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Explicit terms in the small volume expansion of the shift of Neumann Laplacian eigenvalues due to a grounded inclusion in two dimensions

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Abstract

The first terms in the small volume asymptotic expansion of the shift of Neumann Laplacian eigenvalues caused by a grounded inclusion of area ε^2 are derived. A novel explicit formula to compute them from the capacity, the eigenvalues and the eigenfunctions of the unperturbed domain, the size and the position of the inclusion, is given. The key step in the derivation is the filtering of the spectral decomposition of the Neumann function with the residue theorem. As a consequence of the formula, when a bifurcation of a double eigenvalue occurs (as for example in the case of a generic inclusion inside a disk) one eigenvalue decays like $O(1/\log\varepsilon)$, the other like $O(\varepsilon^2)$.

1 Introduction

Consider a planar domain Ω and let ω^2 be an eigenvalue of the negative Laplacian on Ω with homogeneous Neumann boundary conditions. Suppose a small inclusion $D = z + \varepsilon B$ (where $z \in \Omega$, $|B| = |\Omega|$, and ε is small) is inserted inside Ω. This may cause the eigenvalue ω_ε^2 of the perturbed domain $\Omega\setminus D$ (with Neumann condition on $\partial\Omega$ and Dirichlet on ∂D) to vary in value or in multiplicity with respect to ω^2. Asymptotic formulae of the perturbation with respect to the size of the inclusion have been derived in the '80s in [10] and [5]. In particular it has been shown that if ω^2 is simple and u is the associated L^2-normalized eigenfunction, the perturbation is singular and

$$\omega_\varepsilon^2 - \omega^2 = -\frac{2\pi|u(z)|^2}{\log\varepsilon} + o\left(1/\log(\varepsilon)\right). \tag{1}$$

More recently, Gohberg-Sigal theory for meromorphic operators applied to the integral equation formulation of the eigenvalue problem has led to new results (see [3], [4]). In this paper we elaborate on these results to further improve (1), by calculating explicitly the terms up to $O(\varepsilon^2)$ and by generalizing it to the case of multiplicity 2. As a consequence of our derivation, for perturbed eigenvalues $\omega_{\varepsilon,1}^2 < \omega_{\varepsilon,2}^2$ splitted from a double eigenvalue ω^2 of the original domain Ω, it holds

$$\begin{aligned}\omega_{\varepsilon,2} - \omega &= -\frac{C_1}{\log(\varepsilon) + C_2} + O(\varepsilon^2),\\ \omega_{\varepsilon,1} - \omega &= O(\varepsilon^2),\end{aligned}$$

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where C_1 and C_2 do not depend on ε and can be explicitly calculated from the capacity, the eigenvalues, and the value at z of the eigenfunctions of Ω.

More in detail the structure of the paper is as follows. After introducing in Section 1.1 the precise setting of the problem and the notation, in Section 1.2 we recall the equivalent formulation of the Laplacian eigenvalues as characteristic values of an appropriate integral operator. An asymptotic expansion of this integral operator can be obtained by expanding in Taylor series the free space fundamental solution. Gohberg-Sigal theory then provides a link between eigenvalues' shifts and the traces of these integral operators through power sum polynomials. In the core Section 2, explicit terms for the small volume expansion of these power sum polynomials are derived by using properties of layer potentials. The key step in this derivation is the filtering of the spectral decomposition of the Neumann function using the residue theorem to obtain geometric-like series which can be summed. A tentative proposal for formal automated computation of higher order coefficients is given in Section 2.3. Finally in Section 3 some interesting consequences for special cases and a brief validation with numerical experiments are provided.

1.1 Main tools and notation

The eigenvalue problem Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$ with C^1 boundary. It is well known that the eigenvalues of the negative Laplacian on Ω with Neumann boundary condition are non-negative, have finite multiplicity and can be arranged in an increasing divergent sequence

$$0 = \omega_0^2 < \omega_1^2 < \omega_2^2 < \cdots < \omega_k^2 \to \infty.$$

For each index i, let m_i be the multiplicity of ω_i^2. We choose the associated eigenfunctions $u_{i,1}, \ldots, u_{i,m_i}$ to be orthonormal in L^2. For any i, j we thus have

$$\begin{cases} (\Delta + \omega_i^2) u_{i,j} = 0 & \text{in } \Omega, \\ \dfrac{\partial u_{i,j}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\int_\Omega u_{i,j} u_{k,l} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

We will often use the vector notation

$$U_i := (u_{i,1}, \ldots, u_{i,m_i}). \tag{2}$$

Free space fundamental solution The free space fundamental solution for Helmholtz equation $(\Delta + \omega^2)u = 0$ is a function Γ_ω s.t. for any $x, y \in \mathbb{R}^2$, it holds

$$(\Delta_x + \omega^2)\Gamma_\omega(x, y) = \delta_y(x)$$

in the distributional sense, where δ_y is the Dirac delta at y. We adopt as fundamental solution

$$\Gamma_\omega(x, y) := \begin{cases} \dfrac{1}{2\pi} \log|x - y| & \text{if } \omega = 0, \\ \dfrac{1}{4} Y_0(\omega|x - y|) & \text{otherwise,} \end{cases}$$

where Y_0 is the Bessel function of the second kind and order 0; it can be defined by the power series

$$Y_0(t) := \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \left(\frac{t^n}{2^n n!} \right)^2 \log(\eta_n t),$$

with $\log \eta_n :=$ *Euler-Mascheroni constant* $- \log 2 + \sum_{k=1}^{n} \frac{1}{k}$ (for further details see [1, Chapter 9]).

