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Multilevel quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients*

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Abstract

We analyze the convergence rate of a multilevel quasi-Monte Carlo (MLQMC) Finite Element Method for a scalar diffusion equation with log-Gaussian, isotropic coefficients in a bounded, polytopal domain $D \subset \mathbb{R}^d$. The multilevel algorithm Q_L^* which we analyze here was first proposed, in the case of parametric PDEs with sequences of uniformly, i.i.d distributed parameters in [Frances Y. Kuo, Christoph Schwab, and Ian H. Sloan: Multilevel quasi-Monte Carlo finite element methods for a class of elliptic PDEs with random coefficients, *Found. Comput. Math.* **15** (2015) pp. 411–449]. The random coefficient is assumed to admit a representation with *locally supported coefficient functions, as arise for example in spline- or multiresolution representations of the input random field*. The present analysis builds on and generalizes our single-level analysis in [Lukas Herrmann and Christoph Schwab: QMC integration for lognormal-parametric, elliptic PDEs: local supports and product weights, Report 2016-39, Seminar for Applied Mathematics, ETH Zürich]. It also extends the MLQMC error analysis in [Frances Y. Kuo, Robert Scheichl, Christoph Schwab, Ian H. Sloan, and Elisabeth Ullmann: Multilevel quasi-Monte Carlo methods for lognormal diffusion problems, *Math. Comp.* **86** (2017) pp. 2827–2860], to locally supported basis functions in the representation of the Gaussian random field (GRF) in D , and to product weights in QMC integration. In particular, in polytopal domains $D \subset \mathbb{R}^d$, $d = 2, 3$, our analysis is based on weighted function spaces to describe solution regularity with respect to the spatial coordinates. These spaces allow GRFs and PDE solutions whose realizations become singular at edges and vertices of D . This is natural for *non-stationary* GRFs whose covariance operators and associated precision operator are fractional powers of elliptic differential operators in D with boundary conditions on ∂D . In weighted Sobolev spaces in D , first order, Lagrangean Finite Elements on regular, simplicial triangulations of D with suitable mesh refinement yield optimal asymptotic convergence rates. Our analysis yields also bounds for the ε -complexity of the MLQMC algorithm, uniformly with respect to the dimension of the parameter space.

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1 Introduction

The numerical analysis of solution methods for partial differential equations (PDEs) and more general operator equations with random input data has received increasing attention in recent years, in particular with the development of computational uncertainty quantification and computational science and engineering. There, particular models of randomness in the PDEs' input parameters entail particular requirements to efficient computational uncertainty quantification algorithms. A basic case arises when there are a finite (possibly large) number s of random variables whose densities have bounded support and which parametrize uncertain input from function spaces, such as diffusion coefficients or source terms in the forward PDE model: computation of statistical moments of “responses” being (functionals of) solution families of these PDEs. Numerical Bayesian inversion then amounts to numerical integration over a parameter domain of finite parameter space dimension s , which itself is a discretization parameter. Statistical independence and scaling reduces this task to numerical integration over the unit cube $[0, 1]^s$, against a product probability measure. In the context of PDEs, so-called *distributed random inputs* such as spatially heterogeneous diffusion coefficients, uncertain physical domains, etc. imply, via *uncertainty parametrizations* (such as Fourier-, Karhunen-Loève, B-spline or wavelet expansions) in physical domains D , a countably-infinite number of random parameters (being, for example, Fourier- or wavelet coefficients). This, in turn, renders the problem of numerical estimation of response statistics of PDE solutions a problem of infinite-dimensional numerical integration. Assuming statistical independence of the system of (countably many) random input parameters results in the problem of numerical integration against a product probability measure. The case of the uncertain PDE input being a Gaussian random field (GRF) is particular important in applications, and the numerical analysis has received considerable attention in recent years. Here, the numerical estimation of statistical moments of PDE solutions amounts to integrating parametric PDE solutions against Gaussian measures on function spaces of admissible input data. Adopting uncertainty parametrizations of the input GRFs renders the domain Ω of integration a countable product of real lines $\mathbb{R}^{\mathbb{N}}$, endowed with the Gaussian product measure (GM) μ and with the product sigma algebra obtained by completing the finite dimensional cylinders of Borel sets on \mathbb{R} (we refer to [10] for details on GMs on $\mathbb{R}^{\mathbb{N}}$).

Here, as in [24, 33] and the references there, we analyze the combined discretization by quasi-Monte Carlo (QMC) quadratures and the Finite Element (FE) solution of linear, second order elliptic PDEs in a bounded, polytopal domain $D \subset \mathbb{R}^d$, $d = 2, 3$. Unlike the analysis in [24, 33] and the references there, and in [27], where stationarity enters the algorithms and the error analysis in an essential way, here we consider isotropic (i.e. scalar), log-Gaussian diffusion coefficient $a = \exp(Z)$, where Z is a *possibly non-stationary* GRF in D .

We place the present work in perspective with other recent work on the numerical analysis of PDEs with GRF inputs. In [24, 33], an error analysis of single- and multilevel algorithms was developed for Karhunen-Loève type representations of the GRF Z . Except for rather special settings where Karhunen-Loève eigenfunctions are explicitly known (when D is a torus or a sphere [29]), Karhunen-Loève type representations of GRFs are not explicitly available but must be computed numerically. This entails the accurate numerical approximation of a large number of eigenpairs of the covariance operator of the GRF Z in the domain D , a significant computational overhead.

Moreover, the covariance eigenfunctions in Karhunen-Loève representations of GRFs in domains or manifolds D typically have *global support* in D . This was shown in [24, 33] to imply in the error analysis of QMC quadrature rules so-called *product-and-order dependent* (POD) weights. Constructing QMC points with these weights introduces, via the corresponding fast component-by-component (CBC) algorithm, a *quadratic scaling* w.r. to the QMC integration dimension s of the construction cost for QMC rules, see [40, 14] and the references there. For this

reason, the QMC construction cost is not explicitly accounted for in recent complexity estimates of QMC-FE algorithms for PDEs.

To bypass the need for numerical Karhunen-Loève eigenfunction computation, under (strong) assumptions on stationarity of the GRF Z , fast, FFT based numerical methods have been proposed for efficient numerical realization of GRFs. We refer to [15] and the references there for details. FFT based techniques have recently been used in conjunction with QMC and FE for elliptic PDEs with coefficients given by (exponentials of) stationary GRFs Z in [27]. While allowing linear scaling w.r. to the number of FE degrees of freedom in the domain D , *stationarity of the GRF Z* is a key condition for the applicability of FFT-based, so-called “*circulant embedding*” methods. Being essentially Fourier-based techniques, the QMC-FE error analysis in [27] involves QMC weight sequences with POD structure and, hence, quadratic w.r. to QMC integration dimension s scaling of the cost for QMC rule construction via the fast CBC construction (see [27, Equation (3.16), Remark 9]).

In recent years, computational modelling of noisy spatial data has increasingly employed *non-stationary GRFs* in bounded domains D . We mention only recently used random models in so-called “spatial statistics” (see [37, 17] and the references there), GRFs on manifolds such as the sphere (see [29] and the references there), and *deep Gaussian processes* (see [16] and the references there). As proposed in [37], rather general non-stationary GRFs Z in bounded domains or on manifolds D can be modelled and sampled as solutions of *stochastic (integro) PDE* (SPDE). A widely used equation which generalizes the classical Whittle–Matérn [46, 38] covariances of stationary Gaussian random fields reads as

$$(-\operatorname{div}(A(x)\nabla) + \kappa^2(x))^{\alpha/2}Z = \mathcal{W} \quad \text{in } D. \quad (1)$$

Here \mathcal{W} denotes spatial white noise on D , and $\alpha > 0$ is suitably chosen. If $D = \mathbb{R}^d$, $A(x) \equiv \operatorname{Id}$, and $\kappa(x) \equiv \operatorname{const}$, the solution Z to (1) (we assume Z to be centered throughout this paper) is stationary with so-called *Matérn-type covariance*, cp. [45, 46]. For a variable coefficient matrix $A(x)$ and variable $\kappa(x)$, or in bounded domains D with homogeneous Dirichlet or Neumann boundary conditions, equation (1) results in non-stationary, “Matérn-like” Gaussian random fields. On bounded domains D , boundary conditions for Z are mandatory for the unique solvability of the SPDE (1). Boundary conditions on ∂D generally entails nonstationarity of the GRF Z , cp. [37, Section A.4]. Then, FFT-based methods are generally not available, and computation of Karhunen-Loève eigenbases for (1) will entail, again, prohibitive cost. Alternative, *covariance independent* representations of GRFs via multiresolution systems in D allow to circumvent the numerical solution of Karhunen-Loève eigenproblems, the classical example being the Brownian bridge in $D = (0, 1)$, going back to P. Lévy and Ciesielski. The basis functions in corresponding representation systems are well-localized in D (either compactly supported or exponentially decaying) and allow for fast evaluation of the GRF in D , similar to FFTs. While retaining linear scaling w.r. to the spatial resolution of the approximate GRF in D , the hierarchical nature of MRA’s naturally enables *multilevel QMC algorithms* with a discretization level dependent resolution of GRF and QMC integration. In addition, as observed by us recently in [19, 30], the localization of the supports of the representation system in D allows to use QMC quadrature with *product weights*. This, in turn, is known to afford *linear scaling* of the work with respect to the parameter dimension s to compute the QMC generating vectors (see [41, 14] and the references there). To provide a complete error vs. work analysis of a multilevel QMC FE algorithm for the numerical solution of a linear, second order elliptic PDE with GRF input and locally supported basis functions in a bounded, polytopal domain $D \subset \mathbb{R}^d$, $d = 2, 3$, where the GRF satisfies (1) with suitable boundary conditions on ∂D is the purpose of the present paper. Recently, and independent of the present work, in [32] a combined QMC and wavelet-based discretization of log-Gaussian random fields was proposed and error bounds were proposed. The analysis in the present work is considerably more general, in several respects, as

the results in that work. We consider, in particular, multilevel QMC-FE discretizations, and use sharper bounds on the error caused by truncating the expansion of the GRF, from our single-level analysis in [30]. We also generalize, based on [30], the QMC error analysis by admitting Gaussian type weight functions in the anisotropic QMC norms, as opposed to the exponential weights used in [24, 33]. This extends the summability range of Karhunen-Loève and wavelet expansions of the admissible GRFs for the applicability of QMC with product weights, in addition to obviating stationarity, as compared to [24, 33]. Furthermore, in the present paper, we provide a full regularity analysis of the PDE as required for the MLQMC-FE. As is well-known, this requires a form of “mixed regularity” analysis, with possibly sharp, quantitative bounds of the sensitivities of the parametric integrand functions with respect to the coordinates in the GRF Z , in weighted $H^2(D)$ norms. In the present paper, we also develop these norm bounds.

We confine QMC integration error analysis to first order, randomly shifted lattice rules proposed originally in [40], and to continuous, piecewise linear “Courant” FE methods in D . We adopt the setting of our analysis [30] of the single-level QMC-FE algorithm: in a bounded, polytopal domain $D \subset \mathbb{R}^d$, $d = 2, 3$ we consider a model Dirichlet problem

$$-\nabla \cdot (a \nabla u) = f, \quad u \Big|_{\partial D} = 0. \quad (2)$$

As in [30], we assume that the GRF $Z = \log(a) : \Omega \rightarrow L^\infty(D)$ is (formally) represented as

$$Z := \sum_{j \geq 1} y_j \psi_j, \quad (3)$$

where $(\psi_j)_{j \geq 1}$ is a sequence of real-valued, bounded, and measurable functions in D . In particular, with respect to the GM μ the sequence $\mathbf{y} = (y_j)_{j \geq 1}$ has independent and identically distributed (i.i.d.) components and for every $j \geq 1$, y_j is standard normally distributed. That is to say, $y_j \sim \mathcal{N}(0, 1)$, i.i.d. for $j \in \mathbb{N}$. The lognormal coefficient a in (2) is given by

$$a := \exp(Z). \quad (4)$$

The series (3) converges in $L^q(\Omega; L^\infty(D))$, $q \in [1, \infty)$, under the assumption that there exists a positive sequence $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (0, \infty)$ such that

$$K := \left\| \sum_{j \geq 1} \frac{|\psi_j|}{b_j} \right\|_{L^\infty(D)} < \infty. \quad (\mathbf{A1})$$

In the setting of (3) and (4), the expectation with respect to the GM μ of the solution to (2) can be computed with QMC by randomly shifted lattice rules and product weights with dimension-independent convergence rates under the assumption **(A1)** with $p < 2$, cp. [30]. The assumption in **(A1)** can account for locality in the support of the functions ψ_j . An assumption of the type of **(A1)** in the case of so called affine-parametric coefficients in conjunction with the application of QMC with product weights was already discussed by us in [19]. In the present work, we extend the analysis of [30] to a multilevel QMC algorithm with log-Gaussian inputs to reduce the overall work. The perspective of multilevel QMC integration with product weights for random inputs ψ_j with localized supports was originally introduced in [18] for the case of so-called affine-parametric coefficients. Multilevel QMC for elliptic PDEs with affine coefficients was first introduced in [34] (there for globally supported Karhunen-Loève eigenfunctions and with POD weights). As we showed there, localization of supports allows to obtain in certain cases estimates for the work of the evaluation of the multilevel QMC quadrature, which are asymptotically equal to the work to solve one instance of the corresponding deterministic PDE with the same error tolerance also in the case that the FE convergence rate is higher than

$1/d$ with respect to the FE degrees of freedom. The FE convergence rate of first order FE is higher than $1/d$ if, for example, the spatial error is considered in a weaker Sobolev norm. In the present paper, the cost of generating the QMC points using the fast CBC construction of [41, 42] is included into the overall complexity estimate. This is due to product weights affording construction cost of QMC rules which is linear scaling in terms of the dimension of the integration domain.

The outline of this paper is as follows. In Section 2, we recapitulate known results on the well-posedness of problem (2) - (4) under assumption **(A1)**, and on the integrability of random solution with respect to the GM. We also present bounds on the error incurred in the random solution when the expansion (3) is truncated to a finite number s of terms. As we combine QMC quadrature approximation of the GM with continuous, piecewise linear FE discretization of (2) of the random solution in polytopal domains $D \subset \mathbb{R}^d$, $d = 2, 3$, we also review in Section 2 elements of elliptic regularity theory and FE approximation theory in D ; notably, handling corner and (in space dimension $d = 3$) edge singularities induced by D we review weighted Sobolev spaces in D in which (2) admits a full regularity shift. Corresponding weighted spaces also appear in our convergence rate analysis of the expansion (3) of the GRF. In Section 3, we review QMC convergence theory from [40, 30]. Suitable (weighted) spaces on \mathbb{R}^s of integrand functions with mixed first derivatives which ensure (nearly) first order convergence with dimension-independent constants are introduced. Section 4 presents the key mathematical results: parametric regularity analysis for the integrand functions which arise from the dimensionally truncated, FE discretized problem, generalizing the single level QMC analysis in [30] by admitting locally supported functions ψ_j in the representation (3) of the GRF; while similar in spirit to the multilevel analysis in [24], there are significant technical differences due to accounting for local supports of ψ_j , analogous to the recent gpc N -term approximation rate analysis in [7]. The error bounds are then combined in Section 5 to a novel, MLQMC convergence rate bound in terms of the (sequences of) truncation dimensions $(s_\ell)_{\ell=0,\dots,L}$, numbers $(M_\ell)_{\ell \geq 0}$ of FE degrees of freedom and of QMC sample numbers $(N_\ell)_{\ell=0,\dots,L}$, where L denotes the number of discretization levels. Judicious choices of these parameters for concrete MLQMC-FE algorithms are derived in Section 6 by the “usual” error vs. work analysis through optimization, of the error bounds in Section 5, derived analogously to [34, 24]. Numerical experiments of this multilevel QMC algorithm for non-stationary GRF input represented by a multiresolution function system are presented in the univariate case in Section 7.

2 Well-posedness and spatial approximation

2.1 Well-posedness

We consider the variational formulation of the PDE (2) with lognormal coefficient $a = \exp(Z)$, i.e., to find $u : \Omega \rightarrow V$ such that

$$\int_D a \nabla u \cdot \nabla v dx = f(v), \quad v \in V, \quad (5)$$

where $V := H_0^1(D)$ with dual space V^* . We also identify $L^2(D)$ with its dual space $L^2(D)^*$, i.e., $V \subset L^2(D) \simeq L^2(D)^* \subset V^*$. Under the assumption that for some $p_0 \in (0, \infty)$, $(b_j)_{j \geq 1} \in \ell^{p_0}(\mathbb{N})$ it holds that $Z \in L^q(\Omega; L^\infty(D))$ for every $q \in [1, \infty)$, cp. [30, Theorem 2]. Hence, $0 < \text{ess inf}_{x \in D} \{a(x)\} \leq \|a\|_{L^\infty(D)} < \infty$, μ -a.s. . As in previous works [30, 33, 24], in the ensuing error analysis, the quantities

$$a_{\min} := \text{ess inf}_{x \in D} \{a(x)\} \quad \text{and} \quad a_{\max} := \|a\|_{L^\infty(D)}$$

will play an important role. Under Assumption **(A1)**, a_{\min} and a_{\max} are random variables on the probability space $(\Omega, \otimes_{j \geq 1} \mathcal{B}(\mathbb{R}), \mu)$ (see, for example, [10, Example 2.3.5]). Therefore, continuity and coercivity of the random bilinear form $(w, v) \mapsto \int_D a \nabla w \cdot \nabla v dx$ in (5) on $V \times V$ holds with coercivity constant a_{\min} and continuity constant a_{\max} , μ -a.s. By the Lax–Milgram Lemma, a unique solution u to (5) exists μ -a.s. By [30, Proposition 3] (see also [12]), for every $q \in [1, \infty)$,

$$\|u\|_{L^q(\Omega; V)} \leq \|1/a_{\min}\|_{L^q(\Omega)} \|f\|_{V^*} < \infty,$$

where the strong measurability of $u : \Omega \rightarrow V$ follows, since the V -valued solution u depends continuously on the $L^\infty(D)$ -valued coefficient a (by the second Strang lemma).

Numerical approximation of (functionals of) the random solution by QMC quadratures requires a finite dimensional domain of integration. To this end, the expansion of the GRF Z in (3) is truncated to a finite number s of terms: the *s-term truncated lognormal random field* a^s is defined by $a^s := \exp(Z^s) = \exp(\sum_{j=1}^s y_j \psi_j)$, for every $s \in \mathbb{N}$. With a^s , we associate the random variables

$$a_{\min}^s := \operatorname{ess\,inf}_{x \in D} \{a^s(x)\} \quad \text{and} \quad a_{\max}^s := \|a^s\|_{L^\infty(D)}.$$

By u^s we denote the solution of the variational formulation (5) with the s -term truncated, parametric coefficient a^s in place of a , i.e.,

$$u^s : \Omega \rightarrow V \text{ s.t. } \int_D a^s \nabla u^s \cdot \nabla v dx = f(v), \quad v \in V. \quad (6)$$

The *truncation error* can be controlled if the sequence $(b_j)_{j \geq 1}$ is p -summable. Specifically, if $(b_j)_{j \geq 1} \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0, \infty)$, [30, Proposition 7] implies that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for every $G(\cdot) \in V^*$ and for every $s \in \mathbb{N}$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s))| \leq C_\varepsilon \|G(\cdot)\|_{V^*} \|f\|_{V^*} \begin{cases} \sup_{j > s} \{b_j^{1-\varepsilon}\} & \text{if } p_0 > 2, \\ \sup_{j > s} \{b_j^{2-p_0/2}\} & \text{if } p_0 \leq 2. \end{cases} \quad (7)$$

2.2 Elliptic Regularity in D

Approximations of second order, elliptic PDEs with regular, simplicial Finite Elements in a polytope $D \subset \mathbb{R}^d$, $d = 2, 3$, on regular, simplicial families of uniformly refined triangulations may produce suboptimal convergence rates, due to the occurrence of singularities in the parametric solutions u and u^s at vertices and, in space dimension $d = 3$, also at edges. In such domains, linear elliptic PDEs admit regularity shifts in certain weighted Sobolev spaces, cp. [5, 39] which we now recapitulate as we require the precise definition of the weighted norms in D in the ensuing QMC error analysis. We assume the polyhedron resp. polygon D to have straight edges and plane faces and J corners $\mathcal{C} := \{c_1, \dots, c_J\} \subset \partial D$.

