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Abstract

For any Lipschitz domain we construct an arbitrarily small, localized perturbation which splits the spectrum of the Laplacian into simple eigenvalues. We use for this purpose a Hadamard's formula and spectral stability results.

1 Introduction

In the seminal works [8] and [10], respectively Micheletti and Uhlenbeck showed that the eigenvalues of the Dirichlet Laplacian are generically simple in the space of smooth manifolds equipped with the C^k-topology (see also the survey papers [3, Section 4.3], [5, Section 1.3] and references therein for subsequent works). In this paper we prove that a localized version of this result holds as follows, even for non-smooth domains.

Theorem 1. *For any Lipschitz domain Ω, $\varepsilon > 0$, and x in the closure of Ω, there exists a domain $\tilde{\Omega}$ whose symmetric difference with Ω is contained in the ball of radius ε centered at x, and whose (Dirichlet, Neumann, or Robin) Laplacian eigenvalues are all simple. Moreover $\tilde{\Omega}$ can be constructed so that the Lipschitz constant of $\partial\tilde{\Omega}$ is arbitrarily near to the one of $\partial\Omega$.*

More in detail the structure of the paper is the following. In Section 2 we review some preliminary material, in particular regarding spectral stability. In Section 3 we recall a Hadamard's formula and study some independence properties of eigenfunctions and their gradients at the boundary. More in detail, Hadamard's formula provides us with a first-order estimate on the shift of an eigenvalue λ which depends on the value of

$$|\nabla u|^2 - cu^2 \tag{1}$$

at the boundary of the domain considered, where u is an eigenfunction associated to λ and c is a constant which depends only on the choice of boundary conditions. By showing that for two orthogonal eigenfunctions the corresponding values of (1) in any open subset of the boundary must differ at least at a point, we are able to construct a localized perturbation which splits any non-simple eigenvalue. However, even when small, this perturbation might cause the shift and the overlap of other eigenvalues. This possibility is ruled out in Section 4, where uniform bounds for the whole spectrum are adapted to our case from sharp stability estimates from [2]. In conclusion, these bounds allow the construction of a localized perturbation, which consists of a sequence of small "bumps" at the boundary of the domain considered, which proves Theorem 1.

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2 Notations and preliminary results

In this section we fix the main notation which will be used in the paper and recall some preliminary results on eigenvalues and eigenfunctions of the Laplacian. Regarding the notation:

- we say that X is a domain if X is an open, bounded, and connected subset of $\mathbb{R}^N$;
- we say that λ is an eigenvalue of a domain X with associated eigenfunction u (assumed to be not constant zero) if
$$\Delta u + \lambda u = 0 \qquad \text{in } X, \tag{2}$$
and either one of the following homogeneous boundary conditions is satisfied on ∂X:
$$\begin{cases} u = 0 & \text{(Dirichlet)}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{(Neumann)}, \\ \sigma u = \dfrac{\partial u}{\partial \nu} & \text{(Robin)}, \end{cases} \tag{3}$$
where σ is a fixed non-zero constant and ν indicates the outward unit normal vector;
- we indicate as Ω a fixed domain with Lipschitz boundary.

We actually require (2) and (3) to be satisfied only in a weak sense, that is: λ is an eigenvalue of X with associated eigenfunction u, if u is an element of a function space $V(X)$ and

$$Q(u,v) = \lambda \int_X uv, \qquad \text{for every } v \in V(X),$$

where, depending on the choice of boundary conditions, we have

Boundary conditions	$Q(u,v)$	$V(X)$
Dirichlet	$\int_X \nabla u \cdot \nabla v$	$\{u \in H^1(X) : \text{trace of } u \text{ at } \partial X \text{ is } 0\}$
Neumann	$\int_X \nabla u \cdot \nabla v$	$H^1(X)/\mathbb{R}$
Robin	$\int_X \nabla u \cdot \nabla v - \int_{\partial X} \sigma uv$	$H^1(X)$

(4)

where H^1 is the space of square integrable functions with square integrable distributional gradient. However, from elliptic regularity theory, we know that Laplacian eigenfunctions are analytic inside any open domain. Thus (2) is satisfied also in the classical sense. Moreover if Σ is a smooth (that is C^∞) part of ∂X, u is also smooth on Σ (see for example [4, Section 6.3] for proofs of these facts).