Layer potentials Given $\phi \in L^2(\partial\Omega)$, we define the operators

$$\begin{aligned}
&\text{(Single layer potential)} && \mathcal{S}_\Omega^\omega[\phi](x) := \int_{\partial\Omega} \Gamma_\omega(x,y)\phi(y)\, \mathrm{d}\sigma(y) && \text{for } x \in \mathbb{R}^2,\\
&\text{(Double layer potential)} && \mathcal{D}_\Omega^\omega[\phi](x) := \int_{\partial\Omega} \frac{\partial \Gamma_\omega(x,y)}{\partial \nu(y)}\phi(y)\, \mathrm{d}\sigma(y) && \text{for } x \in \mathbb{R}^2 \setminus \partial\Omega,\\
&\text{(Neumann-Poincaré operator)} && \mathcal{K}_\Omega^\omega[\phi](x) := \int_{\partial\Omega} \frac{\partial \Gamma_\omega(x,y)}{\partial \nu(y)}\phi(y)\, \mathrm{d}\sigma(y) && \text{for } x \in \partial\Omega.
\end{aligned}$$

For their properties and extensive applications in the theory of boundary value problems we refer to [6].

Capacity of a set The single layer potential can be used to define the capacity of a set as follows. It can be shown that there exists a unique non-zero couple $(\varphi_{\text{cap}}, a) \in L^2(\partial\Omega) \times \mathbb{R}$ which solves

$$\begin{cases} \mathcal{S}_\Omega^0[\varphi_{\text{cap}}](x) \equiv a & \forall x \in \partial\Omega,\\ \int_{\partial\Omega} \varphi_{\text{cap}} = 1. \end{cases}$$

The logarithmic capacity of $\partial\Omega$ is then defined as $\operatorname{cap}\partial\Omega := e^{2\pi a}$ (for further properties of the capacity we refer to [9, Section 16.4]).

Remark 1.1. Consider the special case when Ω is the unit disk. Writing t for the angle in the usual polar parametrization of the boundary, one can calculate that

$$\mathcal{S}_\Omega^0[e^{int}](\tau) = \begin{cases} 0 & \text{if } n = 0,\\ -\dfrac{1}{2n}e^{in\tau} & \text{otherwise.} \end{cases}$$

Thus we have an explicit expression of $\mathcal{S}_\Omega^0$ in the Fourier basis of $L^2(\partial\Omega)$. Notice however that the fact that $\mathcal{S}_\Omega^0[1] = 0$ causes the non-invertibilty of $\mathcal{S}_\Omega^0$. However, if we consider

$$\phi \mapsto \mathcal{S}_\Omega^0[\phi] + \lambda \int_{\partial\Omega} \phi,$$

we see that, for a constant $\lambda \neq a$, this operator is always invertible from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$. This is still true for a more general domain Ω as in our assumptions (see [12, Theorem 4.11] for more details).

Fundamental solution for a bounded domain The Neumann function N_Ω^ω is defined as the solution of

$$\begin{cases} (\Delta_x + \omega^2)N_\Omega^\omega(x,z) = \delta_z(x) & \text{for } x \in \Omega,\\ \dfrac{\partial N_\Omega^\omega(x,z)}{\partial \nu(x)} = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\omega \in \mathbb{C}$ is not one of the eigenvalues ω_i and $z \in \Omega$. It has the spectral representation

$$N_\Omega^\omega(x,z) = \sum_{j=1}^{\infty} \frac{U_j(x)\cdot U_j(z)}{\omega^2 - \omega_j^2},$$

where the convergence of the series to N_Ω^ω in general is only in L^2 (see [7, *expansion theorems*]).

By integrating N_Ω^ω against test functions in $L^2(\partial\Omega)$ and using the properties of layer potentials one can show that

$$(I/2 - \mathcal{K}_\Omega^\omega)^{-1}\left[\Gamma_\omega(\cdot,z)\right](x) = N_\Omega^\omega(x,z). \tag{3}$$

We recall also that the Neumann function has a logarithmic singularity, in particular

$$N_\Omega^\omega(x,z) = \frac{1}{2\pi}\log|x-z| + R_\Omega^\omega(x,z) \quad \forall x \neq z, \tag{4}$$

with R_Ω^ω continuous on $\Omega \times \Omega$ (for more details on the last two results, see [2, Section 2.3.5] and references therein).

The perturbed eigenvalue problem Let B be a bounded domain with piecewise smooth boundary, with area $|B| = |\Omega|$, and centered at the origin in the sense that

$$\int_{\partial B} y_1 \,\mathrm{d}\sigma(y_1,y_2) = \int_{\partial B} y_2 \,\mathrm{d}\sigma(y_1,y_2) = 0.$$

We fix for the rest of the paper a point $z \in \Omega$, a scaling factor $0 < \varepsilon \ll 1$ and an index $\theta \in \mathbb{N}$. Suppose then that the domain Ω is perturbed by inserting a grounded inclusion $D := z + \varepsilon B$ inside Ω. This causes the eigenvalue ω_θ^2 to split into m_θ (possibly distinct) eigenvalues $\omega_{\varepsilon,1}^2 \leq \cdots \leq \omega_{\varepsilon,m_\theta}^2$ with associated eigenfunctions $u_{\varepsilon,1},\ldots,u_{\varepsilon,m_\theta}$. This means that for $j = 1,\ldots,m_\theta$,

$$\begin{cases} (\Delta + \omega_{\varepsilon,j}^2)u_{\varepsilon,j} = 0 & \text{in } \Omega \setminus D, \\ u_{\varepsilon,j} = 0 & \text{on } \partial D, \\ \dfrac{\partial u_{\varepsilon,j}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

It has been shown in [11] that under our assumptions $\omega_{\varepsilon,j}^2 \to \omega_\theta^2$ as $\varepsilon \to 0$. The problem we will consider is the following:

to find an asymptotic expansion of $\omega_{\varepsilon,j}^2 - \omega_\theta^2$ in terms of ε, for all $j \in \{1,\ldots,m_\theta\}$.

With this purpose we will transform the eigenvalue problem into an equivalent integral formulation.

