|\phi_m\|\right]\right)$$
$$+O\left(a\left[1+\|\phi_l\|+\sum_{m\neq l}\frac{a}{d_{ml}}\|\phi_m\|+\sum_{m\neq l}\frac{a^2}{d_{ml}^2}\|\phi_m\|\right]\right),$$

where we have ignored the contribution of the term $\boldsymbol{L}^{-1}[\boldsymbol{\nabla^1\Phi}^c_{\kappa_0}-\boldsymbol{\nabla^1\Phi}^c_0]\phi_z$ since the term $\boldsymbol{L}^{-1}\boldsymbol{\nabla^1\Phi}^c_0\phi$ is clearly more singular.

To deal with the terms involving V^{rem}, we note that

(3.60)
$$\sum_{m\neq l}\frac{1}{d_{ml}^3}a|V_m^{rem}|=a\;O\Big(a^{3-\gamma}+\Big[a^{4-\gamma}+\frac{a^{5-\gamma}}{d^2}+\frac{a^{5-\gamma}}{d^{3\alpha}}+\frac{a^4}{d^2}+\frac{a^4}{d^{3\alpha}}+\frac{a^{6-\gamma}}{d^3}+\frac{a^{6-\gamma}}{d^{3\alpha+1}}\Big]\|\phi\|\Big)\sum_{m\neq l}\frac{1}{d_{ml}^3}$$
$$=O\Big(a^{4-\gamma}+\Big[a^{5-\gamma}+\frac{a^{6-\gamma}}{d^2}+\frac{a^{6-\gamma}}{d^{3\alpha}}+\frac{a^5}{d^2}+\frac{a^5}{d^{3\alpha}}+\frac{a^{7-\gamma}}{d^3}+\frac{a^{7-\gamma}}{d^{3\alpha+1}}\Big]\|\phi\|\Big)O\left(\frac{1}{d^3}+\frac{1}{d^{3\alpha+1}}\right)$$
$$=O\Big(\frac{a^{4-\gamma}}{d^3}+\frac{a^{4-\gamma}}{d^{3\alpha+1}}+\Big[\frac{a^{5-\gamma}}{d^3}+\frac{a^{5-\gamma}}{d^{3\alpha+1}}+\frac{a^{6-\gamma}}{d^5}+\frac{a^{6-\gamma}}{d^{3\alpha+3}}+\frac{a^{6-\gamma}}{d^{6\alpha+1}}$$
$$+\frac{a^5}{d^5}+\frac{a^5}{d^{3\alpha+3}}+\frac{a^5}{d^{6\alpha+1}}+\frac{a^{7-\gamma}}{d^6}+\frac{a^{7-\gamma}}{d^{3\alpha+4}}+\frac{a^{7-\gamma}}{d^{6\alpha+2}}\Big]\|\phi\|\Big)$$
$$=O\Big(a^{4-\gamma-3t}+a^{4-\gamma-s-t}+\Big[a^{5-\gamma-3t}+a^{5-\gamma-s-t}+a^{6-\gamma-5t}+a^{6-\gamma-s-3t}+a^{6-\gamma-2s-t}$$
$$+a^{5-5t}+a^{5-s-3t}+a^{5-2s-t}+a^{7-\gamma-6t}+a^{7-\gamma-s-4t}+a^{7-\gamma-2s-2t}\Big]\|\phi\|\Big).$$

