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H. Kang and S. Yu

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CH-8092 Zürich
Switzerland

A proof of the Flaherty-Keller formula on the effective property of densely packed elastic composites*

Hyeonbae Kang† Sanghyeon Yu‡

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Abstract

We prove in a mathematically rigorous way the asymptotic formula of Flaherty and Keller on the effective property of densely packed periodic elastic composites with hard inclusions. The proof is based on the primal-dual variational principle, where the upper bound is derived by using the Keller-type test functions and the lower bound by singular functions made of nuclei of strain. Singular functions are solutions of the Lamé system and capture precisely singular behavior of the stress in the narrow region between two adjacent hard inclusions.

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Key words. Flaherty-Keller formula, densely packed composite, effective elastic modulus, primal-dual principle, singular functions

1 Introduction

In a two phase composite where inclusions are close to each other and the material property, such as conductivity and elastic moduli, of the inclusion is of high contrast with that of the matrix, the effective property of the composite becomes singular. Several asymptotic formula (as the distance between inclusions tends to 0) for the effective properties have been found in diverse contexts. We review some of them which are related to present work. For a more complete account of such results, we refer readers to section 10.10 of Milton's book [10].

In [8], Keller considered a square array of circular cylinders when cylinders are nearly touching to each other and have extreme conductivities (perfectly conducting or insulating). It is observed that the effective conductivity blows up in the nearly touching limit and this divergence is due to the fact that the electric current flux is concentrated in the narrow region between adjacent cylinders. In fact, he derived an asymptotic formula for the effective conductivity as the distance of two inclusions tends to 0, and showed its validity numerically.

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†Department of Mathematics, Inha University, Incheon 22212, S. Korea (hbkang@inha.ac.kr)

‡Seminar for Applied Mathematics, ETH Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland (sanghyeon.yu@sam.math.ethz.ch)

Berlyand and Kolpakov [5] considered general non-periodic composites of circular inclusions and showed that its effective conductivity can be well approximated by a discrete network. In the course of such investigation, they proved Keller's asymptotic formula in a rigorous manner using the primal-dual variational principle. Remarkably their network approach has been extended to concentrated suspensions (Stokes system) and asymptotic formula for the effective property of periodic suspension has been derived [3, 4].

Keller's formula for the effective conductivity has been extended to elastic composites by Flaherty and Keller [6] (see (2.7) of this paper). They obtained asymptotic formulas for the effective elastic moduli of a rectangular array of cylinders in the nearly touching limit, when the cylinder is either a hard inclusion or a hole. However, a rigorous proof of their formulas is still missing to the best of our knowledge, and it is the purpose of this paper to provide a rigorous proof of the formula when the cylinder is a hard inclusion.

The proof of this paper is based on the primal-dual variational principle, which is a standard tool. The novelty of the proof lies in the construction of test functions, especially those for the dual principle. The test functions for the dual principle are constructed using singular functions introduced by authors in [7]. These functions are elaborated linear combinations of nuclei of strain and capture precisely stress concentration between two adjacent hard inclusions. Nuclei of strain are columns of the Kelvin matrix of the fundamental solution to the Lamé system and their variants. One important feature is that they are solutions of the Lamé system.

It is worth mentioning that in presence of two inclusions with extreme material property the gradient blows up in between two inclusions. Such gradient blow-up represents either stress concentration or field enhancement. Precise quantitative study on the gradient blow-up in various context has been important theme of active research in last ten years or so. We refer to [7] and references therein for such a development.

This paper is organized as follows. In the next section we set up the problem and introduce the Flaherty-Keller formula. In section 3 the primal-dual variational principle is introduced with a short proof. In section 4 we introduce the Keller-type functions and singular functions as test functions for primal and dual variational principles, respectively. The last section is to prove the Flaherty-Keller formula.

2 The Flaherty-Keller formula

Let L_1 and L_2 be positive numbers and let $Y = (-L_1, L_1) \times (-L_2, L_2)$ denote the period cell. Let $D \subset Y$ be a strictly convex domain containing the origin with the smooth boundary, which represents the two-dimensional cross section of the cylindrical inclusion. We assume that D is symmetric with respect to both x- and y-axes. Following [6] we assume that the periodic inclusions are nearly touching in one direction and they are away from each other in the other direction. We assume that D is close to the vertical boundary of Y, but away from the horizontal boundary. Let $\epsilon/2$ be the distance between D and the vertical boundary of Y, so that ϵ becomes the distance between two adjacent inclusions. See Figure 2.1. It is worth mentioning that if D is close to the horizontal boundary, then we can easily modify the Flaherty-Keller formula which is presented in Theorem 2.1.