Nonstandard notation We will use the following notation:

- we indicate as $\oint$ the normalized complex path integral $\dfrac{1}{2\pi \mathrm{i}}\displaystyle\int$;
- for clarity, we adopt the symbol $\diamond$ to indicate the function variable of an operator evaluated at a point, e.g. $\mathcal{D}_\Omega^\omega[\diamond](z)$ indicates the function $L^2(\partial\Omega) \ni \varphi \mapsto \mathcal{D}_\Omega^\omega[\varphi](z) \in \mathbb{R}$;
- given a multi-index $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$ we indicate as ∂^α the normalized differential operator

$$\frac{1}{\alpha_1!\ldots\alpha_n!}\frac{\partial}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}.$$

1.2 Integral formulation

Define $\mathcal{A}_\varepsilon(\omega)$ as

$$\mathbb{C} \ni \omega \quad \mapsto \quad \mathcal{A}_\varepsilon(\omega) := \begin{pmatrix} I/2 - \mathcal{K}_\Omega^\omega & -\mathcal{S}_D^\omega \\ \mathcal{D}_\Omega^\omega & \mathcal{S}_D^\omega \end{pmatrix},$$

meaning that for any fixed $\omega \in \mathbb{C}$, $\mathcal{A}_\varepsilon(\omega)$ is the operator which takes $\phi \in L^2(\partial\Omega), \psi \in L^2(\partial D)$ to

$$\begin{pmatrix} (I/2 - \mathcal{K}_\Omega^\omega)[\phi] - \mathcal{S}_D^\omega[\psi] \\ \mathcal{D}_\Omega^\omega[\phi] + \mathcal{S}_D^\omega[\psi] \end{pmatrix} \in \begin{matrix} L^2(\partial\Omega) \\ \times \\ L^2(\partial D). \end{matrix}$$

By expanding the fundamental solution in Taylor series in ε, one can show that $\mathcal{A}_\varepsilon$ has the same characteristic values as the operator $\sum_{n=0}^{\infty} \varepsilon^n \mathcal{H}_n$ (the series converges in operator norm), where

$$\mathcal{H}_0 := \begin{pmatrix} I/2 - \mathcal{K}_\Omega^\omega & -\Gamma_\omega(x,z) \int_{\partial B} \diamond(y) \, \mathrm{d}\sigma(y) \\ \mathcal{D}_\Omega^\omega[\diamond](z) & \tilde{\mathcal{S}}_B^\omega \end{pmatrix},$$

$$\mathcal{H}_n := \begin{pmatrix} 0 & (-1)^{n+1} \sum_{|\alpha|=n} (\partial^\alpha \Gamma_\omega)(x,z) \int_{\partial B} y^\alpha \diamond(y) \, \mathrm{d}\sigma(y) \\ \sum_{|\alpha|=n} (\partial^\alpha \mathcal{D}_\Omega^\omega[\diamond])(z) x^\alpha & \mathcal{X}_n \end{pmatrix},$$

with

$$\mathcal{X}_n := \begin{cases} \dfrac{\omega^n}{2^{n+1} n! \pi} \displaystyle\int_{\partial B} \log(\eta_{\frac{n}{2}} \omega \varepsilon |x-y|) |x-y|^n \diamond(y) \, \mathrm{d}\sigma(y) & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases}$$

$$\tilde{\mathcal{S}}_B^\omega := \frac{1}{2\pi} \int_{\partial B} \log(\eta_0 \omega \varepsilon |x-y|) \diamond(y) \, \mathrm{d}\sigma(y). \tag{5}$$

A study of the properties of A_ε can be found in [2, Chapter 1 and Section 3.1]). In the next proposition we collect only the properties which will be used in the following discussion. Recall that $\omega \in \mathbb{C}$ is a characteristic value of A_ε if the null-space of $A_\varepsilon(\omega)$ contains some non-zero function.

Proposition 1.2. *The following results hold:*

1. *$\omega \mapsto \mathcal{A}_\varepsilon(\omega)$ is analytic on $\mathbb{C} \setminus \mathrm{i}\mathbb{R}^-$ and $\omega \mapsto \mathcal{A}_\varepsilon(\omega)^{-1}$ is meromorphic in $\mathbb{C}$;*
2. *ω_θ is a characteristic value of $I/2 - \mathcal{K}_\Omega^\omega$ and a simple pole of $(I/2 - \mathcal{K}_\Omega^\omega)^{-1}$;*
3. *$(\omega_{\varepsilon,j})_{j=1,\dots,m_\theta}$ are characteristic values of $\mathcal{A}_\varepsilon$;*
4. *There is an open neighbourhood V (which we fix for the rest of the paper) of ω_θ s.t. $\omega_{\varepsilon,j} \in V$ for $j = 1, \dots, m_\theta$, and no other characteristic values of $\mathcal{A}_\varepsilon$ are in V.*

Consider now the power sum polynomials

$$p_l := \sum_{j=1}^{m_\theta} (\omega_{\varepsilon,j} - \omega_\theta)^l.$$

By properties of symmetric polynomials we can express $\omega_{\varepsilon,1} - \omega_\theta, \dots, \omega_{\varepsilon,m_\theta} - \omega_\theta$ as roots of a polynomial $z^{m_\theta} + c_1 z^{m_\theta - 1} + \cdots + c_{m_\theta}$, where the coefficients c_k are themselves polynomials in p_j; in particular the coefficients c_k can be recovered from the recurrence relations

$$p_{l+m_\theta} + c_1 p_{l+m_\theta-1} + \cdots + c_{m_\theta} p_l \qquad \text{for } l = 0, \dots, m_\theta - 1.$$

Example 1.3. If $m_\theta = 1$ we have

$$\omega_{\varepsilon,1} - \omega_\theta = p_1,$$

while if $m_\theta = 2$ then

$$\omega_{\varepsilon,2} - \omega_\theta = \frac{p_1 + \sqrt{2p_2 - p_1^2}}{2}, \qquad \omega_{\varepsilon,1} - \omega_\theta = \frac{p_1 - \sqrt{2p_2 - p_1^2}}{2}.$$

Thus we have reduced the problem of finding an asymptotic expansion of $\omega_{\varepsilon,j}^2 - \omega_\theta^2$ to finding an asymptotic expansion of p_l. Before computing the explicit terms in the expansion of p_l we recall some crucial concepts from Gohberg-Sigal theory.