Recall from spectral theory that the eigenvalues of Ω have finite multiplicity and can be arranged in a non-decreasing sequence which tends to infinity, and which we will denote as

$$\lambda_1 \leq \lambda_2 \leq \dots,$$

where each eigenvalue is repeated as many times as its multiplicity.

For future reference we record the following uniqueness result.

Theorem 2. *Let u be such that $\Delta u + \lambda u = 0$ in Ω. If $u = 0$ and $\dfrac{\partial u}{\partial \nu} = 0$ on Σ, an open and smooth subset of $\partial\Omega$, then u is constant zero in the whole Ω.*

We briefly outline the classic argument to prove this fact from Holmgren's uniqueness theorem. Let B be an open ball such that $B \cap \partial\Omega \subseteq \Sigma$. Extending u to 0 in $B \setminus \Omega$, it is easy to check that $-\Delta u = \lambda u$ in the distributional sense in B. By [7, Theorem 5.3.1], u must be zero also in an open set inside Ω. But then $u = 0$ on the whole Ω by analytic continuation.

2.1 Stability of eigenvalues of the Laplacian

We review some results that show that the spectrum of the Laplacian is continuous under domain perturbations, and give some useful quantitative estimates on the eigenvalues' shifts.

First we recall a result of analyticity of eigenvalues and eigenfunctions with respect to a perturbation parameter, which is a consequence of the classic Rellich-Nagy Theorem [9, Theorem 1 at p. 33] (see also [3, Section 4.2] and references therein).

Theorem 3. *Let $(\phi_t)_{t\in[0,t_0]}$ be a family of diffeomorphisms of $\mathbb{R}^N$ such that ϕ_t is analytic in t, ϕ_0 is the identity, and $\phi_t(\Omega) \supseteq \Omega$ for every t. Let λ be an eigenvalue of Ω of multiplicity m. Then there exist $\lambda_t^1 \leq \cdots \leq \lambda_t^m$ and functions $u_t^1, \dots, u_t^m$ such that for $j = 1, \dots, m$,*

- *for any t, λ_t^j is an eigenvalue of Ω_t with associated eigenfunction u_t^j;*
- *for any t, $\int_{\Omega_t} u_t^j u_t^i$ is 1 if $j = i$ and is 0 otherwise;*
- *λ_t^j and u_t^j are analytic in t;*
- *$\lambda_0^j = \lambda$ and u_0^j is an eigenfunction associated to λ.*

Moreover for any $\delta > 0$ small enough, there is a T such that for any $t < T$ the only eigenvalues of $\phi_t(\Omega)$ in $(\lambda - \delta, \lambda + \delta)$ are $\lambda_t^1, \dots, \lambda_t^m$.

For our purposes we will also need a finer estimate on the variation of eigenvalues, as expressed in the following lemma.

Lemma 4. *Let ϕ be a diffeomorphism of $\mathbb{R}^N$. Let λ_n be the n-th eigenvalue of Ω and $\tilde{\lambda}_n$ be the n-th eigenvalue of $\phi(\Omega)$. Then there exists a constant C, which depends only on the Lipschitz constants of $\partial\Omega$ and of ϕ, such that*

$$|\tilde{\lambda}_n - \lambda_n| \leq C \max\{\tilde{\lambda}_n, \lambda_n\}(|\phi - id|_{C^1(\overline{\Omega})}).$$

The proof of this estimate can be obtained by repeating the same argument from the proof of [2, Lemma 6.1], only substituting appropriately the bilinear form and the function space with the ones defined in (4), depending on the boundary conditions considered.