Let (λ, μ) be the pair of Lamé constants of $Y \setminus \overline{D}$ which satisfies the strong ellipticity

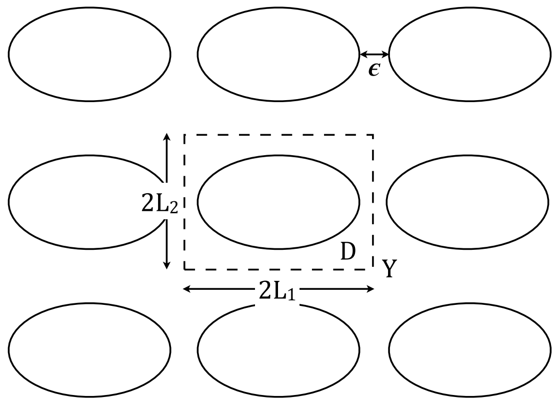

Figure 2.1: Geometry of the composite

conditions

$$\mu > 0 \quad \text{and} \quad \lambda + \mu > 0.$$

Then the elasticity tensor $\mathbb{C} = (C_{ijkl})$ is given by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where δ_{ij} denotes Kronecker's delta. The Lamé operator $\mathcal{L}_{\lambda,\mu}$ of the linear isotropic elasticity is defined by

$$\mathcal{L}_{\lambda,\mu} u := \nabla \cdot \mathbb{C}\widehat{\nabla} u = \mu \Delta u + (\lambda + \mu)\nabla\nabla \cdot u, \tag{2.1}$$

where $\widehat{\nabla}$ denotes the symmetric gradient, namely,

$$\widehat{\nabla} u = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \quad (T \text{ for transpose}).$$

The corresponding co-normal derivative $\partial_\nu u$ either on ∂D or on ∂Y is defined as

$$\partial_\nu u = (\mathbb{C}\widehat{\nabla} u) n, \tag{2.2}$$

where n is the outward unit normal vector field to ∂D or ∂Y, respectively.

Let

$$\Psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.3}$$

For $j = 1, 2$, let $v_j \in H^1(Y \setminus \overline{D})$ be the solution to the following problem:

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_j = 0, & \text{in } Y \setminus \overline{D}, \\ v_j = 0, & \text{on } \partial D, \\ v_j = \pm \dfrac{1}{2}\Psi_j, & \text{on } x = \pm L_1, \\ \partial_\nu v_j = 0, & \text{on } y = \pm L_2. \end{cases} \tag{2.4}$$

We extend v_j to the whole space $\mathbb{R}^2$ so that the extended function, denoted still by v_j, satisfies the following:

$$v_j(x + 2L_1, y) = v_j(x, y) + \Psi_j, \quad v_j(x, y + 2L_2) = v_j(x, y). \tag{2.5}$$

In particular, $\widehat{\nabla} v_j$ is periodic. The effective extensional modulus E_* and the effective shear modulus μ_* are given by (see [6])

$$\begin{aligned} E_* &= \frac{(1+\rho)(1-2\rho)}{1-\rho} \frac{L_1}{L_2} \int_{-L_2}^{L_2} \partial_\nu v_1(L_1, y) \cdot \Psi_1 \, dy, \\ \mu_* &= \frac{L_1}{L_2} \int_{-L_2}^{L_2} \partial_\nu v_2(L_1, y) \cdot \Psi_2 \, dy, \end{aligned}$$

where ρ is Poisson's ratio, namely,

$$\rho = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.6}$$

Let E be Young's modulus of the matrix, namely,

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

The asymptotic formulas for the effective elastic moduli obtained in [6] are as follows:

Theorem 2.1 (Flaherty-Keller formula). *Let κ_0 be the curvature of ∂D at the closest point to the boundary $x = L_1$. The following hold:*

$$E_* = E \frac{L_1}{L_2} \frac{\pi}{\sqrt{\kappa_0}} \frac{1}{\sqrt{\epsilon}} + O(1) \tag{2.7}$$

and

$$\mu_* = \mu \frac{L_1}{L_2} \frac{\pi}{\sqrt{\kappa_0}} \frac{1}{\sqrt{\epsilon}} + O(1), \tag{2.8}$$

as $\epsilon \to 0$.