If A is a finite range operator on a Banach space, its trace $\operatorname{tr} A$ can be defined as the trace of A restricted to the finite dimensional space where A is non zero. In the next proposition we recall some properties of the trace which will be extensively used in the subsequent computations.

Proposition 1.4. *The following results hold:*

1. *Suppose A_1, A_2, A_3, A_4 are finite dimensional operators on a Banach space. Then*

$$\operatorname{tr}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \operatorname{tr} A_1 + \operatorname{tr} A_4.$$

2. *Suppose B, C are operator valued maps defined on U, a neighborhood of a common singularity $\nu \in \mathbb{C}$. If B, C are analytic in $U \setminus \nu$ and have only finite dimensional operators in the negative terms of their Laurent expansion in ν, then $\int_{\partial U} B(\omega)C(\omega)\,\mathrm{d}\omega$ is finite dimensional and*

$$\operatorname{tr}\oint_{\partial U} B(\omega)C(\omega)\,\mathrm{d}\omega = \operatorname{tr}\oint_{\partial U} C(\omega)B(\omega)\,\mathrm{d}\omega.$$

3. *If P_ω is a projection from L^2 to $\mathbb{C}$ and $f_\omega \in L^2$, then*

$$\operatorname{tr}\oint_{\partial V} f_\omega P_\omega\,\mathrm{d}\omega = \oint_{\partial V} P_\omega[f_\omega]\,\mathrm{d}\omega.$$

An application of the argument principle for operator valued maps (for its formulation we refer to [8]) leads to the following crucial representation.

Theorem 1.5. *We can rewrite p_l as*

$$\begin{aligned} p_l = {} & \operatorname{tr}\oint_{\partial V} (\omega-\omega_\theta)^l \mathcal{H}_0(\omega)^{-1}\partial_\omega \mathcal{H}_0(\omega)\,\mathrm{d}\omega \\ & + l\sum_{n=1}^{\infty}\varepsilon^n \sum_{j=1}^{n}\frac{(-1)^j}{j}\operatorname{tr}\oint_{\partial V}(\omega-\omega_\theta)^{l-1}\left(\sum_{\substack{n_1+\cdots+n_j=n\\ n_i\in\mathbb{Z}^+}}\prod_{k=1}^{j}\mathcal{H}_0(\omega)^{-1}\mathcal{H}_{n_k}(\omega)\right)\mathrm{d}\omega. \end{aligned}$$

The previous expression can be obtained by following the same algebraic manipulations in the proof of [2, Theorem 3.9].

2 Computations for explicit formulae

First we highlight the quantities playing a key role in the expansion of p_l in the following definition.

Definition 2.1. Let α, β be multi-indices in $\mathbb{N}^2$. The *generalized capacity* of B of order (α, β), at frequency ω is

$$\mathfrak{c}_{\alpha,\beta}(\omega) := -\int_{\partial B} (\tilde{\mathcal{S}}_B^\omega)^{-1}[\xi^\alpha](y)\, y^\beta\,\mathrm{d}\sigma(y),$$

where $\tilde{\mathcal{S}}_B^\omega$ is defined in (5). We also introduce

$$\mathfrak{t}(\omega) := \frac{U_\theta(z)\cdot \mathcal{D}_\Omega^\omega[U_\theta](z)}{\omega+\omega_\theta}, \qquad \mathfrak{r}(\omega) := \sum_{\substack{j=1\\ j\neq\theta}}^{\infty}\frac{U_j(z)\cdot\mathcal{D}_\Omega^\omega[U_j](z)}{\omega^2-\omega_j^2}, \tag{6}$$

where U_j is the vector of eigenfunctions associated to ω_j^2 as defined in (2).

In the subsequent discussion we will often indicate the generalized capacity of order $(\alpha, 0)$ as $\mathfrak{c}_\alpha$ instead of $\mathfrak{c}_{\alpha,0}$.

Remark 2.2. We collect some useful properties of the quantities introduced in the previous defintion:

1. The generalized capacity of order zero can be rewritten as
$$\mathfrak{c}_0(\omega) = -\int_{\partial B} (\tilde{\mathcal{S}}_B^\omega)^{-1}[1] = -\frac{2\pi}{\log(\eta_0 \omega \varepsilon \operatorname{cap} \partial B)}.$$
2. It holds $\mathfrak{c}_{\alpha,\beta} = 0$ if $|\alpha| + |\beta|$ is odd. This is a consequence of the fact that φ is even/odd if and only if $\mathcal{S}_B^0[\varphi]$ is even/odd (as functions parametrized on ∂B).
3. Althought the series defining $\mathfrak{r}$ in (6) does not converge absolutely, Weyl's law (which states that $\omega_j^2 \sim j$) and the oscillatory nature of the eigenfunctions of Ω evaluated at z suggest that the series converges conditionally.
4. By exploiting the spectral expansion of the Neumann function (4), we have that
$$\mathcal{D}_\Omega^\omega[N_\Omega^\omega(\cdot, z)](z) = \frac{\mathfrak{t}(\omega)}{\omega - \omega_\theta} + \mathfrak{r}(\omega). \tag{7}$$
In the subsequent calculations, this identity will enable us to rewrite the expansion in Theorem 1.5 in terms of $\mathfrak{t}(\omega_\theta)$ and $\mathfrak{r}(\omega_\theta)$.
5. By Green's identity and the defining property of Γ_{ω_θ} and U_θ, we can compute the special value
$$\mathfrak{t}(\omega_\theta) := \frac{|U_\theta(z)|^2}{2\omega_\theta}. \tag{8}$$