For $d = 2$, let $\beta = (\beta_1, \dots, \beta_J)$ be a J -tuple of weight exponents, we define the *corner weight function*

$$\Phi_\beta(x) := \prod_{i=1}^J |x - c_i|^{\beta_i}, \quad x \in D,$$

where $\beta_i \in [0, 1)$, $i = 1, \dots, J$. Here and in the following, the Euclidean norm in \mathbb{R}^d is denoted by $|\cdot|$. The weighted function spaces $L_\beta^q(D)$ and $H_\beta^2(D)$ are defined as closures of $C^\infty(\overline{D})$ with respect to the norms

$$\|v\|_{L_\beta^q(D)} := \|v \Phi_\beta\|_{L^q(D)}, \quad q \in [1, \infty],$$

and

$$\|v\|_{H_\beta^2(D)}^2 := \|v\|_{H^1(D)}^2 + \sum_{|\alpha|=2} \|\partial_x^\alpha v \Phi_\beta\|_{L^2(D)}^2.$$

For $d = 3$, let the polyhedron D have J' straight edges $\mathcal{E} := \{e_1, \dots, e_{J'}\} \subset \partial D$ and define $\mathcal{X}_j := \{k : c_j \in \bar{e}_k\}$ as the index set of edges that meet at corner c_j , $j = 1, \dots, J$. Let r_k denote the distance to the edge e_k and let ρ_j denote the distance to the corner c_j . Let $(V_j : j = 1, \dots, J)$ be a finite, open covering of D such that

$$\bar{D} \subset \bigcup_{j=1}^J V_j, \quad c_i \notin \bar{V}_j, \text{ if } i \neq j, \quad \text{and} \quad \bar{V}_j \cap \bar{e}_k = \emptyset \text{ if } k \notin \mathcal{X}_j.$$

For a real-valued J -tuple $\beta \in [0, 1)^J$ and a real-valued J' -tuple $\delta \in [0, 1)^{J'}$, define the corner-edge weight function

$$\Phi_{(\beta, \delta)}(x) := \sum_{j=1}^J \rho_j^{\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{\delta_k} \mathbb{1}_{V_j}(x), \quad x \in D. \quad (8)$$

With this weight, we associate the weighted Sobolev spaces $L_{\beta, \delta}^2(D)$ and $H_{\beta, \delta}^2(D)$, cp. [39, Section 4.1.2] as closures of $C_0^\infty(\bar{D} \setminus (\mathcal{C} \cup \mathcal{E}))$ with respect to the norms

$$\|v\|_{L_{(\beta, \delta)}^2(D)} := \|v\Phi_{\beta, \delta}\|_{L^2(D)}$$

and for $\iota = 0, 1, 2$,

$$\|v\|_{H_{(\beta, \delta)}^\iota(D)} := \left(\sum_{j=1}^J \sum_{|\alpha| \leq \iota} \int_{D \cap V_j} \rho_j^{2(\beta_j - \iota + |\alpha|)}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2(\delta_j - \iota + |\alpha|)} |\partial^\alpha v|^2 dx \right)^{1/2}.$$

We note that the spaces $L_{\beta, \delta}^2(D)$ and $H_{\beta, \delta}^0(D)$ are isomorphic with equivalent norms: for every $x \in D$,

$$\sum_{j=1}^J \rho_j^{2\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2\delta_k} \mathbb{1}_{V_j}(x) \leq (\Phi_{(\beta, \delta)}(x))^2 \leq J \sum_{j=1}^J \rho_j^{2\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2\delta_k} \mathbb{1}_{V_j}(x),$$

Also, we define the weighted seminorm

$$|v|_{H_{(\beta, \delta)}^2(D)} := \left(\sum_{j=1}^J \sum_{|\alpha|=2} \int_{D \cap V_j} \rho_j^{2\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2\delta_j} |\partial^\alpha v|^2 dx \right)^{1/2}.$$

Lemma 2.1 *For a polygon D (i.e. in spatial dimension $d = 2$), there exists a constant $C > 0$ such that for every $f \in L_{\beta}^2(D)$,*

$$\|f\|_{V^*} \leq C \|f\|_{L_{\beta}^2(D)}.$$

For a polyhedron D (i.e. in spatial dimension $d = 3$), there exists a constant $C > 0$ such that for every $f \in L_{(\beta, \delta)}^2(D)$,

$$\|f\|_{V^*} \leq C \|f\|_{L_{(\beta, \delta)}^2(D)}.$$

Proof. The case $d = 2$ is proven in [31, Lemma 1]. The case $d = 3$ follows by [39, Lemma 4.1.4]. Specifically, in the notation of [39] the assertion of this lemma reads that the embedding $V_{\beta, \delta}^{0,2}(D) \subset V_{\mathbf{0}, \mathbf{0}}^{-1,2}(D)$ is continuous, if $\beta_j < 1$ and $\delta_k < 1$, $j = 1, \dots, J$, $k = 1, \dots, J'$. We note that here the space $V_{\beta, \delta}^{0,2}(D)$ of [39] coincides with our spaces $H_{(\beta, \delta)}^0(D) = L_{(\beta, \delta)}^2(D)$ and the space $V_{\mathbf{0}, \mathbf{0}}^{-1,2}(D)$ is isomorphic to V^* . In the definition of the weighted space $L_{(\beta, \delta)}^2(D) = H_{(\beta, \delta)}^0(D)$, it has been assumed that $\beta_j < 1$ and $\delta_k < 1$, $j = 1, \dots, J$, $k = 1, \dots, J'$. \square

In polygons D in space dimension $d = 2$ and for functions in $H_{\beta}^2(D)$, a full regularity shift for the Laplacian is available, cp. for example [5, Theorem 3.2]: there exists a constant $C > 0$ such that for every $w \in V$ with $\Delta w \in L_{\beta}^2(D)$,

$$\|w\|_{H_{\beta}^2(D)} \leq C \|\Delta w\|_{L_{\beta}^2(D)}, \quad (9)$$

where we assume that the weight exponent sequence β satisfies $\max\{0, 1 - \pi/\omega_i\} < \beta_i < 1$, $i = 1, \dots, J$. Here, ω_i denotes the interior angle of the polygon D at corner c_i , $i = 1, \dots, J$. Since [5] considers the Poisson boundary value problem with a zero order term, i.e., $-\Delta u + u = f$, we note that Lemma 2.1 implies that there exists a constant C such that for every $w \in V \cap H_{\beta}^2(D)$, $\|w\|_{L_{\beta}^2(D)} \leq C \|\Delta w\|_{L_{\beta}^2(D)}$.

In space dimension $d = 3$, when D is a polyhedral domain with plane sides and for functions in $H_{(\beta, \delta)}^2(D) \cap V$, there holds a corresponding regularity shift of the Dirichlet Laplacian by [39, Lemma 4.3.1] and by the inverse mapping theorem, cp. [13, Theorem 5.6-2]: there exists a constant $C > 0$ such that for every $w \in H_{(\beta, \delta)}^2(D) \cap V$ holds

$$\|w\|_{H_{(\beta, \delta)}^2(D)} \leq C \|\Delta w\|_{L_{(\beta, \delta)}^2(D)}, \quad (10)$$

where we assume that

$$\frac{1}{2} - \lambda_j < \beta_j < 1, \quad j = 1, \dots, J, \quad \text{and} \quad 1 - \frac{\pi}{\omega_k} < \delta_k < 1, \quad k = 1, \dots, J',$$

where ω_k is the interior angle between two faces meeting at edge e_k and λ_j is given by

$$\lambda_j := -\frac{1}{2} + \sqrt{\Lambda_j + \frac{1}{4}},$$

where Λ_j is the smallest, strictly positive eigenvalue of the Dirichlet Laplace–Beltrami operator on the intersection of the unit sphere centered at c_j and the infinite, interior polyhedral tangent cone to ∂D with vertex c_j , cp. [39, Section 4.3.1].

2.3 FE convergence theory

Let $\{\mathcal{T}_{\ell}\}_{\ell \geq 0}$ denote a sequence of regular, simplicial triangulations of D with proper mesh refinements near vertices and, if $d = 3$, also near edges of D . Let further $\mathbb{P}^1(K)$ denote the affine functions on a subset K of \mathbb{R}^d . In FE spaces $V_{\ell} := \{v \in V : v|_K \in \mathbb{P}^1(K), K \in \mathcal{T}_{\ell}\}$ of continuous, piecewise linear functions on $\{\mathcal{T}_{\ell}\}_{\ell \geq 0}$, optimal asymptotic convergence rates are achievable, also in the presence of singularities. We state these for subsequent reference, recapitulating from [5, 20, 4, 1] approximation properties in $H^1(D)$ of the subspaces V_{ℓ} .

Specifically, there exists $C > 0$ such that for every $w \in H_{\beta}^2(D)$ for $d = 2$, resp. for every $w \in H_{(\beta, \delta)}^2(D)$ for $d = 3$, and for every $\ell \geq 0$ there exists $w_{\ell} \in V_{\ell}$ such that, with $M_{\ell} := \dim(V_{\ell})$,

$$\|w - w_{\ell}\|_V \leq C M_{\ell}^{-1/d} \begin{cases} \|w\|_{H_{\beta}^2(D)} & \text{if } d = 2, \\ \|w\|_{H_{(\beta, \delta)}^2(D)} & \text{if } d = 3. \end{cases} \quad (11)$$

For $d = 2$, the convergence rate bound (11) is due to [5, Lemmas 4.1 and 4.5] for regular, graded simplicial meshes, resp. due to [20] for simplicial bisection tree meshes. In polyhedral domains D in space dimension $d = 3$, this estimate follows by [4, Theorem 4.6] for every $w \in C_0^{\infty}(\overline{D} \setminus \mathcal{C})$ and follows for every $w \in H_{(\beta, \delta)}^2(D)$, since $C_0^{\infty}(\overline{D} \setminus (\mathcal{C} \cup \mathcal{E}))$ is dense in $H_{(\beta, \delta)}^2(D)$ (see also [3, 36]).

2.4 Combined Dimension Truncation FE error bound

We now derive an error bound for the combined effect of truncating the GRF Z to a finite number of parameters s , and to FE discretization in D of the resulting s -parametric problem (6).

Let accordingly $u^{s, \mathcal{T}_\ell} : \Omega \rightarrow V_\ell$ denote the FE solution, i.e.,

$$\int_D a^s \nabla u^{s, \mathcal{T}_\ell} \cdot \nabla v dx = f(v), \quad \forall v \in V_\ell. \quad (12)$$

For notational convenience, we introduce

$$\bar{\beta} := \begin{cases} \beta & \text{if } d = 2, \\ (\beta, \delta) & \text{if } d = 3. \end{cases} \quad (13)$$

The Banach space $W_{\bar{\beta}}^{1, \infty}(D)$ is the space of all measurable functions $v : D \rightarrow \mathbb{R}$ that have finite $W_{\bar{\beta}}^{1, \infty}(D)$ -norm, where

$$\|v\|_{W_{\bar{\beta}}^{1, \infty}(D)} := \max\{\|v\|_{L^\infty(D)}, \|\nabla v|_{\Phi_{\bar{\beta}}}\|_{L^\infty(D)}\}.$$

In order for the ML algorithm Q_L^* to yield improved (w.r. to the single-level algorithm) error vs. work bounds, we require stronger assumptions than in the single-level analyses of [24, 30] on the function system $(\psi_j)_{j \geq 1}$. This corresponds to what was found for uniform random parameters in [34] and in the lognormal case for ψ_j with global supports in [33]. Let $(\bar{b}_j)_{j \geq 1}$ be a positive sequence such that

$$\left\| \sum_{j \geq 1} \frac{\max\{|\nabla \psi_j|_{\Phi_{\bar{\beta}}}, |\psi_j|\}}{\bar{b}_j} \right\|_{L^\infty(D)} < \infty. \quad (\mathbf{A2})$$

We remark that the assumption **(A2)** is stronger than **(A1)**, which was found sufficient in [30] for the convergence rate analysis of the corresponding single-level QMC-FE algorithm.

Remark 2.1 *When the precision operator of Z is a positive power of a shifted Dirichlet Laplacian on D the Karhunen-Loève eigenfunctions v_j are, by the spectral mapping theorem, eigenfunctions of the Dirichlet Laplacian on D : $-\Delta v_j = \nu_j v_j$, $v_j|_{\partial D} = 0$, $j \in \mathbb{N}$. Here, the eigenvalues ν_j are related to the ones appearing in the Karhunen-Loève expansion of the GRF Z by the spectral mapping theorem. Elliptic regularity shifts for the Dirichlet Laplacian are also known in certain weighted Hölder spaces in D : for $d = 3$, [39, Lemma 4.3.1.2)], implies that $v_j \in W_{(\beta, \delta)}^{1, \infty}(D)$ provided that $1 - \lambda_j < \beta_j < 1$, $j = 1, \dots, J$, and $1 - \pi/\theta_k < \delta_k < 1$, $k = 1, \dots, J'$, where we used here that the weighted $C^{1+\varepsilon}(\bar{D})$ -type space $N_{\beta, \delta}^{1, \varepsilon}(D)$ (in the notation of [39, Sections 4.2 and 4.3]) embeds continuously into $W_{(\beta, \delta)}^{1, \infty}(D)$. Note that this condition on β for the KL eigenfunctions is stronger than in Assumption **(A2)**. Similar statements hold for $d = 2$. Here, singularities at corners and (for $d = 3$) along edges of the Karhunen-Loève eigenfunctions appear as a consequence of regularity shifts for the Dirichlet Laplacian in weighted Hölder spaces. The structure of the weight functions $\Phi_{\bar{\beta}}$ (which depend only on D and on the (Dirichlet) Laplacian) in the assumption **(A2)** on the Karhunen-Loève eigenfunctions is identical to the weights in the elliptic regularity shift (10). In the case of Matérn-type covariance functions, as induced by solutions to (1), there is neither dependence of the functional form of the weight functions on the regularity nor on the positive correlation length of the respective GRF. Note, however, that in general, Karhunen-Loève eigenfunctions have global support in D .*

Assumption **(A2)** implies $W_{\bar{\beta}}^{1,\infty}(D)$ -regularity of the GRF Z and strong approximation by its truncated expansion. This is made precise in the following proposition. Its proof is completely analogous to [30, Theorem 2] and therefore not detailed.

Proposition 2.2 *Let the assumption in **(A2)** be satisfied for some sequence $(\bar{b}_j)_{j \geq 1}$ such that $(\bar{b}_j)_{j \geq 1} \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0, \infty)$. For every $\varepsilon > 0$ and $q \in [1, \infty)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$,*

$$\|Z - Z^s\|_{L^q(\Omega; W_{\bar{\beta}}^{1,\infty}(D))} \leq C \sup_{j > s} \{\bar{b}_j^{1-\varepsilon}\}.$$

Since $(\nabla a)\Phi_{\bar{\beta}} = (a\nabla Z)\Phi_{\bar{\beta}}$ holds in $L^\infty(D)^d$, μ -a.s., Proposition 2.2 and [30, Corollary 6] imply with the Cauchy–Schwarz inequality that for every $q \in [1, \infty)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$,

$$\|a\|_{L^q(\Omega; W_{\bar{\beta}}^{1,\infty}(D))} < \infty \quad \text{and} \quad \|a^s\|_{L^q(\Omega; W_{\bar{\beta}}^{1,\infty}(D))} \leq C < \infty. \quad (14)$$

To obtain an estimate of the Laplacian of u , we note that in any compact subset $\tilde{D} \subset\subset D$ it holds $-a\Delta u = f - \nabla a \cdot \nabla u$, μ -a.s., where we assume that $f \in L_{\bar{\beta}}^2(D)$. This equation may be tested with $-\Delta u \Phi_{\bar{\beta}}^2/a$, which implies with Lemma 2.1

$$\|\Delta u\|_{L_{\bar{\beta}}^2(D)} \leq \frac{\|f\|_{L_{\bar{\beta}}^2(D)}}{a_{\min}} + \|Z\|_{W_{\bar{\beta}}^{1,\infty}(D)} \|u\|_V \leq C \frac{\|f\|_{L_{\bar{\beta}}^2(D)}}{a_{\min}} (1 + \|Z\|_{W_{\bar{\beta}}^{1,\infty}(D)}). \quad (15)$$

An Aubin–Nitsche duality argument, (7), (9), (11), Proposition 2.2, (14), and (15) imply that for every $\varepsilon > 0$ exists a constant $C > 0$ such that for every $s \in \mathbb{N}$, $\ell \in \mathbb{N}_0$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s, \mathcal{T}_\ell))| \leq C \left(\sup_{j > s} \{b_j^t\} + M_\ell^{-2/d} \right) \|f\|_{L_{\bar{\beta}}^2(D)} \|G\|_{L_{\bar{\beta}}^2(D)}, \quad (16)$$

where $t = 2 - p_0/2$ if $p_0 \leq 2$ and $t = 1 - \varepsilon$ otherwise. Recall that $(b_j)_{j \geq 1} \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0, \infty)$. In this setting, $p_0 \in (0, 2)$ is the range of applicability of QMC, cp. [30, Theorem 11].

Remark 2.2 *By interpolation the error estimate in (16) extends to the case that $f \in (V^*, L_{\bar{\beta}}^2(D))_{t,\infty}$ and $G(\cdot) \in (V^*, L_{\bar{\beta}}^2(D))_{t',\infty}$ for some $t, t' \in [0, 1]$. Then the estimate in (16) holds with the term $M_\ell^{-2/d}$ replaced by $M_\ell^{-(t+t')/d}$. To see this, we observe that the real method of interpolation can be applied to the regularity shifts in (9) and in (15). Specifically, the linear operator relating the solution $u \in V$ to its approximation error with a V -bounded, and quasioptimal projector $\Pi_\ell : V \rightarrow V_\ell$, where Π_ℓ is, for example, the $H_0^1(D)$ -projection. From the approximation property in (11), the interpolation couple $L_{\bar{\beta}}^2(D) \subset V^*$ then yields the fractional convergence order. Here and throughout what follows, interpolation spaces shall be understood with respect to the real method of interpolation; we refer to [43, Chapter 1].*

3 QMC integration

With convergence rate bounds on the dimension truncation and the FE discretization error at hand, we address the numerical approximation of the expectations in (16) with respect to the GM μ . Due to dimension truncation, we evaluate its s -variate section, i.e. we integrate w.r. to the GM on \mathbb{R}^s . As in [24], we approximate the s -variate integrals by so-called randomly shifted

lattice rules proposed in [40]. Accordingly, we review QMC error estimates of randomly shifted lattice rules for high-dimensional integrals with respect to the s -variate normal distribution. The construction of generating vectors for such QMC rules in particular with respect to Gaussian and exponentially decaying weight functions with a fast CBC construction have been found in [40]. There, concrete error estimates of the resulting QMC rules in the mean-square sense (with respect to the random shift) have been derived, cp. [40, Theorem 8]. See also [35, Examples 4 and 5] for the estimation of constants appearing in the error bound of [40, Theorem 8] for Gaussian and exponential weight functions, respectively.

The error analysis of randomly shifted lattice rules requires, for sequences of positive weights $\gamma = (\gamma_{\mathbf{u}})_{\mathbf{u}}$, indexed by all finite subsets $\mathbf{u} \subset \mathbb{N}$, the weighted Sobolev space $\mathcal{W}_{\gamma}(\mathbb{R}^s)$ of mixed first order derivatives, which is defined by the following norm

$$\|F\|_{\mathcal{W}_{\gamma}(\mathbb{R}^s)} := \left(\sum_{\mathbf{u} \subset \{1:s\}} \gamma_{\mathbf{u}}^{-1} \int_{\mathbb{R}^{|\mathbf{u}|}} \left| \int_{\mathbb{R}^{s-|\mathbf{u}|}} \partial^{\mathbf{u}} F(\mathbf{y}) \prod_{j \in \{1:s\} \setminus \mathbf{u}} \phi(y_j) d\mathbf{y}_{\{1:s\} \setminus \mathbf{u}} \right|^2 \prod_{j \in \mathbf{u}} w_j^2(y_j) d\mathbf{y}_{\mathbf{u}} \right)^{1/2}. \quad (17)$$

Here, ϕ denotes the univariate normal density

$$\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}.$$

The norm in (17) is considered with respect to Gaussian and exponential weight functions

$$w_{g,j}^2(y) := e^{-\frac{y^2}{2\alpha_{g,j}}}, \quad y \in \mathbb{R}, j \in \mathbb{N}, \quad \text{and} \quad w_{\text{exp},j}^2(y) := e^{-\alpha_{\text{exp},j}|y|}, \quad y \in \mathbb{R}, j \in \mathbb{N},$$

where the parameters $\alpha_{g,j} > 1$ and $\alpha_{\text{exp},j} > 0$ will be determined in the following. If the parameters $\alpha_{g,j}$ or $\alpha_{\text{exp},j}$ are constant with respect to j , we omit j for ease of presentation. In the following we consider the case $\alpha_{g,j} = \alpha_g > 1$ and $\alpha_{\text{exp},j} = \alpha_{\text{exp}} > 0$ for every $j \in \mathbb{N}$. In this work, we consider in (17) product weights $\gamma = (\gamma_{\mathbf{u}})_{\mathbf{u} \subset \mathbb{N}}$, determined by a positive QMC weight sequence $(\gamma_j)_{j \geq 1}$, i.e.,

$$\gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \gamma_j, \quad \mathbf{u} \subset \mathbb{N}, |\mathbf{u}| < \infty.$$

We will denote the QMC approximation in s dimensions with N points by $Q_{s,N}(\cdot)$. It shall approximate integrals with respect to the multivariate normal distribution which we denote for every integrand $F \in L^1(\mathbb{R}^s, \mu)$ by

$$I_s(F) := \int_{\mathbb{R}^s} F(\mathbf{y}) \prod_{j \in \{1:s\}} \phi(y_j) d\mathbf{y}.$$

For a sequence of dimension truncations $(s_{\ell})_{\ell=0,\dots,L}$ and a sequence $(N_{\ell})_{\ell=0,\dots,L}$, $L \in \mathbb{N}_0$, the multilevel QMC quadrature algorithm of [34] is defined by

$$Q_L^*(G(u^L)) := \sum_{\ell=0}^L Q_{s_{\ell}, N_{\ell}}(G(u^{\ell}) - G(u^{\ell-1})), \quad L \geq 0, \quad (18)$$

with the understanding that $G(u^{-1}) := 0$. we used the notation that $u^{\ell} := u^{s_{\ell}, \mathcal{T}_{\ell}}$, $\ell \geq 0$. Multilevel QMC algorithms stemming from randomly shifted lattice rules have been considered in [34, 33]. The following error estimate (see [34, Equation (23)] or [33, Equation (3.2)]) holds due to the independence of the random shifts on the different levels

$$\mathbb{E}^{\Delta} (|I_s(G(u^L)) - Q_L^*(G(u^L))|^2) = \sum_{\ell=0}^L \mathbb{E}^{\Delta} (|I_s(G(u^{\ell} - u^{\ell-1})) - Q_{s_{\ell}, N_{\ell}}(G(u^{\ell} - u^{\ell-1}))|^2), \quad (19)$$

where we apply a randomly shifted lattice rule with respect to (possibly) a different QMC weight sequence on the PDE discretization level $\ell = 0$.

In [30], convergence of randomly shifted lattice rules with product weights is investigated, which relies on parametric regularity estimates of a particular form. We summarize the QMC convergence theory in the following theorem.