3 Primal-dual variational principle

The effective moduli E_* and μ_* can be represented using energy integrals. In fact, since $\widehat{\nabla} v_j$ is periodic and $n|_{x=-L_1} = -n|_{x=L_1}$, we see that

$$\partial_\nu v_j(-L_1, y) = -\partial_\nu v_j(L_1, y).$$

So, we have from the boundary condition of v_j that

$$\begin{aligned} \int_{-L_2}^{L_2} \partial_\nu v_j(L_1, y) \cdot \Psi_j &= \int_{-L_2}^{L_2} \partial_\nu v_j(-L_1, y) \cdot (-\frac{1}{2}\Psi_j) + \int_{-L_2}^{L_2} \partial_\nu v_j(L_1, y) \cdot \frac{1}{2}\Psi_j \\ &= \int_{\partial(Y \setminus \overline{D})} \partial_\nu v_j \cdot v_j = \int_{Y \setminus \overline{D}} \mathbb{C} \widehat{\nabla} v_j : \widehat{\nabla} v_j. \end{aligned}$$

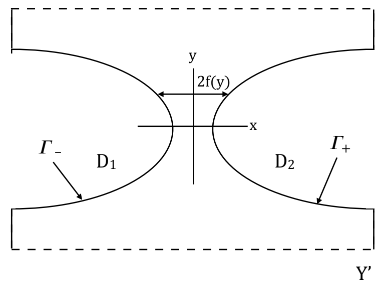

Figure 3.1: Geometry of Y'

Here and throughout the paper, the expression $A : B$ for 2×2 matrices $A = (a_{ij})$ and $B = (b_{ij})$ indicates $\sum_{i,j} a_{ij} b_{ij}$.

Let

$$\mathcal{E}_j := \int_{Y \setminus \overline{D}} \mathbb{C}\widehat{\nabla} v_j : \widehat{\nabla} v_j. \tag{3.1}$$

Then we have

$$E_* = \frac{(1+\rho)(1-2\rho)L_1}{(1-\rho)L_2}\mathcal{E}_1 \quad \text{and} \quad \mu_* = \frac{L_1}{L_2}\mathcal{E}_2. \tag{3.2}$$

It is more convenient to consider the energy integral $\mathcal{E}_j$ in a translated cell $Y_t := Y - (L_1, 0) = (-2L_1, 0) \times (-L_2, L_2)$. Let us denote $D_1 = D - (2L_1, 0)$, $D_2 = D$, and $Y' = Y_t \setminus \overline{D_1 \cup D_2}$. Let $\Gamma_- = (\partial D_1 \cup \{x = -L_1\}) \cap \partial Y'$ and $\Gamma_+ = (\partial D_2 \cup \{x = L_1\}) \cap \partial Y'$. See Figure 3.1 for the configuration of Y'.

Let us denote v_j after translation by the same notation v_j. Then, by periodicity, we have

$$\mathcal{E}_j = \int_{Y'} \mathbb{C}\widehat{\nabla} v_j : \widehat{\nabla} v_j.$$

Note that, for $j = 1, 2$, the function $v_j|_{Y'} \in H^1(Y')$ is the solution to the following equation:

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_j = 0, & \text{in } Y', \\ v_j = 0, & \text{on } \Gamma_-, \\ v_j = \Psi_j, & \text{on } \Gamma_+, \\ \partial_\nu v_j = 0, & \text{on } \{y = \pm L_2\}, \end{cases} \tag{3.3}$$

The following primal-dual variational principle is used:

Lemma 3.1 (primal-dual variational principle)**.** *Let*

$$V_j = \left\{ \varphi \in H^1(Y') : \varphi|_{\Gamma_-} = 0,\, \varphi|_{\Gamma_+} = \Psi_j \right\}, \tag{3.4}$$

$$W_j = \left\{ \sigma \in L^2_s(Y' : \mathbb{R}^{2\times 2}) : \nabla \cdot \sigma = 0,\, \sigma n = 0 \textit{ if } y = \pm L_2 \right\}, \tag{3.5}$$

where $L^2_s(Y' : \mathbb{R}^{2\times 2})$ denotes the collection of the square integrable real symmetric 2×2 matrix valued functions. The following holds:

$$\mathcal{E}_j = \min_{\varphi\in V_j} \int_{Y'} \mathbb{C}\widehat{\nabla}\varphi : \widehat{\nabla}\varphi \tag{3.6}$$

$$= \max_{\sigma\in W_j} -\int_{Y'} \sigma : \mathbb{C}^{-1}\sigma + 2\int_{\Gamma_+} \sigma n\cdot \Psi_j. \tag{3.7}$$

The primal principle (3.6) is used to obtain the upper bound on $\mathcal{E}_j$, while the dual principle (3.7) is used for the lower bound. The primal-dual principle for the Laplace operator was used in [5] to prove the Keller formula for the effective conductivity. The principle for the Lamé system may be well-known. However, we were not able to find a reference, and so we include a proof here.