2.1 Zero order term

Lemma 2.3. *The zero order term in the expansion in powers of ε of p_l is*
$$\left(\frac{\mathfrak{t}(\omega_\theta)}{1/\mathfrak{c}_0(\omega_\theta) - \mathfrak{r}(\omega_\theta)}\right)^l. \tag{9}$$

Proof. By Theorem 1.5, our problem reduces to compute explicitly
$$\operatorname{tr} \oint_{\partial V} (\omega - \omega_\theta)^l \mathcal{H}_0(\omega)^{-1} \partial_\omega \mathcal{H}_0(\omega) \, \mathrm{d}\omega.$$
To make further computations clearer and more concise, we rename
$$\begin{aligned} A &:= I/2 - \mathcal{K}_\Omega^\omega[\diamond](x), \qquad & \Gamma &:= \Gamma_\omega(x, z), \\ N &:= N_\Omega^\omega(x, z), & D &:= \mathcal{D}_\Omega^\omega[\diamond](z), \\ \mathfrak{s} &:= \omega - \omega_\theta. \end{aligned} \tag{10}$$
The characteristic values of $\mathcal{H}_0$ are the $\omega \in \mathbb{C}$ for which there exist $\phi \in L^2(\Omega), \psi \in L^2(B)$, at least one of them non-zero, s.t.
$$\begin{cases} A\phi - \Gamma \int_{\partial B} \psi = 0, \\ D\phi + \tilde{\mathcal{S}}_B^\omega \psi = 0. \end{cases}$$
By recalling from Remark 1.1 that $\tilde{\mathcal{S}}_B^\omega$ is invertible, applying $(\tilde{\mathcal{S}}_B^\omega)^{-1}$ and integrating the second equation of the system we obtain
$$\int_{\partial B} \psi = \mathfrak{c}_0 D\phi.$$

Substituting this back into the first equation, we have that the characteristic values of the system correspond to the characteristic values of the operator

$$H := A - \mathfrak{c}_0 \Gamma D.$$

Therefore the coefficient we are looking for will be given by

$$E := \operatorname{tr} \oint_{\partial V} \mathfrak{s}^l H^{-1} H' \, \mathrm{d}\omega,$$

where $'$ denotes differentiation w.r.t. ω. A straightforward calculation shows that

$$H^{-1} = (I - \mathfrak{c}_0 N D)^{-1} A^{-1} = \sum_{m=0}^{\infty} (\mathfrak{c}_0 N D)^m A^{-1},$$

(where the m exponent indicates m times the composition of $c_0 N D$) and

$$H' = A' - \Gamma' \mathfrak{c}_0 D - \Gamma (\mathfrak{c}_0 D)'.$$

Then

$$H^{-1} H' = A^{-1} A' + \sum_{m=1}^{\infty} (\mathfrak{c}_0 N D)^m A^{-1} A' - (\mathfrak{c}_0 N D)^{m-1} A^{-1} \Gamma' \mathfrak{c}_0 D - (\mathfrak{c}_0 N D)^{m-1} N (\mathfrak{c}_0 D)'.$$

A is analytic in V, A^{-1} has a simple pole at ω_θ, and we are only interested in the case $l \geq 1$, therefore

$$\oint_{\partial V} \mathfrak{s}^l A^{-1} A' = 0.$$

Then, by Point 3 of Proposition 3,

$$\begin{aligned} E &= \sum_{m=1}^{\infty} \operatorname{tr} \oint_{\partial V} \mathfrak{s}^l \left((\mathfrak{c}_0 N D)^m A^{-1} A' - N^{m-1} (\mathfrak{c}_0 D)^m A^{-1} \Gamma' - N^m (\mathfrak{c}_0 D)^{m-1} (\mathfrak{c}_0 D)' \right) \\ &= \sum_{m=1}^{\infty} \oint_{\partial V} \mathfrak{s}^l \left(\mathfrak{c}_0^m (DN)^{m-1} D A^{-1} A' A^{-1} \Gamma - \mathfrak{c}_0^m (DN)^{m-1} D A^{-1} \Gamma' - (\mathfrak{c}_0 D N)^{m-1} (\mathfrak{c}_0 D)' N \right). \end{aligned}$$

Since $(A^{-1})' = -A^{-1} A' A^{-1}$, by applying multiple times the chain rule we obtain

$$E = -\sum_{m=1}^{\infty} \frac{1}{m} \oint_{\partial V} \mathfrak{s}^l \left((\mathfrak{c}_0 D N)^m \right)'.$$

Then, by an integration by parts followed by a binomial expansion of $(DN)^m$, we have that

$$\begin{aligned} E &= \sum_{m=1}^{\infty} \frac{l}{m} \oint_{\partial V} \mathfrak{s}^{l-1} (\mathfrak{c}_0 D N)^m \\ &= \sum_{m=1}^{\infty} \frac{l}{m} \oint_{\partial V} \mathfrak{s}^{l-1} \mathfrak{c}_0^m \left(\frac{\mathfrak{t}}{\mathfrak{s}} + \mathfrak{r} \right)^m \\ &= \sum_{m=1}^{\infty} \frac{l}{m} \sum_{k=0}^{m} \binom{m}{k} \oint_{\partial V} \frac{1}{\mathfrak{s}^{k-l+1}} \mathfrak{c}_0^m \mathfrak{t}^k \mathfrak{r}^{m-k}. \end{aligned} \tag{11}$$

Since the only pole in V of the integrand is ω_θ, by applying the residue theorem we can cancel each addend of the sum on k except the one corresponding to a pole of order 1, obtaining

$$E = l\mathfrak{t}(\omega_\theta)^l \mathfrak{r}(\omega_\theta)^{-l} \sum_{m=l}^{\infty} \frac{1}{m} \binom{m}{l} (\mathfrak{c}_0(\omega_\theta)\mathfrak{r}(\omega_\theta))^m.$$

A final application of the identity

$$\sum_{m=l}^{\infty} \frac{1}{m} \binom{m}{l} x^m = \frac{1}{l} \left(\frac{x}{1-x} \right)^l,$$

leads to the formula in the thesis.