Theorem 3.1 *Let $(\tilde{b}_j)_{j \geq 1}$ be a positive sequence such that for some $F : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ there exists a constant $C > 0$ and a positive function $H(\mathbf{y})$ such that for every $\mathbf{y} \in \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} : \exists s \in \mathbb{N}, y_j = 0 : \forall j > s\}$,*

$$\sum_{\mathbf{u} \subset \mathbb{N}, |\mathbf{u}| < \infty} |\partial^{\mathbf{u}} F(\mathbf{y})|^2 \prod_{j \in \mathbf{u}} \left(\frac{\kappa}{\tilde{b}_j} \right)^2 \leq C H(\mathbf{y})^2.$$

1. *Let $(\tilde{b}_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$. For $\varepsilon \in (0, 3/4 - 1/(2p))$, set $p' = p/4 + 1/2 - \varepsilon p \in (0, 1)$. Consider the Gaussian weight functions $(w_{g,j})_{j \geq 1}$ with parameter α_g and QMC integration weight sequence*

$$\alpha_g \in \left(\frac{p}{2(p-p')}, \frac{p}{p-2(1-p')} \right) \quad \text{and} \quad \gamma_j = \tilde{b}_j^{2p'}, \quad j \geq 1.$$

Then, there exists a constant C (independent of F) such that for $q_0 = 2qq'/(q' - q)$, where $q = p/(2(1-p'))$ and $q' \in (q, \alpha_g/(1-\alpha_g))$,

$$\sqrt{\mathbb{E}^{\Delta} (|I_s(F) - Q_{s,N}(F)|^2)} \leq C (\varphi(N))^{-1/(2p)-1/4+\varepsilon} \|H\|_{L^{q_0}(\mathbb{R}^s, \mu)}.$$

2. *Let $(\tilde{b}_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 1]$. Assume that $H(\mathbf{y}) \leq \eta_1 \exp(\eta_2 \sum_{j \geq 1} \tilde{b}_j |y_j|)$ for some $\eta_1, \eta_2 > 0$. Set $p' = 1 - p/2$. Consider the exponential weight functions $(w_{\text{exp},j})_{j \geq 1}$ with parameter α_{exp} and QMC integration weight sequence*

$$\alpha_{\text{exp}} > 2\eta_2 \quad \text{and} \quad \gamma_j = \tilde{b}_j^{2p'}, \quad j \geq 1.$$

Then, there exists a positive constant C (independent of η_1) such that

$$\sqrt{\mathbb{E}^{\Delta} (|I_s(F) - Q_{s,N}(F)|^2)} \leq C (\varphi(N))^{-1/p+1/2} \eta_1.$$

The Euler totient function is denoted by $\varphi(\cdot)$.

This theorem was, in the case of Gaussian weight functions, obtained in [30, Theorems 9 and 11] and in the case of exponential weight functions in [30, Theorems 9 and 12]. The main ingredient of the proof of [30, Theorem 9] is a parametric regularity estimate of the form assumed in Theorem 3.1. The parametric regularity estimates derived in [24, 33] for globally supported ψ_j afforded bounds for each partial derivative separately. In [30], we used the bound from [7, Theorem 4.1] which does account for local supports and affords control of “bulk” sums of (norms of) solution derivatives with respect to the parameters y_j . We also note that in applications, the sequence $(\tilde{b}_j)_{j \geq 1}$ may be arbitrarily scaled by a factor κ in order to satisfy such a regularity estimate.

4 Parametric regularity

In this section we derive parametric regularity estimates that allow to prove dimension independent convergence rates of multilevel QMC. We extend the argument that results in the estimate in [7, Theorem 4.1] to dimensionally truncated and FE differences.

For every finite $s \in \mathbb{N}$, the truncated fields Z^s, a^s , and u^s , are well-defined regardless of assumption **(A1)**. In particular, $Z^s = \sum_{j=1}^s y_j \psi_j$ is well-defined for every $\mathbf{y} \in \Omega = \mathbb{R}^{\mathbb{N}}$. We may therefore interpret Z^s as a mapping from \mathbb{R}^s to L^∞ such that pointwise evaluation is well-defined for every $\mathbf{y} \in \mathbb{R}^s$. Similarly, a^s and u^s may be interpreted as mappings from \mathbb{R}^s to $L^\infty(D)$ and to V , respectively. In the same way Z, a , and u are mappings with pointwise evaluation from the set

$$U := \{\mathbf{y} \in \Omega : \exists s \in \mathbb{N}, y_j = 0, j > s\}$$

to $L^\infty(D)$ and V , respectively. Note that $\mathbb{R}^s \times \{\mathbf{0}\} \subset U = \bigcup_{s \in \mathbb{N}} \mathbb{R}^s \times \{\mathbf{0}\}$ for every $s \in \mathbb{N}$, where $\mathbf{0} \in \mathbb{R}^{\mathbb{N} \setminus \{1:s\}}$. Hence, the set U of admissible parameters \mathbf{y} is sufficiently rich for the ensuing QMC convergence analysis. The mappings Z^s, a^s , and u^s extend naturally to mappings from U to $L^\infty(D)$ and to V , respectively.

4.1 Dimension truncation

Let $s \in \mathbb{N}$ be a truncation level. For every $\mathbf{y} \in U$, the difference $u(\mathbf{y}) - u^s(\mathbf{y})$ satisfies the variational formulation

$$\int_D a(\mathbf{y}) \nabla(u(\mathbf{y}) - u^s(\mathbf{y})) \cdot \nabla v dx = - \int_D (a(\mathbf{y}) - a^s(\mathbf{y})) \nabla u^s(\mathbf{y}) \cdot \nabla v dx, \quad \forall v \in V. \quad (20)$$

We will mostly (in the proofs) omit the \mathbf{y} dependence in the following. Set $\mathcal{F} := \{\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\tau}| < \infty\}$. For every real-valued sequence $\rho = (\rho_j)_{j \geq 1}$ and $\boldsymbol{\tau} \in \mathcal{F}$, we shall use the notation $\rho^\boldsymbol{\tau} = \prod_{j \geq 1} \rho_j^{\tau_j}$, which is a product with finitely many factors due to $|\boldsymbol{\tau}| = \sum_{j \geq 1} \tau_j < \infty$. For every $\boldsymbol{\tau} \in \mathcal{F}$ and a positive sequence $(\rho_j)_{j \geq 1}$, let us define the following numbers

$$\kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) := \frac{\sqrt{\boldsymbol{\tau}!} \rho^{\boldsymbol{\tau} - \boldsymbol{\nu}} |\psi|^{\boldsymbol{\tau} - \boldsymbol{\nu}}}{\sqrt{\boldsymbol{\nu}!} (\boldsymbol{\tau} - \boldsymbol{\nu})!}, \quad \boldsymbol{\nu} \leq \boldsymbol{\tau}.$$

Also, for given $k, r \in \mathbb{N}$ introduce the set $\Lambda_k := \{\boldsymbol{\tau} \in \mathcal{F} : |\boldsymbol{\tau}| = k, \|\boldsymbol{\tau}\|_{\ell^\infty} \leq r\}$ and for any integer $\ell \leq k - 1$ and for $\boldsymbol{\nu} \in \Lambda_\ell$, introduce

$$R_{\boldsymbol{\nu}, k} := \{\boldsymbol{\tau} \in \Lambda_k : \boldsymbol{\tau} \geq \boldsymbol{\nu}\},$$

where r denotes the maximal order of differentiability to be considered. The following lemma reveals that κ_0 can be interpreted as a discrete measure which charges certain multi-indices of interest with mass less than one, which will be useful in the ensuing analysis.

Lemma 4.1 *Assume that there exists a positive sequence $(\rho_j)_{j \geq 1}$ such that, for some $r \in \mathbb{N}$,*

$$K := \left\| \sum_{j \geq 1} \rho_j |\psi_j| \right\|_{L^\infty(D)} < \frac{\log(2)}{\sqrt{r}}. \quad (21)$$

Then, for every $\boldsymbol{\tau} \in \mathcal{F}$ such that $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$,

$$\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq e^{\sqrt{r}K} - 1 < 1$$

and for every positive integer $\ell \leq k - 1$ and multi-index $\boldsymbol{\nu} \in \Lambda_\ell$,

$$\sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}.$$

The estimates in this lemma are given in [7, Equations (4.12) and (4.14)]. The second estimate of Lemma 4.1 still holds if the smallness assumption in (21) is not guaranteed. We note that the condition (21) is implied by **(A1)** with $\rho_j^{-1} = b_j \bar{K} \sqrt{r} / \log(2)$ provided that $\bar{K} > \|\sum_{j \geq 1} |\psi_j| / b_j\|_{L^\infty(D)}$. For every $s \in \mathbb{N}$, integers $\ell \leq k - 1$, and $\boldsymbol{\nu} \in \Lambda_\ell$, introduce the set

$$R_{\boldsymbol{\nu}, k}^s := \{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k} : \exists j > s \text{ such that } \tau_j > 0\}.$$

Lemma 4.2 *Let the assumptions of Lemma 4.1 hold for a positive sequence $(\rho_j)_{j \geq 1}$ such that $c := \|(\rho_j^{-1})_{j \geq 1}\|_{\ell^\infty(\mathbb{N})} < \infty$. Let us further assume that for some $\eta > 0$*

$$K_\eta := \left\| \sum_{j \geq 1} \rho_j^{1+\eta} |\psi_j| \right\|_{L^\infty(D)} < \infty.$$

Then, for $s \in \mathbb{N}$ and every $\boldsymbol{\tau} \in \mathcal{F}$ such that $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$ and $\tau_j > 0$ for some $j > s$,

$$\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \nu_j = 0 \forall j > s} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq 2(e^{\sqrt{r} K_\eta c^\eta} - 1) c^{-\eta} \sup_{j > s} \{\rho_j^{-\eta}\}.$$

For $s \in \mathbb{N}$, positive integers $\ell \leq k - 1$, and $\boldsymbol{\nu} \in \Lambda_\ell$ such that $\nu_j = 0$, $j > s$,

$$\sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}^s} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq \frac{(\sqrt{r} K_\eta c^\eta)^{k-\ell}}{(k-\ell)!} c^{-\eta} \sup_{j > s} \{\rho_j^{-\eta}\}.$$

Proof. There is $j > s$ such that $\tau_j > 0$. Since κ_0 is a product, by Lemma 4.1,

$$\begin{aligned} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \nu_j = 0 \forall j > s} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) &= \left(\sum_{\boldsymbol{\nu}_{\{1:s\}} \leq \boldsymbol{\tau}_{\{1:s\}}} \kappa_0(\boldsymbol{\tau}_{\{1:s\}}, \boldsymbol{\nu}_{\{1:s\}}) \right) \kappa_0(\boldsymbol{\tau}_{\mathbb{N} \setminus \{1:s\}}, \mathbf{0}_{\mathbb{N} \setminus \{1:s\}}) \\ &\leq 2\kappa_0(\boldsymbol{\tau}_{\mathbb{N} \setminus \{1:s\}}, \mathbf{0}_{\mathbb{N} \setminus \{1:s\}}), \end{aligned}$$

where we used the notation that for every $\mathbf{u} \subset \mathbb{N}$, $\tau_{\mathbf{u}}$ is a multi-index that satisfies $(\tau_{\mathbf{u}})_j = \tau_j$, $j \in \mathbf{u}$, and $(\tau_{\mathbf{u}})_j = 0$ otherwise. With $c = \|(\rho_j^{-1})_{j \geq 1}\|_{\ell^\infty(\mathbb{N})}$, we obtain

$$\begin{aligned} \kappa_0(\boldsymbol{\tau}_{\mathbb{N} \setminus \{1:s\}}, \mathbf{0}_{\mathbb{N} \setminus \{1:s\}}) &\leq \frac{\rho^{\boldsymbol{\tau}_{\mathbb{N} \setminus \{1:s\}}}}{\sqrt{\boldsymbol{\tau}_{\mathbb{N} \setminus \{1:s\}}}} |\psi|^{\boldsymbol{\tau}_{\mathbb{N} \setminus \{1:s\}}} \leq \exp \left(\sqrt{r} \sum_{j > s} \rho_j |\psi_j| \right) - 1 \\ &\leq (e^{\sqrt{r} K_\eta c^\eta} - 1) c^{-\eta} \sup_{j > s} \{\rho_j^{-\eta}\}. \end{aligned}$$

For the proof of the second inequality, we observe that

$$\sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}^s} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq \sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}^s} \frac{\sqrt{\boldsymbol{\tau}!} (\rho^{1+\eta} c^\eta)^{(\boldsymbol{\tau}-\boldsymbol{\nu})} |\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{\boldsymbol{\nu}!} (\boldsymbol{\tau}-\boldsymbol{\nu})!} c^{-\eta} \sup_{j > s} \{\rho_j^{-\eta}\},$$

where we used that for every $\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}^s$ there exists $j > s$ such that $\tau_j - \nu_j > 0$ and that $\rho_j^{-1}/c \leq 1$, $j \leq 1$. By the first statement of Lemma 4.1,

$$\sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}^s} \frac{\sqrt{\boldsymbol{\tau}!} (\rho^{1+\eta} c^\eta)^{(\boldsymbol{\tau}-\boldsymbol{\nu})} |\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{\boldsymbol{\nu}!} (\boldsymbol{\tau}-\boldsymbol{\nu})!} \leq \sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu}, k}^s} \frac{\sqrt{\boldsymbol{\tau}!} (\rho^{1+\eta} c^\eta)^{(\boldsymbol{\tau}-\boldsymbol{\nu})} |\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{\boldsymbol{\nu}!} (\boldsymbol{\tau}-\boldsymbol{\nu})!} \leq \frac{(\sqrt{r} K_\eta c^\eta)^{k-\ell}}{(k-\ell)!},$$

which implies the assertion of the lemma. \square

Theorem 4.3 [Truncation error]

Let the assumptions of Lemmas 4.1 and 4.2 be satisfied for a positive sequence $(\rho_j)_{j \geq 1}$ and $\eta > 0$. There exists a constant $C > 0$ such that for every $s \in \mathbb{N}$ and every $\mathbf{y} \in U$

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u(\mathbf{y}) - u^s(\mathbf{y}))\|_{a(\mathbf{y})}^2 \leq C \left(\left\| \frac{a(\mathbf{y}) - a^s(\mathbf{y})}{a(\mathbf{y})} \right\|_{L^\infty(D)}^2 + \sup_{j > s} \{\rho_j^{-2\eta}\} \right) \|u^s(\mathbf{y})\|_{a(\mathbf{y})}^2.$$

Proof. We divide the index set $\mathcal{F}_r := \{\boldsymbol{\tau} \in \mathcal{F} : \tau_j \leq r, j \in \mathbb{N}\}$ into $\mathcal{F}_1^s := \{\boldsymbol{\tau} \in \mathcal{F}_r : \tau_j = 0 \forall j > s\}$ and $\mathcal{F}_2^s := \{\boldsymbol{\tau} \in \mathcal{F}_r : \exists j > s \text{ s.t. } \tau_j > 0\}$. Obviously, $\mathcal{F}_r = \mathcal{F}_1^s \cup \mathcal{F}_2^s$.

Let $\mathbf{0} \neq \boldsymbol{\tau} \in \mathcal{F}_1^s$ be arbitrary. We observe that for every $v \in V$,

$$\begin{aligned} \int_D a \nabla \partial^{\boldsymbol{\tau}}(u - u^s) \cdot \nabla v dx &= - \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \int_D \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}} a \nabla \partial^{\boldsymbol{\nu}}(u - u^s) \cdot \nabla v dx \\ &\quad - \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau} : \forall j > s \tau_j = \nu_j} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \int_D \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}} (a - a^s) \nabla \partial^{\boldsymbol{\nu}} u^s \cdot \nabla v dx. \end{aligned}$$

Set

$$\sigma_k := \sum_{\boldsymbol{\tau} \in \Lambda_k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u - u^s)\|_a^2$$

and take $v = \partial^{\boldsymbol{\tau}}(u - u^s)$. By a twofold application of the Cauchy–Schwarz inequality and by Lemma 4.1

$$\begin{aligned} \sigma_k &\leq \int_D \sum_{\boldsymbol{\tau} \in \Lambda_k} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} a \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} |\nabla \partial^{\boldsymbol{\nu}}(u - u^s)| \frac{\rho^{\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\nabla \partial^{\boldsymbol{\tau}}(u - u^s)| \\ &\quad + \int_D \sum_{\boldsymbol{\tau} \in \Lambda_k} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} |a - a^s| \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} |\nabla \partial^{\boldsymbol{\nu}} u^s| \frac{\rho^{\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\nabla \partial^{\boldsymbol{\tau}}(u - u^s)| \\ &\leq \int_D \left(\sum_{\boldsymbol{\tau} \in \Lambda_k} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} a \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} |\nabla \partial^{\boldsymbol{\nu}}(u - u^s)|^2 \right)^{1/2} \\ &\quad \times \left(a \sum_{\boldsymbol{\tau} \in \Lambda_k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\nabla \partial^{\boldsymbol{\tau}}(u - u^s)|^2 \right)^{1/2} \\ &\quad + \int_D \left(\sum_{\boldsymbol{\tau} \in \Lambda_k} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} |a - a^s| \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} |\nabla \partial^{\boldsymbol{\nu}} u^s|^2 \right)^{1/2} \\ &\quad \times \left(2 \sum_{\boldsymbol{\tau} \in \Lambda_k} |a - a^s| \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\nabla \partial^{\boldsymbol{\tau}}(u - u^s)|^2 \right)^{1/2} \end{aligned}$$

Further, we apply the Cauchy–Schwarz inequality on the integral and obtain that

$$\begin{aligned} \sigma_k &\leq \left(\int_D \sum_{\boldsymbol{\tau} \in \Lambda_k} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} a \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} |\nabla \partial^{\boldsymbol{\nu}}(u - u^s)|^2 \right)^{1/2} \sqrt{\sigma_k} \\ &\quad + \left(\int_D \sum_{\boldsymbol{\tau} \in \Lambda_k} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} |a - a^s| \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} |\nabla \partial^{\boldsymbol{\nu}} u^s|^2 \right)^{1/2} \sqrt{2 \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)} \sqrt{\sigma_k}}. \end{aligned}$$

By [7, Equation (4.18)] in the proof of [7, Theorem 4.1], for any $\delta \in [\sqrt{r}K/\log(2), 1)$ and for every $\ell \in \mathbb{N}$,

$$\sum_{\tau \in \Lambda_\ell} \frac{\rho^{2\tau}}{\tau!} \|\partial^\tau u^s\|_a^2 \leq \|u^s\|_a^2 \delta^\ell. \quad (22)$$

We change the order of summation in order to apply the second estimate in Lemma 4.1 and insert (22) to obtain with Young's inequality that for any $\varepsilon > 0$

$$\begin{aligned} \sigma_k &\leq (1 + \varepsilon) \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sigma_\ell \\ &\quad + \left(1 + \frac{1}{\varepsilon}\right) \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)}^2 \sum_{\ell=0}^k \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{\tau \in \Lambda_\ell} \frac{\rho^{2\tau}}{\tau!} \|\partial^\tau u^s\|_a^2 \\ &\leq (1 + \varepsilon) \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sigma_\ell + \left(1 + \frac{1}{\varepsilon}\right) 2 \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)}^2 \|u^s\|_a^2 \sum_{\ell=0}^k \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \delta^\ell \\ &\leq (1 + \varepsilon) \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sigma_\ell + \left(1 + \frac{1}{\varepsilon}\right) 4 \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)}^2 \|u^s\|_a^2 \delta^k. \end{aligned}$$

By a change of the order of summation, we obtain that

$$\sum_{k \geq 1} \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sigma_\ell = \sum_{\ell \geq 0} \left(\sum_{k=\ell+1}^{\infty} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \right) \sigma_\ell \leq (e^{\sqrt{r}K} - 1) \sum_{\ell \geq 0} \sigma_\ell. \quad (23)$$

Let us choose $\varepsilon > 0$ such that $\varepsilon < (2 - e^{\sqrt{r}K}) / (e^{\sqrt{r}K} - 1)$, which implies that $(1 + \varepsilon)(e^{\sqrt{r}K} - 1) < 1$. Denote $C^* := (1 - (1 + \varepsilon)(e^{\sqrt{r}K} - 1))^{-1}$. We sum σ_k over $k \geq 1$ and obtain that

$$\sum_{k \geq 1} \sigma_k \leq (1 + \varepsilon)(e^{\sqrt{r}K} - 1) \sum_{\ell \geq 0} \sigma_\ell + \left(1 + \frac{1}{\varepsilon}\right) 4 \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)}^2 \|u^s\|_a^2 \frac{\delta}{1 - \delta}.$$

Since $(1 + \varepsilon)(e^{\sqrt{r}K} - 1) < 1$, we conclude that

$$\sum_{k \geq 1} \sigma_k \leq C^* \sigma_0 + C^* \left(1 + \frac{1}{\varepsilon}\right) 4 \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)}^2 \|u^s\|_a^2 \frac{\delta}{1 - \delta},$$

which implies

$$\sum_{\tau \in \mathcal{F}_1^s} \frac{\rho^{2\tau}}{\tau!} \|\partial^\tau (u - u^s)\|_a^2 \leq C \left(\|u - u^s\|_a^2 + \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)}^2 \|u^s\|_a^2 \right).$$

In the other case $\tau \in \mathcal{F}_2^s$, we observe that for arbitrary $\mathbf{0} \neq \tau \in \mathcal{F}_2^s$,

$$\begin{aligned} \int_D a \nabla \partial^\tau (u - u^s) \cdot \nabla v dx &= - \sum_{\nu \leq \tau, \nu \neq \tau} \binom{\tau}{\nu} \psi^{\tau-\nu} a \nabla \partial^\nu (u - u^s) \cdot \nabla v dx \\ &\quad - \sum_{\nu \leq \tau} \binom{\tau}{\nu} \psi^{\tau-\nu} a \nabla \partial^\nu u^s \cdot \nabla v dx, \quad \forall v \in V. \end{aligned} \quad (24)$$

We used that there is $j > s$ such that $\tau_j > 0$, which implies that for $\nu \neq \tau$ such that $\nu \leq \tau$, either $\tau_j - \nu_j > 0$ yielding $\partial^{\tau-\nu} a^s = 0$ or $\tau_j = \nu_j > 0$ yielding $\partial^\nu u^s = 0$. Moreover, in the

second sum above, we can restrict the index set to those $\boldsymbol{\nu}$ satisfying $\nu_j = 0$ for every $j > s$. In particular, always $\boldsymbol{\nu} \neq \boldsymbol{\tau}$. The estimate of the sum over $\boldsymbol{\tau} \in \mathcal{F}_2^s$ follows with a similar argument using Lemma 4.1 for the first sum and Lemma 4.2 for the second sum of the right hand side of equality (24), where we crucially use that $\boldsymbol{\nu} \neq \boldsymbol{\tau}$, which yield that the sum runs only over $\ell \in \{0, \dots, k-1\}$. Specifically,

$$\sum_{\boldsymbol{\tau} \in \mathcal{F}_2^s} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u - u^s)\|_a^2 \leq C \left(\|u - u^s\|_a^2 + \max_{j>s} \{\rho_j^{-2\eta}\} \|u^s\|_a^2 \right).$$

Since by (20) and by the Cauchy–Schwarz inequality

$$\|u - u^s\|_a \leq \left\| \frac{a - a^s}{a} \right\|_{L^\infty(D)} \|u^s\|_a,$$

the assertion of the theorem follows. \square

Remark 4.1 *The statement of Theorem 4.3 also holds true for the FE solution $u^{\mathcal{T}_\ell}$ and u^{s,\mathcal{T}_ℓ} for every truncation dimension s_ℓ with $\ell \geq 0$.*

4.2 FE discretization

First we show parametric regularity estimates of the solution u . Thus, we bound weighted sums over sensitivities of u in the norm of the smoothness space. For every $\boldsymbol{\tau} \in \mathcal{F}$, we define the quantities

$$\kappa_1(\boldsymbol{\tau}, \boldsymbol{\nu}) := \frac{\sqrt{\boldsymbol{\tau}!} \rho^{\boldsymbol{\tau}-\boldsymbol{\nu}} |\nabla \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}}|_{\Phi_{\boldsymbol{\beta}}}}{\sqrt{\boldsymbol{\nu}!} (\boldsymbol{\tau}-\boldsymbol{\nu})!}, \quad \boldsymbol{\nu} \leq \boldsymbol{\tau}.$$

Similar to Lemma 4.1, the following lemma reveals that κ_1 has the property of a discrete probability density, which will be essential in the ensuing analysis.