3 Hadamard's formula and boundary properties of eigenfunctions

In this section we study some independence properties of Laplacian eigenfunctions and of their gradients at the boundary. We first recall a Hadamard's formula for the variation of eigenvalues under a deformation of the boundary. The dot superscript will indicate differentiation in t.

Lemma 5. *Let $(\phi_t)_{t\in[0,t_0]}$ be a family of diffeomorphisms such that ϕ_t is analytic in t and ϕ_0 is the identity. Suppose that the support of ϕ_t is contained in a fixed open set U for every t, and that $\partial\Omega \cap U$ is smooth. Let λ_t, u_t be an eigenvalue-eigenfunction couple of $\phi_t(\Omega)$, and suppose both are differentiable in t. Then*

$$\dot{\lambda}_0 = \int_{\partial\Omega} \Big(|\nabla u_0|^2 - \lambda_0 u_0^2 + (\partial_{\nu_0} u_0)(H u_0 - 2\partial_{\nu_0} u_0)\Big)\nu_0 \cdot \dot{e}_0, \tag{5}$$

where ν_t indicates the outward unit normal vector, e_t the identity on $\phi_t(\partial\Omega)$, and H is the mean curvature of $\partial\Omega$.

Hereafter we briefly prove this fact in the case of homogeneous Dirichlet or Neumann boundary conditions. The case of Robin conditions requires a finer analysis of the dependence on t of the surfaces $\phi_t(\partial\Omega)$, for which we refer to [1, Identities (69) and (57)].

Proof. Let $(\Omega_t)_{t\in[0,t_0]}$ be a family of domains such that $\Omega_t = \phi_t(\Omega)$ for every t. By the divergence theorem, the distributional gradient of the measure $\chi_{\Omega_t}\mathcal{L}^N$ is given by $\nu_t\Sigma_t^{N-1}$, where χ_{Ω_t} is the characteristic function of Ω_t, $\mathcal{L}^N$ is the N-dimensional Lebesgue measure, and Σ_t^{N-1} is the surface measure on $\partial\Omega_t$. Therefore by the chain rule

$$\frac{d}{dt}(\chi_{\Omega_t}\,\mathcal{L}^N) = \nu_t \cdot \dot{e}_t\,\Sigma_t^{N-1},$$

so we have the following Leibniz' formula:

$$\frac{d}{dt}\left(\int_{\Omega_t} f_t\right) = \int_{\Omega_t} \dot{f}_t + \int_{\partial\Omega_t} f_t \nu_t \cdot \dot{e}_t. \tag{6}$$

Consider now the identity

$$\lambda_t = -\int_{\Omega_t} u_t \Delta u_t = \int_{\Omega_t} |\nabla u_t|^2. \tag{7}$$

Differentiating in t the first equality in (7) and using (6) we obtain

$$2\lambda_t \int_{\Omega_t} \dot{u}_t u_t = -\lambda_t \int_{\partial\Omega_t} u_t^2 \nu_t \cdot \dot{e}_t. \tag{8}$$

In the case of Neumann boundary conditions, differentiating in t the last term in (7), using (6), integrating by parts, and substituting (8), we have that

$$\dot{\lambda}_t = \int_{\partial\Omega_t} (|\nabla u_t|^2 - \lambda_t u_t^2)\nu_t \cdot \dot{e}_t + 2\int_{\partial\Omega_t} \dot{u}_t \frac{\partial u_t}{\partial \nu_t},$$

which gives (5) since $\partial_{\nu_0} u_0 = 0$ on $\partial\Omega_0$. Proceeding in the same way for Dirichlet boundary conditions, only exchanging the roles of the functions in the integration by parts step, we obtain