□

2.2 First order term

Lemma 2.4. *The coefficient of the ε term in the expansion of p_l is null.*

Proof. With the notation introduced in (10),

$$\mathcal{H}_1 = \sum_{|\alpha|=1} \begin{pmatrix} 0 & (\partial^\alpha \Gamma) \int_{\partial B} y^\alpha \diamond \\ (\partial^\alpha D) x^\alpha & 0 \end{pmatrix}.$$

By applying the blockwise inversion formula

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}^{-1} = \begin{pmatrix} (W - XZ^{-1}Y)^{-1} & -W^{-1}X(Z - YW^{-1}X)^{-1} \\ -Z^{-1}Y(W - XZ^{-1}Y)^{-1} & (Z - YW^{-1}X)^{-1} \end{pmatrix}$$

to calculate $\mathcal{H}_0^{-1}$, and rewriting the inverses of sums of operators in a Neumann series, we obtain

$$\mathcal{H}_0^{-1} = \begin{pmatrix} (Nc_0 D)^m A^{-1} & N \int_{\partial B} (-(\tilde{\mathcal{S}}_B^\omega)^{-1}[1] DN \int_{\partial B})^m (\tilde{\mathcal{S}}_B^\omega)^{-1} \\ -(\tilde{\mathcal{S}}_B^\omega)^{-1}[1] D (Nc_0 D)^m A^{-1} & (-(\tilde{\mathcal{S}}_B^\omega)^{-1}[1] DN \int_{\partial B})^m (\tilde{\mathcal{S}}_B^\omega)^{-1} \end{pmatrix}.$$

A straightforward computation leads to

$$\mathcal{H}_0^{-1}\mathcal{H}_1 = \sum_{m=0}^{\infty} \sum_{|\alpha|=1} \begin{pmatrix} \mathfrak{c}_\alpha(...) & (...) \\ (...) & -(\tilde{\mathcal{S}}_B^\omega)^{-1}[1](...) \int_{\partial B} y^\alpha \diamond \end{pmatrix},$$

where we omit most of the terms by writing (...) instead. They indeed do not count towards our calculations, since from the fact that $\mathfrak{c}_\alpha = 0$ for $|\alpha| = 1$, we have that the coefficient of ε is

$$\operatorname{tr} \oint_{\partial V} \mathfrak{s}^{l-1} \mathcal{H}_0^{-1} \mathcal{H}_1 = \operatorname{tr} \oint_{\partial V} \mathfrak{s}^{l-1} \left((\mathcal{H}_0^{-1}\mathcal{H}_1)_{11} + (\mathcal{H}_0^{-1}\mathcal{H}_1)_{22} \right) = \mathfrak{c}_\alpha(\dots) = 0.$$

□

2.3 On higher order terms

In this subsection we propose a method to calculate explicitly any coefficient of the expansion of p_l. To the shorthand notation introduced in (10) we add $a = DN$, $S = \tilde{\mathcal{S}}_B^\omega$, $\phi = -S^{-1}[1]$. Then we can rewrite

$$\mathcal{H}_0^{-1} = \begin{pmatrix} A^{-1} & N\int_{\partial B} S^{-1} \\ \phi D A^{-1} & S^{-1} \end{pmatrix} + \sum_{m=1}^{\infty} \begin{pmatrix} N\mathfrak{c}_0^m a^{m-1} D A^{-1} & N(\mathfrak{c}_0 a)^m \int_{\partial B} S^{-1} \\ \phi(\mathfrak{c}_0 a)^m D A^{-1} & \phi \mathfrak{c}_0^{m-1} a^m \int_{\partial B} S^{-1} \end{pmatrix},$$

$$\mathcal{H}_n = \sum_{|\alpha|=n} \begin{pmatrix} 0 & (-1)^{n+1}(\partial^\alpha \Gamma)\int_{\partial B} y^\alpha \diamond (y)\, \mathrm{d}\sigma(y) \\ (\partial^\alpha D) x^\alpha & \mathcal{X}_n \end{pmatrix}.$$

An explicit computation leads to

$$\begin{aligned} -\mathcal{H}_0^{-1}\mathcal{H}_n = &\sum_{|\alpha|=n} \begin{pmatrix} N c_\alpha \partial^\alpha D & (-1)^n A^{-1}\partial^\alpha \Gamma \int_{\partial B} y^\alpha \diamond - N\int_{\partial B} S^{-1}\mathcal{X}_n \\ c_\alpha \partial^\alpha D & (-1)^n \phi D A^{-1}\partial^\alpha\Gamma \int_{\partial B} y^\alpha \diamond - S^{-1}\mathcal{X}_n \end{pmatrix} \\ &+ \sum_{m=1}^{\infty} \begin{pmatrix} N(\mathfrak{c}_0 a)^m \mathfrak{c}_\alpha \partial^\alpha D & (-1)^n N\mathfrak{c}_0^m a^{m-1} D A^{-1}(\partial^\alpha\Gamma)\int_{\partial B} y^\alpha \diamond - N(\mathfrak{c}_0 a)^m \int_{\partial B} S^{-1}\mathcal{X}_n \\ \phi \mathfrak{c}_0^{m-1} a^m \mathfrak{c}_\alpha \partial^\alpha D & (-1)^n \phi(\mathfrak{c}_0 a)^m D A^{-1}(\partial^\alpha\Gamma)\int_{\partial B} y^\alpha \diamond - \phi\mathfrak{c}_0^{m-1} a^m \int_{\partial B} S^{-1}\mathcal{X}_n \end{pmatrix}. \end{aligned}$$