Lemma 4.4 *Assume that for $r \in \mathbb{N}$*

$$\left\| \sum_{j \geq 1} \rho_j \max\{|\nabla \psi_j|_{\Phi_{\boldsymbol{\beta}}}, |\psi_j|\} \right\|_{L^\infty(D)} =: K < C_r := \sup \left\{ c > 0 : \sqrt{r} c e^{\sqrt{r}c} \leq 1 \right\}. \quad (25)$$

Then for every $\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}}$ such that $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$

$$\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_1(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq \sqrt{r} K e^{\sqrt{r}K} < 1$$

and for every $\ell \leq k-1$ and $\boldsymbol{\nu} \in \Lambda_\ell$,

$$\sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu},k}} \kappa_1(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq (k-\ell) \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}.$$

Proof. We set $k = |\boldsymbol{\tau}|$ and observe with the multinomial theorem

$$\begin{aligned}
\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_1(\boldsymbol{\tau}, \boldsymbol{\nu}) &= \sum_{\ell=1}^k \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, |\boldsymbol{\tau}-\boldsymbol{\nu}|=\ell} \kappa_1(\boldsymbol{\tau}, \boldsymbol{\nu}) \\
&\leq \sum_{\ell=1}^k r^{\ell/2} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, |\boldsymbol{\tau}-\boldsymbol{\nu}|=\ell} \ell \frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}} \max\{|\nabla\psi|_{\Phi_{\bar{\beta}}}, |\psi|\}^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!} \\
&\leq \sum_{\ell=1}^k r^{\ell/2} \ell \sum_{|\mathbf{m}|=\ell} \frac{\rho^{\mathbf{m}} \max\{|\nabla\psi|_{\Phi_{\bar{\beta}}}, |\psi|\}^{\mathbf{m}}}{\mathbf{m}!} \\
&= \sum_{\ell=1}^k \frac{r^{\ell/2}}{(\ell-1)!} \left(\sum_{j \geq 1} \rho_j \max\{|\nabla\psi_j|_{\Phi_{\bar{\beta}}}, |\psi_j|\} \right)^\ell \leq \sqrt{r} K e^{\sqrt{r}K} < 1,
\end{aligned}$$

where we applied that

$$|\nabla\psi^{\boldsymbol{\tau}-\boldsymbol{\nu}}|_{\Phi_{\bar{\beta}}} \leq \sum_{j \geq 1} (\tau_j - \nu_j) |\psi_j|^{\tau_j - \nu_j - 1} |\nabla\psi_j|_{\Phi_{\bar{\beta}}} \prod_{i \neq j} |\psi_i|^{\tau_i - \nu_i} \leq |\boldsymbol{\tau} - \boldsymbol{\nu}| \max\{|\nabla\psi_j|_{\Phi_{\bar{\beta}}}, |\psi|\}^{\boldsymbol{\tau}-\boldsymbol{\nu}}.$$

The second estimate follows similarly. \square

Theorem 4.5 *Let the assumption of Lemma 4.4 be satisfied for a positive sequence $(\rho_j)_{j \geq 1}$, and assume that $r \in \mathbb{N}$ and $K < C_r$. There exists a constant $C > 0$ such that for every $\mathbf{y} \in U$*

$$\begin{aligned}
&\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta\partial^{\boldsymbol{\tau}}u(\mathbf{y})\|_{L^2_{\bar{\beta}}(D)}^2 \\
&\leq C \frac{1}{a_{\min}(\mathbf{y})} \left((1 + \|\nabla Z(\mathbf{y})|_{\Phi_{\bar{\beta}}}\|_{L^\infty(D)}^2) \|u(\mathbf{y})\|_{a(\mathbf{y})}^2 + \|\Delta u(\mathbf{y})\|_{L^2_{\bar{\beta}}(D)}^2 \right).
\end{aligned}$$

Proof. Let $\mathbf{0} \neq \boldsymbol{\tau} \in \mathcal{F}$ be given such that $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$. We observe that for every $v \in C_0^\infty(D)$,

$$-\int_D av \Delta\partial^{\boldsymbol{\tau}}u dx = \int_D \left(\nabla a \cdot \nabla\partial^{\boldsymbol{\tau}}u + \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} (\nabla\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}a \cdot \nabla\partial^{\boldsymbol{\nu}}u + \partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}a \Delta\partial^{\boldsymbol{\nu}}u) \right) v dx.$$

Using the density of $C_0^\infty(D)$ in $L^2_{\bar{\beta}}(D)$, we choose the test function $v = -\Phi_{\bar{\beta}}^2/a \Delta\partial^{\boldsymbol{\tau}}u$, multiply by $\rho^{2\boldsymbol{\tau}}/\boldsymbol{\tau}!$, and apply the Young inequality for arbitrary $\varepsilon > 0$ to obtain

$$\begin{aligned}
\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta\partial^{\boldsymbol{\tau}}u\|_{L^2_{\bar{\beta}}(D)}^2 &= -\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \int_D \left(\frac{\nabla a}{a} \cdot \nabla\partial^{\boldsymbol{\tau}}u + \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \frac{\nabla\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}a}{a} \cdot \nabla\partial^{\boldsymbol{\nu}}u \right) \Delta\partial^{\boldsymbol{\tau}}u \Phi_{\bar{\beta}}^2 dx \\
&\quad - \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \int_D \left(\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \frac{\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}a}{a} \Delta\partial^{\boldsymbol{\nu}}u \right) \Delta\partial^{\boldsymbol{\tau}}u \Phi_{\bar{\beta}}^2 dx \\
&\leq \varepsilon \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta\partial^{\boldsymbol{\tau}}u\|_{L^2_{\bar{\beta}}(D)}^2 \\
&\quad + \frac{1}{4\varepsilon} \int_D \left(|\nabla Z|_{\Phi_{\bar{\beta}}} \frac{\rho^{\boldsymbol{\tau}} |\nabla\partial^{\boldsymbol{\tau}}u|}{\sqrt{\boldsymbol{\tau}!}} + \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_1(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{\boldsymbol{\nu}} |\nabla\partial^{\boldsymbol{\nu}}u|}{\sqrt{\boldsymbol{\nu}!}} \right)^2 dx \\
&\quad + \int_D \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \frac{\rho^{\boldsymbol{\nu}} |\Delta\partial^{\boldsymbol{\nu}}u|_{\Phi_{\bar{\beta}}} \rho^{\boldsymbol{\tau}} |\Delta\partial^{\boldsymbol{\tau}}u|_{\Phi_{\bar{\beta}}}}{\sqrt{\boldsymbol{\nu}!} \sqrt{\boldsymbol{\tau}!}} dx.
\end{aligned} \tag{26}$$

Note that $\nabla((\partial^{\tau-\nu}a)/a) = \nabla\psi^{\tau-\nu}$. Note also the change of the order of summation: for any sequence $(\kappa'(\tau, \nu))$ and for any $k \in \mathbb{N}$

$$\sum_{\tau \in \Lambda_k} \sum_{\nu \leq \tau, \nu \neq \tau} \kappa'(\tau, \nu) = \sum_{\ell=0}^{k-1} \sum_{\nu \in \Lambda_\ell} \sum_{\tau \in R_{\nu, k}} \kappa'(\tau, \nu), \quad (27)$$

which implies with Lemma 4.1 and with the elementary estimate $xy \leq (x^2 + y^2)/2$, $x, y > 0$, that for any $k \geq 1$,

$$\begin{aligned} & \sum_{\tau \in \Lambda_k} \int_D \sum_{\nu \leq \tau, \nu \neq \tau} \kappa_0(\tau, \nu) \frac{\rho^\nu |\Delta \partial^\nu u| \Phi_{\bar{\beta}} \rho^\tau |\Delta \partial^\tau u| \Phi_{\bar{\beta}}}{\sqrt{\nu!} \sqrt{\tau!}} dx \\ & \leq \frac{1}{2} \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{\nu \in \Lambda_\ell} \frac{\rho^{2\nu}}{\nu!} \|\Delta \partial^\nu u\|_{L^2_{\bar{\beta}}(D)}^2 + \frac{1}{2} (e^{\sqrt{r}K} - 1) \sum_{\tau \in \Lambda_k} \frac{\rho^{2\tau}}{\tau!} \|\Delta \partial^\tau u\|_{L^2_{\bar{\beta}}(D)}^2. \end{aligned} \quad (28)$$

Similarly, we obtain with Lemma 4.4

$$\begin{aligned} & \sum_{\tau \in \Lambda_k} \frac{1}{4\varepsilon} \int_D \left(|\nabla Z| \Phi_{\bar{\beta}} \frac{\rho^\tau |\nabla \partial^\tau u|}{\sqrt{\tau!}} + \sum_{\nu \leq \tau, \nu \neq \tau} \kappa_1(\tau, \nu) \frac{\rho^\nu |\nabla \partial^\nu u|}{\sqrt{\nu!}} \right)^2 dx \\ & \leq \frac{1}{2\varepsilon} \frac{\|\nabla Z| \Phi_{\bar{\beta}}\|_{L^\infty(D)}^2}{a_{\min}} \sum_{\tau \in \Lambda_k} \frac{\rho^{2\tau}}{\tau!} \|\partial^\tau u\|_a^2 + \frac{1}{2\varepsilon} \sum_{\tau \in \Lambda_k} \int_D \left(\sum_{\nu \leq \tau, \nu \neq \tau} \kappa_1(\tau, \nu) \frac{\rho^\nu |\nabla \partial^\nu u|}{\sqrt{\nu!}} \right)^2 dx \\ & \leq \frac{1}{2\varepsilon} \frac{1}{a_{\min}} \left(\|\nabla Z| \Phi_{\bar{\beta}}\|_{L^\infty(D)}^2 \sum_{\tau \in \Lambda_k} \frac{\rho^{2\tau}}{\tau!} \|\partial^\tau u\|_a^2 + \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell-1)!} \sum_{\nu \in \Lambda_\ell} \frac{\rho^{2\nu}}{\nu!} \|\partial^\nu u\|_a^2 \right). \end{aligned} \quad (29)$$

As before by the proof of [7, Theorem 4.1 and Equation (4.18)], for any $\delta \in [\sqrt{r}K/\log(2), 1)$ and for every $\ell \in \mathbb{N}_0$,

$$\sum_{\nu \in \Lambda_\ell} \frac{\rho^{2\nu}}{\nu!} \|\partial^\nu u\|_a^2 \leq \delta^\ell \|u\|_a^2. \quad (30)$$

Hence,

$$\begin{aligned} & \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell-1)!} \sum_{\nu \in \Lambda_\ell} \frac{\rho^{2\nu}}{\nu!} \|\partial^\nu u\|_a^2 \leq \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell-1)!} \delta^\ell \|u\|_a^2 \\ & \leq \delta^k \sum_{\ell=0}^{k-1} \log(2) \frac{(\log(2))^{k-\ell-1}}{(k-\ell-1)!} \|u\|_a^2 \\ & \leq \delta^k \log(2) 2 \|u\|_a^2 = \delta^k \log(4) \|u\|_a^2. \end{aligned} \quad (31)$$

We choose $0 < \varepsilon < 1 - e^{\sqrt{r}K}/2$, which implies that $C_\varepsilon := (1 - \varepsilon - (e^{\sqrt{r}K} - 1)/2)^{-1} < 2$. This allows us to subtract $\Delta \partial^\tau u$ -terms summed over Λ_k in (26) and (28) while obtaining a constant $C_\varepsilon^{-1} > 1/2$ which is shifted to the left hand side, i.e.,

$$\begin{aligned} & \sum_{\tau \in \Lambda_k} \frac{\rho^{2\tau}}{\tau!} \|\Delta \partial^\tau u\|_{L^2_{\bar{\beta}}(D)}^2 \leq \frac{C_\varepsilon}{2\varepsilon} \frac{1}{a_{\min}} \left(\|\nabla Z| \Phi_{\bar{\beta}}\|_{L^\infty(D)}^2 + \log(4) \right) \delta^k \|u\|_a^2 \\ & \quad + \frac{C_\varepsilon}{2} \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{\nu \in \Lambda_\ell} \frac{\rho^{2\nu}}{\nu!} \|\Delta \partial^\nu u\|_{L^2_{\bar{\beta}}(D)}^2, \end{aligned}$$

where we have also inserted (29), (30) and (31). We sum over $k \geq 1$ and obtain with (23)

$$\begin{aligned} \sum_{k \geq 1} \sum_{\boldsymbol{\tau} \in \Lambda_k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta \partial^{\boldsymbol{\tau}} u\|_{L^2_{\beta}(D)}^2 &\leq \frac{C_\varepsilon}{2\varepsilon} \frac{1}{a_{\min}} \left(\|\nabla Z| \Phi_{\beta}\|_{L^\infty(D)}^2 + \log(4) \right) \frac{\delta}{1-\delta} \|u\|_a^2 \\ &\quad + \frac{C_\varepsilon}{2} (e^{\sqrt{r}K} - 1) \sum_{\ell \geq 0} \sum_{\boldsymbol{\nu} \in \Lambda_\ell} \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \|\Delta \partial^{\boldsymbol{\nu}} u\|_{L^2_{\beta}(D)}^2, \end{aligned}$$

which implies the assertion as at the end of the proof of Theorem 4.3, since $(C_\varepsilon/2)(e^{\sqrt{r}K} - 1) < 1$. \square

We remark that a related estimated to Theorem 4.5 has been derived in [6, Theorem 6.1] without taking into account the spatial weight function.

Theorem 4.6 *Let the assumption of Lemma 4.4 be satisfied for a positive sequence $(\rho_j)_{j \geq 1}$, $r \in \mathbb{N}$, and $K < C_r$. There exists a constant $C > 0$ such that for every $\mathbf{y} \in U$*

$$\begin{aligned} \sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u(\mathbf{y}) - u^{\mathcal{T}_\ell}(\mathbf{y}))\|_{a(\mathbf{y})}^2 \\ \leq C \left(\frac{(a_{\max}(\mathbf{y}))^2}{(a_{\min}(\mathbf{y}))^4} (1 + \|\nabla Z(\mathbf{y})| \Phi_{\beta}\|_{L^\infty(D)}^2) \right) \|f\|_{L^2_{\beta}(D)}^2 M_\ell^{-2/d}. \end{aligned}$$

Proof. Define the Galerkin projection $\mathcal{P}_h : V \rightarrow V_\ell$ for every $w \in V$ by

$$\int_D a \nabla(w - \mathcal{P}_h w) \cdot \nabla v dx = 0, \quad \forall v \in V_\ell.$$

Since $(\mathcal{I} - \mathcal{P}_h)v = 0$ for every $v \in V_\ell$, it holds that for every $\boldsymbol{\tau} \in \mathcal{F}$,

$$\|\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a \leq \|\mathcal{P}_h \partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a + \|(\mathcal{I} - \mathcal{P}_h) \partial^{\boldsymbol{\tau}} u\|_a. \quad (32)$$

Let $\boldsymbol{\tau} \in \mathcal{F}$ be such that $\|\boldsymbol{\tau}\|_{\ell^\infty(\mathbb{N})} \leq r$ and $|\boldsymbol{\tau}| = k$ for some $k \in \mathbb{N}$. We observe that

$$\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \int_D a |\nabla \mathcal{P}_h \partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})|^2 dx \leq \int_D \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) a \frac{\rho^{\boldsymbol{\nu}} |\nabla \partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})|}{\sqrt{\boldsymbol{\nu}!}} \frac{\rho^{\boldsymbol{\tau}} |\nabla \mathcal{P}_h \partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})|}{\sqrt{\boldsymbol{\tau}!}} dx.$$

A twofold application of the Cauchy–Schwarz inequality using that by the first estimate of Lemma 4.1 for fixed $\boldsymbol{\tau} \in \mathcal{F}$ such that $\|\boldsymbol{\tau}\|_{\ell^\infty(\mathbb{N})} \leq r$ the sequence $(\kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}))_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}}$ is a discrete probability density implies with the change of the order of summation in (27) and the second estimate in Lemma 4.1 the bound

$$\sum_{|\boldsymbol{\tau}|=k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\mathcal{P}_h \partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a^2 \leq \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{|\boldsymbol{\nu}|=\ell} \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \|\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})\|_a^2. \quad (33)$$

By the approximation property in (11), by (32), (33), the Young inequality for any $\varepsilon > 0$, and by the change of the order of summation that implied (23)

$$\begin{aligned} \sum_{k \geq 1} \sum_{|\boldsymbol{\tau}|=k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a^2 &\leq (1 + \varepsilon) \sum_{k \geq 1} \sum_{\ell=0}^{k-1} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{|\boldsymbol{\nu}|=\ell} \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \|\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})\|_a^2 \\ &\quad + \left(1 + \frac{1}{\varepsilon}\right) \sum_{k \geq 1} \sum_{|\boldsymbol{\tau}|=k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|(\mathcal{I} - \mathcal{P}_h) \partial^{\boldsymbol{\tau}} u\|_a^2 \\ &\leq (1 + \varepsilon) (e^{\sqrt{r}K} - 1) \sum_{\ell \geq 0} \sum_{|\boldsymbol{\nu}|=\ell} \frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \|\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})\|_a^2 \\ &\quad + \left(1 + \frac{1}{\varepsilon}\right) C \|a\|_{L^\infty(D)} M_\ell^{-2/d} \sum_{k \geq 1} \sum_{|\boldsymbol{\tau}|=k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta \partial^{\boldsymbol{\tau}} u\|_{L^2_{\beta}(D)}^2. \end{aligned}$$

Hence, we choose $\varepsilon < (2 - e^{\sqrt{r}K}) / (e^{\sqrt{r}K} - 1)$ and conclude with Theorem 4.5 and (15) that there exists a constant $C > 0$ such that

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a^2 \leq C \left(\frac{(a_{\max})^2}{(a_{\min})^4} (1 + \|\nabla Z|_{\Phi_{\bar{\beta}}}\|_{L^\infty(D)}^2) \right) \|f\|_{L^2_{\bar{\beta}}(D)}^2 M_\ell^{-2/d}.$$

□

Remark 4.2 *The parametric regularity estimate in Theorem 4.6 also holds if $f \in (V^*, L^2_{\bar{\beta}}(D))_{t,\infty}$ for some $t \in [0, 1]$ with the FE error bounded by an absolute constant times $M_\ell^{-2t/d}$. This can be shown by interpolation applied in the last and next to last step of the proof of Theorem 4.6, see also Remark 2.2).*

Let $G(\cdot) \in L^2_{\bar{\beta}}(D)$ denote a solution functional of interest. We are interested in the parametric regularity of $G(u - u^{\mathcal{T}_\ell})$. Introduce v_G and $v_G^{\mathcal{T}_\ell}$ to be the solution and respective FE solution to the adjoint problem with right hand side $G(\cdot)$. It holds that

$$G(u - u^{\mathcal{T}_\ell}) = \int_D a \nabla(u - u^{\mathcal{T}_\ell}) \cdot \nabla(v_G - v_G^{\mathcal{T}_\ell}) dx.$$