(3.61)
$$\sum_{m\neq l}\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_m^{rem}|=\frac{a^2}{|\rho_l|}O\Big(a^{3-\gamma}+\Big[a^{4-\gamma}+\frac{a^{5-\gamma}}{d^2}+\frac{a^{5-\gamma}}{d^{3\alpha}}+\frac{a^4}{d^2}+\frac{a^4}{d^{3\alpha}}+\frac{a^{6-\gamma}}{d^3}+\frac{a^{6-\gamma}}{d^{3\alpha+1}}\Big]\|\phi\|\Big)\sum_{m\neq l}\frac{1}{d_{ml}^2}$$
$$=\frac{1}{|\rho_l|}O\Big(a^{5-\gamma}+\Big[a^{6-\gamma}+\frac{a^{7-\gamma}}{d^2}+\frac{a^{7-\gamma}}{d^{3\alpha}}+\frac{a^6}{d^2}+\frac{a^6}{d^{3\alpha}}+\frac{a^{8-\gamma}}{d^3}+\frac{a^{8-\gamma}}{d^{3\alpha+1}}\Big]\|\phi\|\Big)O\left(\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right)$$
$$=O\Big(a^{4-2\gamma}+\Big[a^{5-2\gamma}+\frac{a^{6-2\gamma}}{d^2}+\frac{a^{6-2\gamma}}{d^{3\alpha}}+\frac{a^{5-\gamma}}{d^2}+\frac{a^{5-\gamma}}{d^{3\alpha}}+\frac{a^{7-2\gamma}}{d^3}+\frac{a^{7-2\gamma}}{d^{3\alpha+1}}\Big]\|\phi\|\Big)O\left(\frac{1}{d^2}+\frac{1}{d^{3\alpha}}\right)$$
$$=O\Big(\frac{a^{4-2\gamma}}{d^2}+\frac{a^{4-2\gamma}}{d^{3\alpha}}+\Big[\frac{a^{5-2\gamma}}{d^2}+\frac{a^{5-2\gamma}}{d^{3\alpha}}+\frac{a^{6-2\gamma}}{d^4}+\frac{a^{6-2\gamma}}{d^{3\alpha+2}}+\frac{a^{6-2\gamma}}{d^{6\alpha}}$$
$$+\frac{a^{5-\gamma}}{d^4}+\frac{a^{5-\gamma}}{d^{3\alpha+2}}+\frac{a^{5-\gamma}}{d^{6\alpha}}+\frac{a^{7-2\gamma}}{d^5}+\frac{a^{7-2\gamma}}{d^{3\alpha+3}}+\frac{a^{7-2\gamma}}{d^{6\alpha+1}}\Big]\|\phi\|\Big)$$
$$=O\Big(a^{4-2\gamma-2t}+a^{4-2\gamma-s}+\Big[a^{5-2\gamma-2t}+a^{5-2\gamma-s}+a^{6-2\gamma-4t}+a^{6-2\gamma-s-2t}+a^{6-2\gamma-2s}$$
$$+a^{5-\gamma-4t}+a^{5-\gamma-s-2t}+a^{5-\gamma-2s}+a^{7-2\gamma-5t}+a^{7-2\gamma-s-3t}+a^{7-2\gamma-2s-t}\Big]\|\phi\|\Big).$$

Now if we assume that $0\leq t<\frac{1}{2},\ 0\leq s\leq\frac{3}{2},\ 0\leq\gamma\leq 1,\frac{s}{3}\leq t$ and $s+\gamma\leq 2$, then we can derive the estimate

$$\sum_{m\neq l}\left(\frac{1}{d_{ml}^3}a|V_m^{rem}|+\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_m^{rem}|\right)=O(a+a^{\frac{3}{2}}\|\phi\|). \tag{3.62}$$

For the terms involving $V_{m,1}^{dom}$, we can deduce that

$$\sum_{m\neq l}\frac{1}{d_{ml}^3}a|V_{m,1}^{dom}| = a^{4-\gamma}\|\phi\|O(\frac{1}{d^3}+\frac{1}{d^{3\alpha+1}}) = O([a^{4-\gamma-3t}+a^{4-\gamma-s-t}]\|\phi\|), \tag{3.63}$$

$$\sum_{m\neq l}\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_{m,1}^{dom}| = a^{4-2\gamma}\|\phi\|O(\frac{1}{d^2}+\frac{1}{d^{3\alpha}}) = O([a^{4-2\gamma-2t}+a^{4-2\gamma-s}]\|\phi\|). \tag{3.64}$$

Assuming that $0\leq t<\frac{1}{2},\ 0\leq s\leq\frac{3}{2},\ 0\leq\gamma\leq 1,\ \frac{s}{3}\leq t$ and $s+\gamma\leq 2$, we can derive the estimate

$$\sum_{m\neq l}\left(\frac{1}{d_{ml}^3}a|V_{m,1}^{dom}|+\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_{m,1}^{dom}|\right) = O(a\|\phi\|). \tag{3.65}$$

Similarly for the terms involving $V_{m,2}^{dom}$, we can write

$$\begin{aligned}\sum_{m\neq l}\frac{1}{d_{ml}^3}a|V_{m,2}^{dom}| &= O\left(a^{3-\gamma}a\Big(\sum_{m\neq l}\frac{1}{d_{ml}^3}\Big)\frac{1}{d}\sum_{n=1}^{M}|Q_n|\right) = O\left(a^{4-\gamma}\Big(\frac{1}{d^4}+\frac{1}{d^{3\alpha+2}}\Big)\sum_{n=1}^{M}|Q_n|\right)\\ &= O([a^{4-\gamma-4t}+a^{4-\gamma-s-2t}]\sum_{n=1}^{M}|Q_n|),\end{aligned} \tag{3.66}$$

$$\begin{aligned}\sum_{m\neq l}\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_{m,2}^{dom}| &= O\left(a^{-1-\gamma}a^{3-\gamma}a^2\Big(\sum_{m\neq l}\frac{1}{d_{ml}^2}\Big)\frac{1}{d}\sum_{n=1}^{M}|Q_n|\right) = O\left(a^{4-2\gamma}(\frac{1}{d^3}+\frac{1}{d^{3\alpha+1}})\sum_{n=1}^{M}|Q_n|\right)\\ &= O([a^{4-2\gamma-3t}+a^{4-2\gamma-s-t}]\sum_{n=1}^{M}|Q_n|).\end{aligned} \tag{3.67}$$