Since all elements of the matrix are one-dimensional projection operators on either N, ϕ or 1, the operator $-\mathcal{H}_0^{-1}\mathcal{H}_n$ has the same characteristic values of the projection operator on 1 given by

$$\begin{aligned} &\sum_{|\alpha|=n} \begin{pmatrix} c_\alpha(\partial^\alpha D)N & (-1)^n \int_{\partial B} y^\alpha A^{-1}\partial^\alpha\Gamma - \int_{\partial B} S^{-1}\mathcal{X}_n N \\ c_\alpha(\partial^\alpha D)[1] & (-1)^n \mathfrak{c}_{0,\alpha} D A^{-1}\partial^\alpha \Gamma - S^{-1}\mathcal{X}_n[1] \end{pmatrix} \\ &+ \sum_{m=1}^{\infty} \begin{pmatrix} (\mathfrak{c}_0 a)^m \mathfrak{c}_\alpha (\partial^\alpha D)N & (-1)^n \mathfrak{c}_0^m a^{m-1} D A^{-1}(\partial^\alpha\Gamma)\int_{\partial B} y^\alpha N - (\mathfrak{c}_0 a)^m \int_{\partial B} S^{-1}\mathcal{X}_n N \\ \mathfrak{c}_0^{m-1} a^m \mathfrak{c}_\alpha (\partial^\alpha D)\phi & (-1)^n (\mathfrak{c}_0 a)^m D A^{-1}(\partial^\alpha\Gamma)\mathfrak{c}_{0,\alpha} - \mathfrak{c}_0^{m-1} a^m \int_{\partial B} S^{-1}\mathcal{X}_n \phi \end{pmatrix}. \end{aligned} \tag{12}$$

Suppose that the elements of the matrices in (12) can be rewritten explicitly in terms of sums of powers of the singularity $\mathfrak{s}$. The same approach used to derive the zero order term in the previous section could then be applied to compute explicitly

$$\operatorname{tr} \oint_{\partial V} \mathfrak{s}^{l-1} \mathcal{H}_0^{-1}\mathcal{H}_{n_1} \dots \mathcal{H}_0^{-1}\mathcal{H}_{n_j}.$$

Substituting this value back in the expression for p_l in Theorem 1.5, we could compute, for example with the aid of a computer algebra system, any of the coefficients of the expansion in ε.

However, the task of rewriting explicitly the singularity $\mathfrak{s}$ in (12) for $\alpha \geq 1$ is not trivial, as identities similar to (3) and (7), but involving the terms $(\partial^\alpha D)N$, $A^{-1}\partial^\alpha\Gamma$ and $S^{-1}\mathcal{X}_n N$ would need to be determined.

3 Results for special cases

We collect in this final section some interesting results which follow directly, or with minor algebraic manipulations, from Lemmas 2.3, 2.4 and Example 1.3.

Proposition 3.1. *Suppose ω_θ is simple.*

1. *We have*

$$\omega_{\varepsilon,1} - \omega_\theta = \frac{\mathfrak{t}(\omega_\theta)}{1/\mathfrak{c}_0 - \mathfrak{r}(\omega_\theta)} + O(\varepsilon^2),$$

where, recalling Definition 2.1 and Identity (8),

$$1/\mathfrak{c}_0 = -\frac{\log(\eta_0 \omega_\theta \varepsilon \operatorname{cap} \partial B)}{2\pi}, \quad \mathfrak{t}(\omega_\theta) = \frac{|U_\theta(z)|^2}{2\omega_\theta}, \quad \mathfrak{r}(\omega_\theta) = \sum_{\substack{j=1 \\ j\neq\theta}}^{\infty} \frac{U_j(z) \cdot \mathcal{D}_\Omega^{\omega_\theta}[U_j](z)}{\omega_\theta^2 - \omega_j^2};$$

2. *For ε small enough, we can deduce that $\omega_{\varepsilon,1} \geq \omega_\theta$.*

Proposition 3.2. *If ω_θ has double multiplicity then*

$$\omega_{\varepsilon,2} - \omega_\theta = \frac{\mathfrak{t}(\omega_\theta)}{1/\mathfrak{c}_0 - \mathfrak{r}(\omega_\theta)} + O(\varepsilon^2),$$
$$\omega_{\varepsilon,1} - \omega_\theta = O(\varepsilon^2).$$

We notice that by considering an expansion in powers of $1/\log(\varepsilon)$ we obtain

$$\frac{\mathfrak{t}(\omega_\theta)}{1/\mathfrak{c}_0 - \mathfrak{r}(\omega_\theta)} + O(\varepsilon^2) = -\frac{1}{\log \varepsilon} \frac{\pi |U_\theta(z)|^2}{\omega_\theta} + O\left(\frac{1}{\log(\varepsilon)^2}\right).$$

In particular substituting $2\omega_\theta \simeq \omega_\epsilon + \omega_\theta$ gives back (1).

We also notice that if z is on a nodal set of U_θ (i.e. $u_{\theta,1}, \dots, u_{\theta,m_\theta}$ are all zero at z) then $\mathfrak{t}(\omega_\theta) = 0$, and thus in both the cases of a simple or a double eigenvalue, the splitting order will be $O(\varepsilon^2)$.

Numerical validation for disk domain and disk inclusion

Let Ω be the unit disk and let ω_θ^2 be its first non-zero eigenvalue. It is known that ω_θ is given by the first root of the derivative of the Bessel function J_1 and has double multiplicity. Suppose that also the rescaled inclusion B is a unit disk.

In Figure 1 we compare results obtained through the multipole expansion method with the ε^2 error theoretized by the formula for $\omega_{\varepsilon,1} - \omega_\theta$. The multipole expansion is implemented by setting two polar coordinate systems, one centered in the center of Ω and one in z, and exploiting Graf's summation formula for Bessel functions to rewrite the eigenvalue problem as a root search for the determinant of the coordinate transformation matrix.