Proof of Lemma 3.1. By Green's identity and the boundary conditions in (3.4), we have

$$\int_{Y'} \mathbb{C}\widehat{\nabla}v_j : \widehat{\nabla}\varphi = \int_{\partial Y'} \partial_\nu v_j \cdot \varphi = \int_{\Gamma_+} \partial_\nu v_j\cdot \Psi_j \quad \text{for all } \varphi\in V_j.$$

Therefore, by the Cauchy-Schwartz inequality, we have

$$\int_{Y'} \mathbb{C}\widehat{\nabla}v_j : \widehat{\nabla}v_j = \int_{Y'} \mathbb{C}\widehat{\nabla}v_j : \widehat{\nabla}\varphi \le \frac{1}{2}\left(\int_{D^e} \mathbb{C}\widehat{\nabla}v_j : \widehat{\nabla}v_j + \int_{D^e} \mathbb{C}\widehat{\nabla}\varphi : \widehat{\nabla}\varphi\right).$$

This proves (3.6).

Note that $\mathbb{C}\widehat{\nabla}v_j \in W_j$. If $\sigma\in W_j$, by the divergence theorem and the fact that $\nabla\cdot\sigma = 0$, we have

$$\int_{Y'} \sigma : \widehat{\nabla}v_j = -\int_{Y'} (\nabla\cdot\sigma)\cdot v_j + \int_{\partial Y'} \sigma n\cdot v_j = \int_{|y|=L_2} 0\cdot v_j + \int_{\Gamma_-} \sigma n\cdot 0 + \int_{\Gamma_+} \sigma n\cdot\Psi_j = \int_{\Gamma_+} \sigma n\cdot \Psi_j. \tag{3.8}$$

By the Cauchy-Schwartz inequality, we have

$$\int_{Y'} \mathbb{C} f : g \le \frac{1}{2}\Big(\int_{Y'} \mathbb{C} f : f + \int_{Y'} \mathbb{C} g : g\Big),$$

for all $f, g\in L^2_s(Y' : \mathbb{R}^{2\times 2})$. So, by letting $f = \mathbb{C}^{-1}\sigma$ and $g = \widehat{\nabla}v_j$, we obtain from (3.8)

$$\int_{\Gamma_+} \sigma n\cdot\Psi_j = \int_{Y'} \sigma : \widehat{\nabla}v_j \le \frac{1}{2}\Big(\int_{Y'} \sigma : \mathbb{C}^{-1}\sigma + \int_{Y'} \mathbb{C}\widehat{\nabla}v_j : \widehat{\nabla}v_j\Big).$$

Now, (3.7) follows, and the proof is complete. □

4 Test functions

Here we introduce test functions to be inserted into (3.6) and (3.7). For that we first describe the geometry of two inclusions D_1 and D_2. Since D_1 and D_2 are strictly convex, there are unique points, one on ∂D_1 and the other on ∂D_2, which have the shortest distance. Denote them by $z_1 \in \partial D_1$ and $z_2 \in \partial D_2$. Let κ_0 be the common curvature of ∂D_j at z_j. Let B_j be the disk osculating to D_j at z_j ($j = 1, 2$). Then the common radius r_0 of B_j is given by $r_0 = 1/\kappa_0$. Let R_j be the reflection with respect to ∂B_j and let $p_1 \in B_1$ and $p_2 \in B_2$ be the unique fixed points of the combined reflections $R_1 \circ R_2$ and $R_2 \circ R_1$, respectively. Then one can easily see that p_1 and p_2 are written as

$$p_1 = (-a, 0) \quad \text{and} \quad p_2 = (a, 0). \tag{4.1}$$

where the constant a is given by $a := \sqrt{\epsilon(4r_0 + \epsilon)}/2$, from which one can infer