Proposition 4.7 *For $(\tilde{\rho})_{j \geq 1}$ defined by $\tilde{\rho}_j := \sqrt{2}\rho_j$, $j \in \mathbb{N}$, assume that $(\tilde{\rho})_{j \geq 1}$ satisfies the sparsity assumption in (21) of Lemma 4.1. Then, for every $\mathbf{y} \in U$, there holds*

$$\begin{aligned} \sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\partial^{\boldsymbol{\tau}} G(u(\mathbf{y}) - u^{\mathcal{T}_\ell}(\mathbf{y}))|^2 &\leq 4 \left(\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\tilde{\rho}^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u(\mathbf{y}) - u^{\mathcal{T}_\ell}(\mathbf{y}))\|_{a(\mathbf{y})}^2 \right) \\ &\quad \times \left(\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\tilde{\rho}^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(v_G(\mathbf{y}) - v_G^{\mathcal{T}_\ell}(\mathbf{y}))\|_{a(\mathbf{y})}^2 \right). \end{aligned}$$

Proof. We observe that for every $\boldsymbol{\tau} \in \mathcal{F}$

$$\begin{aligned} \frac{\rho^\boldsymbol{\tau}}{\sqrt{\boldsymbol{\tau}!}} \partial^{\boldsymbol{\tau}} G(u - u^{\mathcal{T}_\ell}) &= \frac{\rho^\boldsymbol{\tau}}{\sqrt{\boldsymbol{\tau}!}} \int_D \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \left[\sum_{\mathbf{m} \leq \boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\mathbf{m}} \psi^{\boldsymbol{\nu}-\mathbf{m}}(\sqrt{a} \nabla \partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell})) \right] \\ &\quad \times (\sqrt{a} \nabla \partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G - v_G^{\mathcal{T}_\ell})) dx \\ &= \int_D \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} \sqrt{\binom{\boldsymbol{\tau}}{\boldsymbol{\nu}}} \left[\sum_{\mathbf{m} \leq \boldsymbol{\nu}} \kappa_0(\boldsymbol{\nu}, \mathbf{m}) \left(\frac{\rho^{\mathbf{m}}}{\sqrt{\mathbf{m}!}} \sqrt{a} \nabla \partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell}) \right) \right] \\ &\quad \times \left(\frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{(\boldsymbol{\tau}-\boldsymbol{\nu})!}} \sqrt{a} \nabla \partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G - v_G^{\mathcal{T}_\ell}) \right) dx. \end{aligned}$$

It holds that $\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} = 2^{\boldsymbol{\tau}}$. By a twofold application of the Cauchy–Schwarz inequality

$$\begin{aligned} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\partial^{\boldsymbol{\tau}} G(u - u^{\mathcal{T}_\ell})|^2 &\leq \left(\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} \sqrt{\binom{\boldsymbol{\tau}}{\boldsymbol{\nu}}} \|\dots\|_{L^2(D)} \frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{(\boldsymbol{\tau}-\boldsymbol{\nu})!}} \|\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G - v_G^{\mathcal{T}_\ell})\|_a \right)^2 \\ &\leq 2^{\boldsymbol{\tau}} \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}} \|\dots\|_{L^2(D)}^2 \frac{\rho^{2(\boldsymbol{\tau}-\boldsymbol{\nu})}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!} \|\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G - v_G^{\mathcal{T}_\ell})\|_a^2. \end{aligned}$$

We define the sequence $(\tilde{\rho})_{j \geq 1}$ by $\tilde{\rho}_j := \sqrt{2}\rho_j$, $j \in \mathbb{N}$. By a change of the order of summation

$$\begin{aligned} & \sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\partial^{\boldsymbol{\tau}} G(u - u^{\mathcal{T}_\ell})|^2 \\ & \leq \sum_{\|\boldsymbol{\nu}\|_{\ell^\infty} \leq r} 2^{\boldsymbol{\nu}} \|\dots\|_{L^2(D)}^2 \sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r, \boldsymbol{\tau} \geq \boldsymbol{\nu}} \frac{\tilde{\rho}^{2(\boldsymbol{\tau}-\boldsymbol{\nu})}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!} \|\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G - v_G^{\mathcal{T}_\ell})\|_a^2. \end{aligned}$$

Since $\sum_{\mathbf{m} \leq \boldsymbol{\nu}} \kappa_0(\boldsymbol{\nu}, \mathbf{m}) \leq 2$ due to Lemma 4.1, by the Cauchy–Schwarz inequality and (27)

$$\begin{aligned} & \sum_{k \geq 0} \sum_{\boldsymbol{\nu} \in \Lambda_k} \int_D \left(\sum_{\mathbf{m} \leq \boldsymbol{\nu}} \kappa_0(\boldsymbol{\nu}, \mathbf{m}) \frac{\tilde{\rho}^{\mathbf{m}}}{\sqrt{\mathbf{m}!}} \sqrt{a} |\nabla \partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell})| \right)^2 dx \\ & \leq 2 \sum_{k \geq 0} \sum_{\boldsymbol{\nu} \in \Lambda_k} \sum_{\mathbf{m} \leq \boldsymbol{\nu}} \kappa_0(\boldsymbol{\nu}, \mathbf{m}) \frac{\tilde{\rho}^{2\mathbf{m}}}{\mathbf{m}!} \|\partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell})\|_a^2 \\ & \leq 2 \sum_{k \geq 0} \sum_{\ell=0}^k \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{\mathbf{m} \in \Lambda_\ell} \frac{\tilde{\rho}^{2\mathbf{m}}}{\mathbf{m}!} \|\partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell})\|_a^2 \\ & = 2 \sum_{\ell \geq 0} \sum_{k \geq \ell} \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!} \sum_{\mathbf{m} \in \Lambda_\ell} \frac{\tilde{\rho}^{2\mathbf{m}}}{\mathbf{m}!} \|\partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell})\|_a^2 \\ & \leq 4 \sum_{\|\mathbf{m}\|_{\ell^\infty} \leq r} \frac{\tilde{\rho}^{2\mathbf{m}}}{\mathbf{m}!} \|\partial^{\mathbf{m}}(u - u^{\mathcal{T}_\ell})\|_a^2, \end{aligned}$$

which proves the assertion together with the previous inequality. \square

The following theorem is directly implied by Theorem 4.6 and Proposition 4.7.

Theorem 4.8 *Let the assumption of Lemma 4.4 be satisfied for a positive sequence $(\rho_j)_{j \geq 1}$, and let $r \in \mathbb{N}$ and assume that $K < C_r/\sqrt{2}$.*

Then there exists a constant $C > 0$ such that for every $\mathbf{y} \in U$

$$\begin{aligned} & \sum_{\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} |\partial^{\boldsymbol{\tau}} G(u(\mathbf{y}) - u^{\mathcal{T}_\ell}(\mathbf{y}))|^2 \\ & \leq C \left(\frac{(a_{\max}(\mathbf{y}))^2}{(a_{\min}(\mathbf{y}))^4} (1 + \|\nabla Z(\mathbf{y})\|_{\Phi_\beta} \|L_\infty(D)\|) \right)^2 M_\ell^{-4/d} \|f\|_{L_\beta^2(D)}^2 \|G\|_{L_\beta^2(D)}^2. \end{aligned}$$

Remark 4.3 *The statement of Theorem 4.8 also holds true for the dimensionally truncated solutions u^s and u^{s, \mathcal{T}_ℓ} , for every truncation dimension $s \in \mathbb{N}$. In particular, the constant C which appears in the error bound is independent of s .*

Remark 4.4 *The parametric regularity estimate in Theorem 4.8 also holds if $f \in (V^*, L_\beta^2(D))_{t, \infty}$ and $G(\cdot) \in (V^*, L_\beta^2(D))_{t', \infty}$ for $t, t' \in [0, 1]$. Then, the FE discretization error contribution to the overall error is bounded by a constant times $M_\ell^{-2(t+t')/d}$. This follows from Remark 4.2.*

5 Multilevel QMC convergence analysis

The sequences $(b_j)_{j \geq 1}$ and $(\bar{b}_j)_{j \geq 1}$ in the assumptions in **(A1)** and **(A2)** will be the input for the QMC weight sequence $(\gamma_j)_{j \geq 1}$ of product weights. In the multilevel QMC quadrature algorithm

Q_L^* in (18), we apply a randomly shifted lattice rule on level $\ell = 0$ with respect to the QMC weight sequence

$$\gamma_j = b_j^{2p'}, \quad j \geq 1, \quad (34)$$

for some $p' \in (0, 1)$ and on the levels $\ell = 1, \dots, L$ with respect to the QMC weight sequence

$$\bar{\gamma}_j = (b_j^{1-\theta} \vee \bar{b}_j)^{2\bar{p}'}, \quad j \geq 1 \quad (35)$$

for some $p' \in (0, 1)$ and some $\theta \in (0, 1)$. Here, for $c_1, c_2 \in \mathbb{R}$, $c_1 \vee c_2 := \max\{c_1, c_2\}$.

Theorem 5.1 *For every $L \in \mathbb{N}_0$ and sequences $(s_\ell)_{\ell=0, \dots, L}$ and $(N_\ell)_{\ell=0, \dots, L}$, the ensuing error estimate holds under the following conditions:*

1. *Gaussian weight functions: $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$ and $(b_j^{1-\theta} \vee \bar{b}_j)_{j \geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for some $\bar{p} \in [p, 2)$ with $\chi = 1/(2p) + 1/4 - \varepsilon$ and $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \bar{\varepsilon}$. The QMC weight sequence in (34) is applied with $p' = p/4 + 1/2 - \varepsilon p$ on the level $\ell = 0$ for $\varepsilon \in (0, 3/4 - 1/(2p))$. The QMC weight sequence in (35) is applied with $\bar{p}' = \bar{p}/4 + 1/2 - \bar{\varepsilon}\bar{p}$ on the levels $\ell = 1, \dots, L$ for $\bar{\varepsilon} \in (0, 3/4 - 1/(2\bar{p}))$.*
2. *Exponential weight functions: $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 1]$ and for $(b_j^{1-\theta} \vee \bar{b}_j)_{j \geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for some $\bar{p} \in [p, 1]$ with $\chi = 1/p - 1/2$ and $\bar{\chi} = 1/\bar{p} - 1/2$. The QMC weight sequence in (34) is applied with $p' = 1 - p/2$ on the level $\ell = 0$. The QMC weight sequence in (35) is applied with $\bar{p}' = 1 - \bar{p}/2$ on the levels $\ell = 1, \dots, L$.*

There exists a constant $C > 0$ that is in particular independent of $(M_\ell)_{\ell \geq 0}$, $(s_\ell)_{\ell=0, \dots, L}$, $(N_\ell)_{\ell=0, \dots, L}$, and of $L \in \mathbb{N}_0$, such that

$$\begin{aligned} \sqrt{\mathbb{E}^\Delta(|\mathbb{E}(G(u)) - Q_L^*(G(u^L))|^2)} &\leq C \left(\sup_{j > s_L} \{b_j^{4-p}\} + M_L^{-4/d} + (\varphi(N_0))^{-2\chi} \right. \\ &\quad \left. + \sum_{\ell=1}^L (\varphi(N_\ell))^{-2\bar{\chi}} \left(\xi_{\ell, \ell-1} \sup_{j > s_{\ell-1}} \{b_j^{2\theta}\} + M_{\ell-1}^{-4/d} \right) \right)^{1/2}, \end{aligned}$$

where $\xi_{\ell, \ell-1} := 0$ if $s_\ell = s_{\ell-1}$ and $\xi_{\ell, \ell-1} := 1$ otherwise.

Proof. By the triangle inequality, for $\ell = 1, \dots, L$,

$$\begin{aligned} &|(I_{s_\ell} - Q_{s_\ell, N_\ell})(G(u^\ell) - G(u^{\ell-1}))| \\ &\leq |(I_{s_\ell} - Q_{s_\ell, N_\ell})(G(u^{s_\ell, \mathcal{T}_\ell}) - G(u^{s_\ell, \mathcal{T}_{\ell-1}}))| + |(I_{s_\ell} - Q_{s_\ell, N_\ell})(G(u^{s_\ell, \mathcal{T}_{\ell-1}}) - G(u^{s_{\ell-1}, \mathcal{T}_{\ell-1}}))| \end{aligned}$$

and

$$\begin{aligned} &|(I_{s_\ell} - Q_{s_\ell, N_\ell})(G(u^{s_\ell, \mathcal{T}_\ell}) - G(u^{s_\ell, \mathcal{T}_{\ell-1}}))| \\ &\leq |(I_{s_\ell} - Q_{s_\ell, N_\ell})(G(u^{s_\ell}) - G(u^{s_\ell, \mathcal{T}_\ell}))| + |(I_{s_\ell} - Q_{s_\ell, N_\ell})(G(u^{s_\ell}) - G(u^{s_\ell, \mathcal{T}_{\ell-1}}))|, \end{aligned}$$

where we recall that $u^\ell := u^{s_\ell, \mathcal{T}_\ell}$, $\ell = 0, \dots, L$. We wish to show the conditions of Theorem 3.1 for integrands $\mathbf{y} \mapsto G(u^{s_\ell}(\mathbf{y})) - G(u^{s_\ell, \mathcal{T}_\ell}(\mathbf{y}))$ and $\mathbf{y} \mapsto G(u^{s_\ell, \mathcal{T}_{\ell-1}}(\mathbf{y})) - G(u^{s_{\ell-1}, \mathcal{T}_{\ell-1}}(\mathbf{y}))$.

Setting

$$\bar{K} := \left\| \sum_{j \geq 1} \frac{\max\{|\nabla \psi_j| \Phi_{\bar{\beta}}, |\psi_j|\}}{\bar{b}_j} \right\|_{L^\infty(D)} < \infty, \quad (36)$$

the conditions of Theorem 3.1 are satisfied for the integrand $\mathbf{y} \mapsto G(u^{s_\ell}(\mathbf{y})) - G(u^{s_\ell, \mathcal{T}_\ell}(\mathbf{y}))$ with the sequence $(\bar{b}_j)_{j \geq 1}$ and $\kappa < C_r/(\sqrt{2}\bar{K})$ by Theorem 4.8 and Remark 4.3 with $r = 1$. Specifically, we apply Theorem 4.8 and Remark 4.3 with $\rho_j = \kappa/\bar{b}_j$, $j \geq 1$.

For the integrand $\mathbf{y} \mapsto G(u^{s_\ell, \mathcal{T}_{\ell-1}}(\mathbf{y})) - G(u^{s_{\ell-1}, \mathcal{T}_{\ell-1}}(\mathbf{y}))$, we apply Theorem 4.3 with $\rho_j = \kappa/b_j^{1-\theta}$, $j \geq 1$. Then, the condition of Theorem 4.3 is satisfied for $\eta = \theta/(1-\theta)$ and $\kappa < \log(2)/K$, where K is as in assumption **(A1)**. Hence, the conditions of Theorem 3.1 are satisfied for the integrand $\mathbf{y} \mapsto G(u^{s_\ell, \mathcal{T}_{\ell-1}}(\mathbf{y})) - G(u^{s_{\ell-1}, \mathcal{T}_{\ell-1}}(\mathbf{y}))$. Since the sequence $(b_j^{1-\theta} \vee \bar{b}_j)_{j \geq 1}$ dominates $(b_j^{1-\theta})_{j \geq 1}$ and $(\bar{b}_j)_{j \geq 1}$, Theorem 3.1 can be applied with $\tilde{b}_j = b_j^{1-\theta} \vee \bar{b}_j$, $j \geq 1$. For the exponential weight functions, we note that $\eta_1 = C(\max_{j > s_{\ell-1}} \{b_j^\theta\} + M_{\ell-1}^{-2/d})$ for a constant $C > 0$ (independent of ℓ), and with $\eta_2 = 8$ in the notation of the second point of Theorem 3.1.

On discretization level $\ell = 0$, the parametric integrand is $\mathbf{y} \mapsto G(u^{s_0, \mathcal{T}_0})$. The conditions of Theorem 3.1 are satisfied with $\tilde{b}_j = b_j$, $j \geq 1$ (see also [30, Theorems 11 and 13]). The assertion follows with (16) and (19). \square

Remark 5.1 *If $f \in (V^*, L^2_{\beta}(D))_{t, \infty}$ and $G(\cdot) \in (V^*, L^2_{\beta}(D))_{t', \infty}$ for some $t, t' \in [0, 1]$, then the error estimate in Theorem 5.1 also holds with an error bounded by an absolute multiple of $M_\ell^{-2(t+t')/d}$ on mesh level ℓ .*

Remark 5.2 *When the GRF Z is stationary in $D \subset \mathbb{R}^d$, the covariance kernel $k(x, x') := \mathbb{E}(Z(x)Z(x'))$ of Z depends only on $x - x'$, cp. [2]. A widely used parametric family of covariances for stationary GRF's was proposed by B. Matérn. Here, the covariance kernel depends on two parameters $\nu, \lambda > 0$, where λ is referred to as correlation length and $Z \in C^t(\bar{D})$, μ -a.s., for every positive real number $t < \nu$. Wavelet type function systems exist which allow to represent the GRF Z in terms of a sequence $(y_j)_{j \geq 1}$ of independent, standard normally distributed y_j , that satisfy Assumption **(A1)** with $b_j \sim j^{-\hat{\beta}/d}$, $j \geq 1$, for every $\hat{\beta} < \nu$, cp. e.g. [8, Corollary 4.3]. In [8], the random field Z in D is constructed by restriction of a GRF defined on suitable product domain that depends on the correlation length λ and which is a superset of D . For a constructive approach to obtain function systems of expansions with i.i.d. coefficients, we refer for example to [22] and the references there. For a discussion of the Hölder regularity and $L^q(\Omega)$ integrability of GRFs expanded in generic wavelets, we refer to [30, Section 9]. There, also if $C^t(\bar{D})$ -regularity of the respective GRF Z holds as an implication by [30, Proposition 18], the generic wavelets satisfy Assumption **(A1)** with $b_j \sim j^{-\hat{\beta}/d}$, $j \geq 1$, for every $\hat{\beta} < t$.*

Remark 5.3 *In the case of single-level QMC, also fractional Hölder regularity of the lognormal coefficient $a = \exp(Z)$ is covered by our theory in [30]. The GRF of the model function system of generic wavelets discussed in [30, Section 9] is for $d = 1$ and for wavelets that are scaled to decay as $\|\psi_j\|_{L^\infty(D)} \sim j^{-1/2-\varepsilon}$, $j \geq 1$, a member of $L^q(\Omega; C^{1/2+\varepsilon'}(\bar{D}))$, for every $q \in [1, \infty)$ and for every $\varepsilon > \varepsilon' > 0$, cp. [30, Proposition 19]. The sequence $(b_j)_{j \geq 1}$ may be chosen such that $b_j \sim j^{-1/2-\varepsilon'}$, for every $j \geq 1$ and for some $\varepsilon' \in (0, \varepsilon)$. For every $p > 2/(1 + 2\varepsilon')$, this sequence $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ is admissible with Gaussian weight functions for every $\varepsilon' > 0$, cp. [30, Theorem 11] and therefore QMC with Gaussian weight functions and product weights is applicable for every $\varepsilon > 0$. However, for $1/2 > \varepsilon > 0$, the convergence theory for QMC with product weights in [30, Theorem 13] does not seem to be applicable with exponential weight functions in this case. Numerical tests in [30, Section 11] for Z the Brownian bridge and Gaussian weight functions also show empirically convergence rates slightly larger than $1/2$. Note that the Brownian bridge is the boundary case of applicability, since there $Z \in C^{1/2-\varepsilon}(\bar{D})$ μ -a.s. for every $0 < \varepsilon \leq 1/2$ and $b_j \sim j^{-1/2}$ using the Lévy–Ciesielski decomposition of Z .*

6 Error vs. work analysis

In the estimate of Theorem 5.1, the error contributions of the QMC quadrature and the spatial approximation by FE and dimension truncation are coupled on the different levels. The numbers

of QMC points per level should minimize the error estimate and a corresponding work measure. In this manuscript, we will consider locally supported functions $(\psi_j)_{j \geq 1}$ as for example certain multiresolution analyses (MRA). Note that this will only affect the choice of the work measure for the assembly of stiffness matrices.

Let us assume that the MRA $(\psi_\lambda)_{\lambda \in \nabla}$ results from a finite number of generating (or “mother”) wavelets by scaling and translation, i.e.,

$$\psi_\lambda(x) := \psi(2^{|\lambda|}x - k), \quad k \in \nabla_{|\lambda|}, x \in D. \quad (37)$$

We use notation that is standard for MRA, i.e., the function system is indexed by $\lambda = (|\lambda|, k) \in \nabla$, where $|\lambda| \in \mathbb{N}_0$ refers to the level and $k \in \nabla_{|\lambda|}$ to the translation. The index set ∇_ℓ has cardinality $|\nabla_\ell| = \mathcal{O}(2^{d\ell})$, $\ell \in \mathbb{N}_0$. Let $j : \nabla \rightarrow \mathbb{N}$ be a suitable enumeration. The overlap on every level $|\lambda| = \ell \in \mathbb{N}_0$ is assumed to be uniformly bounded, i.e., there exists $K > 0$ such that for every $\ell \in \mathbb{N}_0$ and every $x \in D$,

$$|\{\lambda \in \nabla : |\lambda| = \ell, \psi_\lambda(x) \neq 0\}| \leq K.$$

Additionally, for constants $\sigma, \hat{\alpha} > 0$ we introduce the scaling

$$\|\psi_\lambda\|_{L^\infty(D)} \leq \sigma 2^{-\hat{\alpha}|\lambda|}, \quad \lambda \in \nabla. \quad (38)$$

Under this assumption, the work to assemble one sample of the stiffness matrix (i.e. for one QMC point) on discretization level $\ell \in \mathbb{N}_0$ scales for large ℓ as $\mathcal{O}(M_\ell |j^{-1}(s_\ell)|) = \mathcal{O}(M_\ell \log(s_\ell))$.