$$\dot{\lambda}_t = \int_{\partial\Omega_t} (|\nabla u_t|^2 - \lambda_t u_t^2)\nu_t \cdot \dot{e}_t + 2\int_{\partial\Omega_t} u_t \frac{\partial \dot{u}_t}{\partial \nu_t} + 2\dot{\lambda}_t,$$

which gives (5) since $u_0 = 0$ on $\partial\Omega_0$. □

We notice that considering

$$c = \begin{cases} 0 & \text{if } u|_{\partial\Omega} = 0, \\ \lambda_0 & \text{if } \partial_\nu u|_{\partial\Omega} = 0, \\ \lambda_0 + 2\sigma^2 & \text{if } \sigma u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega}, \end{cases} \tag{9}$$

if $\dot{e}_0$ is supported on a flat part of $\partial\Omega$, the integrand in (5) can be rewritten as $|\nabla u|^2 - cu^2$. In the following lemma we study such a quantity, in particular the behavior of its zeros.

Lemma 6. *Let c be an arbitrary constant and let $u, \tilde{u}$ be two orthonormal eigenfunctions associated to the same eigenvalue. Let Σ be an arbitrary smooth open subset of $\partial\Omega$. Then:*

1. *$|\nabla u|^2 - cu^2$ cannot be constant zero on Σ;*
2. *$|\nabla u|^2 - cu^2 - (|\nabla \tilde{u}|^2 - c\tilde{u}^2)$ cannot be constant zero on Σ.*

Proof. The thesis for the case $c = 0$ is given by Theorem 2. Consider $c \neq 0$. Our approach is inspired to the treatment of [6, Chapter 6].

We first prove Point 1. Suppose by contradiction that $|\nabla u|^2 = cu^2$ on Σ. We consider separately the different possible boundary conditions in (3).

i) If the Dirichlet condition holds then $\partial u/\partial\nu = u = 0$ on Σ. By Theorem 2 then $u = 0$ on Ω, a contradiction.

ii) Suppose the Neumann condition holds. The eigenfunction u cannot be constant 0 on Σ, otherwise we would be again in the situation of Case i, so there is a point $x_0 \in \Sigma$ such that $u(x_0) \neq 0$. Let γ_t be a solution in Σ of the ODE

$$\begin{cases} \gamma_0 = x_0, \\ \dot{\gamma}_t = C\nabla u(\gamma_t), \end{cases}$$

with C a constant to be determined. Then

$$\frac{du(\gamma_t)}{dt} = C|\nabla u(\gamma_t)|^2 = Ccu(\gamma_t)^2, \tag{10}$$

if $\gamma_t \in \Sigma$. Therefore by choosing C large enough, there will be a time T at which $\gamma_T \in \Sigma$ and $|u(\gamma_t)| \xrightarrow{t\to T} \infty$, which is a contradiction.

iii) If the Robin condition holds, then

$$cu^2 = |\nabla u|^2 = \sigma^2 u^2 + |\nabla_S u|^2 \quad \text{ on } \Sigma,$$

where $\nabla_S u$ is the surface gradient of u on $\partial\Omega$. If $c \neq \sigma^2$, we can build, as in Case ii, a curve γ on which the eigenfunction u blows up in short time, leading to a contradiction. If $c = \sigma^2$ then $|\nabla_S u| = 0$ on Σ, and this leads to the following chain of implications: u is constant on Σ, $\partial u/\partial\nu$ is constant on Σ, u is constant in Ω by Theorem 2, $\partial u/\partial\nu$ is zero on $\partial\Omega$, u is zero on Ω by Theorem 2, a contradiction.