$$a = \sqrt{\frac{\epsilon}{\kappa_0}} + O(\epsilon^{3/2}). \tag{4.2}$$

Let us consider the narrow region between D_1 and D_2. There exists $L > 0$ independent of ϵ and a strictly convex function $f : [-L, L] \to \mathbb{R}$ such that $z_2 = (f(0), 0)$, $f'(0) = 0$, and ∂D_2 is the graph of $f(y)$ for $|y| < L$, *i.e.*, $(f(y), y) \in \partial D_2$. Note that ∂D_1 is the graph of $-f(y)$ for $|y| < L$ (see Figure 3.1). Since D_2 is symmetric with respect to the y-axis, we have for $|y| < L$,

$$f(y) = \frac{\epsilon}{2} + \frac{\kappa_0}{2} y^2 + O(\epsilon^2 + y^4). \tag{4.3}$$

We denote by Π_L the narrow region between D_1 and D_2, namely,

$$\Pi_L = \{(x, y) \in \mathbb{R}^2 | -f(y) < x < f(y), \, |y| < L\}. \tag{4.4}$$

For the primal problem (3.6), we use the following test function: Let

$$\varphi_j(x, y) = \frac{x + f(y)}{2f(y)} \Psi_j, \quad (x, y) \in \Pi_L. \tag{4.5}$$

Note that $\varphi_j = 0$ on $\Gamma_- \cap \partial\Pi_L$ and $\varphi_j = \Psi_j$ on $\Gamma_+ \cap \partial\Pi_L$. We then extend φ_j to Y' so that $\varphi_j|_{\Gamma_-} = 0$, $\varphi_j|_{\Gamma_+} = \Psi_j$, $\|\varphi_j\|_{H^1(Y' \setminus \Pi_L)} \le C$ for some C independent of ϵ. Then one can see that $\varphi_j \in V_j$.

The function $(x + f(y))/2f(y)$ appearing in the definition of φ_j has been used in [8] for derivation of the effective conductivity, and used in [5] for its proof. For this reason, we call φ_j the Keller-type function. Recently the function φ_j was efficiently used by Bao *et al* [1, 2] to derive the upper bound on the blow-up rate of the gradient in presence of adjacent hard elastic inclusions. The upper bound in two dimensions turns out to be $\epsilon^{-1/2}$ where ϵ is the distance between two inclusions.

We emphasize that the function φ_j is not a solution of the Lamé system and does not seem to fit to the dual principle. In fact, we can modify the function φ_j so that the modified function becomes the solution of the Lamé system, and use it for the dual principle. But it does not yield the correct lower bound. In this paper we use singular functions introduced in [7] as test functions for the dual principle. Let $\Gamma = (\Gamma_{ij})_{i,j=1}^2$ be the Kelvin matrix of fundamental solutions to the Lamé operator $\mathcal{L}_{\lambda,\mu}$, namely,

$$\Gamma_{ij}(x) = \alpha_1 \delta_{ij} \ln |\mathbf{x}| - \alpha_2 \frac{x_i x_j}{|x|^2}, \tag{4.6}$$

where

$$\alpha_1 = \frac{1}{4\pi}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{4\pi}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right). \tag{4.7}$$

Singular functions are defined using the following functions as basic building blocks:

$$\Gamma(x)e_1, \quad \Gamma(x)e_2, \quad \frac{x}{|x|^2}, \quad \frac{x^\perp}{|x|^2}, \tag{4.8}$$

where $\{e_1, e_2\}$ is the standard orthonormal basis in Cartesian coordinates and $x^\perp = (-x_2, x_1)$ if $x = (x_1, x_2)$. These functions are known as nuclei of strain [9]. Singular functions are defined as follows:

$$q_1(x) := \Gamma(x-p_1)e_1 - \Gamma(x-p_2)e_1 + \alpha_2 a \left(\frac{x-p_1}{|x-p_1|^2} + \frac{x-p_2}{|x-p_2|^2}\right), \tag{4.9}$$

and

$$q_2(x) := \Gamma(x-p_1)e_2 - \Gamma(x-p_2)e_2 - \alpha_2 a \left(\frac{(x-p_1)^\perp}{|x-p_1|^2} + \frac{(x-p_2)^\perp}{|x-p_2|^2}\right), \tag{4.10}$$

where a is the number appearing in (4.1).

It is shown in [7] that functions q_1 and q_2 capture the singular behavior of the gradient in presence of adjacent hard elastic inclusions. As a consequence it is proved that the upper bound $\epsilon^{-1/2}$ mentioned above is actually the optimal bound on the gradient blow-up. We emphasize that q_1 and q_2 are solutions to the Lamé system.