In Figures 2 and 3 we compare asymptotic formulae for $\omega_{\varepsilon,2} - \omega_\theta$ with results obtained with the multipole expansion method. The asymptotic formula is implemented numerically by truncating at a finite value the series in the definition of $\mathfrak{r}(\omega_\theta)$ in (6), and approximating the boundary layer integrals in $\mathfrak{r}(\omega_\theta)$ by quadrature.

We remark that the improved resolution of the inclusion size and position opens the possibility of the development of accurate inclusion reconstruction algorithms using the asymptotic formulae from Propositions 3.1 and 3.2. Of particular interest, also in applications, would be to analyze a reconstruction method for small inclusions based only on the knowledge of eigenfrequencies shifts. The implementation of such a method and its applications will be the topic of future studies.

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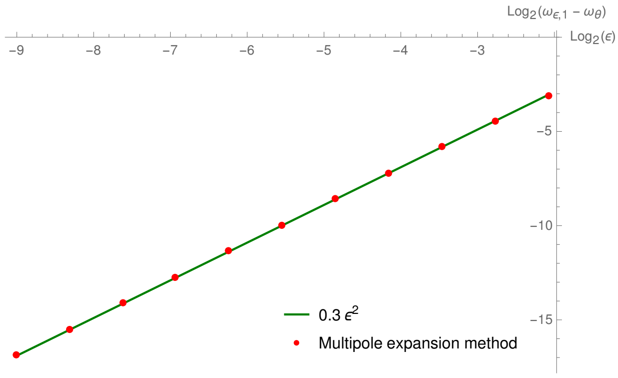

Figure 1: A $\log_2$-$\log_2$ plot of $\omega_{\varepsilon,1} - \omega_\theta$ as the size of the inclusion varies and the center is fixed (at $|z| = .5$).

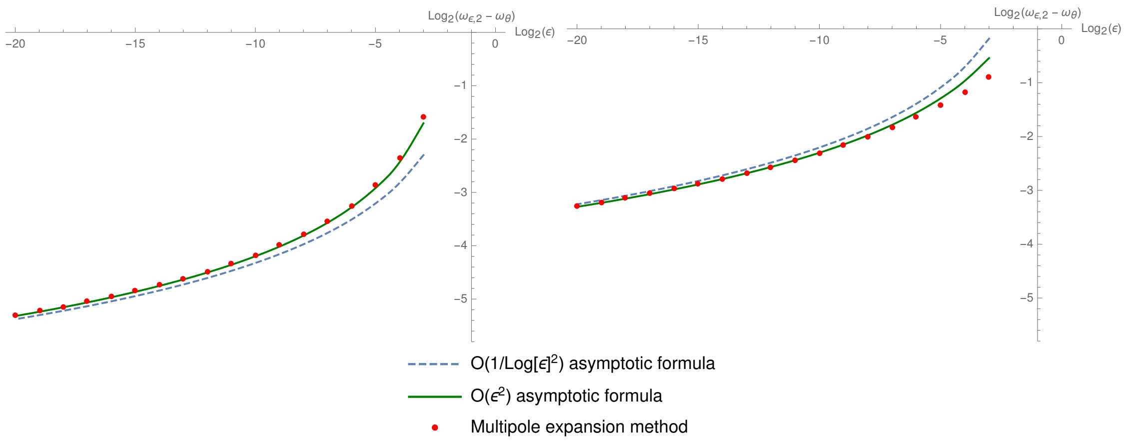

Figure 2: A $\log_2$-$\log_2$ plot of the shift $\omega_{\varepsilon,2} - \omega_\theta$ as the size of the inclusion varies and its center remains constant (left at $|z| = .3$, right at $|z| = .8$).

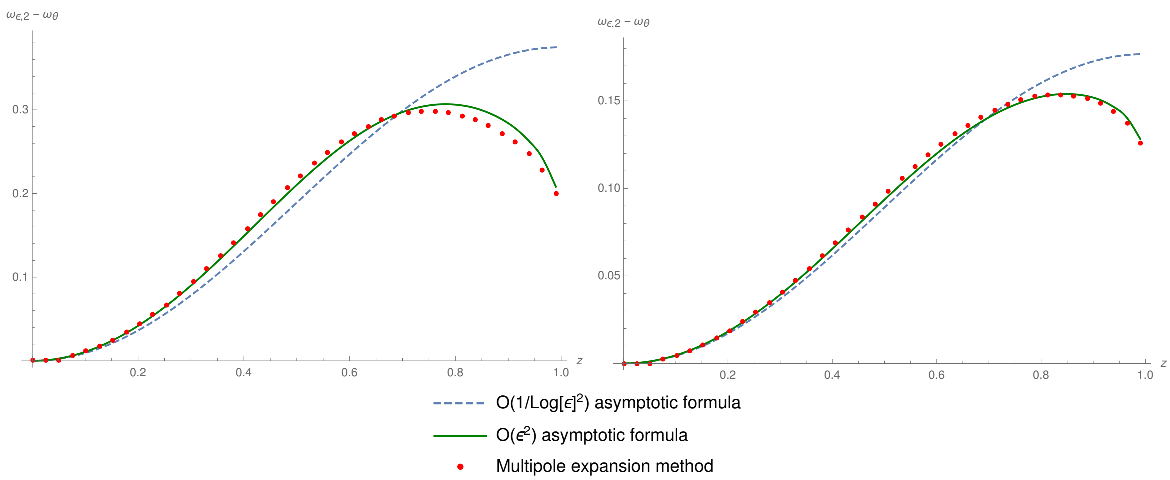

Figure 3: A plot of the shift $\omega_{\varepsilon,2} - \omega_\theta$ as the distance from the origin of the center of the inclusion varies and its size remains constant (left at $\varepsilon = 10^{-2}$, right at $\varepsilon = 10^{-4}$).

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