Proposition 6.1 *For $d = 1$, the work to solve the linear system that corresponds to (12) for one sample is $\mathcal{O}(M_\ell)$, $\ell \in \mathbb{N}_0$.*

Proof. The parametric stiffness matrix is tridiagonal and symmetric, positive definite with probability one. Therefore both, Cholesky decomposition and backsubstitution, can be performed in $\mathcal{O}(M_\ell)$ work and memory (see, e.g., [23, Chapter 4.3.6]). \square

Due to Proposition 6.1 and Remark 6.1, we stipulate availability of a PDE solver with work

$$\text{work}_{\text{PDEsolve}} = \mathcal{O}(M_\ell^{1+\eta}) \quad (\mathbf{A3})$$

for some $\eta \geq 0$ with implied constants independent of $\ell \in \mathbb{N}_0$ and, in particular, of the realization of the PDE coefficients. For $D = (0, 1)^2$ and a sparse direct solver based on *nested dissection* it is known that $\eta = 1/2$, cp. [21]. Note that $\eta = 0$ corresponds to linear complexity as is afforded by multigrid or multilevel preconditioned iterative solvers for elliptic PDEs in the deterministic setting; see, e.g., [11, 47]. The results in [28] on convergence of these methods for log-Gaussian coefficients in the $L^q(\Omega; V)$ -norm, $q \in [1, \infty)$, and $d = 2, 3$ suggest that $\eta = 0$ may not be admissible for MLQMC and $d = 2, 3$.

Remark 6.1 *The uniformity w.r. to the coefficient realizations of the work estimate (A3) is, for the presently considered log-Gaussian diffusion coefficient models, by no means to be taken for granted [28]. Since for $d = 2, 3$ stiffness matrices will not be tridiagonal, usually iterative solvers are used. In [28], strong convergence (in the $L^q(\Omega; V)$ -norm) for iterative methods is shown for every $\eta > 0$ in the general framework of [47], which is nearly optimal complexity (w.r. to the degrees of freedom) of a PDE solver. This is sufficient for single-level QMC and multilevel Monte Carlo. Applicability to multilevel QMC does not seem to be a direct consequence. In practice also direct solvers have been used for $d = 2$ with observed $\eta < 1/2$ using different sparse direct solvers than in [21], e.g., in [33, Figure 5] for $D = (0, 1)^2$, $\eta_{\text{observed}} \approx 0$ and in [29, Figure 3], $\eta_{\text{observed}} \approx 0.3$ for $D = \mathbb{S}^2$ (the two dimensional sphere).*

Under **(A3)** the model for the computational work for the multilevel QMC quadrature reads, for every $L \in \mathbb{N}_0$, as

$$\text{work}_L = \mathcal{O} \left(\sum_{\ell=0}^L s_\ell N_\ell \log(N_\ell) + \sum_{\ell=0}^L N_\ell (M_\ell \log(s_\ell) + M_\ell^{1+\eta}) \right), \quad (39)$$

where the first sum in (39) is the work of the generation of the QMC points which includes the work to obtain the generating vectors by the fast CBC construction, cp. [41, 42]. The work model in (39) depends on the choices for $(s_\ell)_{\ell=0, \dots, L}$, $(N_\ell)_{\ell=0, \dots, L}$, and $(M_\ell)_{\ell \geq 0}$, which we shall not indicate explicitly in our notation and simply write “work_L”. The second sum in (39) is the work of the evaluation of the multilevel QMC quadrature. The sequence

$$b_{j(\lambda)} = b_\lambda := c 2^{-\widehat{\beta}|\lambda|}, \quad \lambda \in \text{rev}\nabla, \quad (40)$$

together with $(\psi_\lambda)_{\lambda \in \nabla}$ defined in (37) and (38) satisfies the assumption in **(A1)** for $\max\{1, d/2\} < \widehat{\beta} < \widehat{\alpha}$ and some $c > 0$. Since $\|\nabla \psi_\lambda\|_{L^\infty(D)} \leq \sigma 2^{-(\widehat{\alpha}-1)|\lambda|} \|\nabla \psi\|_{L^\infty(D)}$, $\lambda \in \nabla$, the sequence

$$\bar{b}_j := b_j^{(\widehat{\beta}-1)/\widehat{\beta}}, \quad j \in \mathbb{N}, \quad (41)$$

and $(\psi_j)_{j \geq 1}$ defined in (37) and (38) satisfy the assumption **(A2)**. In this section we assume that

$$f \in (V^*, L^2_{\widehat{\beta}}(D))_{t, \infty} \quad \text{and} \quad G(\cdot) \in (V^*, L^2_{\widehat{\beta}}(D))_{t', \infty}, \quad t, t' \in [0, 1], \quad (\mathbf{A4})$$

and define $\tau := t + t'$. In the following, we assume that

$$M_\ell \sim 2^{d\ell}, \quad \ell \geq 0. \quad (\mathbf{A5})$$

The ensuing analysis is inspired by [34, Section 3.7] (see also [33, 18]). We will restrict the analysis to one QMC rule with respect to the QMC weight sequence (35) on all levels $\ell = 0, \dots, L$, but remark that in some cases it might be beneficial to use a second one with respect to the QMC weight sequence (34) on the level $\ell = 0$. The multilevel QMC quadrature depends on the algorithmic steering parameters $(N_\ell)_{\ell=0, \dots, L}$, $(s_\ell)_{\ell=0, \dots, L}$, $(M_\ell)_{\ell \geq 0}$, and also on $\theta \in (0, 1)$. The number of degrees of freedom $(M_\ell)_{\ell \geq 0}$ of the FE discretization in D are assumed to be given. The parameter $\theta \in (0, 1)$ is for now left arbitrary. According to the estimate in Theorem 5.1, θ can be used to balance the truncation error with the FE error on the levels $\ell = 0, \dots, L$. We will use this feature to discuss two possible strategies to choose the truncation dimensions $(s_\ell)_{\ell=0, \dots, L}$.

Strategy 1: The contributions in the QMC weight sequence in (35) are equilibrated, i.e., we choose $\theta = 1/\widehat{\beta}$, which implies that $b^{1-\theta} = \bar{b}_j$, $j \in \mathbb{N}$. The truncation dimension s_L is also chosen to equilibrate the respective truncation and FE error in the estimate of Theorem 5.1. We choose

$$s_L \sim 2^{d\lceil L\tau/\widehat{\beta} \rceil}$$

for some

$$1 < \widetilde{\beta} < 2\widehat{\beta} - \frac{d}{2} \quad (42)$$

close to $2\widehat{\beta} - d/2$, where we use that $M_\ell = \mathcal{O}(2^{d\ell})$, $\ell = 0, \dots, L$, and $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for every $p > d/\widehat{\beta}$. On the levels $\ell = 0, \dots, L-1$, we either increase s_ℓ or leave it constant. We choose

$$s_\ell \sim \min\{2^{d\lceil \tau\ell \rceil}, s_L\}, \quad \ell = 0, \dots, L-1.$$

Strategy 2: For particular $(\psi_\lambda)_{\lambda \in \nabla}$ and meshes, it may be interesting to align the level structure $(\psi_\lambda)_{\lambda \in \nabla}$ and the used FE meshes. Therefore, we choose

$$s_\ell \sim M_\ell, \quad \ell = 0, \dots, L.$$

The choice $\theta = \tau/\widehat{\beta}$ equilibrates the truncation and FE error in the estimate of Theorem 5.1 on the levels $\ell = 0, \dots, L$ assuming that $\widehat{\beta} > \tau$. Then, $(b_j^{1-\theta} \vee \bar{b}_j)_{j \geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for every $\bar{p} > d/(\min\{\widehat{\beta} - \tau, \widehat{\beta} - 1\})$.

For either of the strategies and for every $L \in \mathbb{N}_0$, by Theorem 5.1 we obtain the error estimate

$$\text{error}_L^2 = \mathcal{O} \left(M_L^{-2\tau/d} + \sum_{\ell=0}^L (\varphi(N_\ell))^{-2\bar{\chi}} M_\ell^{-2\tau/d} \right). \quad (43)$$

Since the Euler totient function satisfies that $(\varphi(N))^{-1} \leq N^{-1}(e^{\widehat{\gamma}} \log \log N + 3/\log \log(N))$ for every $N \geq 3$, where $\widehat{\gamma} \approx 0.5772$ is the Euler–Mascheroni constant, $(\varphi(N))^{-1} \leq 9/N$ for every $N = 3, \dots, 10^{40}$. We will for simplicity restrict in our analysis the range of N to $N \leq 10^{40}$ and use the bound $(\varphi(N))^{-1} \leq 9/N$. In Strategies 1 and 2, the p -summability of the sequence $(b_j^{1-\theta} \vee \bar{b}_j)_{j \geq 1}$ holds with a strict inequality condition on p , i.e., $(b_j^{1-\theta} \vee \bar{b}_j)_{j \geq 1} \in \ell^{\bar{p}}(\mathbb{N})$, for every $\bar{p} > d/(\widehat{\beta} - 1)$ in the case of Strategy 1 and for every $\bar{p} > d/\min\{\widehat{\beta} - \tau, \widehat{\beta} - 1\}$ in the case of Strategy 2. Since the QMC convergence rate $\bar{\chi}$ depends on the exponent p , there exists $\varepsilon > 0$ such that $\bar{\chi}(1 + \varepsilon)$ is also admissible in (43) due to Theorem 5.1. Using $\log(N) \leq N^\varepsilon/(\varepsilon e)$ for every $N \in \mathbb{N}$, cp. see e.g. the proof of [18, Lemma 1], the factor $N_\ell \log(N_\ell)$ in (39) may be estimated by $N_\ell^{1+\varepsilon}$. Since $N_\ell^{1+\varepsilon}$ appears then in the estimate of the work (39) and in the error estimate (43), it can be substituted by N_ℓ , using the strict inequalities in the above bounds for the admissible indices, and choosing $\varepsilon > 0$ sufficiently small.

We obtain with the choices for $(s_\ell)_{\ell=0, \dots, L}$ in Strategies 1 and 2

$$\text{work}_L = \begin{cases} \mathcal{O} \left(\sum_{\ell=0}^L N_\ell (M_\ell \log(M_\ell) + \max\{M_\ell^{1+\eta}, \min\{M_\ell^\tau, M_L^{\tau/\widehat{\beta}}\}\}) \right), & \text{for Strategy 1,} \\ \mathcal{O} \left(\sum_{\ell=0}^L N_\ell (M_\ell \log(M_\ell) + M_\ell^{1+\eta}) \right), & \text{for Strategy 2.} \end{cases}$$

and

$$\text{error}_L^2 = \mathcal{O} \left(M_L^{-2\tau/d} + \sum_{\ell=0}^L N_\ell^{-2\bar{\chi}} M_\ell^{-2\tau/d} \right).$$

We will distinguish between the cases that $\eta = 0$ and $\eta > 0$ in **(A3)**. We treat Strategy 2 and the case $\eta > 0$ first. As above, $\log(M) \leq M^\eta/(\eta e)$ for every $M \in \mathbb{N}$. To obtain optimal choices for the sample numbers $(N_\ell)_{\ell=0, \dots, L}$, we search for a stationary point of the function

$$g(\xi) := M_L^{-2\tau/d} + \sum_{\ell=0}^L N_\ell^{-2\bar{\chi}} M_\ell^{-2\tau/d} + \xi \sum_{\ell=0}^L N_\ell M_\ell^{1+\eta}$$

with respect to N_ℓ , i.e., we solve the first order necessary condition $\partial g/\partial N_\ell = 0$ (see also [34, Section 3.7]). This gives

$$N_\ell = \left\lceil N_0 M_\ell^{-(2\tau/d+1+\eta)/(1+2\bar{\chi})} \right\rceil, \quad \ell = 1, \dots, L, \quad (44)$$

and with setting $E_\ell = M_\ell^{(1+\eta-\tau/(d\bar{\chi}))2\bar{\chi}/(1+2\bar{\chi})}$, $\ell = 0, \dots, L$,

$$\text{error}_L^2 = \mathcal{O} \left(M_L^{-2\tau/d} + N_0^{-2\bar{\chi}} \sum_{\ell=0}^L E_\ell \right) \quad \text{and} \quad \text{work} = \mathcal{O} \left(N_0 \sum_{\ell=0}^L E_\ell \right), \quad (45)$$

where

$$\sum_{\ell=0}^L E_\ell = \begin{cases} \mathcal{O}(1) & \text{if } 1 + \eta < \tau/(d\bar{\chi}), \\ \mathcal{O}(L) & \text{if } 1 + \eta = \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{(2\bar{\chi}d(1+\eta)-2\tau)L/(1+2\bar{\chi})}) & \text{if } 1 + \eta > \tau/(d\bar{\chi}). \end{cases} \quad (46)$$

The parameter N_0 is chosen to balance the error contributions, i.e., $N_0^{-2\bar{\chi}} \sum_{\ell=0}^L E_\ell = \mathcal{O}(M_L^{-2\tau/d})$, which implies

$$N_0 = \begin{cases} \lceil 2^{\tau L/\bar{\chi}} \rceil & \text{if } 1 + \eta < \tau/(d\bar{\chi}), \\ \lceil 2^{\tau L/\bar{\chi}} L^{1/(2\bar{\chi})} \rceil & \text{if } 1 + \eta = \tau/(d\bar{\chi}), \\ \lceil 2^{(2\tau+d(1+\eta))L/(1+2\bar{\chi})} \rceil & \text{if } 1 + \eta > \tau/(d\bar{\chi}). \end{cases} \quad (47)$$

We conclude that error $_L^2 = \mathcal{O}(M_L^{-2\tau/d})$ can be achieved with

$$\text{work}_L = \begin{cases} \mathcal{O}(2^{\tau L/\bar{\chi}}) & \text{if } 1 + \eta < \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{\tau L/\bar{\chi}} L^{(1+2\bar{\chi})/(2\bar{\chi})}) & \text{if } 1 + \eta = \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{dL(1+\eta)}) & \text{if } 1 + \eta > \tau/(d\bar{\chi}). \end{cases}$$

In the case that $\eta = 0$, the resulting work measure is considered in [34, Section 3.7]. In particular, we obtain by [34, Equations (74) and (77)]

$$N_\ell = \left\lceil N_0 \left(M_\ell^{-1-2\tau/d} \log(s_\ell)^{-1} \right)^{1/(1+2\bar{\chi})} \right\rceil, \quad \ell = 1, \dots, L, \quad (48)$$

and

$$N_0 = \begin{cases} \lceil 2^{\tau L/\bar{\chi}} \rceil & \text{if } d < \tau/\bar{\chi}, \\ \lceil 2^{\tau L/\bar{\chi}} L^{(1+4\bar{\chi})/(\bar{\chi}(2+4\bar{\chi}))} \rceil & \text{if } d = \tau/\bar{\chi}, \\ \lceil 2^{(d+2\tau)L/(1+2\bar{\chi})} L^{1/(1+2\bar{\chi})} \rceil & \text{if } d > \tau/\bar{\chi}. \end{cases} \quad (49)$$

Note that the corresponding work estimates are given on [34, p. 443]. We summarize this analysis as ε -complexity bounds in the following theorem.

Theorem 6.2 [Error vs. work for Strategy 2]

Let the truncation dimensions $(s_\ell)_{\ell=0,\dots,L}$ be chosen according to Strategy 2 assuming $\widehat{\beta} > \max\{\tau, 1\}$. Let the assumptions **(A5)** and **(A3)** be satisfied for $\eta \geq 0$. If $\eta > 0$, the sample numbers for $Q_L^*(\cdot)$ are given by (47) and (44), $L \in \mathbb{N}_0$. If $\eta = 0$, the sample numbers for $Q_L^*(\cdot)$ are given by (49) and (48), $L \in \mathbb{N}_0$. Let f and $G(\cdot)$ satisfy **(A4)**.

1. *Gaussian weight functions:* for $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-\tau), d/(\widehat{\beta}-1)\}, 2)$, $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \varepsilon'$ for $\varepsilon' > 0$ sufficiently small assuming $d/\min\{\widehat{\beta} - \tau, \widehat{\beta} - 1\} < 2$.
2. *Exponential weight functions:* for $\bar{p} \in (\max\{2/3, d/(\widehat{\beta} - \tau), d/(\widehat{\beta} - 1)\}, 1]$, $\bar{\chi} = 1/\bar{p} - 1/2$ assuming $d/\min\{\widehat{\beta} - \tau, \widehat{\beta} - 1\} < 1$.

For an error threshold $1 > \varepsilon > 0$, we obtain

$$\sqrt{\mathbb{E}^\Delta(|\mathbb{E}(G(u)) - Q_L^*(G(u^L))|^2)} = \mathcal{O}(\varepsilon)$$

is achieved with

$$\text{work}_L = \begin{cases} \mathcal{O}(\varepsilon^{-1/\bar{\chi}}) & \text{if } 1 + \eta < \tau/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+2\bar{\chi})/(2\bar{\chi})}) & \text{if } 1 + \eta = \tau/(d\bar{\chi}), \eta > 0, \\ \mathcal{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+4\bar{\chi})/(2\bar{\chi})}) & \text{if } d = \tau/\bar{\chi}, \eta = 0, \\ \mathcal{O}(\varepsilon^{-d/\tau(1+\eta)}) & \text{if } 1 + \eta > \tau/(d\bar{\chi}), \eta > 0, \\ \mathcal{O}(\varepsilon^{-d/\tau} \log(\varepsilon^{-1})) & \text{if } d > \tau/\bar{\chi}, \eta = 0. \end{cases}$$

Here, the implied constants are independent of L , $(s_\ell)_{\ell=0,\dots,L}$, $(N_\ell)_{\ell=0,\dots,L}$, and of $(M_\ell)_{\ell \geq 0}$.

Remark 6.2 *In Strategy 2, there is one parameter respectively one dimension of integration, per spatial degree of freedom, so that $s_\ell \sim M_\ell$, $\ell \geq 0$. This coupling occurs, for example, when circulant embedding is applied to evaluate a GRF on uniformly spaced spatial grid points such that each element of the FE mesh contains at least one of these points to perform a one point quadrature for computing the stiffness matrix. Numerical experiments with a QMC rule using a circulant embedding are presented in [25] and the references there.*

For Strategy 1, we may restrict the analysis to the case $\tau > 1$, since for $\tau \leq 1$ the additional restriction $\tilde{\beta} > \tau$ for Strategy 2 is redundant and Strategy 2 can be applied. For $\eta > 0$, we obtain following the same line of argument as applied in the analysis of Strategy 2

$$N_\ell = \left\lceil N_0 \left(M_\ell^{2\tau/d} \max\{M_\ell^{1+\eta}, \min\{M_\ell^\tau, M_L^{\tau/\tilde{\beta}}\}\} \right)^{-1/(1+2\tilde{\chi})} \right\rceil, \quad \ell = 1, \dots, L, \quad (50)$$

where also (45) holds with

$$E_\ell = \left(M_\ell^{-\tau/(d\tilde{\chi})} \max\{M_\ell^{1+\eta}, \min\{M_\ell^\tau, M_L^{\tau/\tilde{\beta}}\}\} \right)^{2\tilde{\chi}/(1+2\tilde{\chi})}, \quad \ell = 0, \dots, L.$$

We observe that

$$\begin{aligned} & \sum_{\ell=0}^L \left(M_\ell^{-\tau/(d\tilde{\chi})} \max\{M_\ell^{1+\eta}, M_L^{\tau/\tilde{\beta}}\} \right)^{2\tilde{\chi}/(1+2\tilde{\chi})} \\ &= \begin{cases} \mathcal{O}(2^{dL(\tau/\tilde{\beta})2\tilde{\chi}/(1+2\tilde{\chi})}) & \text{if } 1 + \eta \leq \tau/(d\tilde{\chi}), \\ \mathcal{O}(2^{dL \max\{1+\eta-\tau/(d\tilde{\chi}), \tau/\tilde{\beta}\}2\tilde{\chi}/(1+2\tilde{\chi})}) & \text{if } 1 + \eta > \tau/(d\tilde{\chi}), \end{cases} \end{aligned}$$

where we used that $\max\{x, y\} \leq x + y$ for every $x, y \in [0, \infty)$. The respective estimate for the sum over $(M_\ell^{-\tau/(d\tilde{\chi})} \max\{M_\ell^{1+\eta}, M_L^{\tau/\tilde{\beta}}\})^{2\tilde{\chi}/(1+2\tilde{\chi})}$ is given in (46) with $\max\{1 + \eta, \tau\}$ in place of $1 + \eta$ (also in the conditions of the three cases). To estimate $\sum_{\ell=0}^L E_\ell$, we use the identity that $\max\{x, \min\{y, z\}\} = \min\{\max\{x, y\}, \max\{x, z\}\}$ for every $x, y, z \in \mathbb{R}$, and apply the superadditivity of the minimum to obtain that

$$\sum_{\ell=0}^L E_\ell = \begin{cases} \mathcal{O}(1) & \text{if } \max\{\tau, 1 + \eta\} < \tau/(d\tilde{\chi}), \\ \mathcal{O}(L) & \text{if } \max\{\tau, 1 + \eta\} = \tau/(d\tilde{\chi}), \\ \mathcal{O}(2^{dL(1+\eta-\tau/(d\tilde{\chi}))2\tilde{\chi}/(1+2\tilde{\chi})}) & \text{if } 1 + \eta \geq \tau, 1 + \eta > \tau/(d\tilde{\chi}), \\ \mathcal{O}(2^{dL \min\{\tau-\tau/(d\tilde{\chi}), \max\{(1+\eta-\tau/(d\tilde{\chi}), \tau/\tilde{\beta})\}\}2\tilde{\chi}/(1+2\tilde{\chi})}) & \text{if } 1 + \eta < \tau, 1 < d\tilde{\chi}. \end{cases}$$