We now prove Point 2. Suppose by contradiction that $|\nabla u|^2 - |\nabla\tilde{u}|^2 = c(u^2 - \tilde{u}^2)$ on Σ. Let $x_0 \in \Sigma$ be a point where $u(x_0)$ and $\tilde{u}(x_0)$ are different (existence of such a point is guaranteed by the smoothness of eigenfunctions on Σ and Theorem 2). Let $f_t = u(\gamma_t)$, $\tilde{f}_t = -\tilde{u}(\tilde{\gamma}_t)$, where γ and $\tilde{\gamma}$ solve

$$\begin{cases} \dot{\gamma}_t = C\nabla u(\gamma_t), \\ \dot{\tilde{\gamma}}_t = -C\nabla\tilde{u}(\gamma_t), \\ \gamma_0 = \tilde{\gamma}_0 = x_0, \end{cases}$$

and C is a constant to be determined. Then

$$\dot{f}_t + \dot{\tilde{f}}_t = Cc(f_t^2 + \tilde{f}_t^2).$$

Therefore $\dot{f}_t \geq Ccf_t^2$ or $\dot{\tilde{f}}_t \geq Cc\tilde{f}_t^2$ for t in a small neighborhood of 0. In conclusion, a choice of C large enough would lead to blow up in short time of u or $\tilde{u}$, which is impossible. □

4 Splitting of the spectrum

With the tools developed so far we can construct a localized boundary deformation which splits the eigenvalues perturbed from one eigenvalue as follows.

Proposition 7. *Let $x \in \partial\Omega$, B a ball centered at x, and $\Sigma = B \cap \partial\Omega$. Suppose Σ is flat, that is Σ is contained in a hyperplane. Then, under the same hypotheses and notation of Theorem 3, we can construct a family of diffeomorphisms $(\phi_t)_{t\in(0,t_0)}$ such that ϕ_t is the identity outside B, $|\phi_t - id|_{C^1}$ is arbitrarily small, and $\lambda_t^i \neq \lambda_t^j$ for any $i \neq j$ and for all $t \in (0, t_0)$.*

Proof. Let c be as in (9). By Point 2 of Lemma 6, there exists y on Σ such that

$$(|\nabla u_0^i|^2 - c(u_0^i)^2)(y) \neq (|\nabla u_0^j|^2 - c(u_0^j)^2)(y). \tag{11}$$

Then, by choosing a deformation of the boundary ϕ_t which is the identity outside an appropriately small neighborhood of y, we have

$$\int_{\partial\Omega} (|\nabla u_0^i|^2 - c(u_0^i)^2)\nu \cdot \dot{\phi}_0 \neq \int_{\partial\Omega} (|\nabla u_0^j|^2 - c(u_0^j)^2)\nu \cdot \dot{\phi}_0. \tag{12}$$

Such a perturbation can be constructed in many ways; for the sake of completeness, we give an explicit example hereafter.

By eventually reducing to a smaller B and applying an invertible affine transformation, we can assume that $y = 0$ and $\Sigma = \{z \in B_1 : z_N = 0\}$, where B_1 is the unit ball. Let $\hat{z}$ indicate $(z_1, \dots, z_{N-1})$ and let

$$\rho_c(\hat{z}) = \begin{cases} c^2 \exp\left(\dfrac{1}{|\hat{z}/c|^2 - 1}\right) & \text{if } |\hat{z}| < c, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that by construction $|\rho_c|_{C^1} \leq c$ for any $c \leq 1$. Let $\phi_t(z)$ be the extension of the map $z \mapsto (\hat{z}, t\rho_c(\hat{z}))$ from Σ to a smooth function which is the identity outside B and such that $|\phi_t - id|_{C^1} \leq |\rho_c|_{C^1}$. By construction, $\nu \cdot \dot{\phi}_0 = \rho_c(\hat{z})$ on Σ. Then by choosing c small enough, by the smoothness of u on Σ and by (11), we have that (12) holds. Moreover we remark that it holds

$$|\phi_t - id|_{C^1} \leq c. \tag{13}$$

In conclusion, by Lemma 5, (12) implies that $\dot{\lambda}_0^i \neq \dot{\lambda}_0^j$. Since λ_t^i and λ_t^j are both analytic in t, there exists a small t_0 such that $\lambda_t^i \neq \lambda_t^j$ for $t \in (0, t_0)$. □