To construct test functions in W_j, let

$$m_1 := \frac{\pi(\lambda+2\mu)}{\sqrt{\kappa_0}} \quad \text{and} \quad m_2 := \frac{\pi\mu}{\sqrt{\kappa_0}}, \tag{4.11}$$

and let

$$\sigma_j^S := \frac{m_j}{\sqrt{\epsilon}} \mathbb{C}\widehat{\nabla} q_j. \tag{4.12}$$

Since q_j is a solution of the Lamé system, σ_j^S satisfies $\nabla \cdot \sigma_j^S = 0$. But the function σ_j^S does not belong to W_j because

$$\sigma_j^S n|_{y=\pm L_2} = \pm \sigma_j^S e_2 \neq 0. \tag{4.13}$$

To see this, recall that, for a displacement field u, its associated stress tensor $\sigma = \mathbb{C}\widehat{\nabla} u$ is represented by $\sigma = (\sigma_{ij})$ where

$$\begin{aligned}
\sigma_{11} &= (\lambda+2\mu)\partial_1 u_1 + \lambda \partial_2 u_2, \\
\sigma_{22} &= \lambda \partial_1 u_1 + (\lambda+2\mu)\partial_2 u_2, \\
\sigma_{12} &= \sigma_{21} = \mu(\partial_2 u_1 + \partial_1 u_2).
\end{aligned}$$

So we obtain

$$\sigma_j^S e_2 = \frac{m_j}{\sqrt{\epsilon}} \big(\mu(\partial_2 q_{j1} + \partial_1 q_{j2}),\ \lambda \partial_1 q_{j1} + (\lambda + 2\mu)\partial_2 q_{j2}\big)^T, \tag{4.14}$$

from which one can easily see that (4.13) holds.

We now modify σ_j^S by adding a function. Let

$$F_j(x,y) = -\frac{y + L_2}{2L_2}\big[\sigma_j^S(x, L_2) + \sigma_j^S(x, -L_2)\big]e_2 + \sigma_j^S(x, -L_2)e_2, \tag{4.15}$$

$$G_j(x) = \frac{1}{2L_2}\int_0^x \big[\sigma_j^S(x', L_2) + \sigma_j^S(x', -L_2)\big]e_2\, dx'. \tag{4.16}$$

Let σ_j^c be the 2×2 matrix-valued function having G_j and F_j as its columns, namely,

$$\sigma_j^c(x,y) = \begin{bmatrix} G_j(x) & F_j(x,y) \end{bmatrix}, \quad (x,y) \in Y'.$$

Then, one can check that

$$\nabla \cdot \sigma_j^c = 0, \quad \sigma_j^c n|_{y=\pm L_2} = -\sigma_j^S n|_{y=\pm L_2}. \tag{4.17}$$

In fact, since

$$\partial_1(\sigma_j^c e_1) = \partial_1 G_j(x,y) = \frac{1}{2L_2}\big[\sigma_j^S(x, L_2) + \sigma_j^S(x, -L_2)\big]e_2,$$

$$\partial_2(\sigma_j^c e_2) = \partial_2 F_j(x,y) = -\frac{1}{2L_2}\big[\sigma_j^S(x, L_2) + \sigma_j^S(x, -L_2)\big]e_2,$$

we have

$$\nabla \cdot \sigma_j^c = \partial_1(\sigma_j^c e_1) + \partial_2(\sigma_j^c e_2) = 0.$$

If $y = \pm L_2$, we see from (4.15) that

$$\sigma_j^c n = \sigma_j^c e_2 = F_j(x, \pm L_2) = \mp \sigma_j^S(x, \pm L_2)e_2 = -\sigma_j^S n.$$

Let

$$\sigma_j := \sigma_j^S + \sigma_j^c. \tag{4.18}$$

Then, from (4.17) and the fact that $\nabla \cdot \sigma_j^S = 0$, we easily see that $\sigma_j \in W_j$.