As above, N_0 is chosen to balance the error, i.e., $N_0 \sim M_L^{\tau/(d\tilde{\chi})} (\sum_{\ell=0}^L E_\ell)^{1/(2\tilde{\chi})}$. Specifically,

$$N_0 = \begin{cases} \lceil 2^{L\tau/\tilde{\chi}} \rceil & \text{if } \max\{\tau, 1 + \eta\} < \tau/(d\tilde{\chi}), \\ \lceil 2^{L\tau/\tilde{\chi}} L^{1/(2\tilde{\chi})} \rceil & \text{if } \max\{\tau, 1 + \eta\} = \tau/(d\tilde{\chi}), \\ \lceil 2^{(2\tau+d(1+\eta))L/(1+2\tilde{\chi})} \rceil & \text{if } 1 + \eta \geq \tau, 1 + \eta > \tau/(d\tilde{\chi}), \\ \lceil 2^{dL \min\{\tau-\tau/(d\tilde{\chi}), \max\{(1+\eta-\tau/(d\tilde{\chi}), \tau/\tilde{\beta})\}\}/(1+2\tilde{\chi}) + L\tau/\tilde{\chi}} \rceil & \text{if } 1 + \eta < \tau, 1 < d\tilde{\chi}. \end{cases} \quad (51)$$

For $\eta = 0$,

$$N_\ell = \left\lceil N_0 \left(M_\ell^{2\tau/d} \max\{M_\ell \log(M_\ell), \min\{M_\ell^\tau, M_L^{\tau/\tilde{\beta}}\}\} \right)^{-1/(1+2\tilde{\chi})} \right\rceil, \quad \ell = 1, \dots, L, \quad (52)$$

and $E_\ell = (M_\ell^{-\tau/(d\bar{\chi})} \max\{M_\ell \log(M_\ell), \min\{M_\ell^\tau, M_L^{\tau/\tilde{\beta}}\}\})^{-1/(1+2\bar{\chi})}$, $\ell = 0, \dots, L$. We obtain similarly using $\tau > 1$,

$$\sum_{\ell=0}^L E_\ell = \begin{cases} \mathcal{O}(1) & \text{if } 1 < 1/(d\bar{\chi}), \\ \mathcal{O}(L) & \text{if } 1 = 1/(d\bar{\chi}), \\ \mathcal{O}(2^{(d-\tau/\bar{\chi})L2\bar{\chi}/(1+2\bar{\chi})} L^{2\bar{\chi}/(1+2\bar{\chi})}) & \text{if } d - \tau/\bar{\chi} \geq d\tau/\tilde{\beta}, \\ \mathcal{O}(2^{d(\tau/\tilde{\beta})L2\bar{\chi}/(1+2\bar{\chi})}) & \text{if } d - \tau/\bar{\chi} < d\tau/\tilde{\beta} < \tau(d - 1/\bar{\chi}), \\ \mathcal{O}(2^{\tau(d-1/\bar{\chi})L2\bar{\chi}/(1+2\bar{\chi})}) & \text{if } d - \tau/\bar{\chi} < d\tau/\tilde{\beta}, \tau(d - 1/\bar{\chi}) \leq d\tau/\tilde{\beta}. \end{cases}$$

Again by $N_0 \sim M_L^{\tau/(d\bar{\chi})} (\sum_{\ell=0}^L E_\ell)^{1/(2\bar{\chi})}$,

$$N_0 = \begin{cases} \lceil 2^{L\tau/\bar{\chi}} \rceil & \text{if } 1 < 1/(d\bar{\chi}), \\ \lceil 2^{L\tau/\bar{\chi}} L^{1/(2\bar{\chi})} \rceil & \text{if } 1 = 1/(d\bar{\chi}), \\ \lceil 2^{(2\tau+d)L/(1+2\bar{\chi})} \rceil & \text{if } d - \tau/\bar{\chi} \geq d\tau/\tilde{\beta}, \\ \lceil 2^{(\tau/\bar{\chi} + d\tau/(\tilde{\beta}(1+2\bar{\chi})))L} \rceil & \text{if } d - \tau/\bar{\chi} < d\tau/\tilde{\beta} < \tau(d - 1/\bar{\chi}), \\ \lceil 2^{(2\tau+d\tau)L/(1+2\bar{\chi})} \rceil & \text{if } d - \tau/\bar{\chi} < d\tau/\tilde{\beta}, \tau(d - 1/\bar{\chi}) \leq d\tau/\tilde{\beta}. \end{cases} \quad (53)$$

Explicit error vs. work estimates are summarized as ε -complexity bounds in the following theorem, where we recall that work = $N_0 \sum_{\ell=0}^L E_\ell = M_L^{\tau/(d\bar{\chi})} (\sum_{\ell=0}^L E_\ell)^{(1+2\bar{\chi})/(2\bar{\chi})}$.

Theorem 6.3 [Error vs. work for Strategy 1]

Let the truncation dimension $(s_\ell)_{\ell \geq 1}$ be chosen according to Strategy 1 assuming $\hat{\beta} > 1$ and $\tau > 1$. Let the assumptions **(A5)** and **(A3)** be satisfied for $\eta \geq 0$. The sample numbers for $Q_L^*(\cdot)$ are given by (51) and (50) for $\eta > 0$ and by (53) and (52) for $\eta = 0$, $L \in \mathbb{N}_0$. Let f and $G(\cdot)$ satisfy **(A4)**.

1. Gaussian weight functions: for $\bar{p} \in (\max\{2/3, d/(\hat{\beta} - 1)\}, 2)$, $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \varepsilon'$ for $\varepsilon' > 0$ sufficiently small assuming $d/(\hat{\beta} - 1) < 2$.
2. Exponential weight functions: for $\bar{p} \in (\max\{2/3, d/(\hat{\beta} - 1)\}, 1]$, $\bar{\chi} = 1/\bar{p} - 1/2$ assuming $d/(\hat{\beta} - 1) < 1$.

For an error threshold $\varepsilon > 0$, we obtain

$$\sqrt{\mathbb{E}^\Delta(|\mathbb{E}(G(u)) - Q_L^*(G(u^L))|^2)} = \mathcal{O}(\varepsilon)$$

is achieved with

$$\text{work} = \begin{cases} \mathcal{O}(\varepsilon^{-1/\bar{\chi}}) & \text{if } \max\{\tau, 1 + \eta\} < \tau/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+2\bar{\chi})/(2\bar{\chi})}) & \text{if } \max\{\tau, 1 + \eta\} = \tau/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-d/\tau(1+\eta)}) & \text{if } 1 + \eta \geq \tau, 1 + \eta > \tau/(d\bar{\chi}), \eta > 0, \\ & \text{or } d - \tau/\bar{\chi} \geq d\tau/\tilde{\beta}, \eta = 0, \\ \mathcal{O}(\varepsilon^{-d \min\{1, \max\{(1+\eta)/\tau, 1/\tilde{\beta} + 1/(d\bar{\chi})\}\}}) & \text{if } 1 + \eta < \tau, 1 < d\bar{\chi}, \eta > 0, \\ \mathcal{O}(\varepsilon^{-(1/\bar{\chi} + 1/\tilde{\beta})}) & \text{if } d - \tau/\bar{\chi} < d\tau/\tilde{\beta} < \tau(d - 1/\bar{\chi}), \eta = 0, \\ \mathcal{O}(\varepsilon^{-d}) & \text{if } d - \tau/\bar{\chi} < d\tau/\tilde{\beta}, \tau(d - 1/\bar{\chi}) \leq d\tau/\tilde{\beta}, \eta = 0. \end{cases}$$

Here, $\tilde{\beta}$ is as in (42) chosen close to $0 < \hat{\beta} - d/2$ such that $\tilde{\beta} < 2\hat{\beta} - d/2$ and all implied constants are independent of L , $(s_\ell)_{\ell=0, \dots, L}$, $(N_\ell)_{\ell=0, \dots, L}$, and $(M_\ell)_{\ell \geq 0}$.

Remark 6.3 In the case that Z is a stationary GRF and has a Matérn covariance with smoothness parameter $\nu > 1$, see Remark 5.2, a function system exists such that the assumption **(A1)** is satisfied with $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for every $p > d/\nu$. The statement of [8, Corollary 4.3] maybe extended to gradients of the function system, which would imply that the assumption **(A2)** could be satisfied with $(\bar{b}_j)_{j \geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for every $\bar{p} > d/(\nu - 1)$. Error vs. work estimates achieved by MLQMC with product weights then hold according to Theorems 6.2 and 6.3. We note that Strategy 2 is applicable provided that $\nu > \tau$, where $\tau \in [0, 2]$ is the approximation rate of the considered FEs.

7 Numerical experiments

We illustrate the complexity estimates and algorithmic details on GRF generation in locally supported representation systems with numerical tests. To this end, we consider the following class of Gaussian random fields. We admit Gaussian random fields Z which are pathwise, weak solutions to the SPDE (1), where \mathcal{W} denotes spacial white noise on D . See e.g. [2] for details on this. In (1), we assume that $A(x) \in \mathbb{R}^{d \times d}$ is symmetric for a.e. $x \in D$ and there exists $\bar{A} > 0$ such that

$$\operatorname{ess\,inf}_{x \in D} \xi^\top A(x) \xi \geq \bar{A} \xi^\top \xi, \quad \forall \xi \in \mathbb{R}^d,$$

and $\operatorname{ess\,inf}_{x \in D} \kappa(x) > 0$.

We recall from Section 1 that if $D = \mathbb{R}^d$, $A(x) \equiv \operatorname{Id}$, and if also $\kappa(x) \equiv \operatorname{const}$, then the stationary solution Z to (1) is well known to have Matérn covariance. As proposed in [37], the SPDE (1) can be used to define and numerically sample non-stationary GRFs in bounded domains and for general coefficients κ and A , which accomodates non-stationary Gaussian random fields Z . As in [37], both in stationary and non-stationary cases (see [37, Section 3.2]), we shall refer to solutions to (1) as *Matérn fields*.

We choose $D = (0, 1)$ with periodic boundary conditions, which can be identified with the one-dimensional sphere or the one-dimensional torus \mathbb{T}^1 . To obtain a series expansion of Z with i.i.d. standard normally distributed coefficients, i.e., the form of (3), we discretize (1) by biorthogonal and continuous, piecewise linear spline prewavelets as in [44]. Let $(V_\ell)_{\ell \geq 0}$ be a sequence of FE spaces of piecewise affine functions on uniformly refined meshes with meshwidth $h_\ell = 2^{-\ell-2}$ and $\dim(V_\ell) = 2^{\ell+2}$. Each FE space is spanned by the usual hat functions, i.e., $V_\ell = \operatorname{span}\{\varphi_1^\ell, \dots, \varphi_{2^{\ell+2}}^\ell\}$, $\ell \geq 0$. We shall use the following representation system for the Gaussian random field in D : for every $\ell \in \mathbb{N}$, define the spline-prewavelets as in [44, Equation (4.7)] by

$$\phi_{\ell,k} := \sum_{\iota=1}^5 a_\iota \varphi_{2k-4+\iota}^\ell, \quad k = 1, \dots, N_\ell, a = (1/2, -3, 5, -3, 1/2), \quad (54)$$

where $N_\ell = 2^{\ell+1}$ and negative indices of φ_k^ℓ are taken modulo $2^{\ell+2}$. We define the wavelet spaces $W^\ell := \operatorname{span}\{\phi_{\ell,1}, \dots, \phi_{\ell,N_\ell}\}$, $\ell \geq 1$, with the understanding that $W^0 := V_0$. For $\ell = 0$, we define $\phi_{\ell,k} = \varphi_k^\ell$, $k = 1, \dots, N_0 := 4$. These spaces are L^2 -orthogonal across levels, i.e., $\int_D w_1 w_2 dx = 0$ for all $w_1 \in W^{\ell_1}$, $w_2 \in W^{\ell_2}$ s.t. $\ell_1, \ell_2 \in \mathbb{N}_0$ and $\ell_1 \neq \ell_2$. Hence, we obtain the multilevel splitting

$$V^\ell = W^0 \oplus W^1 \oplus \dots \oplus W^\ell, \quad \ell \geq 1.$$

We note that $\{\phi_{\ell,k} : \ell \geq 0, k = 1, \dots, N_\ell\}$ is a Riesz basis of $L^2(\mathbb{T}^1)$ and of $H^1(\mathbb{T}^1)$. Upon proper scaling, there hold stable norm equivalences in scale of spaces $H^t(\mathbb{T}^1)$ for $t \in [0, 3/2)$, cp. [44, Proposition 4.1]. There are constants C_1, C_2 such that for every $L \geq 0$ and for every

$$v = \sum_{\ell=0}^L \sum_{k=1}^{N_\ell} v_{\ell,k} \phi_{\ell,k}$$

$$C_1 \sum_{\ell=0}^L 2^{2(t-1/2)\ell} \sum_{j=1}^{N_\ell} |v_{\ell,j}|^2 \leq \|v\|_{H^t(\mathbb{T}^1)}^2 \leq C_2 \sum_{\ell=0}^L 2^{2(t-1/2)\ell} \sum_{j=1}^{N_\ell} |v_{\ell,j}|^2.$$

Let us define the sequence space $\ell_1^2 := \{\mathbf{c} \in \mathbb{R}^{\mathbb{N}} : \sum_{\ell \geq 0} 2^\ell \sum_{k=1}^{N_\ell} |\mathbf{c}_{\ell,k}|^2 < \infty\}$ that corresponds to $H^1(\mathbb{T}^1)$. Note that when the white noise \mathcal{W} is applied to prewavelets a random vector \mathbf{W} results, which is normally distributed with zero mean and has covariance, which is determined by the mass operator \mathbf{M} , i.e., $\text{cov}(\mathcal{W}(\phi_{\ell_2,k_2}), \mathcal{W}(\phi_{\ell_1,k_1})) = \int_D \phi_{\ell_2,k_2} \phi_{\ell_1,k_1} dx =: \mathbf{M}_{\ell_1,k_1,\ell_2,k_2}$, $\ell_1, \ell_2 \geq 0$, $k_1 \in \{1, \dots, N_{\ell_1}\}$, $k_2 \in \{1, \dots, N_{\ell_2}\}$. Due to the orthogonality of the prewavelets across levels the bi-infinite mass matrix is block diagonal with the mass matrices of the spaces W^ℓ on the diagonal. This means there is no correlation between the different levels. This fact will be convenient in sampling realizations of \mathbf{W} with a block Cholesky algorithm; its complexity is $\mathcal{O}(N_\ell)$ on every block. Note that the random vectors \mathbf{W} and $\mathbf{L}\mathbf{y}$ have the same distribution for an i.i.d. standard normally distributed \mathbf{y} and any operator \mathbf{L} that satisfies $\mathbf{L}\mathbf{L}^\top = \mathbf{M}$. For example, \mathbf{L} can be the operator that results by applying the Cholesky algorithm to every block of \mathbf{M} . For $\alpha = 2$, the operator in (1) is local and reads $(-\text{div}(A(x)\nabla) + \kappa^2(x))$. It can be equivalently represented as a bi-infinite matrix in a prewavelet basis. We denote the resulting (bi-infinite) matrix by \mathbf{A} , where $\mathbf{A}_{\ell_1,k_1,\ell_2,k_2} = \int_D (\nabla \phi_{\ell_2,k_2})^\top A \nabla \phi_{\ell_1,k_1} + \kappa^2 \phi_{\ell_2,k_2} \phi_{\ell_1,k_1} dx$, $\ell_1, \ell_2 \geq 0$, $k_1 \in \{1, \dots, N_{\ell_1}\}$, $k_2 \in \{1, \dots, N_{\ell_2}\}$.

For $\alpha = 2$, the variational formulation of (1) with respect to this prewavelet basis is: for a given parameter vector \mathbf{y} , find $\mathbf{Z}(\mathbf{y}) \in \ell_1^2$ such that

$$\mathbf{A}\mathbf{Z}(\mathbf{y}) = \mathbf{L}\mathbf{y},$$

where the parametric coefficients $\mathbf{Z}(\mathbf{y})$ and the Gaussian random field Z are related by

$$Z(\mathbf{y}) = \sum_{\ell \geq 0} \sum_{j=1}^{N_\ell} \mathbf{Z}(\mathbf{y})_{\ell,j} \phi_{\ell,j}$$

Let us define $\mathbf{e}(\ell, k)$ by $\mathbf{e}(\ell, k)_{\ell',k'} := 1$ if $\ell = \ell'$ and $k = k'$ and zero otherwise. A piecewise linear multiresolution representation of the Matérn field Z with i.i.d. coefficients and a function system $(\psi_{\ell,k}^{\alpha/2})_{\ell \geq 0, k=1, \dots, N_\ell}$ (here $\alpha/2 = 1$) is now obtained by

$$Z(\mathbf{y}) = \sum_{\ell \geq 0} \sum_{k=1}^{N_\ell} y_{\ell,k} \psi_{\ell,k}^1, \quad \text{with} \quad \psi_{\ell,k}^1 = \sum_{\ell' \geq 0} \sum_{k'=1}^{N_{\ell'}} (\mathbf{A}^{-1} \mathbf{L} \mathbf{e}(\ell, k))_{\ell',k'} \phi_{\ell',k'}. \quad (55)$$

For $\alpha > 2$, let $\{\alpha/2\}$ be the fractional part of $\alpha/2$ and $\lfloor \alpha/2 \rfloor = \alpha/2 - \{\alpha/2\}$. The SPDE (1) is rewritten recursively as

$$\begin{aligned} (-\text{div}(A(x)\nabla) + \kappa^2(x))^{\{\alpha/2\}} Z &= Z_0 \\ (-\text{div}(A(x)\nabla) + \kappa^2(x)) Z_i &= Z_{i-1}, \quad i = 1, \dots, \lfloor \alpha/2 \rfloor, \end{aligned} \quad (56)$$

where $Z_0 := \mathcal{W}$. If $\{\alpha/2\} = 0$, $(-\text{div}(A(x)\nabla) + \kappa^2(x))^0$ is understood as the identity operator. For $\alpha \in 2\mathbb{N}$, there is no fractional PDE to be solved in (56) and by standard Galerkin techniques

$$\psi_{\ell,k}^{\alpha/2} = \sum_{\ell' \geq 0} \sum_{k'=1}^{N_{\ell'}} ((\mathbf{A}^{-1} \mathbf{M})^{\alpha/2-1} \mathbf{A}^{-1} \mathbf{L} \mathbf{e}(\ell, k))_{\ell',k'} \phi_{\ell',k'}, \quad \ell \geq 0, k = 1, \dots, N_\ell. \quad (57)$$

Generally, for $\alpha \geq 2$,

$$\psi_{\ell,k}^{\alpha/2} = \mathbf{A}_{\{\alpha/2\}}^{-1} \mathbf{M} \psi_{\ell,k}^{\lfloor \alpha/2 \rfloor} \quad \ell \geq 0, k = 1, \dots, N_\ell,$$

where $\mathbf{A}_{\{\alpha/2\}}$ is the wavelet representation of the fractional operator $(-\operatorname{div}(A(x)\nabla) + \kappa^2(x))^{\{\alpha/2\}}$. In the case that $\{\alpha/2\} > 0$, a fractional PDE with a non-local operator needs to be discretized. Efficient FE methods for this have been recently analyzed in [9]. We remark that the application of first order prewavelets is sufficient here, since the convergence rate of randomly shifted lattice rules is limited by one and also first order FE is considered to discretize (2) in space.

Remark 7.1 *The application of the sparse operator \mathbf{L} introduces a weight sequence $(2^{-(1/2)^\ell})_{\ell \geq 0, j=1, \dots, N_\ell}$ and the application of the inverse of \mathbf{A} introduces an additional weight sequence $(2^{-\ell})_{\ell \geq 0, k=1, \dots, N_\ell}$ on the parameter vector \mathbf{y} or $e(\ell, j)$. The theory of pseudodifferential operators and wavelet compression suggests that $(\psi_{\ell,k}^{\alpha/2})_{\ell \geq 0, k=1, \dots, N_\ell}$ satisfies Assumption (A1) and Assumption (A2) with $b_{\ell,k} = 2^{-\hat{\beta}\ell}$ and $\bar{b}_{\ell,k} = 2^{-(\hat{\beta}-1)\ell}$, $\ell \geq 1$, $k = 1, \dots, N_\ell$, for all $1 < \hat{\beta} < \alpha - 1/2$. See ahead Figures 1(a) and 1(b) for an illustration of this property. In practical implementations the infinite series of the $\psi_{\ell,k}^{\alpha/2}$'s in (55) needs to be truncated. A detailed analysis of these aspects in a more general setting will be presented elsewhere.*

For a scaling parameter $\theta > 0$ to be specified, the parametric PDE (2) with log-Matérn input $a = \exp(\theta Z)$ and right hand side $f \in L^2(\mathbb{T}^1)$ such that $\int_{\mathbb{T}^1} f dx = 0$ is discretized by the FE spaces $(V_\ell)_{\ell \geq 0}$, which are spanned by the standard hat functions. We recall the variational formulation: for all $\ell \geq 0$, find $u^{h_\ell} \in V_\ell$ such that

$$\int_{\mathbb{T}^1} a (u^{h_\ell})' v' dx = \int_{\mathbb{T}^1} f v dx, \quad \forall v \in V_\ell, \quad \text{and} \quad \int_{\mathbb{T}^1} u^{h_\ell} dx = 0. \quad (58)$$

Due to periodicity, no essential boundary conditions enter the variational formulation (58). The vanishing mean condition on the solution is sufficient to ensure well-posedness, since the kernel of the precision operator comprises exactly the constant functions.

We will adopt Strategy 2 from Section 6, which means that on every discretization level $\ell \geq 0$ we set $s_\ell = M_\ell$. We also truncate the expansion of the $\psi_{\ell,k}^1$'s in (55) and (57) and the infinite matrices \mathbf{A} to M_ℓ terms. Note that the bi-infinite matrices \mathbf{M} and \mathbf{L} do not need to be truncated as they are block-diagonal. The work to compute $Z^{s_\ell}(\mathbf{y})$ for a given \mathbf{y} in a continuous, piecewise linear representation is $\mathcal{O}(M_\ell \log(M_\ell))$, since it amounts to the solution of a PDE discretized by prewavelets and the application of the Cholesky algorithm to sparse band matrices. The work for the approximate solution of the PDE discretized with the hat functions on mesh-level ℓ scales as $\mathcal{O}(M_\ell)$.