Remark 8. The flatness assumption of Σ, although making the argument simpler, is not really necessary in the proof of Proposition 7, as one might build a boundary deformation such that (12) holds even if Σ is not flat; the idea would be the same, only some care would be required to manage the mean curvature term which is present in (5). On the other hand, if our aim is to find a local perturbation as in Theorem 1, the flatness assumption is not restrictive. In fact, if Σ is not contained in a hyperplane, by eventually considering a smaller B and changing basis, we can assume that Σ is the graph of a Lipschitz function ϕ such that $\phi(0) = x = 0$. Let B_r, B_R be two balls centered in 0 such that $B_r \subset B_R \subset B$, and let η be a smooth function which is 0 in B_r and 1 outside B_R. Then the graph of $\phi\eta$ will be flat in B_r. Notice also that as $r \to 0$, η can be chosen so that the Lipschitz constant of $\phi\eta$ converges to the Lipschitz constant of ϕ. Thus for any $\delta > 0$, we can build a Lipschitz domain which differs from Ω only in B, is flat in B_r (for a certain r which depends on δ), and whose Lipschitz constant differs from the Lipschitz constant of Ω by less than δ.

We further remark that although Proposition 7 shows how to split one eigenvalue, the perturbation chosen might cause a couple of two other eigenvalues to overlap, creating a new repeated eigenvalue. To avoid this problem we need a finer control on the behavior of the whole spectrum; this is what is achieved in the following lemma.

Lemma 9. *Consider $\varepsilon > 0$, x a point on the boundary $\partial\Omega$, and λ_r the first eigenvalue of Ω of multiplicity $m \geq 2$. Then for any $M > 0$ there exists a Lipschitz domain $\tilde{\Omega}$, whose eigenvalues we indicate as $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$, such that:*

1. *the symmetric difference $\tilde{\Omega} \triangle \Omega$ is contained in the ball of radius ε centered at x;*
2. *for all $i \leq r+m+1$, it holds $|\tilde{\lambda}_i - \lambda_i| \leq M d_r$, where d_r is the minimum positive number of the set $\{\lambda_{j+1} - \lambda_j : j = 1, \dots, r+m\}$;*
3. *the multiplicity of $\tilde{\lambda}_r$ is strictly smaller than the multiplicity of λ_r;*
4. *for all $i > r+m$, it holds $\tilde{\lambda}_i > \lambda_r$.*

Proof. Let B_ε be the ball of radius ε centered at x and let $\Sigma = B_\varepsilon \cap \partial\Omega$. With the same construction of Remark 8 and of the proof of Proposition 7, we can build $(\Omega_t)_{t\in(0,t_0)}$ a family of perturbations of Ω obtained by a deformation of the boundary of Ω localized in B_ε. Let $\lambda_1^t, \lambda_2^t, \dots$ indicate the sequence of eigenvalues of Ω_t, with associated eigenfunctions $u_1^t, u_2^t, \dots$. By Theorem 3 we can assume that λ_i^t, u_i^t are analytic in t, that $\lambda_i^0 = \lambda_i$, and that $u_r^0, \dots, u_{r+m}^0$ is an orthonormal basis for the eigenspace of λ_r. By Proposition 7, there are two distinct indices i and j among $\{r, \dots, r+m\}$, such that for t_0 small enough

$$\lambda_i^t \neq \lambda_j^t, \qquad \text{for } t \in (0, t_0). \tag{14}$$

By the eigenvalue stability estimate of Lemma 4, there is a t_0 small enough such that

$$|\lambda_i^t - \lambda_i| \leq M d_r, \quad \forall t < t_0, \forall i \in \{1, \dots, r+m+1\}. \tag{15}$$