The following evaluations of integrals are obtained in [7] (Lemma 4.7 and 4.8):

$$\int_{\partial D_i} \partial_\nu q_j \cdot \Psi_k = (-1)^i \delta_{jk}, \quad i,j,k = 1,2, \tag{4.19}$$

$$\int_{\partial D_1 \cup \partial D_2} \partial_\nu q_j \cdot q_j = m_j^{-1}\sqrt{\epsilon} + O(\epsilon), \quad j = 1,2. \tag{4.20}$$

We emphasize the signs in the above are opposite to those in Lemma 4.7 and 4.8 in [7] since the normal vector n in $\partial_\nu q_j = (\mathbb{C}\widehat{\nabla} q_j)n$ in this paper directed outward to $\partial Y'$ (and hence inward to ∂D_i). We also invoke an estimate from [7] (Lemma 3.4):

$$|q_j(x)| + |\nabla q_j(x)| \lesssim \sqrt{\epsilon} \quad \text{for all } x \in \mathbb{R}^2 \setminus (D_1 \cup D_2 \cup \Pi_L), \quad j = 1,2. \tag{4.21}$$

Here and afterwards, the expression $A \lesssim B$ implies that there is a constant C independent of ϵ such that $A \le CB$.

One can see from (4.2), (4.14) and (4.21) that $|\sigma_j^S n|_{y=\pm L_2}| \lesssim 1$. So we have

$$|\sigma_j^c(x,y)| \lesssim 1, \quad \text{for } (x,y) \in Y'. \tag{4.22}$$

5 Proof of Theorem 2.1

We first derive an upper bound using the primal principle (3.6).

Lemma 5.1 (Upper bound)**.** *We have*

$$\mathcal{E}_1 \le (\lambda + 2\mu)\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\epsilon}} + O(1) \tag{5.1}$$

and

$$\mathcal{E}_2 \le \mu\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\epsilon}} + O(1). \tag{5.2}$$

Proof. To prove (5.1), we use (3.6) with $\varphi = \varphi_1$, where φ_1 is the function defined with (4.5). Note that since $\|\varphi_j\|_{H^1(Y'\setminus\Pi_L)} \le C$ for some C independent of ϵ, it suffices to estimate $\int_{\Pi_L} \mathbb{C}\widehat{\nabla}\varphi_1 : \widehat{\nabla}\varphi_1$.

Let ∂_1 and ∂_2 denote partial derivatives with respect to x and y variables, respectively. Straightforward computations yield

$$\begin{aligned}
\mathbb{C}\widehat{\nabla}\varphi_1 : \widehat{\nabla}\varphi_1 &= \begin{bmatrix} (\lambda+2\mu)\frac{1}{2f(y)} & \mu\partial_2\frac{x}{2f(y)} \\ \mu\partial_2\frac{x}{2f(y)} & 0 \end{bmatrix} : \begin{bmatrix} \frac{1}{2f(y)} & \partial_2\frac{x}{4f(y)} \\ \partial_2\frac{x}{4f(y)} & 0 \end{bmatrix} \\
&= \frac{\lambda+2\mu}{4}\frac{1}{f(y)^2} + \frac{\mu}{4}\frac{x^2 f'(y)^2}{f(y)^4}.
\end{aligned} \tag{5.3}$$

So we have

$$\int_{\Pi_L} \mathbb{C}\widehat{\nabla}\varphi_1 : \widehat{\nabla}\varphi_1 = \frac{\lambda+2\mu}{4}\int_{\Pi_L}\frac{1}{f(y)^2} + \frac{\mu}{2}\int_{\Pi_L}\frac{x^2 f'(y)^2}{f(y)^4} =: I + II. \tag{5.4}$$

Thanks to the Taylor expansion (4.3) of f, we have

$$\begin{aligned}
\frac{4}{\lambda+2\mu} I &= \int_{-L}^{L}\int_{-f(y)}^{f(y)} \frac{1}{f(y)^2}dxdy = \int_{-L}^{L}\frac{2}{f(y)}dy \\
&= \int_{-L}^{L}\frac{2}{(\epsilon+\kappa_0 y^2)/2}dy + \int_{-L}^{L}\frac{2}{f(y)} - \frac{2}{(\epsilon+\kappa_0 y^2)/2}dy \\
&= \int_{-\infty}^{\infty}\frac{2}{(\epsilon+\kappa_0 y^2)/2}dy + O(1) + \int_{-L}^{L}\frac{O(y^4)}{f(y)(\epsilon+\kappa_0 y^2)/2}dy \\
&= \frac{4\pi}{\sqrt{\kappa_0}\sqrt{\epsilon}} + O(1).
\end{aligned}$$

We also have

$$|II| \lesssim \int_{\Pi_L}\frac{(\epsilon+y^2)^2 y^2}{(\epsilon+y^2)^4} \lesssim \int_{-L}^{L}\frac{y^2}{\epsilon+y^2}dy \lesssim 1. \tag{5.5}$$

Therefore we obtain

$$\int_{\Pi_L} \mathbb{C}\widehat{\nabla}\varphi_1 : \widehat{\nabla}\varphi_1 = \frac{\pi(\lambda+2\mu)}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\epsilon}} + O(1).$$

So, (5.1) follows from the primal principle (3.6).