Again assuming that $0\leq t<\frac{1}{2},\ 0\leq s\leq\frac{3}{2},\ 0\leq\gamma\leq 1,\ \frac{s}{3}\leq t$ and $s+\gamma\leq 2$, we can derive the estimate

$$\sum_{m\neq l}\left(\frac{1}{d_{ml}^3}a|V_{m,2}^{dom}|+\frac{1}{|\rho_l|}\frac{1}{d_{ml}^2}a^2|V_{m,2}^{dom}|\right) = O(a^{\frac{1}{2}+}\sum_{n=1}^{M}|Q_n|). \tag{3.68}$$

Using (2.68), (3.62), (3.65), (3.68) in (3.59), we can deduce that for $l=1,\dots,M$,

$$\begin{aligned}\|\phi_l\| &= O(a^{-\gamma})+O\left([a^{1-2t}+a^{-\gamma}]M\max|\boldsymbol{C}_m|\right)\\ &\quad+O\left([a^{1-2t}+a^{-\gamma}]M\max|\boldsymbol{C}_m|\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\right)\|\phi\|\\ &\quad+O(a)+O(a^{\frac{3}{2}})\|\phi\|+O(a)\|\phi\|+O\left(a^{\frac{1}{2}+}M\max|\boldsymbol{C}_m|\left[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\right]\right)\\ &\quad+O\left(a^{3-\gamma}\sum_{m\neq l}\frac{1}{d_{ml}^4}\right)\|\phi\|+O\left(a^{1-\gamma}\left[1+\|\phi\|+a\|\phi\|\left(\sum_{m\neq l}\frac{1}{d_{ml}}\right)+a^2\|\phi\|\left(\sum_{m\neq l}\frac{1}{d_{ml}^2}\right)\right]\right).\end{aligned}$$

Now if $M\max|\boldsymbol{C}_m| = O(a^{-h})$, then we obtain

$$\begin{aligned}\|\phi_l\| &= O(a^{-\gamma})+O\left(a^{-\gamma}\cdot a^{-h}\right)+O\left(a^{-\gamma-h}\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\right)\|\phi\|\\ &\quad+O(a)+O(a^{\frac{3}{2}})\|\phi\|+O(a)\|\phi\|+O\left(a^{-h+\frac{1}{2}+}\left[1+a+\left(a^2+a^{3-3t}+a^{3-s-t}+a^{4-2s}\right)\|\phi\|\right]\right)\\ &\quad+O\left(a^{3-\gamma}\sum_{m\neq l}\frac{1}{d_{ml}^4}\right)\|\phi\|+O\left(a^{1-\gamma}\left[1+\|\phi\|+a\|\phi\|\left(\sum_{m\neq l}\frac{1}{d_{ml}}\right)+a^2\|\phi\|\left(\sum_{m\neq l}\frac{1}{d_{ml}^2}\right)\right]\right)\\ &= O(a^{-\gamma})+O(a^{-\gamma-h})+O((a^{2-\gamma-h}+a^{3-3t-\gamma-h}+a^{3-s-t-\gamma-h}+a^{4-2s-\gamma-h})\|\phi\|)\end{aligned}$$

$$+ O(a^{1-2t})\|\phi\| + O(a^{1-\gamma})\|\phi\| + O(a^{2-\gamma-s})\|\phi\|.$$

Therefore provided $h < \frac{1}{2}$,

$$\|\phi\| = O(a^{-\gamma}) + O(a^{-\gamma-h}) + O(a^{0+})\|\phi\| + O(a^{1-2t})\|\phi\| + O(a^{1-\gamma})\|\phi\| + O(a^{2-\gamma-s})\|\phi\|,$$

whence it follows that

$$\|\phi\| = O(a^{-\gamma}) + O(a^{-\gamma-h}).$$

Note that if $\gamma = 1$ or $\gamma + s = 2$, to deduce the last step we need to assume that the constant C_ρ in (1.5) is large enough.

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