In our numerical tests, $D = (0, 1)$, $f(x) = \sin(2\pi x)$ and

$$\kappa^2(x) = \bar{\kappa}^2 \left(1 + \frac{1}{2} \sin(2\pi x) \right), \quad A(x) = \operatorname{Id}, \quad \alpha \in \{2, 4\}.$$

The parameter ν such that $\alpha = \nu + d/2$ determines the path regularity of realizations of the Matérn field. In this sense, we test the cases $\nu \in \{3/2, 7/2\}$. The correlation length scale parameter λ is defined by

$$\lambda := \frac{2\sqrt{\nu}}{\bar{\kappa}}.$$

For $\kappa(x) \equiv \bar{\kappa}$ this corresponds to the correlation length parameter used in numerical experiments in [33, Section 4]. We aim at testing for different values of λ without greatly affecting the variance of Z . Thus, we set the scaling θ of the (non-stationary) Gaussian random field Z to

$$\theta = \frac{\sigma_0}{\sigma(\alpha, \bar{\kappa})}$$

with $\sigma_0 > 0$ still at our disposal. In the stationary (“Matern”) case, i.e., when $A(x) \equiv \text{Id}$ and $\kappa(x) \equiv \bar{\kappa} > 0$, elementary Fourier analysis reveals that the marginal variance is given by

$$\sigma^2(\alpha, \bar{\kappa}) := \frac{1}{\bar{\kappa}^{2\alpha}} + \sum_{i \geq 1} \frac{2}{((2\pi i)^2 + \bar{\kappa}^2)^\alpha}.$$

The function systems $(\psi_{\ell,k}^{\alpha/2} : \ell \geq 0, k = 1, \dots, N_\ell)$ are well localized and satisfy a decay condition which suits our MLQMC analysis with product weights. Let us illustrate this numerically. To this end, we define

$$b_{\text{ref}} := (2^{-(3/2)\ell})_{\ell \geq 0, k=1, \dots, N_\ell}.$$

In Figure 1(a), we plot this reference sequence and $\|\psi_{\ell,j}^1\|_{L^\infty(\mathbb{T}^1)}$ for several choices of the correlation length $\lambda > 0$. Figure 1(b) shows a plot of $\psi_{\ell,k}^1$. For the illustrations in both figures, the expansion in (55) has been truncated to the maximal level $L = 11$. For $\alpha = 2$, Figures 1(a) and 1(b) suggest that $(\psi_{\ell,k}^1)_{\ell \geq 0, k=1, \dots, N_\ell}$ satisfies Assumption **(A1)** and Assumption **(A2)** with weight sequence $b_{\text{ref}}^{1-\varepsilon}$ for every $\varepsilon > 0$.

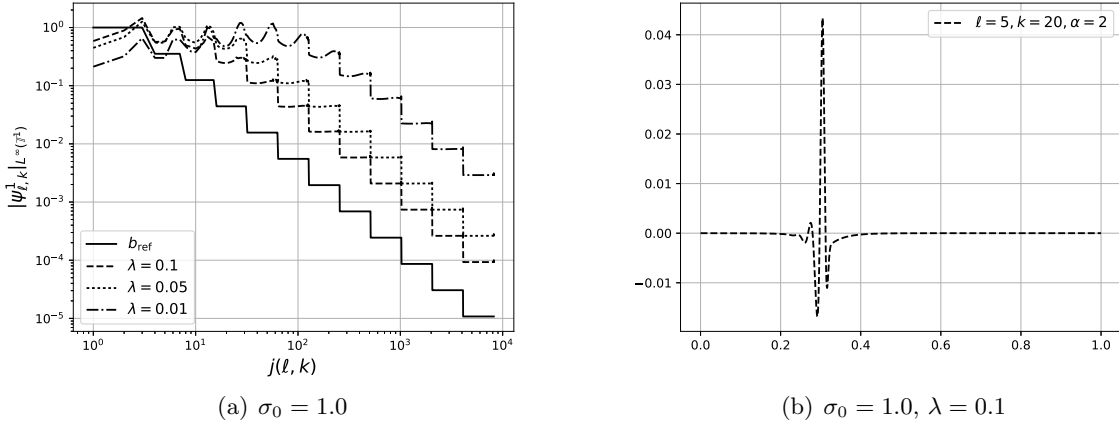


Figure 1: Quantitative properties of $(\psi_{\ell,k}^{\alpha/2})_{\ell \geq 0, k=1, \dots, N_\ell}$ for $\sigma_0 = 1.0$, $\lambda \in \{0.1, 0.05, 0.01\}$, $\alpha = 2$ ($\nu = 3/2$). On the left: decay of $\|\psi_{\ell,k}^1\|_{L^\infty(\mathbb{T}^1)}$. On the right: localization of $\psi_{\ell,k}^1$.

Remark 7.2 For stationary Gaussian random fields, the theory from [24, 33] with globally supported Karhunen-Loève basis functions and QMC integration with POD weights is applicable; in particular, in the special case that $A(x) \equiv \text{Id}$ and $\kappa(x) \equiv \text{const}$, and that periodic boundary conditions are imposed on ∂D . Then, the Gaussian random field Z is stationary, and the SPDE (1) can be numerically solved with Fourier methods, as the precision operator is a power of a shifted Laplacean which is diagonalized by the Fourier system. We refer to [26, 15] for details on this. Unlike the product weights for QMC integration which were derived in the present work, the appearance of QMC weights with POD structure in [24, 33] implies that the construction cost for these QMC integration methods scales as $O(s^2 N + s N \log(N))$ [40].

In our numerical tests, we consider the functional $G(v) := v(x_0)$ with $x_0 = 0.7$, which is not a node in any of our FE meshes for all levels $\ell \geq 0$. Note that $G(\cdot) \in H^{-1/2-\varepsilon}(\mathbb{T}^1)$ for every $\varepsilon > 0$. Since the QMC rate $\bar{\chi}$ is restricted to $[1/2, 1)$ the complexity estimate in Theorem 6.2 in the regime $1 < \tau/\bar{\chi}$ (here $d = 1$, $\eta = 0$) does not seem to benefit from $\tau > 1$. Thus, QMC sample numbers are chosen according to (48) and (49) with $\tau = 1$ and $\bar{\chi}$ to be specified, i.e.,

$$N_\ell = \left\lceil 2^{(\tau/\bar{\chi})(L+1)} \left(2^{-(2\tau+1)(\ell+1)} (\ell+1)^{-1} \right)^{1/(2\bar{\chi}+1)} \right\rceil, \quad \ell = 1, \dots, L, \quad N_0 = \left\lceil 2^{(\tau/\bar{\chi})(L+1)} \right\rceil \quad (59)$$

They are rounded up to the next odd prime number. We compute the generating vectors by the fast CBC algorithm, Gaussian weight functions, and product weights according to Theorem 3.1 and the sequence

$$\bar{b}_{\text{ref}}^{\alpha/2} = (c2^{-(\nu-1)\ell})_{\ell \geq 0, k=1, \dots, N_\ell}$$

for some $c > 0$ (see Remark 7.1 and Figure 1(a)). According to Theorem 3.1, also the value p' is needed for the product weights, which are an input in the CBC algorithm to construct the QMC points. For $\alpha = 2$, we use the boundary values $p' = 1$ and $\varepsilon = 0$ in Theorem 3.1. We note that $\nu = 3/2$ is the boundary case of applicability of our MLQMC convergence theory from Theorem 6.2. We expect a convergence rate $\bar{\chi} \approx 1/2$. In our single-level QMC experiments, we observed in [30, Figure 1] that the QMC rate for a boundary case of applicability (i.e. $(b_j)_{j \geq 1} \notin \ell^p(\mathbb{N})$ for every $p \in (0, 2)$, but $(b_j)_{j \geq 1} \in \ell^2(\mathbb{N})$) was always larger than 0.65. So for $\alpha = 2$ ($\nu = 3/2$), we use QMC sample numbers (59) with $\bar{\chi} = 0.65$. For $\alpha = 4$, $\bar{b}_{\text{ref}}^{\alpha/2} \in \ell^{2/3}(\mathbb{N})$. Thus, we use $p' = 2/3 = p$ and the boundary value $\varepsilon = 0$ in Theorem 3.1. For $\alpha = 4$, the sample numbers are chosen with $\bar{\chi} = 0.9$.

In Figures 2(a) and 2(b), we visualize error vs. work for $\alpha = 2$, i.e., $\nu = 3/2$, $L = 2, \dots, 11$ with reference solution on level $L_{\text{ref}} = 12$. For $L = 11$, there are $s_L = 8192$ dimensions on the finest level and for the reference solution the highest occurring dimension is $s_{L_{\text{ref}}} = 16384$. The asymptotic work model is here and in the following taken according to Theorem 6.2 as

$$\text{work}_L = 2^{(\tau/\bar{\chi})L}.$$

In all numerical tests, the mean square error is approximated by the empirical variance of R samples Q_j corresponding to R i.i.d. realizations of the random shift, with the unbiased estimator

$$\sqrt{\frac{1}{R-1} \sum_{j=1}^R (Q_j - \bar{Q})^2} \approx \sqrt{\mathbb{E}^\Delta(|\mathbb{E}(G(u)) - Q_L^*(G(u^L))|^2)}.$$

The reference value \bar{Q} is the average over R i.i.d. random shifts of $Q_{L_{\text{ref}}}^*(G(u^{L_{\text{ref}}}))$ with $L_{\text{ref}} = 12$. In all numerical tests we use $R = 20$. For $\alpha = 2$, this is the boundary case $p = 2$ of Theorem 6.2 and we expect a convergence rate of the error as a function of the work of $\approx 1/2$. The empirically observed rate is a least squares fit taking into account the four data pairs corresponding to finer resolution. The MLQMC algorithms converge even for very small correlation length $\lambda > 0$, which is presented in Figures 2(a) and 2(b) for two choices of σ_0 . Specifically, we observe that for correlation length $\lambda \in \{0.1, 0.05, 0.01\}$, there seems to be a pre-asymptotic regime until the (non-dimensional) correlation length λ can be resolved by the FE discretization in D .

In Figure 3(a), we study the error versus the variance of the Matérn field Z . We control this variance by the parameter σ_0 , and monitor the convergence rate of the error as a function of the work. The test is carried out for $\alpha = 4$, i.e., $\nu = 7/2$, fixed correlation length $\lambda = 0.1$, and $L = 2, \dots, 11$ with reference solution on level $L_{\text{ref}} = 12$. Thus, $s_{11} = 8192$ dimensions on the highest considered level and $s_{L_{\text{ref}}} = 16384$ dimensions of the reference solution. The empirically observed rate is a least squares fit taking into account the six data pairs corresponding to finer resolution. We observe that the convergence rate seems to be influenced by the variance of Z , the size of the fluctuations. This was also observed in previous numerical experiments in [31, 30, 33, 24].

All numerical tests in Figures 2(a) and 2(b) were performed with generating vectors that do not depend on the correlation length. However, Figure 1(a) suggests that the pre-asymptotic spacial decay of the functions $\psi_{\ell,k}^1$ depends on the correlation length $\lambda > 0$, which shall be reflected in the decay of the respective QMC integration weight sequence $(\bar{b}_{\ell,k})_{\ell \geq 0, k=1, \dots, N_\ell}$. In Figure 3(b), we present numerical results with generating vectors that are *informed* by the

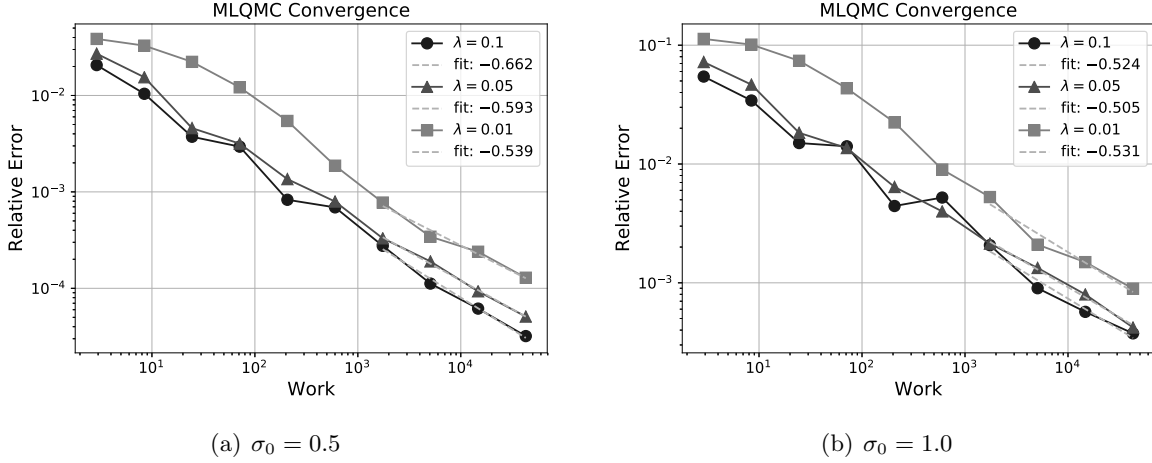


Figure 2: MLQMC convergence with $\nu = 3/2$, $\alpha_g = 1.05$, $c = 0.1$, $\lambda \in \{0.1, 0.05, 0.01\}$. On the left: $\sigma_0 = 0.5$. On the right: $\sigma_0 = 1.0$.

particular pre-asymptotic decay. The test is carried out for $\alpha = 2$, i.e., $\nu = 3/2$, $\sigma_0 = 1.0$, and $L = 2, \dots, 9$ with reference solution on level $L_{\text{ref}} = 10$. The empirically observed rate is a least squares fit taking into account the four data pairs corresponding to finer resolution. Specifically, we compute the generating vectors with the sequence $(\bar{b}_{\ell,k})_{\ell \geq 0, k=1, \dots, N_\ell}$, which sequence is given by $\bar{b}_{\ell,k} = cb_{\ell,k}^{(\hat{\beta}-1)/\hat{\beta}}$ (according to (41)) for $\hat{\beta} = 3/2$ and

$$b_{\ell,k} = \|\psi_{\ell,k}\|_{L^\infty(D)}, \quad \ell \geq 0, k = 1, \dots, N_\ell.$$

In Figure 3(b), we observe a very similar behavior compared to Figure 2(b). We thus conclude that the observed pre-asymptotic regime is indeed due to the inability of the FE method to resolve the small correlation length until the MLQMC algorithm contains levels with sufficiently fine meshes.

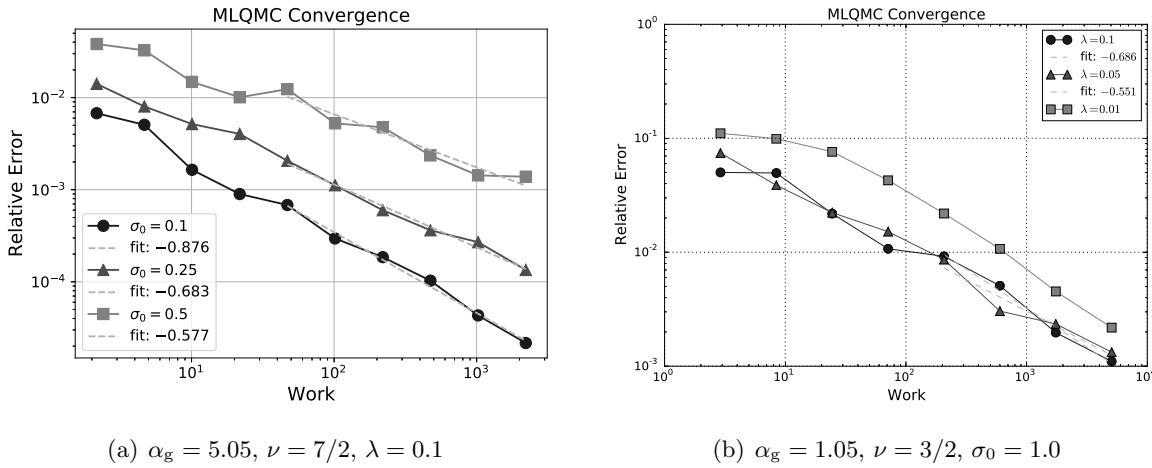


Figure 3: MLQMC convergence with $\nu \in \{3/2, 7/2\}$ and $c = 0.1$. On the left: $\nu = 7/2$, $\lambda = 0.1$, and $\sigma_0 \in \{0.1, 0.25, 0.5\}$. On the right: $\nu = 3/2$, $\sigma_0 = 1.0$, and $\lambda \in \{0.1, 0.05, 0.01\}$ with *informed* QMC generating vectors.

8 Conclusions

For linear, second order diffusion equations (2) in a polygonal or polyhedral domain D , and with diffusion coefficient $a = \exp(Z)$, where the Gaussian random field Z in D is represented in terms of a series expansion with “localized supports”, taking values in weighted Hölder spaces in D , we extended the convergence rate and error versus work analysis of combined QMC quadratures and multilevel FE approximation from [24, 33] in several directions. We considered randomly shifted lattice QMC rules introduced in [40] for numerical integration of PDE outputs against a dimension-truncated Gaussian measure. The present work extends previous results to possibly non-stationary Gaussian random fields Z , accounts explicitly for a discretization-level dependent truncation of the representation of the Gaussian random field, and accounts for possibly low path regularity of Z and of the random solution u in a polytopal physical domain D . In particular, Z and u are admitted in weighted Hölder and Sobolev spaces in the polytope $D \subset \mathbb{R}^d$, the weights allowing singularities in realizations of Z and of u due to corners (and edges in dimension $d = 3$) of D . This allows, in particular, Gaussian random fields Z whose covariance is non-stationary, with associated *Matérn SPDE* in D with Dirichlet or Neumann boundary conditions, as proposed recently in [37]. Whereas in [24, 33, 34], globally (in D) supported ϕ_j 's were admitted (implying QMC quadratures with so-called “POD” weights), the present analysis shows that for multilevel representation systems $(\psi_j)_{j \geq 1}$ with localized supports, QMC quadratures with product weights are admissible and, in a sense, natural. We also provided a novel QMC error analysis with Gaussian weighted function spaces for QMC integration on the unbounded integration domain. It extends the range of summability exponents from $p \in (0, 1]$ obtained by exponential weight function as considered in [24, 33] to $p \in (2/3, 2)$ (Theorem 3.1, item 1.), while still retaining dimension independent convergence rate up to $1/(2p) + 1/4$.

We proved that for Gaussian random fields Z whose spatial variation is parametrized by a representation system $(\psi_j)_{j \geq 1}$ of functions $\psi_j(x)$ defined in D with “localized supports” that QMC combined with continuous, piecewise linear FE in D on families of regular, simplicial triangulations of D with suitable mesh refinement near vertices and (in space dimension $d = 3$) edges of D , allow for parameter-dimension independent error vs. work bounds. The case of full elliptic regularity in function spaces in D without spatial weights and $s_\ell = s_L$, $\ell = 0, \dots, L$, is considered in [33, Corollary 2 and Section 5]. In the present work, we admitted bounded, polytopal domains D where ∂D consists of straight lines (in space dimension $d = 2$) or of plane faces (in space dimension $d = 3$) which require weighted spaces for the spatial coordinate, and we consider level dependent truncations s_ℓ of the input GRF. In particular, the level dependent truncations s_ℓ of the GRF allow in certain cases an ε -complexity of MLQMC-FE with product weights for elliptic PDEs with log-Gaussian coefficients, which is asymptotically equivalent to the ε -complexity of QMC in the case that integrand evaluation would be available at unit cost. The parametric regularity results in weighted function spaces in D remain valid, however, also for polytopal D with piecewise smoothly curved boundaries as considered in [39]. Since the assumed localization of the supports of the ψ_j in D was shown to allow for QMC integration rules with so-called product weights, the present model of the computational work (39) includes the cost of the generation of the QMC points. This cost is dominated by the cost of the fast CBC construction of generating vectors. It was considered a pre-computation in [34, 33] and the (quadratic w.r. to the parameter dimensions s_ℓ) work count for the (precomputed) CBC construction was omitted from the work counts in that reference. We also note that the same generating vectors can be used for different right hand sides f . However, if the function system $(\psi_j)_{j \geq 1}$ is altered, e.g., by changing certain parameters of a considered class of $(\psi_j)_{j \geq 1}$ that relate to the smoothness or spatial correlation, due to modeling considerations of the lognormal diffusion coefficient, then the generating vectors need in general to be recomputed. The QMC error analysis being based on product weights, the work of the fast CBC construction of generating vectors due to R.

Cools and D. Nuyens [41] and the generation of QMC points scales linearly with respect to the parameter dimensions s_ℓ . Therefore, we conclude in certain cases the same asymptotic error vs. work bounds of the considered multilevel QMC algorithm than in the case of the respective deterministic, elliptic PDE; also in the case that the FE error has a rate higher than $1/d$ with respect to the dimension of the FE spaces. We considered only homogeneous Dirichlet boundary conditions on all of ∂D in (2) and, in the numerical experiments section, only even integer order precision operators. This was for ease of exposition only: the parametric regularity analysis of Section 4 and the elliptic regularity results in Section 2.2 remain valid verbatim for problems with Neumann or mixed boundary conditions. In particular, an analogous structure of the corner- and edge-weights in (8) can be used to characterize elliptic regularity shifts in scales of weighted Sobolev- and Hölder spaces in D for these boundary conditions. Precision operators of fractional and odd integer order in (1), i.e. when $\alpha \in \mathbb{R} \setminus (2\mathbb{N})$, can be treated in exactly the same fashion, using recently developed methods for the efficient numerical solution of the SPDE (1). We refer to [9] and the references there for details of the corresponding algorithms.

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