One can prove (5.2) in the exactly same manner using φ_2 defined in (4.5). □

We now derive a lower bound using the dual variational princicple (3.7).

Lemma 5.2 (Lower bound)**.** *We have*

$$\mathcal{E}_1 \geq (\lambda + 2\mu)\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\epsilon}} + O(1) \tag{5.6}$$

and

$$\mathcal{E}_2 \geq \mu\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\epsilon}} + O(1). \tag{5.7}$$

Proof. Let σ_j be the function in W_j defied by (4.18). Then we have

$$\begin{aligned}
&-\int_{Y'} \sigma_j : \mathbb{C}^{-1}\sigma_j + 2\int_{\Gamma_+} \sigma_j n \cdot \Psi_j \\
&= \left[-\int_{Y'} \sigma_j^S : \mathbb{C}^{-1}\sigma_j^S + 2\int_{\Gamma_+} \sigma_j^S n \cdot \Psi_j \right] + \left[-\int_{Y'} \sigma_j^c : \mathbb{C}^{-1}\sigma_j^c + 2\int_{\Gamma_+} \sigma_j^c n \cdot \Psi_j \right] \\
&=: I_j + II_j.
\end{aligned}$$

From (4.22), it is clear that $|II_j| \lesssim 1$.

Now we estimate I_j. From the definition (4.12) of σ_j^S, we have

$$I_j = -\frac{m_j^2}{\epsilon}\int_{Y'} \mathbb{C}\widehat{\nabla} q_j : \widehat{\nabla} q_j + \frac{2m_j}{\sqrt{\epsilon}}\int_{\Gamma_+} \partial_\nu q_j \cdot \Psi_j. \tag{5.8}$$

Since q_j is a solution of the Lamé system, we obtain by the divergence theorem that

$$\int_{Y'} \mathbb{C}\widehat{\nabla} q_j : \widehat{\nabla} q_j = \int_{\partial Y'} \partial_\nu q_j \cdot q_j.$$

Let $A := (\partial D_1 \cup \partial D_2) \setminus \partial Y'$ and $B := \partial Y' \setminus (\partial D_1 \cup \partial D_2)$ so that $\partial Y' = (\partial D_1 \cup \partial D_2) \setminus A \cup B$, and

$$\int_{\partial Y'} \partial_\nu q_j \cdot q_j = \int_{\partial D_1 \cup \partial D_2} - \int_A + \int_B \partial_\nu q_j \cdot q_j.$$

Since A and B are away from Π_L, we infer from (4.21) that

$$\left| -\int_A + \int_B \partial_\nu q_j \cdot q_j \right| \lesssim \epsilon.$$

It then follows from (4.20) that

$$\int_{Y'} \mathbb{C}\widehat{\nabla} q_j : \widehat{\nabla} q_j = m_j^{-1}\sqrt{\epsilon} + O(\epsilon). \tag{5.9}$$

Similarly, we write $\Gamma_+ = (\partial D_2 \cup (\Gamma_+ \setminus \partial D_2)) \setminus (\partial D_2 \setminus \Gamma_+)$. Since $\Gamma_+ \setminus \partial D_2$ and $\partial D_2 \setminus \Gamma_+$ are away from Π_L, we infer that

$$\int_{\Gamma_+} \partial_\nu q_j \cdot \Psi_j = \int_{\partial D_2} \partial_\nu q_j \cdot \Psi_j + O(\sqrt{\epsilon}).$$

So, (4.19) yields

$$\int_{\Gamma_+} \partial_\nu q_j \cdot \Psi_j = 1 + O(\sqrt{\epsilon}). \tag{5.10}$$

It then follows from (5.8), (5.9) and (5.10) that

$$I_j = m_j\sqrt{\epsilon} + O(1).$$

Now (5.6) and (5.7) follow from (4.11). This completes the proof. □

Lemma 5.1 and 5.2 certainly lead us to Theorem 2.1.

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