Let C, C' indicate two constants which depend only on the dimension N, the Lipschitz constant of $\partial\Omega$ and the area of Ω. By Weyl's asymptotic law, $\lambda_n = Cn^{2/N} + o(n^{2/N})$ for any n. Then, from the uniform estimate of Lemma 4, for $i > r+m$ it holds

$$\lambda_i^t - \lambda_r \geq (\lambda_i^t - \lambda_i) + \lambda_i - \lambda_r \geq C'(-Cci^{2/N} + i^{2/N} - r^{2/N}),$$

where $c > 0$ is a bound on the deformation magnitude (which we can choose arbitrarily small) as in (13). Therefore for t_0 and c small enough,

$$\lambda_i^t - \lambda_r > 0, \quad \forall t < t_0, \forall i > r+m. \tag{16}$$

In conclusion, taking $\tilde{\Omega} := \Omega_t$ for a certain t small enough, Point 1 of the thesis holds by construction while Points 2-3-4 are consequences of (15)-(14)-(16). □

The construction in the previous proof gives us a method to split the first non-simple eigenvalue without altering the simplicity of smaller eigenvalues. In fact by taking $M < 1/2$, from Points 2 and 4 of Lemma 9 we have that the eigenvalues $\tilde{\lambda}_i$ perturbed from λ_i:

- lie in disjoint neighborhoods of λ_i, for $i < r$;
- are not further than $d_r/2$ from λ_i, for $r \leq i \leq r+m$;
- are larger than λ_r, for $i > r+m$.

Therefore $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{r-1}$ must still be simple. We can iterate this procedure to split the whole spectrum as in the following proof.

Proof of Theorem 1. Let B_ε be the ball of radius ε centered at x. Consider first the case when x is on the boundary of $\partial\Omega$. Let $\Sigma = B_\varepsilon \cap \partial\Omega$. As in Remark 8, for any $\delta > 0$, we can modify Σ into Σ' so that an open subset of Σ' is contained in a hyperplane and the Lipschitz constant of Σ' differs from the Lipschitz constant of Σ by less than δ. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of disjoint balls of radius $c2^{-n}$ with centers on Σ' and contained in B_ε, with c small enough so that $\Sigma' \cap \bigcup_n B_n$ is flat. In each B_n we deform Σ' with a diffeomorphism ϕ_n built as in the proof of Proposition 7. We obtain this way a sequence of domains $(\Omega_n)_{n\in\mathbb{N}}$ such that the thesis of Lemma 9 holds with $\Omega, \tilde{\Omega}, B, M$ replaced respectively by $\Omega_n, \Omega_{n+1}, B_n, M_n$ for each n, where for M_n we take a positive constant smaller than $1/2^{n+1}$. Additionally, we can take ϕ_n such that $|\phi_n - id|_{C^1} \leq \delta/n$. And thus as $n \to \infty$, Ω_n converges to a domain $\tilde{\Omega}$ with Lipschitz constant not farther than δ from the Lipschitz constant of Ω.

Let r_n be the index of the first non-simple eigenvalue of Ω_n. By Points 2 and 4 of Lemma 9 we have that all eigenvalues with index smaller than r_n are simple for any n. Moreover r_n is a non-decreasing sequence of integers which cannot be definitely constant; in fact by Point 3 of Lemma 9, r_{n+j} can be equal to r_n for at most $j \in \{1, \ldots, r_n\}$. Therefore $r_n \to \infty$ as $n \to \infty$, and thus $\tilde{\Omega}$ can have only simple eigenvalues.

Consider now the case when x is in the interior of Ω. Then by cutting an appropriately shaped hole inside Ω, for example by considering $\Omega \setminus F$ where F is a rescaling and translation of Ω so that $F \subseteq B_\varepsilon$, we can then build a deformation of the boundary ∂F exactly as in the previous steps, so that the spectrum of the perturbed domain is simple and its Lipschitz constant is arbitrarily near to the one of Ω. □

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